1997

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Recommended Citation

Adams, W.W., Loustaunau, P., Palamodov, VP., & Struppa, D.C. (1997). Hartog's phenomenon for polyregular functions and projective dimension of related modules over a polynomial ring. *Annales de l'Institut Fourier*, 47(2), 623-640. Retrieved from http://aif.cedram.org/aif-bin/item?id=AIF_1997__47_2_623_0

# HARTOG'S PHENOMENON FOR POLYREGULAR FUNCTIONS AND PROJECTIVE DIMENSION OF RELATED MODULES OVER A POLYNOMIAL RING

W.W. ADAMS, P. LOUSTAUNAU, V.P. PALAMODOV, D.C. STRUPPA

ABSTRACT. In this paper we prove that the projective dimension of $\mathcal{M}_n = R^4/\langle A_n\rangle$ is $2n-1$, where $R$ is the ring of polynomials in $4n$ variables with complex coefficients, and $\langle A_n\rangle$ is the module generated by the columns of a $4\times 4n$ matrix which arises as the Fourier transform of the matrix of differential operators associated with the regularity condition for a function of $n$ quaternionic variables. As a corollary we show that the sheaf $\mathcal{R}$ of regular functions has flabby dimension $2n-1$, and we prove a cohomology vanishing theorem for open sets in the space $\mathrm{H}^n$ of quaternions. We also show that $\mathrm{Ext}^j(\mathcal{M}_n, R) = 0$, for $j = 1, \ldots, 2n-2$ and $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R) \neq 0$, and we use this result to show the removability of certain singularities of the Cauchy–Fueter system.

RÉSUMÉ. Soit $R$ l'anneau des polynomes de $4n$ variables. Soit $A_n$ la transformation de Fourier de la matrice d'opérateurs différentiels associée à la condition de régularité imposée à une fonction de $n$ variables quaterniones. Soit aussi $\langle A_n\rangle$ le module défini par les colonnes de $A_n$. Dans cet article nous prouvons que la dimension projective du module $\mathcal{M}_n = R^4/\langle A_n\rangle$ est $2n-1$. Nous prouvons ensuite, dans un corollaire, que la dimension flasque du faisceau $\mathcal{R}$ des fonctions régulières est $2n-1$, et nous prouvons que certains groups de cohomologie sont zéro pour les ouverts de l'espace $\mathrm{H}^n$ de quaternions. Nous prouvons que $\mathrm{Ext}^j(\mathcal{M}_n, R) = 0$, pour $j = 1, \ldots, 2n-2$ et que $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R) \neq 0$, et nous utilisons ce résultat pour prouver que certaines singularités du system de Cauchy-Fueter peuvent être éliminées.

## 1. INTRODUCTION

In a recent paper, [1], the authors have studied the Cauchy–Fueter system with the purpose of analyzing the singularities of regular functions of several quaternionic variables. We now recall the basic set-up of our problem. Let $\boldsymbol{f} = (f_0, f_1, f_2, f_3)$ be a vector whose components are $\mathcal{C}^\infty$ functions in $4n$ real variables ( $\xi_{i0}, \xi_{i1}, \xi_{i2}, \xi_{i3}$;

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*Date*: December 11, 1995.

$i = 1, \dots, n$). We say that $\boldsymbol{f}$ is *left regular* if,

$$
\left\{
\begin{aligned}
&\frac{\partial f_0}{\partial \xi_{i0}} - \frac{\partial f_1}{\partial \xi_{i1}} - \frac{\partial f_2}{\partial \xi_{i2}} - \frac{\partial f_3}{\partial \xi_{i3}} = 0\\
&\frac{\partial f_0}{\partial \xi_{i1}} + \frac{\partial f_1}{\partial \xi_{i0}} - \frac{\partial f_2}{\partial \xi_{i3}} + \frac{\partial f_3}{\partial \xi_{i2}} = 0\\
&\frac{\partial f_0}{\partial \xi_{i2}} + \frac{\partial f_1}{\partial \xi_{i3}} + \frac{\partial f_2}{\partial \xi_{i0}} - \frac{\partial f_3}{\partial \xi_{i1}} = 0\\
&\frac{\partial f_0}{\partial \xi_{i3}} - \frac{\partial f_1}{\partial \xi_{i2}} + \frac{\partial f_2}{\partial \xi_{i1}} + \frac{\partial f_3}{\partial \xi_{i0}} = 0
\end{aligned}
\right. \tag{1}
$$

for $i = 1, \dots, n$. We can view $\boldsymbol{f}$ as a function $\boldsymbol{f} : \mathrm{H}^n \longrightarrow \mathrm{H}$, where H is the space of quaternions. If we let $\boldsymbol{q} = (q_1, \dots, q_n)$ be the variable in $\mathrm{H}^n$, then Condition (1) is equivalent to

$$\frac{\partial \boldsymbol{f}}{\partial \bar{q}_i} = 0, \qquad i = 1, \dots, n. \tag{2}$$

By taking the Fourier transform of the matrix of differential operators associated to Equation (1), one is led to consider the matrix

$$A_n = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix},$$

where

$$U_i = \begin{bmatrix} x_{i0} & x_{i1} & x_{i2} & x_{i3} \\ -x_{i1} & x_{i0} & x_{i3} & -x_{i2} \\ -x_{i2} & -x_{i3} & x_{i0} & x_{i1} \\ -x_{i3} & x_{i2} & -x_{i1} & x_{i0} \end{bmatrix},$$

for $i = 1, \dots, n$, and where the variables $x_{ij}$ are the dual variables of the variables $\xi_{ij}$.

Let $R = \mathrm{C}[x_{i0}, x_{i1}, x_{i2}, x_{i3} \mid i = 1, \dots, n]$ and let $\wp_n$ be the maximal ideal of $R$ generated by the $4n$ variables. Given a matrix $A$, we denote by $\langle A \rangle$ the $R$-module generated by the columns of $A$. In [1] we showed, among other things, that

$$\mathrm{pd}(R^4/\langle A_1 \rangle) = 1, \quad \text{and} \quad \mathrm{pd}(R^4/\langle A_2 \rangle) = 3,$$

where $\mathrm{pd}(M)$ denotes the projective dimension of an $R$-module $M$. In this paper we prove that, for every $n \geq 1$,

$$\mathrm{pd}(R^4/\langle A_n \rangle) = 2n - 1,$$

$$\mathrm{Ext}^j(R^4/\langle A_n \rangle, R) = 0, 0 \leq j \leq 2n-2, \text{ and } \mathrm{Ext}^{2n-1}(R^4/\langle A_n \rangle, R) \neq 0.$$

In Section 3 we show how this result has interesting and unexpected consequences for the theory of regular functions. In particular we prove that if $\mathcal{R}$ is the sheaf of regular functions, then its flabby dimension, $\mathrm{fl.dim}(\mathcal{R})$, is $2n-1$. In particular this shows that if $U$ is any open set in $\mathrm{H}^n$ and $p \geq 2n-1$, then $H^p(U, \mathcal{R}) = 0$. This result is a quaternionic version of the famous result of Malgrange for holomorphic functions (see [7]) and we do not see how it could have been proved by purely analytic methods. We also give some results on the removability of singularities of the Cauchy–Fueter system.

We note that we can compute the projective dimension of $R^4/\langle A_n\rangle$ for $n = 1, 2, 3$ using the software CoCoA[1] which gives the explicit minimal free resolutions of $R^4/\langle A_n\rangle$ for $n = 1, 2, 3$. In the present paper we give an algebraic proof of the equality $\mathrm{pd}(R^4/\langle A_n\rangle) = 2n-1$ using techniques from commutative and computational algebra, in particular Gröbner bases. For any particular $n$ CoCoA could be used, in principle, to compute a minimal resolution of $R^4/\langle A_n\rangle$, however, running CoCoA on a Sparc 10, we were able to compute only the cases $n = 1, 2, 3$ (the machine crashed at $n = 4$, and the file of the 4 matrices which define the free resolution for $n = 3$ is 128kbytes!)

## 2. Projective Dimension of $R^4/\langle A_n\rangle$

In this section we compute the projective dimension of the $R$-module $\mathcal{M}_n = R^4/\langle A_n\rangle$. Since nothing is changed in the proofs below until we get to Theorem 2.6 we will assume until then that $R$ is the polynomial ring in the given variables over any field $k$.

If $n = 1$, it is straightforward to see that the syzygy module of $A_1$ is zero, and so $\mathrm{pd}(\mathcal{M}_1) = 1$. ¿From now on we will assume that $n > 1$.

We will use the Auslander-Buchsbaum formula (see, for example, [4, Theorem 19.9 and Exercise 19.8])

$$\mathrm{pd}(\mathcal{M}_n) = \mathrm{depth}(\wp_n, R) - \mathrm{depth}(\wp_n, \mathcal{M}_n).$$

We recall that, for an ideal $I$ of $R$ and an $R$-module $M$, the *depth of $I$ on $M$*, denoted $\mathrm{depth}(I, M)$, is the length of any maximal $M$-regular sequence in $I$. The polynomials $f_1, \ldots, f_s \in I$ form an *$M$-regular sequence* if

1. $f_\nu$ is a non-zerodivisor on $M/\langle f_1, \ldots, f_{\nu-1}\rangle M$, for $\nu = 1, \ldots, s$;
2. $M \neq \langle f_1, \ldots, f_s\rangle M$.

See, for example, [4] for a thorough development of the notion of depth. Clearly, $\mathrm{depth}(\wp_n, R) = 4n$, so we only need to compute $\mathrm{depth}(\wp_n, \mathcal{M}_n)$. To do this we will exhibit a maximal $\mathcal{M}_n$-regular sequence in $\wp_n$.

This will be accomplished using the theory of Gröbner bases (see, for example, [2] for a detailed presentation of Gröbner bases). Related ideas were used in [8, ch. II, Section 2.4]. We first need a Gröbner basis for $\langle A_n\rangle$ . We use the degree reverse lexicographic (degrevlex) term ordering on $R$ with

$$x_{10} > x_{20} > \cdots > x_{n0} > x_{11} > \cdots > x_{n1} > x_{12} > \cdots > x_{n3}, \tag{3}$$

and the TOP (TOP stands for term over position) ordering on $R^4$ with $\boldsymbol{e}_1 > \boldsymbol{e}_2 > \boldsymbol{e}_3 > \boldsymbol{e}_4$, where $\boldsymbol{e}_i$ is the $i$th column of the $4 \times 4$ identity matrix. That is, for monomials $X = x_{10}^{\alpha_{10}} \cdots x_{n3}^{\alpha_{n3}}$ and $Y = x_{10}^{\beta_{10}} \cdots x_{n3}^{\beta_{n3}}$, we have

$$X\boldsymbol{e}_r > Y\boldsymbol{e}_s \Longleftrightarrow \begin{cases} \deg(X) = \displaystyle\sum_{\substack{i=1,\ldots,n\\ j=0,1,2,3}} \alpha_{ij} > \deg(Y) = \displaystyle\sum_{\substack{i=1,\ldots,n\\ j=0,1,2,3}} \beta_{ij} & \text{or} \\ \deg(X) = \deg(Y) \text{ and } \alpha_{ij} < \beta_{ij} \text{ for the index } ij, & \\ \qquad \text{last with respect to (3), such that } \alpha_{ij} \neq \beta_{ij} & \text{or} \\ X = Y \text{ and } r < s. & \end{cases}$$

[1] CoCoA is a special purpose system for doing computations in commutative algebra. It is the ongoing product of a research team in Computer Algebra at the University of Genova, Italy. It is freely available, and more information can be obtained by sending an e-mail message to `cocoa@dima.unige.it`.

**Lemma 2.1.** *The reduced Gröbner basis for the $R$-module $\langle A_n\rangle$ is given by the columns of $A_n$ together with the columns of the $\binom{n}{2}$ matrices $U_rU_s-U_sU_r$. Moreover the module generated by the leading terms of all the elements of $\langle A_n\rangle$, denoted $\mathrm{Lt}(A_n)$, is*

$$\mathrm{Lt}(A_n)=\langle x_{i0}\boldsymbol{e}_\ell, x_{r2}x_{s1}\boldsymbol{e}_\ell\rangle_{\substack{i=1,\ldots,n\\ 1\le r<s\le n\\ \ell=1,2,3,4}}.$$

*Proof.* It is easy to verify the statement for $n=2,3$, and 4 using CoCoA. Let $n>4$. The S-polynomial of any two columns of $A_n$ can be computed and reduced as in the case $n=2$, and so the S-polynomials generated by the columns of $A_n$ give rise to the vectors in the columns of the matrices $U_rU_s-U_sU_r$. To verify that the columns of $A_n$ together with the columns of all distinct $U_rU_s-U_sU_r$ form the reduced Gröbner basis of $\langle A_n\rangle$, we need to verify that all the S-polynomials generated by these vectors reduce to zero. An S-polynomial generated by a column of $A_n$ and a column of $U_rU_s-U_sU_r$ is computed and reduced as in the case $n=2$ or 3, depending on whether the column of $A_n$ comes from $U_r$, $U_s$, or neither. An S-polynomial generated by two columns of $U_rU_s-U_sU_r$ is computed and reduced as in the case $n=2$. An S-polynomial generated by a column of $U_rU_s-U_sU_r$ and a column of $U_tU_u-U_uU_t$ is computed and reduced as in the case $n=3$ or 4, depending on whether one or none of the indices $r,s$, and $t,u$ is the same.

For the statement about $\mathrm{Lt}(A_n)$, we first note that, for $1\le r<s\le n$, $U_rU_s-U_sU_r=$

$$\begin{bmatrix} 0 & -x_{r3}x_{s2}+x_{r2}x_{s3} & -x_{r3}x_{s1}+x_{r1}x_{s3} & x_{r2}x_{s1}-x_{r1}x_{s2}\\ x_{r3}x_{s2}-x_{r2}x_{s3} & 0 & x_{r2}x_{s1}-x_{r1}x_{s2} & x_{r3}x_{s1}-x_{r1}x_{s3}\\ -x_{r3}x_{s1}+x_{r1}x_{s3} & x_{r2}x_{s1}-x_{r1}x_{s2} & 0 & x_{r3}x_{s2}-x_{r2}x_{s3}\\ x_{r2}x_{s1}-x_{r1}x_{s2} & x_{r3}x_{s1}-x_{r1}x_{s3} & -x_{r3}x_{s2}+x_{r2}x_{s3} & 0\end{bmatrix}.$$

The result then follows immediately from the definition of the term ordering. □

This result allows us to start an $\mathcal{M}_n$-regular sequence in $\wp_n$.

**Corollary 2.2.** *The variables $x_{11},x_{n2},x_{i3}, i=1,\ldots,n$ form an $\mathcal{M}_n$-regular sequence of length $n+2$.*

*Proof.* We note that the variables $x_{11},x_{n2},x_{i3}, i=1,\ldots,n$ are precisely the variables which do not appear in any of the leading terms of the elements of the reduced Gröbner basis of $\langle A_n\rangle$ given in Lemma 2.1. In general, if $D$ is a submodule of $R^4$ and a variable $x_{ij}$ does not appear in any of the leading terms in a Gröbner basis for $D$, then $x_{ij}$ is a non-zero divisor on $R^4/D$. This is because if $\boldsymbol{0}\neq\boldsymbol{g}\in R^4$ and $\boldsymbol{g}$ is reduced with respect to the Gröbner basis of $D$ and $x_{ij}\boldsymbol{g}$ is in $D$ then $x_{ij}\mathrm{lt}(\boldsymbol{g})$ must be divisible by the leading term of one of the elements of the given Gröbner basis and so then $\mathrm{lt}(\boldsymbol{g})$ must also be divisible by the same leading term, contradicting the fact that $\boldsymbol{g}$ is reduced. □

To enlarge this regular sequence, we consider the module

$$\begin{aligned}\mathcal{M}_n^* &= \mathcal{M}_n/\langle x_{11},x_{n2},x_{i3}, i=1,\ldots,n\rangle\mathcal{M}_n\\ &\simeq R^4/\langle A_n+\langle x_{11},x_{n2},x_{i3}, i=1,\ldots,n\rangle R^4\rangle\\ &= R^4/\langle U_i, U_rU_s-U_sU_r, x_{11}\boldsymbol{e}_\ell, x_{n2}\boldsymbol{e}_\ell, x_{i3}\boldsymbol{e}_\ell\rangle_{\substack{i=1,\ldots,n\\ 1\le r<s\le n\\ \ell=1,2,3,4}}.\end{aligned}$$

Let $B = \langle U_i, U_rU_s - U_sU_r, x_{11}\boldsymbol{e}_\ell, x_{n2}\boldsymbol{e}_\ell, x_{i3}\boldsymbol{e}_\ell\rangle_{\substack{i=1,\dots,n\\ 1\le r<s\le n\\ \ell=1,2,3,4}}$. We note that the columns of $U_rU_s - U_sU_r$ can be reduced, using $x_{r3}\boldsymbol{e}_\ell$, and $x_{s3}\boldsymbol{e}_\ell$, $\ell = 1,2,3,4$, to the matrix

$$\begin{bmatrix} 0 & 0 & 0 & x_{r2}x_{s1} - x_{r1}x_{s2} \\ 0 & 0 & x_{r2}x_{s1} - x_{r1}x_{s2} & 0 \\ 0 & x_{r2}x_{s1} - x_{r1}x_{s2} & 0 & 0 \\ x_{r2}x_{s1} - x_{r1}x_{s2} & 0 & 0 & 0 \end{bmatrix}.$$

So we have

$$B = \langle U_i, (x_{r2}x_{s1} - x_{r1}x_{s2})\boldsymbol{e}_\ell, x_{11}\boldsymbol{e}_\ell, x_{n2}\boldsymbol{e}_\ell, x_{i3}\boldsymbol{e}_\ell\rangle_{\substack{i=1,\dots,n\\ 1\le r<s\le n\\ \ell=1,2,3,4}}. \tag{4}$$

We note that the generators of $B$ given in (4) form a Gröbner basis for $B$. To see this, note that the only S-polynomials we need to consider are those computed using a column of $U_i$ and one of $(x_{r2}x_{s1} - x_{r1}x_{s2})\boldsymbol{e}_\ell$, $x_{11}\boldsymbol{e}_\ell$, $x_{n2}\boldsymbol{e}_\ell$, or $x_{i3}\boldsymbol{e}_\ell$. The leading terms of the columns of $U_i$ are $x_{i0}\boldsymbol{e}_\ell$ which are relatively prime to $x_{11}\boldsymbol{e}_\ell$, $x_{n2}\boldsymbol{e}_\ell$, and $x_{i3}\boldsymbol{e}_\ell$, and it is easy to verify that the corresponding S-polynomials reduce to zero. The leading term of $(x_{r2}x_{s1} - x_{r1}x_{s2})\boldsymbol{e}_\ell$ is $x_{r2}x_{s1}\boldsymbol{e}_\ell$, and so it is relatively prime to $x_{i0}$. Again, it is easy to verify that the corresponding S-polynomials reduce to zero.

**Proposition 2.3.** *The polynomials $x_{21} + x_{12}, x_{31} + x_{22}, \dots, x_{n1} + x_{n-1,2}$ form a maximal $\mathcal{M}_n^*$-regular sequence in $\wp_n$.*

*Proof.* In order to show that the polynomials $x_{21} + x_{12}, x_{31} + x_{22}, \dots, x_{n1} + x_{n-1,2}$ form a $\mathcal{M}_n^*$-regular sequence in $\wp_n$, we need to show that the polynomial $x_{\nu+1,1} + x_{\nu 2}$ is a non-zero divisor on $R^4/B_{\nu-1}$ (for $\nu = 1,2,\dots,n-1$), where $B_{\nu-1} = \langle B, (x_{21} + x_{12})\boldsymbol{e}_\ell, (x_{31} + x_{22})\boldsymbol{e}_\ell, \dots, (x_{\nu 1} + x_{\nu-1,2})\boldsymbol{e}_\ell\rangle_{\ell=1,2,3,4}$ (and $B_0 = B$). Then to show that the sequence is maximal we will show that every element of $\wp_n$ is a zero divisor on $R^4/B_{n-1}$.

In order to do this we first find a Gröbner basis for $B_{\nu-1}$ for $1 \le \nu \le n$. This basis will consist of the following vectors:

a) The columns of $U_i$ for $1 \le i \le n$
b) $x_{12}x_{s-1,2}\boldsymbol{e}_\ell$ for $2 \le s \le \nu$
c) $x_{12}x_{s1}\boldsymbol{e}_\ell$ for $\nu + 1 \le s \le n$
d) $x_{r2}x_{n1}\boldsymbol{e}_\ell$ for $1 \le r < n$
e) $(x_{r2}x_{s-1,2} - x_{r-1,2}x_{s2})\boldsymbol{e}_\ell$ for $2 \le r < s \le \nu$
f) $(x_{r2}x_{s1} + x_{r-1,2}x_{s2})\boldsymbol{e}_\ell$ for $2 \le r \le \nu < s < n$
g) $(x_{r2}x_{s1} - x_{r1}x_{s2})\boldsymbol{e}_\ell$ for $\nu < r < s < n$
h) $x_{11}\boldsymbol{e}_\ell$
i) $x_{n2}\boldsymbol{e}_\ell$
j) $x_{i3}\boldsymbol{e}_\ell$ for $1 \le i \le n$
k) $(x_{r1} + x_{r-1,2})\boldsymbol{e}_\ell$ for $1 < r \le \nu$,

where $\ell = 1,2,3,4$. These vectors are obtained from the vectors in the generating set for $B$ given in Equation (4) by substituting 0 for $x_{11}$ and $x_{n2}$, and $-x_{r-1,2}$ for $x_{r1}$. Thus the given vectors do form a generating set for the module $B_{\nu-1}$. That this set of vectors forms a Gröbner basis with respect to the given order can be verified by checking that all the corresponding S-polynomials in fact reduce to 0. Note that all of the vectors above are written with their leading term first. Also note that in the extreme cases for $\nu$, i.e. $\nu = 1$ and $\nu = n-1$, the ranges in many of the above contain no $r$ or $s$. We denote this Gröbner basis of $B_{\nu-1}$ by $G_{\nu-1}$.

We now verify that for $\nu = 1, 2, \ldots, n-1$, $x_{\nu+1,1} + x_{\nu 2}$ is a non-zero divisor on $R^4/B_{\nu-1}$. The verification will be made in the case where all of the vectors in the above list appear. The extreme cases of $\nu = 1$ and $\nu = n-1$ are the same but avoid some of the complications of the following. So assume that we have a vector $\boldsymbol{g}$ in $R^4 - B_{\nu-1}$ such that $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g} \in B_{\nu-1}$. We may assume that $\boldsymbol{g}$ is reduced with respect to $G_{\nu-1}$. In particular this means that $\boldsymbol{g}$ can only contain the variables $x_{r1}$ for $\nu+1 \leq r \leq n$ and $x_{s2}$ for $1 \leq s \leq n-1$ (that the variables $x_{11}, x_{n2}, x_{i3}$ $(1 \leq i \leq n)$, and $x_{r1}$ $(1 < r \leq \nu)$ do not appear follows immediately from the vectors in h), i), j), and k) in the above list for the Gröbner basis for $B_{\nu-1}$; that the variables $x_{i0}$ for $1 \leq i \leq n$ do not appear follows from the fact that in the matrices $U_i$ for $1 \leq i \leq n$ there is a leading term of the form $x_{i0}\boldsymbol{e}_\ell$ for $1 \leq i \leq n$ and $\ell = 1, 2, 3, 4$ and no $x_{i0}$ in any other coordinate of that vector in $U_i$.

We, of course, have that $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g}$ reduces to zero by $G_{\nu-1}$. Only the vectors in b), c), d), e), f), and g) in the list for the Gröbner basis $G_{\nu-1}$ above can ever be used to reduce $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g}$. Now $\boldsymbol{g}$ must have a non-zero coordinate, say $\boldsymbol{g}\boldsymbol{e}_\ell$ (for some $\ell = 1, 2, 3, 4$). Then, due to the nature of the vectors in the Gröbner basis $G_{\nu-1}$ that can be use to reduce $(x_{\nu+1,1} + x_{\nu 2})\boldsymbol{g}$, we see that $(x_{\nu+1,1} + x_{\nu 2})g$ must reduce to zero using the polynomials in the list below:

b) $x_{12}x_{s-1,2}$ for $2 \leq s \leq \nu$
c) $x_{12}x_{s1}$ for $\nu+1 \leq s \leq n$
d) $x_{r2}x_{n1}$ for $1 \leq r < n$
e) $x_{r2}x_{s-1,2} - x_{r-1,2}x_{s2}$ for $2 \leq r < s \leq \nu$
f) $x_{r2}x_{s1} + x_{r-1,2}x_{s2}$ for $2 \leq r \leq \nu < s < n$
g) $x_{r2}x_{s1} - x_{r1}x_{s2}$ for $\nu < r < s < n$.

Denote this list of polynomials by $H_{\nu-1}$. Note that $g$ is reduced with respect to $H_{\nu-1}$ and only involves the variables $x_{r1}$ for $\nu+1 \leq r \leq n$ and $x_{s2}$ for $1 \leq s \leq n-1$. Thus one of the leading power products in $H_{\nu-1}$ must divide $\mathrm{lp}((x_{\nu+1,1} + x_{\nu 2})g) = x_{\nu+1,1}\mathrm{lp}(g)$ and cannot divide $\mathrm{lp}(g)$. These polynomials come from the polynomials in c) and f) in the list for $H_{\nu-1}$ above, and so we see that $x_{r2}$ must divide $\mathrm{lp}(g)$ for one of $r = 1, \ldots, \nu$. Since $g$ is reduced with respect to $H_{\nu-1}$ we see, using the polynomials in c), d), and f) in the list for $H_{\nu-1}$, that no $x_{r1}$ can divide $\mathrm{lp}(g)$. Thus

$$g = x_{12}^{a_1}x_{22}^{a_2}\cdots x_{n-1,2}^{a_{n-1}} + h,$$

where all of the terms in $h$ are smaller than $\mathrm{lp}(g) = x_{12}^{a_1}\cdots x_{n-1,2}^{a_{n-1}}$. Moreover one of the $a_r$, for $1 \leq r \leq \nu$ is non-zero. Then

$$(x_{\nu+1,1} + x_{\nu 2})g =$$
$$x_{\nu+1,1}x_{12}^{a_1}\cdots x_{n-1,2}^{a_{n-1}} + x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}} + (x_{\nu+1,1} + x_{\nu 2})h.$$

If $a_1 \geq 1$ then using the monomial in c) in the list for $H_{\nu-1}$ we have that $(x_{\nu+1,1} + x_{\nu 2})g$ reduces to $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}} + (x_{\nu+1,1} + x_{\nu 2})h$. If $a_1 = 0$, then one of $a_2, \ldots, a_\nu$ is greater than zero, say $a_j \geq 1$ $(2 \leq j \leq \nu)$, and so using the polynomial $x_{\nu+1,1}x_{j2} + x_{j-1,2}x_{\nu+1,2}$ in f) in the list for $H_{\nu-1}$ we have that $(x_{\nu+1,1} + x_{\nu 2})g$ reduces to

$$-x_{12}^{a_1}\cdots x_{j-2,2}^{a_{j-2}}x_{j,2}^{a_j-1}x_{j-1,2}^{a_{j-1}+1}\cdots x_{\nu,2}^{a_\nu}x_{\nu+1,2}^{a_{\nu+1}+1}x_{\nu+2,2}^{a_{\nu+2}}\cdots x_{n-1,2}^{a_{n-1}}$$
$$+x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}} + (x_{\nu+1,1} + x_{\nu 2})h.$$

We see that the second term in this last expression is larger than the first term in the degrevlex ordering, since $a_{\nu+1}+1 > a_{\nu+1}$ (the degrees of the two monomials are the same). Thus the leading power product in the reduced polynomial just obtained from $(x_{\nu+1,1}+x_{\nu 2})g$ is either $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}$ or $x_{\nu+1,1}\mathrm{lp}(h)$.

**Claim.** Let $X$ be a term of $h$ such that

$$x_{\nu+1,1}X > x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}.$$

Assume that $x_{\nu+1,1}X$ can be reduced using $H_{\nu-1}$. Then $x_{\nu+1,1}X$ can be reduced to a term $Y$, using $H_{\nu-1}$, such that

$$Y < x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}.$$

Assuming the Claim we complete the proof that $x_{\nu+1,1}+x_{\nu 2}$ is a non-zero divisor on $R^4/B_{\nu-1}$ as follows. We first observe that $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}$ cannot be reduced using $H_{\nu-1}$. To see this we note that since only the variables $x_{s2}$ appear we could only possibly use the polynomials in b) or e) in the list for $H_{\nu-1}$; then since $g$ is reduced with respect to $H_{\nu-1}$, $x_{\nu 2}$ would have to appear in the polynomial used to do the reduction, but this variable does not appear in any of the polynomials in b) and e). Thus if $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}$ is the leading term we have a contradiction. Otherwise $x_{\nu+1,1}\mathrm{lp}(h)$ is the leading term and so must be reducible using $H_{\nu-1}$. Letting $X=\mathrm{lt}(h)$ in the Claim and setting $h'=h-\mathrm{lt}(h)$ we reduce $(x_{\nu+1,1}+x_{\nu 2})g$ to

$$x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}+Y+x_{\nu 2}X+(x_{\nu+1,1}+x_{\nu 2})h'.$$

Since $\mathrm{lp}(g) > \mathrm{lp}(h)$ we see that the leading term of this last expression is either $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}$ or $x_{\nu+1,1}\mathrm{lp}(h')$. If it is the latter then $x_{\nu+1,1}\mathrm{lp}(h')$ must be reducible using $H_{\nu-1}$. Thus the argument may be repeated until we obtain an expression which must reduce to zero using $H_{\nu-1}$ but whose leading term is $x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}$ and we have again arrived at a contradiction.

It remains to prove the Claim. As above, we see that $x_{\nu+1,1}X$ can only be reduced using the polynomials in c) and f) in the list for $H_{\nu-1}$ above. Thus $x_{r2}$ must divide $X$ for some $r=1,\ldots,\nu$. Moreover no variable $x_{r1}$ can divide $X$ since $X$ cannot be reduced using $H_{\nu-1}$. Thus $X=x_{12}^{b_1}x_{22}^{b_2}\cdots x_{n-1,2}^{b_{n-1}}$. Now if we can use the monomial $x_{12}x_{\nu+1,1}$ in c) then $x_{\nu+1,1}X$ reduces to 0 and the Claim is true. Otherwise $x_{r2}$ divides $X$ for some $r$ such that $2\le r\le \nu$. For the reduction of $x_{\nu+1,1}X$ we replace $x_{r2}x_{\nu+1,1}$ by $-x_{r-1,2}x_{\nu+1,2}$. Thus we need to show that

$$\frac{X}{x_{r2}}x_{r-1,2}x_{\nu+1,2} < x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}} \tag{5}$$

under the hypotheses

$$X=x_{12}^{b_1}\cdots x_{n-1,2}^{b_{n-1}} < X = x_{12}^{a_1}\cdots x_{n-1,2}^{a_{n-1}}$$

and

$$x_{\nu+1,1}X > x_{12}^{a_1}\cdots x_{\nu-1,2}^{a_{\nu-1}}x_{\nu 2}^{a_\nu+1}x_{\nu+1,2}^{a_{\nu+1}}\cdots x_{n-1,2}^{a_{n-1}}.$$

These two hypotheses guarantee that all terms present have the same degree. From the first we choose $\ell$ such that

$$b_{\ell+1}=a_{\ell+1},\ldots,b_{n-1}=a_{n-1}, b_\ell > a_\ell.$$

Then the second hypothesis guarantees that $\ell\le\nu$. but then the left side of Equation (5), $\frac{X}{x_{r2}}x_{r-1,2}x_{\nu+1,2}$, has the exponent $b_{\nu+1}+1$ for $x_{\nu+1,2}$ while the right side has the exponent $a_{\nu+1} = b_{\nu+1}$ and both sides have equal exponents for all $x_{r2}$ for $r > \nu + 1$. Thus (5) is true and this completes the proof of the Claim.

It remains to show that every element of $\wp_n$ is a zero divisor on $R^4/B_{n-1}$. We first need to reduce the Gröbner basis given above for $B_{n-1}$. First note that this Gröbner basis is given by the following vectors:

a) The columns of $U_i$ for $1 \le i \le n$
b) $x_{12}x_{s-1,2}\boldsymbol{e}_\ell$ for $2 \le s \le n$
d) $x_{r2}x_{n1}\boldsymbol{e}_\ell$ for $1 \le r < n$
e) $(x_{r2}x_{s-1,2} - x_{r-1,2}x_{s2})\boldsymbol{e}_\ell$ for $2 \le r < s < n$
h) $x_{11}\boldsymbol{e}_\ell$
i) $x_{n2}\boldsymbol{e}_\ell$
j) $x_{i3}\boldsymbol{e}_\ell$ for $1 \le i \le n$
k) $(x_{r1} + x_{r-1,2})\boldsymbol{e}_\ell$ for $1 < r \le n$,

where $\ell = 1, 2, 3, 4$. We first note that we can use the vector in k) with $r = n$ to reduce the vectors in d) to $x_{r2}x_{n-1,2}\boldsymbol{e}_\ell$ for $1 \le r < n$. We now look at the vectors in e). If $s = n-1$ then use $x_{r-1,2}x_{n-1,2}\boldsymbol{e}_\ell$ to reduce $(x_{r2}x_{n-2,2} - x_{r-1,2}x_{n-1,2})\boldsymbol{e}_\ell$ to $x_{r2}x_{n-2,2}\boldsymbol{e}_\ell$. Then with this last vector we reduce the vector in e) with $s = n-2$, $(x_{r2}x_{n-3,2} - x_{r-1,2}x_{n-2,2})\boldsymbol{e}_\ell$ to $x_{r2}x_{n-3,2}\boldsymbol{e}_\ell$. Continue in this fashion and we obtain the reduced Gröbner basis for $B_{n-1}$ consisting of the following vectors

1. The columns of $U_i$ for $1 \le i \le n$
2. $x_{s2}x_{r2}\boldsymbol{e}_\ell$ for $1 \le r \le s \le n-1$.
3. $x_{11}\boldsymbol{e}_\ell$
4. $x_{n2}\boldsymbol{e}_\ell$
5. $x_{i3}\boldsymbol{e}_\ell$ for $1 \le i \le n$
6. $(x_{r1} + x_{r-1,2})\boldsymbol{e}_\ell$ for $1 < r \le n$,

where $\ell = 1, 2, 3, 4$. Denote this Gröbner basis by $G$.

So let $f \in \wp_n$ be non-zero. If $f\boldsymbol{e}_1 \in B_{n-1}$ then $f(\boldsymbol{e}_1 + B_{n-1}) = 0$ and so $\boldsymbol{e}_1 \notin B_{n-1}$ implies that $f$ is a zero divisor. So assume that $f\boldsymbol{e}_1 \notin B_{n-1}$. Then $f(f\boldsymbol{e}_1 + B_{n-1}) = f^2\boldsymbol{e}_1 + B_{n-1}$, and so it suffices to show that for any $f \in \wp_n$, $f^2\boldsymbol{e}_1 \in B_{n-1}$; that is, show that $f^2\boldsymbol{e}_1$ reduces to zero using $G$. Since $f \in \wp_n$ every term in $f^2$ is of degree 2 or higher. Then, using the columns of the $U_i$'s, we can reduce $f^2\boldsymbol{e}_1$ to a vector $\boldsymbol{f}_1$ with no variables $x_{i0}$ $(1 \le i \le n)$ in it and with all terms of degree 2 or higher. Then, using the last four types of vectors itemized in $G$ above, we can reduce $\boldsymbol{f}_1$ to a vector $\boldsymbol{f}_2$ containing only the variables $x_{r2}$ $(1 \le r \le n-1)$ and with all terms of degree 2 or higher. Finally, using the vectors in 2) above, we see $\boldsymbol{f}_2$ reduces to zero. □

We can now obtain the formula for the projective dimension of $\mathcal{M}_n$.

**Theorem 2.4.**

$$\mathrm{pd}(\mathcal{M}_n) = 2n - 1.$$

*Proof.* By the Auslander-Buchsbaum formula we have

$$\mathrm{pd}(\mathcal{M}_n) = \mathrm{depth}(\wp_n, R) - \mathrm{depth}(\wp_n, \mathcal{M}_n) = 4n - \mathrm{depth}(\wp_n, \mathcal{M}_n).$$

By Corollary 2.2 and Proposition 2.3 we have

$$\mathrm{depth}(\wp_n, \mathcal{M}_n) = 2n + 1.$$

□

**Remark 2.5.** We now have a free resolution of $R^4/\langle A_n\rangle$

(6) $0 \longrightarrow R^{r_{2n-1}} \xrightarrow{C} R^{r_{2n-2}} \xrightarrow{B} \cdots \longrightarrow R^{r_2} \longrightarrow R^{4n} \longrightarrow R^4 \longrightarrow R^4/\langle A_n\rangle \longrightarrow 0$

(by the well-known Quillen-Suslin Theorem, we know that every projective $R$-module is free, see [9]). By taking the dual of Resolution (6) we obtain a complex

$$0 \longrightarrow R^4 \longrightarrow R^{4n} \longrightarrow R^{r_2} \longrightarrow \cdots \longrightarrow R^{r_{2n-2}} \xrightarrow{C^t} R^{r_{2n-1}} \longrightarrow 0, \tag{7}$$

whose homology groups are, by definition, $\mathrm{Ext}^i(\mathcal{M}_n, R)$. We see that the last homology, $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R)$, in (7) is not zero. Indeed, if the map $R^{r_{2n-2}} \longrightarrow R^{r_{2n-1}}$ is onto, then we obtain a matrix $D$ with $D^t$ defining a map $R^{r_{2n-1}} \longrightarrow R^{r_{2n-2}}$ such that $C^tD^t = I$, the identity. So we get that $DC = I$ as well and the map $C$ in Resolution (6) splits, $R^{r_{2n-2}} = \mathrm{im}C \oplus \ker D$. Since $\ker D$ is free and $B$ restricted to $\ker D$ is one to one we have obtained a shorter free resolution for $R^4/\langle A_n\rangle$ than (7), which violates Theorem 2.4.

It is actually possible to say more than this.

**Theorem 2.6.** *Sequence (7) is exact except at the last spot, i.e.*

$$\mathrm{Ext}^j(\mathcal{M}_n, R) = 0, \text{ for all } j = 0, \dots, 2n-2$$

*and*

$$\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R) \neq 0.$$

*Proof.* We will prove in the next Proposition that the characteristic variety $V(\mathcal{M}_n)$ of $\mathcal{M}_n$ (which can be defined, in view of [8, Proposition 2, p. 139], as the set of points where the rank of $A_n$ is strictly less than 4) has dimension $2n+1$. But then, by [8, Corollary 1, p. 377], we have immediately that $\mathrm{Ext}^j(\mathcal{M}_n, R) = 0$ for $j < 2n-1$ and $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R) \neq 0$. □

**Proposition 2.7.** *The characteristic variety $V(\mathcal{M}_n)$ of $\mathcal{M}_n$ has dimension $2n+1$.*

*Proof.* As observed above, the characteristic variety $V_n = V(\mathcal{M}_n)$ is the subset of points $\zeta \in \mathbb{C}^{4n}$ where the rank of the matrix $A_n(\zeta)$ is strictly less than 4. We show that the algebraic set $V_n$ has dimension $2n+1$ in a neighborhood of an arbitrary point $\zeta^0 \neq 0$ in $V_n$.

We write $\zeta = (\zeta_1, \dots, \zeta_n) \in V_n$, where $\zeta_1, \dots, \zeta_n \in \mathbb{C}^4$. We may assume that $\zeta_1 \neq 0$. Finally we write $\zeta_i = (\xi_{i0}, \xi_{i1}, \xi_{i2}, \xi_{i3})$, where $\xi_{ij} \in \mathbb{C}$, for $i = 1, \dots, n$.

We can consider each vector $\zeta_i$ as the element $\zeta_i = \xi_{i0} + \xi_{i1}\underline{\mathrm{i}} + \xi_{i2}\underline{\mathrm{j}} + \xi_{i3}\underline{\mathrm{k}}$ of the complexified quaternionic algebra $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, where $\{\underline{1}, \underline{\mathrm{i}}, \underline{\mathrm{j}}, \underline{\mathrm{k}}\}$ is the standard basis for the $\mathbb{C}$-vector space $\mathbb{H}_{\mathbb{C}}$. This is an associative $\mathbb{C}$-algebra with involution $\zeta_i^* = \xi_{i0} - \xi_{i1}\underline{\mathrm{i}} - \xi_{i2}\underline{\mathrm{j}} - \xi_{i3}\underline{\mathrm{k}}$. It is easy to see that the columns of $A_n(\zeta)$ correspond to the quaternions

$$\zeta_1^*, \zeta_1^*\underline{\mathrm{i}}, \zeta_1^*\underline{\mathrm{j}}, \zeta_1^*\underline{\mathrm{k}}, \zeta_2^*, \zeta_2^*\underline{\mathrm{i}}, \zeta_2^*\underline{\mathrm{j}}, \zeta_2^*\underline{\mathrm{k}}, \dots, \zeta_n^*, \zeta_n^*\underline{\mathrm{i}}, \zeta_n^*\underline{\mathrm{j}}, \zeta_n^*\underline{\mathrm{k}}.$$

The determinant of the first four columns of $A_n(\zeta)$ is easily computed to be $(\zeta_1^*\zeta_1)^2$, where $\zeta_1^*\zeta_1 = \xi_{10}^2 + \xi_{11}^2 + \xi_{12}^2 + \xi_{13}^2$. The equation $\zeta_1^*\zeta_1 = 0$ defines a quadratic cone $V_1$ in $\mathbb{C}^4$ of dimension three.

For $\eta \in \mathbb{H}_{\mathbb{C}}$ we define four complex subspaces of $\mathbb{H}_{\mathbb{C}}$ as follows. Set

$$L_\eta = \{\eta q | q \in \mathbb{H}_{\mathbb{C}}\} \text{ and } L_\eta^- = \{q \in \mathbb{H}_{\mathbb{C}} | \eta q = 0\},$$

$$R_\eta = \{q\eta | q \in \mathbb{H}_{\mathbb{C}}\} \text{ and } R_\eta^- = \{q \in \mathbb{H}_{\mathbb{C}} | q\eta = 0\}.$$

One of the first two spaces is the image of left multiplication in $\mathbb{H}_{\mathbb{C}}$ by $\eta$ and the other is the kernel of this map so we have that $\dim_{\mathbb{C}} L_\eta + \dim_{\mathbb{C}} L_\eta^- = 4$. We also have $\dim_{\mathbb{C}} R_\eta + \dim_{\mathbb{C}} R_\eta^- = 4$.

For $\eta \neq 0$ and $\eta \in V_1$, i.e. $\eta^*\eta = 0$, we see that $\dim_{\mathbb{C}} L_\eta = 2$. This follows since the matrix of the map of left multiplication by $\eta$ with respect to the basis $\{\underline{1}, \underline{i}, \underline{j}, \underline{k}\}$ is the first four columns of $A_n$ with $\eta^*$ substituted in and the three by three subdeterminants of this matrix are readily computed to all be multiples of $\eta^*\eta$, while $\eta \neq 0$ easily implies that not all of the two by two subdeterminants are zero. Moreover, since $L_\eta \subseteq L_{\eta^*}^-$, we conclude from looking at the dimensions that $L_\eta = L_{\eta^*}^-$. We similarly get $\dim_{\mathbb{C}} R_\eta = 2$ and $R_\eta = R_{\eta^*}^-$.

We now show that

$$\zeta \in V_n \text{ if and only if } \zeta_1 \in V_1 \text{ and } \zeta_j \in R_{\zeta_1} (2 \leq j \leq n).$$

First assume that $\zeta_1 \in V_1$ and $\zeta_j \in R_{\zeta_1}(2 \leq j \leq n)$. Then $\zeta_j = q_j \zeta_1$, for some $q_j \in \mathbb{H}_{\mathbb{C}}$. So $\zeta_j^* \nu \in L_{\zeta_1^*}$ where $\nu = 1, \underline{i}, \underline{j}, \underline{k}$ and so we see that the column space of $A_n(\zeta)$ is contained in the two dimensional space $L_{\zeta_1^*}$ and so the rank of $A_n(\zeta)$ is $2 < 4$, and so $\zeta \in V_n$.

Conversely assume that $\zeta \in V_n$. Since $\dim_{\mathbb{C}} L_{\zeta_1^*} = 2$ we may assume, by symmetry, that $\zeta_1^*$ and $\zeta_1^* \underline{i}$ are linearly independent and so form a basis for $L_{\zeta_1^*}$. Fix a $j$. Since the rank of $A_n(\zeta)$ is less than 4, we have that $\zeta_1^*, \zeta_1^*\underline{i}, \zeta_j^*, \zeta_j^*\underline{i}$ are linearly dependent, and so there are complex numbers $a_1, b_1, c_1, d_1$, not all zero, such that

$$\zeta_1^*(a_1 + b_1\underline{i}) = \zeta_j^*(c_1 + d_1\underline{i}).$$

Since $\zeta_1^*$ and $\zeta_1^*\underline{i}$ are linearly independent we must have one of $c_1, d_1$ non-zero. Now if $d_1 = 0$ then $\zeta_j = q^*\zeta_1 \in R_{\zeta_1}$ where $q = c_1^{-1}(a_1 + b_1\underline{i})$ and we would be done. So assume that $d_1 \neq 0$. Similarly we would be done unless we had complex numbers $a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3$ with $d_2 \neq 0, d_3 \neq 0$ such that

$$\zeta_1^*(a_2 + b_2\underline{i}) = \zeta_j^*(c_2 + d_2\underline{j})$$

and

$$\zeta_1^*(a_3 + b_3\underline{i}) = \zeta_j^*(c_3 + d_3\underline{k}).$$

Multiplying these last three displayed equations on the left by $\zeta_1$, recalling that $\zeta_1\zeta_1^* = 0$, we obtain $\zeta_1\zeta_j^*(c_1 + d_1\underline{i}) = 0$, $\zeta_1\zeta_j^*(c_2 + d_2\underline{j}) = 0$, and $\zeta_1\zeta_j^*(c_3 + d_3\underline{k}) = 0$. That is, we have

$$c_1 + d_1\underline{i}, c_2 + d_2\underline{j}, c_3 + d_3\underline{k} \in L_{\zeta_1\zeta_j^*}^-.$$

Now $(\zeta_1\zeta_j^*)(\zeta_1\zeta_j^*)^* = 0$ and so $c_1 + d_1\underline{i}, c_2 + d_2\underline{j}, c_3 + d_3\underline{k}$ linearly independent over $\mathbb{C}$ implies $\zeta_1\zeta_j^* = 0$, since $\zeta_1\zeta_j^* \neq 0$ implies $\dim_{\mathbb{C}} L_{\zeta_1\zeta_j^*}^- = 2$. Thus we have

$$\zeta_j^* \in L_{\zeta_1}^- = L_{\zeta_1^*}.$$

We conclude that $\zeta_j^* = \zeta_1^* q$ for some $q \in \mathbb{H}_{\mathbb{C}}$ and thus $\zeta_j = q^*\zeta_1 \in R_{\zeta_1}$, as desired.

The dimension of $V_n$ follows immediately:

$$\dim_{\mathbb{C}} V_n = \dim_{\mathbb{C}} V_1 + (n-1)\dim_{\mathbb{C}} R_{\zeta_1} = 3 + 2(n-1) = 2n + 1.$$

$\square$

## 3. Application to the Theory of Regular Functions

Let $P(D) = [P_{ij}(D)]$ be the $4n \times 4$ matrix of differential operators defining the Cauchy–Fueter system (1). We have that

$$P(D) : \left[\mathcal{E}\left(\mathrm{R}^{4n}\right)\right]^4 \longrightarrow \left[\mathcal{E}\left(\mathrm{R}^{4n}\right)\right]^{4n}$$

and we denote by $\mathcal{R} = \mathcal{E}^P$ the sheaf of $\mathcal{C}^\infty$ solutions of $P$, i.e. the sheaf of regular functions (see Section 1). Note that, since $P$ is an elliptic system (see, e.g., [5]), we have $\mathcal{R} = \mathcal{E}^P = \mathcal{D}'^P$, where $\mathcal{D}'$ is the sheaf of distributions. However a fundamental result of Bengel-Harvey-Komatsu (see [6]) shows that we also have $\mathcal{R} = \mathcal{B}^P$, where $\mathcal{B}$ is the sheaf of hyperfunctions. This fact immediately allows us to prove the following

**Theorem 3.1.** *The sheaf $\mathcal{R}$ has flabby dimension equal to $2n-1$.*

*Proof.* We first note that the matrix $A_n$ is the transpose of the Fourier transform of $P(D)$, so from Theorem 2.4 we have the complex (7) which gives us

$$(8) \qquad 0 \longrightarrow \mathcal{B}^P \longrightarrow \mathcal{B}^4 \xrightarrow{P(D)} \mathcal{B}^{4n} \longrightarrow \mathcal{B}^{r_2} \longrightarrow \cdots \longrightarrow \mathcal{B}^{r_{2n-2}} \longrightarrow \mathcal{B}^{r_{2n-1}} \longrightarrow 0,$$

which is a resolution of the sheaf $\mathcal{B}^P$. This result is essentially due to Ehrenpreis-Malgrange-Palamodov, but in the hyperfunction setting it was actually proved by Komatsu (see [6] for details and references). Since $\mathcal{R} = \mathcal{B}^P$, as we noted above, and since $\mathcal{B}$ is flabby, Resolution (8) proves that $\mathrm{fl.dim}(\mathcal{R}) \leq 2n-1$. On the other hand, the flabby dimension cannot be strictly less than $2n-1$, since (see [6, Theorem 1.2]) this would imply the vanishing of $H^{2n-1}(\mathrm{H}^n, \mathrm{H}^n \backslash K; \mathcal{R})$ for every compact convex set $K$ in $\mathrm{H}^n$. This would imply that $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R) = 0$, which would contradict Remark 2.5 or Theorem 2.6. We have therefore proved that $\mathrm{fl.dim}(\mathcal{R}) = 2n-1$. □

**Remark 3.2.** For $n = 1, 2$ this result is implicitly contained in [5] and [1] even though it was not explicitly stated.

**Remark 3.3.** Theorem 3.1 generalizes to the sheaf of germs of regular functions the well-known fact that $\mathrm{fl.dim}(\mathcal{O}) = n$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions. Such a result was probably hard to imagine before our computations in [1].

As we have shown in [5], all open sets $U$ in H are cohomologically trivial in the sense that

$$H^p(U, \mathcal{R}) = 0 \qquad p \geq 1.$$

In [1], on the other hand, we showed that this result fails for $n > 1$, since a Hartog's phenomenon occurs. This situation clearly mirrors what happens for the sheaf $\mathcal{O}$ of holomorphic functions. In that case, the most important result, due to Malgrange [7], states that, for any open set $U \subseteq \mathrm{C}^n$,

$$H^p(U, \mathcal{O}) = 0 \qquad p \geq n.$$

In our case, the analog of such a statement is an immediate corollary of Theorem 3.1.

**Corollary 3.4.** *If $U$ is any open set in $\mathrm{H}^n$, then*

$$H^p(U, \mathcal{R}) = 0 \qquad p \geq 2n-1.$$

Once again, we believe this result to be quite unexpected. We do not know of any analytic proof for it.

Let us now explain the significance of these results for the construction of a theory of quaternionic hyperfunctions. As it is well-known, see for example [10], the theory of hyperfunctions is based on two key facts: one is that the flabby dimension of $\mathcal{O}$ is $n$ and the other is the fact that $\mathrm{R}^n$ is purely $n$-codimensional in $\mathrm{C}^n$ (see [6] or [10]). These facts allowed Sato to define $\mathcal{B}$ as the $n$-th derived sheaf $\mathcal{H}^n_{\mathrm{R}^n}(\mathcal{O})$ of $\mathcal{O}$ restricted to $\mathrm{R}^n$. It is therefore clear that the present paper provides us with the first step towards a similar construction. The difficulty will be to figure out which subset $S$ of $\mathrm{H}^n$ should be chosen to restrict the derived sheaf. In [5], we took

$$S = \tilde{\mathrm{H}} = \{q = x_0 + \boldsymbol{i}x_1 + \boldsymbol{j}x_2 + \boldsymbol{k}x_3, x_0 = 0\} \subseteq \mathrm{H}$$

which is purely 1-codimensional and we were able to reconstruct the entire theory.

We conclude by pointing out other interesting byproducts of the results from Section 2. To begin with, one can use our arguments from [1] to completely restore the duality theorem which prompted our interest in this investigations.

**Theorem 3.5.** *Let $K$ be a compact convex set in $\mathrm{H}^n$. Then if $\mathcal{S}$ denotes the sheaf of distribution solutions to the system associated to the matrix $C^t$ which appears in (7), then*

$$H^{2n-1}(\mathrm{H}^n, \mathrm{H}^n \backslash K; \mathcal{S}) \simeq [H^0(K, \mathcal{R})]'.$$

On the other hand, the vanishing of so many Ext-modules also gives more information on removability of singularities of the Cauchy–Fueter system.

**Theorem 3.6.** *Let $\Omega$ be a convex connected open set in $\mathrm{H}^n = \mathrm{R}^{4n}$, and let $K$ be a compact subset of $\Omega$. Let $\Sigma_1, \ldots, \Sigma_{2n-2}$ be closed half spaces in $\mathrm{R}^{4n}$ and set $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_{2n-2}$. Then every regular function $f \in \Omega \backslash (K \cup \Sigma)$ extends to a regular function $\tilde{f} \in \Omega \backslash \Sigma$ which coincides with $f$ in $\Omega \backslash (K' \cup \Sigma)$, for $K'$ a compact subset of $\Omega$.*

*Proof.* This is an immediate consequence of our Theorem 2.6 and of [8, Theorem 4, p. 405]. □

**Theorem 3.7.** *Let $L$ be a subspace of $\mathrm{H}^n = \mathrm{R}^{4n}$ of dimension $2n+2$. Then for every compact $K$ contained in $L$, and every connected open set $\Omega$, relatively compact in $K$, every regular function defined in the neighborhood of $K \backslash \Omega$ can be extended to a regular function defined in a neighborhood of $K$.*

*Proof.* This result follows again from our Theorem 2.6 together with [8, Theorem 3, p. 403] if we can prove that none of the varieties associated to the module $\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R)$ is hyperbolic with respect to $L$. However, [8, Corollary 2, p. 377] shows that $V(\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R))$ is contained in $V(\mathcal{M}_n)$, and since $\mathcal{M}_n$ is elliptic, we can conclude that every variety in $V(\mathrm{Ext}^{2n-1}(\mathcal{M}_n, R))$ is elliptic and therefore cannot be hyperbolic. This concludes the proof. □

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