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Shock Study in Fully Relativistic Isothermal Flows. II

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Shock study in fully relativistic isothermal flows. II

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Abstract. The isothermal shocks and their stabilities in fully relativistic accretion wedge flows onto black holes are studied. The jump condition across the shock is modified by the relativistic effects when the sound speed is comparable to the speed of light. With a new kind of instability analysis, it is found that only one of the two possible shocks is stable. The results are applied to the QPO behavior in galactic black hole candidates such as Cygnus X-1.

Key words: black hole physics – relativity – shock waves – stars: Cyg X-1

1. Introduction

Standing shocks are believed to form in the inner accretion disks. Both numerically (Hawley et al. 1984a,b) and analytical calculations (Chakrabarti 1989, 1990; Abramowicz & Chakrabarti 1990) confirm this existence. In the analytical works, the governing equations are based on the pseudo-Newtonian potential model (Paczynski & Wiita 1980). In the present work, we follow the same outline but use the fully relativistic, hydrodynamic equations to analyze the shock conditions and the possible shock locations in isothermal, one dimensional accretion onto black holes. Relativistic adiabatic shocks have been studied by, among others, Taub (1948), Lichnerowicz (1967) and Thorne (1973). Chang & Ostriker (1985), on the other hand, investigated the shocks in accretion flows onto black holes by forcing heating and cooling to generate the multiple sonic points, which is the necessary condition for the formation of shocks. Here, we focus on isothermal shocks following Chakrabarti (1989).

We are undertaking the present work for the following reasons: (a) To make possible comparison about shocks between the pseudo-Newtonian model and the fully relativistic model. (b) To analyze the stability of the shocks in a different way from

that used by Chakrabarti (1989). (c) To apply the results in astrophysical problems such as the QPO behavior in black hole candidates.

2. Governing equations

For the steady, inviscid, axisymmetrical flows along the equatorial plane applicable to a thin disk, we obtain the dimensionless equations in the Schwarzschild metric (Kafatos & Yang 1994; herein referred to as Paper I):

$$\frac{du}{dX} = \frac{N}{D}, \quad (1)$$

$$\frac{dv}{dX} = Bv \left[\frac{1}{u} \frac{du}{dX} + \frac{2}{X} \right] - \frac{v}{X}, \quad (2)$$

where the numerator N in (1) has the form

$$N = (X - 3) \frac{v^2}{X^2} - \frac{1}{X^2} + \frac{2B}{X} \left[1 - \frac{2}{X} + u^2 \right], \quad (3)$$

and the denominator D is

$$D = (1 - B)u - \frac{B}{u} \left(1 - \frac{2}{X} \right). \quad (4)$$

The dimensionless quantities are defined as $X = \frac{r}{r_g}$, $u = \frac{u^r}{c}$, and $v = \frac{u^\phi r}{c}$, with $r_g = GM_{\text{bh}}/c^2$ and c as the length and velocity scales. Here u^r and u^ϕ are the radial and azimuthal components of the four-velocity and M_{bh} is the mass of the central black hole. The parameter $B = \frac{n}{w} \frac{dp}{dn}$, equals the square of the local sound speed. Here w is the enthalpy per unit proper volume, n the particle number density and p the gas pressure in the frame in which the fluid is at rest. In the above derivation (cf. Paper I), we have used the continuity equation

$$nuX^2 = \text{constant}. \quad (5)$$

Moreover, since we henceforth focus on isothermal flows, for which $p = Kn$, B is a constant reflecting the constant temperature. Finally, we assume that the increase of M_{bh} due to the accretion process itself is negligible.

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3. Solution structure and parameters

The exact solutions to Eqs. (1) and (2) can be found analytically. The solutions reflect the fact that the specific angular momentum l is conserved, where

$$l = \frac{vX}{\sqrt{s^2(1+v^2)+u^2}}, \quad (6)$$

and

$$\frac{(s^2+u^2)(uX^2)^{-2B}}{1-s^2l^2/X^2} = \text{constant}. \quad (7)$$

In the above expressions $s^2 \equiv 1 - 2/X$.

Accretion flows onto black holes must be transonic (Chakrabarti 1990). The critical points are determined by the conditions $D = N = 0$ and the analytical solutions. The radial velocity in the frame of reference corotating with the fluid (Lu 1985) is defined as

$$\hat{u}^2 = \frac{u^2}{s^2+u^2}. \quad (8)$$

Therefore, when $D = 0$, $u_c^2 = Bs^2/(1-B)$ and $\hat{u}_c^2 = B$, i.e., at the critical point, the radial velocity in the corotating frame is, as expected, equal to the sound speed whereas at the Schwarzschild radius $\hat{u} = 1$ as expected. In other words, the critical points are sonic points and the flows at those points are transonic. The Mach number is defined as

$$M \equiv \frac{\hat{u}}{\sqrt{B}}, \quad (9)$$

and, as expected, $M_c = 1$ at the critical point. The relationship between u and M is then

$$u = \frac{Ms\sqrt{B}}{\sqrt{1-M^2B}}, \quad (10)$$

and Eq. (7) can be rewritten as

$$\frac{(1-M^2B)^{B-1}s^2}{(1-s^2l^2/X^2)(M^2Bs^2X^4)^B} = \text{constant}. \quad (11)$$

For a given B value, the sonic points are not always unique. For $B < B_c = 0.017548$ which corresponds to $T \lesssim 1.8 \times 10^{11}$ K, there exists a range of angular momenta $l_A < l < l_B$ for which there are more than one critical point (Paper I). A further investigation demonstrates that only two of them are physical. Hereafter, the critical points or sonic points examined are only the physical ones. We name them the *outer* sonic point and the *inner* sonic point, respectively, according to their radial distance r from the black hole. The temperature limit obtained above is close to the maximum virial value allowed for any flow and specifically for a hot, two-temperature, ion dominated disk in the Schwarzschild metric, $T_i \sim 10^{12}$ K.

Following Chakrabarti's work, the constant in Eq. (11) is determined by the specific energy E . When the values of temperature (or, equivalently, the parameter B) and angular momentum l are given, the values of E completely determine the

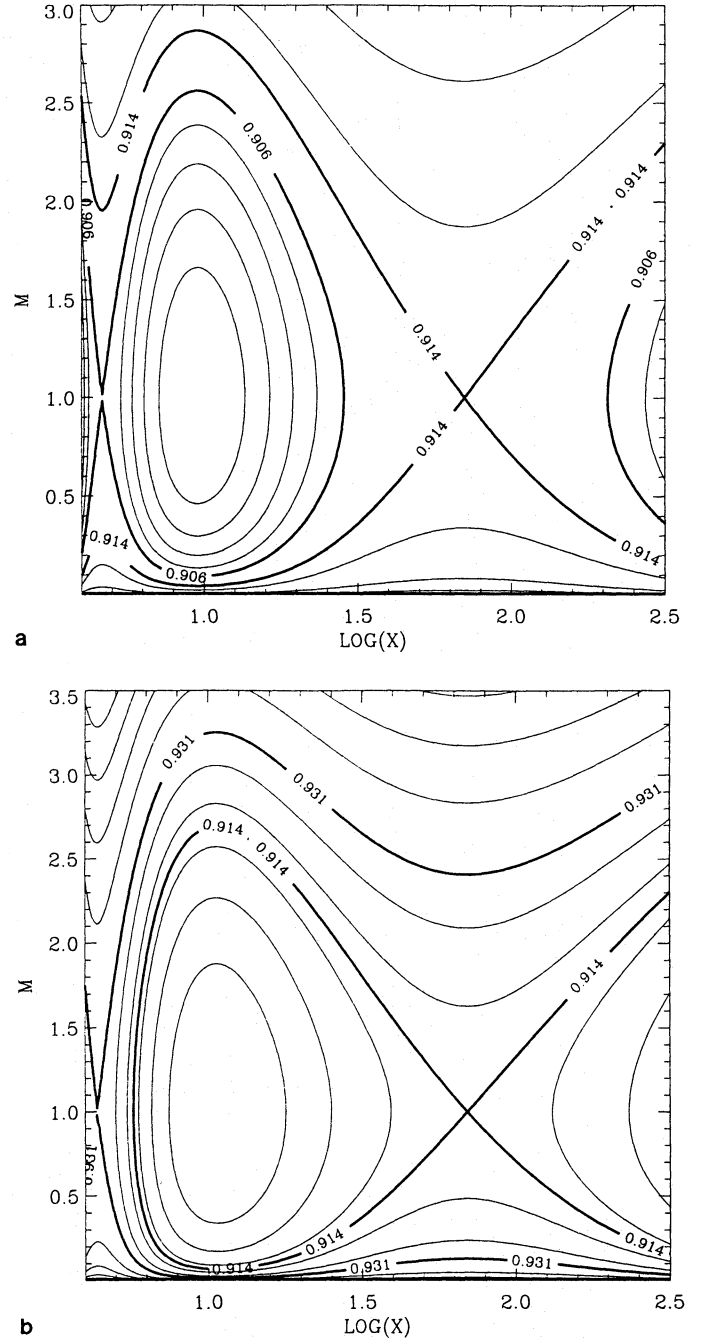


Fig. 1. Flow pattern. The energy contours in the M - X plane. The parameters are: $B = 0.0059113$; **a** $l = 3.74$, and **b** $l = 3.82$, respectively

flow. Figure 1 shows two typical energy contours as functions of the Mach number M and the radial coordinate X . The thicker solid lines passing through the saddle type sonic points ($M = 1$) represent transonic solutions. One of them, subsonic at far field and supersonic near the black hole denotes the stationary accretion flow solution. There exists a critical angular momentum l_c ($l_A < l_c < l_B$) for which the energy at the two sonic points is the same. Moreover, when $l < l_c$, the energy at the outer sonic

point (E_0) is larger than that at the inner one (E_i) (see Fig. 1a). When $l > l_C$, on the other hand, then $E_i > E_0$ (see Fig. 1b).

4. Shocks and the associated jump conditions

The existence of multiple sonic points is evident in Fig. 1a and b. Although only one of the transonic shock-free solutions is physical in the sense that this solution is valid in the entire radial domain of accretion, from the event horizon to infinity, there also exists a possibility for a shock to appear between the two sonic points. In other words, there is a possibility that the subsonic flow becomes supersonic after passing through the outer sonic point. Then the flow may go through a stationary shock to become subsonic. Finally, the subsonic flow may accelerate to become supersonic as it passes through the inner sonic point. The total flow field is then subsonic at infinity and supersonic near the black hole, a physically reasonable solution for accretion onto black holes.

The jump conditions across a stationary shock in the relativistic flow are discussed below. Across a stationary shock, the mass conservation gives

$$[[nu]] = 0, \quad (12)$$

which is the same condition as that in non-relativistic flows. Here, $[[f]]$ denotes the difference $f_+ - f_-$, and “-” and “+” denote the values before and after the shock, respectively. The momentum conservation condition (Anile & Russo 1986; Landau & Lifshitz 1987) is given by $[[T^{rr}]] = 0$, where T^{rr} is the rr component of the four energy-momentum tensor. In the problem discussed here, the momentum jump condition is reduced to

$$\left[\left[n \left(\frac{u^2}{B} + 1 - \frac{2}{X} \right) \right] \right] = 0. \quad (13)$$

From the jump conditions given in (12) and (13), we can solve for u_+ as a function of u_- . One of the solutions is the trivial one $u_+ = u_-$ and the other is

$$u_+ = \frac{Bs^2}{u_-}. \quad (14)$$

When gravity is weak, $s^2 \rightarrow 1$, and the flow velocity is much smaller than the speed of light, then $\hat{u} \rightarrow u$, and Eq. (14) reduces to $u_+u_- = B$ which is identical to the condition $M_+M_- = 1$ (cf. Eq. (12) of Chakrabarti 1989). Therefore, the jump conditions given above are consistent with those obtained in the pseudo-Newtonian model in the classical limit.

Using Eq. (10) to replace u , the above relations give

$$\frac{M_+}{\sqrt{1 - M_+^2 B}} = \frac{\sqrt{1 - M_-^2 B}}{M_-}, \quad (15)$$

or

$$M_+ = \frac{\sqrt{1 - M_-^2 B}}{\sqrt{(1 - B^2)M_-^2 + B}}. \quad (16)$$

It is worth noting that Eq. (16) which describes the relation between the Mach numbers after and before the shock is independent of the actual location of the shock. This means that gravitational effects (the terms involving s^2) do not affect the shock conditions. This is because the shock is a local phenomenon. Therefore, no matter what kind of gravitational law one chooses, the resultant shock condition is the same. However, as shown by the shock condition (Eq. (16)), special relativistic effects do affect the condition when the sound speed is comparable to the light speed ($B \sim 1$). When $B \ll 1$, the condition reduces to the pseudo-Newtonian result $M_+ = 1/M_-$.

Following Chakrabarti (1989), we define

$$\frac{M}{\sqrt{1 - M^2 B}} + \frac{\sqrt{1 - M^2 B}}{M} = 2C, \quad (17)$$

and because of Eq. (15), in the above equation, M can be either M_+ or M_- , that is, C is a constant across the shock. For a given C , we obtain the results

$$M_+^2 = \frac{2C^2 + B - 1 - 2C\sqrt{C^2 - 1}}{(1 - B)^2 + 4C^2 B}, \quad (18)$$

and

$$M_-^2 = \frac{2C^2 + B - 1 + 2C\sqrt{C^2 - 1}}{(1 - B)^2 + 4C^2 B}. \quad (19)$$

When $C = 1$, $M_+^2 = M_-^2 = \frac{1}{1+B}$. Using these two relations, we write the two energy equations for the special energy values at the sonic points. Solving then the two coupled equations, we obtain the locations of shocks and their corresponding Mach numbers. For accretion onto black holes, we are interested in only those shocks existing between the two sonic points.

The locations of the possible shocks are numerically investigated in the same way as done by Chakrabarti (1989). Figure 2 shows the energy contours which pass through the two sonic points in the $C-X$ plane. The intersection points are the possible shock locations. A detailed study yields that, for $l < l_{C1}$, there are two such locations. One of them is inside the inner sonic point and the other is outside the outer sonic point. Therefore, in these cases, there are no physically allowed shocks for accretion onto black holes. When, however, $l_{C1} < l < l_C$, there are two possible shock locations between the sonic points, X_{s2} and X_{s3} where we have followed the nomenclature of Chakrabarti (1989). We find that our results for the relativistic conical flows are qualitatively the same as those in Chakrabarti's vertically integrated pseudo-Newtonian model but with some quantitative differences (Chakrabarti 1989). For example, when we choose the following values of the parameters $B = 0.000754987$, $l = 3.64$, we find that there are no shocks between the sonic points. These conditions are equivalent to $l = 1.82$, $K = 0.08689$ in Chakrabarti (1989), for which Chakrabarti (1989) finds possible shocks between the sonic points.

It should be mentioned that in Chakrabarti (1989), a 1.5 dimensional system was used which took into account the existence of vertical hydrostatic equilibrium. In this work, only a

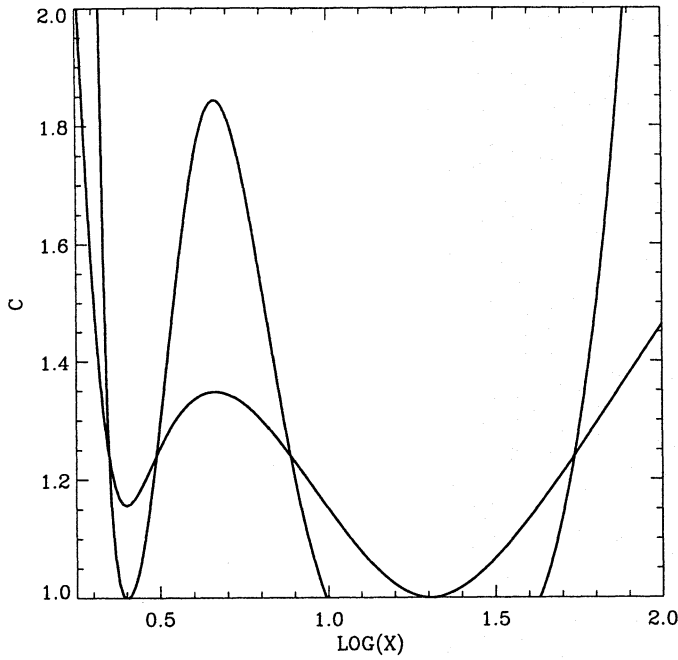


Fig. 2. The energy contours passing sonic points in the C - X plane. The parameters are: $B = 0.0059113$; $l = 3.76$

one-dimensional conical model is used. Even if the two treatments are identical for certain limiting conditions, the difference between the models may result in quantitative differences for the global properties such as the shock locations, as shown above. However, since the shock discussed here is a local phenomenon and the thickness of the disk is not expected to change across the isothermal shock, whether we consider vertical condition or not should not affect the jump conditions across the shock. As a consequence, any differences between the jump conditions in the two models must be due to the relativistic treatment adopted here. The following provides an example.

The shock strength can be measured by the ratio of the energy before the shock to that after the shock where we find that

$$E_-/E_+ = [(1 - BM_-^2)B + M_-^2]^{-1} \left(\frac{1 - M_-^2 B}{M_-^2} \right)^{2B-1}. \quad (20)$$

Physically, it is required that $E_-/E_+ > 1$ for dissipative shocks. The difference between E_- and E_+ is accounted for as loss of energy at the location of the shock due to dissipative effects. This requirement limits the range of Mach numbers for which isothermal shocks are possible. In non-relativistic gas dynamics, the condition is $M_- > 1$, that is, shocks only appear in supersonic flows. The corresponding conditions in relativistic flows are investigated here.

From Eq. (14), it is possible to find that when $u_+ = u_- = \sqrt{Bs}$, $M_+ = M_- = 1/\sqrt{1+B}$ which results in $E_-/E_+ = 1$. This is similar to $M_- = M_+ = 1$ in the non-relativistic case. However, since $1/\sqrt{1+B} < 1$, it follows that the minimum value of the Mach number which equals unity for the presence of shocks in a relativistic flow is not appropriate. To find the real

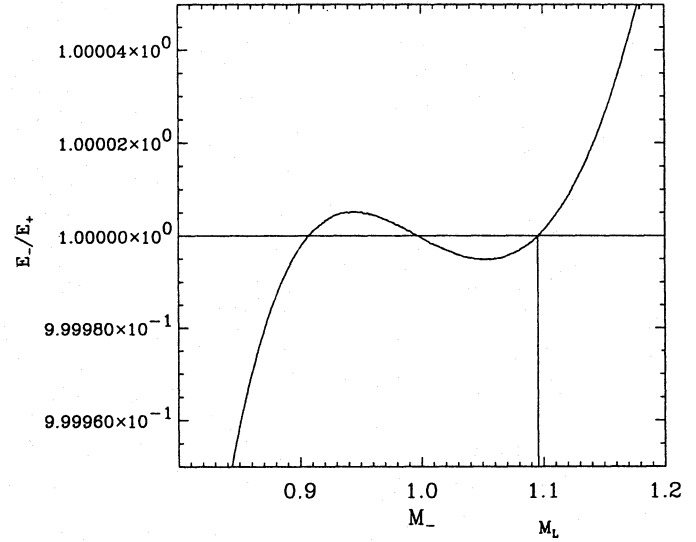


Fig. 3. Variation of energy ratio E_-/E_+ with the Mach number M_- ($B = 0.0059113$)

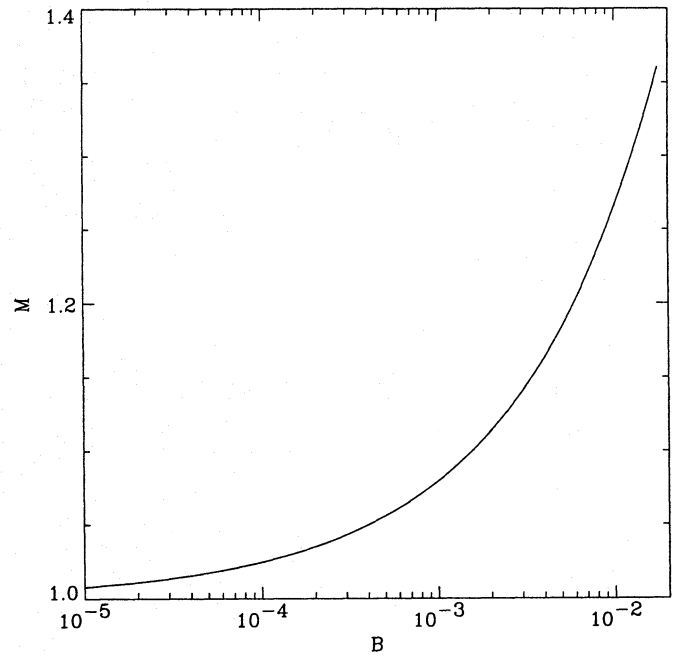


Fig. 4. Variation of the critical Mach number M_L with parameter B

condition, we plot E_-/E_+ as a function of M_- for a given value of B as shown in Fig. 3. It is clear from this figure that at $M_- = 1/\sqrt{1+B} < 1$, $E_-/E_+ = 1$. Nevertheless, in the vicinity of this value, E_-/E_+ is a decreasing function of M_- , and therefore, $M_- = 1/\sqrt{1+B}$ is not the lower limit of M_- for shocks to exist. Also from the Fig. 3, we find that E_-/E_+ reaches a minimum value (< 1) and then increases with M_- . M_L denotes the point at which $E_-/E_+ = 1$ again, and $E_-/E_+ > 1$ for $M_- > M_L$. Thus, this value of M_L (> 1 and ~ 1.1 in the case shown) is the physically lower limit of the Mach number above which shocks become possible in a flow. Figure 4 shows the variation

of M_L with the parameter B . When $B \rightarrow 0$, $M_L \rightarrow 1$, i.e., the relativistic condition in that limit reduces to the situation of non-relativistic flows. In addition, it should be pointed out that M_+ is a monotonically decreasing function of M_- (see Eq. (16)). Because $M_+ = 1/\sqrt{1+B}$ when $M_- = 1/\sqrt{1+B}$, it follows that $M_+ < 1/\sqrt{1+B} < 1$ when $M_- > M_L$. In other words, as in the non-relativistic case, we still obtain the result that across the shock, the flow changes from supersonic to subsonic.

The existence of M_L in relativistic isothermal shock conditions can be considered as a significant departure from the classical results. The value of M_L depends on the value of B and approaches unity as $B \rightarrow 0$. Since the value $B_c = 0.017548$, is small, the difference due to special relativistic effects is not very large as shown by Fig. 4, but nevertheless, there is a qualitative difference. This qualitative difference can also be illustrated by using the Mach number relation at the critical point. At that point, $M_- = 1$, and Eq. (16) gives $M_+ = \sqrt{1-B}/\sqrt{1-B^2+B} \neq 1$. This is totally different from the classical shock condition for which $M_- = 1$ leads to $M_+ = 1$.

It should be pointed out that an isothermal relativistic shock may start from strength arbitrary close to unity measured by the ratios of pressure or particle density after and before the shock. But for the accretion onto a black hole, it is required that the flow after a shock passes through the inner sonic point. This condition and the energy loss condition (energy cannot increase across a shock) make the shock first appear in a location where $M = M_L$.

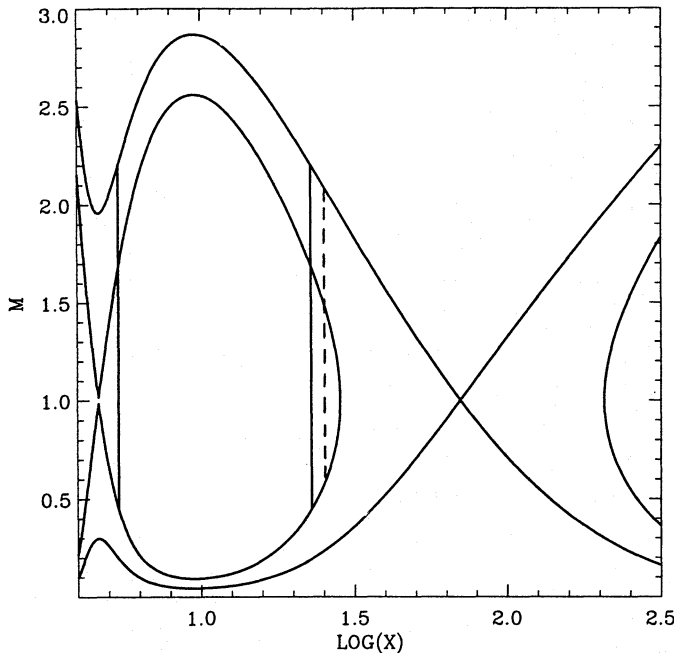


Fig. 5. Shock locations and their stabilities. Curves are the energy contours passing sonic points in M - X plane. Two solid straight lines are the possible stationary shock locations. Dashed line denotes the shock location after a small perturbation on the location of the stationary shock at X_{s3} . The parameters are: $B = 0.0059113$; $l = 3.74$

5. Stability of shocks

Instead of following Chakrabarti's (1989) stability analysis in the present fully relativistic case, we investigate this problem in a direct, physical way rooted from corrugation stability (Anile & Russo 1986; Whitham 1974; see also Chakrabarti & Molteni 1993). However, we need 2-D (in the r - ϕ plane) unsteady governing equations to fully explore the corrugation stability. Our corrugation analysis is, therefore, also limited. Let us consider a shock as shown in Fig. 5 at $X = X_s$. Across the shock, the momentum flux $F \equiv n(u^2/B + 1 - 2/X)$ (a constant factor is eliminated in defining F) is conserved. That is, the amounts of the momentum flowing in and out of the shock are the same, and the shock is stationary. If due to some perturbation, the shock is moved to $X_s + \Delta X$, a new location, the momentum flux may not be in balance. Defining

$$\Delta F = F_+ - F_- = \left(\frac{dF}{dX} \Big|_+ - \frac{dF}{dX} \Big|_- \right) \Delta X \equiv \Delta \cdot \Delta X,$$

we may discuss the stability according to the sign of the value Δ . If $\Delta > 0$, when $\Delta X > 0$, there is more momentum flux flowing out of the shock than the momentum flux flowing in, and the shock should move to the right; when $\Delta X < 0$, the imbalance of the momentum flux would cause the shock to move to the left. In short, when $\Delta > 0$, the change of the momentum flux due to ΔX results in a further increase of ΔX . Therefore, the shock is unstable. On the contrary, when $\Delta < 0$, the change due to ΔX results in the further decrease of ΔX , and the shock is stable.

Before we analyze the stability of the shocks for the fully relativistic condition, we first apply the method described above to pseudo-Newtonian flows. In this case, using the same notations as those in Chakrabarti (1989), we find that $F = (K^2 + u^2)\Sigma$, where K is the sound speed, u the radial velocity and Σ the surface density. At the shock location, it is found that

$$\frac{du_+}{dX} = -\frac{du_-}{dX}, \quad (21)$$

and Δ is reduced to

$$\Delta = \frac{\dot{\mu}}{X} \left(M_-^2 + \frac{1}{M_-^2} - 2 \right) \frac{du_-}{dX} \quad (22)$$

where $\dot{\mu}$ denotes the accretion rate. Since the real function $x + 1/x > 2$ for all $x \neq 1$, we have $M_-^2 + \frac{1}{M_-^2} - 2 > 0$ and

$\text{sign}(\Delta) = \text{sign}\left(\frac{du_-}{dX}\right)$. From this we can conclude that when $\frac{du_-}{dX} > 0$, the shock is unstable; when $\frac{du_-}{dX} < 0$, it is stable. In this sense, the shock at X_{s2} is unstable and the shock at X_{s3} is stable, i.e., a unique shock location is obtained. In a similar analysis, Chakrabarti & Molteni (1993) find that the shock at X_{s3} is stable but their analysis does not give a definitive conclusion about the shock at X_{s2} .

Going back to our fully relativistic case, we get

$$\Delta = \frac{2n_-}{X^2} \left(\frac{u_-^2}{Bs^2} - 1 \right) + \frac{n_- u_-}{B} \left(\frac{du_+}{dX} - \frac{du_-}{dX} \right)$$

$$-\frac{s^2 n_-}{u_-} \left(\frac{u_-^4}{B^2 s^4} \frac{du_+}{dX} - \frac{du_-}{dX} \right). \quad (23)$$

Because of relativistic effects, a relation similar to Eq. (21) cannot be found, i.e., $\frac{du_+}{dX} \neq -\frac{du_-}{dX}$ unlike the condition $\frac{du_+}{dX} = -\frac{du_-}{dX}$ in the pseudo-Newtonian model. Consequently, it is difficult to estimate the actual sign of Δ . We, therefore, have evaluated Δ through numerical calculations.

One significant difference between the fully relativistic result of the present work and the pseudo-Newtonian one is that since $u_- > Bs^2$ when $\frac{du_+}{dX} = \frac{du_-}{dX} = 0$, $\Delta > 0$. Therefore, near that point, the shock in the relativistic situation is unstable. Even so, the numerical evaluation demonstrates that the effect from the first term of RHS of Eq. (23) is very weak and the stabilities of the shocks behave in the same way as those in the pseudo-Newtonian model. In other words, the shock at X_{s2} is unstable, but the shock at X_{s3} is stable except when X_{s3} is very close to the point at which $\frac{du_-}{dX} = 0$.

Now, we can choose the angular momentum l as a parameter in our problem to discuss the transonic flows in black hole accretion with $B < B_c$. There are some special values of angular momentum, l_A , l_{C1} , l_C and l_B as defined above, separating the different regions (cf. Paper I). When $l < l_A$, the sonic point is unique, and the resultant transonic flow is an unique shock-free solution to equations (1) and (2). Also the accretion rate, as expected, decreases with increasing angular momentum. When $l_A < l < l_B$, there are two physically acceptable sonic points. But the shock-free solution can pass through only the outer sonic point when $l < l_C$. Inversely, when $l > l_C$, the shock-free solution can only pass through only the inner one. Actually, the accretion is limited to a certain angular momentum range $0 \leq l < l_D$ with l_D generally being much smaller than l_B (Paper I).

Although the governing Eqs. (1) and (2) admit shock-free stationary solutions for the entire range of $0 \leq l < l_D$, stationary solutions with shocks are also possible. For accretion onto black holes, dissipative shocks may appear only when $l_{C1} < l < l_C$. When $l > l_C$, if we use the isothermal shock conditions, we then find that $E_-/E_+ < 1$. It is impossible for this to happen for any physical shock unless there were other mechanisms to generate the extra energy needed at the shock. Shocks could exist in wind flows in this case, but wind solutions are generally not physically allowable for black holes due to the existence of the inner boundary condition where near the black hole itself $\hat{u} \rightarrow 1$ (unless other local physics could initiate such wind flows, e.g., evaporation from a corona).

6. Discussion and conclusions

The multiplicity of the sonic points is a necessary condition for the existence of shocks, but not a sufficient one. It has been found that for a special class of adiabatic accretion flows, namely fully ionized pure hydrogen gas with gas dominated pressure, no multiple sonic points are present no matter what the temperature of the disk is, either ‘‘hot’’ or ‘‘cool’’ (Paper I). For isothermal flows, only when the temperature is below 1.8×10^{11} K ($B <$

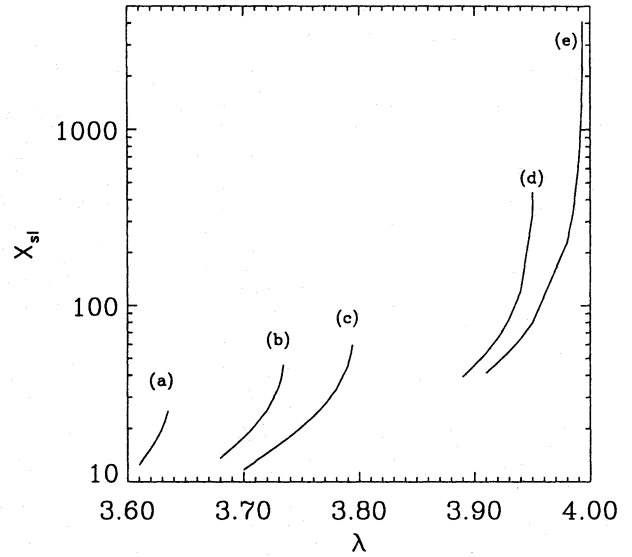


Fig. 6. Variations of the stable shock locations X_{s1} with the angular momentum under different temperatures. (a) $T = 10^{11}$ K, (b) $T = 7 \times 10^{10}$ K, (c) $T = 5.2 \times 10^{10}$ K, (d) $T = 10^{10}$ K, (e) $T = 10^9$ K

0.017548), are shocks possible. Accordingly, one may conclude that even in the condition of forced heating or cooling, which leads to the variation of the polytropic index (Chang & Ostriker 1985), the shock is unlikely to form if maximum temperatures in hot, two-temperature disks ($B \sim 0.1$) are achieved.

By using the fully relativistic equations to study the shocks, we find that when the sound speed is comparable to the speed of light, the effects of special relativity become non-negligible. The pseudo-Newtonian model (Chakrabarti 1989) does not include these effects. The major difference resulting from this effect is the minimum Mach number for shocks. If relativity does not apply, the minimum value is $M_- = 1$. Conversely, with the relativistic effects, one must have $M_- = M_L(B) > 1$, before a shock may form.

The method of stability analysis adopted here is totally different from that used by Chakrabarti (1989), but is similar to that by Chakrabarti & Molteni (1993). For accretion onto black holes, our stability analysis shows that only one of the two possible shocks between the sonic points is stable. As a result, the location of the possible shock is unique. Figure 6 demonstrates the variations of the stable shock locations with the angular momentum for different temperatures.

Standing shocks have been proposed by Chakrabarti & Wita (1992) to modify the observed UV spectra of AGN accretion disks. For the quasar 2130+099, they find that $T \gtrsim 5 \times 10^4$ K for $r \lesssim 6r_g$. Our theoretical isothermal disks examined here would be considerably hotter, although still not as hot as the two-temperature disks postulated by Becker et al. (1994) to explain the γ -ray emission from the blazar 3C 279 (Hartman et al. 1992). If indeed temperatures as low as $T \sim 10^8$ K hold for 3C 273 (Chang & Ostriker 1985), our theoretical disks would apply to this AGN. The location of the stable shock is very sensitive to the values of the specific angular momentum l for such

temperatures. For example, when l varies from 3.900 to 3.999, the location changes from $X \sim 36$ to $X \sim 10,254$. In those cases, and assuming the mass of the black hole $M_{\text{bh}} \sim 10^9 M_{\odot}$, the variability time scale based on the shock locations ranges from about one day to about three years. If we know the variability time scale, the angular momentum could in principle be accurately determined from the above discussion.

It has been found that a number of galactic black hole candidates including Cygnus X-1 and J0422+35 show QPO behavior (Kouveliotou et al. 1993). Since no solid surface for the black hole can be found in contrast to the case for neutron stars, it is difficult to understand theoretically such behavior. We speculate that QPO behavior may be explained in transonic, isothermal accretion disks as acoustic waves interacting with the shock. One frequency would be $\Omega(X_s)$, the orbital frequency at the sonic radius; the other frequency would be $\Omega(X_{\text{sl}})$, the orbital frequency at radius X_{sl} where the shock is located. This gives an estimate on the QPO frequency $\nu_{\text{QPO}} = 2 \times 10^4 l [1 - (\frac{X_{\text{sl}}}{X_s})^2] / X_{\text{sl}}^2$ for stellar black holes of mass $10 M_{\odot}$. Using the values obtained from observations, one may determine the parameters such as temperature and angular momentum. For instance, if we pick that the time scale for the variability of Cygnus X-1 is ~ 50 ms, we find that the corresponding temperature is $T_0 = 4.8 \times 10^9$ K and $X_s = 1,028$. Furthermore, the observed QPO frequency for Cygnus X-1 is $\nu_{\text{QPO}} = 0.04 H_Z$ (Kouveliotou et al. 1993), from which we find that $X_{\text{sl}} = 829.4$, and the corresponding angular momentum $l \approx 3.9736$.

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