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# Capacity Analysis for Gaussian and Discrete Memoryless Interference Networks

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# *Abstract*

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Interference is an important issue for wireless communication systems where multiple uncoordinated users try to access to a common medium. The problem is even more crucial for next-generation cellular networks where frequency reuse becomes ever more intense, leading to more closely placed co-channel cells. This thesis describes our attempt to understand the impact of interference on communication performance as well as optimal ways to handle interference. From the theoretical point of view, we examine how interference affects the fundamental performance limits, and provide insights on how interference should be treated for various channel models under different operating conditions. From the practical design point of view, we provide solutions to improve the system performance under unknown interference using multiple independent receptions of the same information.

For the simple two-user Gaussian interference channel, we establish that the simple Frequency Division Multiplexing (FDM) technique suffices to provide the optimal sum-rate within the largest computable subregion of the general achievable rate region for a certain interference range.

For the two-user discrete memoryless interference channels, we characterize different interference regimes as well as the corresponding capacity results. They include one-sided weak interference and mixed interference conditions. The sum-rate capacities are derived in both cases. The conditions, capacity expressions, as well as the capacity achieving schemes are analogous to those of the Gaussian channel model. The study also leads to new outer bounds that can be used to resolve the capacities of several new discrete memoryless interference channels.

A three-user interference up-link transmission model is introduced. By examining how interference affects the behavior of the performance limits, we capture the differences and similarities between the traditional two-user channel model and the channel model with more than two users. If the interference is very strong, the capacity region is just a simple extension of the two-user case. For the strong interference case, a line segment on the boundary of the

capacity region is attained. When there are links with weak interference, the performance limits behave very differently from that of the two-user case: there is no single case that is found of which treating interference as noise is optimal. In particular, for a subclass of Gaussian channels with mixed interference, a boundary point of the capacity region is determined. For the Gaussian channel with weak interference, sum capacities are obtained under various channel coefficients and power constraint conditions. The optimalities in all the cases are obtained by decoding part of the interference.

Finally, we investigate a topic that has practical ramifications in real communication systems. We consider in particular a diversity reception system where independently copies of low density parity check (LDPC) coded signals are received. Relying only on non-coherent reception in a highly dynamic environment with unknown interference, soft-decision combining is achieved whose performance is shown to improve significantly over existing approaches that rely on hard decision combining.

# Capacity Analysis for Gaussian and Discrete Memoryless Interference Networks

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# Abbreviations

| | |
|---|---|
| **BC** | **B**roadcast **C**hannel |
| **DMDIC** | **D**iscrete **M**emoryless **D**egraded **I**nterference **C**hannel |
| **DMIC** | **D**iscrete **M**emoryless **I**nterference **C**hannel |
| **DMZIC** | **D**iscrete **M**emoryless **Z** **I**nterference **C**hannel |
| **DSC** | **D**ARPA **S**pectrum **C**hallenge |
| **eICIC** | **e**nhanced **I**nter**C**ell **I**nterference **C**oordination |
| **FDG** | **F**unctional **D**irectional **G**raph |
| **FDM** | **F**requency **D**ivision **M**ultiplexing |
| **GDoF** | **G**eneralized **D**egree **o**f **F**reedom |
| **GIC** | **G**aussian **I**nterference **C**hannel |
| **GZIC** | **G**aussian **Z** **I**nterference **C**hannel |
| **HDC** | **H**ard **D**ecision **C**ombining |
| **HK** | **H**an and **K**obayashi |
| **IC** | **I**nterference **C**hannel |
| **ICIC** | **I**nter**C**ell **I**nterference **C**oordination |
| **LDPC** | **L**ow **D**ensity **P**arity **C**heck |
| **LTE** | **L**ong **T**erm **E**volution |
| **LTE-A** | **L**ong **T**erm **E**volution - **A**dvanced |
| **MAC** | **M**ultiple **A**ccess **C**hannel |
| **MAZIC** | **M**ultiple **A**ccess **Z** **I**nterference **C**hannel |

| | |
|---|---|
| **SDC** | **S**oft **D**ecision **C**ombining |
| **TDM** | **T**ime **D**ivision **M**ultiplexing |

# Notations

| | |
|---|---|
| $X$ | random variable or matrix |
| $\mathbf{x}$ | a vector |
| $A \succeq B$ | A-B is positive semi-definite |
| $\mathbf{I}$ | identity matrix |
| $\mathbf{0}$ | zero matrix |
| $tr(X)$ | trace of a matrix $X$ |
| $X^T$ | transpose of a matrix $X$ |
| $X^n$ | $[X_1, X_2, \cdots, X_n]$ a vector with the element $X_i$ |
| $X_i^j$ | $[X_i, \cdots, X_j]$ |
| $\mathcal{X}$ | a set |
| $\lvert \cdot \rvert$ | the size (cardinality) of a set |
| $cvx(\cdot)$ | convex hull |
| $\overline{\mathrm{co}}(\cdot)$ | closure of convex hull |
| $\mathcal{N}(\mu, \mathbf{\Sigma})$ | Gaussian distribution with mean $\mu$ and covariance matrix $\Sigma$ |
| $H(\cdot)$ | entropy function |
| $h(\cdot)$ | differential entropy function |
| $h_2(\cdot)$ | entropy function of a binary random variable |
| $I(\cdot;\cdot)$ | mutual information |
| $E[\cdot]$ | expectation |
| $\log(\cdot)$ | logarithm function of base 2 |

# Chapter 1

# Introduction

## 1.1 Introducing the background

The mathematical theory of communication was born out of Claude E. Shannon's classic paper in 1948 [1]. Shannon introduced the concept of channel capacity: the maximum rate that can be achieved over a channel with asymptotically small probability of error. The simple yet elegant notion has established for the first time that reliable communications at a nonzero rate through noisy channels are possible. Shannon's original work has been the cornerstone of all the major breakthroughs in telecommunications, especially in systems where point-to-point communications can be studied in an isolated manner.

However, telecommunication systems have evolved dramatically over the past couple of decades and current and future wireless systems often involve multiple transceiver pairs. As such, interference is inevitable as multiple users try to access a common medium. In most of the existing systems, the interference is dealt with either by interference avoidance, in which the communication links are orthogonalized in time or frequency, or by treating interference as noise, often assisted by power control at the transmitters. These approaches are typically not optimal, and often lead to the loss of degrees of freedom. One long-standing problem in information theory is to study the theoretical communication limits when operating in the presence of interference. The basic model that captures the essence of interference is the so-called interference channel (IC), which mathematically abstracts the situation where the transmitters communicate concurrently with their intended receivers while generating interference to unintended receivers.

Despite decades of intensive research, the capacity region of the two-user IC remains unknown except for a few special cases. Nevertheless, recent progresses have been made towards characterizing the sum-rate capacity for certain Gaussian ICs (GIC). A GIC, as to be defined in the next chapter, is a linear channel model where the received signal at each receiver is a superposition of the intended signal, interference, and additive Gaussian noise. Close examination of the literature reveals a strong parallel, both in terms of capacity region and capacity achieving coding schemes, between two classes of interference channels: the discrete memoryless interference channel (DMIC) and the GIC. This is manifested in the analogous results when the interference is strong or very strong relative to the strength of the intended signal. This bears the question about whether some very recent breakthrough in characterizing the sum-rate capacity may also have counterpart in the DMIC?

Current and future cellular systems have seen exploding demand of wireless data transmissions; as such, the need for spectrum reuse increases, which leads to ever decreasing cell sizes and densely placed co-channel cells. Inter-cell interference can no long be neglected for both down-link and up-link transmissions. The conventional way of modeling the down-link transmission as a broadcast channel (BC) and up-link transmission as a multiple-access channel (MAC) is no longer applicable. Indeed, inter-cell interference coordination (ICIC) has been a active area of research in Long-Term Evolution (LTE) standard, and later extended to the enhanced ICIC (eICIC) in LTE-advanced (LTE-A). Naturally, fundamental performance limits of either BC or MAC in the presence of co-channel interference becomes highly relevant research problems from a practical perspective.

At the other end of the spectrum, there exist systems where active interference management (e.g., interference cancellation) is impossible due to the lack of user coordination or spectrum pre-planning. An example is the DARPA Spectrum Challenge that took place from January 2013 through March 2014. For such applications, treating interference as noise is often out of practical necessity despite of its sub-optimality for the interference network. Therefore, the design objective to ensure robustness to the unknown interference is to attain the desired balance and reliability in a highly dynamic transmission environment.

This thesis addresses interference in multi-user wireless systems from both a theoretical perspective and a practical design viewpoint.

- For a two-user GIC, we examine the potential optimality in terms of sum-rate of orthogonal transmissions in the moderate interference regime.
- For discrete memoryless interference channels, we derive parallel capacity results that are inspired by recent breakthroughs in characterizing the sum-rate capacity of GICs.
- Compound ICs that capture the co-channel interference in up-link transmissions are studied; fundamental performance limits are characterized under different interference regimes.
- Diversity combining using soft-decision output of non-coherently modulated and LDPC coded signals is studied that provides robust performance in the presence of unknown interference.

## 1.2 Main contributions

This dissertation is devoted to the understanding and the management of interference in multi-user networks. Major contributions are summarized below.

1. The largest computable achievable region of the classic two-user GIC is examined. We show that the naïve FDM/TDM turns out to be sum rate optimal for a certain range of channel parameters within this class of computable achievable region.

2. The one-sided weak interference condition for the classic two-user discrete memoryless interference channel is established, whose channel property resembles that of the Gaussian interference channel with one-sided weak interference. Under the definition of one-sided weak interference, the sum-rate capacity is derived, which is achieved by letting the transceiver pair subject to interference communicate at a rate such that its message can be decoded at the unintended receiver using single user detection. This capacity achieving scheme, as well as the resulting capacity expression, are analogous to that of the Gaussian case. In addition, it is established that this class of discrete memoryless interference channels is equivalent in capacity region to certain degraded interference channels. It yields an outer-bound of the capacity region using the associated degraded broadcast channels. The same technique is then used to determine the mixed interference condition, and the sum-rate capacity under the defined condition. The obtained outer-bound and sum-rate capacities can resolve the capacities of several new discrete memoryless interference channels.

3. The capacity of an up-link network with co-channel interference is studied. By modeling such networks using a multiple-access interference channel with one-sided interference, we have obtained an inner bound to the capacity region for both the discrete memoryless case and the Gaussian case. The capacities are examined under different interference conditions: the strong interference, the mixed strong interference, the mixed interference, and the weak interference. The capacity region for the discrete memoryless channel model with strong and very strong interference has been established. For the Gaussian setting, we have 1) determined the capacity region for the very strong interference case, and for the case in which one interference link is strong and the other one is very strong; 2) obtained a boundary line segment of the capacity

region for the strong interference case. For the mixed interference case, a boundary point of the capacity region has been obtained. For the weak interference case, a sum-rate upper bound has been established which gives rise to a sum-rate capacity result under certain power constraints. Different from that of the two-user IC, partial interference cancellation plays an essential part even in the weak interference regime for the multiple-access interference channel model.

4. The last research topic in the thesis addresses a practical physical layer design problem. It is motivated by the prevalent use of diversity receptions which multiple copies of the same information are often available at the receiver end. A novel non-coherent combining scheme is proposed, and the performance is simulated. It is shown that the designed scheme using soft-decoding output is used that provides the noticeable improvement over existing combining techniques, and is robust against fading and interference environment.

## 1.3 Outline of thesis

The thesis intends to make progress toward a better understanding on addressing interference problem along two directions: theoretical limitations through capacity analysis (Chapters 3, 4, 5, and 6); and improved solutions for practical design of information reception in wireless interference networks (Chapter 7).

The rest of the thesis is organized as follows.

We start with a comprehensive overview of state of the art in interference channels in Chapter 2, which motivated most of the topics in this thesis.

Starting with the achievable rates for the two-user GIC, Chapter 3 shows that the widely used technique - FDM/TDM, is actually optimal within the largest computable subregion of the general achievable rate region in a specific parameter range.

Chapter 4 is focused on extending the current sum-rate capacity results to the discrete memoryless channel model. In particular, the condition for one-sided weak interference is defined for the discrete memoryless channel model. This condition, which parallels that of the GIC, leads to the sum-rate capacity. We move on to outer-bound the capacity region by establishing the equivalence between this class of channels and certain degraded interference channel. After that, some examples are provided showing the sum-rate capacities and outer-bounds for a vast number of discrete memoryless interference channels. Some of the examples lead to the whole capacity region by using the proposed techniques.

In Chapter 5, we further extend the technique to define the mixed interference for the discrete memoryless channel model. For this case, we also derive the sum-rate capacity. Then, examples are provided to use the new proposed techniques to resolve some simple channels, in which some of them can lead to the full capacity region.

In Chapter 6, a three-user interference network is addressed. It is a much complex case than the classic two-user case. Exact capacity results are derived for strong, very strong, mixed and weak interference cases. In particular, for all the cases, we provide some insightful discussions to analyze the distinction and analogies between the two-user and multiple-user cases.

A non-coherent soft combining scheme is proposed in Chapter 7, in order to enhance the information reception over wireless channels suffering from intense interference.

Chapter 8 concludes this dissertation and points out potential future research topics.

# Chapter 2

# Preliminaries and Background Theory

## 2.1 Two-user interference channels

### 2.1.1 Discrete memoryless interference channels

A discrete interference channel is specified by its input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and the channel transition probabilities

$$\begin{aligned} p(y_1|x_1x_2) &= \sum_{y_2 \in \mathcal{Y}_2} p(y_1y_2|x_1x_2), && (2.1) \\ p(y_2|x_1x_2) &= \sum_{y_1 \in \mathcal{Y}_1} p(y_1y_2|x_1x_2). && (2.2) \end{aligned}$$

The model is depicted in Fig. 2.1.

The discrete IC is said to be *memoryless* if

$$p(y_1^n y_2^n|x_1^n x_2^n) = \prod_{i=1}^{n} p(y_{1i}y_{2i}|x_{1i}x_{2i}). \tag{2.3}$$

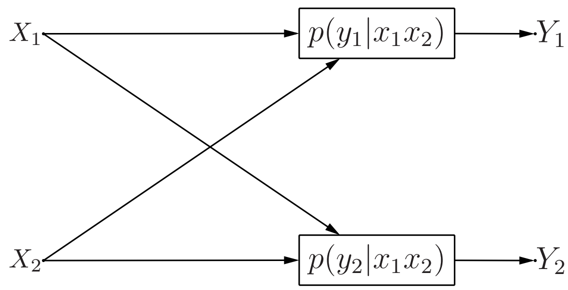



FIGURE 2.1: The discrete interference channel model.

Let $\lceil x \rceil$ be the smallest integer that is greater than or equal to $x$. A $(n, 2^{\lceil nR_1 \rceil}, 2^{\lceil nR_2 \rceil}, \lambda_1, \lambda_2)$ *code* for a DMIC with independent information consists of two message sets $\mathcal{M}_1 = \{1, 2, \cdots, 2^{\lceil nR_1 \rceil}\}$ and $\mathcal{M}_2 = \{1, 2, \cdots, 2^{\lceil nR_2 \rceil}\}$ for senders 1 and 2 respectively, two encoding functions

$$f_1 : \mathcal{M}_1 \to \mathcal{X}_1^n, \quad f_2 : \mathcal{M}_2 \to \mathcal{X}_2^n,$$

and two decoding functions

$$\varphi_1 : \mathcal{Y}_1^n \to \mathcal{M}_1, \quad \varphi_2 : \mathcal{Y}_2^n \to \mathcal{M}_2.$$

The average probabilities of error are defined as

$$\begin{aligned} \lambda_1 &= \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{w_1=1}^{2^{\lceil nR_1 \rceil}} \sum_{w_2=1}^{2^{\lceil nR_2 \rceil}} Pr\{\varphi_1(\mathbf{y_1}) \neq \mathbf{w_1} | \mathbf{W_1} = \mathbf{w_1}, \mathbf{W_2} = \mathbf{w_2}\}, \\ \lambda_2 &= \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{w_1=1}^{2^{\lceil nR_1 \rceil}} \sum_{w_2=1}^{2^{\lceil nR_2 \rceil}} Pr\{\varphi_2(\mathbf{y_2}) \neq \mathbf{w_2} | \mathbf{W_1} = \mathbf{w_1}, \mathbf{W_2} = \mathbf{w_2}\}. \end{aligned}$$

A rate pair $(R_1, R_2)$ is said to be *achievable* for a DMIC if there exists a sequence of $(n, 2^{\lceil nR_1 \rceil}, 2^{\lceil nR_2 \rceil}, \lambda_1, \lambda_2)$ codes such that $\lambda_1, \lambda_2 \to 0$ as $n \to \infty$. The capacity region of a

DMIC is defined as the closure of the set of all achievable rate pairs.

### 2.1.2 Gaussian interference channels

A two-user GIC, in its general form, has its outputs expressed as

$$\begin{aligned} Y_1 &= h_{11}X_1 + h_{21}X_2 + Z_1, \\ Y_2 &= h_{12}X_1 + h_{22}X_2 + Z_2, \end{aligned}$$

where $X_i$ is the transmitted signal; $Z_i$ is Gaussian noise; $h_{ij}$ is the channel coefficient from the $i$th transmitter to the $j$th receiver.

Carleial has shown that the received signals can be simplified into the standard from [3], with equivalent capacity region:

$$Y_1 = X_1 + aX_2 + Z_1, \tag{2.4}$$

$$Y_2 = bX_1 + X_2 + Z_2, \tag{2.5}$$

where $a$ and $b$ are the channel coefficients corresponding to the interference links; $X_i$ and $Y_i$ are the transmitted and received signals with the input sequence $X_{i1}, X_{i2}, \cdots, X_{in}$ subject to power constraints $\sum_{j=1}^{n} \mathcal{E}[X_{ij}^2] \leq nP_i,\ i = 1, 2;$ $Z_1$ and $Z_2$ are Gaussian noises with zero mean and unit variance and are independent of $X_1, X_2$. The channel model is shown in Fig. 2.2.

We review below the state-of-the-art results for the two types of interference channels.

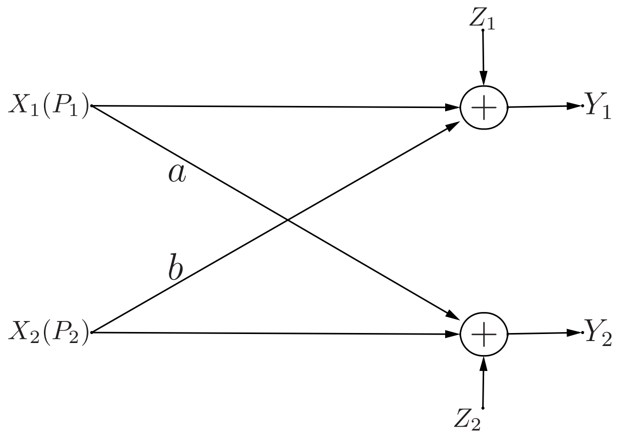



FIGURE 2.2: The Gaussian interference channel model.

### 2.1.3 The achievable rate region for the general two-user IC

The best achievable rate region for a two-user IC is still the Han-Kobayashi (HK) region [4]. It utilizes superposition coding at the transmitter and simultaneous decoding at the receiver and the obtained rate region remains to be the largest to this date. Each encoder splits its messages into two parts, which are referred to as private messages and common messages. At the receiver sides, partial interference cancellation is facilitated by allowing the common message to be decoded at the unintended receiver side. The general HK region, denoted by $\mathcal{R}_{HK}$, is defined as

$$\mathcal{R}_{HK} = \text{closure of} \bigcup_{Z \in \mathcal{P}(Z)} \mathcal{R}(Z) \tag{2.6}$$

where $\mathcal{P}(Z)$ is the set of all $Z = QU_1W_1U_2W_2X_1X_2Y_1Y_2 \in \mathcal{P}(Z)$ such that

- $U_1, W_1, U_2, W_2$ are conditionally independent given $Q$;
- $X_1 = f_1(U_1W_1|Q), X_2 = f_2(U_2W_2|Q)$ where $f_1$ and $f_2$ are deterministic encoders;
- $p(y_1, y_2|x_1, x_2)$ is the channel transition probability.

For each $q \in \mathcal{Q}$, where $\|\mathcal{Q}\| \leq 11$, $f_i(\cdot \mid q) : \mathcal{U}_i \times \mathcal{W}_i = \mathcal{X}_i, i = 1, 2$ are arbitrary deterministic functions. $\mathcal{R}(Z)$ is the set of all achievable $(R_1, R_2)$ such that $R_1 = S_1 + T_1, R_2 = S_2 + T_2$, and $S_1$, $T_1$, $S_2$ and $T_2$ are defined in [4, Equations (3.2)-(3.15)]. For the sake of completeness, we include the inequalities in the following.

$$S_1 \leq I(U_1; Y_1|W_1W_2Q), \tag{2.7}$$

$$T_1 \leq I(W_1; Y_1|U_1W_2Q), \tag{2.8}$$

$$T_2 \leq I(W_2; Y_1|U_1W_1Q), \tag{2.9}$$

$$S_1 + T_1 \leq I(U_1W_1; Y_1|W_2Q), \tag{2.10}$$

$$S_1 + T_2 \leq I(U_1W_2; Y_1|W_1Q), \tag{2.11}$$

$$T_1 + T_2 \leq I(W_1W_2; Y_1|U_1Q), \tag{2.12}$$

$$S_1 + T_1 + T_2 \leq I(U_1W_1W_2; Y_1|Q), \tag{2.13}$$

$$S_2 \leq I(U_2; Y_2|W_1W_2Q), \tag{2.14}$$

$$T_2 \leq I(W_2; Y_2|U_2W_1Q), \tag{2.15}$$

$$T_1 \leq I(W_1; Y_2|U_2W_2Q), \tag{2.16}$$

$$S_2 + T_2 \leq I(U_2W_2; Y_2|W_1Q), \tag{2.17}$$

$$S_2 + T_1 \leq I(U_2W_1; Y_2|W_2Q), \tag{2.18}$$

$$T_1 + T_2 \leq I(W_1W_2; Y_2|U_2Q), \tag{2.19}$$

$$S_2 + T_1 + T_2 \leq I(U_2W_1W_2; Y_2|Q). \tag{2.20}$$

For the Gaussian channel model, additional power constraints are added to the distribution of $X_1$ and $X_2$:

- $\mathcal{E}(X_1^2) \leq P_1, \mathcal{E}(X_2^2) \leq P_2$.

### 2.1.4 The capacity region for the very strong interference case

Carleial [3] defined the very strong interference for a GIC in the standard form as one satisfying

$$a^2 \geq 1 + P_1, \tag{2.21}$$

$$b^2 \geq 1 + P_2 \tag{2.22}$$

in Eqs. (2.4) and (2.5). In this case, interference can be decoded first and subtracted from the received signals, resulting in interference-free signals for the intended receivers. This sequential decoding scheme under the very strong interference condition achieves the following rate region

$$\mathcal{R}(P_1, P_2) = \left\{ (R_1, R_2) \left| \begin{array}{l} 0 \leq R_1 \leq \frac{1}{2}\log(1 + P_1) \\ 0 \leq R_2 \leq \frac{1}{2}\log(1 + P_2) \end{array} \right. \right\}.$$

This rate region is also a natural outer bound, hence is indeed the capacity region of the GIC under very strong interference, and is achieved with Gaussian input. For *Gaussian input*, the condition in (2.21) and (2.22) implies that

$$I(X_1; Y_1|X_2) \leq I(X_1; Y_2), \tag{2.23}$$

$$I(X_2; Y_2|X_1) \leq I(X_2; Y_1). \tag{2.24}$$

Sato in [5] imposes the above condition on a DMIC with the additional requirement that it hold for all product input and obtained the capacity region for a DMIC with very strong interference to be

$$\mathcal{R} = \left\{ (R_1, R_2) \left| \begin{array}{l} 0 \leq R_1 \leq I(X_1; Y_1|X_2) \\ 0 \leq R_2 \leq I(X_2; Y_2|X_1) \end{array} \right. \right\}.$$

Sato alluded in [5] that (2.23) and (2.24) hold for all product input may be too restrictive, i.e., "This is a sufficient condition for the coincidence of the bounds, but may not be necessary." In [6], it was established indeed that for a DMIC, the very strong interference can be relaxed to be such that conditions (2.23) and (2.24) need to be satisfied only for input distribution achieving the boundary points of the capacity region. This simple generalization broadens the class of DMIC with very strong interference and is also consistent with the GIC counterpart - it was shown in [6] that (2.23) and (2.24) may be violated with non-Gaussian input even if (2.21) and (2.22) are satisfied.

### 2.1.5 The capacity region for the strong interference case

Han and Kobayashi [4, Theorem 5.1] and Sato [7] independently obtained the capacity region of a GIC under strong interference, defined to be that satisfying $a \geq 1$ and $b \geq 1$ in Eqs. (2.4) and (2.5), as the following

$$\mathcal{R}(P_1, P_2) = \left\{ (R_1, R_2) \;\middle|\; \begin{array}{c} 0 \leq R_1 \leq \frac{1}{2}\log(1+P_1) \\ 0 \leq R_2 \leq \frac{1}{2}\log(1+P_2) \\ R_1 + R_2 \leq \min\{\frac{1}{2}\log(1+P_1+a^2P_2), \\ \frac{1}{2}\log(1+b^2P_1+P_2)\} \end{array} \right\}. \tag{2.25}$$

Clearly, this capacity region coincides with that of a compound multiple-access channel (MAC) where both receivers are expected to decode both messages. Notice that in the case of $a^2 \geq 1 + P_1$ and $b^2 \geq 1 + P_2$, the sum rate bound in (2.25) is inactive thus (2.25) includes (2.23) as its special case. Nevertheless, to achieve (2.25) under the strong interference condition, joint decoding instead of sequential decoding is required at each receiver.

In [7] Sato also conjectured the condition as well as the capacity region of DMICs under strong interfernce, which was eventually proved by Costa and El Gamal in 1987 [8]. The strong interference for a DMIC is referred to the condition that the inputs $X_1$ and $X_2$ and corresponding outputs $Y_1$ and $Y_2$ satisfy

$$I(X_1; Y_1|X_2) \leq I(X_1; Y_2|X_2), \tag{2.26}$$

$$I(X_2; Y_2|X_1) \leq I(X_2; Y_1|X_1), \tag{2.27}$$

for all product probability distribution on $\mathcal{X}_1 \times \mathcal{X}_2$.

The corresponding capacity region was shown to be the union of the rate pairs $(R_1, R_2)$ satisfying

$$\mathcal{R} = \left\{ (R_1, R_2) \left| \begin{array}{c} 0 \leq R_1 \leq I(X_1; Y_1|X_2Q) \\ 0 \leq R_2 \leq I(X_2; Y_2|X_1Q) \\ R_1 + R_2 \leq \min\{I(X_1X_2; Y_1|Q), \\ I(X_1X_2; Y_2|Q)\} \end{array} \right. \right\}, \tag{2.28}$$

where $Q$ is a time-sharing parameter of cardinality 4, and the union is over all probability distributions of the form $p(q)p(x_1|q)p(x_2|q)p(y_1y_2|x_1x_2)$, with $p(y_1y_2|x_1x_2)$ specified by the channel. It was established in [6] that the condition in (2.23) and (2.24) are consistent with the strong interference condition for a GIC. That is, for a GIC in stardard form, $a \geq 1$ and $b \geq 1$ is equivalent to (2.23) and (2.24) for all product input distribution for a GIC.

### 2.1.6 Sum-rate capacity results for GICs

It was proved in [9–11] using some extremal power inequalities that the sum-rate capacity for GICs whose channel parameters satisfy

$$a(b^2P_1 + 1) + b(a^2P_2 + 1) \leq 1$$

is

$$C_{\text{sum}} = \frac{1}{2}\log\left(1 + \frac{P_1}{1 + a^2 P_2}\right) + \frac{1}{2}\log\left(1 + \frac{P_2}{1 + b^2 P_1}\right).$$

Clearly, this sum-rate capacity is achieved using the simple scheme of treating interference as noise at each receiver; hence the corresponding interference regime is referred to as the noisy interference.

Sason [12] proved that the sum-rate capacity for GICs with one-sided weak interference, defined to be that satisfying $a \leq 1$ and $b = 0$ in Eqs. (2.4) and (2.5), is

$$C_{\text{sum}} = \frac{1}{2}\log(1 + P_2) + \frac{1}{2}\log\left(1 + \frac{P_1}{1 + a^2 P_2}\right).$$

This sum-rate capacity is achieved by letting the transceiver pair subject to interference communicate at a rate such that its message can be decoded at the unintended receiver using single user detection, and the interference-free transceiver pair communicate at the maximum rate. The GIC with one-sided interference is often referred to as the Gaussian Z interference channel (GZIC).

Motahari and Khandani [11] established that the sum-rate capacity for GICs with mixed interference ($a \leq 1$ and $b \geq 1$) is

$$C_{\text{sum}} = \min\left\{\frac{1}{2}\log\left(1 + \frac{P_1}{1 + a^2 P_2}\right), \frac{1}{2}\log\left(1 + \frac{b^2 P_1}{1 + P_2}\right)\right\} + \frac{1}{2}\log(1 + P_2).$$

To achieve this sum-rate capacity, the transceiver pair subject to strong interference communicates at a rate as if there is no interference, while the transceiver pair subject to weak interference communicates at a rate such that its message can be decoded at both receivers using single user detection.

The results for the strong and the very strong interference cases reveal a strong parallel, both in terms of capacity region and capacity achieving encoding schemes, between DMICs and GICs. However, no analogous sum-rate capacity results have been reported for discrete memoryless interference channels with noisy, mixed, and one-sided weak interference.

## 2.2 Managing interference to achieve capacities

Clearly, capacity studies for ICs reveal the not so surprising fact that interference ought to be treated differently depending on its relative strength. When the interference is *very strong*, i.e., stronger than the intended signal plus the noise, the interference is fully decoded first which is subsequently subtracted, leading to a cleaner version of the intended message that is only affected by the channel noise.

When the interference is *strong*, i.e., stronger than the intended communication link, it is better to decode the full interference with the intended receivers.

When the interference is *very weak*, i.e., weaker than the intended signal and the noise floor, it is proven that the simple way of treating interference as noise gives the best performance. This strategy is also widely used in many communication systems, in which it is so designed that the interference is typically very weak. Therefore, it makes engineering sense to ignore the presence of the interference.

When the interference is moderate, rate splitting at the transmitters derived the best achievable rate regions [4]. Rate splitting, together with superposition, enables the unintended receiver to decode part of the interference. As splitting can be done with varying weights, this approach essentially includes the two extreme cases as its special cases: completely decoding interference and completely ignoring the interference. While general optimality has yet been established, rate splitting, facilitated by time sharing, gives the largest achievable rate region to date.

Orthogonal transmissions, such as Frequency division multiplexing (FDM) and time division multiplexing (TDM), are also commonly used in communication systems. The available degrees of freedom are divided into non-overlapping slots in time or frequency. Different users are assigned orthogonal channels. Hence, the interference is completely avoided. Though it gives rises to spectral inefficiency, it is easy to implement in practice because of its simplicity and it is known to be optimal under certain conditions.

## 2.3 Modeling interference in current wireless networks

In a cellular system, co-channel cells are strategically placed to ensure that interference is kept at a minimum. As such, the down-link transmission within each cell is typically modeled as a broadcast channel (BC) while up-link transmission is modeled as a multiple access channel (MAC). This effectively isolates each cell from all the other co-channel cells and makes it feasible to characterize the performance limits as the capacity regions for the Gaussian BC and the Gaussian MAC have been completely determined (see [13]).

However, for current and next-generation wireless cellular networks, the intra-cell interference is mitigated by separating subscribers in orthogonal time, frequency or spatial dimensions, by user scheduling, orthogonal frequency-division multiple access (OFDMA), and beam-forming coordinations. On the other hand, the inter-cell interference caused by transmission in neighboring cells remains a major impairment that limits throughput. This issue becomes even more acute as the cell size is shrinking and the cell density is increasing for improved frequency reuse. In addition, hierarchical cellular structures such as pico-cells, femto-cells, etc. heavily overlap with macro-cell deployment. The inter-cell interference is no longer negligible in both down-link and up-link transmissions. It is therefore of interest to examine the fundamental limits on up-link and down-link transmission models with co-channel interference.

For down-link transmissions, the Gaussian broadcast-interference channel model has been studied in [14–16] with an emphasis on the one-sided interference model. The capacity region of such a channel with very strong and slightly strong interference, and some boundary points on the capacity region of that with moderate and weak interferences were determined. It was shown that the capacity is achieved by fully decoding the interference when it is strong, partially decoding the interference when it is moderate, and treating the interference as noise when it is weak.

Up-link transmission models have also been investigated in [17] and [18], both of which considered the two-sided interference between the two cells. The authors in [17] derived the capacity region for the very strong and some of the strong interference cases, and provided an upper-bound of the sum-rate for the weak interference case which is nearly optimal in low signal-to-noise ratio regime, while [18] characterized the capacity region in the form of interference alignment under the weak symmetric interference assumption.

The authors in [19] studied the two-user Gaussian X channel and characterized the sum capacity and generalized degrees of freedom (GDoF) for the symmetric case. However, the major differences between the X channel and the traditional IC are: 1) the message in an X channel is split into two parts with each part intended for one receiver, while the message in a IC is intended for only one receiver, and 2) the interference at each receiver can be dependent on the desired signal in an X channel, while it is always independent in an IC. The dirty multiple-access channel was explored in [20]. Unlike the IC model, the interference signal is the state information that is available at the transmitter side, and no separate interferer is involved. The proposed lattice strategies strongly depend on the state information. Other similar models include the interference-multiple-access channel considered in [21], where one of the receivers is required to decode messages from both users in the same physical channel as that of the two-user interference channel. A semi-deterministic channel was considered and the gap between the inner bound and the outer bound was characterized. Moreover, a class of multiple access interference channels was studied in [22], which consists of a multiple-access channel and a point-to-point link. The transmitter of the point-to-point link mutually interferes with only one of the transmitters of the multiple-access channel. The gaps were characterized for semi-deterministic model as well as the scalar Gaussian model.

All the listed works are mostly focused on interference alignment or lattice codes approach. Capacity studies of these channel models are scarcely reported. In particular, the performance limit of uplink models with weak interference is less well studied.

## 2.4 Useful properties of Markov chains

The following properties of Markov chains are useful throughout the dissertation [23]:

- Decomposition: $X - Y - ZW \Longrightarrow X - Y - Z$;
- Weak Union: $X - Y - ZW \Longrightarrow X - YW - Z$;
- Contraction: $(X - Y - Z)$ and $(X - YZ - W) \Longrightarrow X - Y - ZW$.

# Chapter 3

# The Sum Rate Optimality of the Naïve FDM for the Gaussian Interference Channel within the Computable Han-Kobayashi Subregion

## 3.1 General Han-Kobayashi Inner Bound and its subregion

As introduced in Chapter 2, $\mathcal{G}^*$ in (2.6) is not computable due to the unknown optimal distributions as well as the involvement of the time sharing variable. As such, some alternative achievable subregions which are more amenable to evaluation were given in [4]:

- With time sharing replaced by convex hull formulation

$$\mathcal{G} \;=\; cvx \bigcup_{Z\in\mathcal{P}(P_1,P_2)} \mathcal{R}(Z),$$

where $Z \in \mathcal{P}(P_1, P_2)$ if and only if $Z \in \mathcal{P}^*(P_1, P_2)$ and $\mathcal{Q} = \phi$.

- With Gaussian input, fixed $f_1$ and $f_2$, and without time sharing

$$\mathcal{G}' \;=\; cvx \bigcup_{Z\in\mathcal{P}'(P_1,P_2)} \mathcal{R}(Z),$$

where $Z \in \mathcal{P}'(P_1, P_2)$ if and only if $Z \in \mathcal{P}^*(P_1, P_2)$, $\mathcal{Q} = \phi$ , $U_1, U_2, W_1, W_2$ are all Gaussian and $X_1 = U_1 + W_1, X_2 = U_2 + W_2$.

Only $\mathcal{G}'$ can be computed, as $\mathcal{G}$ still requires exhausting all possible distributions. The following propositions can be found in [24]:

**Proposition 3.1.** *For a symmetric GIC with* $0 < a = b < 1$*, the maximum sum rate of* $\mathcal{G}'$ *is described as follows:*

$$C_{\mathcal{G}'}(P_1,P_2) = \begin{cases} \frac{1}{2}\log\left(1+\frac{P_1}{1+aP_2}\right)+\frac{1}{2}\log\left(1+\frac{P_2}{1+aP_1}\right), & if \quad P_1 \le \frac{1-a}{a^2}, P_2 \le \frac{1-a}{a^2} \\ \frac{1}{2}\log(1+aP_1+P_2), & if \quad P_1 \le \frac{1-a}{a^2}, P_2 \ge \frac{1-a}{a^2} \\ \frac{1}{2}\log(1+aP_2+P_1), & if \quad P_1 \ge \frac{1-a}{a^2}, P_2 \le \frac{1-a}{a^2} \\ \max\left\{r(\hat{P}_1), r\left(\min\left(P_1, \frac{P_2}{k}\right)\right), R_s\right\}, & if \quad P_1 \ge \frac{1-a}{a^2}, P_2 \ge \frac{1-a}{a^2} \end{cases} \tag{3.1}$$

*where*

$$\hat{P}_1 = (1-a) + (1-a^2)P_1 \tag{3.2}$$

$$k = \frac{1+(1+a)P_2}{1+(1+a)P_1}, \tag{3.3}$$

$$r(p) = \frac{1}{2}\log\left(\frac{1+aP_2+P_1}{1+aP_2-akp}\right) + \frac{1}{2}\log\left(\frac{1+aP_1+P_2-ap-kp}{1+aP_1-ap}\right), \tag{3.4}$$

$$R_s = \begin{cases} r(P^*); & \textit{if } AP^{*2}+BP^*+C=0 \textit{ and } P^* \in \left[\hat{P}_1, \min\left\{P_1, \frac{P_2}{k}\right\}\right], \\ 0; & \textit{otherwise} \end{cases} \tag{3.5}$$

$$A = -a^2k(a+k), \tag{3.6}$$

$$B = 2a^2k(1+aP_1+P_2), \tag{3.7}$$

$$C = -a^3kP_1^2 - a^2P_2^2 + (ak-2a^2k)P_1 - aP_2 + (k-ak). \tag{3.8}$$

## 3.2 The Modified FDM/TDM Method and the New Achievable Rate Region

Sato first introduced the idea of modified (or non-naïve) FDM/TDM method for the degraded GIC [5], where the total bandwidth is divided into two sub-bands; in each sub-band one user is transmitting at the maximum rate while the other is transmitting at a rate that both users can reliably decode its message. Later, Sason used the same idea for the general GIC in [12] and obtained a new achievable rate region:

$$\mathcal{D} = \bigcup_{\alpha,\lambda_1,\lambda_2\in[0,1]} \left\{ (R_1,R_2) \;\middle|\; \begin{array}{l} R_1 \le \alpha R_1^{(1)} + \bar{\alpha} R_1^{(2)} \\ R_2 \le \bar{\alpha} R_2^{(1)} + \alpha R_2^{(2)} \end{array} \right\}, \tag{3.9}$$

where

$$\begin{aligned}
\bar{\alpha} &\triangleq 1-\alpha, && (3.10)\\
R_1^{(1)} &= \gamma\left(\frac{\lambda_1 P_1}{\alpha}\right), && (3.11)\\
R_1^{(2)} &= \min\left\{\gamma\left(\frac{\bar{\lambda}_1 P_1}{\bar{\alpha}+a\bar{\lambda}_2 P_2}\right), \gamma\left(\frac{b\bar{\lambda}_1 P_1}{\bar{\alpha}+\bar{\lambda}_2 P_2}\right)\right\}, && (3.12)\\
R_2^{(1)} &= \gamma\left(\frac{\bar{\lambda}_2 P_1}{\bar{\alpha}}\right), && (3.13)\\
R_2^{(2)} &= \min\left\{\gamma\left(\frac{\lambda_2 P_2}{\alpha+b\lambda_1 P_1}\right), \gamma\left(\frac{a\lambda_2 P_2}{\alpha+\lambda_1 P_1}\right)\right\}. && (3.14)
\end{aligned}$$

and

$$\gamma(p) \triangleq \frac{1}{2}\log(1+p). \tag{3.15}$$

Based on the HK subregion $\mathcal{G}'$ and the same non-naïve FDM/TDM method, an improvement of Sason's region was given in [24], repeated below.

$$\mathcal{S} = \bigcup_{\substack{\alpha\in[0,0.5]\\ \lambda_1,\lambda_2\in[0,1]}} \left\{ (R_1,R_2) \;\middle|\; \begin{array}{l} R_1 \le \alpha R_1^{(1)} + \bar{\alpha} R_1^{(2)},\\ R_2 \le \alpha R_2^{(1)} + \bar{\alpha} R_2^{(2)};\\ \left(R_1^{(1)}, R_2^{(1)}\right) \in \mathcal{G}'\left(\frac{\lambda_1 P_1}{\alpha}, \frac{\lambda_2 P_2}{\alpha}\right),\\ \left(R_1^{(2)}, R_2^{(2)}\right) \in \mathcal{G}'\left(\frac{\bar{\lambda}_1 P_1}{\bar{\alpha}}, \frac{\bar{\lambda}_2 P_2}{\bar{\alpha}}\right). \end{array} \right\}, \tag{3.16}$$

where $\mathcal{G}'(p_1,p_2)$ is the HK subregion $\mathcal{G}'$ for a GIC with power constraint $p_1$ and $p_2$. One can show that $\mathcal{S}$ is an achievable region, since it is a subset of the general HK region. From the FDM/TDM point of view, $\mathcal{S}$ is obtained by dividing the total bandwidth into two sub-bands, one with $\alpha$ and the other with $\bar{\alpha}$ fraction of the total bandwidth. The power is allocated into each sub-band with a factor $\lambda_i$. Naturally $\mathcal{D} \subset \mathcal{S}$.

One can generalize $\mathcal{S}$ by dividing the bandwidth into more than two subbands. In fact, the following rate region $\mathcal{S}^\dagger$ is introduced in [24]:

$$\mathcal{S}^\dagger = cvx \bigcup_{\lambda_i,\alpha_i,\beta_i \in \mathbf{I}^3, \lambda_i \geq \lambda_{i+1}} \left\{ \sum_{i=1}^{3} \lambda_i \mathcal{G}'(\frac{\alpha_i P_1}{\lambda_i}, \frac{\beta_i P_2}{\lambda_i}) \right\}, \tag{3.17}$$

where $x_i \in \mathbf{I}^n$ means $\sum_{i=1}^n x_i = 1, x_i \geq 0$.

**Proposition 3.2.** *$\mathcal{S}^\dagger$ is unchanged by dividing the frequency band into more than three subbands.*

This proposition states that the three-band division is sufficient for a two user GIC. This is because that the power allocation is only in two dimensions. The complete proof for the above two propositions can be found in [24]. The same result was rediscovered in [11], *albeit* for the more general $m$ user GIC case.

## 3.3 Main Results

**Theorem 3.3.** *For a symmetric GIC with $0 < a = b < 1$ and power constraints $P_1$, $P_2$, the naïve FDM/TDM is optimal in the range $.5 < a = b < 1$ within $\mathcal{S}^\dagger$.*

First, from the definition of $\mathcal{S}^\dagger$, the maximal sum rate that $\mathcal{S}^\dagger$ can achieve is

$$C_{\mathcal{S}^\dagger} = \max_{\lambda_i,\alpha_i,\beta_i \in \mathbf{I}^3} \left\{ \sum_{i=1}^{3} \lambda_i \mathcal{G}'(\frac{\alpha_i P_1}{\lambda_i}, \frac{\beta_i P_2}{\lambda_i}) \right\}. \tag{3.18}$$

Define $C_{\mathcal{F}}$ to be the achievable sum rate using the naïve FDM for the above symmetric GIC, i.e.,

$$C_{\mathcal{F}}(P_1, P_2) = \frac{1}{2}\log(1 + P_1 + P_2). \tag{3.19}$$

From the concavity of the function $\frac{1}{2}\log(1+P_1+P_2)$, if in each subband, over certain range of parameter $a$,

$$C_{\mathcal{G}'}(p_{1i},p_{2i}) < \frac{1}{2}\log(1+p_{1i}+p_{2i}),$$

then,

$$\sum_{\substack{i=1 \\ \lambda_i,\alpha_i,\beta_i\in\mathbf{I}^3}}^{3} \lambda_i C_{\mathcal{G}'}(p_{1i},p_{2i}) \le \sum_i \frac{\lambda_i}{2}\log(1+p_{1i}+p_{2i}) = \sum_i \lambda_i C_{\mathcal{F}}(p_{1i},p_{2i}) \le C_{\mathcal{F}}(P_1,P_2).$$

Therefore, $C_{\mathcal{S}'} \le C_{\mathcal{F}}$.

To this end, we only need to show $C_{\mathcal{G}'}(p_{1i},p_{2i}) \le C_{\mathcal{F}}(p_{1i},p_{2i})$ for $a \in [.5,1]$.

Since we have an explicit expression for $C_{\mathcal{G}'}$, We now compare $C_{\mathcal{F}}(P_1,P_2)$ with $C_{\mathcal{G}'}(P_1,P_2)$ for all range of power constraint.

1. $P_1 \le \frac{1-a}{a^2}, P_2 \le \frac{1-a}{a^2}$. In this case, each receiver treats interference as noise to achieve $C_{\mathcal{G}'}$ (cf. first line of Eq. (3.1)). For the sake of simplicity,

   $$f_1 \triangleq \frac{1}{2}\log(1+\frac{P_1}{1+aP_2}) + \frac{1}{2}\log(1+\frac{P_2}{1+aP_1}).$$

   It is easy to compute

   $$\frac{\partial f_1}{\partial a} < 0,$$

   for any $0<a<1$. i.e., $f_1$ is a monotone decreasing function of $a$. Moreover,

   $$f_1 \ge C_{\mathcal{F}}$$

   if and only if

   $$0 < a \le \frac{\sqrt{1+P_1+P_2}-1}{P_1+P_2}.$$

Furthermore, $\frac{\sqrt{1+P_1+P_2}-1}{P_1+P_2}$ is a decreasing function of $(P_1+P_2)$, and

$$\lim_{P_1\to 0, P_2\to 0} \frac{\sqrt{1+P_1+P_2}-1}{P_1+P_2} = .5.$$

Therefore, FDM/TDM is better than treating interference as noise when $.5 < a < 1$ within $\mathcal{G}'$.

2. $P_1 \leq \frac{1-a}{a^2}$, $P_2 \geq \frac{1-a}{a^2}$. This is a simple case, the maximal achievable sum rate characterized by Eq. (3.1) is (cf. second line of Eq. (3.1))

$$\frac{1}{2}\log(1+aP_1+P_2) \triangleq f_2.$$

Clearly, $f_2 \leq C_{\mathcal{F}}$ for the entire range $a \in [0,1]$.

3. $P_1 \geq \frac{1-a}{a^2}$, $P_2 \leq \frac{1-a}{a^2}$. This is the same as the above case.

4. $P_1 \geq \frac{1-a}{a^2}$, $P_2 \geq \frac{1-a}{a^2}$. $C_{\mathcal{G}'}$ in this case is (cf. last line of Eq. (3.1))

$$\max\{r(\hat{P}_1), r(min(P_1, \frac{P_2}{k})), R_s\} \triangleq f_3,$$

where $r(\cdot)$, $k$ and $R_s$ are defined in Eq. (3.1). There are three terms inside the operation $\max(\cdot)$. Let us solve them one by one.

   (a) $r(min(P_1, \frac{P_2}{k})) \triangleq f_3^1$.

$$\begin{aligned} r(P_1) &= \frac{1}{2}\log\left(\frac{1+aP_2+P_1}{1+aP_2-kP_1}\right) + \frac{1}{2}\log\left(1+P_2-kP_1\right) \\ &= \frac{1}{2}\log(1+aP_1+P_2). \end{aligned}$$

$$r(\frac{P_2}{k}) = \frac{1}{2}\log(1+P_1+aP_2).$$

Therefore,

$$f_3^1 = \begin{cases} \frac{1}{2}\log(1 + aP_1 + P_2), & if P_1 < P_2, \\ \frac{1}{2}\log(1 + aP_2 + P_1), & otherwise. \end{cases}$$

Those two functions have been discussed in above cases, thus $f_3^1 \leq C_{\mathcal{F}}$ for the entire range $a \in [0, 1]$.

(b) $r(\hat{P}_1) \triangleq f_3^2$.

$$\begin{aligned} f_3^2 &= \frac{1}{2}\log\left(\frac{1 + P_1 + aP_2}{1 + aP_2 - ak\hat{P}_1}\right) + \frac{1}{2}\log\left(\frac{1 + aP_1 + P_2 - a\hat{P}_1 - k\hat{P}_1}{1 + aP_1 - a\hat{P}_1}\right) \\ &= \frac{1}{2}\log\left(\frac{1 + P_1 + aP_2}{1 - a + a^2 + a^3 P_2}\right) + \frac{1}{2}\log\left(\frac{a^2(1 + P_2 + aP_1)}{1 - a + a^2 + a^3 P_1}\right). \end{aligned}$$

Then we can verify that $f_3^2$ is indeed an increasing and then decreasing function over $a \in [0, 1]$ when $P_1 \geq \frac{1-a}{a^2}$ and $P_2 \geq \frac{1-a}{a^2}$, and,

$$\frac{\partial r(\hat{P}_1)}{\partial a}|_{a=.5} < 0 \quad and \quad r(\hat{P}_1)|_{a=.5} < C_{\mathcal{F}}.$$

From these two observations, one get

$$r(\hat{P}_1) < C_{\mathcal{F}}.$$

(c) $R_s$. First, we can solve the equation of $P^*$ in Eq. (3.1). Since $-\frac{B}{2A} = P_1 + \frac{1}{1+a}$, we want the root $P^* \leq P_1$, therefore,

$$P^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = P_1 + \frac{1}{1 + a} - \frac{1}{a(1 + a)\sqrt{k}}.$$

By the same way, one can get the differential of $R_s|_{a>.5}$ and compare $R_s|_{a=.5}$ and $C_{\mathcal{F}}$. $R_s$ for the range $a \in [.5, 1]$ is a decreasing function of a if $P^* \in [\hat{P}_1, \frac{P_2}{k}]$, $R_s|_{a=1} = C_{\mathcal{F}}$ and $R_s|_{a=.5} < C_{\mathcal{F}}$. The final result gives $R_s < C_{\mathcal{F}}$, if $a \in [.5, 1]$.

From all the above cases, TDM/FDM is optimal in each subband for the parameter range $a \in [.5, 1]$. The proof is completed.

## 3.4 Summary

The work was motivated by careful examination of the achievable sum rates of various computable rate regions for the Gaussian interference channel. As illustrated in Fig. 2, the largest computable rate region, $\mathcal{S}^{\dagger}$, becomes flat in its sum rate with $a$ exceeding a certain threshold. This constant sum rate turns out to be precisely that achieved by the naïve FDM, which we proved in this chapter. Additionally, it is also easy to establish that for completely symmetric GIC, i.e., both channel coefficients and power constraints are identical, the maximum sum rate using $\mathcal{S}'$ results in equal power allocation among subbands for both users, except using the naïve FDM.

This work established the sum rate optimality of the naïve FDM within a specific computable subregion of the general HK region. At the present, it is only known that the naive FDM is sum rate optimal within the general HK region at a singular point (i.e., $a = 1$) at which the naïve FDM actually achieves the sum rate capacity.

# Chapter 4

# The DMIC with One-sided Weak Interference

In Chapter 2, we have presented the sum-rate capacity result for the two-user GIC with one-sided weak interference. Unlike the strong and very strong interference chase, the discrete momeoryless counterpart has not been studied yet. This chapter attempts to derive parallel sum-rate capacity result for DMICs with one-sided weak interference.

The rest of the chapter is organized as follows. Section 4.1 presents the channel model, and defines the notation of "one-sided" as well as "one-sided weak" for the discrete memoryless channel model. The sum-rate capacity is derived in Section 4.2. In Section 4.3, the equivalence between the DMIC with one-sided weak interference and the discrete degraded interference channel (DMDIC) is established which allows one to construct an outer-bound of the capacity region for the DMZIC using the capacity region of the associated degraded broadcast channel. Several specific DMIC examples are studied in Section 4.4 whose capacites or capacity bounds are obtained. Section 4.5 summarizes this chapter.

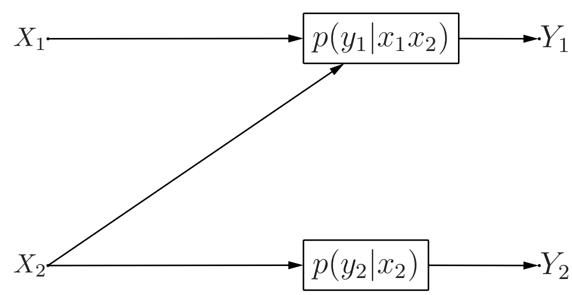



FIGURE 4.1: The DMIC with one-sided interference model.

## 4.1 Channel model

**Definition 4.1.** For the DMIC defined in Chapter 1, if for all $x_1$, $x_2$, $y_2$,

$$p(y_2|x_2) = p(y_2|x_1x_2), \tag{4.1}$$

or equivalently,

$$X_1 - X_2 - Y_2 \tag{4.2}$$

forms a Markov chain, this DMIC is said to have one-sided interference.

A general one-sided DMIC is depicted in Fig. 4.1. Clearly, the Markov chain condition (4.2) holds for the GIC with $b = 0$ in (2.5). As with the Gaussian case, we refer to the DMIC with one-sided interference as simply discrete memoryless Z interference channel (DMZIC). From the definition, it follows that $X_1$ and $Y_2$ are independent for all input distribution $p(x_1)p(x_2)$.

To define DMZIC with weak interference, we first revisit some properties of Gaussian ZIC with weak interference. Costa [25] has shown that a Gaussian ZIC with weak interference is equivalent in its capacity region to a degraded Gaussian ZIC satisfying the Markov chain

$$X_2 - (X_1, Y_2) - Y_1. \tag{4.3}$$

This motivates us to define DMZIC with weak interference as follows.

**Definition 4.2.** A DMZIC is said to have *weak interference* if the channel transition probability factorizes as

$$p(y_1y_2|x_1x_2) = p(y_2|x_2)p'(y_1|x_1y_2), \tag{4.4}$$

for some $p'(y_1|x_1y_2)$, or, equivalently, the channel is stochastically degraded.

A stochastic DMZIC with weak interference is shown in Fig. 4.2. In the absence of receiver cooperation, a stochastically degraded interference channel is equivalent in its capacity to a physically degraded interference channel. As such, we will assume in the following that the channel is physically degraded, i.e., the DMZIC with weak interference admits the Markov chain $X_2 - (X_1, Y_2) - Y_1$.

The channel transition probability $p(y_1y_2|x_1x_2)$ for this class of channels factorizes as

$$\begin{aligned} p(y_1y_2|x_1x_2) &= p(y_2|x_1x_2)p(y_1|x_1x_2y_2) \\ &= p(y_2|x_2)p(y_1|x_1y_2). \end{aligned} \tag{4.5}$$

As a consequence, the following inequality holds

$$I(U;Y_2) \geq I(U;Y_1|X_1), \tag{4.6}$$

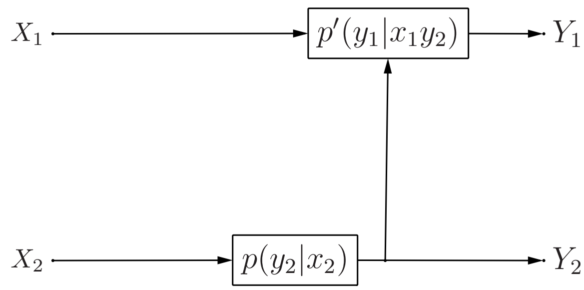



FIGURE 4.2: The DMZIC with weak interference channel model.

for all input distributions $p(x_1)p(u)p(x_2|u)$. To prove this inequality, it suffices to show that the Markov chain $X_1Y_1 - Y_2 - U$ holds. First of all, from the memoryless condition, we have the following Markov chain:

$$U - X_1X_2 - Y_1Y_2.$$

By the weak union property, we obtain the Markov chain

$$Y_1 - X_1X_2Y_2 - U.$$

Together with the weak interference condition

$$Y_1 - X_1Y_2 - X_2,$$

The following Markov chain can be attained by the contraction rule:

$$Y_1 - X_1Y_2 - UX_2.$$

Then, by decomposition rule, the Markov chain

$$Y_1 - X_1Y_2 - U$$

holds. Together with the Markov chain

$$X_1 - Y_2 - U$$

from the independence between $X_1$ and $(U, Y_2)$, we establish the desired Markov chain by the contraction rule.

We note that this condition is indeed what is needed in establishing the sum-rate capacity of this channel and was used in [26] to define the weak interference for DMZIC. The definition used in this paper, while stronger than necessary, is much more intuitive and easier to verify.

The above definition of weak interference leads to the following sum-rate capacity result.

## 4.2 Sum-rate capacity

**Theorem 4.3.** *The sum-rate capacity of a DMZIC with weak interference as defined above is*

$$C_{sum} = \max_{p(x_1)p(x_2)} \{I(X_1; Y_1) + I(X_2; Y_2)\}. \tag{4.7}$$

*Proof.* This sum-rate is achieved by two receivers decoding their own messages while treating any interference, if present, as noise.

For the converse, we have

$$
\begin{aligned}
& n(R_1+R_2)-n\epsilon \\
\overset{(a)}{\le}\ & I(X_1^n;Y_1^n)+I(X_2^n;Y_2^n) \\
\overset{(b)}{=}\ & \sum_{i=1}^{n}\big(H(Y_{1i}|Y_1^{i-1})-H(Y_{1i}|Y_1^{i-1}X_1^n)+H(Y_{2i}|Y_2^{i-1})-H(Y_{2i}|Y_2^{i-1}X_2^n)\big) \\
\overset{(c)}{\le}\ & \sum_{i=1}^{n}\big(H(Y_{1i})-H(Y_{1i}|Y_1^{i-1}X_1^nY_2^{i-1})+H(Y_{2i}|Y_2^{i-1})-H(Y_{2i}|Y_2^{i-1}X_{2i})\big) \\
\overset{(d)}{=}\ & \sum_{i=1}^{n}\big(H(Y_{1i})-H(Y_{1i}|X_1^nY_2^{i-1})+I(X_{2i};Y_{2i}|U_i)\big) \\
\overset{(e)}{=}\ & \sum_{i=1}^{n}\big(H(Y_{1i})-H(Y_{1i}|X_{1i}Y_2^{i-1})+I(X_{2i};Y_{2i}|U_i)\big) \\
=\ & \sum_{i=1}^{n}(I(U_iX_{1i};Y_{1i})+I(X_{2i};Y_{2i}|U_i)) \\
=\ & \sum_{i=1}^{n}(I(X_{1i};Y_{1i})+I(U_i;Y_{1i}|X_{1i})+I(X_{2i};Y_{2i}|U_i)) \\
\overset{(f)}{\le}\ & \sum_{i=1}^{n}(I(X_{1i};Y_{1i})+I(U_i;Y_{2i})+I(X_{2i};Y_{2i}|U_i)) \\
=\ & \sum_{i=1}^{n}(I(X_{1i};Y_{1i})+I(U_iX_{2i};Y_{2i})) \\
\overset{(g)}{=}\ & \sum_{i=1}^{n}(I(X_{1i};Y_{1i})+I(X_{2i};Y_{2i})), \qquad (4.8)
\end{aligned}
$$

where $(a)$ follows the Fano's nequality, $(b)$ is from the chain rule and the definition of mutual information, $(c)$ is because of the fact that conditioning reduces entropy, and that $Y_{2i}$ is independent of any other random variables given $X_{2i}$, $(d)$ is due to the memoryless property of the channel and the fact that $Y_{1i}$ is independent of any other random variables given $X_{1i}$ and $Y_{2i}$, then $(X_{1,i}^n, Y_{1i})-(X_1^{i-1},Y_2^{i-1})-Y_1^{i-1}$ forms a Markov chain. By the weak union property, the Markov chain $Y_{1i}-(X_1^n,Y_2^{i-1})-Y_1^{i-1}$ holds. Moreover, define $U_i \triangleq Y_2^{i-1}$

for all $i$. $U_i$ is independent of $X_{1i}$ since

$$\begin{aligned} p(x_{1i}, y_2^{i-1}) &= \sum_{x_2^{i-1}} p(x_{1i} x_2^{i-1}, y_2^{i-1}) = \sum_{x_2^{i-1}} p(x_{1i}, x_2^{i-1}) p(y_2^{i-1} | x_2^{i-1}, x_{1i}) \\ &= \sum_{x_2^{i-1}} p(x_{1i}) p(x_2^{i-1}) p(y_2^{i-1} | x_2^{i-1}) = p(x_{1i}) p(y_2^{i-1}); \end{aligned}$$

$(e)$ is because of the Markov chain $(X_1^{i-1}, X_{1,i+1}^n) - (X_{1i}, Y_2^{i-1}) - Y_{1i}$. This can be established using the *functional dependence graph (FDG)* [27]. The formal proof is included in Appendix A. Alternatively, we first note that the Markov chain

$$(X_1^{i-1}, X_{1,i+1}^n, Y_2^{i-1}) - (X_{1i}, Y_{2i}) - Y_{1i}$$

holds, since given $X_{1i}$ and $Y_{2i}$, $Y_{1i}$ is independent of $X_1^{i-1}, X_{1,i+1}^n, Y_2^{i-1}$. By the weak union property, the following Markov chain is obtained:

$$(X_1^{i-1}, X_{1,i+1}^n) - (X_{1i}, Y_2^i) - Y_{1i}.$$

The independence between $Y_2^n$ and $X_1^n$ gives the Markov chain

$$(X_1^{i-1}, X_{1,i+1}^n) - X_{1i} - Y_2^i.$$

The above two Markov chains lead to the following Markov chain:

$$(X_1^{i-1}, X_{1,i+1}^n) - X_{1i} - (Y_{1i}, Y_2^i)$$

by the contraction property. Again, using the weak union property and then the decomposition property, we obtain the Markov chain

$$(X_1^{i-1}, X_{1,i+1}^n) - (X_{1i}, Y_2^{i-1}) - Y_{1i}$$

as desired. Since $U_i$ and $X_{1i}$ are independent, then $p(x_1x_2u) = p(x_1)p(u, x_2)$, thus $(f)$ comes from (4.6). Finally, $(g)$ follows from the Markov chain $U_i - X_{2i} - Y_{2i}$. Finally, by introducing a time-sharing random variable $Q$, one obtains

$$\begin{aligned} R_1 + R_2 &\leq I(X_1; Y_1|Q) + I(X_2; Y_2|Q) + \epsilon \\ &= \sum_{q \in \mathcal{Q}} p(q) \{I(X_1; Y_1|Q = q) + I(X_2; Y_2|Q = q)\} + \epsilon \\ &\leq \sum_{q \in \mathcal{Q}} p(q) \left\{ \max_{p(x_1)p(x_2)} \{I(X_1; Y_1) + I(X_2; Y_2)\} \right\} + \epsilon \\ &= \max_{p(x_1)p(x_2)} \{I(X_1; Y_1) + I(X_2; Y_2)\} + \epsilon. \end{aligned} \tag{4.9}$$

□

*Remark* 4.4. From the strong interference condition (2.27), it is perhaps tempting to define the condition for weak interference as

$$I(X_2; Y_1|X_1) \leq I(X_2; Y_2), \tag{4.10}$$

for all product input distribution on $\mathcal{X}_1 \times \mathcal{X}_2$. Notice that the right-hand side is same as $I(X_2; Y_2|X_1)$ given that this is one-sided interference channel. The Markov chain (4.3) is a sufficient, but not necessary, condition for the mutual information condition (4.10). An example is provided in Appendix B such that the mutual information condition holds but the Markov chain is not valid. This is different from that of the Gaussian case; it can be shown that the coefficient $a \leq 1$ in a Gaussian ZIC is a sufficient and necessary condition for (4.10) to hold. It is yet unknown if condition (4.10) is sufficient for the sum-rate capacity result (4.7) to hold for the DMZIC.

*Remark* 4.5. For a DMZIC with weak interference, an achievable rate region, $\mathcal{C}$, is given by the set of all nonnegative rate pairs $(R_1, R_2)$ that satisfy

$$R_1 \leq I(X_1; Y_1|U_2Q), \tag{4.11}$$

$$R_2 \leq I(X_2; Y_2|Q), \tag{4.12}$$

$$R_1 + R_2 \leq I(U_2X_1; Y_1|Q) + I(X_2; Y_2|U_2Q), \tag{4.13}$$

where the input distribution factorizes as:

$$p(qu_2x_1x_2) = p(q)p(x_1|q)p(u_2|q)p(x_2|u_2, q). \tag{4.14}$$

Furthermore, the region remains invariant if we impose the constraints $\|\mathcal{Q}\| \leq 5$, $\|\mathcal{U}_2\| \leq \|\mathcal{X}_2\| + 3$. This can be readily obtained from the Han-Kobayashi region of the general two-user IC [4, 28].

In the next lemma, we provide a simpler description for the above achievable rate region.

**Lemma 4.6.** *The region $\mathcal{C}$ is equivalent to the set of all rate pairs $(R_1, R_2)$ satisfying*

$$R_1 \leq I(X_1; Y_1|U_2'Q), \tag{4.15}$$

$$R_2 \leq I(U_2'; Y_1|Q) + I(X_2; Y_2|U_2'Q), \tag{4.16}$$

*where the input distribution factorizes as (4.14). Furthermore, the region remains invariant if we impose the constraints $\|\mathcal{Q}\| \leq 4$, $\|\mathcal{U}_2'\| \leq \|\mathcal{X}_2\| + 3$.*

*Proof.* Let $E$ denote the set defined in the above lemma. The fact that $E \subseteq \mathcal{C}$ follows simply by setting $U_2 = U_2'$ and noticing that (4.15) and (4.16) imply (4.11)-(4.13). To prove that $\mathcal{C} \subseteq E$, we first note that for a given $p(qu_2x_1x_2)$, $\mathcal{C}$ is a pentagon with two extreme points in the first quadrant given by

$$\begin{aligned} p_1 &= (I(X_1;Y_1|U_2,Q=q), \\ &\quad I(U_2;Y_1|Q=q)+I(X_2;Y_2|U_2,Q=q)), \end{aligned} \tag{4.17}$$

$$\begin{aligned} p_2 &= (I(U_2X_1;Y_1|Q=q)-I(U_2;Y_2|Q=q), \\ &\quad I(X_2;Y_2|Q=q)). \end{aligned} \tag{4.18}$$

It suffices to show that, for any given $p(qu_2x_1x_2)$ in (4.14), the corresponding $p_1$ and $p_2$, belongs to the set $E$.

That $p_1 \in E$ follows from setting $U_2 = U_2'$. To show that $p_2 \in E$, we use the following inequality

$$\begin{aligned} & I(U_2X_1;Y_1|Q=q)-I(U_2;Y_2|Q=q) \\ &= I(U_2;Y_1|X_1Q=q)-I(U_2;Y_2|Q=q)+I(X_1;Y_1|Q=q) \\ &\overset{(a)}{\leq} I(X_1;Y_1|Q=q) \\ &\overset{(b)}{\leq} I(X_1;Y_1|U_2,Q=q), \end{aligned}$$

where $(a)$ follows from (4.6); $(b)$ is due to the independence between $X_1$ and $U_2$ conditioned on $Q$. Hence, $\mathcal{C} \subseteq E$. $\square$

## 4.3 Capacity outer bound for DMZICs with weak interference

### 4.3.1 Outer bound of the capacity region

Costa proved in [25] that a GZIC with weak interference is equivalent in capacity region to a degraded GIC. As such, Sato's outer-bound on degraded GIC [29] applies to that of GZIC with weak interference. Sato's outer-bound is in essence the capacity region of a related Gaussian broadcast channel, which is a natural outer-bound to the interference channel due to its implied transmitter cooperation. In this section, we use the same technique to obtain a capacity outer-bound for DMZIC with weak interference, i.e., that satisfies the Markov chain $X_2 - (X_1, Y_2) - Y_1$. Specifically, for any such DMZIC with weak interference, one can find an equivalent (in capacity region) DMDIC whose capacity region is bounded by that of an associated degraded broadcast channel.

We begin with the equivalence between the DMZIC with weak interference and the DMDIC.

**Theorem 4.7.** *A DMZIC with weak interference with inputs $(X_1, X_2)$ and outputs $(Y_1, Y_2)$ is equivalent, in capacity region, to a DMIC with the same inputs and outputs $(Y_1, Y_2')$, where $Y_2' = f(X_1, Y_2)$ such that the Markov chain $(X_1, X_2) - Y_2' - Y_1$ holds and $H(Y_2'|X_1) = H(Y_2)$ (It is shown in Fig. 4.3).*

The complete proof is given in Appendix C.

**Theorem 4.8.** *For a DMZIC that satisfies the Markov chain $X_2 - X_1Y_2 - Y_1$, the capacity region is outer-bounded by*

$$\mathcal{R}_{OB} = \overline{co}\left\{ \bigcup_{p(u)p(x_1x_2|u)} (R_1, R_2) \;\middle|\; \begin{array}{l} R_1 \leq I(U; Y_1), \\ R_2 \leq I(X_1X_2; Y_2'|U) \end{array} \right\},$$

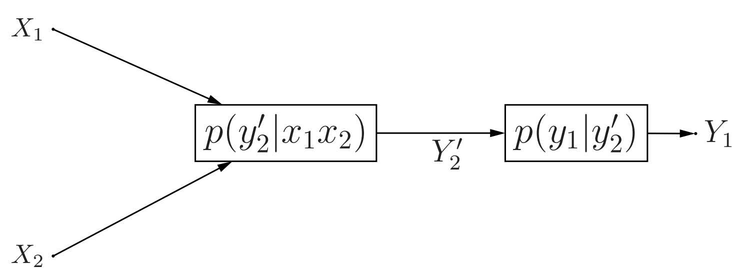



FIGURE 4.3: The DMDIC model.

*where* $\overline{co}\{\cdot\}$ *denotes the closure of the convex hull operation,* $U - X_1X_2 - Y_2' - Y_1$ *forms a Markov chain and the rate region remains invariant if we impose* $\|\mathcal{U}\| \leq \min\{\|\mathcal{Y}_2'\|, \|\mathcal{X}_1\| \cdot \|\mathcal{X}_2\|\} + 1$.

*Proof.* As we have the equivalent DMDIC. By treating $X_1, X_2$ as a group, we can outer-bound the capacity region of the DMDIC by the capacity region of the associated broadcast channel. □

*Remark* 4.9. A trivial choice of $Y_2'$ is a bijection of $X_1$ and $Y_2$. It is easy to verify that the Markov chain $(X_1', X_2') - Y_2' - Y_1'$ holds for such $Y_2'$. However, other $Y_2'$ can be constructed, as long as the Markov chain $(X_1', X_2') - Y_2' - Y_1'$ is satisfied. Nevertheless, the associated broadcast channels would have the same the capacity region. In the following, an example is shown that other choices of $Y_2'$ are available.

### 4.3.2 Capacity region of a subclass of DMZICs

As we have proved that there always exists at least one DMDIC that has the same capacity region with any DMZIC with weak interference, the capacity region of the DMZIC with weak interference can be resolved sometimes through its equivalent DMDIC.

Liu and Ulukus has proposed a single-letter characterization for the capacity region of a class of DMDIC [30, Section II]. For this class of DMDICs, encoder cooperation does not enlarge the capacity region. In the following, we apply this technique to DMZICs with weak interference to resolve the capacity region of a subclass of DMZICs.

Let $T$ denote the $|\mathcal{Y}_1| \times (|\mathcal{X}_1||\mathcal{Y}_2|)$ matrix of transition probabilities $p'(y_1|x_1y_2)$, and $T'$ is the compact form of $T$, in the sense that it only keeps all the distinct columns. If $T'$ is different from $T$, it is clear that we can find a $Y_2'$ other than the one-to-one mapping of $(X_1, Y_2)$ that represents each column of $T'$, i.e., $|\mathcal{Y}_2'|$ is the number of columns in $T'$. Hereafter, $Y_2'$ corresponds to the one with the cardinality being the number of columns in $T'$. Let $V_{\overline{x}_1}$ be the $|\mathcal{Y}_2'| \times |\mathcal{X}_2|$ matrix of transition probability $p(y_2'|\overline{x}_1x_2)$ for each $x_2 \in \mathcal{X}_2$.

**Lemma 4.10.** *If a DMZIC with weak interference with channel transition probability* $(p(y_2|x_2), p'(y_1|x_1y_2))$ *satisfies the following conditions,*

1. $T'$ *is input symmetric*[1]. *Let the input symmetry group be* $\mathcal{G}$.
2. *For any realization pairs* $x_1'$, $x_1'' \in \mathcal{X}_1$, *there exists a permutation group* $G \in \mathcal{G}$, *such that*
$$V_{x_1'} = GV_{x_1''}.$$
3. $H(Y_2'|X_1 = x_1, X_2 = x_2) = \eta$, *independent of* $x_1$ *and* $x_2$.
4. $p(y_2'|x_1x_2)$ *satisfies*
$$\sum_{x_1} p(y_2'|x_1x_2) = \frac{|\mathcal{X}_1|}{|\mathcal{Y}_2'|}, \textit{for } x_2 \in \mathcal{X}_2, y_2' \in \mathcal{Y}_2'.$$

[1] For an $m \times n$ stochastic matrix $T'$ (an $n$-input, $m$-output channel), the input symmetry group $\mathcal{G}$ is defined as the set of permutation matrices $G$ such that the column permutations of $T'$ with $G$ may be achieved with corresponding row permutations. $T'$ is input symmetric, if $\mathcal{G}$ is transitive, i.e., any element of $\{1, 2, \cdots, n\}$ can be mapped to every other element of $\{1, 2, \cdots, n\}$ by some member of $\mathcal{G}$.

5. *Let* $\mathbf{p}_{x_1,x_2}$ *be the* $|\mathcal{Y}_2'|$*-dimensional vector of probabilities* $p(y_2'|x_1x_2)$ *for a given* $x_1$*,* $x_2$*. Then, there exists an* $\tilde{x}_1 \in \mathcal{X}_1$*, such that*

$$\begin{aligned}&\left\{\sum_{x_1,x_2} a_{x_1,x_2}\mathbf{p}_{x_1,x_2} : \sum_{x_1,x_2} a_{x_1,x_2} = 1, a_{x_1,x_2} \geq 0\right\} \\ \subseteq \quad &\left\{G\left(\sum_{x_2} b_{x_2}\mathbf{p}_{\tilde{x}_1,x_2}\right) : \sum_{x_2} b_{x_2} = 1, b_{x_2} \geq 0, G \in \mathcal{G}\right\},\end{aligned}$$

*the single-letter characterization of the capacity region is*

$$\bigcup_{p(u)p(x_1x_2|u)} \left\{ (R_1, R_2) \,\middle|\, \begin{array}{l} R_1 \leq \tau - \sum_{u\in\mathcal{U}} p_U(u)H(Y_1|U=u), \\ R_2 \leq \sum_{u\in\mathcal{U}} p_U(u)H(Y_2'|U=u) - \eta \end{array} \right\},$$

*where the auxiliary random variable* $U$ *satisfies the Markov chain* $U-(X_1,X_2)-(Y_1,Y_2)$*, and its cardinality is bounded by*

$$|\mathcal{U}| \leq \min\left(|\mathcal{Y}_1|, |\mathcal{Y}_2'|, |\mathcal{X}_1||\mathcal{X}_2|\right);$$

$\tau = \max_{\Delta_{|\mathcal{Y}_2'|}} H(T'\mathbf{p})$*, i.e., the maximum entropy of* $Y_1$ *over all possible distributions of* $Y_2'$*.*

The conditions look complex, but they will be more clear in the examples provided in Section 4.4. The proof follows exact the same fashion as in [30], and is omitted here.

An interesting observation is that when $T = T'$, i.e., $Y_2'$ is a one-to-one mapping from $(X_1, Y_2)$, the above lemma cannot apply. The reason is that Condition 5 is impossible to satisfy. To illustrate this point, we set $Y_2' = (X_1, Y_2)$ without loss of generality. Suppose $|\mathcal{X}_1| = k$, $|\mathcal{Y}_2| = l$, $|\mathcal{X}_2| = m$, then the transition probability matrix of $p(y_2'|x_1x_2)$ is given by

$$\begin{bmatrix} [p(y_2|x_2)]_{l\times m} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & [p(y_2|x_2)]_{l\times m} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & [p(y_2|x_2)]_{l\times m} \end{bmatrix}_{kl\times km}.$$

$\mathbf{p}_{x_1,x_2}$ corresponds to a column in this matrix, while $\mathbf{p}_{\tilde{x}_1,x_2}$ corresponds to a column within a sub-matrix with respect to $\tilde{x}_1$. Given one particular $\tilde{x}_1'$, it is impossible to express the convex combination of all columns with only permutation operations. It is because that there is always $(k-1)l$ 0's while the convex combination of all columns does not necessarily contain 0 as its entry.

## 4.4 Examples

**Example 4.1.** *Consider a DMZIC with input and output alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ and is defined by the equations: $y_1 = x_1 \cdot x_2$, $y_2 = x_2$, shown in Fig. 4.4 Etkin and Ordentlich in [31] established the capacity region for this binary multiplier channel via a new outer-bounds derived in their paper. As this channel satisfies the weak interference condition in this paper, we can immediately get the sum-rate capacity to be*

$$\begin{aligned} & \max_{p(x_1)p(x_2)} I(X_1;Y_1) + I(X_2;Y_2) \\ = & \max_{p(x_1)p(x_2)} \{H(X_1 \cdot X_2) - \Pr\{X_1 = 1\}H(X_2) + H(X_2)\} \\ \approx & \ 1.3881. \end{aligned}$$

**Example 4.2.** *Let $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ and*

$$\begin{aligned} Y_1 &= X_1 \oplus Y_2, \\ Y_2 &= X_2 \oplus Z, \end{aligned}$$

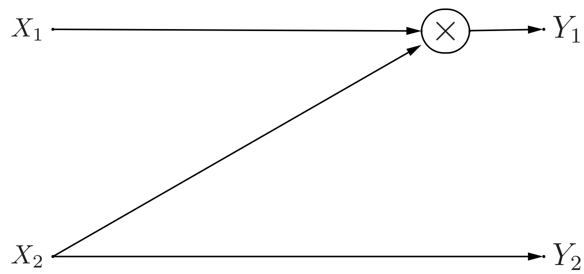



FIGURE 4.4: The binary multiplier channel model.

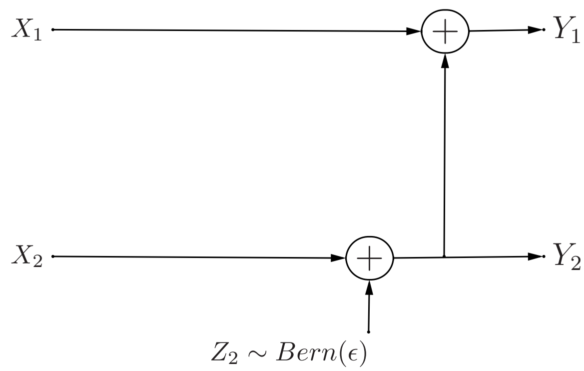



FIGURE 4.5: The binary degraded additive DMZIC model.

*where $\oplus$ denotes the modulo 2 sum and $Z \sim Bern(\epsilon)$. This channel is depicted in Fig. 4.5. Clearly, the Markov chain $X_2 - X_1Y_2 - Y_1$ is satisfied. Let $p = \Pr(X_2 = 1)$. Then,*

$$\begin{aligned} I(X_2; Y_2) &= h_2\left(\epsilon(1-p) + (1-\epsilon)p\right) - h_2(\epsilon), \\ I(X_1; Y_1) &= H(Y_1) - h_2\left(\epsilon(1-p) + (1-\epsilon)p\right). \end{aligned}$$

*The sum-rate capacity is*

$$C_{sum} = \max_{p(x_1)p(x_2)} \{I(X_1;Y_1) + I(X_2;Y_2)\} = 1 - h_2(\epsilon),$$

*which is achieved by any* $p(x_1)p(x_2)$ *such that* $H(Y_1) = 1$*. Additionally, both points* $(0, 1 - h_2(\epsilon))$ *and* $(1-h_2(\epsilon), 0)$ *are trivially achievable. Therefore, the capacity region of this channel is the triangle connecting the two rate pairs* $(0, 1 - h_2(\epsilon))$ *and* $(1 - h_2(\epsilon), 0)$*.*

*This channel does not belong to any class of channels that have been studied in the literature. The property of* $H(Y_1|X_1) = H(Y_2)$ *is similar to the deterministic interference channel definition [32]. However,* $Y_2$ *is not a deterministic function of* $X_2$*.*

*This channel is equivalent, in the capacity region, to the following interference channel:*

$$\begin{aligned} Y_1 &= X_1 \oplus X_2 \oplus Z, \\ Y_2 &= X_1 \oplus X_2 \oplus Z. \end{aligned}$$

*This can be proved in a similar way to that used in [25] for proving the equivalence between the Gaussian ZIC and the Gaussian degraded IC. Notice that the capacity region of the discrete additive degraded IC is solved by Benzel in [33], the capacity region of the DMZIC can be obtained through the equivalent discrete additive degraded IC, i.e., the closure of the convex hull of all the nonnegative* $(R_1, R_2)$ *satisfying the following inequalities:*

$$R_1 \leq I(X_1;Y_1),$$
$$R_2 \leq I(X_2;Y_2),$$

*for all possible product input distribution on* $\mathcal{X}_1 \times \mathcal{X}_2$*.*

Actually, the capacity region can be resolved for any modulo sum channel.

**Example 4.3.** *Let* $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{S} = \{0, 1, \cdots, s-1\}$ *and*

$$\begin{aligned} Y_1 &= X_1 \oplus X_2 \oplus V_1 \oplus V_2, \\ Y_2 &= X_2 \oplus V_2, \end{aligned}$$

*where* $V_1$ *and* $V_2$ *are independent noise random variables defined over* $\mathcal{S}$ *with distributions*

$$\mathbf{p}_i = (p_i(0), p_i(1), \cdots, p_i(s-1)), i = 1, 2.$$

*This is a DMZIC with weak interference, as we can write* $Y_1 = X_1 \oplus Y_2 \oplus V_1$*. We can check that* $T'$ *is circulant, Conditions* $1-5$ *are satisfied.*

**Example 4.4.** *Let* $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ *and*

$$\begin{aligned} Y_1 &= X_1 \cdot Y_2, \\ Y_2 &= X_2 \oplus Z. \end{aligned}$$

*This channel is similar to Example 4.2 except that* $Y_1$ *is replaced by a multiplicative channel, and is shown in Fig. 4.6.*

*The Markov chain* $X_2 - X_1Y_2 - Y_1$ *holds and the capacity region of this channel can be obtained in a manner similar to that of [31]. We first upper-bound the two individual rates* $R_1$ *and* $R_2$*. From the proof of Theorem 4.3, it is straightforward to obtain*

$$R_1 - \epsilon_1 \leq I(UX_1; Y_1|Q),$$

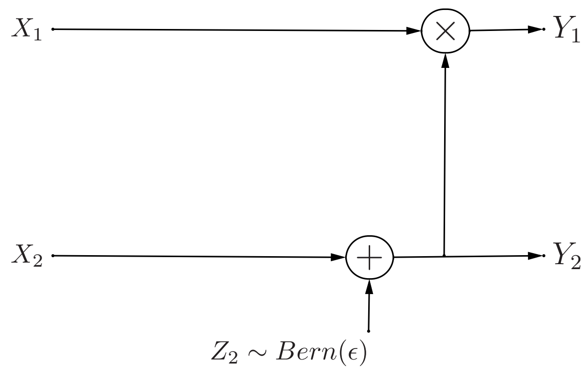



FIGURE 4.6: The binary degraded multiplicative DMZIC model.

*where $U$ is an auxiliary random variable satisfying $p(ux_1x_2) = p(x_1)p(ux_2)$. For $R_2$,*

$$
\begin{aligned}
n(R_2 - \epsilon_2) &\leq I(X_2^n; Y_2^n) \\
&\leq \sum_{i=1}^{n} \left( H(Y_{2i}|Y_2^{i-1}) - H(Y_{2i}|X_2^n Y_2^{i-1}) \right) \\
&\leq \sum_{i=1}^{n} \left( H(Y_{2i}) - H(Y_{2i}|X_{2i}) \right) \\
&= \sum_{i=1}^{n} I(X_{2i}; Y_{2i}) \\
&= nI(X_2; Y_2|Q). \qquad (4.19)
\end{aligned}
$$

*Let $p_{1,q} = \Pr(x_1 = 1|Q = q)$, $p_{2,q} = \Pr(x_2 = 1|Q = q)$, $p^y_{2,q} = \Pr(y_2 = 1|Q = q)$, $r_q = H(Y_2|U, q)$, note that*

$$p^y_{2,q} = p_{2,q}(1 - \epsilon) + (1 - p_{2,q})\epsilon,$$

*and*

$$r_q \leq h_2(p_{2,q}),$$

*for each $q$. Then,*

$$\begin{aligned}
R_1 - \epsilon_1 &\leq I(UX_1; Y_1|Q) \\
&= \sum_{q=1}^{\|\mathcal{Q}\|} [H(Y_1|q) - \sum_{x_1=0}^{1} p(x_1|q) H(Y_1|x_1, U, q)] \\
&= \sum_{q=1}^{\|\mathcal{Q}\|} [H(Y_1|q) - p(x_1 = 1|q) H(Y_2|U, q)] \\
&= \sum_{q=1}^{\|\mathcal{Q}\|} [h_2(p_{1,q} p_{2,q}^{y}|q) - p(x_1 = 1|q) r_q]
\end{aligned}$$

*and*

$$\begin{aligned}
R_2 - \epsilon_2 &\leq I(X_2; Y_2|Q) \\
&= H(Y_2|Q) - H(Y_2|X_2 Q) \\
&= h_2(p_{2,q}^{y}) - h_2(\epsilon).
\end{aligned}$$

*Compared with the expressions in [31, Eqs. (15) and (16)], the only difference is the constant $h_2(\epsilon)$, which does not affect the optimization. Therefore, the optimization process there can be directly applied here. It follows that the capacity region of this channel is the convex hull of $\mathcal{R}'$, where*

$$\mathcal{R}' = \bigcup_{0 \leq p_1, p_2 \leq 1} \left\{ (R_1, R_2) \;\middle|\; \begin{array}{c} R_1 \leq I(X_1; Y_1) = h_2(p_1 p_{y_2}) - p_1 h_2(p_{y_2}) \\ R_2 \leq I(X_2; Y_2) = h_2(p_{y_2}) - h_2(\epsilon) \end{array} \right\},$$

*where $p_{y_2} = \epsilon(1 - p_2) + (1 - \epsilon) p_2$. Clearly, the sum-rate capacity is*

$$\max_{p_1 p_2} \left\{ (p_1 p_{y_2}) + (1 - p_1) h_2(p_{y_2}) - h_2(\epsilon) \right\}.$$

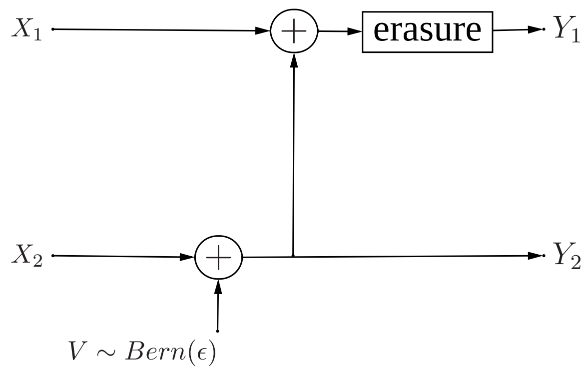



FIGURE 4.7: The degraded DMZIC model with an erasure channel.

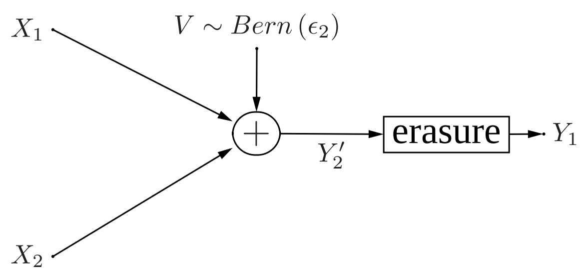



FIGURE 4.8: The equivalent DMDIC model with an erasure channel.

**Example 4.5.** $\|\mathcal{X}_1\| = \|\mathcal{X}_2\| = \|\mathcal{Y}_2\| = 2$, $\|\mathcal{Y}_1\| = 3$.

$$\begin{aligned} Y_1 &= \begin{cases} X_1 \oplus Y_2, & \text{with probability } 1-\delta \\ e, & \text{with probability } \delta \end{cases}, \\ Y_2 &= X_2 \oplus V_1, \end{aligned}$$

*where $V_1 \sim Bern(\epsilon)$. The channel is shown in Fig. 4.7. Clearly, $Y_1$ is the output of a erasure channel with input $X_1 \oplus Y_2$ and erasure proability $\delta$.*

*Define* $Y_2' = X_1 \oplus Y_2$*. Thus, the DMIC with inputs* $X_1, X_2$ *and outputs* $Y_1, Y_2'$ *is a degraded DMIC, depicted in Fig. 4.8. The capacity region of this degraded DMIC has been solved by Liu and Ulukus [30], and can be expressed as*

$$\mathcal{R}_I = \overline{co}\left\{ \bigcup_{p(x_1)p(x_2)} ((R_1, R_2) : R_1 \leq I(X_1; Y_1), R_2 \leq I(X_2; Y_2'|X_1)) \right\}.$$

*The corresponding capacity region for the DMZIC is*

$$\mathcal{R}_Z = \overline{co}\left\{ \bigcup_{p(x_1)p(x_2)} ((R_1, R_2) : R_1 \leq I(X_1; Y_1), R_2 \leq I(X_2; Y_2)) \right\}. \tag{4.20}$$

That $\mathcal{R}_Z$ being the capacity region comes from the fact that $I(X_2; Y_2'|X_1) = I(X_2; Y_2)$ while $\mathcal{R}_I$ is naturally an outer-bound.

**Example 4.6.** *Let* $\|\mathcal{X}_1\| = \|\mathcal{X}_2\| = \|\mathcal{Y}_1\| = \|\mathcal{Y}_2\| = 2$ *and the channel transition probability be given by*

$$p(y_1 y_2|x_1 x_2) = p(y_2|x_2)p(y_1|x_1 y_2),$$

*where* $p(y_2|x_2)$ *and* $p(y_1|x_1 y_2)$ *are specified in Table 4.1. The channel is shown in Fig. 4.9.*

Table 4.1: Channel Transition Probabilities

| $p(y_2\|x_2)$ | $y_2 = 0$ | $y_2 = 1$ | $p(y_1\|x_1 y_2)$ | $y_1 = 0$ | $y_1 = 1$ |
|---|---|---|---|---|---|
| $x_2 = 0$ | .1 | .9 | $x_1 y_2 = 00$ or $11$ | .75 | .25 |
| $x_2 = 1$ | .9 | .1 | $x_1 y_2 = 01$ or $10$ | 0 | 1 |

*By Theorem 4.3, the sum-rate capacity is*

$$\mathcal{C}_{sum} = \max_{p(x_1)p(x_2)} I(X_1; Y_1) + I(X_2; Y_2) \approx .531.$$

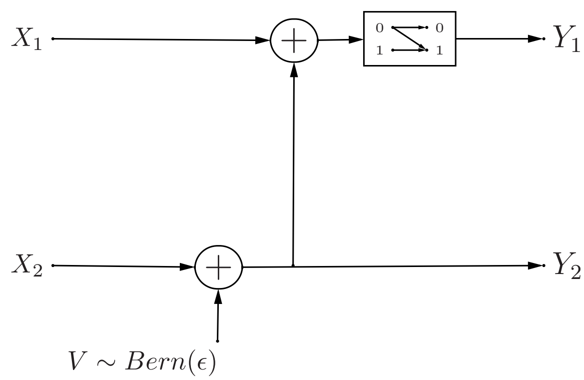



FIGURE 4.9: The DMZIC model with a binary Z channel channel degradation.

*In addition, a simple outer bound can be constructed as follows*

$$R_1 \leq I(X_1; Y_1|X_2), \tag{4.21}$$

$$R_2 \leq I(X_2; Y_2), \tag{4.22}$$

$$R_1 + R_2 \leq I(X_1; Y_1) + I(X_2; Y_2). \tag{4.23}$$

*We now use Theorem 4.8 to obtain a new outer bound. Construct $Y_2'$ as follows*

$$Y_2' = \begin{cases} 0, & \textit{if } x_1 y_2 = 00 \textit{ or } 11, \\ 1, & \textit{otherwise.} \end{cases}$$

*Then $p(y_2'|x_1x_2)$ is given in Table 4.2.*

TABLE 4.2: $P(Y_2'|X_1X_2)$

| $p(y_2'\|x_1x_2)$ | $y_2' = 0$ | $y_2' = 1$ |
|---|---|---|
| $x_1x_2 = 00$ | .1 | .9 |
| $x_1x_2 = 01$ | .9 | .1 |
| $x_1x_2 = 10$ | .9 | .1 |
| $x_1x_2 = 11$ | .1 | .9 |

*Using Theorem 4.8, the capacity region of the DMZIC is outer-bounded by that of the associated discrete memoryless degraded broadcast channel:*

$$\mathcal{R}_{OB} = \overline{co}\left\{ \bigcup_{p(u)p(x_1x_2|u)} (R_1, R_2) \;\middle|\; \begin{array}{l} R_1 \leq I(U;Y_1), \\ R_2 \leq I(X_1X_2;Y_2'|U) \end{array} \right\},$$

*Let* $R_2$ *to be fixed at* $x$*, then*

$$\begin{aligned} \max_{R_2=x} R_1 &= \max_{H(Y_2'|U)=x+h_2(.1)} H(Y_1) - H(Y_1|U) \\ &\leq \log(|\mathcal{Y}_1|) - f_T(x + h_2(.1)), \end{aligned}$$

*where* $f_T(\cdot)$ *is a function defined by Witsenhausen and Wyner [34]. Fig. 4.10 depicts the new outer-bound specified by*

$$\mathcal{R}'_{OB} = \left\{ (R_1, R_2) | \begin{array}{c} R_1 \leq \log|\mathcal{Y}_1| - f_T(x + h_2(.1)), \\ R_2 \leq x \end{array} \right\}. \tag{4.24}$$

*This new outer-bound significantly improves upon the simple outer-bound (4.21)-(4.23).*

## 4.5 Summary

In this chapter, we have derived the sum-rate capacity for a class of discrete memoryless interference channels whose channel property resembles that of the Gaussian interference channel with one-sided and weak interference. Capacity outer bounds are also derived for this class of channels. The capacity expressions as well as the encoding schemes that achieve the sum-rate capacity are analogous to the Gaussian interference channel counterpart. These results allow us to obtain capacity results for several new discrete memoryless interference channels.

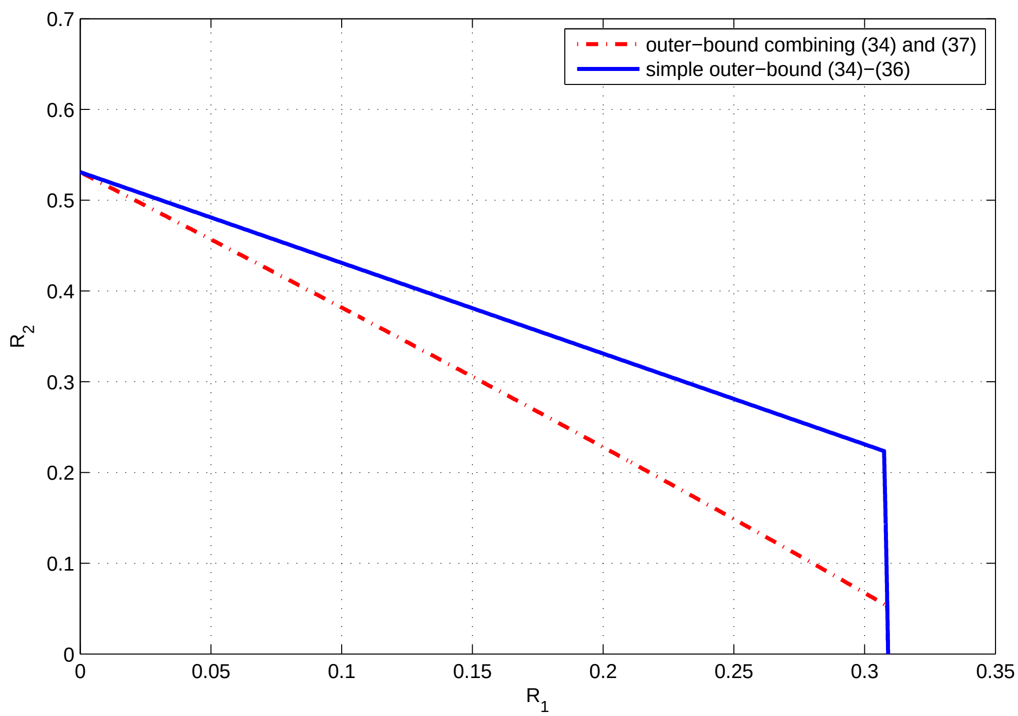



FIGURE 4.10: Comparison of the outer-bounds.

# Chapter 5

# The DMIC with Mixed Interference

In this chapter, we further extend the technique used in the previous chapter to the mixed interference case. Section 5.1 defines the mixed interference for the DMIC, and then derives the sum-rate capacity result for this class of channels. Examples are provided in Section 5.2. Finally, Section 5.3 concludes this chapter.

## 5.1 Mixed interference and sum-rate capacity for the DMIC

For the GIC with mixed interference ($a \leq 1$ and $b \geq 1$ in (2.4) and (2.5)), one can construct an equivalent GIC with degradedness defined by the Markov chain $X_2 - (X_1, Y_2) - Y_1$:

$$
\begin{aligned}
Y_1' &= (1 - ab)X_1 + aY_2 + Z_1', \\
Y_2 &= bX_1 + X_2 + Z_2,
\end{aligned}
$$

where $Z_1' \sim \mathcal{N}(0, 1 - a^2)$. This motivates us to define DMIC with mixed interference in an analogous fashion, which leads directly to its sum-rate capacity described in Theorem 5.2.

**Definition 5.1.** A DMIC is said to have *mixed interference* if it satisfies the Markov chain

$$X_2 - (X_1, Y_2) - Y_1 \tag{5.1}$$

and

$$I(X_1; Y_1|X_2) \leq I(X_1; Y_2|X_2) \tag{5.2}$$

for all possible product distributions on $\mathcal{X}_1 \times \mathcal{X}_2$.

**Theorem 5.2.** *The sum-rate capacity of a DMIC with mixed interference, i.e., one that satisfies the two conditions (5.1) and (5.2), is*

$$C_{sum} = \max_{p(x_1)p(x_2)} \left\{ I(X_2; Y_2|X_1) + \min\{I(X_1; Y_1), I(X_1; Y_2)\} \right\}. \tag{5.3}$$

*Proof.* In order to achieve this sum rate, user 1 transmits its message at a rate such that both receivers can decode it by treating the signal from user 2 as noise; user 2 transmits at the interference-free rate since receiver 2 is able to subtract the interference from user $X_1$ prior to decoding its own message.

For the converse, we prove the following two sum-rate bounds separately:

$$n(R_1 + R_2) \leq \sum_{i=1}^{n} I(X_{1i}X_{2i}; Y_{2i}), \tag{5.4}$$

$$n(R_1 + R_2) \leq \sum_{i=1}^{n} I(X_{1i}; Y_{1i}) + I(X_{2i}; Y_{2i}|X_{1i}). \tag{5.5}$$

For (5.4), the derivation follows the same steps as Costa and El Gamal's result[8]. For (5.5), we apply similar techniques used in the proof of Theorem 4.3. First, notice that (5.1) implies

$$I(U; Y_1|X_1) \leq I(U; Y_2|X_1) \tag{5.6}$$

for any $U$ whose joint distribution with $X_1, X_2, Y_1, Y_2$ is

$$p(u, x_1, x_2, y_1, y_2) = p(u)p(x_1x_2|u)p(y_1y_2|x_1x_2). \tag{5.7}$$

Therefore,

$$\begin{aligned}
& n(R_1 + R_2) - n\epsilon \\
\overset{(a)}{\leq}\; & I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n|X_1^n) \\
=\; & \sum_{i=1}^{n} \big(H(Y_{1i}|Y_1^{i-1}) - H(Y_{1i}|Y_1^{i-1}X_1^n) + H(Y_{2i}|Y_2^{i-1}X_1^n) - H(Y_{2i}|Y_2^{i-1}X_2^nX_1^n)\big) \\
\overset{(b)}{\leq}\; & \sum_{i=1}^{n} \big(H(Y_{1i}) - H(Y_{1i}|Y_1^{i-1}X_1^nY_2^{i-1}) + H(Y_{2i}|U_iX_{1i}). - H(Y_{2i}|X_{2i}X_{1i}U_i)\big) \\
=\; & \sum_{i=1}^{n}(I(U_iX_{1i}; Y_{1i}) + I(X_{2i}; Y_{2i}|U_iX_{1i})) \\
=\; & \sum_{i=1}^{n}(I(X_{1i}; Y_{1i}) + I(U_i; Y_{1i}|X_{1i}) + I(X_{2i}; Y_{2i}|U_iX_{1i})) \\
\overset{(c)}{\leq}\; & \sum_{i=1}^{n}(I(X_{1i}; Y_{1i}) + I(U_i; Y_{2i}|X_{1i}) + I(X_{2i}; Y_{2i}|U_iX_{1i})) \\
\overset{(d)}{=}\; & \sum_{i=1}^{n}(I(X_{1i}; Y_{1i}) + I(X_{2i}; Y_{2i}|X_{1i})),
\end{aligned}$$

where $(a)$ is because of the independence between $X_1^n$ and $X_2^n$; $(b)$ is from the fact that conditioning reduces entropy and by defining $U_i \triangleq (X_1^{i-1}X_{1,i+1}^n, Y_2^{i-1})$; $(c)$ is from (5.6); and $(d)$ is because of the memoryless property of the channel and (5.7). From (5.4) and (5.5), we have

$$R_1 + R_2 \leq \sum_{i=1}^{n} \min\{I(X_{1i}X_{2i}; Y_{2i}), I(X_{1i}; Y_{1i}) + I(X_{2i}; Y_{2i}|X_{1i})\}. \tag{5.8}$$

Finally, by introducing the time-sharing random variable $Q$ and following the same process as in (4.9), one obtains (5.3) as desired. $\square$

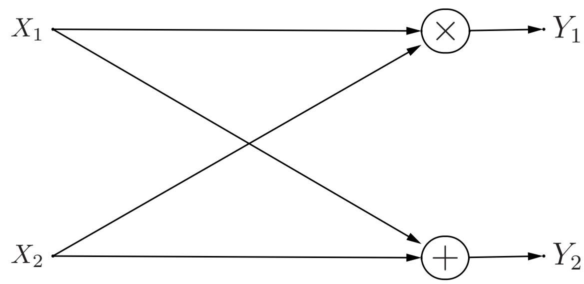



FIGURE 5.1: The DMIC with mixed interference example model.

We give the following example where the obtained sum-rate capacity helps determine the capacity region of a DMIC.

## 5.2 Example

**Example 5.1.** *Consider the following deterministic channel:*

$$\begin{aligned} Y_1 &= X_1 \cdot X_2, \\ Y_2 &= X_1 \oplus X_2, \end{aligned}$$

*where the input and output alphabets* $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$*. This channel model is depicted in Fig. 5.1 Notice that this channel does not satisfy the condition of the deterministic interference channel in [32]. Obviously, the Markov chain (5.1) holds. Moreover,*

$$\begin{aligned} I(X_1; Y_1|X_2) &= H(Y_1|X_2) = p(x_2 = 1)H(X_1), \\ I(X_1; Y_2|X_2) &= H(Y_2|X_2) = H(X_1). \end{aligned}$$

*Therefore,*

$$I(X_1;Y_1|X_2) \leq I(X_1;Y_2|X_2),$$

*for all possible input product distributions on* $\mathcal{X}_1 \times \mathcal{X}_2$*. Thus, this is a DMIC with mixed interference. On applying Theorem 5.2, we compute the sum-rate capacity to be*

$$\begin{aligned} \mathcal{C}_{sum} &= \max_{p(x_1)p(x_2)} [\min(I(X_1X_2;Y_2), I(X_1;Y_1) + I(X_2;Y_2|X_1))] \\ &= 1. \end{aligned} \quad (5.9)$$

*Given that* $(1,0)$ *and* $(0,1)$ *are both trivially achievable, the above sum-rate capacity leads to the capacity region for this DMIC to be* $\{(R_1, R_2) : R_1 + R_2 \leq 1\}$.

## 5.3 Summary

In this chapter, we combined the weak interference condition from the previous chapter together with the strong condition derived by Costa and El Gamal [8] to form the mixed interference definition. Then the analogous sum-rate capacity result is derived under this condition. Several new DMICs were studied whose sum-rate capacities or the capacity region were resolved.

# Chapter 6

# Capacity Analysis of Multiple-Access-Z-Interference Channels

In this chapter, we focus on a 3-user uplink model with one-sided interference, where we attempt to derive exact capacity results for strong, very strong, mixed and weak interference cases.

The rest of the chapter are organized as follows. We give the problem formulation in Section 6.1. Section 6.2 gives an achievable rate region for the discrete memoryless MAZIC and the result is extended to the Gaussian case. Capacity results for the strong, mixed and weak interference cases are derived in Sections 6.3, 6.4 and 6.5 respectively. Section 6.6 concludes the paper.

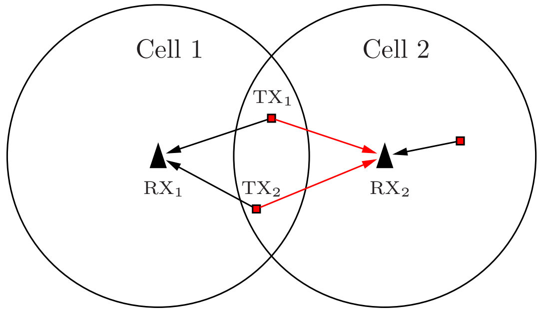



FIGURE 6.1: Two-cell uplink transmission.

## 6.1 Model formulation

Fig. 6.2 is an abstract model of the above network. Transmitters 1 and 2 and receiver 1 form a MAC. Transmitter 3 and receiver 2 form a single-user channel and receiver 2 is subject to interference from transmitters 1 and 2. Specifically, the channel outputs are given by

$$Y_1 = X_1 + X_2 + Z_1, \tag{6.1}$$

$$Y_2 = \sqrt{a}X_1 + \sqrt{b}X_2 + X_3 + Z_2, \tag{6.2}$$

where $X_i$ and $Y_j$ are the transmitted and received signals of transmitter $i$ and receiver $j$, respectively, for $i = 1, 2, 3$ and $j = 1, 2$. For each $j$, $Z_j$ is Gaussian noise with zero mean and unit variance and we assume all the noise terms are independent of each other and over time. For channels with arbitrary coefficients and noise variances, standard normalization can be applied such that its capacity is equivalent to the above channel, i.e., the gains for $X_1$, $X_2$ in $Y_1$ and $X_3$ in $Y_2$ are all assumed to be 1. The channel coefficients $a$ and $b$ are fixed and known at both the transmitters and the receivers. Without loss of generality, we assume $a, b > 0$, i.e., they are strictly positive. For transmitter $i$, the user/channel input sequence $X_{i1}, X_{i2}, \cdots, X_{in}$ is subject to a block power constraint $\sum_{k=1}^{n} \mathcal{E}[X_{ik}^2] \leq nP_i$. We denote the rates for messages $W_1$, $W_2$ and $W_3$ by $R_1$, $R_2$ and $R_3$, respectively. The channel

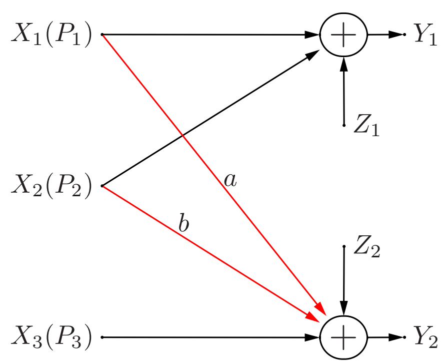



FIGURE 6.2: The Gaussian Multiple-Access-Z-interference Channel model

defined here is referred to as a Multiple-Access-Z-Interference channel (MAZIC). Unlike the two-user Z-interference channel (ZIC), there are more than one interference signal from multiple independent senders. For example, in the Gaussian case, the interference signals are multiplied by different coefficients. One cannot claim equivalence to degraded channels as in the two-user ZIC case. As such, capacity analysis becomes more complicated. Our goal is to obtain capacity results for the strong, mixed[1] and weak interference cases for the MAZIC.

Mathematically, a discrete memoryless MAZIC is defined by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, p, \mathcal{Y}_1, \mathcal{Y}_2)$, where $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3$ are finite input alphabet sets; $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are finite output alphabet sets; and $p(y_1y_2|x_1x_2x_3)$ is the channel transition probability. As the receivers do not cooperate, the capacity depends only on the marginal channel transition probabilities. Thus we can only consider two marginal distributions $(p(y_1|x_1x_2), p(y_2|x_1x_2x_3))$. The channels are memoryless, i.e.,

$$p(y_1^n y_2^n|x_1^n x_2^n x_3^n) = \prod_{i=1}^{n} p(y_{1i}y_{2i}|x_{1i}x_{2i}x_{3i}), \tag{6.3}$$

[1]Here, the notion of mixed interference refers to the strengths of the two interference links with coefficients $\sqrt{a}$ and $\sqrt{b}$. It differs from the classical notion of mixed interference where the interference is imposed on two different receivers.

where $x_i^n = [x_{i1}, x_{i2}, \cdots, x_{in}]$ and $y_j^n = [y_{j1}, y_{j2}, \cdots, y_{jn}]$, for $i = 1, 2$, and $j = 1, 2, 3$. The message for transmitter $i$ is $W_i \in \{1, 2, \cdots, 2^{nR_i}\}$, $i = 1, 2, 3$. A $(2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)$ code consists of three encoders:

$$f_1 \ : \ \{1, 2, \cdots, 2^{nR_1}\} \to \mathcal{X}_1^n, \tag{6.4}$$

$$f_2 \ : \ \{1, 2, \cdots, 2^{nR_2}\} \to \mathcal{X}_2^n, \tag{6.5}$$

$$f_3 \ : \ \{1, 2, \cdots, 2^{nR_3}\} \to \mathcal{X}_3^n, \tag{6.6}$$

and two decoders:

$$g_1 \ : \ \mathcal{Y}_1^n \to \{1, 2, \cdots, 2^{nR_1}\} \times \{1, 2, \cdots, 2^{nR_2}\}, \tag{6.7}$$

$$g_2 \ : \ \mathcal{Y}_2^n \to \{1, 2, \cdots, 2^{nR_3}\}. \tag{6.8}$$

The error probability is defined as

$$P_e = \Pr\{g_1(Y_1^n) \neq (W_1, W_2), \text{ or } g_2(Y_2^n) \neq W_3\}. \tag{6.9}$$

Assuming $W_1$, $W_2$ and $W_3$ are all uniformly distributed, a rate triple $(R_1, R_2, R_3)$ is achievable if there exist a sequence of $(2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)$ codes for $n$ sufficiently large such that $P_e \to 0$ when $n \to \infty$. Throughout this paper, we make the assumption that all the transmitters implement deterministic encoders instead of stochastic encoders as one can easily prove, following the same approach as that of [35], that stochastic encoders do not increase the capacity for a MAZIC.

## 6.2 An achievable region for the general MAZIC

We use superposition coding and joint decoding to derive an achievable rate region. Consider the independent messages $W_1$ and $W_2$ generated by transmitters 1 and 2, respectively. We split them into

$$\begin{aligned} W_1 &= [W_{1c}, W_{1p}], \\ W_2 &= [W_{2c}, W_{2p}], \end{aligned}$$

where $W_{1c}$ and $W_{2c}$ denote the common messages that are to be decoded at both receivers 1 and 2; and $W_{1p}$ and $W_{2p}$ represent the private messages that are to be decoded only at receiver 1.

We first introduce the auxiliary random variables $Q$, $U_1$, and $U_2$, where $Q$ is a time-sharing random variable, and $U_1$ and $U_2$ contain the information $W_{1c}$ and $W_{2c}$ respectively. The distribution of $(Q, U_1, U_2, X_1, X_2, X_3)$ factorizes as

$$p(qu_1u_2x_1x_2x_3) = p(q)p(u_1|q)p(x_1|u_1, q)p(u_2|q)p(x_2|u_2, q)p(x_3|q). \tag{6.10}$$

The following achievable rate region can be obtained whose proof is given in Appendix D.

**Theorem 6.1.** *For a discrete memoryless MAZIC, an achievable rate region is given by the set of all nonnegative rate triples* $(R_1, R_2, R_3)$ *that satisfy*

$$\begin{aligned}
R_1 &\leq I(X_1; Y_1|X_2Q), && (6.11)\\
R_2 &\leq I(X_2; Y_1|X_1Q), && (6.12)\\
R_3 &\leq I(X_3; Y_2|U_1U_2Q), && (6.13)\\
R_1 + R_2 &\leq I(X_1X_2; Y_1|Q), && (6.14)\\
R_1 + R_3 &\leq I(X_1; Y_1|U_1X_2Q) + I(U_1X_3; Y_2|U_2Q), && (6.15)\\
R_2 + R_3 &\leq I(X_2; Y_1|U_2X_1Q) + I(U_2X_3; Y_2|U_1Q), && (6.16)\\
R_1 + R_2 + R_3 &\leq I(X_1X_2; Y_1|U_1U_2Q) + I(U_1U_2X_3; Y_2|Q), && (6.17)\\
R_1 + R_2 + R_3 &\leq I(X_1X_2; Y_1|U_1Q) + I(U_1X_3; Y_2|U_2Q), && (6.18)\\
R_1 + R_2 + R_3 &\leq I(X_1X_2; Y_1|U_2Q) + I(U_2X_3; Y_2|U_1Q), && (6.19)\\
R_1 + 2R_2 + R_3 &\leq I(X_2; Y_1|U_2X_1Q) + I(X_1X_2; Y_1|U_1Q) + I(U_1U_2X_3; Y_2|Q), && (6.20)\\
2R_1 + R_2 + R_3 &\leq I(X_1; Y_1|U_1X_2Q) + I(X_1X_2; Y_1|U_2Q) + I(U_1U_2X_3; Y_2|Q), && (6.21)
\end{aligned}$$

*where the input distribution factors as (4.14). Furthermore, the region remains the same if we impose the constraints* $\|\mathcal{Q}\| \leq 12$, $\|U_1\| \leq \|X_1\| + 5$, *and* $\|U_2\| \leq \|X_2\| + 5$.

The MAC and the Z-interference channel (ZIC) are two special cases of a MAZIC. On setting $X_3U_1U_2 = \emptyset$, we obtain the capacity region for the MAC:

$$\begin{aligned}
R_1 &\leq I(X_1; Y_1|X_2Q),\\
R_2 &\leq I(X_2; Y_1|X_1Q),\\
R_1 + R_2 &\leq I(X_1X_2; Y_1|Q).
\end{aligned}$$

Alternatively, on setting $U_2X_2 = \emptyset$, we obtain Han and Kobayashi's achievable rate region for the ZIC [4, 28, 36]:

$$\begin{aligned} R_1 &\leq I(X_1;Y_1|Q), \\ R_3 &\leq I(X_3;Y_2|U_1Q), \\ R_1 + R_3 &\leq I(X_1;Y_1|U_1Q) + I(U_1X_3;Y_2|Q). \end{aligned}$$

Theorem 6.1 allows us to obtain a computable achievable region for Gaussian MAZICs.

*Remark* 6.2. The MAZIC model looks similar to the many-to-one interference channel studied in [37]. A key difference, however, is that receiver $Y_1$ in the MAZIC setting is a MAC receiver. If one applies the same lattice codes described in [37], receiver 1 can only decode the sum but not the individual messages from transmitters 1 and 2. To avoid such a scenario, one may employ orthogonal transmissions for transmitters 1 and 2. Together with the fact that at each level, only user 3 or users $1, 2$ transmit as in the approach introduced in [37], it will result in a simple TDM (Time-division Multiplexing) scheme, where only one user transmits at a time. This is obviously included in the achievable rate region described in Theorem 6.1. The same situation would occur if one applies interference alignment methodologies introduced in [38–40]. The interference channel in [40] is symmetric and each receiver is required to decode its own message, while in our model the channel is asymmetric and receiver 1 is required to decode messages from both transmitters 1 and 2. Therefore, applying the lattice coding scheme in [40] to our problem and aligning $X_1$ and $X_2$ at receiver 2 requires additional work to make both $X_1$ and $X_2$ distinguishable at receiver 1. Instead, the real interference alignment [41] can be directly used here to achieve 1.5 degrees of freedom (DoF). The detailed coding scheme is presented in the following:

For user $i$, the transmitter selects a constellation $\mathcal{U}_i$ to send the data stream. The constellation points are chosen from integer points $\mathcal{U}_i \subset \mathbb{Z}$, and $\mathcal{U}_i \subset [-\frac{1}{2}P^{\frac{1-\epsilon}{2(2+\epsilon)}}, \frac{1}{2}P^{\frac{1-\epsilon}{2(2+\epsilon)}}]$, where $\epsilon > 0$ is arbitrarily small. To adjust the power, the transmitter multiplies the signal by a constant, the transmitted signals are $x_1 = A\sqrt{b}u_1$, $x_2 = A\sqrt{a}u_2$ and $x_3 = Au_3$, where the constant $A$ is chosen such that the power constraint is satisfied at all three transmitters.

The received signals in this case are

$$\begin{aligned} y_1 &= A\sqrt{b}u_1 + A\sqrt{a}u_2 + z_1, \quad (6.22) \\ y_2 &= A\sqrt{ab}u_1 + A\sqrt{ab}u_2 + Au_3 + z_2 \\ &= A\sqrt{ab}(u_1 + u_2) + Au_3 + z_2. \quad (6.23) \end{aligned}$$

It is clear that users 1 and 2 have distinct signals at the intended receiver $Y_1$ while they are aligned at receiver 2. Since the rational dimension of the interference signal is 1, the multiplexing gain of the intended data stream is $\frac{1}{2}$. This is where the number $\frac{1-\epsilon}{2(2+\epsilon)}$ comes from.

Note that the sum rate inner bound proposed here is not DoF optimal. However, DoF may not be a good fit in terms of the performance metrics under our problem setting, as DoF becomes optimal only for large enough $P_1$ and $P_2$. The application studied herein is the uplink transmission with inter-cell interference. As mobile units have limited power constraints, it is better to consider lower power scenario and use the achievable sum rate derived by random coding schemes. In addition, it will be shown in Lemma 6.23 that for weak interference and bounded power constraints, treating interference as noise based on random coding methodology achieves a sum rate that is within half a bit of the sum capacity.

*Remark* 6.3. The generalized Han-Kobayashi achievable scheme is introduced for the 3-user full interference network in [42]. The codebook structure uses rate splitting and superposition coding as in the traditional Han-Kobayashi scheme [4, 28, 36], as well as Marton coding developed for the broadcast channel [43]. This layer of Marton coding is used to explore the different interference structures in different receiving signals. However, in our MAZIC model, another layer of Marton coding is not necessary, since only receiver $Y_2$ is subject to interference.

*Remark* 6.4. We would like to explain the bound on $R_1+2R_2+R_3$ in detail in the context of deterministic model, where the transmitter is modeled as $n$ bit vector for each user, and each input bit is viewed as a "level". Suppose we increase the rate of user 2 by a small amount $\delta$ (take 1 level more). The slope implies that $2\delta$ amount of rate for $R_3$ is needed to balance it out if one want to keep $R_1$ unchanged, or we need to give away 2 levels. Intuitively speaking, this is because one may need to change the transmission scheme of $X_1$ in order to keep $R_1$ the same, in addition to the impact from increasing $R_2$. For example, in the deterministic model setting, increase $R_2$ by taking over a level which is interfered by $X_1$ at receiver 1, as well as $X_3$ at receiver 2. To maintain $R_1$, user 1 needs to take over another level which is interfered with $X_3$ at receiver 2. As a result, increasing $R_2$ by one bit would essentially make $R_3$ to sacrifice two bits, in order to maintain $R_1$ at the same rate, and facilitate $R_2$ to increase.

**Corollary 6.5.** *For any nonnegative pair $[\alpha, \beta] \in [0, 1]$, the non-negative rate triples $(R_1, R_2, R_3)$ satisfying the conditions (6.24)-(6.34) are achievable for a Gaussian MAZIC.*

$$R_1 \leq \frac{1}{2}\log(1 + P_1), \tag{6.24}$$

$$R_2 \leq \frac{1}{2}\log(1 + P_2), \tag{6.25}$$

$$R_3 \leq \frac{1}{2}\log\left(1 + \frac{P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.26}$$

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1 + P_1 + P_2\right), \tag{6.27}$$

$$R_1 + R_3 \leq \frac{1}{2}\log\left(1 + \alpha P_1\right) + \frac{1}{2}\log\left(1 + \frac{a\bar{\alpha}P_1 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.28}$$

$$R_2 + R_3 \leq \frac{1}{2}\log\left(1 + \beta P_2\right) + \frac{1}{2}\log\left(1 + \frac{b\bar{\beta}P_2 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.29}$$

$$R_1 + R_2 + R_3 \leq \frac{1}{2}\log\left(1 + \alpha P_1 + \beta P_2\right) + \frac{1}{2}\log\left(1 + \frac{a\bar{\alpha}P_1 + b\bar{\beta}P_2 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.30}$$

$$R_1 + R_2 + R_3 \leq \frac{1}{2}\log\left(1 + \alpha P_1 + P_2\right) + \frac{1}{2}\log\left(1 + \frac{a\bar{\alpha}P_1 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.31}$$

$$R_1 + R_2 + R_3 \leq \frac{1}{2}\log\left(1 + P_1 + \beta P_2\right) + \frac{1}{2}\log\left(1 + \frac{b\bar{\beta}P_2 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.32}$$

$$R_1 + 2R_2 + R_3 \leq \frac{1}{2}\log\left(1 + \beta P_2\right) + \frac{1}{2}\log\left(1 + \alpha P_1 + P_2\right) + \frac{1}{2}\log\left(1 + \frac{a\bar{\alpha}P_1 + b\bar{\beta}P_2 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right), \tag{6.33}$$

$$2R_1 + R_2 + R_3 \leq \frac{1}{2}\log\left(1 + \alpha P_1\right) + \frac{1}{2}\log\left(1 + P_1 + \beta P_2\right) + \frac{1}{2}\log\left(1 + \frac{a\bar{\alpha}P_1 + b\bar{\beta}P_2 + P_3}{1 + a\alpha P_1 + b\beta P_2}\right). \tag{6.34}$$

*Proof.* Corollary 6.5 follows directly from Theorem 6.1 by choosing $\|\mathcal{Q}\| = 1$, $X_1 \sim \mathcal{N}(0, P_1)$, $X_2 \sim \mathcal{N}(0, P_2)$, and $X_1 = U_1 + V_1$, $X_2 = U_2 + V_2$, where $U_1$, $U_2$, $V_1$ and $V_2$ are independent random variables with $U_1 \sim \mathcal{N}(0, \alpha P_1)$, $U_2 \sim \mathcal{N}(0, \beta P_2)$, $V_1 \sim \mathcal{N}(0, \bar{\alpha}P_1)$ and $V_2 \sim \mathcal{N}(0, \bar{\beta}P_2)$. □

In the following, we discuss capacity results for different interference regimes for MAZICs.

## 6.3 MAZICs with strong interference

### 6.3.1 Discrete case

Similar to [8], the discrete MAZIC with strong interference is defined as a discrete memoryless MAZIC satisfying

$$I(X_1;Y_1|X_2) \le I(X_1;Y_2|X_2X_3), \tag{6.35}$$

$$I(X_2;Y_1|X_1) \le I(X_2;Y_2|X_1X_3), \tag{6.36}$$

$$I(X_1X_2;Y_1) \le I(X_1X_2;Y_2|X_3), \tag{6.37}$$

for all product distributions on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$.

The above single letter conditions imply multi-letter conditions as stated below.

**Lemma 6.6.** *For a discrete memoryless interference channel, if (6.35)-(6.37) are satisfied for all product probability distributions on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$, then*

$$I(X_1^n;Y_1^n|X_2^n) \le I(X_1^n;Y_2^n|X_2^nX_3^n), \tag{6.38}$$

$$I(X_2^n;Y_1^n|X_1^n) \le I(X_2^n;Y_2^n|X_1^nX_3^n), \tag{6.39}$$

$$I(X_1^nX_2^n;Y_1^n) \le I(X_1^nX_2^n;Y_2^n|X_3^n). \tag{6.40}$$

*Proof.* From the channel model, we have

$$\begin{aligned}
I(X_1^n;Y_1^n|X_2^nX_3^n) &= I(X_1^n;Y_1^n|X_2^n), \\
I(X_2^n;Y_1^n|X_1^nX_3^n) &= I(X_2^n;Y_1^n|X_1^n), \\
I(X_1^nX_2^n;Y_1^n|X_3^n) &= I(X_1^nX_2^n;Y_1^n).
\end{aligned}$$

The rest of the proof can be established using techniques similar to that of [8], hence is omitted. □

The above lemma leads to the following theorem.

**Theorem 6.7.** *For a discrete memoryless MAZIC with conditions (6.35)-(6.37) for all product probability distributions on* $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$*, the capacity region is given by the set of all the nonnegative rate triples* $(R_1, R_2, R_3)$ *that satisfy*

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|X_2Q), && (6.41)\\ R_2 &\leq I(X_2; Y_1|X_1Q), && (6.42)\\ R_3 &\leq I(X_3; Y_2|X_1X_2Q), && (6.43)\\ R_1 + R_2 &\leq I(X_1X_2; Y_1|Q), && (6.44)\\ R_2 + R_3 &\leq I(X_2X_3; Y_2|X_1Q), && (6.45)\\ R_1 + R_3 &\leq I(X_1X_3; Y_2|X_2Q), && (6.46)\\ R_1 + R_2 + R_3 &\leq I(X_1X_2X_3; Y_2|Q), && (6.47) \end{aligned}$$

*where the input distribution factors as*

$$p(qx_1x_2x_3) = p(q)p(x_1|q)p(x_2|q)p(x_3|q). \quad (6.48)$$

*Furthermore, the region remains invariant if we impose the constraint* $\|\mathcal{Q}\| \leq 8$*.*

The proof is given in Appendix E.

### 6.3.2 Gaussian case

For a Gaussian MAZIC, the strong interference is defined as the case where $a \geq 1$ and $b \geq 1$, which are sufficient and necessary conditions for (6.35) and (6.36), respectively. However, it is hard to find a sufficient and necessary conditions for (6.37), and there are counter examples in which condition (6.37) is violated even if $a \geq 1$ and $b \geq 1$. That is, there exist input distributions such that (6.37) does not hold with $a \geq 1$ and $b \geq 1$. We provide a counter example in [44, Example 1].

While Theorem 6.5 still applies, a better rate splitting strategy can be devised for this case. If $(R_1, R_2, R_3)$ is an achievable rate triple, then receiver 2 can reliably recover $X_1$ and $X_2$ at these rates. Therefore, receiver 2 can decode whatever receiver 1 decodes. Thus, if we choose the private message sets for users 1 and 2 to be empty, i.e., $\alpha = \beta = 0$, we obtain an achievable rate region.

In the following, we give an outer-bound on the capacity region.

**Corollary 6.8.** *For a Gaussian MAZIC with conditions $a, b \geq 1$, an outer-bound on the capacity region is given by the set of all the nonnegative rate triples $(R_1, R_2, R_3)$ that satisfy*

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log\left(1 + P_1\right), && (6.49)\\
R_2 &\leq \frac{1}{2}\log\left(1 + P_2\right), && (6.50)\\
R_3 &\leq \frac{1}{2}\log\left(1 + P_3\right), && (6.51)\\
R_1 + R_2 &\leq \frac{1}{2}\log\left(1 + P_1 + P_2\right), && (6.52)\\
R_2 + R_3 &\leq \frac{1}{2}\log\left(1 + bP_2 + P_3\right), && (6.53)\\
R_1 + R_3 &\leq \frac{1}{2}\log\left(1 + aP_1 + P_3\right). && (6.54)
\end{aligned}$$

The proof of this corollary is very similar to the proof of Theorem 6.7, except for the bound on $R_1 + R_2 + R_3$. The reason is that with $a \geq 1$ and $b \geq 1$, $I(X_1X_2; X_1 + X_2 + Z_1) \leq I(X_1X_2; \sqrt{a}X_1 + \sqrt{b}X_2 + Z_2)$ is generally not true for every possible input distribution, hence we do not have (6.37). Therefore, inequality (6.47) cannot be obtained.

Next, let us consider one interference link being strong, for example, $1 \leq a \leq 1 + P_3$. In this case, we can easily get the following outer-bound:

$$
\begin{aligned}
R_1 &\leq \frac{1}{2}\log(1 + P_1), && (6.55)\\
R_2 &\leq \frac{1}{2}\log(1 + P_2), && (6.56)\\
R_3 &\leq \frac{1}{2}\log(1 + P_3), && (6.57)\\
R_1 + R_2 &\leq \frac{1}{2}\log(1 + P_1 + P_2), && (6.58)\\
R_1 + R_3 &\leq \frac{1}{2}\log(1 + aP_1 + P_3). && (6.59)
\end{aligned}
$$

On the other hand, by setting $\alpha = \beta = 0$ in the achievable region for Gaussian MAZICs in Corollary 6.5, one would have an achievable rate region with all nonnegative rate triples $(R_1, R_2, R_3)$ that satisfy

$$
\begin{aligned}
R_1 &\leq \frac{1}{2}\log(1 + P_1), && (6.60)\\
R_2 &\leq \frac{1}{2}\log(1 + P_2), && (6.61)\\
R_3 &\leq \frac{1}{2}\log(1 + P_3), && (6.62)\\
R_1 + R_2 &\leq \frac{1}{2}\log(1 + P_1 + P_2), && (6.63)\\
R_1 + R_3 &\leq \frac{1}{2}\log(1 + aP_1 + P_3), && (6.64)\\
R_2 + R_3 &\leq \frac{1}{2}\log(1 + bP_2 + P_3), && (6.65)\\
R_1 + R_2 + R_3 &\leq \frac{1}{2}\log(1 + aP_1 + bP_2 + P_3). && (6.66)
\end{aligned}
$$

The following theorem summarizes the cases where some segment of the line: the intersection of the two hyperplanes defined by

$$R_1 + R_2 = \frac{1}{2}\log(1 + P_1 + P_2), \tag{6.67}$$
$$R_1 + R_3 = \frac{1}{2}\log(1 + aP_1 + P_3) \tag{6.68}$$

is on the boundary of the capacity region.

**Theorem 6.9.** *For a Gaussian MAZIC with $1 \le a \le 1 + P_3$, if*

$$b \ge \frac{1 + aP_1 + P_3}{1 + P_1}, \tag{6.69}$$

*a segment of the line defined by (6.67) and (6.68), which starts at*

$$\left(\frac{1}{2}\log(1 + P_1), \frac{1}{2}\log\left(1 + \frac{P_2}{1 + P_1}\right), \frac{1}{2}\log\left(1 + \frac{P_3}{1 + aP_1}\right)\right), \tag{6.70}$$

*and ends at*

$$\begin{pmatrix} \frac{1}{2}\log(1 + P_1 + P_2) - \frac{1}{2}\log\left(1 + \frac{bP_2}{1+aP_1+P_3}\right), \\ \frac{1}{2}\log\left(1 + \frac{bP_2}{1+aP_1+P_3}\right), \\ \frac{1}{2}\log\left(\frac{1+aP_1+bP_2+P_3}{1+P_1+P_2}\right) \end{pmatrix}, \tag{6.71}$$

*is on the boundary of the capacity region of the channel.*

*Proof.* Consider the rate triple $(R_1, R_2, R_3)$ on the line defined by (6.67) and (6.68). Any achievable rate triple on this line that also satisfies (6.65) and (6.66) must appear on the boundary of the capacity region as it belongs to both the inner and outer bounds.

Consider the rate triple defined by (6.70). It is achievable if

$$\frac{1}{2}\log\left(1 + \frac{P_2}{1 + P_1}\right) \le \frac{1}{2}\log\left(1 + \frac{bP_2}{1 + aP_1 + P_3}\right), \tag{6.72}$$

i.e.,

$$b \geq \frac{1 + aP_1 + P_3}{1 + P_1}, \tag{6.73}$$

as receiver 1 first decodes $X_2$, subtracts it, and then decodes $X_1$; receiver 2 also first decodes $X_2$, subtracts it, and then decodes $X_3$ by treating $X_1$ as noise.

The other rate triple defined by (6.71) satisfies (6.66) with equality, and satisfies (6.65) if $1 \leq a \leq 1 + P_3$ and $b \geq \frac{1+aP_1+P_3}{1+P_1}$.

Therefore, the line segment between these two rate triples (6.70) and (6.71) is on the boundary of the capacity region, and is achieved by time sharing. □

Fig. 6.3 gives an example where a line segment defined by (6.67) and (6.68) is on the boundary of the capacity region.

Increasing $b$ even further for the case of $a \geq 1$ will ensure that (6.65) and (6.66) are never active. Specifically, we have

**Corollary 6.10.** *For a Gaussian MAZIC with $a > 1$ and $b > 1 + aP_1 + P_3$, the capacity region is the set of all nonnegative rate triples $(R_1, R_2, R_3)$ that satisfies*

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log(1 + P_1), && (6.74)\\
R_2 &\leq \frac{1}{2}\log(1 + P_2), && (6.75)\\
R_3 &\leq \frac{1}{2}\log(1 + P_3), && (6.76)\\
R_1 + R_2 &\leq \frac{1}{2}\log(1 + P_1 + P_2), && (6.77)\\
R_1 + R_3 &\leq \frac{1}{2}\log(1 + aP_1 + P_3). && (6.78)
\end{aligned}$$

*Proof.* With $a \geq 1$ and $b \geq 1 + aP_1 + P_3$, (6.65) and (6.66) are redundant in the achievable region. As a result, the inner-bound and outer-bound coincide with each other. □

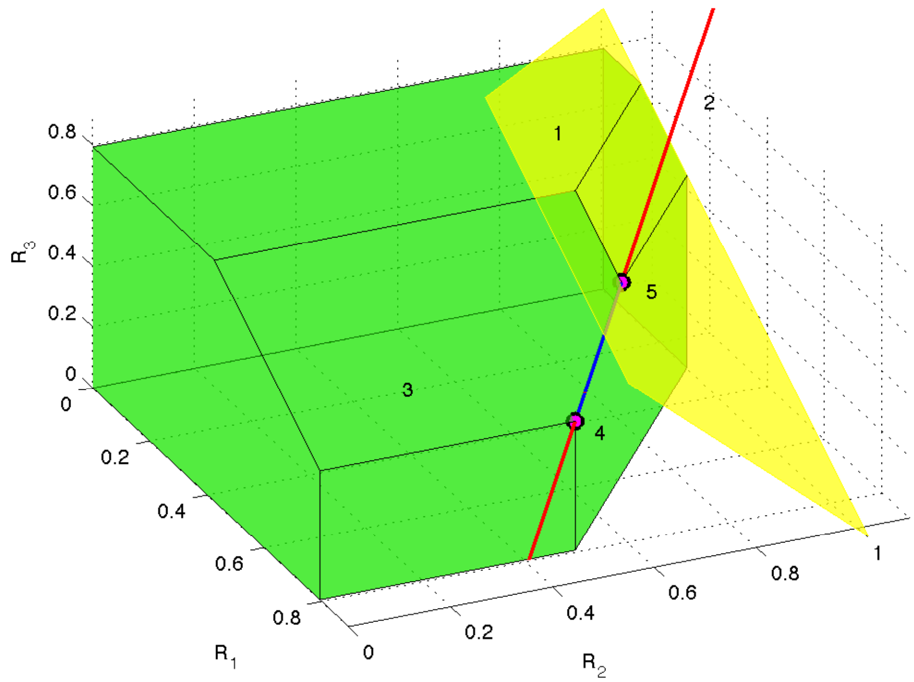



FIGURE 6.3: The line 2 defined in Eq. (6.67) and Eq. (6.68) appears as the boundary line of the capacity region. (Plane 1 is defined by $R_1 + R_2 + R_3 = \frac{1}{2}\log(1 + aP_1 + bP_2 + P_3)$; Region 3 is defined by inequalities (6.60)-(6.66)); Points 4 and 5 are the two endpoints of the line segment that is on the capacity region. For this example, the corresponding channel parameters are: $a = 1.2$, $b = 3$, $P_1 = P_3 = 2$, $P_2 = 3$.

*Remark* 6.11. In general, for the strong interference case, we can conclude that the sum-capacity is within a constant factor of 1.5 times the sum-rate achieved by the Han-Kobayashi scheme. To show this, notice that the inner bound specified by (6.60)-(6.66) differs from the outer-bound specified by (6.49)-(6.54) only in the additional sum-rate bound (6.66). Suppose that the sum-rate specified by (6.66) is achievable and we can then choose the upper-bound defined by $\frac{1}{2}\{(6.52) + (6.53) + (6.54)\}$. Is is easy to verify that the sum-rate upper-bound is less than 1.5 times the achievable sum-rate defined by (6.66). If, on the other hand, (6.66) is not achievable, this implies that the inner and outer bounds coincide thus the sum capacity is determined by (6.52), (6.53) and (6.54). Thus in both cases, the sum-capacity is within a factor of 1.5 times the achievable sum-rate.

*Remark* 6.12. As a special case, we can define the very strong interference case as a discrete memoryless MAZIC satisfying

$$I(X_1;Y_1|X_2) \leq I(X_1;Y_2|X_2), \tag{6.79}$$

$$I(X_2;Y_1|X_1) \leq I(X_2;Y_2|X_1), \tag{6.80}$$

$$I(X_1X_2;Y_1) \leq I(X_1X_2;Y_2), \tag{6.81}$$

for all product distributions on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. By Theorem 6.7, one can immediately obtain the capacity region of the MAZIC with very strong interference as the set of all nonnegative rate triples $(R_1, R_2, R_3)$ that satisfy

$$R_1 \leq I(X_1;Y_1|X_2Q), \tag{6.82}$$

$$R_2 \leq I(X_2;Y_1|X_1Q), \tag{6.83}$$

$$R_3 \leq I(X_3;Y_2|X_1X_2Q), \tag{6.84}$$

$$R_1 + R_2 \leq I(X_1X_2;Y_1|Q), \tag{6.85}$$

with the input distribution factoring as

$$p(qx_1x_2x_3) = p(q)p(x_1|q)p(x_2|q)p(x_3|q). \tag{6.86}$$

The region remains invariant if we impose the constraint $\|\mathcal{Q}\| \leq 5$.

Similarly, we can define the very strong interference for a Gaussian MAZIC as $a, b \geq 1 + P_3$. Notice that the condition $a, b \geq 1 + P_3$ is not a sufficient condition for (6.79) and (6.80), as discussed in [6, Theorem 2]. A counter example is also provided in [6, Appendix]. Again, it is a special case of the strong interference case, therefore, the capacity region can be readily obtained from Corollary 6.8, which is the set of all nonnegative rate triples $(R_1, R_2, R_3)$ that satisfy

$$
\begin{aligned}
R_1 &\leq \frac{1}{2}\log\left(1+P_1\right), && (6.87)\\
R_2 &\leq \frac{1}{2}\log\left(1+P_2\right), && (6.88)\\
R_3 &\leq \frac{1}{2}\log\left(1+P_3\right), && (6.89)\\
R_1+R_2 &\leq \frac{1}{2}\log\left(1+P_1+P_2\right). && (6.90)
\end{aligned}
$$

## 6.4 The MAZICs with mixed interference

### 6.4.1 Discrete case

The discrete MAZIC with mixed interference is defined as a discrete memoryless MAZIC satisfying

$$
\begin{aligned}
p(y_1y_2|x_1x_2x_3) &= p(y_1|x_1x_2)p(y_2|x_1x_2x_3)\\
&= p(y_1|x_1x_2)p'(y_2|x_3x_1y_1), && (6.91)
\end{aligned}
$$

for some $p'(y_2|x_3x_1y_1)$, and

$$I(X_2;Y_1|X_1) \leq I(X_2;Y_2|X_1X_3), \tag{6.92}$$

for all input distributions that factorizes as $p(x_1)p(x_2)p(x_3)$[2].

Condition (6.91) means that we can find another discrete memoryless MAZIC with $(p(y_1|x_1x_2), p'(y_2|x_3x_1y_1))$ that has the same capacity region as the original MAZIC. Furthermore, the alternative

[2]Condition (6.91) is referred to the link of weak interference, and condition (6.92) is referred to the link of strong interference.

MAZIC admits the Markov chain

$$X_1 - (X_2, X_3, Y_1) - Y_2. \tag{6.93}$$

For this class of channels, we can outer-bound the capacity region as follows.

**Theorem 6.13.** *For a discrete memoryless MAZIC with mixed interference, an outer-bound to the capacity region can be expressed as a set of nonnegative rate pairs* $(R_1, R_2)$ *satisfying the following inequalities:*

$$R_1 \leq I(X_1; Y_1|X_2U_1Q), \tag{6.94}$$

$$R_2 \leq I(X_2; Y_1|X_1Q), \tag{6.95}$$

$$R_3 \leq I(X_3; Y_2|X_1X_2Q), \tag{6.96}$$

$$R_3 \leq I(U_1X_3; Y_1|Q), \tag{6.97}$$

$$R_1 + R_2 \leq I(X_1X_2; Y_1|Q), \tag{6.98}$$

$$R_2 + R_3 \leq I(X_2X_3; Y_2|X_1Q), \tag{6.99}$$

*where the input distribution is factorized as* $p(q)p(u_1|q)p(x_1|u_1q)p(x_2|u_1q)p(x_3|q)$.

*Proof.* Inequalities (6.95) and (6.96) are trivial outer-bounds, and (6.98) is the same as the sum-rate upper-bound for the MAC. Moreover, (6.99) is the same as the sum-rate upper-bound for the two-user IC with strong interference [8]. It remains to show (6.94) and (6.97).

First, let us consider

$$\begin{aligned}
& n(R_1 - \epsilon) \\
\stackrel{(a)}{\le} \; & I(X_1^n; Y_1^n) \\
\stackrel{(b)}{\le} \; & I(X_1^n; Y_1^n | X_2^n) \\
= \; & \sum_{i=1}^n I(X_1^n; Y_{1i} | X_2^n Y_1^{i-1}) \\
= \; & \sum_{i=1}^n \left\{ H(Y_{1i} | X_2^n Y_1^{i-1}) - H(Y_{1i} | X_2^n Y_1^{i-1} X_1^n) \right\} \\
\stackrel{(c)}{=} \; & \sum_{i=1}^n \left\{ H(Y_{1i} | X_2^{i-1} X_{2i} Y_1^{i-1}) - H(Y_{1i} | X_{1i} X_{2i}) \right\} \\
\stackrel{(d)}{\le} \; & \sum_{i=1}^n \{ H(Y_{1i} | X_{2i} U_{1i}) - H(Y_{1i} | X_{1i} X_{2i} U_{1i}) \} \\
= \; & \sum_{i=1}^n I(X_{1i}; Y_{1i} | X_{2i} U_{1i}),
\end{aligned}$$

where $(a)$ comes from Fano's inequality; $(b)$ is because of the independence between $X_1^n$ and $X_2^n$; $(c)$ is because that conditioning reduces entropy and the channel is assumed to be memoryless; for $(d)$, first we identify $U_{1i} = (X_2^{i-1}, Y_1^{i-1})$ and also the memoryless property induces the Markov chain $U_{1i} - (X_{1i}, X_{2i}) - Y_{1i}$.

Now, let us show $X_{1i} - U_{1i} - X_{2i}$. Due to the memoryless property, the following Markov chain holds:

$$(X_{1i} X_{2i}) - (X_1^{i-1}, X_2^{i-1}) - Y_1^{i-1}.$$

By weak union property, we obtain the following Markov chain:

$$X_{2i} - (X_{1i}, X_1^{i-1}, X_2^{i-1}) - Y_1^{i-1}.$$

Together with the Markov chain $X_{2i} - X_2^{i-1} - X_{1i}X_1^{i-1}$, which due to the independence between $X_1^i$ and $X_2^i$, we obtain the following Markov chain by the contraction property:

$$X_{2i} - X_2^{i-1} - (X_{1i}, X_1^{i-1}, Y_1^{i-1}). \tag{6.100}$$

Hence, we get the Markov chain

$$X_{2i} - (X_2^{i-1}, Y_1^{i-1}) - X_{1i} \tag{6.101}$$

by the weak union and then the decomposition property.

Next, we consider

$$\begin{aligned}
& n(R_3 - \epsilon) \\
\overset{(a)}{\leq} \quad & I(X_3^n; Y_2^n) \\
\overset{(b)}{\leq} \quad & I(X_3^n; Y_2^n | X_2^n) \\
= \quad & \sum_{i=1}^{n} I(X_3^n; Y_{2i} | X_2^n Y_2^{i-1}) \\
= \quad & \sum_{i=1}^{n} \left\{ H(Y_{2i} | X_2^n Y_2^{i-1}) - H(Y_{2i} | X_2^n X_3^n Y_2^{i-1}) \right\} \\
\overset{(c)}{\leq} \quad & \sum_{i=1}^{n} \left\{ H(Y_{2i} | X_{2i}) - H(Y_{2i} | X_2^n X_3^n Y_1^{i-1} Y_1^{i-1}) \right\} \\
\overset{(d)}{=} \quad & \sum_{i=1}^{n} \left\{ H(Y_{2i} | X_{2i}) - H(Y_{2i} | X_2^n X_3^n Y_1^{i-1}) \right\} \\
\overset{(e)}{=} \quad & \sum_{i=1}^{n} \left\{ H(Y_{2i} | X_{2i}) - H(Y_{2i} | X_{2i} X_{3i} X_2^{i-1} Y_1^{i-1}) \right\} \\
= \quad & \sum_{i=1}^{n} \left\{ I(X_{3i} U_{1i}; Y_{2i} | X_{2i}) \right\},
\end{aligned}$$

where $(a)$ follows the Fano's Inequality, $(b)$ is from the independence between $X_2^n$ and

$X_3^n$; $(c)$ is because of the fact that conditioning reduces entropy; $(d)$ is due to the memoryless property of the channel, and the degradedness condition $X_1 - (X_2, X_3, Y_1) - Y_2$, hence $Y_2^{i-1}$ is independent of any other random variables given $X_2^{i-1}$, $X_3^{i-1}$ and $Y_1^{i-1}$, then $(X_{2,i}^n, X_{3,i}^n, Y_{2i}) - (X_2^{i-1}, X_3^{i-1}, Y_1^{i-1}) - Y_2^{i-1}$ forms a Markov chain. By the weak union property, the Markov chain $Y_{2i} - (X_2^n, X_3^n, Y_1^{i-1}) - Y_2^{i-1}$ holds; $(e)$ is because of the Markov chain $(X_{2,i+1}, X_3^{i-1}, X_{3,i+1}^n) - (X_2^i, X_{3i}, Y_1^{i-1}) - Y_{2i}$. The easiest way to prove it is using the *FDG (Functional Dependence Graphs) and d-Separation* [45, Section I-C]. Alternatively, we first note that the Markov chain

$$(X_2^{i-1}, X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n, Y_1^{i-1}) - (X_{1i}, X_{2i}, X_{3i}) - (Y_{1i}, Y_{2i})$$

holds because of the memoryless property of the channel. By the decomposition property, the following Markov chain is obtained:

$$(X_2^{i-1}, X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n, Y_1^{i-1}) - (X_{1i}, X_{2i}, X_{3i}) - Y_{2i}$$

Further by the weak union property, we obtain the following Markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_{1i}, X_2^i, X_{3i}, Y_1^{i-1}) - Y_{2i}. \tag{6.102}$$

On the other hand, again because of the memoryless property of the channel, the Markov chain

$$(X_{1i}, X_{2i}, X_{3i}, X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_1^{i-1}, X_2^{i-1}) - Y_1^{i-1}$$

holds. Using the weak union property, we obtain the Markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_{1i}, X_{2i}, X_{3i}, X_1^{i-1}, X_2^{i-1}) - Y_1^{i-1}.$$

Together with the markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_2^{i-1}X_{2i}X_{3i}) - (X_1^{i-1}, X_{1i})$$

due to the independence among $X_1^n$, $X_2^n$ and $X_3^n$, we attain the Markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_2^{i-1}, X_{2i}, X_{3i}) - (X_1^{i-1}, X_{1i}, Y_1^{i-1})$$

by the contraction property. Then by the weak union property and the decomposition property, the Markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_2^{i-1}, X_{2i}, X_{3i}, Y_1^{i-1}) - X_{1i} \tag{6.103}$$

holds. Combine (6.102) with (6.103) by the contraction property, we have the Markov chain

$$(X_{2,i+1}^n, X_3^{i-1}, X_{3,i+1}^n) - (X_2^{i-1}, X_{2i}, X_{3i}, Y_1^{i-1}) - (X_{1i}, Y_{2i})$$

as desired. The rest of the proof is done by introducing the timesharing variable $Q$, similar to the proof of the capacity region for MACs [13]. □

### 6.4.2 Gaussian case

The mixed interference case corresponds to the condition $a \leq 1, b \geq 1$ or $a \geq 1, b \leq 1$ for the Gaussian MAZICs. As mentioned before, the notion of "mixed" differs from that of the classical two-user GIC with mixed interference: here the two interferences go to the same receiver.

First of all, we can extend the outer-bound for the general discrete memoryless MAZICs to the Gaussian case.

**Corollary 6.14.** *For a Gaussian MAZIC with mixed interference ($a \le 1$ and $b \ge 1$), an outer-bound to the capacity region can be expressed as a set of nonnegative rate pairs $(R_1, R_2)$ satisfying the following inequalities:*

$$
\begin{aligned}
R_1 &\le \frac{1}{2}\log(1+\alpha P_1), && (6.104)\\
R_2 &\le \frac{1}{2}\log(1+P_2), && (6.105)\\
R_3 &\le \frac{1}{2}\log(1+P_3), && (6.106)\\
R_3 &\le \frac{1}{2}\log(1+\frac{a(1-\alpha)P_1+P_3}{1+a\alpha P_1}), && (6.107)\\
R_1+R_2 &\le \frac{1}{2}\log(1+P_1+P_2), && (6.108)\\
R_2+R_3 &\le \frac{1}{2}\log(1+bP_2+P_3), && (6.109)
\end{aligned}
$$

*Proof.* This is a direct extension of Theorem 6.13. Inequalities (6.105), (6.106), (6.108) and (6.109) comes from the corresponding inequality in Theorem 6.13 and the fact that given the variance of random variables, Guassian distribution will maximize the entropy.

As for (6.107),

$$
\begin{aligned}
R_3 &\le I(UX_3;Y_2|X_2Q)\\
&= h(Y_2|X_2Q) - h(Y_2|X_2X_3UQ)\\
&= h(\sqrt{a}X_1+X_3+Z_2|Q) - h(\sqrt{a}X_1+Z_2|UQ)\\
&\overset{(a)}{\le} \frac{1}{2}\log[(2\pi e)(1+aP_1+P_3)] - \frac{1}{2}\log a\\
&\quad -h(X_1+Z_1+Z_2'|UQ)\\
&\overset{(b)}{\le} \frac{1}{2}\log[(2\pi e)(1+aP_1+P_3)] - \frac{1}{2}\log a\\
&\quad -\frac{1}{2}\log\left(2^{2h(X_1+Z_1|UQ)}+(2\pi e)(\frac{1-a}{a})\right)\\
&\overset{(c)}{\le} \frac{1}{2}\log(1+aP_1+P_3) - \frac{1}{2}\log\left[a2^{2R_1}+1-a\right],
\end{aligned}
$$

where $(a)$ is by the fact that Gaussian distribution maximizes the entropy for a given variance, and $Z_2' \sim \mathcal{N}\left(0, \frac{1}{a} - 1\right)$, independent of all other random variables; $(b)$ is from the entropy power inequality; $(c)$ is because that from (6.94),

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|X_2UQ) = h(Y_1|X_2UQ) - h(Z_1) \\ &= h(Y_1|X_2UQ) - \frac{1}{2}\log(2\pi e). \end{aligned}$$

Furthermore, since

$$\begin{aligned} 0 \leq R_1 &\leq h(Y_1|X_2UQ) - h(Z_1) \\ &= h(X_1 + Z_1|UQ) - h(Z_1) \\ &\leq h(X_1 + Z_1|Q) - h(Z_1) \\ &\leq \frac{1}{2}\log(1 + P_1), \end{aligned}$$

there exists an $\alpha \in [0, 1]$, such that

$$R_1 = \frac{1}{2}\log(1 + \alpha P_1). \tag{6.110}$$

Then,

$$\begin{aligned} R_3 &\leq \frac{1}{2}\log(1 + aP_1 + P_3) - \frac{1}{2}\log(1 + a\alpha P_1) \\ &= \frac{1}{2}\log\left(1 + \frac{a(1-\alpha)P_1 + P_3}{1 + a\alpha P_1}\right). \end{aligned}$$

□

*Remark* 6.15. The outer-bound in Theorem 6.13 is an extension of Kramer's second outer-bound [46, Thoerem 2] to the discrete memoryless case. To see this, we can consider a special case of Corollary 6.14 by choosing $R_2 = 0$, such that the remaining transmitters 1 and 3, and receivers 1 and 2, form a Gaussian ZIC. The outer bound in Corollary 6.14

reduces to that consisting of only (6.104), (6.106), and (6.107) with the input distribution factorized as $p(q)p(u|q)p(x_1|uq)p(x_3|q)$. If we choose $\beta = \frac{a\alpha P_1}{P}$, where $P = aP_1 + P_3$, we can rewrite the outer bound as:

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1+\frac{\beta P}{a}), \\ R_3 &\leq \frac{1}{2}\log(1+\frac{(1-\beta)P}{1+\beta P}). \end{aligned}$$

This is exactly Kramer's second outer bound on the capacity region of a Gaussian ZIC [46, Theorem 2]. Therefore, the outer bound in Theorem 6.13 is a generalization of Kramer's outer bound to the discrete memoryless case, and an extension from the ZIC to the MAZIC.

In the following, we consider a subclass of Gaussian MAZICs with mixed interference, and we determine some boundary points of the capacity region.

**Lemma 6.16.** *For a Gaussian MAZIC satisfying conditions $a \leq 1$ and $b \geq 1 + aP_1 + P_3$, an achievable rate region is given by the set of all nonnegative rate triples $(R_1, R_2, R_3)$ that satisfy*

$$R_1 \leq \frac{1}{2}\log\left(1+P_1\right), \tag{6.111}$$

$$R_2 \leq \frac{1}{2}\log\left(1+P_2\right), \tag{6.112}$$

$$R_3 \leq \frac{1}{2}\log\left(1+\frac{P_3}{1+a\alpha P_1}\right), \tag{6.113}$$

$$R_1 + R_2 \leq \frac{1}{2}\log\left(1+P_1+P_2\right), \tag{6.114}$$

$$R_1 + R_3 \leq \frac{1}{2}\log\left(1+\alpha P_1\right) + \frac{1}{2}\log\left(1+\frac{a\bar{\alpha}P_1+P_3}{1+a\alpha P_1}\right), \tag{6.115}$$

$$R_1 + R_2 + R_3 \leq \frac{1}{2}\log\left(1+\alpha P_1+P_2\right) + \frac{1}{2}\log\left(1+\frac{a\bar{\alpha}P_1+P_3}{1+a\alpha P_1}\right), \tag{6.116}$$

*for $\alpha \in [0, 1]$.*

*Proof.* If $b \geq 1 + aP_1 + P_3$, we know that receiver 2 can decode user 2's message by treating its own signal as well as the interference from user 1 as noise. Therefore, there is no need to use rate splitting for user 2, i.e., $\beta = 0$. On applying Corollary 6.5 and removing all the redundant inequalities, we get Lemma 6.16. $\square$

*Remark* 6.17. $\frac{1}{2}\log(1+\alpha P_1+P_2) + \frac{1}{2}\log\left(1+\frac{a\bar{\alpha}P_1+P_3}{1+a\alpha P_1}\right)$ is an increasing function of $\alpha$ if $a(1+P_2) \leq 1$. Thus, the maximal achievable sum rate for the above achievable rate region is attained when $\alpha = 1$, which equals $R_s = \frac{1}{2}\log(1+P_1+P_2) + \frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1}\right)$. However, since the expression of $R_s$ is generally not a concave function of $P_1$, we can achieve a larger sum rate than $R_s$ by time sharing.

From Lemma 6.16 and Corollary 6.14, we can directly get a corner point on the capacity region.

**Corollary 6.18.** *For a Gaussian MAZIC with $a \leq 1$ and $b \geq \frac{1+aP_1+P_3}{(1+P_1)}$, the rate triple $(R_1^*, R_2^*, R_3^*)$ is on the boundary of the capacity region, where*

$$R_1^* = \frac{1}{2}\log(1+P_1), \tag{6.117}$$

$$R_2^* = \frac{1}{2}\log\left(1+\frac{P_2}{1+P_1}\right), \tag{6.118}$$

$$R_3^* = \frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1}\right). \tag{6.119}$$

It is easy to see that this boundary point is achieved by fully decoding the interference from transmitter 2 and treating the interference from transmitter 1 as noise.

*Remark* 6.19. For the general MAZIC with mixed interference, we conclude that the sum-capacity is within 1.5 times the sum-rate achieved by the Han-Kobayshi scheme. To verify it, one chooses the upper-bound to be

$$\begin{aligned} &\frac{1}{4}\left(\log(1+P_1+P_2) + \log\left(1+\frac{P_3}{1+aP_1}\right)\right. \\ +\;& \left.\min\{\log(1+bP_2+P_3), \log(1+P_2)+\log(1+P_3)\}\right), \end{aligned}$$

which comes from the sum-capacity of the MAC, 2-user ZIC with weak interference, and 2-user ZIC with strong interference, respectively.

As for the achievable sum-rate, we can apply TDM for users 1 and 2. During half of the time, let user 1 be silent, user 2 and 3 transmit at the sum-capacity of the 2-user ZIC with strong interference, while in the other half, let user 2 be silent, user 1 and 3 form a 2-user ZIC with weak interference. Consequently, the sum-rate achieved is

$$\frac{1}{4}\log(1+P_1)+\frac{1}{4}\log\left(1+\frac{P_3}{1+aP_1}\right)+\frac{1}{4}\min\{\log\left(1+bP_2+P_3\right),\log(1+P_2)+\log(1+P_3)\}.$$

## 6.5 The MAZICs with weak interferences

### 6.5.1 Discrete memoryless case

**Definition 6.20.** A discrete memoryless MAZIC is said to have *weak interferences* if the channel transition probability factorizes as

$$p(y_1y_2|x_1x_2x_3) = p(y_1|x_1x_2)p'(y_2|x_2x_3y_1), \tag{6.120}$$

$$p(y_1y_2|x_1x_2x_3) = p(y_1|x_1x_2)p''(y_2|x_1x_3y_1) \tag{6.121}$$

for some $p'(y_2|x_2x_3y_1)$ and $p''(y_2|x_1x_3y_1)$, or, equivalently, the channel is stochastically degraded.

In the absence of receiver cooperation, a stochastically degraded interference channel is equivalent in its capacity to a physically degraded interference channel. As such, we will assume in the following that the channel is physically degraded, i.e., the MAZIC admits the Markov chains $X_1-(X_2,X_3,Y_1)-Y_2$ and $X_2-(X_1,X_3,Y_1)-Y_2$. As a consequence, the following two inequalities hold

$$I(U_1;Y_2|X_2X_3) \leq I(U_1;Y_1|X_2), \tag{6.122}$$

$$I(U_2;Y_2|X_1X_3) \leq I(U_2;Y_1|X_1) \tag{6.123}$$

for all input distributions $p(x_3)p(u_1)p(x_1|u_1)p(x_2|u_1)$ and $p(x_3)p(u_2)p(x_1|u_2)p(x_2|u_2)$ respectively.

The above definition of weak interference leads to the following outer-bound.

**Theorem 6.21.** *The capacity region of a discrete memoryless MAZIC with weak interferences is outer-bounded by the region determined by the following inequalites:*

$$R_1 \leq I(X_1;Y_1|X_2U_1Q), \tag{6.124}$$

$$R_2 \leq I(X_2;Y_1|X_1U_2Q), \tag{6.125}$$

$$R_3 \leq I(X_3;Y_2|X_1X_2Q), \tag{6.126}$$

$$R_3 \leq I(X_3U_1;Y_2|X_2Q), \tag{6.127}$$

$$R_3 \leq I(X_3U_2;Y_2|X_1Q), \tag{6.128}$$

$$R_1+R_2 \leq I(X_1X_2;Y_1|Q), \tag{6.129}$$

*where the input distribution* $p(u_1u_2x_1x_2x_3) = p(u_1u_2)p(x_1|u_1u_2)p(x_2|u_1u_2)p(x_3)$.

The proof is similar to that of Theorem 4 and is hence omitted. We note that the auxiliary random variables are defined as $U_{1i} = (X_2^{i-1}, Y_1^{i-1})$ and $U_{2i} = (X_1^{i-1}, Y_1^{i-1})$.

### 6.5.2 Gaussian case

The weak interference case for the Gaussian MAZIC corresponds to the condition with $a, b \leq 1$.

First, Theorem 6.21 can be extended to the Gaussian case.

**Corollary 6.22.** *For a Gaussian MAZIC satisfying conditions $a, b \leq 1$, an outer bound to the capacity region is given by the set of all nonnegative rate triples $(R_1, R_2, R_3)$ such that*

$$\begin{aligned}
R_1 &\leq \frac{1}{2}\log(1+\alpha P_1),\\
R_2 &\leq \frac{1}{2}\log(1+\beta P_2),\\
R_3 &\leq \frac{1}{2}\log(1+P_3),\\
R_3 &\leq \frac{1}{2}\log\left(1+\frac{a(1-\alpha)P_1+P_3}{1+a\alpha P_1}\right),\\
R_3 &\leq \frac{1}{2}\log\left(1+\frac{b(1-\beta)P_2+P_3}{1+b\beta P_2}\right),\\
R_1+R_2 &\leq \frac{1}{2}\log(1+P_1+P_2).
\end{aligned}$$

The proof is very similar to that of Corollary 6.14, hence is omitted here.

For a two-user Gaussian ZIC, treating interference as noise is optimal in terms of sum-capacity for the weak interference case. One may conjecture that a similar result holds for the Gaussian MAZIC if both interferences are weak ($a, b \leq 1$). Indeed, similar sum-rate capacity result holds for the case with $0 \leq a = b \leq 1$, i.e., for the Gaussian MAZICs satisfying $0 \leq a = b \leq 1$, the sum-rate capacity is

$$C = \frac{1}{2}\log(1+P_1+P_2) + \frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1+bP_2}\right). \tag{6.130}$$

This is a direct extension of the sum-capacity result of the two-user Gaussian ZICs with weak interference by viewing $X_1$ and $X_2$ as a group. Notice that the DoF in this case is 1, implying that DoF $K/2$ is not always achievable for a $K$-user interference channel.

The above sum-capacity result is not true in general with asymmetric interference. However, it is within .5 bit of the sum capacity for $a, b \leq 1$, and bounded power constraints.

**Lemma 6.23.** *If $a \le 1, b \le 1$, and*

$$P_1 \le \bar{P}_1 \triangleq \begin{cases} \frac{\frac{2\sqrt{a}}{1+a}-\sqrt{b}}{\sqrt{b}-\sqrt{a}} & \text{if } b > a \\ \frac{\sqrt{b}}{\sqrt{a}-\sqrt{b}} & \text{if } b < a \end{cases} \tag{6.131}$$

*then the following achievable sum rate is within half a bit of the sum capacity:*

$$R_1 + R_2 + R_3 = \frac{1}{2}\log(1 + P_1 + P_2) + \frac{1}{2}\log\left(1 + \frac{P_3}{1 + aP_1 + bP_2}\right). \tag{6.132}$$

*By swapping $a$ and $b$ and indices 1 and 2, we obtain the same constant gap when $P_2 \le \bar{P}_2$.*

*Proof.*

$$\begin{aligned}
& n(R_1 + R_2 + R_3 - \epsilon) \\
\overset{(a)}{\le}\ & I\left(X_1^n X_2^n; X_1^n + X_2^n + Z_1^n\right) + I\left(X_3^n; \sqrt{a}X_1^n + \sqrt{b}X_2^n + X_3^n + Z_2^n\right) \\
\overset{(b)}{\le}\ & I\left(X_1^n; X_1^n + Z_1^n, \sqrt{a}X_1^n + N_1^n\right) + I\left(X_2^n; X_1^n + X_2^n + Z_1^n, \sqrt{a}X_1^n + \sqrt{b}X_2^n + N_2^n\right) \\
& + I\left(X_3^n; \sqrt{a}X_1^n + \sqrt{b}X_2^n + X_3^n + Z_2^n\right) \\
\overset{(c)}{=}\ & h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_1^n\right) - h\left(Z_1^n N_1^n\right) + h\left(X_1^n + X_2^n + Z_1^n \,\middle|\, \sqrt{a}X_1^n + \sqrt{b}X_2^n + N_2^n\right) \\
& - h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_2^n\right) + h\left(\sqrt{a}X_1^n + \sqrt{b}X_2^n + X_3^n + Z_2^n\right) \\
\overset{(d)}{\le}\ & h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_1^n\right) - nh\left(Z_1 N_1\right) + nh\left(X_{1G} + X_{2G} + Z_1 \,\middle|\, \sqrt{a}X_{1G} + \sqrt{b}X_{2G} + N_2\right) \\
& - h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_2^n\right) + nh\left(\sqrt{a}X_{1G} + \sqrt{b}X_{2G} + X_{3G} + Z_2\right) \\
\overset{(e)}{\le}\ & n\Big[h(X_{1G} + Z_1, \sqrt{a}X_{1G} + N_1) - h(Z_1 N_1) + h(X_{1G} + X_{2G} + Z_1 | \sqrt{a}X_{1G} + \sqrt{b}X_{2G} + N_2) \\
& - h(X_{1G} + Z_1, \sqrt{a}X_{1G} + N_2) + h(\sqrt{a}X_{1G} + \sqrt{b}X_{2G} + X_{3G} + Z_2)\Big]
\end{aligned}$$

$$\begin{aligned}
&= I\left(X_{1G}; X_{1G}+Z_1, \sqrt{a}X_{1G}+N_1\right) + I\left(X_{2G}; X_{1G}+X_{2G}+Z_1, \sqrt{a}X_{1G}+\sqrt{b}X_{2G}+N_2\right) \\
&\quad +I\left(X_{3G}; \sqrt{a}X_{1G}+\sqrt{b}X_{2G}+X_{3G}+Z_2\right) \\
&\overset{(f)}{=} I\left(X_{1G}; X_{1G}+Z_1, \sqrt{a}X_{1G}+N_1\right) + I\left(X_{2G}; X_{1G}+X_{2G}+Z_1\right) \\
&\quad +I\left(X_{3G}; \sqrt{a}X_{1G}+\sqrt{b}X_{2G}+X_{3G}+Z_2\right). \qquad (6.133)
\end{aligned}$$

where (a) is from Fano's inequality; in (b) we let

$$\begin{bmatrix} Z_1 \\ N_i \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & \rho_i \\ \rho_i & 1 \end{bmatrix}\right), \quad \rho_i^2 < 1, \quad i = 1, 2. \qquad (6.134)$$

Equality (c) follows from the fact that $N_2$ and $Z_2$ have the same marginal distribution. Inequality (d) follows by the fact that Gaussian distribution maximizes conditional entropy under a sum power constraint [47, Lemma 2], where

$$X_{iG} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{P_i}\right), \quad i = 1, 2. \qquad (6.135)$$

In (d), we consider

$$\begin{aligned}
& h\left(X_1^n+Z_1^n, \sqrt{a}X_1^n+N_1^n\right) - h\left(X_1^n+Z_1^n, \sqrt{a}X_1^n+N_2^n\right) \\
&= h\left(X_1^n+Z_1^n \middle| N_1^n-\sqrt{a}Z_1^n\right) - h\left(X_1^n+Z_1^n \middle| N_2^n-\sqrt{a}Z_1^n\right) + h\left(N_1^n-\sqrt{a}Z_1^n\right) \\
&\quad -h\left(N_2^n-\sqrt{a}Z_1^n\right) \\
&= h\left(X_1^n+U_1^n\right) - h\left(X_1^n+U_2^n\right) + h\left(N_1^n-\sqrt{a}Z_1^n\right) - h\left(N_2^n-\sqrt{a}Z_1^n\right) \qquad (6.136)
\end{aligned}$$

where $U_i^n, i = 1, 2$, is an i.i.d. Gaussian sequence with zero mean and variance

$$\mathrm{Var}\left(U_i\right) = \mathrm{Var}\left(Z_1^n \middle| N_i^n - \sqrt{a}Z_1^n\right) = 1 - \frac{\left(\rho_i - \sqrt{a}\right)^2}{1+a-2\rho_i\sqrt{a}}. \qquad (6.137)$$

If $\rho_1 = 0$, then

$$\mathrm{Var}\left(U_1\right) \leq \mathrm{Var}\left(U_2\right) \tag{6.138}$$

under condition

$$0 \leq \rho_2 \leq \frac{2\sqrt{a}}{1+a}. \tag{6.139}$$

Using (6.138) and the extremal inequality [48], expression (6.136) is maximized by $X_{1G}$:

$$\begin{aligned} & h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_1^n\right) - h\left(X_1^n + Z_1^n, \sqrt{a}X_1^n + N_2^n\right) \\ \leq \;\; & nh\left(X_{1G} + Z_1, \sqrt{a}X_{1G} + N_1\right) - nh\left(X_{1G} + Z_1, \sqrt{a}X_{1G} + N_2\right). \end{aligned} \tag{6.140}$$

(e) holds if

$$\rho_2 = \left(\sqrt{b} - \sqrt{a}\right) P_1 + \sqrt{b} \tag{6.141}$$

which implies that $X_{2G} \to X_{1G} + X_{2G} + Z_1 \to \sqrt{a}X_{1G} + \sqrt{b}X_{2G} + N_2$ form a Markov chain [47, Lemma 5]. Combining (6.139) and (6.141), we obtain (6.131). The gap between the upper bound (6.133) and lower bound (6.132) is

$$I\left(X_{1G}; \sqrt{a}X_{1G} + N_1 \,|X_{1G} + Z_1\right) = \frac{1}{2}\log\left(1 + \frac{aP_1}{1+P_1}\right) < \frac{1}{2}\log(1+a) \leq \frac{1}{2}. \tag{6.142}$$

□

Next, We present the following theorem that gives a sum-rate upper-bound.

**Theorem 6.24.** *Any achievable rate triplet* $(R_1, R_2, R_3)$ *for the Gaussian MAZIC with* $0 \leq a \leq b \leq 1$ *must satisfy the following constraint*

$$\begin{aligned}
& n(R_1+R_2+R_3) \\
\leq\ & \min_{a\leq\sigma^2\leq 1}\left\{\frac{n}{2}\log\left((P_1+P_2+1)(aP_1+bP_2+\sigma^2)-(\sqrt{a}P_1+\sqrt{b}P_2+\sqrt{a})^2\right)\right. \\
& \left.-\frac{n}{2}\log(aP_1+bP_2+1)-\frac{n}{2}\log(\sigma^2-a)+\frac{n}{2}\log(aP_1+bP_2+P_3+1)\right\}.
\end{aligned}$$

*Proof.*

$$\begin{aligned}
& n(R_1+R_2+R_3)-n\epsilon \\
\overset{(a)}{\leq}\ & I(X_1^nX_2^n;Y_1^n)+I(X_3^n;Y_2^n) \\
=\ & I(X_1^n;X_1^n+Z_1^n)+I(X_2^n;X_1^n+X_2^n+Z_1^n)+I(X_3^n;\sqrt{a}X_1^n+\sqrt{b}X_2^n+X_3^n+Z_2^n) \\
\overset{(b)}{\leq}\ & I(X_1^n;X_1^n+Z_1^n)+I(X_2^n;X_1^n+X_2^n+Z_1^n,\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n) \\
& +I(X_3^n;\sqrt{a}X_1^n+\sqrt{b}X_2^n+X_3^n+Z_2^n) \\
=\ & h(X_1^n+Z_1^n)-h(Z_1^n)+h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n) \\
& +h(X_1^n+X_2^n+Z_1^n|\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n)-h(\sqrt{a}X_1^n+N_1^n)-h(X_1^n+Z_1^n|\sqrt{a}X_1^n+N_1^n) \\
& +h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+X_3^n+Z_2^n)-h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+Z_2^n) \\
=\ & h(X_1^n+Z_1^n)-h(\sqrt{a}X_1^n+N_1^n)+h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n)-h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+Z_2^n) \\
& -h(Z_1^n)+h(X_1^n+X_2^n+Z_1^n|\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n) \\
& +h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+X_3^n+Z_2^n)-h(Z_1^n-\frac{1}{\sqrt{a}}N_1^n|\sqrt{a}X_1^n+N_1^n) \\
=\ & h(X_1^n+Z_1^n)-h(\sqrt{a}X_1^n+N_1^n|Z_1^n-\frac{1}{\sqrt{a}}N_1^n)+h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n) \\
& -h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+Z_2^n)-h(Z_1^n)+h(X_1^n+X_2^n+Z_1^n|\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n) \\
& +h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+X_3^n+Z_2^n)-h(Z_1^n-\frac{1}{\sqrt{a}}N_1^n)
\end{aligned}$$

$$\stackrel{(c)}{\leq} \frac{n}{2}\log\Big((P_1+P_2+1)(aP_1+bP_2+\sigma^2)-(\sqrt{a}P_1+\sqrt{b}P_2+\sqrt{a})^2\Big)$$
$$-\frac{n}{2}\log(aP_1+bP_2+1)-\frac{n}{2}\log(\sigma^2-a)+\frac{n}{2}\log(aP_1+bP_2+P_3+1)$$

where $(a)$ is from Fano's inequality; $(b)$ is by giving side information $\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n$ to the second mutual information where $N_1^n$ is an i.i.d. Gaussian random variables whose covariance matrix with $Z_1$ is

$$Cov\begin{bmatrix} Z_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{bmatrix};$$

$(c)$ is the result of applying the extremal inequality [48] to the first two terms, and to the third and forth terms respectively. for the first two terms,

$$h(X_1^n+Z_1^n)-h(\sqrt{a}X_1^n+N_1^n|Z_1^n-\frac{1}{\sqrt{a}}N_1^n)\leq\frac{n}{2}\log(1+P_1)-\frac{n}{2}\log(aP_1+a)=-\frac{n}{2}\log a,$$

since the use of the extremal inequality requires $Var(N_1|Z_1-\frac{1}{\sqrt{a}}N_1)\geq a\Rightarrow\rho\sigma=\sqrt{a}$. For the third and fourth terms,

$$h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n)-h(\sqrt{a}X_1^n+\sqrt{b}X_2^n+Z_2^n)$$
$$\leq \frac{n}{2}\log(aP_1+bP_2+\sigma^2)-\frac{n}{2}\log(aP_1+bP_1+1)$$

as the use of the extremal inequality requires $\sigma^2\leq 1$.

For the conditional entropy $h(X_1^n+X_2^n+Z_1^n|\sqrt{a}X_1^n+\sqrt{b}X_2^n+N_1^n)$, identically and independently distributed (i.i.d) zero-mean Gaussian $X_1^n$ and $X_2^n$ are the maximizing distributions [49]. □

**Corollary 6.25.** *For the Gaussian MAZICs satisfying $0 \leq a \leq b \leq 1$, if the power constraints satisfy*

$$\begin{aligned} P_1 &= \frac{1-\sqrt{ab}}{\sqrt{ab}-a}, \\ P_3 &\geq b-1+(b-a)P_1 = \sqrt{\frac{b}{a}}-\sqrt{ab}, \end{aligned}$$

*the sum-rate capacity is*

$$C = \frac{1}{2}\log(1+P_1) + \frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right). \tag{6.143}$$

*Proof.* For the achievability part, let receiver 1 decode messages from users 1 and 2, and receiver 2 decode messages from users 2 and 3, we have the following achievable rate triplets $(R_1, R_2, R_3)$:

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1+P_1), & (6.144)\\ R_2 &\leq \frac{1}{2}\log\left(1+\frac{bP_2}{1+aP_1}\right), & (6.145)\\ R_3 &\leq \frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1}\right), & (6.146)\\ R_1+R_2 &\leq \frac{1}{2}\log(1+P_1+P_2), & (6.147)\\ R_2+R_3 &\leq \frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right). & (6.148) \end{aligned}$$

Apply Fourier-Motzkin elimination with respect to $S = R_1+R_2+R_3$, the resulting achievable sum-rate is

$$\begin{aligned} & R_1+R_2+R_3 \\ \leq\ & \min\left\{\frac{1}{2}\log(1+P_1)+\frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right), \frac{1}{2}\log(1+P_1+P_2)+\frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1}\right)\right\}, \end{aligned}$$

if $(b-a)P_1 \le 1-b+P_3$,

$$\frac{1}{2}\log(1+P_1+P_2)+\frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1}\right) \ge \frac{1}{2}\log(1+P_1)+\frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right).$$

hence, $\frac{1}{2}\log(1+P_1)+\frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right)$ is an achievable sum-rate, and is achieved by user 1 decoding $X_2$ first, subtracting it off, and then decoding $X_1$; and user 2 decoding $X_2$ and $X_3$ simultaneously by treating $X_1$ as noise.

For the converse part, at the last step of the proof of Theorem 6.24, if we further let the Gaussian variables $X_2^n - (\sqrt{a}X_1^n + \sqrt{b}X_2^n + N_1^n) - (X_1^n + X_2^n + Z_1^n)$ form a Markov chain, then

$$P_1 = \frac{\sqrt{ab}-\sigma^2}{a-\sqrt{ab}}. \tag{6.149}$$

The sum-rate upper-bound becomes

$$\frac{1}{2}\log(1+P_1)+\frac{1}{2}\log\left(1+\frac{bP_2}{aP_1+\sigma^2}\right)+\frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1+bP_2}\right).$$

Let $\sigma^2=1$, (6.149) becomes $P_1=\frac{1-\sqrt{ab}}{\sqrt{ab}-a}$, naturally, this requires $a\le b$, and $\sqrt{ab}\le 1$ such that (6.149) is non-negative. This is because $a>b$ is infeasible as it implies $\sqrt{ab}\le a$, i.e., (6.149) is negative when $\sigma^2=1$. □

It is perhaps not intuitive that the sum-rate (6.143) is optimal only if $P_1=\frac{1-\sqrt{ab}}{\sqrt{ab}-a}$. Specifically, given that this sum-rate capacity is achieved when the interference from $X_1$ is treated as noise at $Y_2$, it might be expected that with smaller $P_1$, the same scheme should also be optimal. We show that this is not true.

First, for $a\le 1$,

$$\frac{1-b}{b-a} \le \frac{1-\sqrt{ab}}{\sqrt{ab}-a}.$$

But for $P_1 \leq \frac{1-b}{b-a}$, the achievable sum-rate

$$\frac{1}{2}\log(1+P_1+P_2)+\frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1+bP_2}\right) \tag{6.150}$$

is greater than the sum-rate (6.143).

Now consider any $P_1$ with $\frac{1-b}{b-a} \leq P_1 \leq \frac{1-\sqrt{ab}}{\sqrt{ab}-a}$. The following function is an achievable sum-rate for $P_1 \leq \frac{1-\sqrt{ab}}{\sqrt{ab}-a}$. However, it is easy to show that $f$ is not concave in $P_1$ around the point $\frac{1-b}{b-a}$. Therefore, sum-rates strictly larger than (6.143) can be achieved for $\frac{1-b}{b-a} \leq P_1 \leq \frac{1-\sqrt{ab}}{\sqrt{ab}-a}$ using time-sharing.

$$f(P_1)=\begin{cases} \frac{1}{2}\log(1+P_1+P_2)+\frac{1}{2}\log\left(1+\frac{P_3}{1+aP_1+bP_2}\right), & \text{if } P_1 \leq \frac{1-b}{b-a}, \\ \frac{1}{2}\log(1+P_1)+\frac{1}{2}\log\left(1+\frac{bP_2+P_3}{1+aP_1}\right), & \text{if } \frac{1-b}{b-a} \leq P_1 \leq \frac{1-\sqrt{ab}}{\sqrt{ab}-a}. \end{cases}$$

Next, let us consider an even simpler case, where one of the cross link gain vanishes, for

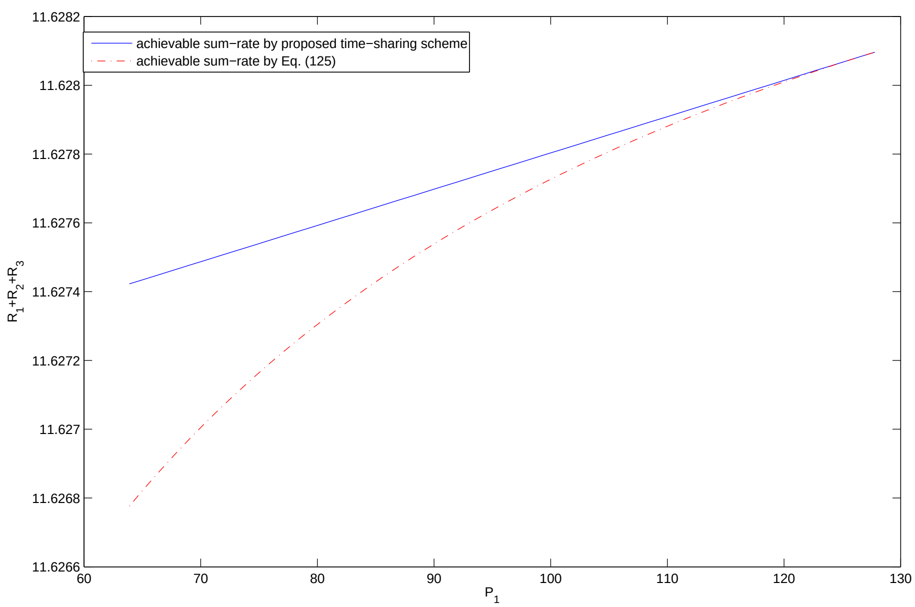



FIGURE 6.4: The Comparison of the sum-rates achieved by proposed time-sharing scheme and Eq. (6.143) when $\frac{1-b}{b-a} \leq P_1 \leq \frac{1-\sqrt{ab}}{\sqrt{ab}-a}$.

example, $a = 0$. With only one weak interference link, we are able to obtain a boundary curve of the capacity region.

**Theorem 6.26.** *For a Gaussian MAZIC with $a = 0$ and $\frac{1+P_3}{1+P_1} \le b \le 1$ ($P_3 \le P_1$), then the following rate triple is always on the boundary of the capacity region:*

$$\left(\frac{1}{2}\log\left(1+\frac{P_1}{1+\bar{\beta}P_2}\right), \frac{1}{2}\log(1+\bar{\beta}P_2)+\frac{1}{2}\log\left(1+\frac{\beta P_2}{1+P_1+\bar{\beta}P_2}\right), \frac{1}{2}\log(1+P_3)\right), \tag{6.151}$$

*where $\beta \in [0,1]$ and satisfy*

$$\frac{1}{2}\log(1+\bar{\beta}P_2)+\frac{1}{2}\log\left(1+\frac{\beta P_2}{1+P_1+\bar{\beta}P_2}\right) \le \frac{1}{2}\log\left(1+\frac{bP_2}{1+P_3}\right). \tag{6.152}$$

*Proof.* By setting $\alpha = 1$, the general achievable rate region in Corollary 6.5 reduces to

$$\begin{aligned} R_1 &\le \frac{1}{2}\log(1+P_1), \\ R_2 &\le \frac{1}{2}\log(1+P_2), \\ R_3 &\le \frac{1}{2}\log\left(1+\frac{P_3}{1+b\beta P_2}\right), \\ R_1+R_2 &\le \frac{1}{2}\log(1+P_1+P_2), \\ R_2+R_3 &\le \frac{1}{2}\log(1+\beta P_2)+\frac{1}{2}\log\left(1+\frac{b\bar{\beta}P_2+P_3}{1+b\beta P_2}\right), \\ R_1+R_2+R_3 &\le \frac{1}{2}\log(1+P_1+\beta P_2)+\frac{1}{2}\log\left(1+\frac{b\bar{\beta}P_2+P_3}{1+b\beta P_2}\right). \end{aligned}$$

If let $R_3 = \frac{1}{2}\log(1+P_3)$, the achievable rate region reduces to

$$R_1 \le \frac{1}{2}\log(1+P_1), \tag{6.153}$$

$$R_2 \le \frac{1}{2}\log\left(1+\frac{bP_2}{1+P_3}\right), \tag{6.154}$$

$$R_1+R_2 \le \frac{1}{2}\log(1+P_1+P_2). \tag{6.155}$$

If $b \geq \frac{1+P_3}{1+P_1}$ ($P_3 \leq P_1$), inequality (6.155) is always active. Therefore, the rate triple (6.151) is always achievable.

For the converse, (6.155) is a natural upper-bound for $R_1 + R_2$. □

*Remark* 6.27. In the general MAZIC with weak interference, we conclude that the sum-capacity is within 1.5 times the sum-rate achieved by Han-Kobayashi scheme as well. To show this, we choose the upper-bound to be

$$\begin{aligned} & \frac{1}{4}\log(1+P_1+P_2) + \frac{1}{4}\log(1+P_1) + \frac{1}{4}\log\left(1+\frac{P_3}{1+aP_1}\right) \\ + \quad & \frac{1}{4}\log(1+P_2) + \frac{1}{4}\log\left(1+\frac{P_3}{1+bP_2}\right). \end{aligned}$$

This upper bound follows directly from the sum-capacity results of 2-user ZIC with weak interference and MAC. For the lower-bound, choose

$$\frac{1}{4}\log(1+P_1) + \frac{1}{4}\log\left(1+\frac{P_3}{1+aP_1}\right) + \frac{1}{4}\log(1+P_2) + \frac{1}{4}\log\left(1+\frac{P_3}{1+bP_2}\right),$$

which is achieved using a simple TDM scheme: split the time in half and let transmitters 1 and 3 transmit in one half while 2 and 3 in the other half. The above sum rate follows directly from the sum capacity of the ZIC with weak interference.

## 6.6 Summary

In this chapter we have studied the capacity of an uplink network with co-channel interference. By modeling such networks using a multiple access interference channel with one-sided interference, we have obtained an inner bound to the capacity region for both the discrete memoryless case and the Gaussian case. The capacity region for the discrete memoryless channel model with strong and very strong interference has been established; for the Gaussian MAZIC, we have determined the capacity region for the very strong interference case, and for the case in which one interference link is strong and the other one is very strong; for the strong interference case, we have obtained a boundary line segment of the capacity region. For the mixed interference case, a boundary point of the capacity region has been obtained. For the weak interference case, we have obtained the sum-rate capacity for the symmetric channel coefficients whose result is analogous to that of the two user Gaussian one-sided interference channel. For the general case, a sum-rate upper bound has been obtained which gives rise to a sum-rate capacity result under certain power constraint conditions. Furthermore, it does not change the capacity results if we allow more users intended for receiver 2 without interfering receiver 1. In this case, $R_3$ is replaced by the sum-rate of all those added users.

# Chapter 7

# Diversity Combining of Non-coherently Modulated LDPC Codes in Wireless Communications

The previous chapters are focused on the information theoretic limits on various channel models with interference. However, in the real world, exploiting the interference structure and partially decoding it require strong coordination of users that are often impossible to accommodate. For example, the devices in a network may not have the same physical layer protocol even though they share the same spectrum.

This is especially true in networks where spectrum pre-planning is not possible and users share the frequency band in a rather independent manner. This is what motivated DARPA to hold a spectrum challenge from January 2013 through March 2014 [2].

## 7.1 DARPA spectrum challenge and software radio implementation

The DARPA spectrum challenge (DSC) consists of two separate tournaments: competitive and cooperative tournaments. Both need to transmit a fixed amount of data (173 Mb) at a given frequency band (5 MHz) and time duration (180 seconds). In the competitive tournament, the goal is to complete transmission ahead of the other team, selfish behavior is thus encouraged. The cooperative tournament, however, is one that involves teamwork among different groups, in which the ultimate goal is the collective throughput as a team, instead of individual throughput.

In the real implementation, the individuals or teams form a two-user and 3-user interference channel physically, corresponding to different tournaments. However, one important takeout message is that implementing interference decoding/subtraction is infeasible at a reasonable cost, because of the lack of coordination. As such, one is left only with interference avoidance or co-existence with interference (i.e., treating interference as noise).

Given the clear goals of two tournaments, we set out our basic strategies as follows:

- Channel codes are definitely needed, especially for the competitive tournament, where an interference intense environment is expected. We chose the low density parity check (LDPC) codes mainly for the competitive mode, facilitated by the BCH code as an outer code. With a customizable coding rates, the concatenation of the two codes has very strong decoding capability that operates reliably with SNR at or below 0 dB. As for the cooperative mode, interference avoidance is a better strategy to help teammates also score. In this case, coding speed is much more critical than capability, as a cleaner environment is expected with spectrum sensing kicking in. For this purpose, we chose Reed-Solomon (RS) code.

- The sliding window feedback that is used in TCP/IP is adapted for wireless application that takes into account the possibility of lost feedback.
- In the cooperative mode, in order to better avoid teammates, in addition to leave time holes, frequency division multiplexing (FDM) is employed to free some frequency space for others.

In the absence of interference cancellation, the challenges are unique when the goal is to maximize your capability to correctly decode the information based on the received signals. In the following, we address one practical design problem regarding improvement of the information reception when sever interference is present in the wireless network.

## 7.2 Current state-of-the-art techniques in the practical system design

In recent communication systems, multiple independently received signal copies of the same message are often available at the receiver. This includes retransmission in packet based systems, multi-channel environment, and multi-antenna receivers that have become prevalent in almost all current and future wireless systems. For example, in the DSC, we implemented retransmission schemes based on the sliding window mechanism. As a result, multiple copies of the same packet may be received. Sometimes, the interference is so disastrous that there is no single packet among those multiple copies can be decoded correctly. One natural question coming out of this is that, can one do some kind of combining of multiple copies? Instead of making use of the received copies independently, can one achieve a better performance by combining the information from independent copies in an optimal manner?

The focus of this chapter is on diversity combining schemes for communication systems with non-coherent modulation while employing LDPC codes. For coherent modulation, it is well known that maximal ratio combining (MRC) is the optimal linear combining technique. Other practical alternatives for coherent systems include equal gain combining and selection combining which strikes a balance between performance and channel knowledge requirement. However, with a highly dynamic transmission may put exacting demand in tracking channel state information. As such non-coherent modulations, e.g., differentially encoded signals, are often used instead. For systems with non-coherent modulation, diversity combining is often limited to hard decision combining (e.g., majority rule) which limits its applications to certain number of diversity branches and is also inferior in performance to that of soft decision combining used in coherent modulated systems.

We show however, for non-coherently modulated systems employing LDPC codes, there is a natural way to implement soft decision diversity combining that is embedded in the decoding process. This soft decision diversity combining is shown to have much improved performance over hard decision combining that requires independent LDPC decoding prior to employing majority rule diversity combining.

### 7.2.1 Non-coherent modulation

Phase shift keying (PSK) and quadrature amplitude modulation (QAM) are most widely used modulation schemes in digital communications. For both modulations, the phase of the signal carries the information that needs to be recovered at the receiver. Coherent modulation uses the absolute phase of the signal to represent the information whereas non-coherently modulated schemes, e.g., those implementing differential encoding, embeds the information in the phase difference between consecutive symbols. As such, for coherent modulation, one needs to keep track of the channel state, especially the channel phase information whereas for differentially encoded signals, there is no such need so long as the channel coefficients do not vary much from time to time.

Tracking channel state information puts additional burden on the communication system, which can become quite stringent with fast fading channels. Thus in many practical systems such as satellite and radio relay communications as well as in some cellular systems, non-coherent modulations are widely used [50].

### 7.2.2 LDPC codes

LDPC codes are a class of linear block codes with a particular characteristic in terms of their parity-check matrix. Specifically, the fraction of nonzero entries is small. LDPC codes provide a performance close to the Shannon limit for a number of important channels. In other words, one can not expect to have codes that perform better than LDPC in terms of transmission rate and reliability trade-off. Furthermore, the decoding algorithms have linear time complex. These advantages, i.e., the superior error correction performance and simplicity in implementation makes it the most widely used error correction codes in existing and future wireless communications systems, including the digital television broadcast standard (DVB-S2) [51], ITU-T G.hn standard [52]. LDPC is also used for 10GBase-T Ethernet, which sends data at 10 gigabits per second over twisted-pair cables. As of 2009, LDPC codes are also part of the Wi-Fi 802.11 standard as an optional part of 802.11n [53] and 802.11ac, in the High Throughput (HT) PHY specification.

The decoding algorithms for LDPC codes can be classified into two main categories: hard-decision decoding and soft-decision decoding. The difference between the two lies in the inputs that are taken in by the algorithm. For the hard-decision decoding, the inputs are decoded symbols from the demodulator, while likelihood ratio values are inputs for the soft-decision algorithms. Soft-decision decoding, based on the concept of belief propagation, yields a better decoding performance and is therefore the preferred method where possible.

## 7.3 System Model and Diversity Combining Techniques

The RF signal received over the air goes through the typical RF circuit chain and reduces to the corresponding baseband signal, denoted as a sequence $\mathbf{y} = \{y_0, y_1, \cdots, y_K\}$ that can be written as

$$y_k = h_k x_k + n_k, \tag{7.1}$$

where $k = 1, 2, \cdots$ is the symbol index, $h_k$ is the channel coefficient, and $n_k$ is a complex additive white Gaussian noise (AWGN) with zero mean and variance $2\sigma^2$. The channel coefficient $h_k$ can be written as $h_k = \rho_k e^{j\phi_k}$. For soft decision based demodulator, the outputs are the log likelihood ratios (LLR). For example, in binary modulation schemes, the LLR is expressed as the following.

$$I_k = \log \frac{p(y_k, y_{k-1} | m_k = 0)}{p(y_k, y_{k-1} | m_k = 1)},$$

where $m_k$ denotes the coded bit before differential modulation.

Suppose there are $L$ independent branches available, and let $LLR(l)$ $(1 \leq l \leq L)$ denote each diversity reception after non-coherent demodulator. Various receiver architectures can be developed for taking advantage of such multiple receptions.

A simple and direct approach is to employ a distinct LDPC decoder for each independent branch. The transmitted information bits are estimated by having the decoders take the majority vote. It is refered as hard-decision combining (HDC) hereafter. The procedure is shown in Fig. 7.1. However, the combined codeword after the vote may not be legitimate. Furthermore, it becomes ambiguous for some of the decisions when the number of branches goes even. More importantly, having multiple LDPC decoders takes up tremendous computation resourses.

To overcome the computational complexity, it is realistic to combine the receiving signals before the LDPC decoder, as depicted in Fig. 7.2. Therefore, only one decoder is needed despite of the number of branches. But in this way, the problem becomes how to combine the multiple symbol decisions to feed the only decoder. One reasonable solution is to select the best LLR out of multiple copies. In other words, the LLR with the largest magnitude is selected for each bit, i.e.,

$$SC = LLR(l^*), l^* = \arg\max_{1 \leq l \leq L} |LLR(l)|. \tag{7.2}$$

We refer this methodology as selection combining (SC). As switching among the branches for the one with the highest confidence, the SC method is expected to have better performance than a single decoding without combining. However, if the channel conditions in each branch are alike, the LLRs are not of much difference. Therefore, the performance improvement is limited, especially in highly noisy channel conditions.

In the following , we propose the likelihood ratio combining (LRC) technique for taking advantage of all received copies via diverse channels. The LLRs for each bit are added with equal gain, that is

$$LLR = \sum_{l=1}^{L} LLR(l). \tag{7.3}$$

The operation of addition is considered because of the independence of each received version of the signals. By adding LLRs together, the estimation of each bit from each branch is weighted by their own confidence of their decisions. Hence, better performance is obtained by integrating all the available informantion together. In the following section, we compare the performance of all the techniques mentioned above.

## 7.4 Performance Comparison

We use DVB standard LDPC codes with code rate 1/4. The message length and codeword length are 16200 and 64800 bits. As for the modulation, we adopt BPSK with differential encoding and non-coherent detection. The ways to calculate the LLR are from [54, Section

III.B]. We simulated with 2 types of channels, Gassian channels with unknown fixed channel coefficient (both magnitude and phase), and Rayleigh fading channels with randomized channel coefficient. Moreover, 3 branches of received signals are assumed, in which the channel conditions are alike. The performance of different combining techniques are compared in Fig. 7.3 and Fig. 7.4.

The BER (bit error rate) curve ends when the decodes message bits are free of errors. From the simulation results, the LRC apparently has the best performance, and 3 $dB$ effective SNR gain is obtained in Gaussian channels compared to the reception without combining as well as that with HDC. SC has reasonable improvements over no combining and HDC, but is not as good as LRC. Similar performance is obtained in Rayleigh fading channels.

## 7.5 Summary

Performance of LDPC codes with non-coherent combining has been studied. Simulation results have shown that the simple addition of all LLRs from available independent copies provides the best SNR gain over other combining techniques, such as hard-decision combining and selection combining. The described LRC is also robust in Rayleigh fading channels.

Figure 7.1: Communication system diagram with hard decision combining

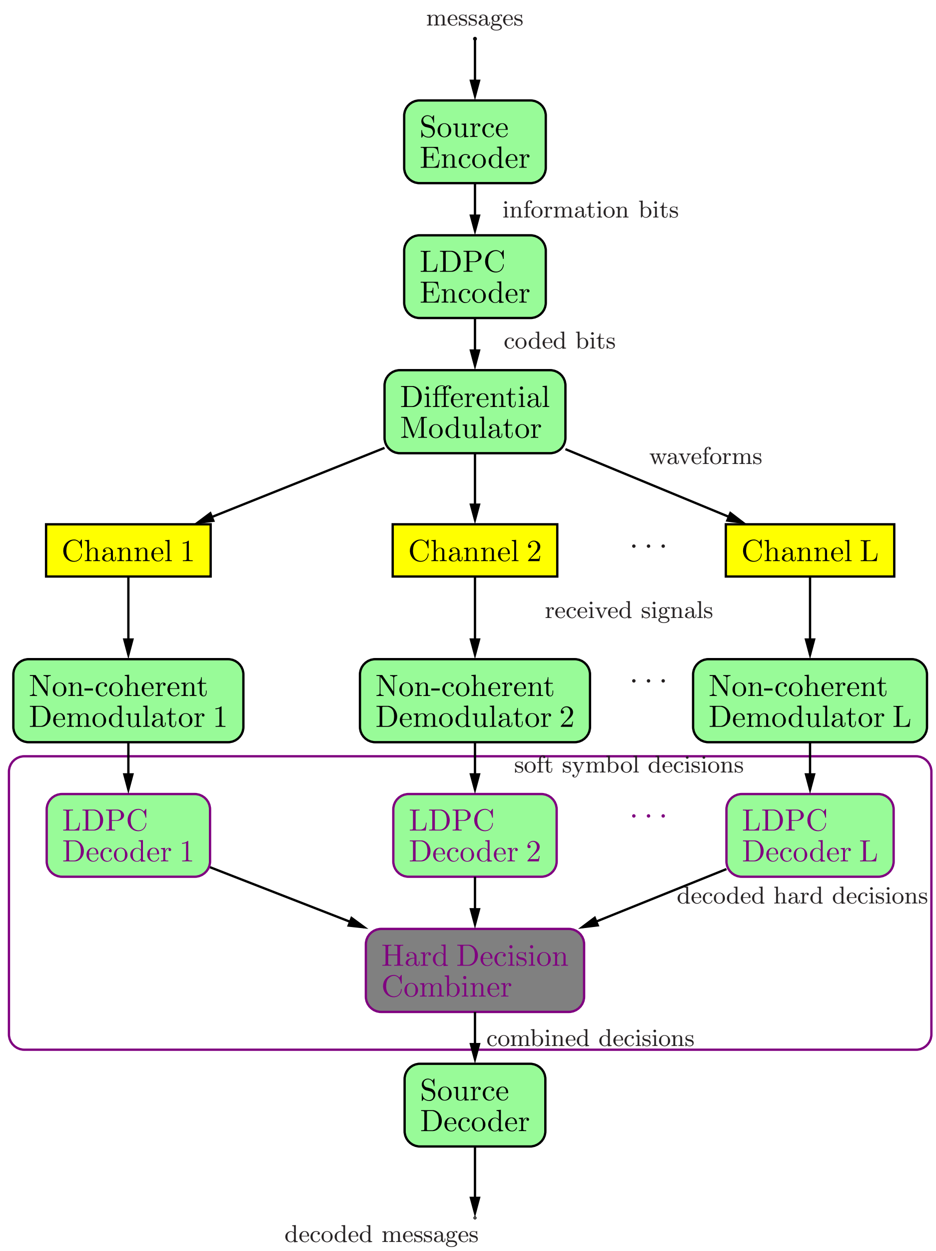


FIGURE 7.2: Communication system diagram with diversity combining before the LDPC decoder.

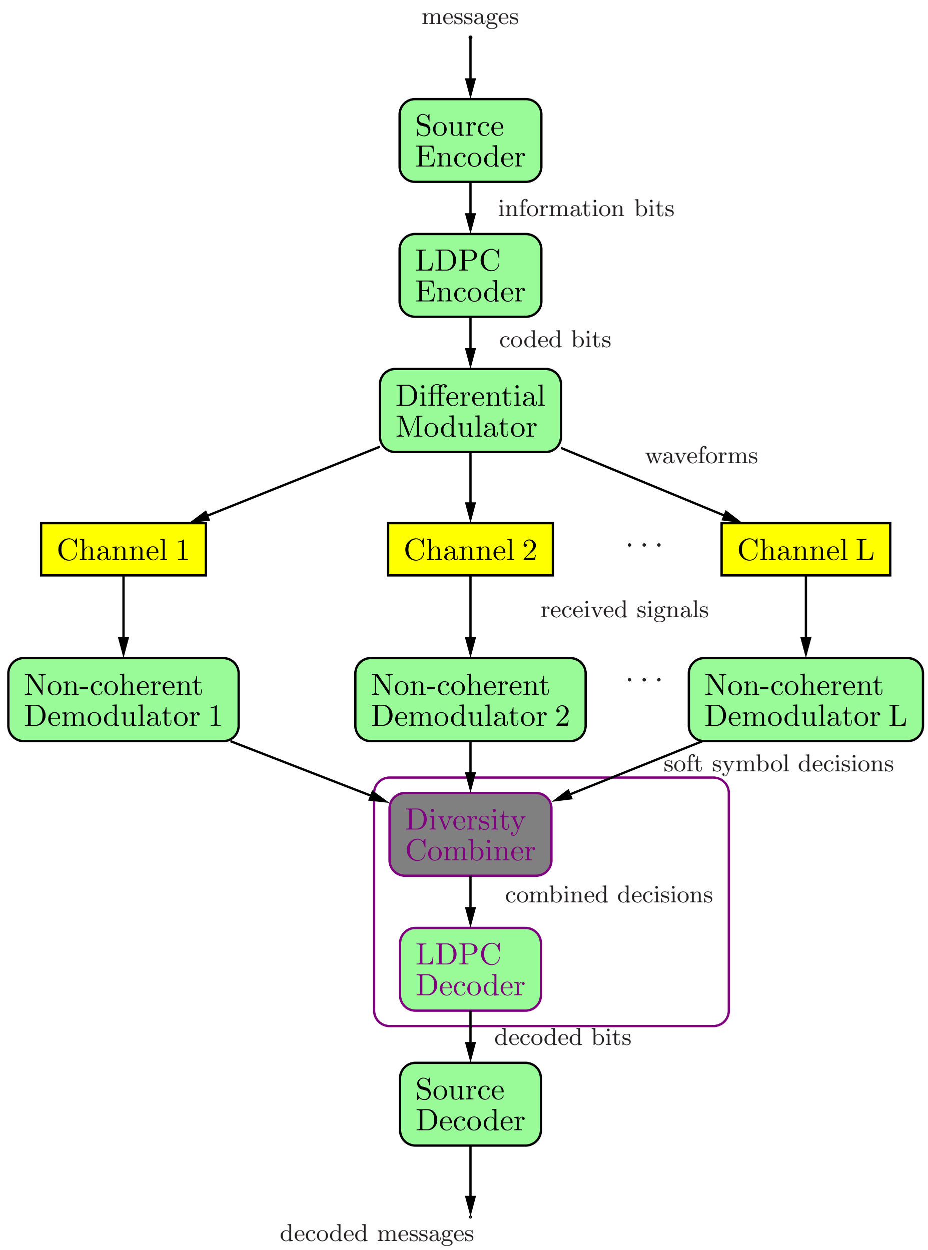


FIGURE 7.3: BER comparison for combining over 3 Gaussian channels

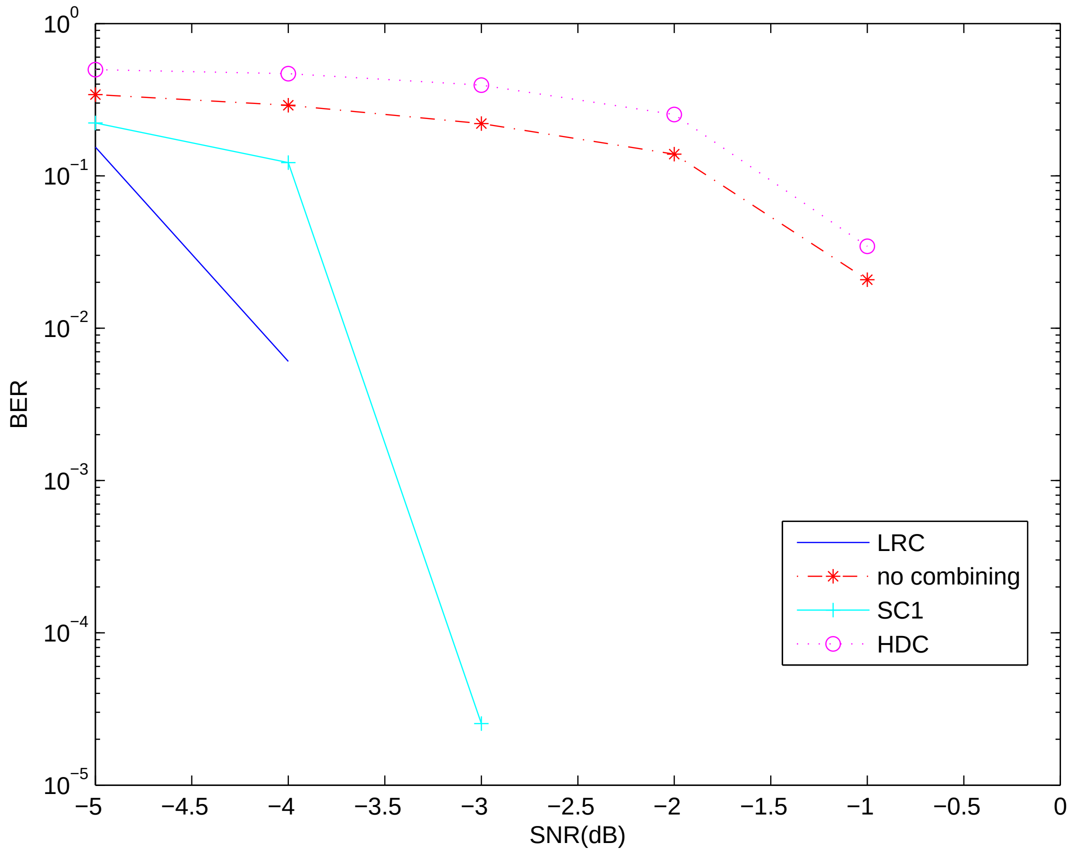
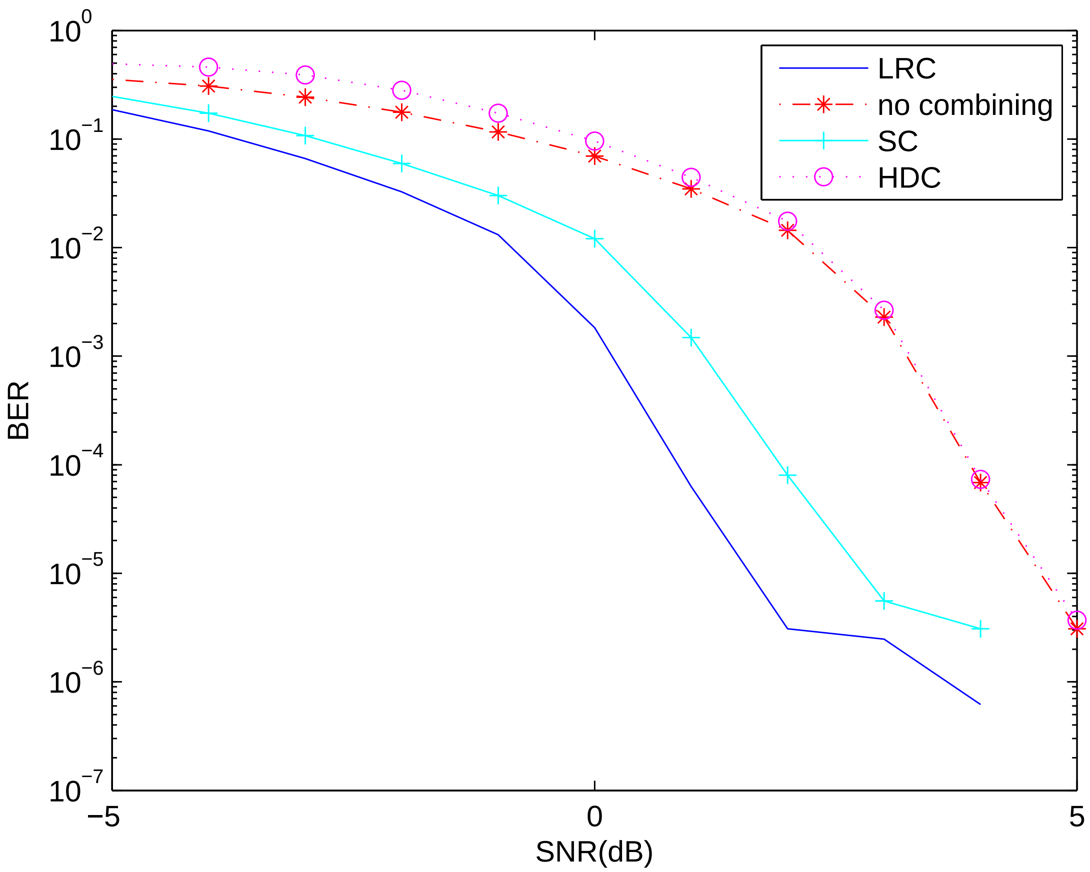


FIGURE 7.4: BER comparison for combining over 3 Rayleigh fading channels.



# Chapter 8

# Conclusions and Future Research Directions

## 8.1 Conclusions

This dissertation studies the effect of interference in communication networks. The primary focus is on the theoretical performance limits for various channel models; yet we also touch upon implementation issues where system design needs to have robust performance in the presence of unknown interference.

First, for a simple two-user Gaussian interference channel, we establish that within the computable subregion of the Han-Kobayashi achievable rate region, frequency division multiplexing (FDM) suffices to achieve the best sum-rate.

Next, motivated by recent progresses of the exact sum-rate capacity characterizations of two-user Gaussian interference channels, the two-user discrete memoryless interference channel model is considered. The condition of the interference link being weak is extended to the discrete memoryless case, which is characterized by a stochastic degradedness. The channel property resembles that for the Gaussian z-interference channel with weak interference. Under this proposed weak interference definition, the sum-rate capacity is characterized for the discrete memoryless interference channel with one-sided weak interference. Subsequently, an outer-bound of the capacity region is also derived for this class of channels. The results lead to the exact single-letter characterization of the capacity region of a subclass within the discrete memoryless z-interference channel with weak interference.

The same technique is then applied to obtain the sum-rate capacity of discrete memoryless interference channels with mixed interference. Similarly, the capacity region of some new discrete memoryless interference channels are also characterized.

For both of these cases, the capacity expressions as well as the encoding schemes that achieve the sum-rate capacity are analogous to the Gaussian interference channel counterpart.

We then consider channel models that are directly motivated by current cellular systems. In particular, a 3-user up-link model with co-channel interference is considered. By studying such a model, we aim to get insight on how to manage interference in an optimal manner for multi-user communication systems. In this model, analyses of both the discrete memoryless case and the Gaussian case are obtained. We have established an inner bound to the capacity region for both cases. Capacity upper-bounds are also provided, in order to attain the exact capacity results, and capacity achieving schemes. In the process of examining different cases according to the interference strength over two cross links, we have the following observations:

- It is harder to characterize the conditions for interference being weak or strong for the discrete memoryless channels when the number of the involved users becomes larger. The difficulty is due to the characterizations of the combined effects of different interference sources. This is not pronounced in the two-user interference channel case, in which only one interference source is present at each receiver. One particular example provided in Chapter 6 is that in the Gaussian case, $a, b \geq 1$ does not necessarily yield the general optimality of joint decoding with interfering messages.

- The question of whether treating interference as noise is optimal for systems with more than two users is also more complicated. In two-user Gaussian interference channels, the noisy or very weak interference regime is where the simple scheme of treating interference as noise is optimal in achieving the sum-rate capacity. Although similar genie-aided techniques are developed for our uplink model, treating interference as noise no longer achieves the sum-rate capacity. Instead, the sum-rate capacity is attained by partially joint decoding with one of the interference messages.

These findings highlight the difficulties to manage the interference for complex communication networks.

Apart from above theoretical analysis, a simple diversity combining using soft-decision for non-coherently modulated and LDPC coded signals was described in Chapter 7. The scheme, while intuitive and simple, exhibits a much more robust performance in the presence of interference compared with the typical hard decision combining for non-coherently modulated signals.

## 8.2 Potential research topics

In the following, we describe some interesting ideas that worth pursuing in the future.

### 8.2.1 Beyond the weak and strong interference for the DMZICs

In the standard two-user Gaussian interference channel model, it is clear that the interference has a clear boundary point that separates strong and weak interference. In the case of

Gaussian interference channel in its standard form, the point corresponds to the interference link gain of 1. In particular, for the one-sided interference case, when the only interference link is weak, the sum-rate capacity is achieved by that the receiver subject to interference treats interference as noise, and the other transceiver pair communicates at rate in the point-to-point channel. For the strong interference case, the capacity region is achieved by joint decoding interference with the intended message.

However, for the discrete memoryless channel model, the weak and strong interference cases do not cover all the scenarios. To illustrate this point, we provide the following example.

**Example 8.1.** *All the input and output alphabets are binary. Let $f_{ij}$ represent $p(y_1 = 1|x_1 = i, x_2 = j)$, $g_j$ represent $p(y_2 = 1|x_2 = j)$, $p_i = \Pr\{X_i = 1\}$, and $\bar{p}_i = 1 - p_i$ $(i, j \in \{0, 1\})$.*

- *For the interference being weak, one needs to satisfy the Markov chain (4.3), i.e.,*

$$p(y_1|x_1x_2y_2) = p(y_1|x_1y_2).$$

*Then we would have,*

$$p(y_1|x_1x_2) = \sum_{y_2} p(y_1y_2|x_1x_2) = \sum_{y_2} p(y_2|x_2)p(y_1|x_1y_2).$$

*To be more concrete, we have*

$$\begin{aligned} f_{00} &= g_0h_{01} + (1 - g_0)h_{00}, \\ f_{01} &= g_1h_{01} + (1 - g_1)h_{00}, \\ f_{10} &= g_0h_{11} + (1 - g_0)h_{10}, \\ f_{11} &= g_1h_{11} + (1 - g_1)h_{11}, \end{aligned}$$

*with* $h_{ij}$ *representing* $p(y_1 = 1|x_1 = i, y_2 = j)$, $i, j \in \{0, 1\}$. *Solving the above equations about* $h_{ij}$, *we obtain*

$$\begin{aligned} h_{00} &= \frac{g_1 f_{00} - g_0 f_{01}}{g_1 - g_0}, \\ h_{01} &= \frac{(1-g_0)f_{01} - (1-g_1)f_{00}}{g_1 - g_0}, \\ h_{10} &= \frac{g_1 f_{10} - g_0 f_{11}}{g_1 - g_0}, \\ h_{11} &= \frac{(1-g_0)f_{11} - (1-g_0)f_{10}}{g_1 - g_0}. \end{aligned} \tag{8.1}$$

- *For the interference being strong, one needs to satisfy the inequality*

$$I(X_2; Y_1|X_1) \geq I(X_2; Y_2)$$

*for all input product distribution on* $\mathcal{X}_1 \times \mathcal{X}_2$. *That is*

$$\begin{aligned} & h_2(\bar{p}_2 g_0 + p_2 g_1) - p_2 h_2(g_1) - \bar{p}_2 h_2(g_0) \\ \leq\ & \bar{p}_1 h_2(\bar{p}_2 f_{00} + p_2 f_{01}) + p_1 h_2(\bar{p}_2 f_{10} + p_2 f_{11}) \\ & -\bar{p}_1 \bar{p}_2 h_2(f_{00}) - \bar{p}_1 p_2 h_2(f_{01}) - p_1 \bar{p}_2 h_2(f_{10}) - p_1 p_2 h_2(f_{11}) \end{aligned} \tag{8.2}$$

*for all* $p_1, p_2 \in [0, 1]$.

*Now let* $f_{00} = 0.1$, $f_{01} = 0.2$, $f_{10} = 0.3$, $f_{11} = 0.25$, *and* $g_0 = 0.1$, *then the weak interference range is that* $g_1 \in [0.25, 1]$, *and the strong interference range is that* $g_1 \in [0.038, 0.2]$. *There exists a gap between the weak and the strong interference ranges.*

One natural question to ask is that whether time sharing or rate splitting is going to be optimal in this uncharacterized interference region.

### 8.2.2 Defining the noisy (very weak) interference case for the two-user discrete memoryless channel model

In Chapters 4 and 5, we extend the one-sided weak interference and mixed interference cases to the two-user discrete memoryless interference channel, respectively; the case of noisy interference case is not addressed. One can also think about extending the noisy interference case. The difficulty is that in the Gaussian case to the discrete memoryless case, the definition of noisy interference regime is rather complicated, there is no simple way to define it analogously for the discrete memoryless case. One feasible approach is to find some examples, that treating interference as noise is actually sum-rate optimal. The corresponding conditions for those channels might provide a clue for defining the general noisy interference condition for the discrete memoryless channel model.

# Appendix A

# FDG Proof of the Markov Chain $(X_1^{i-1}, X_{1,i+1}^n) - (X_{1i}, Y_2^{i-1}) - Y_{1i}$

We use [45, Definition 1] to prove in the following that $(X_{1i}, Y_2^{i-1})$ d-separate $(X_1^{i-1}, X_{1,i+1}^n)$ from $Y_{1i}$.

1. Consider the subgraph consisting of the vertices appeared in the Markov chain to be proved, as well as the edges and vertices encountered when moving backward one or more edges starting fro any of the vertices in the Markov chain. The subgraph is depicted in Fig. A.1.

2. Delete all edges coming out of the vertices in $(X_{1i}, Y_2^{i-1})$. It results in the Fig. A.2.

3. Now there is no edges connecting $(X_1^{i-1}, X_{1,i+1}^n)$ and $Y_{1i}$. We prove that the Markov chain

$$(X_1^{i-1}, X_{1,i+1}^n) - (X_{1i}, Y_2^{i-1}) - Y_{1i}$$

holds.

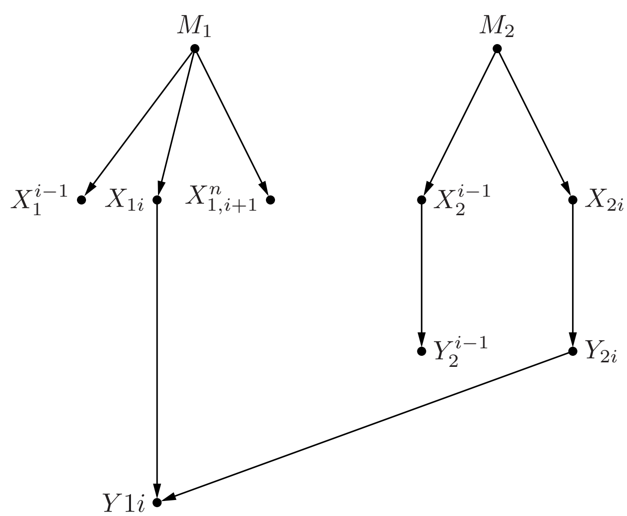



FIGURE A.1: FDG subgraph.

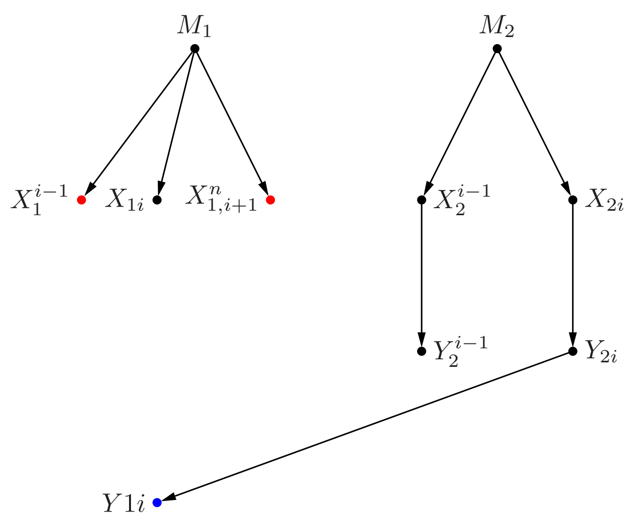



FIGURE A.2: Result of d-separation.

# Appendix B

# A Counter Example for the Equivalence between the Two Different Conditions

This example explains that a DMZIC that satisfies the mutual information condition (4.10) does not necessarily imply the Markov chain relationship (4.3).

**Example B.1.** *Consider a DMIC with binary inputs and outputs. Let $f_{ij}$ represent $p(y_1 = 1|x_1 = i, x_2 = j)$, $g_j$ represent $p(y_2 = 1|x_2 = j)$, $p_i = \Pr\{X_i = 1\}$, and $\bar{p}_i = 1 - p_i$ $(i, j \in \{0, 1\})$. From the mutual information condition (4.10)*

$$I(X_2; Y_2) \geq I(X_2; Y_1|X_1),$$

*we have*

$$H(Y_2) - H(Y_2|X_2) \geq H(Y_1|X_1) - H(Y_1|X_1, X_2),$$

*i.e.,*

$$
\begin{aligned}
& h_2(\bar{p}_2 g_0 + p_2 g_1) - p_2 h_2(g_1) - \bar{p}_2 h_2(g_0) \\
\geq\ & \bar{p}_1 h_2(\bar{p}_2 f_{00} + p_2 f_{01}) + p_1 h_2(\bar{p}_2 f_{10} + p_2 f_{11}) \\
& -\bar{p}_1 \bar{p}_2 h_2(f_{00}) - \bar{p}_1 p_2 h_2(f_{01}) - p_1 \bar{p}_2 h_2(f_{10}) - p_1 p_2 h_2(f_{11}),
\end{aligned}
$$

*for all $p_1, p_2 \in [0,1]$. As the right hand side is linear of $p_1$, it suffices to find $\{f_{ij}\}$ and $\{g_j\}$ such that,*

$$
\begin{aligned}
h_2(\bar{p}_2 g_0 + p_2 g_1) - p_2 h_2(g_1) - \bar{p}_2 h_2(g_0) &\geq \bar{h}_2(\bar{p}_2 f_{00} + p_2 f_{01}) - \bar{p}_2 h_2(f_{00}) - p_2 h_2(f_{01}), \\
h_2(\bar{p}_2 g_0 + p_2 g_1) - p_2 h_2(g_1) - \bar{p}_2 h_2(g_0) &\geq h_2(\bar{p}_2 f_{10} + p_2 f_{11}) - p_2 h_2(f_{11}) - \bar{p}_2 h_2(f_{10}).
\end{aligned}
$$

*Upon obtaining the above inequality, one can make specific choices of $\{f_{ij}\}$ and $\{g_j\}$ to make the above inequality hold for all possible $p_2$ ranging from 0 to 1. For example, it is easy to verify that a valid choice is*

$$
\begin{aligned}
f_{00} &= .1, f_{01} = .3, f_{10} = .5, f_{11} = .25, \\
g_0 &= .1, g_1 = .5.
\end{aligned}
$$

*In the following, we prove by contradiction that this channel does not satisfy the markov chain condition (4.3).*

*Suppose that the markov chain (4.3) is satisfied,*

$$p(y_1|x_1 x_2 y_2) = p(y_1|x_1 y_2).$$

*Then we would have,*

$$p(y_1|x_1 x_2) = \sum_{y_2} p(y_1 y_2|x_1 x_2) = \sum_{y_2} p(y_2|x_2) p(y_1|x_1 y_2).$$

*Solving this equation, we get*

$$p(y_1 = 1|x_1 = 1, y_2 = 1) = -\frac{1}{16},$$

*which contradicts the fact that channel transit probability can never be negative.*

# Appendix C

# The equivalence between the DMDIC and the DMZIC

In this appendix, we prove that the following two channels have the same capacity region.

- Channel 1: The DMZIC with inputs $(X_1, X_2)$ and outputs $(Y_1, Y_2)$. In addition, the Markov chain $X_2 - (X_1, Y_2) - Y_1$ holds.
- Channel 2: The DMDIC with inputs $(X_1, X_2)$ and outputs $(Y_1, Y_2')$, where $X_1, X_2$ and $Y_1$ are identical with the above DMZIC, while $Y_2' = f(X_1, Y_2)$ such that the Markov chain $(X_1, X_2) - Y_2' - Y_1$ holds and $H(Y_2'|X_1) = H(Y_2)$.

First, we show that if a rate pair is achievable for channel 2, it is also achievable for channel 1. Notice that $Y_1$ is identical for both channels, it suffice to show that $H(W_2|Y_2^n) \leq n\epsilon_{2n}$ if

$H(W_2|Y_2'^n) \leq n\epsilon_{2n}$. To prove it, we have

$$
\begin{aligned}
& H(W_2|Y_2^n) \\
\leq\ & H(W_2 X_1^n|Y_2^n) \\
=\ & H(X_1^n|Y_2^n) + H(W_2^n|X_1^n Y_2^n) \\
=\ & H(W_2^n|X_1^n Y_2^n Y_2'^n) \\
\leq\ & H(W_2|Y_2'^n) \\
\leq\ & n\epsilon_{2n},
\end{aligned}
$$

where we make use of the independence of $X_1^n$ and $Y_2^n$, $Y_2'$ is a function of $X_1$ and $Y_2$, and the fact that conditioning reduces entropy.

To this point, to establish the equivalence, it is left to show that if a rate pair is achievable in channel 1, then it is also achievable in channel 2. It thus suffices to prove that

$$
\begin{cases}
H(W_1|Y_1^n) \leq n\epsilon_{1n}, \\
H(W_2|Y_2^n) \leq n\epsilon_{2n},
\end{cases}
$$

implies

$$
\begin{cases}
H(W_1|Y_1^n) \leq n\epsilon_{1n}, \\
H(W_2|Y_2'^n) \leq n\epsilon'_{2n}.
\end{cases}
$$

Therefore, the same code that works for channel 1 also works for channel 2. Notice that $Y_1$ is identical for both channels. Therefore, the first decoders can use the same decoding rule to achieve the same rate.

In the following, we prove that the second decoder of channel 2 can perform as well as that of channel 1. We have the following sequence of inequalities

$$\begin{aligned}
H(W_2|Y_2'^n) &\leq H(W_1W_2|Y_2'^n) \\
&= H(W_1|Y_2'^n) + H(W_2|W_1Y_2'^n) \\
&\overset{(a)}{\leq} H(W_1|Y_1^n) + H(W_2|W_1X_1^nY_2'^n) \\
&\overset{(b)}{\leq} n\epsilon_{1n} + H(W_2|W_1X_1^nY_2^nY_2'^n) \\
&\overset{(c)}{=} n\epsilon_{1n} + H(W_2|W_1X_1^nY_2^n) \\
&\overset{(d)}{=} n\epsilon_{1n} + H(W_2|Y_2^n) \\
&\leq n\epsilon_{1n} + n\epsilon_{2n},
\end{aligned}$$

where (a) follows since $Y_1$ is a degraded version of $Y_2'$, by the Markov chain $(X_1, X_2) - Y_2' - Y_1$; (b) is because that $Y_2' = f(X_1, Y_2)$ satisfying $H(Y_2'|X_1) = H(Y_2)$. This is equivalent to requiring the existence of function $h(\cdot, \cdot)$ such that $Y_2 = h(X_1, Y_2')$ [8]. Therefore, $Y_2$ is completely determined by $X_1$ and $Y_2'$; (c) is from the fact that $Y_2' = f(X_1, Y_2)$; (d) is due to the independence of $(W_1, X_1^n)$ and $(W_2, Y_2^n)$. By defining $\epsilon_{2n}' = \epsilon_{1n} + \epsilon_{2n}$, we proved the desired equivalence.

# Appendix D

# Proof of Theorem 6.1

Fix $p(q)p(u_1|q)p(x_1|u_1q)p(u_2|q)p(x_2|u_2q)p(x_3|q)$.

Codebook generation: Randomly generate a time sharing sequence $q^n$ according to $\prod_{i=1}^n p(q_i)$. Randomly generate $2^{nR_3}$ sequences $x_3^n(m_3)$, $m_3 \in [1 : 2^{nR_3}]$, according to $\prod_{i=1}^n p(x_{3i}|q_i)$. For $j = 1, 2$, randomly generate $2^{nT_i}$ sequences $u_j^n(l_j)$, $l_j \in [1 : 2^{nT_j}]$, each according to $\prod_{i=1}^n p_{U_j|Q}(u_{ji}|q_i)$. For each $u_j^n(l_j)$, randomly generate $2^{nS_j}$ sequences $x_j^n(l_j, k_j)$, $k_j \in [1 : 2^{nS_j}]$, each according to $\prod_{i=1}^n p_{X_j|U_j,Q}(x_j|u_{ji}(l_j), q_i)$. The codebook is available at all transmitters and receivers.

Encoding: For user $j$, $j = 1, 2$, to send message $m_j = (l_j, k_j)$, encoder $j$ transmits $x_j^n(l_j, k_j)$. For user 3, to send message $m_3$, encoder 3 transmits $x_3^n(m_j)$.

Decoding: Upon receiving $y_1^n$, decoder 1 finds the unique message tuple $(\hat{l}_1, \hat{l}_2, \hat{k}_1, \hat{k}_2)$ such that

$$\begin{aligned} &(q^n, u_1^n(\hat{l}_1), u_2^n(\hat{l}_2), x_1^n(\hat{l}_1, \hat{k}_1), x_2^n(\hat{l}_2, \hat{k}_2), y_1^n) \\ \in &A_\epsilon^{(n)}(QU_1U_2X_1X_2Y_1). \end{aligned} \tag{D.1}$$

If no such unique tuple exists, the decoder declares an error.

Upon receiving $y_2^n$, decoder 2 finds the unique message $\hat{m}_3$ such that

$$(q^n, u_1^n(l_1), u_2^n(l_2), x_3^n(\hat{m}_3)) \in A_\epsilon^{(n)}(QU_1U_2X_3Y_2), \tag{D.2}$$

for at least one $l_1 \in [1 : 2^{nT_1}]$ and at least one $l_2 \in [1 : 2^{nT_2}]$. If no such unique $\hat{m}_3$ exists, the decoder declares an error.

Analysis of the probability of error: By the symmetry of the codebook generation, we assume that the transmitted indices are $l_1 = l_2 = k_1 = k_2 = m_3 = 1$. For user 1, we define the following event:

$$\begin{aligned} & E^1_{l_1 l_2 k_1 k_2} \\ =& \{(q^n, u_1^n(l_1), u_2^n(l_2), x_1^n(l_1, k_1), x_2^n(l_2, k_2), y_1^n) \\ & \in A_\epsilon^{(n)}(QU_1U_2X_1X_2Y_1)\Big\}. \end{aligned} \tag{D.3}$$

The error probability at receiver 1 is

$$\begin{aligned} P_{e1}^n \quad &= \Pr\left\{ {E^1_{1111}}^c \bigcup \cup_{(l_1 l_2 k_1 k_2) \neq (1,1,1,1)} E^1_{l_1 l_2 k_1 k_2} \right\} \\ &\leq \Pr({E^1_{1111}}^c) + \sum_{l_1 \neq 1, l_2 = k_1 = k_2 = 1} \Pr(E^1_{l_1 111}) \\ &+ \sum_{l_2 \neq 1, l_1 = k_1 = k_2 = 1} \Pr(E^1_{1 l_2 11}) + \sum_{k_1 \neq 1, l_1 = l_2 = k_2 = 1} \Pr(E^1_{11 k_1 1}) \\ &+ \sum_{k_2 \neq 1, l_1 = l_2 = k_1 = 1} \Pr(E^1_{111 k_2}) + \sum_{l_1, l_2 \neq 1, k_1 = k_2 = 1} \Pr(E^1_{l_1 l_2 11}) \\ &+ \sum_{l_1, k_1 \neq 1, l_2 = k_2 = 1} \Pr(E^1_{l_1 1 k_1 1}) + \sum_{l_1, k_2 \neq 1, l_2 = k_1 = 1} \Pr(E^1_{l_1 11 k_2}) \\ &+ \sum_{l_2, k_1 \neq 1, l_1 = k_2 = 1} \Pr(E^1_{1 l_2 k_1 1}) + \sum_{l_2, k_2 \neq 1, l_1 = k_1 = 1} \Pr(E^1_{1 l_2 1 k_2}) \\ &+ \sum_{k_1, k_2 \neq 1, l_1 = l_2 = 1} \Pr(E^1_{11 k_1 k_2}) + \sum_{l_1, l_2, k_1 \neq 1, k_2 = 1} \Pr(E^1_{l_1 l_2 k_1 1}) \\ &+ \sum_{l_1, l_2, k_2 \neq 1, k_1 = 1} \Pr(E^1_{l_1 l_2 1 k_2}) + \sum_{l_1, k_1, k_2 \neq 1, l_2 = 1} \Pr(E^1_{l_1 1 k_1 k_2}) \\ &+ \sum_{l_2, k_1, k_2 \neq 1, l_1 = 1} \Pr(E^1_{1 l_1 k_1 k_2}) + \sum_{l_1, l_2, k_1, k_2 \neq 1} \Pr(E^1_{l_1 l_2 k_1 k_2}) \end{aligned}$$

It is obvious that $\Pr({E^1_{1111}}^c) \to 0$ when $n \to \infty$. From the joint typicality we have

$$
\begin{aligned}
&\sum_{l_1 \neq 1, l_2 = k_1 = k_2 = 1} \Pr(E^1_{l_1 111}) \\
\leq & 2^{nT_1} \sum_{(q^n, u_1^n, u_2^n, x_1^n, x_2^n, y_1^n) \in A_\epsilon^{(n)}} p(u_1^n, x_1^n | q^n) p(q^n u_2^n x_2^n y_1^n) \\
\leq & 2^{nT_1} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(U_1X_1|Q)-2\epsilon)} \\
& 2^{-n(H(QU_2X_2Y_1)-\epsilon)} \\
= & 2^{n(T_1 - I(U_1X_1;Y_1|U_2X_2Q)+4\epsilon)} = 2^{n(T_1 - I(X_1;Y_1|X_2Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_2 \neq 1, l_1 = k_1 = k_2 = 1} \Pr(E^1_{1 l_2 11}) \\
\leq & 2^{nT_2} \sum_{(q^n, u_1^n, u_2^n, x_1^n, x_2^n, y_1^n) \in A_\epsilon^{(n)}} p(u_2^n, x_2^n | q^n) p(q^n u_1^n x_1^n y_1^n) \\
\leq & 2^{nT_2} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(U_2X_2|Q)-2\epsilon)} \\
& 2^{-n(H(QU_1X_1Y_1)-\epsilon)} \\
= & 2^{n(T_2 - I(U_2X_2;Y_1|U_1X_1Q)+4\epsilon)} = 2^{n(T_2 - I(X_2;Y_1|X_1Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k_1 \neq 1, l_1 = l_2 = k_2 = 1} \Pr(E^1_{11 k_1 1}) \\
\leq & 2^{nS_1} \sum_{(q^n, u_1^n, u_2^n, x_1^n, x_2^n, y_1^n) \in A_\epsilon^{(n)}} p(x_1^n | u_1^n, q^n) p(q^n u_1^n u_2^n x_2^n y_1^n) \\
\leq & 2^{nS_1} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(X_1|U_1Q)-2\epsilon)} \\
& 2^{-n(H(QU_1U_2X_2Y_1)-\epsilon)} \\
= & 2^{n(S_1 - I(X_1;Y_1|U_1U_2X_2Q)+4\epsilon)} = 2^{n(S_1 - I(X_1;Y_1|U_1X_2Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k_2\neq 1, l_1=l_2=k_1=1} \Pr(E^1_{111k_2})\\
\leq &2^{nS_2} \sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(x_2^n|u_2^n,q^n)p(q^n u_1^n u_2^n x_1^n y_1^n)\\
\leq &2^{nS_2} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(X_2|U_2Q)-2\epsilon)}\\
&2^{-n(H(QU_1U_2X_1Y_1)-\epsilon)}\\
= &2^{n(S_2-I(X_2;Y_1|U_1U_2X_1Q)+4\epsilon)} = 2^{n(S_2-I(X_2;Y_1|U_2X_1Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_1,l_2\neq 1, k_1=k_2=1} \Pr(E^1_{l_1l_211})\\
\leq &2^{n(T_1+T_2)} \sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,u_2^n,x_2^n|q^n)p(q^n y_1^n)\\
\leq &2^{n(T_1+T_2)} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(U_1X_1U_2X_2|Q)-2\epsilon)}\\
&2^{-n(H(QY_1)-\epsilon)}\\
= &2^{n(T_1+T_2-I(U_1X_1U_2X_2;Y_1|Q)+4\epsilon)} = 2^{n(T_1+T_2-I(X_1X_2;Y_1|Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_1,k_1\neq 1, l_2=k_2=1} \Pr(E^1_{l_11k_11})\\
\leq &2^{n(S_1+T_1)} \sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n|q^n)p(q^n u_2^n x_2^n y_1^n)\\
\leq &2^{n(S_1+T_1)} 2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)} 2^{-n(H(U_1X_1Q)-2\epsilon)}\\
&2^{-n(H(QU_2X_2Y_1)-\epsilon)}\\
= &2^{n(S_1+T_1-I(U_1X_1;Y_1|U_2X_2Q)+4\epsilon)} = 2^{n(S_1+T_1-I(X_1;Y_1|X_2Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_1,k_2\neq 1,l_2=k_1=1} \Pr(E^1_{l_1 1 1 k_2})\\
\leq &2^{n(S_2+T_1)} \sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_2^n,x_1^n,x_2^n|u_1^n q^n)p(q^n u_1^n y_1^n)\\
\leq &2^{n(S_2+T_1)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_2X_1X_2|U_1Q)-2\epsilon)}\\
&2^{-n(H(QU_1Y_1)-\epsilon)}\\
= &2^{n(S_2+T_1-I(U_2X_1X_2;Y_1|U_1Q)+4\epsilon)} = 2^{n(S_2+T_1-I(X_1X_2;Y_1|U_1Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_2,k_1\neq 1,l_1=k_2=1} \Pr(E^1_{1 l_2 k_1 1})\\
\leq &2^{n(S_1+T_2)} \sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,x_2^n|u_2^n q^n)p(q^n u_2^n y_1^n)\\
\leq &2^{n(S_1+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_1X_1X_2|U_2Q)-2\epsilon)}\\
&2^{-n(H(QU_2Y_1)-\epsilon)}\\
= &2^{n(S_1+T_2-I(U_1X_1X_2;Y_1|U_2Q)+4\epsilon)} = 2^{n(S_1+T_2-I(X_1X_2;Y_1|U_2Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_2,k_2\neq 1,l_1=k_1=1} \Pr(E^1_{1 l_2 1 k_2})\\
\leq &2^{n(S_2+T_2)}\\
&\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_2^n,x_2^n|q^n)p(q^n u_1^n x_1^n y_1^n)\\
\leq &2^{n(S_2+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_2X_2|Q)-2\epsilon)}\\
&2^{-n(H(QU_1X_1Y_1)-\epsilon)}\\
= &2^{n(S_2+T_2-I(U_2X_2;Y_1|U_1X_1Q)+4\epsilon)}\\
= &2^{n(S_2+T_2-I(X_2;Y_1|X_1Q)+4\epsilon)}
\end{aligned}
$$

$$\sum_{k_1,k_2\neq 1,l_1=l_2=1} \Pr(E^1_{11k_1k_2})$$
$$\leq 2^{n(S_1+S_2)}$$
$$\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(x_1^n|u_1^nq^n)p(x_2^n|u_2^nq^n)p(q^nu_1^nu_2^ny_1^n)$$
$$\leq 2^{n(S_1+S_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(X_1X_2|U_1U_2Q)-2\epsilon)}$$
$$2^{-n(H(QU_1U_2Y_1)-\epsilon)}$$
$$=2^{n(S_1+S_2-I(X_1X_2;Y_1|U_1U_2Q)+4\epsilon)}$$
$$=2^{n(S_1+S_2-I(X_1X_2;Y_1|U_1U_2Q)+4\epsilon)}$$

$$\sum_{l_1,l_2,k_1\neq 1,k_2=1} \Pr(E^1_{l_1l_2k_11})$$
$$\leq 2^{n(S_1+T_1+T_2)}$$
$$\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,u_2^n,x_2^n|q^n)p(q^ny_1^n)$$
$$\leq 2^{n(S_1+T_1+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_1X_1U_2X_2|Q)-2\epsilon)}$$
$$2^{-n(H(QY_1)-\epsilon)}$$
$$=2^{n(S_1+T_1+T_2-I(U_1X_1U_2X_2;Y_1|Q)+4\epsilon)}$$
$$=2^{n(S_1+T_1+T_2-I(X_1X_2;Y_1|Q)+4\epsilon)}$$

$$\sum_{l_1,l_2,k_2\neq 1,k_1=1} \Pr(E^1_{l_1l_21k_2})$$
$$\leq 2^{n(T_1+S_2+T_2)}$$
$$\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,u_2^n,x_2^n|q^n)p(q^ny_1^n)$$
$$\leq 2^{n(T_1+S_2+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_1X_1X_2U_2|Q)-2\epsilon)}$$
$$2^{-n(H(QY_1)-\epsilon)}$$
$$=2^{n(S_1+T_1+S_2-I(U_1U_2X_1X_2;Y_1|Q)+4\epsilon)}$$
$$=2^{n(T_1+S_2+T_2-I(X_1X_2;Y_1|Q)+4\epsilon)}$$

$$
\begin{aligned}
&\sum_{l_1,k_1,k_2\neq 1,l_2=1} \Pr(E^1_{l_1 1 k_1 k_2})\\
\leq& 2^{n(S_1+T_1+S_2)}\\
&\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,x_2^n|u_2^n q^n)p(q^n u_2^n y_1^n)\\
\leq& 2^{n(S_1+T_1+S_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_1X_1X_2|U_2Q)-2\epsilon)}\\
&2^{-n(H(QU_2Y_1)-\epsilon)}\\
=&2^{n(S_1+T_1+S_2-I(U_1X_1X_2;Y_1|U_2Q)+4\epsilon)}\\
=&2^{n(S_1+T_1+S_2-I(X_1X_2;Y_1|U_2Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_2,k_1,k_2\neq 1,l_1=1} \Pr(E^1_{1 l_2 k_1 k_2})\\
\leq& 2^{n(S_1+S_2+T_2)}\\
&\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_2^n,x_1^n,x_2^n|u_1^n q^n)p(q^n u_1^n y_1^n)\\
\leq& 2^{n(S_1+S_2+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_2X_1X_2|U_1Q)-2\epsilon)}\\
&2^{-n(H(QU_1Y_1)-\epsilon)}\\
=&2^{n(S_1+S_2+T_2-I(U_2X_1X_2;Y_1|U_1Q)+4\epsilon)}\\
=&2^{n(S_1+S_2+T_2-I(X_1X_2;Y_1|U_1Q)+4\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{l_1,l_2,k_1,k_2\neq 1} \Pr(E^1_{l_1 l_2 k_1 k_2})\\
\leq& 2^{n(S_1+T_1+S_2+T_2)}\\
&\sum_{(q^n,u_1^n,u_2^n,x_1^n,x_2^n,y_1^n)\in A_\epsilon^{(n)}} p(u_1^n,x_1^n,u_2^n,x_2^n|q^n)p(q^n y_1^n)\\
\leq& 2^{n(S_1+T_1+S_2+T_2)}2^{n(H(QU_1U_2X_1X_2Y_1)+\epsilon)}2^{-n(H(U_1X_1U_2X_2|Q)-2\epsilon)}\\
&2^{-n(H(QY_1)-\epsilon)}\\
=&2^{n(S_1+T_1+S_2+T_2-I(U_1U_2X_1X_2;Y_1|Q)+4\epsilon)}\\
=&2^{n(S_1+T_1+S_2+T_2-I(X_1X_2;Y_1|Q)+4\epsilon)}
\end{aligned}
$$

Putting them together, we have

$$
\begin{aligned}
&P_{e1}^n \\
\leq &\epsilon + 2^{n(T_1 - I(X_1;Y_1|X_2Q)+4\epsilon)} + 2^{n(T_2 - I(X_2;Y_1|X_1Q)+4\epsilon)} \\
&+2^{n(T_1 - I(X_1;Y_1|U_1X_2Q)+4\epsilon)} + 2^{n(S_2 - I(X_2;Y_1|U_2X_1Q)+4\epsilon)} \\
&+2^{n(T_1+T_2 - I(X_1X_2;Y_1|Q)+4\epsilon)} + 2^{n(S_1+T_1 - I(X_1;Y_1|X_2Q)+4\epsilon)} \\
&+2^{n(S_2+T_1 - I(X_1X_2;Y_1|U_1Q)+4\epsilon)} + 2^{n(S_1+T_2 - I(X_1X_2;Y_1|U_2Q)+4\epsilon)} \\
&+2^{n(S_2+T_2 - I(X_2;Y_1|X_1Q)+4\epsilon)} + 2^{n(S_1+S_2 - I(X_1X_2;Y_1|U_1U_2Q)+4\epsilon)} \\
&+2^{n(S_1+T_1+T_2 - I(X_1X_2;Y_1|Q)+4\epsilon)} + 2^{n(T_1+S_2+T_2 - I(X_1X_2;Y_1|Q)+4\epsilon)} \\
&+2^{n(S_1+T_1+S_2 - I(X_1X_2;Y_1|U_2Q)+4\epsilon)} + 2^{n(S_1+S_2+T_2 - I(X_1X_2;Y_1|U_1Q)+4\epsilon)} \\
&+2^{n(S_1+T_1+S_2+T_2 - I(X_1X_2;Y_1|Q)+4\epsilon)}
\end{aligned}
$$

For user 2, we define the following event:

$$
E_{l_1 l_2 m_3}^2 = \left\{ (q^n, u_1^n(l_1), u_2^n(l_2), x_3^n(m_3), y_2^n) \in A_\epsilon^{(n)}(QU_1U_2X_3Y_2) \right\}.
$$

The error probability at receiver 2 is

$$
\begin{aligned}
P_{e2}^n \quad =&Pr\left\{ {E_{111}^2}^c \bigcup \cup_{m_3\neq 1, any(l_1,l_2)} E_{l_1 l_2 m_3}^2 \right\} \\
\leq &\Pr\left({E_{111}^2}^c\right) + \sum_{m_3\neq 1, l_1=l_2=1} \Pr\left(E_{11m_3}^2\right) \\
&+ \sum_{l_1, m_3\neq 1, l_2=1} \Pr\left(E_{l_1 1 m_3}^2\right) + \sum_{l_2, m_3\neq 1, l_1=1} \Pr\left(E_{1 l_2 m_3}^2\right) \\
&+ \sum_{l_1,l_2,m_3\neq 1} \Pr\left(E_{l_1 l_2 m_3}^2\right).
\end{aligned}
$$

Again, it is obvious that $\Pr({E^2_{111}}^c) \to 0$ when $n \to \infty$. From the joint typicality we have

$$\begin{aligned}
&\sum_{m_3 \neq 1, l_1 = l_2 = 1} \Pr\left(E^2_{11m_3}\right) \\
\leq & 2^{nR_3} \sum_{(q^n, u_1^n, u_2^n, x_3^n, y_2^n) \in A_\epsilon^{(n)}} p(x_3^n|q^n) p(q^n, u_1^n, u_2^n, y_2^n) \\
\leq & 2^{nR_3} 2^{n(H(QU_1U_2X_3Y_2)+\epsilon)} 2^{-n(H(X_3|Q)-2\epsilon)} 2^{-n(H(QU_1U_2Y_2)-\epsilon)} \\
= & 2^{n(R_3 - I(X_3;Y_2|U_1U_2Q)+4\epsilon)}
\end{aligned}$$

$$\begin{aligned}
&\sum_{l_1, m_3 \neq 1, l_2 = 1} \Pr\left(E^2_{l_1 1 m_3}\right) \\
\leq & 2^{n(T_1+R_3)} \sum_{(q^n, u_1^n, u_2^n, x_3^n, y_2^n) \in A_\epsilon^{(n)}} p(u_1^n, x_3^n|q^n) p(q^n, u_2^n, y_2^n) \\
\leq & 2^{n(T_1+R_3)} 2^{n(H(QU_1U_2X_3Y_2)+\epsilon)} 2^{-n(H(U_1,X_3|Q)-2\epsilon)} 2^{-n(H(QU_2Y_2)-\epsilon)} \\
= & 2^{n(T_1+R_3 - I(U_1X_3;Y_2|U_2Q)+4\epsilon)}
\end{aligned}$$

$$\begin{aligned}
&\sum_{l_2, m_3 \neq 1, l_1 = 1} \Pr\left(E^2_{1 l_2 m_3}\right) \\
\leq & 2^{n(T_2+R_3)} \sum_{(q^n, u_1^n, u_2^n, x_3^n, y_2^n) \in A_\epsilon^{(n)}} p(u_2^n, x_3^n|q^n) p(q^n, u_1^n, y_2^n) \\
\leq & 2^{n(T_2+R_3)} 2^{n(H(QU_1U_2X_3Y_2)+\epsilon)} 2^{-n(H(U_2,X_3|Q)-2\epsilon)} 2^{-n(H(QU_1Y_2)-\epsilon)} \\
= & 2^{n(T_2+R_3 - I(U_2X_3;Y_2|U_1Q)+4\epsilon)}
\end{aligned}$$

$$\begin{aligned}
&\sum_{l_1, l_2, m_3 \neq 1} \Pr\left(E^2_{l_1 l_2 m_3}\right) \\
\leq & 2^{n(T_1+T_2+R_3)} \sum_{(q^n, u_1^n, u_2^n, x_3^n, y_2^n) \in A_\epsilon^{(n)}} p(u_1^n, u_2^n, x_3^n|q^n) p(q^n, y_2^n) \\
\leq & 2^{n(T_1+T_2+R_3)} 2^{n(H(QU_1U_2X_3Y_2)+\epsilon)} 2^{-n(H(U_1U_2X_3|Q)-2\epsilon)} \\
& 2^{-n(H(QY_2)-\epsilon)} \\
= & 2^{n(T_1+T_2+R_3 - I(U_1U_2X_3;Y_2|Q)+4\epsilon)}
\end{aligned}$$

Therefore, for receiver 2,

$$
\begin{aligned}
&P_{e2}^n \\
\leq &\epsilon + 2^{n(R_3 - I(X_3;Y_2|U_1U_2Q)+4\epsilon)} + 2^{n(T_1+R_3-I(U_1X_3;Y_2|U_2Q)+4\epsilon)} \\
&+2^{n(T_2+R_3-I(U_2X_3;Y_2|U_1Q)+4\epsilon)} \\
&+2^{n(T_1+T_2+R_3-I(U_1U_2X_3;Y_2|Q)+4\epsilon)}
\end{aligned}
$$

In order that $P_{e1}^n$, $P_{e2}^n \to 0$, from above inequalities, we must have

$$T_1 \leq I(X_1;Y_1|X_2Q), \tag{D.4}$$

$$T_2 \leq I(X_2;Y_1|X_1Q), \tag{D.5}$$

$$S_1 \leq I(X_1;Y_1|U_1X_2Q), \tag{D.6}$$

$$S_2 \leq I(X_2;Y_1|U_2X_1Q), \tag{D.7}$$

$$T_1+T_2 \leq I(X_1X_2;Y_1|Q), \tag{D.8}$$

$$S_1+T_1 \leq I(X_1;Y_1|X_2Q), \tag{D.9}$$

$$S_2+T_1 \leq I(X_1X_2;Y_1|U_1Q), \tag{D.10}$$

$$S_1+T_2 \leq I(X_1X_2;Y_1|U_2Q), \tag{D.11}$$

$$S_2+T_2 \leq I(X_2;Y_1|X_1Q), \tag{D.12}$$

$$S_1+S_2 \leq I(X_1X_2;Y_1|U_1U_2Q), \tag{D.13}$$

$$S_1+T_1+T_2 \leq I(X_1X_2;Y_1|Q), \tag{D.14}$$

$$T_1+S_2+T_2 \leq I(X_1X_2;Y_1|Q), \tag{D.15}$$

$$S_1+T_1+S_2 \leq I(X_1X_2;Y_1|U_2Q), \tag{D.16}$$

$$S_1+S_2+T_2 \leq I(X_1X_2;Y_1|U_1Q), \tag{D.17}$$

$$S_1+T_1+S_2+T_2 \leq I(X_1X_2;Y_1|Q), \tag{D.18}$$

$$R_3 \leq I(X_3;Y_2|U_1U_2Q), \tag{D.19}$$

$$T_1+R_3 \leq I(U_1X_3;Y_2|U_2Q), \tag{D.20}$$

$$T_2+R_3 \leq I(U_2X_3;Y_2|U_1Q), \tag{D.21}$$

$$T_1+T_2+R_3 \leq I(U_1U_2X_3;Y_2|Q). \tag{D.22}$$

Using Fourier-Motzkin elimination on (D.4)-(D.22) and getting rid of redundant inequalities, we obtain (6.11)-(6.21). The cardinality bounds on the auxiliary random variables are from the Caratheodory Theorem.

# Appendix E

# Proof of Theorem 6.7

The achievability part follows directly from Theorem 6.1 by setting $U_1 = U_2 = \emptyset$. For the converse, (6.41), (6.42) and (6.44) form an outer bound on the capacity region of the corresponding MAC with $X_1$ and $X_2$ as inputs and $Y_1$ as output. Moreover, (6.43) is a natural bound on $R_3$. Therefore, we only need to prove (6.45)-(6.47). First,

$$
\begin{aligned}
& n(R_2+R_3)-n\epsilon \\
= \;& H(W_2)+H(W_3)-n\epsilon \\
\stackrel{(a)}{\leq} \;& I(X_2^n;Y_1^n)+I(X_3^n;Y_2^n) \\
\stackrel{(b)}{\leq} \;& I(X_2^n;Y_1^n|X_1^n)+I(X_3^n;Y_2^n|X_1^n) \\
\stackrel{(c)}{\leq} \;& I(X_2^n;Y_2^n|X_1^nX_3^n)+I(X_3^n;Y_2^n|X_1^n) \\
= \;& I(X_2^nX_3^n;Y_2^n|X_1^n) \\
= \;& H(Y_2^n|X_1^n)-H(Y_2^n|X_1^nX_2^nX_3^n) \\
= \;& \sum_{i=1}^{n}\left\{H(Y_{2i}|Y_2^{i-1}X_1^n)-H(Y_{2i}|Y_2^{i-1}X_1^nX_2^nX_3^n)\right\} \\
\stackrel{(d)}{\leq} \;& \sum_{i=1}^{n}\{H(Y_{2i}|X_{1i})-H(Y_{2i}|X_{1i}X_{2i}X_{3i})\} \\
= \;& I(X_{2i}X_{3i};Y_{2i}|X_{1i}),
\end{aligned}
$$

where $(a)$ is from Fano's inequality; $(b)$ is because of the mutual independence among $X_1^n$, $X_2^n$ and $X_3^n$; $(c)$ is due to (6.39); and $(d)$ uses the fact that conditioning reduces entropy and the memoryless property. Similarly, we can prove the bound on $R_1 + R_3$. We further have

$$
\begin{aligned}
& n(R_1 + R_2 + R_3) - n\epsilon \\
= \;& H(W_1, W_2) + H(W_3) - n\epsilon \\
\overset{(a)}{\leq} \;& I(X_1^n X_2^n; Y_1^n) + I(X_3^n; Y_2^n) \\
\overset{(b)}{\leq} \;& I(X_1^n X_2^n; Y_1^n) + I(X_3^n; Y_2^n) \\
\overset{(c)}{\leq} \;& I(X_1^n X_2^n; Y_2^n | X_3^n) + I(X_3^n; Y_2^n) \\
= \;& I(X_1^n X_2^n X_3^n; Y_2^n) \\
= \;& H(Y_2^n) - H(Y_2^n | X_1^n X_2^n X_3^n) \\
= \;& \sum_{i=1}^{n} \left\{ H(Y_{2i} | Y_2^{i-1}) - H(Y_{2i} | Y_2^{i-1} X_1^n X_2^n X_3^n) \right\} \\
\overset{(d)}{\leq} \;& \sum_{i=1}^{n} \{ H(Y_{2i} - H(Y_{2i} | X_{1i} X_{2i} X_{3i})) \} \\
= \;& I(X_{1i} X_{2i} X_{3i}; Y_{2i}).
\end{aligned}
$$

By introducing a time-sharing random variable $Q$, we obtain Theorem 6.7. The cardinality of $\mathcal{Q}$ can be verified using the Caratheodory theorem.

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