Discrete Sparse Fourier Hermite Approximations in High Dimensions

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Abstract

In this dissertation, the discrete sparse Fourier Hermite approximation of a function in a specified Hilbert space of arbitrary dimension is defined, and theoretical error bounds of the numerically computed approximation are proven. Computing the Fourier Hermite approximation in high dimensions suffers from the well-known curse of dimensionality. In short, as the ambient dimension increases, the complexity of the problem grows until it is impossible to numerically compute a solution. To circumvent this difficulty, a sparse, hyperbolic cross shaped set, that takes advantage of the natural decaying nature of the Fourier Hermite coefficients, is used to serve as an index set for the approximation. The Fourier Hermite coefficients must be numerically estimated since the Fourier Hermite coefficients are nearly impossible to compute exactly, except in trivial cases. However, care must be taken to compute them numerically, since the integrals involve oscillatory terms. To closely approximate the integrals that appear in the approximated Fourier Hermite coefficients, a multiscale quadrature method is used. This quadrature method is implemented through an algorithm that takes advantage of the natural properties of the Hermite polynomials for fast results.

The definitions of the sparse index set and of the quadrature method used will each introduce many interdependent parameters. These parameters give a user many degrees

of freedom to tailor the numerical procedure to meet his or her desired speed and accuracy goals. Default guidelines of how to choose these parameters for a general function f that will significantly reduce the computational cost over any naive computational methods without sacrificing accuracy are presented. Additionally, many numerical examples are included to support the complexity and accuracy claims of the proposed algorithm.

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Chapter 1

Introduction

A common exercise encountered while studying mathematics is to decompose an certain object in terms of a given basis. Some classical examples include decomposing functions in terms of the bases formed by the piecewise linear hat functions, by wavelets, or by trigonometric polynomials. Such a decomposition is of interest to applied mathematicians, scientists and engineers, but is also powerful for pure mathematicians. In this dissertation, the basis consisting of the set of polynomials that are pairwise orthogonal in the Hilbert space $L^2_\omega(\mathbb{R}^d)$ with the d-dimensional Gaussian weight function ω is used. Given the representation of a function in terms of this Fourier Hermite basis, a sparse, discrete approximation is described, and bounds for the approximation error and computational complexity are discussed. Additionally, in this dissertation it is shown that it is possible to achieve a good approximation error estimate while maintaining a low computational cost. The primary approximation error result is given in Theorem 5.6, and the primary computational cost estimate is given in Theorem 6.10.

1.1 Motivation

The Hermite decomposition of a function is of particular interest and usefulness for several reasons. Every decomposition system is limited by the properties of the basis set. The collection of the Hermite polynomials enjoy two unique properties that make them a very good representation system in a wide variety of settings. First, the orthogonality over the entire plane $\mathbb{R}^d$ makes them an appropriate description basis for objects that do not have a boundary, periodicity, or asymptotic decay at infinity. This makes them particularly powerful when used to numerically compute the solution of types of differential equations that may be encountered in modeling large systems over an infinite time domain, including applications in environmental science, climate change, and astrophysics. The second mathematical property that makes the Hermite polynomial decomposition desirable over other possible basis sets is that the Hermite polynomials are orthogonal with respect to the Gaussian weight function. Systems involving the Gaussian weight are pervasive throughout the mathematical sciences including statistics, analysis, and differential equations and are necessary to describe phenomena throughout all disciplines in the natural sciences.

The Hermite polynomials are necessary for the solution of Schrödinger's equation, which is a non-linear second-order differential equation used in the model of a quantum simple harmonic oscillator in quantum mechanics. The quantum simple harmonic oscillator is an important model in physics [7]. A simple version of the model is taught in undergraduate physics courses, yet it is still an area of active research [21], [23].

Additionally, the Hermite polynomials enjoy a form of rotational invariance. If a function has a representation in terms of finitely many normalized Hermite polynomials, then any rotation of the function has a representation in terms of the same number of normalized Hermite polynomials [17]. Because of this property, the Hermite polynomials have been employed in many areas of image processing including medical imaging, optics and satellite images [8], [17], [19], [20].

1.2 Problem Statement and Challenges

The problem motivating this work is to find a fast, accurate approximation to an arbitrary given function in terms of the Hermite polynomials. Specifically, for any $f \in L^2_\omega(\mathbb{R}^d)$, where $\omega(x) := e^{-x^2}$ is the Gaussian weight function, can a good approximation to the full Fourier-Hermite representation

$$f(x) = \sum_{n \in \mathbb{N}_0^d} c_n H_n(x),$$

where H_n is a Hermite polynomial and c_n is the corresponding transform coefficient, be found quickly?

Many challenges are encountered in attempting to achieve this task. First, the high-dimensional nature of the problem suffers from the so-called 'curse of dimensionality' [4]. To reduce its effect, the number of terms in the sum over the infinite lattice $\mathbb{N}_0^d$ must be reduced while simultaneously ensuring that the truncated sum very nearly approximates the full series. Second, the transform coefficients are defined via integrals over the entire plane $\mathbb{R}^d$, and involve highly oscillatory terms. An accurate and realistic quadrature approach must be taken to ensure the numerically computed coefficients will not contribute enough error to destroy the overall accuracy of the sum. Finally, how can the methods used be designed to optimize the tradeoffs between accuracy and computational complexity?

1.3 Previous Work

Discretization of problems using sparse grids has been a popular method to overcome the curse of dimensionality [3], [5], [12], [13], [16], [18]. It has been shown to be effective in reducing the computational complexity of problems while maintaining a decent approximation order. Bungartz and Griebel [5] published an important review article discussing interpolation methods on sparse grids. I employed a hyperbolic cross shaped sparse grid index set, a strategy that has proven successful for other authors as well. Professor Yuesheng Xu and Ying Jiang [12] used the hyperbolic cross shaped index set to compute the sparse Fourier expansion of a d dimensional function. They showed that approximating the Fourier expansion of a function f in a Sobolev space with regularity s by considering only terms corresponding with elements in the hyperbolic cross shaped index set yielded relative error on the order of $\mathcal{O}(N^{-s})$, where N is the number of index points of the sparse grid in a single dimension. Jie Shen and Li-Lian Wang [25] also have studied methods of approximating an expansion of a function in terms of Hermite functions over the hyperbolic cross shaped index set. They showed that for a function f in a Korobov-type space with smoothness parameter s, the spectral approximation has relative error $\mathcal{O}(s^{(d-1)s}\left(\frac{s}{2}\right)^{ds/2} N^{-s/2})$.

Many scholars who have studied or used Fourier-Hermite approximations have used the Gaussian quadrature to compute the transform coefficients [2], [9], [11], [17]. This seems like a natural choice, since the Gaussian quadrature theory and orthogonal polynomials go hand in hand. Indeed, in theoretical settings, the results have a simple elegance and admit very good approximation error. In practical settings, however, this method has a number of shortcomings. In order to guarantee a small error using Gaussian quadrature methods, it is necessary for the function f to be approximated to be in the space $C^{2N}(\mathbb{R}^d)$ for a large parameter N. This is a very strict requirement, since functions describing real world phenomena are often not that smooth. Moreover, Gaussian quadrature methods require knowledge about the data at prescribed nodes, namely the Hermite-Gaussian nodes which are the zeros of the Hermite polynomial of degree N. These nodes are not uniformly spaced, but in real world applications, data are almost always sampled at regular intervals. So using a Gaussian quadrature method to approximate the coefficients is not a practical approach for applied settings.

Many authors have used a Fourier-Hermite expansion for applications in image processing. Therefore, very little work has been done in finding fast methods to approximate the transform coefficients in higher dimensions. Instead, most research in this area has been focused in only one or two spatial dimensions [2], [11], [17].

1.4 New Contributions

In this dissertation, I shall extend some known results to more general settings, and I shall also prove new results that are improvements on existing methods.

Most authors consider the Fourier-Hermite approximations of functions in the standard square integrable space $L^2(\mathbb{R}^d)$ with small values $d = 1, 2, 3$. I performed the analysis in the much larger space $L^2_\omega(\mathbb{R}^d)$, where ω is the Gaussian weight defined by $\omega(x) = e^{-||x||_2^2}$ and for arbitrary positive integers d. Working in the larger space has some practical advantages. Mainly, functions in the weighted space L^2_ω must not necessarily possess an asymptotic decay property. In fact, a function $f \in L^2_\omega(\mathbb{R}^d)$ may have the property $f(x) \to \infty$ as $||x|| \to \infty$.

This subtle change in the class of functions to be considered has a profound effect on the applicability of the results. My results can be applied toward numerically solving differential equations where the solutions do not have boundary conditions or asymptotic decay. Such problems arise in astrophysics, quantum mechanics, models describing natural phenomena such as population overcrowding or climate change, and pure mathematics. Additionally, since I abandoned the Gaussian quadrature approach for approximating the coefficients, I am able to impose a much more relaxed smoothness requirement on the functions to be considered, thereby enlarging the space of candidate functions even more.

As mentioned above, much of the interest in Hermite polynomial decompositions has come from the image processing community. Therefore much of the research has been limited to few spatial dimensions. However, these low dimension problems that are processed only on defined limited domains do not take advantage of all of the nice properties of the Hermite expansions. The Hermite polynomials are unique in that they are ω-orthogonal over *all* of $\mathbb{R}^d$, but image processing uses only locally orthogonal properties. However, the global orthogonality may be very useful in approximating functions or solutions to PDEs that do not have any boundary conditions in a very large dimensional settings. Therefore, I shall prove theoretical approximation errors and develop computational algorithms with relatively low computational complexity using an arbitrarily large dimension d.

As discussed earlier, I believe the Gaussian quadrature method is often not a good approach for computing the transform coefficients in high dimensions for practical settings. Therefore I used a truncated multiscale piecewise polynomial interpolation method to approximate the coefficients. Although using heirarchical approaches is not a novel idea, this is a new approach for this particular problem. Since I am employing this method on functions over unbounded domains, it is necessary to introduce many interdependent parameters whose values must be chosen carefully to balance each other, the computational cost of evaluating all of the coefficients, and the overall accuracy achieved by the approximation. As new contributions to the field, I shall develop the framework for finding these parameters, prove the approximation and complexity orders using these values, and develop algorithms to find the coefficient values quickly.

Chapter 2

Sparse Fourier Hermite Expansions

I shall begin by listing some basic definitions, properties, and some fundamental results involving the Hermite polynomials. Then I shall define the decomposition of a function in terms of the Hermite polynomial basis. This decomposition, given in Definition 2.9, and finding ways to approximate it quickly and accurate is the focus of this dissertation.

2.1 Preliminaries

In the following, a vector will be denoted by boldface type or a bar over the variable. For the sake of readability, if it is clear from the context that the variable is multidimensional, standard typeface will be used.

For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, the square of the standard ℓ_2 norm is defined as $||\mathbf{x}||_2^2 := \sum x_j^2$ and the standard ℓ_1 norm as $|\mathbf{x}| := \sum |x_j|$. The d-dimensional Gaussian weight is defined via

$$\omega(\mathbf{x}) := e^{-||\mathbf{x}||_2^2} = e^{-(x_1^2+x_2^2+\cdots+x_d^2)}.$$

The weighted L^2 space over $\mathbb{R}^d$ is defined by

$$L^2_\omega(\mathbb{R}^d) := \left\{\text{Lebesgue measurable } f : \mathbb{R}^d \to \mathbb{R} \text{ with } ||f||_\omega < \infty\right\},$$

where $||f||_\omega$ is the Gaussian weighted norm induced by the inner product

$$\langle f, g\rangle_\omega := \int_{\mathbb{R}^d} f(x)g(x)\omega(x)\, \mathrm{d}x.$$

The error function is defined as

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t, \tag{2.1}$$

and the complimentary error function as

$$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t, \tag{2.2}$$

for any $x \in \mathbb{R}$. Notice $\lim_{x\to\infty} \mathrm{erf}(x) = 1$.

The following notations will be needed. The d-dimension factorial is given by $\mathbf{n}! := (n_1!)(n_2!)\cdots(n_d!)$, and the d-dimension Kronecker delta is defined via

$$\delta_{\mathbf{n},\mathbf{m}} := \begin{cases} 1 & \text{if } n_j = m_j \text{ for all } j = 1, 2, \ldots, d \\ 0 & \text{otherwise} \end{cases}.$$

For two d-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$, the notation $\mathbf{x}^{\mathbf{y}}$ means $\prod_{j=1}^{d} x_j^{y_j}$, and for a scalar x and vector $\mathbf{y}$, the notation $x^{\mathbf{y}}$ means $x^{|\mathbf{y}|}$. The following notation for higher order derivatives in d dimensions will also be needed in subsequent sections:

$$\frac{\partial^{\mathbf{n}}}{\partial \mathbf{x}^{\mathbf{n}}} = \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}}.$$

The set of the nonnegative integers will be denoted by $\mathbb{N}_0$.

Definition 2.1. *Let $n \in \mathbb{N}_0$. The n^{th} univariate Hermite polynomial $H_n : \mathbb{R} \to \mathbb{R}$ is given by*

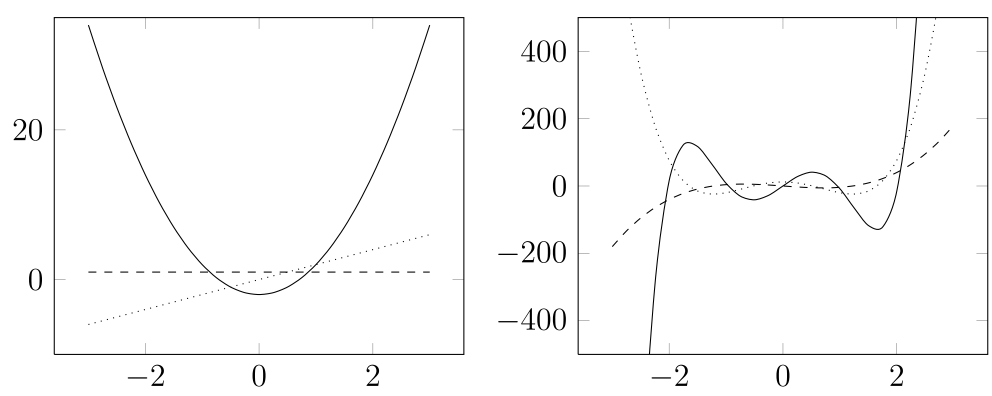
$$H_n(x) := (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{-x^2} \right).$$

The first few Hermite polynomials are

$$H_0(x) = 1, \qquad H_1(x) = 2x,$$

$$H_2(x) = 4x^2 - 2, \qquad H_3(x) = 8x^3 - 12x,$$

$$H_4(x) = 16x^4 - 48x^2 + 12, \qquad H_5(x) = 32x^5 - 160x^3 + 120x.$$



(a) H_0 (- -), H_1 (..) and H_2 (–) (b) H_3 (- -), H_4 (..) and H_5 (–)

Figure 2.1: A few Hermite polynomials

The Hermite polynomials form an orthogonal set in $L^2_\omega(\mathbb{R})$. It is well-known [27] that

$$\langle H_n, H_m\rangle_\omega := \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = n!2^n\sqrt{\pi}\delta_{n,m},$$

where $\delta_{n,m}$ is 1 if $m = n$ and zero otherwise. In addition, the Hermite polynomials enjoy the following properties [27]:

- Three-Term Recurrence: $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.
- $H_n'(x) = 2nH_{n-1}(x)$, and $H_n^{(k)}(x) = 2^k\frac{n!}{(n-k)!}H_{n-k}(x)$.
- H_n is a solution to the differential equation $u'' - 2xu' + 2nu = 0$
- $\{H_n : n \in \mathbb{N}_0\}$ forms a complete orthogonal system in $L^2_\omega(\mathbb{R})$.

A d-dimensional Hermite polynomial is simply a tensor product of the univariate polynomials.

Definition 2.2. *For* $\mathbf{n} \in \mathbb{N}_0^d$ *and* $\mathbf{x} \in \mathbb{R}^d$*, we define the* $\mathbf{n}$*th Hermite polynomial as*

$$H_{\mathbf{n}}(\mathbf{x}) := \prod_{j=1}^{d} H_{n_j}(x_j).$$

Alternatively, one may write

$$H_{\mathbf{n}}(\mathbf{x}) = (-1)^{|\mathbf{n}|}e^{||\mathbf{x}||_2^2}\frac{\partial^{|\mathbf{n}|}}{\partial \mathbf{x}^{\mathbf{n}}}\left(e^{-||\mathbf{x}||_2^2}\right).$$

A few three dimensional Hermite polynomials are

$H_{(0,0,0)}(x,y,z) = 1,$ $H_{(2,1,1)}(x,y,z) = 16x^2yz - 8yz,$

$H_{(2,2,0)}(x,y,z) = 16x^2y^2 - 8x^2 - 8y^2 + 4,$ $H_{(3,0,1)}(x,y,z) = 16x^3z - 24xz,$

$H_{(0,4,0)}(x,y,z) = 16y^4 - 48y^2 + 12.$

The d-dimensional Gaussian weight ω is exactly equal to the product of its univariate cases. This means that for functions where the variables can be separated (as with the Hermite polynomials), then the inner product can be easily computed one spatial dimension at a time. In more detail, for any $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^d$,

$$\begin{aligned}
\langle H_{\mathbf{m}}, H_{\mathbf{n}} \rangle_\omega &= \int_{\mathbb{R}^d} H_{\mathbf{m}}(\mathbf{x}) H_{\mathbf{n}}(\mathbf{x}) \omega(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \prod_{k=1}^{d} \int_{\mathbb{R}} H_{m_k}(x) H_{n_k}(x) \omega(x) \, \mathrm{d}x \\
&= \prod_{k=1}^{d} \langle H_{m_k}, H_{n_k} \rangle_\omega \\
&= \prod_{k=1}^{d} n_k! 2^{n_k} \sqrt{\pi} \delta_{m_k, n_k} \\
&= \mathbf{n}! 2^{\mathbf{n}} \pi^{d/2} \delta_{\mathbf{m},\mathbf{n}}.
\end{aligned}$$

This section concludes with some basic theorems involving the Hermite polynomials.

Theorem 2.3. *Let* $\mathbf{n} \in \mathbb{N}^d$. *Define* $\mathbf{n} - 1 := (n_j - 1)_{j=1}^d$. *Then*

$$\frac{\partial}{\partial \mathbf{x}} \left(H_{\mathbf{n}-1}(\mathbf{x}) \omega(\mathbf{x}) \right) = (-1)^d H_{\mathbf{n}}(\mathbf{x}) \omega(\mathbf{x}).$$

Proof. First consider the univariate case. Observe that $\omega'(x) = -2x\omega(x)$. Then for $n = 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(H_0(x)\omega(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(\omega(x)) = -2x\omega(x) = -H_1(x)\omega(x).$$

Using the three term recurrence, for any integer $n \geq 2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(H_{n-1}(x)\omega(x)\right) &= H'_{n-1}(x)\omega(x) + H_{n-1}(x)\omega'(x)\\ &= 2(n-1)H_{n-2}(x)\omega(x) - 2xH_{n-1}(x)\omega(x)\\ &= -[2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)]\omega(x)\\ &= -H_n(x)\omega(x).\end{aligned}$$

Then for the multivariate case:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{x}}\left(H_{\mathbf{n}-1}(\mathbf{x})\omega(\mathbf{x})\right) &= \left(\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_d}\right)H_{\mathbf{n}-1}(\mathbf{x})\omega(\mathbf{x})\\ &= \prod_{j=1}^{d}\frac{\mathrm{d}}{\mathrm{d}x_j}H_{n_j-1}(x_j)\omega(x_j)\\ &= \prod_{j=1}^{d}(-1)H_{n_j}(x_j)\omega(x_j)\\ &= (-1)^d H_{\mathbf{n}}(\mathbf{x})\omega(\mathbf{x}).\end{aligned}$$

□

Corollary 2.4. *Let $n \in \mathbb{N}_0$. Then*

$$\int_0^x H_{n+1}(t)\omega(t)\mathrm{d}t = (-1)\left(H_n(x)\omega(x) - H_n(0)\right). \tag{2.3}$$

Theorem 2.5. *For any* $\mathbf{n} \in \mathbb{N}_0^d$,

$$\frac{\partial}{\partial \mathbf{x}} \left(H_{\mathbf{n}}(\mathbf{x}) \right) = 2^d \left(\prod_{j=1}^{d} n_j \right) H_{\mathbf{n}-1}(\mathbf{x}).$$

Proof. Consider a single dimension first. The theorem is easy to verify for $n = 1, 2$. In those cases,

$$H_1'(x) = (2x)' = 2 = 2H_0(x), \quad \text{(for } n = 1\text{)},$$

and

$$H_2'(x) = (4x^2 - 2)' = 8x = 2nH_1(x), \quad \text{(for } n = 2.\text{)}$$

Suppose the theorem is true for some positive integers $n-2$ and $n-1$. Then the statement of the theorem can be shown to be true for n by an application of the three term recurrence.

$$\begin{aligned} H_n'(x) &= (2xH_{n-1}(x) - 2(n-1)H_{n-2}(x))' \\ &= 2H_{n-1}(x) + 2xH_{n-1}'(x) - 2(n-1)H_{n-2}'(x) \\ &= 2H_{n-1}(x) + 4(n-1)xH_{n-2}(x) - 4(n-1)(n-2)H_{n-3}(x) \\ &= 2H_{n-1}(x) + 2(n-1)\left(2xH_{n-2}(x) - 2(n-2)H_{n-3}(x)\right) \\ &= 2H_{n-1}(x) + 2(n-1)H_{n-1}(x) \\ &= 2nH_{n-1}(x), \end{aligned}$$

as desired. Therefore, by induction, the theorem is true when $d = 1$ and for all $n \geq 1$.

In several dimensions, the statement of the theorem can be shown to be true using a product of the univariate case. In fact,

$$\frac{\partial}{\partial \mathbf{x}}\left(H_{\mathbf{n}}(\mathbf{x})\right)=\prod_{j=1}^{d}\frac{d}{dx_j}H_{n_j}(x_j)=\prod_{j=1}^{d}2n_jH_{n_j}(x_j)=2^d\left(\prod_{j=1}^{d}n_j\right)H_{\mathbf{n}-1}(\mathbf{x}).$$

□

Theorem 2.6. *Denote the largest root of H_n by x_n^*. For any $n \in \mathbb{N}_0$,*

$$H_n(x) \leq 2^n x^n,$$

for all $x \geq x_n^*$.

Proof. Proceed by induction on n.

The cases $n = 0, 1$ are trivially true since $H_0 \equiv 1$ and $H_1(x) = 2x \leq 2^1 x^1$.

Suppose the statement of the theorem is true for some positive integer n. Define a function $f(x) := 2^{n+1}x^{n+1}$. Since $x_{n+1}^* \geq 0$, we know that $0 = H_{n+1}(x_{n+1}^*) \leq f(x_{n+1}^*)$. Furthermore, it is well known that the zeros of orthogonal polynomials are interlaced. This fact guarantees that the largest root of H_{n+1} is larger than the largest root of H_n for every nonnegative integer n.

Thus for all $x \geq x^*_{n+1} > x^*_n$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} H_{n+1}(x) &= (n+1)H_n(x) \\ &\leq (n+1)2^n x^n, \text{ (via the induction step)} \\ &\leq (n+1)2^{n+1}x^n \\ &= \frac{\mathrm{d}}{\mathrm{d}x} f(x).\end{aligned}$$

Therefore $H_{n+1}(x) \leq f(x)$ for all $x \geq x^*_{n+1}$ for this n.

Induction on n promises that the statement is true for all $n \in \mathbb{N}_0$. □

Theorem 2.7. *For any $n \in \mathbb{N}_0$, the integral of the square of the n^{th} Hermite polynomial can be written a a combination of products of Hermite polynomials and the error function via*

$$\int_0^x H_n^2(t)\omega(t)\mathrm{d}t = \sum_{k=0}^{n-1} \frac{n!2^k}{(n-k)!} H_{n-k}(x)H_{n-k-1}(x)\omega(x) + n!2^{n-1}\sqrt{\pi}\,\mathrm{erf}(x).$$

Proof. Using Theorem 2.3, for any integer $k \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(H_{k-1}(x)\omega(x)\right) = -H_k(x)\omega(x).$$

For any non-negative integer n, the product $H_n(x)H_{n-1}(x)$ is an odd polynomial (Theorem A.3), and therefore for any $n \geq 1$, $H_n(0)H_{n-1}(0) = 0$. One can use integration by parts to see that

$$\begin{aligned}
\int_0^x H_n^2(t)\omega(t)\mathrm{d}t &= \int_0^x (H_n(t))(H_n(t)\omega(t))\mathrm{d}t \\
&= \int_0^x (H_n(t))(-H_{n-1}(t)\omega(t))'\mathrm{d}t \\
&= H_n(t)H_{n-1}(t)\omega(t)|_0^x - \int_0^x (-2n)H_{n-1}^2(t)\omega(t)\mathrm{d}t \\
&= H_n(x)H_{n-1}(x)\omega(x) + 2n\int_0^x H_{n-1}^2(t)\omega(t)\mathrm{d}t.
\end{aligned}$$

Continue to apply the above procedure to get

$$\begin{aligned}
\int_0^x H_n^2(t)\omega(t)\mathrm{d}t &= \sum_{k=0}^{n-1} \frac{n!2^k}{(n-k)!} H_{n-k}(x)H_{n-k-1}(x)\omega(x) + n!2^n \int_0^x \omega(x)\mathrm{d}x \\
&= \sum_{k=0}^{n-1} \frac{n!2^k}{(n-k)!} H_{n-k}(x)H_{n-k-1}(x)\omega(x) + n!2^{n-1}\sqrt{\pi}\,\mathrm{erf}(x).
\end{aligned}$$

□

Corollary 2.8. *Let $n \in \mathbb{N}_0$. Then*

$$\int_0^\infty H_{n+1}^2(t)\omega(t)\mathrm{d}t = n!2^{n-1}\sqrt{\pi}.$$

Proof. In Theorem 2.7, let x tend to positive infinity.

$$\begin{aligned}\int_0^\infty H_{n+1}^2(t)\omega(t)\mathrm{d}t &= \lim_{x\to\infty}\int_0^x H_{n+1}^2(t)\omega(t)\mathrm{d}t \\ &= \lim_{x\to\infty}\left(\sum_{k=0}^{n-1}\frac{n!2^k}{(n-k)!}H_{n-k}(x)H_{n-k-1}(x)\omega(x) + n!2^{n-1}\sqrt{\pi}\,\mathrm{erf}(x)\right) \\ &= \sum_{k=0}^{n-1}\frac{n!2^k}{(n-k)!}\lim_{x\to\infty}\left(H_{n-k}(x)H_{n-k-1}(x)\omega(x)\right) + n!2^{n-1}\sqrt{\pi}.\end{aligned}$$

The interchange of the sum and limit will be justified since the sum is finite.

Note that

$$\begin{aligned}0 &\le \lim_{x\to\infty}\left(H_{n-k}(x)H_{n-k-1}(x)\omega(x)\right) = \lim_{x\to\infty}\frac{H_{n-k}(x)H_{n-k-1}(x)}{e^{-x^2}} \\ &\le \lim_{x\to\infty}\frac{(2x)^{2n-2k-1}}{e^{-x^2}},\end{aligned}$$

by Theorem 2.6.

By L'Hôpital's Rule for limits ([26]), $\lim_{x\to\infty}\frac{(2x)^{2n-2k-1}}{e^{-x^2}} = 0$ for any integers n and k. Therefore,

$$\int_0^\infty H_{n+1}^2(t)\omega(t)\mathrm{d}t = n!2^{n-1}\sqrt{\pi}.$$

□

2.2 Fourier Hermite Series

In the previous section, it was observed that d-dimensional Hermite polynomials are orthogonal in $L^2_\omega(\mathbb{R}^d)$ because the univariate Hermite polynomials are orthogonal in $L^2_\omega(\mathbb{R})$. It is well known that the set of the Hermite polynomials forms a complete orthogonal set in $L^2_\omega(\mathbb{R}^d)$, and therefore also forms a basis. This property will be used throughout the remainder of this dissertation to find a good approximation of a given function $f \in L^2_\omega(\mathbb{R}^d)$ in terms of only a few well-chosen elements in this basis.

Definition 2.9. *For any $f \in L^2_\omega(\mathbb{R}^d)$, the Fourier-Hermite expansion of f is given by*

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^d} c_{\mathbf{n}} H_{\mathbf{n}}(\mathbf{x}),$$

where the coefficients $c_{\mathbf{n}}$ are defined via

$$c_{\mathbf{n}} = \frac{\langle f, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}}.$$

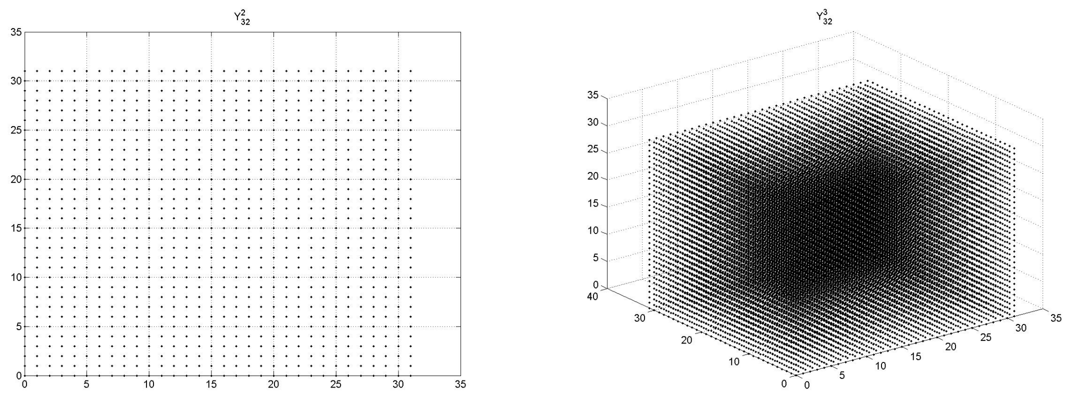
It is impractical to try to calculate the Hermite-Fourier expansion over the entire index set $\mathbb{N}_0^d$. One would like to find the coefficients only for small values of $|\mathbf{n}|$, and would be justified in doing so. The coefficients $c_{\mathbf{n}}$ from Definition 2.9 decay very rapidly as $|\mathbf{n}|$ increases. For each f, there exists a finite positive constant C such that $c_{\mathbf{n}} \leq C/(\mathbf{n}!2^{\mathbf{n}})$. This is due to the fact that since $f \in L^2_\omega(\mathbb{R}^d)$, the norm $||f||_\omega$ is bounded above by a finite positive constant C_1. Then

$$c_{\mathbf{n}} := \frac{\langle f, H_{\mathbf{n}}\rangle_\omega}{||H_{\mathbf{n}}||^2_\omega} \leq \frac{||f||_\omega ||H_{\mathbf{n}}||_\omega}{||H_{\mathbf{n}}||^2_\omega} \leq \frac{C_1}{||H_{\mathbf{n}}||_\omega} = \frac{C}{\mathbf{n}!2^{\mathbf{n}}}.$$

An approximation to the Hermite-Fourier expansion of f over a finite full grid index set is considered first.

Fix $N \in \mathbb{N}_0$, and let $\mathbb{Y}_N^d$ be the full grid index set defined by

$$\mathbb{Y}_N^d = \left\{ \mathbf{n} \in \mathbb{N}_0^d : n_j < N, \ \forall j \in \{1, 2, \ldots, d\} \right\}.$$



(a) $\mathbb{Y}_{32}^2$ (b) $\mathbb{Y}_{32}^3$

Figure 2.2: The Full Grid Index Sets $\mathbb{Y}_{32}^2$ and $\mathbb{Y}_{32}^3$.

Definition 2.10. *For a fixed positive integer N, the* full grid Fourier-Hermite approximation *of order N of the function $f \in L^2_\omega(\mathbb{R}^d)$ is defined by*

$$\mathcal{F}_N f(\mathbf{x}) := \sum_{\mathbf{n} \in \mathbb{Y}_N^d} c_{\mathbf{n}} H_{\mathbf{n}}(\mathbf{x}).$$

The coefficients $c_{\mathbf{n}}$ are the same as in Defintion 2.9.

The full grid Fourier-Hermite approximation is just a truncation of the full Fourier-Hermite expansion as in Defintion 2.9. It is also called the *spectral approximation* of f. Using a Korobov-like norm defined below, one can prove that the spectral approximation is a good approximation of f.

For a fixed number $s > 0$ and functions $\phi, \psi \in L^2_\omega(\mathbb{R}^d)$, define

$$\langle \phi, \psi \rangle_{\kappa^s} := \sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{\langle \phi, H_\mathbf{n} \rangle_\omega \langle \psi, H_\mathbf{n} \rangle_\omega}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}} \prod_{j=1}^{d} (1 + n_j)^{2s}.$$

By Theorem A.1, $\langle \cdot, \cdot \rangle_{\kappa^s}$ defines an inner product. Denote the induced norm on $L^2_\omega(\mathbb{R}^d)$ by $|| \cdot ||_{\kappa^s}$, and let the space $\mathcal{H}_s(\mathbb{R}^d) \subset L^2_\omega(\mathbb{R}^d)$ denote the collection of all functions f such that $||f||_{\kappa^s} < \infty$.

Theorem 2.11. *For any $s > 0$, $f \in L^2_\omega(\mathbb{R}^d)$ and $N \in \mathbb{N}$,*

$$||f - \mathcal{F}_N f||_\omega \leq N^{-s} ||f||_{\kappa^s}.$$

Proof. The Fourier-Hermite expansion of f has the form $\sum_{\mathbf{n} \in \mathbb{N}_0^d} c_\mathbf{n} H_\mathbf{n}(\mathbf{x})$. Since the Hermite polynomials are an orthogonal basis of $L^2_\omega(\mathbb{R}^d)$, a generalization of Parseval's identity can be used (See Theorem A.5).

$$||f - \mathcal{F}_N f||^2_\omega = \left|\left| \sum_{\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{Y}_N^d} c_\mathbf{n} H_\mathbf{n} \right|\right|^2_\omega = \sum_{\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{Y}_N^d} |c_\mathbf{n}|^2 ||H_\mathbf{n}||^2_\omega.$$

The right hand side simplifies to $\sum_{\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{Y}_N^d} \frac{|\langle f, H_\mathbf{n} \rangle_\omega|^2}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}}$. Then multiply and divide by the factor $\prod_{j=1}^d (1 + n_j)^{2s}$ to obtain

$$||f - \mathcal{F}_N f||^2_\omega = \sum_{\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{Y}_N^d} \frac{|\langle f, H_\mathbf{n} \rangle_\omega|^2}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}} \prod_{j=1}^{d} (1 + n_j)^{2s} \prod_{j=1}^{d} (1 + n_j)^{-2s}.$$

Since no $\mathbf{n}$ occurring in the summation index set is an element of $\mathbb{Y}_N^d$, it follows that

$n_{j_a} \geq N$ for some j_a. Then

$$\prod_{j=1}^{d} (1+n_j)^{-2s} = \prod_{j=1}^{d} \frac{1}{(n_j+1)^{2s}} \leq \frac{1}{(1+n_{j_a})^{2s}} \leq \frac{1}{(1+N)^{2s}} \leq \frac{1}{N^{2s}},$$

We have shown $||f - \mathcal{F}_N f||_{\omega}^2 \leq N^{-2s}||f||_{\kappa^s}^2$ as desired. □

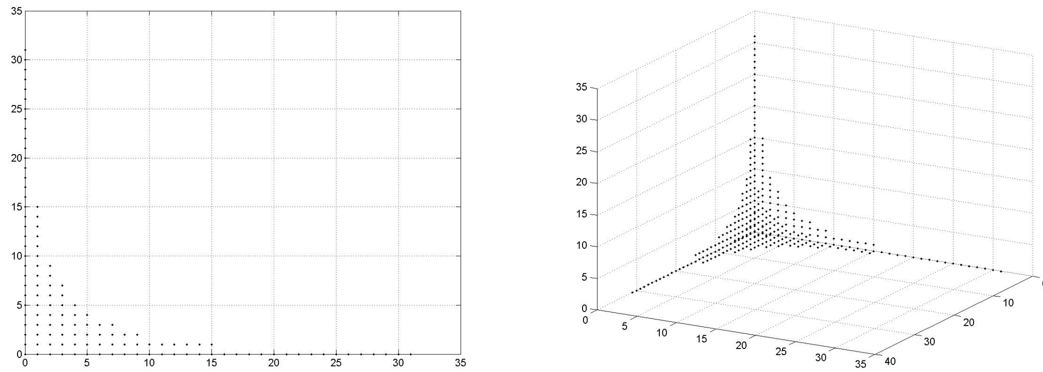
2.3 Hyperbolic Cross Sparse-Grid Approximation

Although having infinitely many coefficients to compute can be avoided by truncating the Fourier-Hermite expansion and summing only over the index set $\mathbb{Y}_N^d$, in high dimensions the cardinality of this index set is still too large to allow quick computation of the projections, since the full grid index set $\mathbb{Y}_N^d$ contains N^d elements. The size of the index set used for the truncation needs to be drastically reduced to allow timely computations, but must be designed to keep the approximation results obtained with the full grid approximation. Considering the nature of decay of the coefficients $c_{\mathbf{n}}$, a hyperbolic cross index set is used to satisfy these requirements.

Fix $N \in \mathbb{N}_0$, and let $\mathbb{L}_N^d$ be the sparse grid index set defined by

$$\mathbb{L}_N^d = \left\{ n \in \mathbb{N}_0^d : \prod_{j=1}^{d} (n_j + 1) \leq N \right\}.$$

The sparse grid index set contains $\mathcal{O}(N \log^{d-1} N)$ elements ([12]). In the case $d = 2$ or 3 and $N = 32$, which are very moderate values for d and N, comparison between Figures 2.2 and 2.3 illustrate the huge computational advantage in using the sparse grid index set over the full grid index set.



(a) $\mathbb{L}^2_{32}$ (b) $\mathbb{L}^3_{32}$

Figure 2.3: The Sparse Grid Index Sets $\mathbb{L}^2_{32}$ and $\mathbb{L}^3_{32}$.

Definition 2.12. *The sparse grid Fourier-Hermite approximation of $f \in L^2_\omega(\mathbb{R}^d)$ of order N is*

$$f_N = \sum_{\mathbf{n} \in \mathbb{L}^d_N} c_\mathbf{n} H_\mathbf{n}. \tag{2.4}$$

The coefficients $c_\mathbf{n}$ are the same as in Definition 2.9.

The sparse grid approximation competes well with the full grid approximation in terms of approximation order. Compare the following with Theorem 2.11.

Theorem 2.13. *For any $s > 0$ and $f \in L^2_\omega(\mathbb{R}^d)$,*

$$||f - f_N||_\omega \leq N^{-s} \, ||f||_{\kappa^s} \, .$$

Proof. The outline of the proof follows that of Theorem 2.11. Since the Hermite polynomials are orthogonal relative to the Gaussian weight ω, one may use Theorem A.5 to obtain the following:

$$\begin{aligned}
||f - f_N||_\omega^2 &= \left|\left| \sum_{n \in \mathbb{N}_0^d} c_n H_n - \sum_{n \in \mathbb{L}_N^d} c_n H_n \right|\right|_\omega^2 \\
&= \sum_{n \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d} |c_n|^2 ||H_n||_\omega^2 \\
&= \sum_{n \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d} \left| \frac{\langle f, H_n \rangle_\omega}{||H_n||_\omega^2} \right|^2 ||H_n||_\omega^2 \\
&= \sum_{n \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d} \frac{\langle f, H_n \rangle_\omega^2}{||H_n||_\omega^2} \\
&= \sum_{n \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d} \frac{\langle f, H_n \rangle_\omega^2}{||H_n||_\omega^2} \prod_{j=1}^{d} (1 + n_j)^{2s} \prod_{j=1}^{d} (1 + n_j)^{-2s}.
\end{aligned}$$

The set $\mathbb{L}_N^d$ is the collection of all multi-indices $\mathbf{n}$ such that $n_j > 0$ and $\prod_{j=1}^{d}(1 + n_j) \leq N$. Therefore for all $\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d$,

$$\prod_{j=1}^{d} (1 + n_j) \geq N \quad \Rightarrow \quad \prod_{j=1}^{d} (1 + n_j)^{-2s} \leq N^{-2s}.$$

Thus

$$||f - f_N||_\omega^2 \leq \sum_{n \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d} \frac{\langle f, H_n \rangle_\omega^2}{||H_n||_\omega^2} \prod_{j=1}^{d} (1 + n_j)^{2s} N^{-2s} \leq ||f||_{\kappa^s}^2 N^{-2s}.$$

□

The following regularity result holds.

Theorem 2.14. *For fixed* $0 < k < s$, *and* $f \in L_\omega^2(\mathbb{R}^d)$,

$$||f - f_N||_{\kappa^k} \leq N^{k-s} ||f||_{\kappa^s}.$$

Proof. Let $\sum_{\mathbf{n}\in\mathbb{N}_0^d} c_{\mathbf{n}} H_{\mathbf{n}}$ be the Fourier-Hermite expansion of f. Then

$$||f - f_N||^2_{\kappa^k} = \sum_{n\in\mathbb{N}_0^d} \frac{\langle f - \tilde{f}_N, H_{\mathbf{n}}\rangle^2_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^{d} (n_j + 1)^{2k}.$$

Since $f - f_N = \sum_{\mathbf{n}\in\mathbb{N}_0^d\setminus\mathbb{L}_N^d} c_{\mathbf{n}} H_{\mathbf{n}}$, it follows that

$$\langle f - f_N, H_{\mathbf{n}}\rangle_\omega = \begin{cases} c_{\mathbf{n}}||H_{\mathbf{n}}||^2_\omega, & \text{if } \mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{L}_N^d \\ \\ 0, & \text{if } \mathbf{n} \in \mathbb{L}_N^d \end{cases}.$$

Also, note that

$$|c_{\mathbf{n}}|^2||H_{\mathbf{n}}||^4_\omega = \frac{\langle f, H_{\mathbf{n}}\rangle^2_\omega}{||H_{\mathbf{n}}||^4_\omega}||H_{\mathbf{n}}||^4_\omega = \langle f, H_{\mathbf{n}}\rangle^2_\omega.$$

Therefore

$$\begin{aligned} ||f - f_N||^2_{\kappa^k} &= \sum_{\mathbf{n}\in\mathbb{N}_0^d\setminus\mathbb{L}_N^d} \frac{|c_{\mathbf{n}}|^2||H_{\mathbf{n}}||^4_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^{d}(1 + n_j)^{2k} \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^d\setminus\mathbb{L}_N^d} \frac{\langle f, H_{\mathbf{n}}\rangle^2_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^{d}(1 + n_j)^{2k} \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^d\setminus\mathbb{L}_N^d} \frac{\langle f, H_{\mathbf{n}}\rangle^2_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^{d}(1 + n_j)^{2s} \prod_{j=1}^{d}(1 + n_j)^{2(k-s)}. \end{aligned}$$

Since the index set for the summation includes no values from $\mathbb{L}_N^d$, it follows that

$\prod_{j=1}^{d}(1+n_j) \geq N$. Therefore

$$\begin{aligned} ||f - f_N||_{\kappa^k}^2 &\leq N^{2(k-s)} \sum_{\mathbf{n}\in\mathbb{N}_0^d \setminus \mathbb{L}_N^d} \frac{\langle f, H_{\mathbf{n}}\rangle_\omega^2}{\mathbf{n}! 2^{\mathbf{n}} \pi^{d/2}} \prod_{j=1}^{d}(1+n_j)^{2s} \\ &= N^{2(k-s)} ||f||_{\kappa^s}^2 \end{aligned}$$

as desired. □

Chapter 3

Discretization Part 1: Removing the Tails

One difficulty encountered in approximating Fourier-Hermite approximations has now been overcome. Summations over an infinite index set must be avoided, so in the previous chapters the index set was truncated first to a finite full grid and then to a finite sparse grid, and the error arising from the truncations has been analyzed. However, to compute the full grid and sparse grid approximations as in Definitions 2.10 and 2.12, one needs to exactly compute the coefficients $c_{\mathbf{n}}$ which, if even possible, are difficult and computationally consuming. The next few chapters of this dissertation focus on the development of a scheme for the approximation of the integrals

$$\int_{\mathbb{R}^d} f(\mathbf{x}) H_{\mathbf{n}}(\mathbf{x}) \omega(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{3.1}$$

appearing in the coefficients $c_{\mathbf{n}}$ of the Fourier-Hermite expansions of f in such a way to keep the approximation order obtained in the previous chapter. To accomplish this, first the integrands are considered only over a compact subset I^d of $\mathbb{R}^d$, then a quadrature method based on multiscale interpolation is used to approximately evaluate the integral

$$\int_{I^d} f(\mathbf{x}) H_{\mathbf{n}}(\mathbf{x}) \omega(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{3.2}$$

One may be tempted to choose an arbitrarily large set I^d, which would indeed guarantee the integral in Equation (3.2) is approximately equal to the integral in Equation (3.1). However, this would be foolish since it would force a more computationally exhaustive quadrature method to be used to numerically compute the integral in (3.2). Therefore the set I^d should be chosen just large enough to ensure the two integrals are within an acceptable margin or error.

In this chapter, I shall show how to find an appropriate compact set I^d and prove that the error between the two integrals in Equations (3.1) and (3.2) is bounded. The next chapter will develop and analyze a quadrature method to numerically compute the integral in Equation (3.2).

3.1 Definitions and Strategies

In general, for any quadrature method used, the following definition applies.

Definition 3.1. *For any function $f \in L^2_\omega(\mathbb{R}^d)$, let $Q(f, \mathbf{n})$ be a quadrature rule used to approximate the integral $\langle f, H_{\mathbf{n}} \rangle_\omega$ with error $\varepsilon(f, \mathbf{n})$. The* discrete sparse Fourier-Hermite approximation of f *is*

$$f_{Q,N} := \sum_{\mathbf{n}\in\mathbb{L}_N^d} c_{Q,\mathbf{n}} H_{\mathbf{n}}, \ \textit{with}\ c_{Q,\mathbf{n}} = \frac{Q(f,\mathbf{n})}{||H_{\mathbf{n}}||_\omega^2}.$$

Notice that this is similar to Definition 2.12, but with each $\langle f, H_{\mathbf{n}}\rangle_\omega$ replaced with $Q(f,\mathbf{n})$.

Theorem 3.2. *Let* f_N *be as in Definition 2.12 and* $f_{Q,N}$ *be as in Definition 3.1. Then*

$$||f_N - f_{Q,N}||_\omega^2 \le \sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{\varepsilon^2(f,\mathbf{n})}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}}.$$

Proof. Use the generalized Parseval's identity. (Theorem A.5.)

$$\begin{aligned} ||f_N - f_{Q,N}||_\omega^2 &= \sum_{\mathbf{n}\in\mathbb{L}_N^d} |c_{\mathbf{n}} - c_{Q,\mathbf{n}}|^2 ||H_{\mathbf{n}}||_\omega^2 \\ &= \sum_{\mathbf{n}\in\mathbb{L}_N^d} \left| \frac{\langle f, H_{\mathbf{n}}\rangle_\omega}{||H_{\mathbf{n}}||_\omega^2} - \frac{Q(f,\mathbf{n})}{||H_{\mathbf{n}}||_\omega^2} \right|^2 ||H_{\mathbf{n}}||_\omega^2 \\ &= \sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{|\langle f, H_{\mathbf{n}}\rangle_\omega - Q(f,\mathbf{n})|^2}{||H_{\mathbf{n}}||_\omega^2} \\ &= \sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{\varepsilon^2(f,\mathbf{n})}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}}. \end{aligned}$$

□

This theorem agrees with intuition; the error should depend on how well the quadrature method performs, but should also be smaller if higher degree polynomials are used in the approximation. Now it remains to pick a quadrature method that not only has a very good error term ε, but that also is computationally cheap.

3.2 Gaussian Quadrature

Since the integrands include a sequence of orthogonal polynomials, one's first instinct may be to use the Gaussian quadrature method to approximate the integrals. In one dimension, for a function $g \in C^{2N}(\mathbb{R})$, the Gaussian quadrature method guarantees

$$\int_{-\infty}^{\infty} g(x)\omega(x)\,\mathrm{d}x = \sum_{i=0}^{N-1} A_i g(x_i) + \frac{N!\sqrt{\pi}}{(2N)!2^N} g^{(2N)}(\xi), \tag{3.3}$$

for some $\xi \in \mathbb{R}$, and with the nodes x_i being the zeros of the Nth Hermite polynomial and the weights given by $A_i = \dfrac{2^{N-1}(N-1)!\sqrt{\pi}}{N^2[H_{N-1}(x_i)]^2}$ [1], [22]. Simply replace g by $f \cdot H_n$ for the desired result. Denote the Hermite-Gaussian quadrature rule by $Q_{\mathrm{G}}(f,n)$. Then Theorem 3.2 becomes

$$\begin{aligned} ||f_N - \hat{f}_N||_\omega^2 &\leq \sum_{n=0}^{N-1} \frac{\varepsilon_G^2(f,n)}{n!2^n\sqrt{\pi}} \\ &= \sum_{n=0}^{N-1} \frac{1}{n!2^n\sqrt{\pi}} \left(\frac{N!\sqrt{\pi}}{(2N)!2^N}\right)^2 \left|(fH_n)^{(2N)}(\xi)\right|^2 \\ &= \sum_{n=0}^{N-1} \frac{1}{n!2^n\sqrt{\pi}} \left(\frac{N!\sqrt{\pi}}{(2N)!2^N}\right)^2 \left|\sum_{i=0}^{2N} \binom{2N}{i} f^{(2N-i)}(\xi) H_n^{(i)}(\xi)\right|^2. \end{aligned}$$

One may stop there and notice a glaring problem with this method, even in the one dimensional case. Summations of high order derivatives of f are being taken. Therefore this method requires f to be very smooth. However, this cannot be guaranteed, and even if this condition is met, f may have very large total variation. If f is highly oscillatory, then the error term may blow up beyond repair. Authors have widely ignored this fact, instead opting to say merely that this method will yield an exact result if fH_n is a polynomial of degree less than or equal to $2N$. Of course, this implies that f must already be a polynomial, so writing it in terms of Hermite polynomials is a trivial exercise. It becomes clear that a better course would be to abandon this method, and opt for ones that are equipped to handle a wider class of functions.

3.3 A Two-Fold Strategy: Selecting the Location of the Cut

If $f \in L^2_\omega(\mathbb{R}^d)$, then it is reasonable to expect the bulk of the integral $\langle f, H_{\mathbf{n}}\rangle_\omega$ to come from some bounded domain in $\mathbb{R}^d$, and for the contribution from the so-called 'tail' portion

$$\int\limits_{|\mathbf{x}|\gg 1} f(\mathbf{x})H_{\mathbf{n}}(\mathbf{x})\omega(\mathbf{x})\ \mathrm{d}\mathbf{x}$$

to be small. Eventually, the weight ω should be the dominant function in the integrand, and thus the integrand should become quite small. See Figure 3.1 for an illustration of this phenomenon.

Fix a positive number A to be determined later, and let $I = [-A, A]$. Split the integral into two pieces and approximate each separately. That is,

$$\int\limits_{\mathbb{R}^d} f(\mathbf{x})H_{\mathbf{n}}(\mathbf{x})\omega(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int\limits_{I^d} f(\mathbf{x})H_{\mathbf{n}}(\mathbf{x})\omega(\mathbf{x})\ \mathrm{d}\mathbf{x} + \int\limits_{\mathbb{R}^d\setminus I^d} f(\mathbf{x})H_{\mathbf{n}}(\mathbf{x})\omega(\mathbf{x})\ \mathrm{d}\mathbf{x}.$$

A quadrature method will be used to approximate the 'inside' integral, and the 'tail' integral will be estimated. This procedure has two sources of error- the error from truncating the interval of integration and the error from the quadrature method used to approximate the 'inside' integral. In this chapter, I shall show how to bound the error from the 'tail' integral. In the subsequent chapters, I shall show how an appropriate quadrature method can be developed to estimate the 'inside' integral well while keeping the computational complexity low.

For motivation, let us investigate the error from the tail portions in one dimension first. The inner product to be approximated can be decomposed as

$$\langle f, H_n \rangle_\omega = \int\limits_I f(x) H_n(x) \omega(x) \, \mathrm{d}x + \int\limits_{\mathbb{R}\setminus I} f(x) H_n(x) \omega(x) \, \mathrm{d}x.$$

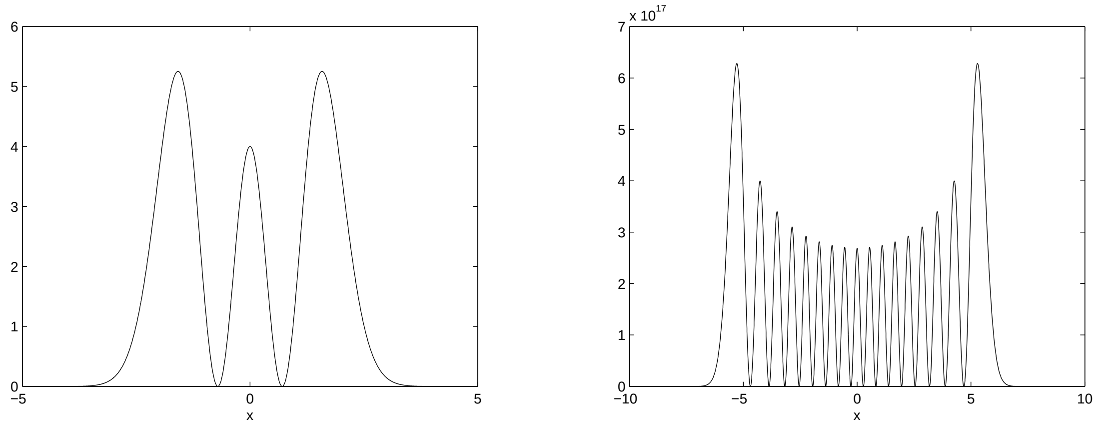
If this integral is approximated by

$$\langle f, H_n \rangle_\omega \approx \int\limits_I \mathcal{P} f(x) H_n(x) \omega(x) \mathrm{d}x,$$

where the portion of the integral over $\mathbb{R} \setminus I$ is discarded, and $\mathcal{P} f$ is an approximation of f to be defined later, then the error ε as in Definition 3.1 has two components. It can be written as

$$\varepsilon(f, n) = \varepsilon_{\text{middle}}(f, n) + \varepsilon_{\text{tail}}(f, n),$$

where $\varepsilon_{\text{middle}}(f, n)$ is the error from the quadrature method over I, and $\varepsilon_{\text{tail}}(f, n)$ is the value of the integral over $\mathbb{R} \setminus I$ which is to be discarded in the computation of the discretized coefficient. It is desirable for the two sources of error to have the same order. Since A is to be fixed in a manner that depends only on the fixed values N and s, it is not included in the notation of the error.



(a) $N = 2$ (b) $N = 2^4$.

Figure 3.1: Plots of $H_N^2(x)\omega(x)$ for different choices of N.

The value A where to cut off the tails of the integrands must be selected carefully. If A is too large, then computing the quadrature on $[-A, A]$ may be too cumbersome. If A is too small, then disregarding the tails may contribute significant error. This section is dedicated to selecting a balanced choice of A, and to approximating the error $\varepsilon_{\text{tail}}$.

Theorem 3.3. *If A is chosen such that for some positive constant c*

$$\frac{1}{||H_n||_\omega^2} \int\limits_{\mathbb{R}^d \setminus I^d} H_n^2(x)\omega(x)\mathrm{d}x \leq \frac{c}{N^{2s+1}\log_2^{d-1} N}, \tag{3.4}$$

for all $n \in \mathbb{L}_N^d$, then there exists $c_1 > 0$ such that

$$\sum_{n\in\mathbb{L}_N^d} \frac{\varepsilon_{tail}^2(f,n)}{||H_n||_\omega^2} \leq c_1 N^{-2s}||f||_\omega^2. \tag{3.5}$$

Proof. Let S be a Lebesgue measurable set. Define the indicator function on S via

$$\chi_S(x) := \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases}$$

Since $\varepsilon_{\text{tail}}(f,n) = \langle f, H_n \chi_{(\mathbb{R}^d \setminus I^d)} \rangle_\omega \leq ||f||_\omega ||H_n \chi_{(\mathbb{R}^d \setminus I^d)}||_\omega$,

$$\begin{aligned}
\sum_{n \in \mathbb{L}_N^d} \frac{\varepsilon_{\text{tail}}^2(f,n)}{||H_n||_\omega^2} &\leq ||f||_\omega^2 \sum_{n \in \mathbb{L}_N^d} \frac{||H_n \chi_{(\mathbb{R}^d \setminus I^d)}||_\omega^2}{||H_n||_\omega^2} \\
&= ||f||_\omega^2 \sum_{n \in \mathbb{L}_N^d} \frac{1}{||H_n||_\omega^2} \int_{\mathbb{R}^d \setminus I^d} H_n^2(x)\omega(x)\mathrm{d}x \\
&\leq c||f||_\omega^2 \sum_{n \in \mathbb{L}_N^d} \frac{1}{N^{2s+1} \log_2^{d-1} N}, \text{ (by assumption)} \\
&\leq \frac{c_1 ||f||_\omega^2}{N^{2s+1} \log_2^{d-1} N} \left(N \log_2^{d-1} N \right) \\
&= c_1 N^{-2s} ||f||_\omega^2.
\end{aligned}$$

Note: The value of c_1 is equal to the value of c times the constant occurring in the size of the index set $\mathbb{L}_N^d$. □

Now a guideline for choosing A satisfying the assumption of Theorem 3.3 must be developed.

Theorem 3.4. *If A is chosen such that*

$$A \geq \max \left\{ \operatorname{erfc} \left(\frac{1}{N^{2s+1} \log_2^{d-1} N} \right), x_{N-1}^* \right\}$$

then there exists a positive constant c such that for all $n \in \mathbb{L}_N^d$,

$$\frac{1}{||H_n||_\omega^2} \int\limits_{\mathbb{R}^d \setminus I^d} H_n^2(x)\omega(x)\mathrm{d}x \leq \frac{c}{N^{2s+1}\log_2^{d-1} N}. \tag{3.6}$$

Proof. If $n = (n_1, n_2, \ldots, n_d) \in \mathbb{L}_N^d$, then denote the first $d-1$ coordinates as $\hat{n}_d = (n_1, n_2, \ldots, n_{d-1})$.

The set $\mathbb{R}^d \setminus I^d$ can be decomposed as $\left[(\mathbb{R}^{d-1} \setminus I^{d-1}) \times I\right] \cup \left[\mathbb{R}^{d-1} \times (\mathbb{R} \setminus I)\right]$. For if $x_d \in I$, then $\hat{x}_d$ must satisfy $\hat{x}_d \in \mathbb{R}^{d-1} \setminus I^{d-1}$. However, if $x_d \in \mathbb{R} \setminus I$, then $\mathbf{x} \in \mathbb{R}^d \setminus I^d$ for any $\hat{x}_d \in \mathbb{R}^{d-1}$.

Using this notation and observation, the left side of Equation (3.6) can be written as

$$\begin{aligned}
&\frac{1}{||H_n||_\omega^2} \int\limits_{\mathbb{R}^d \setminus I^d} H_n^2(x)\omega(x)\mathrm{d}x = \\
&\quad = \frac{1}{||H_n||_\omega^2} \int\limits_{(\mathbb{R}^{d-1} \setminus I^{d-1}) \times I} H_n^2(x)\omega(x)\mathrm{d}x + \frac{1}{||H_n||_\omega^2} \int\limits_{\mathbb{R}^{d-1} \times (\mathbb{R} \setminus I)} H_n^2(x)\omega(x)\mathrm{d}x \\
&\quad = \left(\frac{1}{||H_{\hat{n}_d}||_\omega^2} \int\limits_{\mathbb{R}^{d-1} \setminus I^{d-1}} H_{\hat{n}_d}^2(x)\omega(x)\mathrm{d}x\right)\left(\frac{1}{||H_{n_d}||_\omega^2} \int\limits_{I} H_{n_d}^2(x)\omega(x)\mathrm{d}x\right) + \\
&\quad + \left(\frac{1}{||H_{\hat{n}_d}||_\omega^2} \int\limits_{\mathbb{R}^{d-1}} H_{\hat{n}_d}^2(x)\omega(x)\mathrm{d}x\right)\left(\frac{1}{||H_{n_d}||_\omega^2} \int\limits_{\mathbb{R} \setminus I} H_{n_d}^2(x)\omega(x)\mathrm{d}x\right) \\
&\quad \leq \frac{1}{||H_{\hat{n}_d}||_\omega^2} \int\limits_{\mathbb{R}^{d-1} \setminus I^{d-1}} H_{\hat{n}_d}^2(x)\omega(x)\mathrm{d}x + \frac{1}{||H_{n_d}||_\omega^2} \int\limits_{\mathbb{R} \setminus I} H_{n_d}^2(x)\omega(x)\mathrm{d}x.
\end{aligned}$$

Notice the first term in the above inequality is similar to the original term in the string of inequalities. Therefore, let us continue in this manner to get

$$\begin{aligned}
&\frac{1}{||H_n||_\omega^2} \int\limits_{\mathbb{R}^d \setminus I^d} H_n^2(x)\omega(x)\mathrm{d}x = \\
&\quad \leq \frac{1}{||H_{\hat{n}_d}||_\omega^2} \int\limits_{\mathbb{R}^{d-1} \setminus I^{d-1}} H_{\hat{n}_d}^2(x)\omega(x)\mathrm{d}x + \frac{1}{||H_{n_d}||_\omega^2} \int\limits_{\mathbb{R} \setminus I} H_{n_d}^2(x)\omega(x)\mathrm{d}x \\
&\quad \leq \cdots \\
&\quad \leq \sum_{k=1}^{d} \frac{1}{||H_{n_k}||_\omega^2} \int\limits_{\mathbb{R} \setminus I} H_{n_k}^2(x)\omega(x)\mathrm{d}x \\
&\quad = \sum_{k=1}^{d} \frac{2}{||H_{n_k}||_\omega^2} \int_A^\infty H_{n_k}^2(x)\omega(x)\mathrm{d}x.
\end{aligned}$$

Since the constant c can (and is expected to) depend on d, it suffices to pick a value A such that

$$\frac{1}{n!2^n} \int_A^\infty H_n^2(x)\omega(x)\mathrm{d}x \leq \frac{1}{N^{2s+1}\log_2^{d-1} N},$$

for all $n = 0, 1, \ldots, N-1$. Figure 3.2 suggests this should be possible for a moderate value of A.

If $n = 0$, this reduces to finding A such that

$$\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(A) \leq \frac{1}{N^{2s+1}} \log_2^{d-1} N.$$

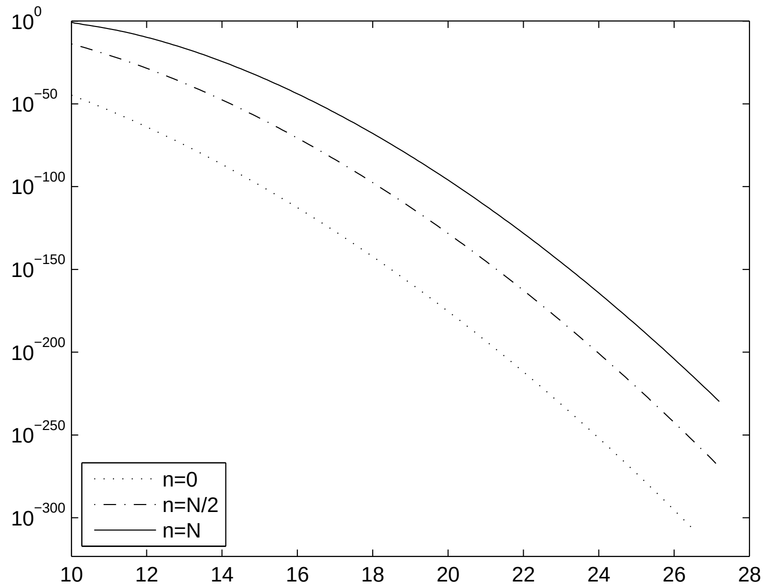


Figure 3.2: Plots of $\dfrac{1}{n!2^n}\displaystyle\int_A^\infty H_n^2(x)\omega(x)\mathrm{d}x$ for various n with $N = 50$.

For $n > 0$, Theorem 2.7 and Corollary 2.8 can be used.

$$\begin{aligned}
&\frac{1}{n!2^n}\int_A^\infty H_n^2(x)\omega(x)\mathrm{d}x = \\
&= \frac{1}{n!2^n}\int_0^\infty H_n^2(x)\omega(x)\mathrm{d}x - \frac{1}{n!2^n}\int_0^A H_n^2(x)\omega(x)\mathrm{d}x \\
&= \frac{1}{n!2^n}\left(n!2^{n-1}\sqrt{\pi}\,\mathrm{erfc}\,A - \sum_{k=0}^{n-1}\frac{n!2^k}{(n-k)!}H_{n-k}(A)H_{n-k-1}(A)\omega(A)\right) \\
&= \frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(A) - \sum_{k=0}^{n-1}\frac{1}{(n-k)!2^{n-k}}H_{n-k}(A)H_{n-k-1}(A)\omega(A) \\
&\leq \frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(A),
\end{aligned}$$

provided $A \geq x_n^*$, the largest root of H_n.

After the observations that erfc is a decreasing function and $x_n^* \leq x_{n+1}^*$ for all $n \in \mathbb{N}$, one can conclude that the desired result holds for any A satisfying

$$A \geq \max\left\{\operatorname{erfc}\left(\frac{1}{N^{2s+1}\log_2^{d-1} N}\right), x_{N-1}^*\right\}. \tag{3.7}$$

The value of c that holds in the above computations is $c = \dfrac{2d}{\sqrt{\pi}}$. □

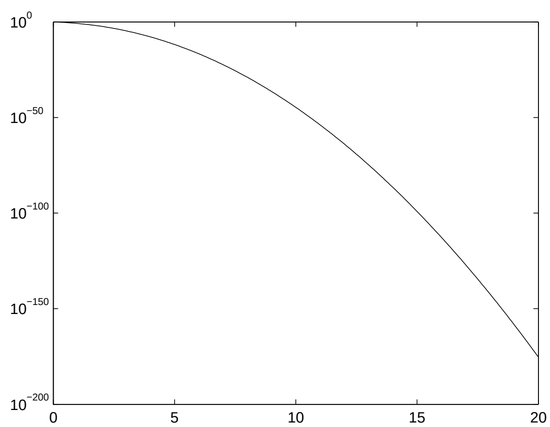


Figure 3.3: Semi-log plot of $y = \operatorname{erfc}(x)$.

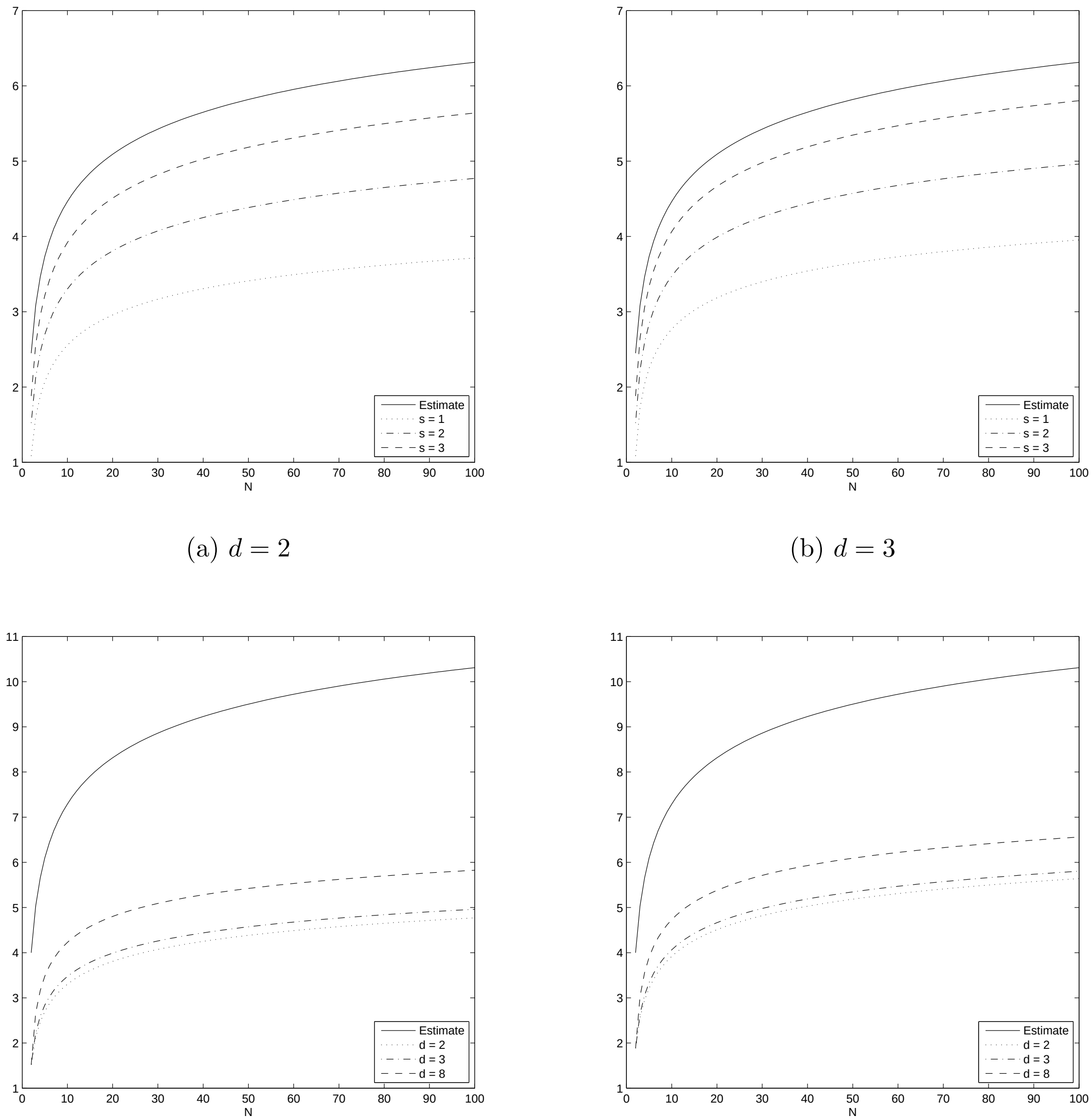
Although obtaining an estimate of A satisfying

$$\operatorname{erfc}(A) \leq \frac{c}{N^{2s+1}\log_2^{d-1} N} \tag{3.8}$$

in terms of N, d and s is difficult, MATLAB can compute the value of A numerically using the `erfcinv` function. Figure 3.3 contains a semi-log plot of the complimentary error function. It is easy to see that even though the magnitude of the right hand side of the inequality (3.8) is very small, it is reasonable to expect a modest value of A will satisfy the inequality. A loose upper bound estimate of the value of A that satisfies Equation (3.8) is

$$A \approx \sqrt{2\max\{s,d\}\log_2 N}.$$

Several plots illustrating this approximation can be seen in Figure 3.4. These plots demonstrate that as the number and degree of Hermite polynomials used in the Fourier Hermite approximation of f grows, the value of A which guarantees and accurate approximation of the truncated coefficients increases at a sub-logarithmic rate.



(a) $d = 2$ (b) $d = 3$

(c) $s = 2$ (d) $s = 3$

Figure 3.4: Plots of erfcinv $\left(\dfrac{1}{N^{2s+1}\log_2^{d-1} N}\right)$ and an estimate for various s and d against values of N.

An approximation for the largest root of H_n is given in [24] as

$$x_n^* \approx \sqrt{2(n+1) - (2(n+1))^{1/3}}.$$

Therefore the condition of A in Equation (3.7) can be updated, and we shall choose the parameter A in the following manner:

$$A = \max\left\{\sqrt{(2(N+1) - (2(N+1))^{1/3})}, \operatorname{erfcinv}\left(\frac{1}{N^{2s+1}\log_2^{d-1} N}\right)\right\}.$$

This value of A is very modest, so it will not force the quadrature method, which will be developed in the next chapter, to consume any unnecessary computational resources. However, it depends on N, s, and d and is defined to be sufficiently large to ensure that when the integral in Equation (3.1) is replaced with the integral in Equation (3.2) for every value of $\mathbf{n} \in \mathbb{L}_N^d$, then the resulting entire Fourier Hermite approximates a function f within N^{-s}.

Chapter 4

Discretization Part 2: Approximate the Body

There are several viable possibilities for approximating the integral

$$\int_{I^d} f(x)H_n(x)\omega(x)\mathrm{d}x$$

after the interval I has been determined. After briefly introducing a univariate quadrature method based on linear spline interpolation, I shall then discuss in depth a multivariate multiscale method based on piecewise polynomial interpolation.

Although the former method is easy to implement and provides elegant results, the latter is less specific and applies to a wider class of functions f of which to approximate. The multiscale method is the one adopted by the subsequent chapters of this dissertation, and it will be used in the development of algorithm and computational results to be presented.

4.1 Linear Spline Interpolation

One possibility is using a quadrature method based on linear spline interpolation. Let T be a natural number, and choose $T+1$ nodes inside $I = [-A, A]$.

$$-A = t_0 < t_1 < \cdots < t_T = A$$

Let B_i be the i^{th} linear B-spline defined by

$$B_i(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i} & \text{if } t_i \le x \le t_{i+1}, \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & \text{if } t_{i+1} \le x \le t_{i+2}, \\ 0 & \text{otherwise.} \end{cases}$$

The graph of a typical B_i is given in Figure 4.1. For $i = 0$ or $i = T$, B_i is defined with the obvious modifications.

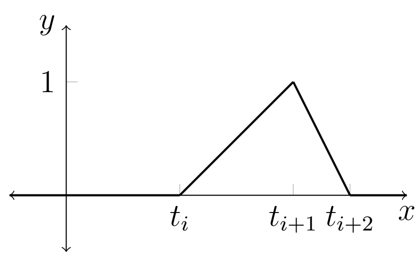


Figure 4.1: Plot of a typical $B_i(x)$.

The piecewise linear spline interpolant of the function f is defined by

$$S_f(x) := \sum_{i=0}^{T} f(t_i) B_i(x).$$

Denote the interpolation error by

$$E_f(x) := f(x) - S_f(x).$$

Let $|\cdot|_{\infty,I^d} : L^2_\omega(\mathbb{R}^d) \to \mathbb{R}$ be defined via

$$|f|_{\infty,I^d} := \sup_{x\in I^d} |f(x)|, \text{ for any } f \in L^2_\omega(\mathbb{R}^d).$$

From Theorem A.2, $|\cdot|_{\infty,I^d}$ defines a seminorm.

It is well-known [10, 15] that the linear spline interpolation error is by bounded by $\frac{\Delta^2}{8}|f''|_{\infty,I}$, where $\Delta = \max\{t_{i+1} - t_i | i = 0 : T-1\}$. Using this method, the 'middle' error $\varepsilon_{\text{middle}}$ as defined in the previous chapter takes the form

$$\varepsilon_{\text{middle}}(f,n) := \left| \int_I E_f(x) H_n(x) \omega(x)\, dx \right|. \tag{4.1}$$

Theorem 4.1. *Let $f \in L^2_\omega(\mathbb{R}^d) \cap C^2(I^d)$. For any $n \in \mathbb{L}^d_N$, there exists a positive constant C such that the quadrature error $\varepsilon_{middle}(f,n)$ as in Equation* (4.1) *with maximum step size Δ, is bounded above by*

$$\varepsilon_{middle} \le C\Delta^2\sqrt{n!2^n}|f''|_{\infty,I}.$$

Proof. First, the weighted L^2 norm of the interpolation error function will be bounded. In fact, it is observed that

$$\begin{aligned}
||E_f||_\omega^2 &= \int_{-\infty}^{\infty} E_f^2(x)\omega(x)\,dx \\
&\leq \int_{-\infty}^{\infty} \frac{\Delta^4}{64}|f''|_{\infty,I}^2\omega(x)\,dx \\
&= \frac{\Delta^4}{64}|f''|_{\infty,I}^2 \int_{-\infty}^{\infty} \omega(x)\,dx \\
&= \frac{\Delta^4\sqrt{\pi}}{64}|f''|_{\infty,I}.
\end{aligned}$$

This inequality is now used below.

$$\begin{aligned}
\varepsilon_{\text{middle}}(f,n) &= \left|\int_I E_f(x)H_n(x)\omega(x)\,dx\right| \\
&= |\langle E_f\chi_I, H_n\rangle_\omega| \\
&\leq ||E_f\chi_I||_\omega\,||H_n||_\omega \\
&\leq \Delta^2|f''|_{\infty,I}||H_n||_\omega \\
&= C\Delta^2\sqrt{n!2^n}|f''|_{\infty,I}.
\end{aligned}$$

Note that the constant C does not depend on n. □

Let $|\cdot|_{W^{2,\infty}(I^d)}$ denote the Sobolev semi-norm on $L^2_\omega(\mathbb{R}^d)\cap C^2(\mathbb{R}^d)$ defined via

$$|f|_{W^{2,\infty}(I^d)} := |f''|_{\infty,I^d}, \text{ for any } f \in L^2_\omega(\mathbb{R}^d)\cap C^2(\mathbb{R}^d).$$

Corollary 4.2. *Let $f \in L^2_\omega(\mathbb{R})\cap C^2(I)$. For any $n \in \mathbb{L}_N$, there exists a positive constant C*

such that for any fixed maximum step size Δ,

$$\sum_{n=0}^{N-1} \frac{\varepsilon^2_{middle}(f,n)}{n!2^n\sqrt{\pi}} \le C\Delta^4 N|f|^2_{W^{2,\infty}(I)}.$$

Proof. The proof is a direct consequence of Lemma 4.1.

$$\begin{aligned}\sum_{n=0}^{N-1} \frac{\varepsilon^2_{\text{middle}}(f,n)}{n!2^n\sqrt{\pi}} &\le \sum_{n=0}^{N-1} \frac{\left(C_1\Delta^2\sqrt{n!2^n}|f''|_{\infty,I}\right)^2}{n!2^n\sqrt{\pi}} \\ &= C\Delta^4|f''|^2_{\infty,I}\sum_{n=0}^{N-1} 1 \\ &= C\Delta^4 N|f''|^2_{\infty,I} \\ &= C\Delta^4 N|f|^2_{W^{2,\infty}(I)}.\end{aligned}$$

□

Using the piecewise linear spline method for obtaining a quadrature method in one dimension can give a good error estimate. Define an approximation to the Fourier-Hermite series of a function f as

$$\tilde{f}_N := \sum_{n\in\mathbb{L}_N} \tilde{c}_n H_n,$$

where the coefficients $\tilde{c}_n$ are found via truncation as in the previous chapter, and linear spline interpolation. I.e.

$$\tilde{c}_n := \frac{1}{||H_n||^2_\omega}\int_I S_f(x)H_n(x)\omega(x)\mathrm{d}x.$$

Theorem 4.3. *Let* $f \in L^2_\omega(\mathbb{R}) \cap C^2(I)$*. Fix a constant* $s > 0$*. Choose the parameter* A *as in Theorem 3.3, and choose the nodes* $\{t_i\}$ *such that* $\Delta \le N^{-(2s+1)/4}$*. There exists a positive* *constant C such that*

$$||f_N - \tilde{f}_N||_\omega \leq CN^{-s}\sqrt{||f||_\omega^2 + |f|_{W^{2,\infty}(I)}^2}.$$

Proof. Since $0 \leq (a-b)^2$ for any real values a and b, it follows that

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2$$
$$\Rightarrow a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$$
$$\Rightarrow (a+b)^2 \leq 2(a^2 + b^2).$$

Therefore, $(\varepsilon_{\text{tail}}(f,n) + \varepsilon_{\text{middle}}(f,n))^2 \leq 2(\varepsilon_{\text{tail}}^2(f,n) + \varepsilon_{\text{middle}}^2(f,n))$. Combine this with Theorem 3.2 to get

$$||f_N - \hat{f}_N||_\omega^2 \leq 2\left(\sum_{n=0}^{N-1} \frac{\varepsilon_{\text{tail}}^2(f,n)}{n!2^n\sqrt{\pi}} + \sum_{n=0}^{N-1} \frac{\varepsilon_{\text{middle}}^2(f,n)}{n!2^n\sqrt{\pi}}\right).$$

From Lemma 4.2 and Theorem 3.3, for some positive constants C_1 and C_2,

$$\begin{aligned} ||\tilde{f}_N - \hat{f}_N||_\omega^2 &\leq C_1 N^{-2s}||f||_\omega^2 + C_2\Delta^4 N|f|_{W^{2,\infty}(I)}^2 \\ &\leq C_1 N^{-2s}||f||_\omega^2 + C_2 N^{-2s}|f|_{W^{2,\infty}(I)}^2. \end{aligned}$$

Therefore, there exists a positive constant C such that

$$||\tilde{f}_N - \hat{f}_N||_\omega^2 \leq CN^{-2s}\left(||f||_\omega^2 + |f|_{W^{2,\infty}(I)}^2\right).$$

□

Now we are ready to present a theorem describing the error from replacing the exact infinitely expanded Fourier-Hermite series with the sparse-grid truncation computed with approximated coefficients.

Theorem 4.4. *Let $f \in L^2_\omega(\mathbb{R}) \cap C^2(I)$. Fix a constant $s > 0$. Choose the parameter A as in Theorem 3.3, and choose the nodes $\{t_i\}$ such that $\Delta \leq N^{-(2s+1)/4}$. There exists a positive constant C such that*

$$||f - \tilde{f}_N||_\omega \leq CN^{-s}\left(||f||_{\kappa^s} + \sqrt{||f||^2_\omega + |f|^2_{W^{2,\infty}(I)}}\right).$$

Proof. From the triangle inequality,

$$||f - \tilde{f}_N||_\omega \leq ||f - f_N||_\omega + ||f_N - \tilde{f}_N||_\omega.$$

Combine Theorems 2.13 and 4.3 to obtain the result

$$||f - \tilde{f}_N||_\omega \leq CN^{-s}\left(||f||_{\kappa^s} + \sqrt{||f||^2_\omega + |f|^2_{W^{2,\infty}(I)}}\right).$$

□

It is expected to have three contributions to the error since there are three steps to the approximation. First, the index set is truncated so that only finitely many terms of the Fourier-Hermite series are computed. Then the coefficients of those terms must be computed. To this end, a two-step method is introduced. The second contribution to the error is from throwing out the tails of the interval over which the integrals are computed. Finally, the third contribution to the error is from the quadrature method used on the remaining portion of the interval. These results are good, but using linear spline interpolation is not always an appropriate choice. In order for the above described method to work, the function f must be continuous and possess a smooth derivative of order at least two. A more general approach based on piecewise polynomial interpolation, which does not assume continuity, is developed in the next few sections.

4.2 Heirarchical Lagrange Interpolation

The ideas from this section can be found in [6], but are included here for completeness.

Another possibility for approximating the middle portion of the integral is using a quadrature method based on piecewise polynomial interpolation. To accommodate a fast algorithm, it is desirable for the piecewise polynomial interpolation to have a multiscale scheme. To achieve this, the interpolating polynomials will be related to a refinable set.

Choose two contractive mappings ϕ_0 and ϕ_1 on I defined by

$$\phi_0(x) := \frac{x-A}{2}, \quad \text{and} \quad \phi_1(x) := \frac{x+A}{2}.$$

Define $\Psi := \{\phi_0, \phi_1\}$. Then I is an invariant set relative to the mappings Ψ. A subset $V \subset I$ is called *refinable* relative to the mappings Ψ if $\Psi(V) \subset V$. For example, the following sets are refinable relative to Ψ:

$$\left\{\frac{-A}{3}, \frac{A}{3}\right\}, \left\{\frac{-A}{5}, \frac{A}{5}\right\}, \{-A, 0, A\}, \text{ and } \left\{\frac{-A}{7}, \frac{3A}{7}, \frac{5A}{7}\right\}.$$

Let V_0 be a refinable set relative to the mappings Ψ of size m. Say

$$V_0 = \{-A < v_{0,1} < v_{0,1} < \cdots < v_{0,m-1} < A\}.$$

Associated with V_0 are the Lagrange cardinal polynomials of degree $m-1$ defined by

$$\ell_{0,r}(x) := \prod_{q=0,q\neq r}^{m-1} \frac{x - v_{0,q}}{v_{0,r} - v_{0,q}}.$$

One may now approximate f by a combination of these cardinal polynomials. In effect, f would be approximated by its interpolating polynomial of degree $m-1$ with nodes in V_0. However, for a higher degree of accuracy, one may want to refine the sets and instead use a piecewise polynomial interpolation. This is a desirable method. To achieve this accuracy without sacrificing computational speed, a multiscale method will be developed.

Define a sequence of sets by $V_j := \Psi^j(V_0)$. It is necessary for each to be refinable.

Theorem 4.5. *Let $\Psi = \{\phi_i\}$ be a collection of contractible mappings, and let V_0 be a set that is refinable relative to Ψ. Then each $V_j = \Psi^j(V_0)$ is also refinable relative to Ψ.*

Proof. By assumption, $\Psi(V_0) \subset V_0$. Suppose $\Psi^j(V_0) \subset \Psi^{j-1}(V_0)$ for some $j \in \mathbb{N}_0$. Then

$$\Psi^{j+1}(V_0) = \Psi(\Psi^j(V_0)) \subset \Psi(\Psi^{j-1}(V_0)) = \Psi^j(V_0).$$

Therefore, by induction on j, each set $V_j := \Psi^j(V_0)$ is refinable. □

These refinable sets will be used as a foundation of the multiscale construction of the piecewise polynomial interpolation of f. The j^{th} layer of the multiscale piecewise polynomial will introduce information about f only on the set $V_j \setminus V_{j-1}$. The multiscale method is excellent for interpolating oscillatory functions. If a higher order of accuracy is needed, then it is easy to refine the interpolation. Only the finer details need to be added while keeping all of the work that has already been done.

Define two linear operators $\mathcal{T}_\rho : L^\infty(I) \to L^\infty(I)$ by

$$(\mathcal{T}_\rho f)(x) := \begin{cases} f \circ \phi_\rho^{-1}(x), & x \in \phi_\rho(I) \\ 0, & x \notin \phi_\rho(I) \end{cases}$$

for $\rho = 0, 1$. These linear operators will be used successively to define new spaces.

Let $F_0 = \text{span}\{\ell_{0,r} | r \in \{0, 1, \ldots, m-1\}\}$, and define

$$F_M := \mathcal{T}_0 F_{M-1} \oplus \mathcal{T}_1 F_{M-1}. \tag{4.2}$$

Since $F_{M-1} \subset F_M$ for each $M \in \mathbb{M}$, F_M may be decomposed as $F_M = F_{M-1} \oplus G_M$. Repeatedly apply this equation and let $G_0 = F_0$ to obtain

$$F_M = G_M \oplus G_{M-1} \oplus \cdots \oplus G_0. \tag{4.3}$$

This is the multiscale decomposition of F_M; to obtain F_M, one needs only to find the new basis functions that are not already included in F_{M-1}.

It is now necessary to describe the bases of F_M and G_M. First, consider how each cardinal function $\ell_{0,r}$ behaves under an application of $\mathcal{T}_0$ or $\mathcal{T}_1$. Both will yield a horizontal compression by a factor of 1/2, and $\mathcal{T}_0$ gives a horizontal shift of $-A/2$, $\mathcal{T}_1$ a horizontal shift of $A/2$. As the linear operators are successively applied, the effects are compounded. Let $\mathbb{P}_M = \{\rho_1, \rho_2, \ldots, \rho_M \mid \rho_i = 0 : 1\}$, and define $\mathcal{T}_{\mathbb{P}_M} = \mathcal{T}_{\rho_M}\mathcal{T}_{\rho_{M-1}} \cdots \mathcal{T}_{\rho_1}$. Then $F_M = \operatorname{span}\{\mathcal{T}_{\mathbb{P}_M}(\ell_{0,r}) : r \in \{0, 1, \ldots, m-1\}\}$. Let us rewrite these basis functions in a more manageable way. Let $\eta(\mathbb{P}_M) = \sum_{i=1}^{M} \rho_i 2^i$. Then the functions $\ell_{M,r}$ for $r = 0, 1, \ldots, 2^N m - 1$ form a basis for F_N, and can be obtained from the functions $\ell_{0,r'}$ by

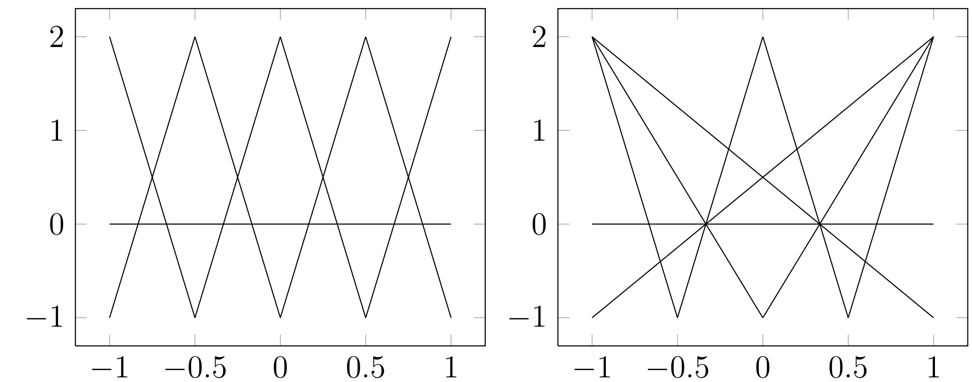
$$\ell_{M,r} = \mathcal{T}_{\mathbb{P}_M}\ell_{0,r'},\ r = m\eta(\mathbb{P}_M) + r'. \tag{4.4}$$

Plots of all of the basis functions in F_2 using the nodal form as in equation (4.2) and the multiscale form as in equation (4.3) are given in the Figures 4.2 and 4.3. Plots of the individual basis functions are given near the end of the chapter in Figures 4.4 and 4.5. Figure 4.2 uses $m = 2$ and the initial refinable set $V_0 = \left\{\frac{-A}{3}, \frac{A}{3}\right\}$, which is symmetric about the origin. For this choice of V_0, F_2 consists of the following Lagrange cardinal polynomials:

$$F_2 = \{\ell_{0,0}, \ell_{0,1}, \ell_{1,0}, \ell_{1,3}, \ell_{2,0}, \ell_{2,3}, \ell_{2,4}, \ell_{2,7}\}.$$

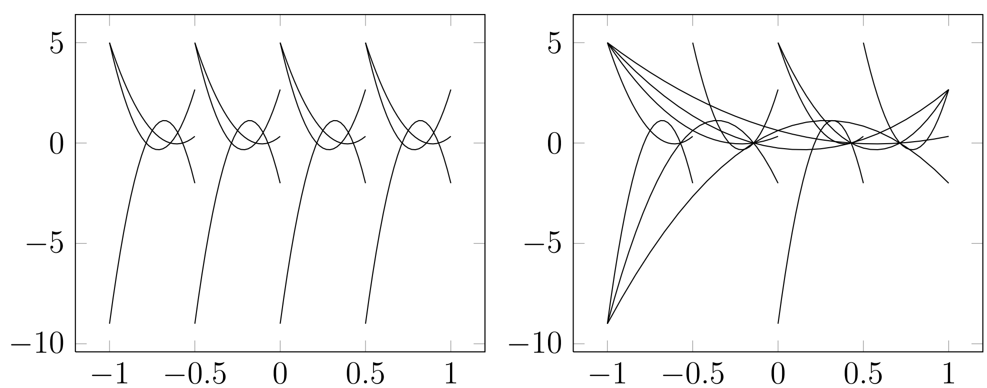
Figure 4.3 uses $m = 3$ and a skewed refinable set. Both use $A = 1$ for convenience.

The collection of all of the interpolation nodes used for the basis functions $\ell_{M,r}$ is exactly the set V_M. Because of the refinability of the sets V_M, many of the basis functions $\ell_{M,r}$



(a) Nodal basis (b) Multiscale basis

Figure 4.2: Nodal Basis and Multiscale Basis for F_2, taking $V_0 = \{-A/3, A/3\}$ and $A = 1$.



(a) Nodal basis of F_2 (b) Multiscale basis of F_2

Figure 4.3: Nodal Basis and Multiscale Basis for F_2, taking $V_0 = \left\{\dfrac{-A}{7}, \dfrac{3A}{7}, \dfrac{5A}{7}\right\}$ and $A = 1$.

can be obtained from combinations of the basis functions $\ell_{M-1,r'}$ of F_{M-1}. The rest of the basis functions for F_M are associated with the nodes in V_M but not in V_{M-1}. Define $W_j = \{r : v_{j,r} \in V_j \setminus V_{j-1}\}$ for $j \geq 1$ and $W_0 = \{0, 1, \ldots, m-1\}$. One can now write $G_j = \text{span}\{\ell_{j,r} : r \in W_j\}$.

4.3 Multiscale Piecewise Polynomial Interpolation

Applying a multiscale piecewise polynomial interpolation method to approximate the integral

$$\int_I f(x)H_n(x)\omega(x)\mathrm{d}x$$

is a new contribution to this area of study. The main result from this subsection is found in Theorem 4.6.

Define the interpolation projection $\mathcal{P}_M : C(I) \to F_M$ of a function $f : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{P}_M f := \sum_{r=0}^{2^M m-1} f(v_{M,r})\ell_{M,r}, \quad \text{for all } f \in C(I). \tag{4.5}$$

This is the 'nodal' interpolation operator. To develop the multiscale decomposition of the interpolation operator, define the difference operator as $\mathcal{Q}_j := \mathcal{P}_j - \mathcal{P}_{j-1}$ for $j \geq 1$, and $\mathcal{Q}_0 = \mathcal{P}_0$. Then the interpolation projection can be written as $\mathcal{P}_M f = \sum_{j=0}^{M} \mathcal{Q}_j f$, for all $f \in C(I)$. Notice that the operators Q_j are associated with the Lagrange cardinal functions with nodes in the sets W_j. These basis functions enjoy the property $\ell_{i,j}(v_{i,j}) = 1$. However, due to scaling, $\ell_{i,j}(v_{k,l})$ is not always zero if $(i,j) \neq (k,l)$. (See figure 4.2.) Therefore, to write an explicit formula for each $\mathcal{Q}_j f$, a new functional must be introduced taking this property into account. Suppose one wanted to write $\mathcal{P}_M$ and each $\mathcal{Q}_j$ as

$$\mathcal{Q}_j f = \sum_{r \in W_j} \eta_{j,r}(f)\ell_{j,r} \tag{4.6}$$

and

$$\mathcal{P}_M f = \sum_{j=0}^{M} \sum_{r \in W_j} \eta_{j,r}(f)\ell_{j,r}. \tag{4.7}$$

Then it is proven in [12] that $\eta_{j,r}$ must be defined via

$$\eta_{0,r}(f) = f(v_{0,r}), \text{ and } \eta_{j,r}(f) = f(v_{j,r}) - \sum_{q=0}^{m-1} f\left(v_{j-1,m\lfloor \frac{r}{2m} \rfloor + q}\right) a_{q,r \mod 2m}, \tag{4.8}$$

for $j \neq 0$, where $a_{q,k} = \ell_{0,q}(v_{1,k})$.

The formula in equation (4.7) is the multiscale piecewise polynomial interpolation of f. It interpolates f exactly on each node $v_{M,r}$ for $r \in \mathbb{Z}_{2^M m}$, and it defines a polynomial of degree $m-1$ on each subinterval $I_{j,r}$, where $I_{j,i} := [-A + i2^{-j+1}A,\ -A + (i+1)2^{-j+1}A]$ for each $j = 0, 1, \ldots, M$ and $i \in \mathbb{Z}_{2^j}$.

Recall that the goal is to replace f in the following integral

$$\int_I f(x)H_n(x)\omega(x)\, dx \tag{4.9}$$

by the piecewise polynomial to obtain the approximation

$$\int_I (\mathcal{P}_M f)(x)H_n(x)\omega(x)\, dx. \tag{4.10}$$

Define a space that consists of all functions that are smooth on each subinterval $I_{j,i}$ via

$$X_M^m(I) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ such that } f^{(\alpha)} \in C(I_{M,i}), \forall i \in \mathbb{Z}_{2^M}, \alpha \leq m \right\}, \tag{4.11}$$

with seminorm

$$|f|_{X_M^m(I)} := \max\left\{ |f^{(\alpha)}|\big|_{I_{M,i}} : \alpha \leq m, i \in \mathbb{Z}_{2^M} \right\}.$$

Notice functions in $X_M^m(I)$ are not necessarily continuous, only piecewise continuous.

Theorem 4.6. *Fix $M \in \mathbb{N}_0$. For all $f \in L^2_\omega(\mathbb{R}) \cap X_M^m(I)$, there exists a positive constant C such that the error in estimating the integral in equation (4.9) by the integral in (4.10) can be bounded by*

$$\left| \int_I \left(f - \mathcal{P}_M f\right)(x) H_n(x) \omega(x) \, \mathrm{d}x \right| \leq C \frac{A^m 2^{-mM+n/2}\sqrt{n!}\, |f|_{X_M^m(I)}}{(m-1)!}.$$

Proof. On each subinterval $I_{M,i}$, $\mathcal{P}_M f$ is a polynomial of degree $m-1$. Fix $i \in \mathbb{Z}_{2^M}$ and let $x \in I_{M,i}$. Then there exists some constant $\xi_x \in I_{M,i}$ such that

$$\begin{aligned}
|(\mathcal{P}_M f)(x) - f(x)| &= \frac{1}{(m-1)!} \left|f^{(m)}(\xi_x)\right| \cdot \prod_{j=0}^{m-1} \left|x - v_{M,2^i m+j}\right| \\
&\leq \frac{|f|_{X_M^m(I)}}{(m-1)!} \prod_{j=0}^{m-1} \left|x - v_{M,2^i m+j}\right| \\
&\leq \frac{|f|_{X_M^m(I)}}{(m-1)!} \prod_{j=0}^{m-1} |I_{M,i}| \\
&\leq (2^{-M+1}A)^m \frac{|f|_{X_M^m(I)}}{(m-1)!}
\end{aligned}$$

Now insert this estimate into the integral.

$$\begin{aligned}\left|\int_I \left(f - \mathcal{P}_M f\right)(x) H_n(x)\omega(x)\,\mathrm{d}x\right| &= \left|\sum_{i=0}^{2^M-1} \int_{I_{M,i}} \left(f - \mathcal{P}_M f\right)(x) H_n(x)\omega(x)\,\mathrm{d}x\right| \\ &\leq \sum_{i=0}^{2^M-1} \frac{A^m |f|_{X_M^m(I)}}{2^{m(M+1)}(m-1)!} \int_{I_{M,i}} |H_n(x)|\,\omega(x)\,\mathrm{d}x \\ &= \frac{A^m |f|_{X_M^m(I)}}{2^{m(M+1)}(m-1)!} \int_I |H_n(x)|\,\omega(x)\,\mathrm{d}x\end{aligned}$$

The remaining integral can be estimated using Jensen's inequality.

$$\int_I |H_n(x)|\,\omega(x)\,dx \leq \left(\int_I H_n^2(x)\omega(x)\,dx\right)^{1/2} \leq \sqrt{n!2^n\sqrt{\pi}}.$$

Therefore one can write for a positive constant $C = \pi^{1/4}$,

$$\left|\int_I \left(f - \mathcal{P}_M f\right)(x) H_n(x)\omega(x)\,\mathrm{d}x\right| \leq C\frac{A^m 2^{-mM+n/2}\sqrt{n!}}{(m-1)!} |f|_{X_M^m(I)} .$$

□

Since A was chosen so that the bulk of each $f(x)H_n(x)\omega(x)$ occurs in the interval I, a quadrature method based on the piecewise polynomial interpolation gives a good approximation even over all of $\mathbb{R}$.

Corollary 4.7. *Fix $M \in \mathbb{N}_0$. Fix a positive integer m, and choose some $0 < s < m$. Let A be chosen as in Theorem 3.3. For all $f \in L^2_\omega(\mathbb{R}) \cap X_M^m(I)$, there exists a positive constant C*

such that

$$\left| \int_{\mathbb{R}} \left(f - \mathcal{P}_M f\right)(x) H_n(x) \omega(x) \, \mathrm{d}x \right| \leq C \left(N^{-s} ||f||_\omega + \frac{A^m 2^{-mM+n/2} \sqrt{n!}}{(m-1)!} |f|_{X_M^m(I)} \right).$$

Proof. Each of the cardinal functions $\ell_{M,i}$ is identically zero outside of the interval I. Therefore $\mathcal{P}_M f(x) = 0$ for all $x \in \mathbb{R} \setminus I$. Combine the results of Theorems 3.3 and 4.6. □

An observation about the preceding Corollary is in order. Supposing A, N, and m are fixed, in order to achieve a decent discrete approximation as in (4.9), one must choose an appropriately large value for M. This is expected though; the refinement of subintervals must be sufficiently small for an acceptable approximation of f. Guidelines for choosing M are formalized in the next chapter.

Now that the theory has been developed for the discrete sparse Fourier Hermite approximation based on multiscale piecewise polynomial interpolation in one dimension, it can extended to higher dimensions by the use of the tensor product. The following section does this.

4.4 High Dimension Discrete Sparse Fourier Hermite Approximations

The formulae in the preceding section can be extended to a high dimensional setting by use of the tensor product. This section contains several important theorems which will facilitate the analysis of the discrete sparse Fourier Hermite approximation of a function, which is defined in Definition 4.13.

The d dimensional Lagrange polynomials and the d dimensional functionals η are given by

$$\ell_{N,\mathbf{r}} := \ell_{N,r_1} \otimes \ell_{N,r_2} \otimes \cdots \otimes \ell_{N,r_d},$$

$$\ell_{\mathbf{j},\mathbf{r}} := \ell_{j_1,r_1} \otimes \ell_{j_2,r_2} \otimes \cdots \otimes \ell_{j_d,r_d},$$

$$\eta_{\mathbf{j},\mathbf{r}} := \eta_{j_1,r_1} \otimes \eta_{j_2,r_2} \otimes \cdots \otimes \eta_{j_d,r_d},$$

for any $M \in \mathbb{N}$ and any $\mathbf{j}, \mathbf{r} \in \mathbb{N}_0^d$. Let $W_M^d = W_M \times W_M \times \cdots \times W_M$, and $\mathbf{v}_{M,\mathbf{r}} = (v_{N,r_1}, v_{N,r_2}, \ldots, v_{N,r_d}) \in W_N^d$. We define $\mathcal{P}_M^d := \otimes_{k=1}^d \mathcal{P}_M$.
Notice that $\mathcal{P}_M^d f = \sum_{\mathbf{v}_{M,\mathbf{r}} \in V_M^d} f(\mathbf{v}_{M,\mathbf{r}}) \ell_{M,\mathbf{r}}$. This is the full grid interpolation projection of f onto F_M^d. Also define $\mathcal{Q}_\mathbf{j} := \mathcal{Q}_{j_0} \otimes \mathcal{Q}_{j_1} \otimes \cdots \otimes \mathcal{Q}_{j_d}$. For the multiscale interpolation, use the high dimensional tensor product version of equations (4.6) and (4.7). It is proven in [12] that

$$\mathcal{P}_M^d f = \sum_{\mathbf{j} \in \mathbb{Z}_{M+1}^d} \sum_{\mathbf{r} \in W_\mathbf{j}} \eta_{\mathbf{j},\mathbf{r}}(f) \ell_{\mathbf{j},\mathbf{r}}, \tag{4.12}$$

and

$$\mathcal{Q}_\mathbf{j} f = \sum_{\mathbf{r} \in W_\mathbf{j}} \eta_{\mathbf{j},\mathbf{r}}(f) \ell_{\mathbf{j},\mathbf{r}}. \tag{4.13}$$

It follows immediately that the tensor product multiscale Lagrange interpolation of f can be written as

$$\mathcal{P}_M^d f = \sum_{\mathbf{j} \in \mathbb{Z}_{M+1}^d} \mathcal{Q}_\mathbf{j} f.$$

To reduce computational cost, I shall again use the ideas from Section 2.2, and compute the sum over a hyperbolic cross index set. For a fixed positive integer M, consider the sparse grid index set $\mathbb{S}_M^d$ defined by

$$\mathbb{S}_M^d := \left\{ \mathbf{j} \in \mathbb{Z}_{M+1}^d : \sum_{k=1}^{d} j_k \leq M+1 \right\}.$$

Definition 4.8. *For a function $f : \mathbb{R}^d \to \mathbb{R}^d$, the* multiscale sparse grid interpolation *of f is given by*

$$\mathcal{S}_M^d f := \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in W_{\mathbf{j}}} \eta_{\mathbf{j},\mathbf{r}}(f) \ell_{\mathbf{j},\mathbf{r}}. \tag{4.14}$$

Note that $\mathcal{S}_M^d f = \sum_{\mathbf{j} \in \mathbb{S}_M^d} \mathcal{Q}_{\mathbf{j}} f$. In higher dimensions, the multiscale sparse grid interpolant $\mathcal{S}_M^d$ will be used instead of $\mathcal{P}_M^d$. To establish the approximation order, first some definitions and lemmas are needed.

For any $j \in \mathbb{S}_M$ and $i \in \mathbb{Z}_{2^j}$, define $I_{j,i} = -A + i2^{-j+1}A$.

Define the collection of sets

$$V_{\mathbf{j}} := \Psi^{j_1}(V_0) \times \Psi^{j_2}(V_0) \times \cdots \times \Psi^{j_d}(V_0),$$

for any $\mathbf{j} \in \mathbb{N}_0^d$. For any $\mathbf{j} \in \mathbb{S}_M^d$ and $\mathbf{i} \in \mathbb{Z}_{2^{\mathbf{j}}}^d$, define $\Omega_{\mathbf{j},\mathbf{i}} := \otimes_{k=1}^d [I_{j_k,i_k}, I_{j_k,i_k+1}]$.

The operator $\mathcal{Q}_{\mathbf{j}}$ is bounded. The following theorem appears in [12].

Theorem 4.9. *There exists a positive constant c such that for all $f \in C(I^d)$ and $\mathbf{j} \in \mathbb{N}_0^d$,*

$$||\mathcal{Q}_{\mathbf{j}} f||_\infty \leq c||f||_\infty \tag{4.15}$$

Extend the space $X_M^m(I)$ to higher dimensions via the following:

$$X_M^m(I^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f^{(\alpha)}|_{\Omega_{\mathbf{j},\mathbf{i}}} \in C(\Omega_{\mathbf{j},\mathbf{i}}),\ \forall \mathbf{j} \in \mathbb{S}_M^d, \mathbf{r} \in W_{\mathbf{j}},\ |\alpha|_\infty \leq m \right\}, \tag{4.16}$$

with seminorm

$$|f|_{X_M^m(I^d)} := \max \left\{ |f^{(\alpha)}|_{\infty,\Omega_{\mathbf{j},\mathbf{i}}} : \alpha \in \mathbb{N}_0^d, |\alpha|_\infty \leq m, \forall \mathbf{j} \in \mathbb{S}_M^d, \mathbf{r} \in W_{\mathbf{j}} \right\}.$$

Then the operators $\mathcal{Q}_{\mathbf{j}}$ are bounded for f in this new space as shown in the next lemma.

Lemma 4.10. *For fixed d and M in $\mathbb{N}$, there exists a positive constant c such that for all $f \in X_M^m(I^d)$ and $\mathbf{j} \in \mathbb{N}_0^d$ with $|\mathbf{j}| > 0$,*

$$||\mathcal{Q}_{\mathbf{j}} f||_\infty \leq c \left(\frac{A}{2} \right)^m 2^{-m|\mathbf{j}|} |f|_{X_M^m(I^d)}. \tag{4.17}$$

The proof closely follows that of Lemma 2.10 in [12].

With these tools, now the multiscale sparse grid Lagrange interpolation error can be investigated.

Theorem 4.11. *For fixed positive integers d and M, there exists a positive constant c such that for all $f \in X_M^m(I^d)$,*

$$\left|\left| \left(\mathcal{P}_M^d - \mathcal{S}_M^d \right) f \right|\right|_\infty \leq c A^m M^{d-1} 2^{-m(M+d)} |f|_{X_M^m(I^d)}.$$

Proof. It follows immediately from (4.12), (4.14), and (4.13) that

$$\left(\mathcal{P}_M^d - \mathcal{S}_M^d\right) f = \sum_{\mathbf{j}\in\mathbb{Z}_{M+1}^d \setminus \mathbb{S}_M^d} \mathcal{Q}_{\mathbf{j}} f.$$

Therefore

$$\begin{aligned} || \left(\mathcal{P}_M^d - \mathcal{S}_M^d\right) f||_\infty &\leq \sum_{\mathbf{j}\in\mathbb{Z}_{M+1}^d \setminus \mathbb{S}_M^d} ||\mathcal{Q}_{\mathbf{j}} f||_\infty \\ &\leq cA^m 2^{-md} ||f||_{X_M^m(I^d)} \sum_{\mathbf{j}\in\mathbb{Z}_{M+1}^d \setminus \mathbb{S}_M^d} 2^{-m|\mathbf{j}|} \\ &\leq cA^m M^{d-1} 2^{-m(M+d)} ||f||_{X_M^m(I^d)}. \end{aligned}$$

The last inequality follows from the estimate $\sum_{\mathbf{j}\in\mathbb{Z}_{M+1}^d \setminus \mathbb{S}_M^d} 2^{-m|\mathbf{j}|} \leq cM^{d-1}2^{-mM}$ from Lemma 3.7 in [5]. □

Now this estimate can be used to bound the difference between f and its sparse multiscale interpolation $\mathcal{S}_M^d f$.

Theorem 4.12. *For fixed positive integers d and M, there exists a positive constant c such that for all functions defined on I^d with the property $f \in X_M^m(I^d)$,*

$$\left|\left|f - \mathcal{S}_M^d f\right|\right|_{\infty, I^d} \leq cA^m M^{d-1} 2^{-m(M+d)} |f|_{X_M^m(I^d)}.$$

Proof. Since $\mathcal{P}_M^d f$ is precisely piecewise polynomial interpolation of f, there exists a positive constant c such that

$$||f - \mathcal{P}_M^d f||_{\infty,I^d} \leq c2^{-mM}|f|_{X_M^m(I^d)}.$$

Now, by the triangle inequality

$$||f - \mathcal{S}_M^d f||_{\infty,I^d} \leq ||f - \mathcal{P}_M^d f||_{\infty,I^d} + ||\mathcal{P}_M^d f - \mathcal{S}_M^d f||_{\infty},$$

and then by substituting the above result with the one from Theorem 4.11 yields the desired result. □

Now equipped with facts about $\mathcal{S}_M^d f$, we are ready to define the *discrete sparse Fourier Hermite approximation of f.*

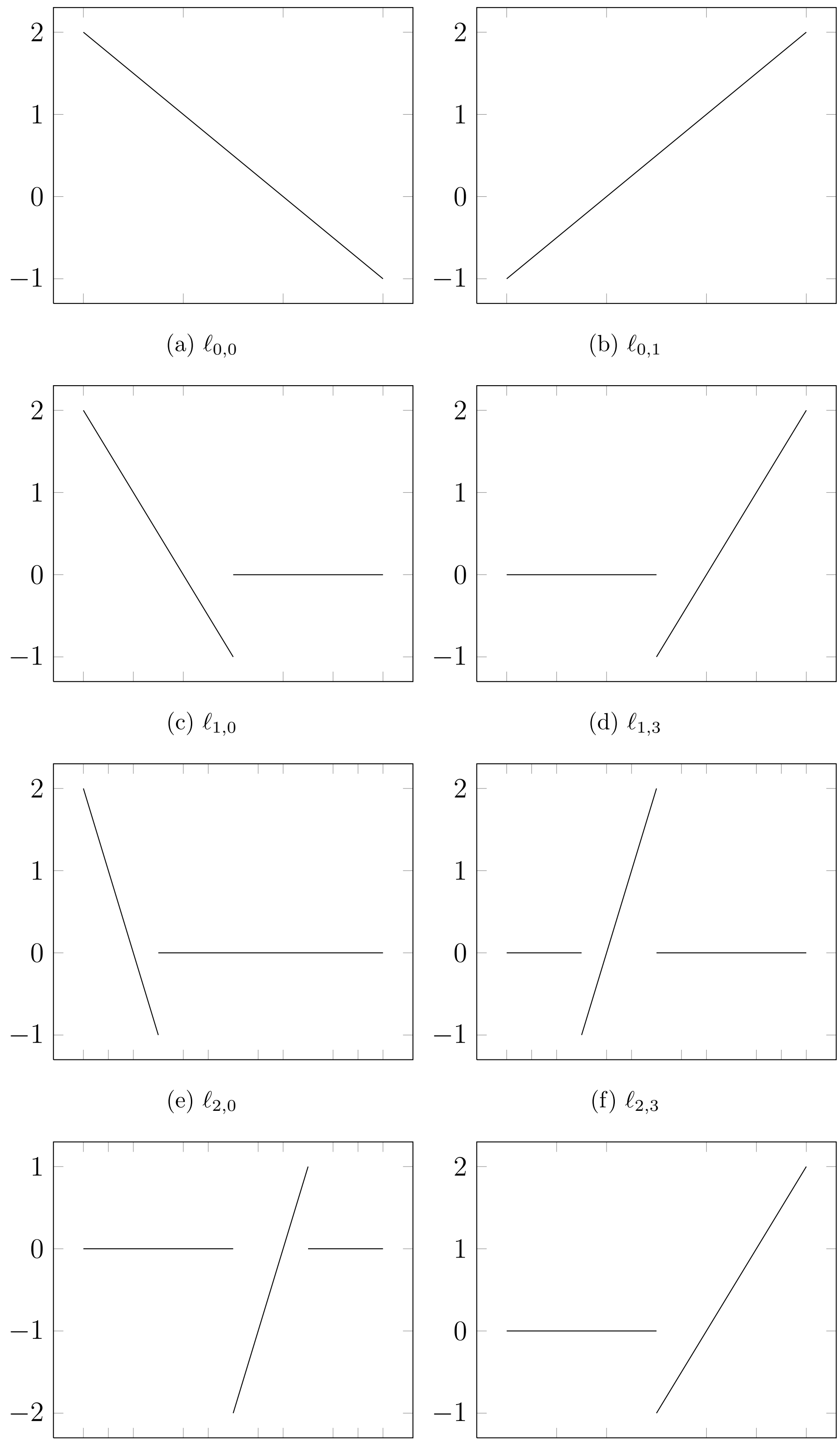
Definition 4.13. *For fixed positive integers M, N, and m, and for any function $f \in \mathbb{L}_\omega^2(\mathbb{R}^d) \cap X_M^m(I^d)$, the* discrete sparse Fourier Hermite approximation of f *is given by*

$$\check{f}_{M,N} := \sum_{\mathbf{n}\in\mathbb{L}_N^d} \check{c}_{M,\mathbf{n}} H_{\mathbf{n}},$$

where the discrete coefficients can be found via the formula

$$\check{c}_{M,\mathbf{n}} := \frac{\langle \mathcal{S}_M^d f, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}}. \tag{4.18}$$

Developing framework that leads to and supports the discrete sparse Fourier Hermite approximation has been the main purpose of this paper so far. From here out, the discussion will focus on its approximation analysis, numerical implementation, and use in applications.



(a) $\ell_{0,0}$ (b) $\ell_{0,1}$

(c) $\ell_{1,0}$ (d) $\ell_{1,3}$

(e) $\ell_{2,0}$ (f) $\ell_{2,3}$

(g) $\ell_{2,4}$ (h) $\ell_{2,7}$

Figure 4.4: Plots of all of the Lagrange cardinal polynomials in F_2, where $V_0 = \{-1/3, 1/3\}$.

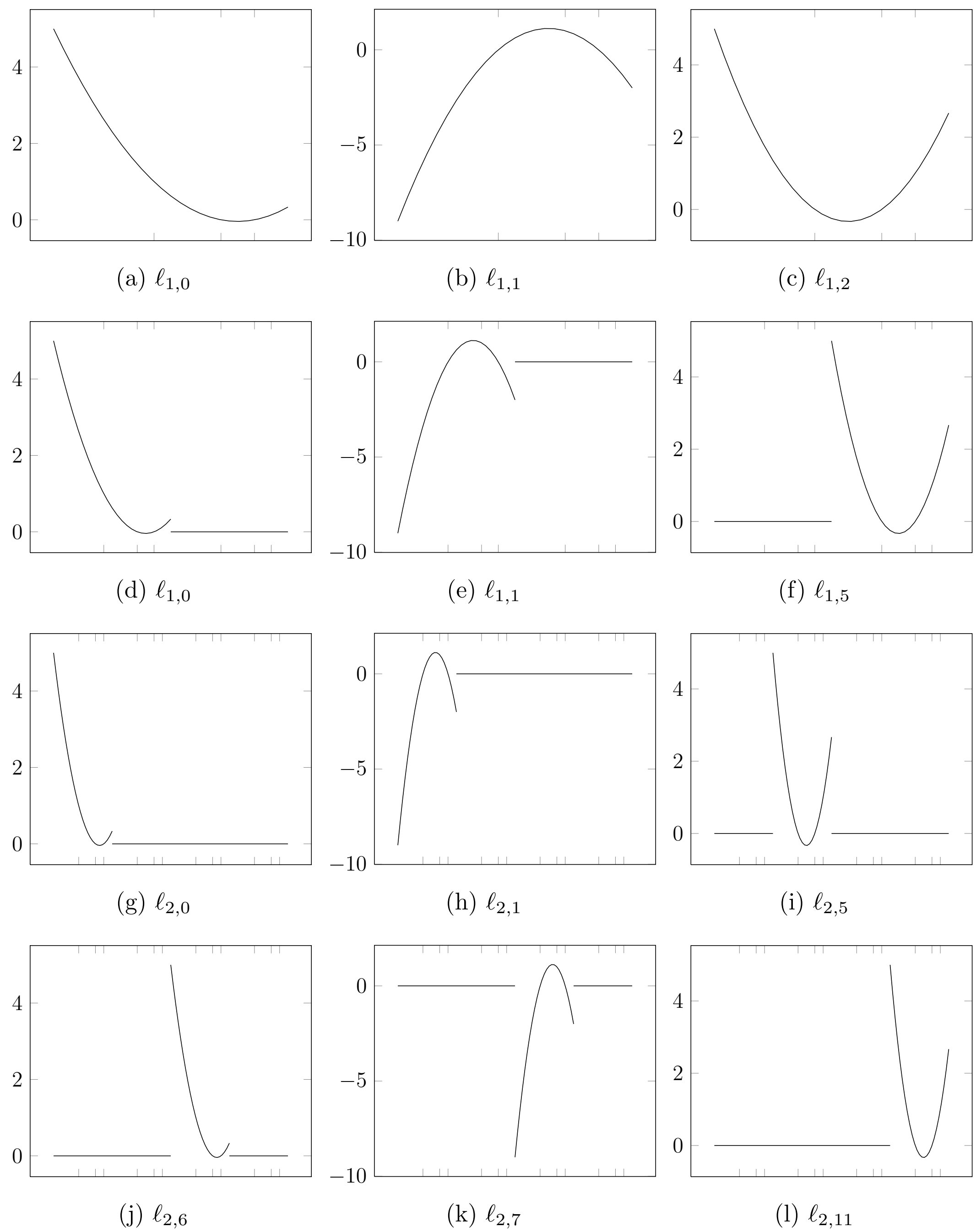


Figure 4.5: Plots of all of the Lagrange cardinal polynomials in F_2, where $V_0 = \left\{\frac{-1}{7}, \frac{3}{7}, \frac{5}{7}\right\}$.

Chapter 5

Discrete Sparse Fourier Hermite Approximations

In the previous chapter, I focused on approximating the coefficients of the Fourier Hermite series using a quadrature method featuring truncation and piecewise polynomial interpolation. Before delving into computational issues in approximating the coefficients, I shall verify that they do not contribute too much error to the overall approximated series.

Fix two positive integers M and N. The integer N will determine the number of terms appearing in the sparse representation of the Fourier Hermite series, and the integer M will determine the number of piecewise polynomial segments used for the quadrature method to approximate the coefficients.

Definition 5.1. *The* discrete sparse Fourier Hermite approximation *of a given function*

$f \in \mathcal{H}_s(\mathbb{R}^d) \cap X_M^m(I^d)$ *is given by*

$$\check{f}_{M,N} := \sum_{\mathbf{n} \in \mathbb{L}_N^d} \check{c}_{M,\mathbf{n}} H_{\mathbf{n}},$$

where the coefficients $\check{c}_{M,\mathbf{n}}$ *are computed using the methods described in the previous chapter. That is,*

$$\check{c}_{M,\mathbf{n}} = \frac{1}{||H_{\mathbf{n}}||_\omega^2} \int_{I^d} \mathcal{S}_M^d f(\mathbf{x}) H_{\mathbf{n}}(\mathbf{x}) \omega(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

It is of course desirable for $\check{f}_{M,N}$ to approximate the Fourier Hermite series of f well. There are three main sources of error: from using a sparse index set, from truncating the intervals over which the integrals for the coefficients are computed, and from using a quadrature method to compute the integral over the remaining interval. To simplify notation when considering the error, denote the sparse Fourier Hermite series that do not have the third source of error by $\check{f}_N$, with Fourier coefficients $\check{c}_{\mathbf{n}}$. That is,

$$\check{f}_N := \sum_{\mathbf{n} \in \mathbb{L}_N^d} \check{c}_{\mathbf{n}} H_{\mathbf{n}},$$

where

$$\check{c}_{\mathbf{n}} := \frac{1}{||H_{\mathbf{n}}||_\omega^2} \int\limits_{I^d} f(x) H_{\mathbf{n}}(x) \omega(x) \, \mathrm{d}x.$$

Using the triangle inequality, the error between f and $\check{f}_{M,N}$ can be bounded by

$$||f - \check{f}_{M,N}||_\omega \leq ||f - f_N||_\omega + ||f_N - \check{f}_N||_\omega + ||\check{f}_N - \check{f}_{M,N}||_\omega. \tag{5.1}$$

The error of the first term was discussed in Chapter 2. The error of the second and third terms will depend on the parameters A, m and M. These parameters will need to be chosen in such a way that each term in (5.1) contributes approximately the same order of error. The purpose of this chapter is to show how these parameters can be chosen in a way that will ensure the total approximation error is small.

Recall from Theorem 2.13 that for any $f \in \mathcal{H}_s(\mathbb{R}^d) \subset L^2_\omega(\mathbb{R}^d)$ with $s > 0$,

$$||f - f_N||_\omega \leq N^{-s}||f||_{\kappa^s}. \tag{5.2}$$

The following two lemmas bound the error from the second norm appearing in equation (5.1)

Lemma 5.2. *For any subset S of $\mathbb{R}^d$, let χ_S denote the characteristic function that is equal to 1 on S and 0 everywhere else. Define the interval $I := [-A, A]$, where the parameter A is chosen as in Theorems 3.3 and 3.4. Let $f \in L^2_\omega(\mathbb{R}^d)$. For a fixed positive integer N,*

$$||f_N - \check{f}_N||_\omega \leq ||f||_\omega \left(\sum_{\mathbf{n} \in \mathbb{L}^d_N} \frac{||H_\mathbf{n}\chi_{\mathbb{R}^d \setminus I^d}||^2_\omega}{||H_\mathbf{n}||^2_\omega} \right)^{1/2}.$$

Proof. By the definitions of f_N and $\check{f}_N$, $||f_N - \check{f}_N||^2_\omega = \sum_{\mathbf{n} \in \mathbb{L}^d_N} (c_\mathbf{n} - \check{c}_\mathbf{n})^2 \, ||H_\mathbf{n}||^2_\omega$.

The difference between the exact and truncated coefficients can be bounded by

$$\begin{aligned} |c_\mathbf{n} - \check{c}_\mathbf{n}| &= \frac{1}{||H_\mathbf{n}||^2_\omega} \left| \int_{\mathbb{R}^d \setminus I^d} f(x) H_\mathbf{n}(x) \omega(x) \, \mathrm{d}x \right| \\ &= \frac{1}{||H_\mathbf{n}||^2_\omega} \left| \langle f, H_\mathbf{n}\chi_{(\mathbb{R}^d \setminus I^d)} \rangle_\omega \right| \\ &\leq \frac{||f||_\omega ||H_\mathbf{n}\chi_{(\mathbb{R}^d \setminus I^d)}||_\omega}{||H_\mathbf{n}||^2_\omega}. \end{aligned}$$

Therefore $||f_N - \check{f}_N||_\omega^2 \le ||f||_\omega^2 \sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{||H_\mathbf{n}\chi_{(\mathbb{R}^d\setminus I^d)}||_\omega^2}{||H_\mathbf{n}||_\omega^2}$. □

Lemma 5.3. *Let $f \in \mathcal{H}_s(\mathbb{R}^d)$, with $0 < s$, and choose the parameter A as in Theorems 3.3 and 3.4. Then there exists a positive constant c such that*

$$||f_N - \check{f}_N||_\omega \le CN^{-s}||f||_\omega. \tag{5.3}$$

Proof. Notice that $\langle f, H_n\chi_{(\mathbb{R}^d\setminus I^d)}\rangle_\omega = \varepsilon_{\text{tail}}(f, n)$. Then apply Theorem 3.3. □

Finally, the third term in equation (5.1) is bounded as in the following lemma.

Lemma 5.4. *Let $f \in L^2_\omega(\mathbb{R}^d) \cap X_M^m(I^d)$. Choose the parameter A as described above. For fixed $M, N \in \mathbb{N}_0$,*

$$||\check{f}_N - \check{f}_{M,N}||_\omega \le C||f - \mathcal{S}_M^d f||_{\infty, I^d} \left(\sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{||H_\mathbf{n}\chi_{I^d}||_\omega^2}{||H_\mathbf{n}||_\omega^2}\right)^{1/2}.$$

Proof. The expansions of $\check{f}_N$ and $\check{f}_{M,N}$ differ only in their coefficients:

$$||\check{f}_N - \check{f}_{M,N}||_\omega^2 = \left|\left|\sum_{\mathbf{n}\in\mathbb{L}_N^d} (\check{c}_\mathbf{n} - \check{c}_{M,\mathbf{n}})H_\mathbf{n}\right|\right|_\omega^2 = \sum_{\mathbf{n}\in\mathbb{L}_N^d} |\check{c}_\mathbf{n} - \check{c}_{M,\mathbf{n}}|^2||H_\mathbf{n}||_\omega^2.$$

By the definitions of $\check{c}_\mathbf{n}$ and $\check{c}_{M,\mathbf{n}}$,

$$|\check{c}_\mathbf{n} - \check{c}_{M,\mathbf{n}}| = \frac{1}{||H_\mathbf{n}||_\omega^2}\left|\int_{I^d} \left(f - \mathcal{S}_M^d f\right)(x)H_\mathbf{n}(x)\omega(x)\,\mathrm{d}x\right|$$

$$= \frac{\left|\langle (f - \mathcal{S}_M^d f)\chi_{I^d}, H_\mathbf{n}\chi_{I^d}\rangle_\omega\right|}{||H_\mathbf{n}||_\omega^2}.$$

Then, by Cauchy Schwarz, the numerator of the previous quantity is bounded above by $||(f - \mathcal{S}_M^d f)\chi_{I^d}||_\omega ||H_{\mathbf{n}}\chi_{I^d}||_\omega$. Finally,

$$\begin{aligned} ||(f - \mathcal{S}_M^d f)\chi_{I^d}||_\omega^2 &= \int_{I^d} (f - \mathcal{S}_M^d f)^2(x)\omega(x)\ \mathrm{d}x \\ & ||f - \mathcal{S}_M^d f||_{\infty, I^d}^2 \int\limits_{I^d} \omega(x)\ \mathrm{d}x \\ &\leq C||f - \mathcal{S}_M^d f||_{\infty, I^d}^2, \end{aligned}$$

for some positive constant C. □

Theorem 5.5. *Suppose $f \in \mathcal{H}_s(\mathbb{R}^d) \cap X_M^m(I^d)$ such that $0 < s < m$. Let A be chosen as in Theorems 3.3 and 3.4. Choose k such that*

$$A \leq \frac{C \log_2^k N}{k^{(d-1)/m} N^{1+1/m}},$$

for some positive constant C, and set $M = \lceil \log_2(\log_2^k N) \rceil$.

Then

$$||\check{f}_N - \check{f}_{M,N}||_\omega \leq C N^{-s} |f|_{X_M^m(I^d)}.$$

Proof. Let ε be chosen such that $s + \varepsilon \leq m$.

From Lemma 5.4 and Theorem 4.12,

$$\begin{aligned}
||\check{f}_N - \check{f}_{M,N}||_\omega &\le C||f - \mathcal{S}_M^d f||_{\infty,I^d} \left(\sum_{\mathbf{n}\in\mathbb{L}_N^d} \frac{||H_\mathbf{n}\chi_{I^d}||_\omega^2}{||H_\mathbf{n}||_\omega^2} \right)^{1/2} \\
&\le C||f - \mathcal{S}_M^d f||_{\infty,I^d} \left(\sum_{\mathbf{n}\in\mathbb{L}_N^d} 1 \right)^{1/2} \\
&\le C\sqrt{N}(\log_2^{\frac{d-1}{2}} N)||f - \mathcal{S}_M^d f||_{\infty,I^d} \\
&\le C\sqrt{N}(\log_2^{\frac{d-1}{2}} N)A^m M^{d-1} 2^{-mM} |f|_{X_M^m(I^d)}.
\end{aligned}$$

Since $M \ge \log_2(\log_2^k N)$, it follows that $2^{-mM} \le \log_2^{-mk} N$. Therefore

$$\begin{aligned}
||\check{f}_N - \check{f}_{M,N}||_\omega &\le C\sqrt{N}(\log_2^{\frac{d-1}{2}} N)A^m M^{d-1} 2^{-mM} |f|_{X_M^m(I^d)} \\
&\le C\sqrt{N}(\log_2^{\frac{d-1}{2}} N)A^m k^{d-1}(\log_2^{d-1}\log_2 N)(\log_2^{-mk} N)|f|_{X_M^m(I^d)} \\
&\le C\left(\log_2^{3(d-1)/2} N\right) \frac{\sqrt{N}k^{d-1}}{\log_2^{mk} N} A^m |f|_{X_M^m(I^d)}.
\end{aligned}$$

Then using the condition on A and the assumption on m gives the result

$$\begin{aligned}
||\check{f}_N - \check{f}_{M,N}||_\omega &\le C\left(\log_2^{d-1} N\right) N^{-m} |f|_{X_M^m(I^d)} \\
&\le C\left(\log_2^{d-1} N\right) N^{-(m+\varepsilon)} |f|_{X_M^m(I^d)} \\
&\le CN^{-s}|f|_{X_M^m(I^d)}.
\end{aligned}$$

□

It seems natural that the number of refinements of the multiscale interpolation method used to help compute integrals over the interval I^d should depend on the size of I^d. This is why it is necessary for the parameter k, which depends on N, to be present in the choice of M. However, the astute reader will have noticed that several of the accuracy results in previous chapters were not tight. Therefore choosing k dependent on N is sufficient to guarantee the presented accuracy, but is not necessary. Indeed, as we will see, the upper bound for the given value of k has very slow growth compared to N. As suggested in Figure 5.1, the size of k is several orders of magnitude smaller than N. Additionally, in most applications, the size of N will be fixed, or at least have an upper bound. Therefore in practical settings, one is free to choose the parameter k, independent of N.

For large N, the size of A will be determined by the location of x_N^*, the largest root of H_N. As was remarked earlier, $x_N^* \approx \sqrt{2N-(2N)^{1/3}}$. Using this bound for A, one can see k will satisfy the assumption in Theorem 5.4 if

$$N^{(3m+1)/2}k^{d-1} \leq c\log_2^{mk} N, \tag{5.4}$$

for some positive constant c and for all N larger than some N_0. An approximation for k that satisfies Equation (5.4) is

$$k = \frac{(3m+2)\log_2 N}{2m\log_2\log_2 N}. \tag{5.5}$$

A plot of this estimate of k versus N, for N ranging from 10 to 10^6, is given in Figure 5.1. One can see that although k depends on N, it's growth is *very very* slow.

Combining Equations (5.1), (5.2) and (5.3) with Theorem 5.5 yields the following Theo-

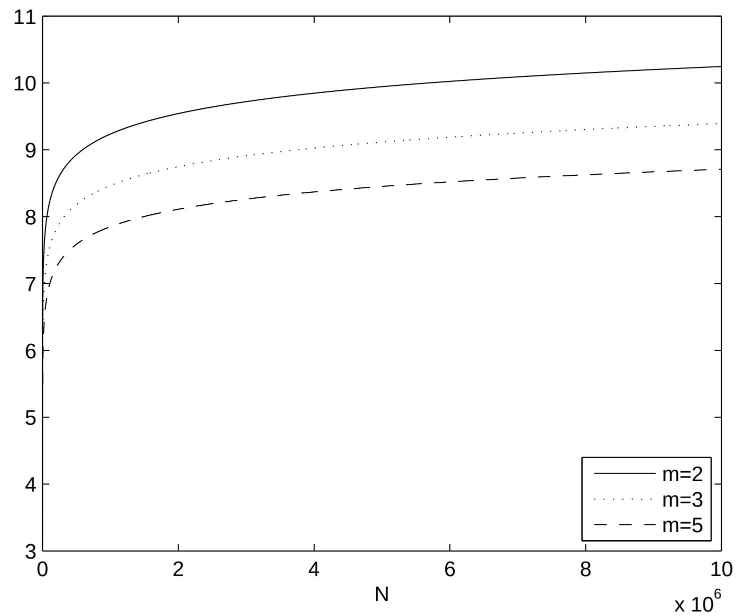


Figure 5.1: Plot of the estimate $k \approx \dfrac{(3m+2)\log_2 N}{2m\log_2\log_2 N}$ for different m.

rem, which is a primary result of this dissertation.

Theorem 5.6. *Suppose $f \in \mathcal{H}_s(\mathbb{R}^d) \cap X_M^m(I^d)$ such that $0 < s < m$. Fix $N \in \mathbb{N}_0$, and choose the parameter A as in Theorems 3.3, 5.5, and k and M as in Theorem 5.5. Then there exists a positive constant C such that*

$$\left|\left|f - \check{f}_{M,N}\right|\right|_\omega \le CN^{-s}\left(||f||_{\kappa^s} + ||f||_\omega + |f|_{X_M^m(I^d)}\right).$$

Chapter 6

Implementation

For a given function $f \in L^2_\omega(\mathbb{R}^d)$ and a positive integer N, the construction of the discrete sparse Fourier Hermite approximation as in Definition 5.1 requires computing the coefficients $\check{c}_{M,N}$. The topic of this section is to find a fast, reliable, accurate way to compute the multiscale discretization of the coefficients in the sparse Fourier Hermite expansion of f.

This chapter switches discussion between univariate and multivariate cases, so the boldface notation will be used to denote vectors.

The terms involved in the Fourier Hermite expansion have very large magnitude, so for numerical considerations, the problem will be recast. The normalized Hermite polynomials are defined as

$$h_{\mathbf{n}} := \frac{H_{\mathbf{n}}}{||H_{\mathbf{n}}||_\omega}. \tag{6.1}$$

Note that several different inner products and norms have been used throughout this dissertation. 'Normalization' here refers to the fact that $||h_{\mathbf{n}}||_\omega = 1$. Using the normalized

Hermite polynomials, the Fourier Hermite series of f can be written as

$$f = \sum_{\mathbf{n} \in \mathbb{N}_0^d} d_{\mathbf{n}} h_{\mathbf{n}}, \tag{6.2}$$

where $d_{\mathbf{n}} = \langle f, h_{\mathbf{n}} \rangle_\omega$. Notice this expression is equivalent to the expansion (2.11). The coefficients $c_{\mathbf{n}}$ have two copies of the norm $||H_{\mathbf{n}}||_\omega$ in their denominators. To derive the expansion (6.2) from (2.11), move one copy of $||H_{\mathbf{n}}||_\omega$ from the coefficient to the polynomial part $H_{\mathbf{n}}$. That is, for each $\mathbf{n} \in \mathbb{N}_0^d$,

$$d_{\mathbf{n}} h_{\mathbf{n}} = \langle f, h_{\mathbf{n}} \rangle_\omega h_{\mathbf{n}} = \frac{\langle f, H_{\mathbf{n}} \rangle_\omega H_{\mathbf{n}}}{||H_{\mathbf{n}}||_\omega^2} = c_{\mathbf{n}} H_{\mathbf{n}}.$$

Similarly, the discrete sparse Fourier Hermite approximation of f in Definition 5.1 can be written as

$$\check{f}_{M,N} = \sum_{\mathbf{n} \in \mathbb{L}_N^d} \check{d}_{M,\mathbf{n}} h_{\mathbf{n}}, \tag{6.3}$$

with the coefficients given by $\check{d}_{M,\mathbf{n}} := \langle \mathbb{S}_M^d f, h_{\mathbf{n}} \rangle_\omega$. Observe $\check{d}_{M,\mathbf{n}} = ||H_{\mathbf{n}}||_\omega \check{c}_{M,\mathbf{n}}$.

From Equation (4.14) and the above observation, it holds that

$$\langle \mathbb{S}_M^d f, h_{\mathbf{n}} \rangle_\omega = \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in \mathbb{W}_{\mathbf{j}}^d} \eta_{\mathbf{j},\mathbf{r}}(f) \langle \ell_{\mathbf{j},\mathbf{r}}, h_{\mathbf{n}} \rangle_\omega.$$

Let

$$A_{\mathbf{j},\mathbf{r}}(\mathbf{n}) := \int_{I^d} \ell_{\mathbf{j},\mathbf{r}}(x) h_{\mathbf{n}}(x) \omega(x) \mathrm{d}x$$

for each $\mathbf{j} \in \mathbb{S}_M^d$, $\mathbf{r} \in \mathbb{W}_{\mathbf{j}}$ and $\mathbf{n} \in \mathbb{N}_0^d$. Then the coefficients in computing the discrete sparse

Fourier Hermite approximation (6.3) can be computed via

$$\check{d}_{M,\mathbf{n}} = \langle \mathbb{S}_M^d, h_{\mathbf{n}} \rangle_\omega = \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in \mathbb{W}_{\mathbf{j}}^d} \eta_{\mathbf{j},\mathbf{r}}(f) A_{\mathbf{j},\mathbf{r}}(\mathbf{n}),$$

and the discrete sparse Fourier Hermite approximation of f can be computed via

$$\check{f}_{M,N}(x) = \sum_{\mathbf{n} \in \mathbb{L}_N^d} \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in \mathbb{W}_{\mathbf{j}}} \eta_{\mathbf{j},\mathbf{r}}(f) A_{\mathbf{j},\mathbf{r}}(\mathbf{n}) h_{\mathbf{n}}(x). \tag{6.4}$$

To achieve the goal of finding a fast and good discrete sparse Fourier Hermite approximation, one must have algorithms to quickly and accurately compute the quantities $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$ and $\eta_{\mathbf{j},\mathbf{r}}(f)$. First, the collection $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$ will be considered.

It is fortunate that the terms $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$, which approximate high dimensional oscillatory integrals, do not need to be further approximated by a quadrature formula. Since the Lagrange cardinal polynomials, the Hermite polynomials, and the Gaussian weight function used in the definition of $A_{j,r}(n)$ are all defined in higher dimensions using tensor products, it follows that

$$A_{\mathbf{j},\mathbf{r}}(\mathbf{n}) = \prod_{k=1}^{d} \int_{I^d} \ell_{j_k,r_k}(x_k) h_{n_k}(x_k) \omega(x_k) \, \mathrm{d}x_k = \prod_{k=1}^{d} A_{j_k,r_k}(n_k). \tag{6.5}$$

As such, an evaluation scheme needs to be developed only for the univariate case.

Therefore, for now, consider the case when $d = 1$. Each $\ell_{j,r}$ is a polynomial of degree $m-1$ on a subinterval of $[-A, A]$, and is equal to zero everywhere else. We use this property along with Theorem 2.3 to compute the integrals $A_{j,r}(n)$ quickly and without introducing any additional quadrature error.

To shorten notation, denote the subintervals of $I = [-A, A]$ by $I_{i,j} = -A + i2^{-j+1}A$ for any $i \in \mathbb{N}_{2^j}$ and $j \in \mathbb{N}_0$. Then

$$\operatorname{supp}(\ell_{j,r}) = \left[I_{\lfloor \frac{r}{m} \rfloor, j} \ , \ I_{\lfloor \frac{r}{m} \rfloor + 1, j} \right],$$

and

$$A_{j,r}(n) = \int_{I_{\lfloor \frac{r}{m} \rfloor, j}}^{I_{\lfloor \frac{r}{m} \rfloor + 1, j}} \ell_{j,r}(x) h_n(x) \omega(x) \, \mathrm{d}x. \tag{6.6}$$

Since each $\ell_{j,r}$ is a polynomial of degree $m - 1$, Equation (6.6) can now be written as

$$A_{j,r}(n) = \sum_{p=0}^{m-1} b_{j,r}(p) \left[F_{p,n} \left(I_{\lfloor \frac{r}{m} \rfloor + 1, j} \right) - F_{p,n} \left(I_{\lfloor \frac{r}{m} \rfloor, j} \right) \right], \tag{6.7}$$

where $\ell_{j,r}(x) = \sum_{p=0}^{m-1} b_{j,r}(p) x^p$ for $x \in \operatorname{supp}(\ell_{j,r})$, and the functionals $F_{p,n}(x)$ are defined as follows.

Definition 6.1. *For each $p \in \{0, 1, \ldots, m-1\}$, $n \in \mathbb{L}_N^1$ and $x \in I$, define*

$$F_{p,n}(x) = \int_0^x t^p h_n(t) \omega(t) \, \mathrm{d}t.$$

The functionals can be computed exactly in an iterative way.

Theorem 6.2. *For each $j \in \{0, 1, \ldots, M\}$, $r \in W_j$ and $x \in I$, the following equations hold.*

1. *For all x,*

$$F_{0,0}(x) = \frac{\sqrt[4]{\pi}}{2} \operatorname{erf}(x). \tag{6.8}$$

2. *For all x,*

$$F_{1,0}(x) = \frac{1 - \omega(x)}{2\sqrt[4]{\pi}} \tag{6.9}$$

3. *For all x and all $p \geq 2$,*

$$F_{p,0}(x) = \frac{-x^{p-1}\omega(x)}{2\sqrt[4]{\pi}} + \left(\frac{p-1}{2\sqrt[4]{\pi}}\right) F_{p-2,0}(x). \tag{6.10}$$

4. *For all x and integers $n > 0$,*

$$F_{0,n}(x) = \left. \frac{-h_{n-1}(t)\omega(t)}{\sqrt{2n}} \right|_0^x . \tag{6.11}$$

5. *For all x and integers $n \geq 1$,*

$$F_{1,n}(x) = \sqrt{\frac{n+1}{2}} F_{0,n+1}(x) + \sqrt{\frac{n}{2}} F_{0,n-1}(x) \tag{6.12}$$

6. *For all x and integers $p \geq 0$,*

$$F_{p,1}(x) = \sqrt{2} F_{p+1,0}(x). \tag{6.13}$$

7. *For all x, integers $p \geq 0$ and integers $n \geq 2$,*

$$F_{p,n}(x) = \sqrt{\frac{2}{n}} F_{p+1,n-1}(x) - \sqrt{\frac{n-1}{n}} F_{p,n-2}(x). \tag{6.14}$$

8. *For all x, integers $n \geq 1$ and integers $p \geq 2$,*

$$F_{p,n}(x) = \frac{-x^{p-1}h_n(x)\omega(x)}{2} + \left(\frac{p-1}{2}\right) F_{p-2,n}(x) + \sqrt{\frac{n}{2}} F_{p-1,n-1}(x). \tag{6.15}$$

Proof. 1. From the definition of erf x, $F_{0,0}(x) = \frac{\sqrt[4]{\pi}}{2}\,\mathrm{erf}(x)$.

2. Equation (6.9) follows from the observation $t\omega(t) = -\frac{1}{2}\omega'(t)$.

3. Use integration by parts to derive Equation (6.10). If $p \geq 2$, then

$$\begin{aligned}
F_{p,0}(x) &= \frac{1}{\sqrt[4]{\pi}} \int_0^x t^p \omega(t)\mathrm{d}t \\
&= \frac{1}{\sqrt[4]{\pi}} \int_0^x t^{p-1} \cdot t\omega(t)\mathrm{d}t \\
&= \frac{-1}{2\sqrt[4]{\pi}} \int_0^x t^{d-1}\omega'(t)\mathrm{d}t \\
&= \left.\frac{-t^{p-1}\omega(t)}{2\sqrt[4]{\pi}}\right|_0^x + \frac{p-1}{2\sqrt[4]{\pi}} \int_0^x t^{p-2}\omega(t)\mathrm{d}t \\
&= \frac{-x^{p-1}\omega(x)}{2\sqrt[4]{\pi}} + \frac{p-1}{2\sqrt[4]{\pi}} F_{p-2,0}(x).
\end{aligned}$$

4. Equation (6.11) follows from Corollary 2.4, and that

$$\frac{H_{n-1}}{||H_n||_\omega} = \frac{H_{n-1}}{\sqrt{n!2^n\sqrt{\pi}}} = \frac{H_{n-1}}{\sqrt{2n}||H_{n-1}||_\omega} = \frac{h_{n-1}}{\sqrt{2n}}.$$

Then

$$F_{0,n}(x) = \int_0^x h_n(t)\omega(t)\mathrm{d}t = \int_0^x \frac{H_n(t)\omega(t)}{||H_n||_\omega}\mathrm{d}t$$
$$= \left.\frac{-H_{n-1}(t)\omega(t)}{||H_n||_\omega}\right|_0^x = \left.\frac{-h_{n-1}(t)\omega(t)}{\sqrt{2n}}\right|_0^x.$$

5. Equation (6.12) follows from the observation that for all $n \geq 1$,

$$tH_n(t) = \frac{H_{n+1}}{2}(t) + nH_{n-1}(t)$$

following from the three term recurrence of the Hermite polynomials. Then

$$th_n(t) = \frac{H_{n+1}(t)}{2\sqrt{n!2^n\sqrt{\pi}}} + \frac{nH_{n-1}(t)}{\sqrt{n!2^n\sqrt{\pi}}} = \sqrt{\frac{n+1}{2}}h_{n+1} + \sqrt{\frac{n}{2}}h_{n-1}(t).$$

6. Since $H_1(t) = 2t$,

$$F_{1,k}(x) = \int_0^x t^p h_1(t)\omega(t)\mathrm{d}t = \sqrt{\frac{2}{\sqrt{\pi}}}\int_0^x t^{p+1}\omega(t)\mathrm{d}t$$
$$= \sqrt{2}\int_0^x t^{p+1}h_0(t)\omega(t)\mathrm{d}t = \sqrt{2}F_{p+1,0}(x).$$

7. Using the recurrence relation $H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t)$ for $n \geq 2$,

$$\begin{aligned}
F_{p,n}(x) &= \int_0^x t^p h_n(t)\omega(t)\mathrm{d}t \\
&= \int_0^x t^p \left(\frac{2tH_{n-1}(t) - 2(n-1)H_{n-2}(t)}{||H_n||_\omega} \right) \omega(t)\mathrm{d}t \\
&= \sqrt{\frac{2}{n}} \int_0^x t^{p+1} h_{n-1}(t)\omega(t)\mathrm{d}t - \sqrt{\frac{n-1}{n}} \int_0^x t^p h_{n-2}(t)\omega(t)\mathrm{d}t \\
&= \sqrt{\frac{2}{n}} F_{p+1,n-1}(x) - \sqrt{\frac{n-1}{n}} F_{p,n-2}(x).
\end{aligned}$$

8. The final formula can be recovered using integration by parts. Notice

$$h_n'(t) = \frac{H_n'(t)}{||H_n||_\omega} = \frac{2nH_{n-1}(t)}{||H_n||_\omega} = \frac{2nH_{n-1}(t)}{\sqrt{2n}||H_{n-1}||_\omega} = \sqrt{2n} h_{n-1}(t).$$

Then, if $p \geq 2$,

$$\begin{aligned}
F_{p,n}(x) &= \int_0^x t^p h_n(t)\omega(t)\mathrm{d}t \\
&= \frac{-1}{2} \int_0^x t^{p-1} h_n(t)\omega'(t)\mathrm{d}t \\
&= \frac{-t^{p-1}h_n(t)\omega(t)}{2}\Bigg|_0^x + \frac{1}{2}\int_0^x \left[t^{p-1}h_n(t)\right]' \omega(t)\mathrm{d}t \\
&= \frac{-t^{p-1}h_n(t)\omega(t)}{2}\Bigg|_0^x + \frac{1}{2}\int_0^x \left[(p-1)t^{p-2}h_n(t) + \sqrt{2n}t^{p-1}h_{n-1}\right] \omega(t)\mathrm{d}t \\
&= \frac{-t^{p-1}h_n(t)\omega(t)}{2}\Bigg|_0^x + \frac{p-1}{2}F_{p-2,n}(x) + \sqrt{\frac{n}{2}}F_{p-1,n-1}(x) \\
&= \frac{-x^{p-1}h_n(x)\omega(x)}{2} + \frac{p-1}{2}F_{p-2,n}(x) + \sqrt{\frac{n}{2}}F_{p-1,n-1}(x).
\end{aligned}$$

□

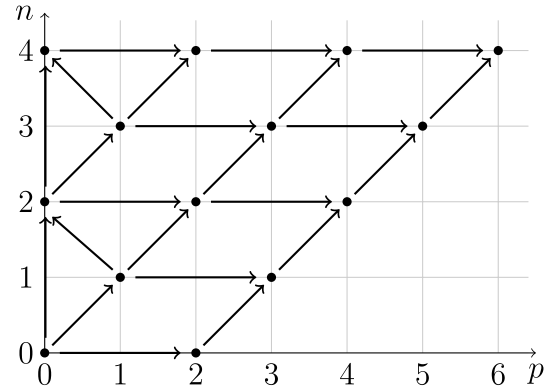


Figure 6.1: Levels of dependence in the recurrence relation from equation (6.15) for $d = 1$.

Algorithm 6.3. *For a fixed x, compute the collection $\{F_{p,n}(x)\}_{p=0,\, n=0}^{m-1,\, N-1}$ in a single dimension as follows.*

Step1. Compute $\omega(x)$ and $h_n(x)$ for each $n = 0, 1, \ldots, N-1$ using the three term recurrence relation.

Step 2. Compute $F_{0,0}(x)$ and $F_{1,0}(x)$ using Equations (6.8) *and* (6.9).

Step 3. Compute $F_{p,0}(x)$ for $p = 2, 3, \ldots, m-1$ using Equation (6.10).

Step 4. Compute $F_{0,n}(x)$ for $n = 2, 3, \ldots, N-1$ using Equation (6.11).

Step 5. Compute $F_{p,1}(x)$ for $p = 0, 1, \ldots, m-2$ using Equation (6.13).

Step 6. Compute $F_{m-1,1}(x)$ using Equation (6.15).

Step 7. For each n = 2,3,...,N-1, compute $F_{p,n}(x)$ for $p = 0, 1, \ldots, m-2$ using Equation (6.14)*, and compute $F_{m-1,n}(x)$ using Equation* (6.15).

If $m = 2$, then Algorithm 6.3 can be simplified.

Algorithm 6.4. *For a fixed x, compute the collection $\{F_{p,n}(x)\}_{p=0,\, n=0}^{1,\, N-1}$ in a single dimension as follows.*

Step 1. Compute $\omega(x)$ and $h_n(x)$ for each $n = 0, 1, \ldots, N-1$ using the three term recurrence relation.

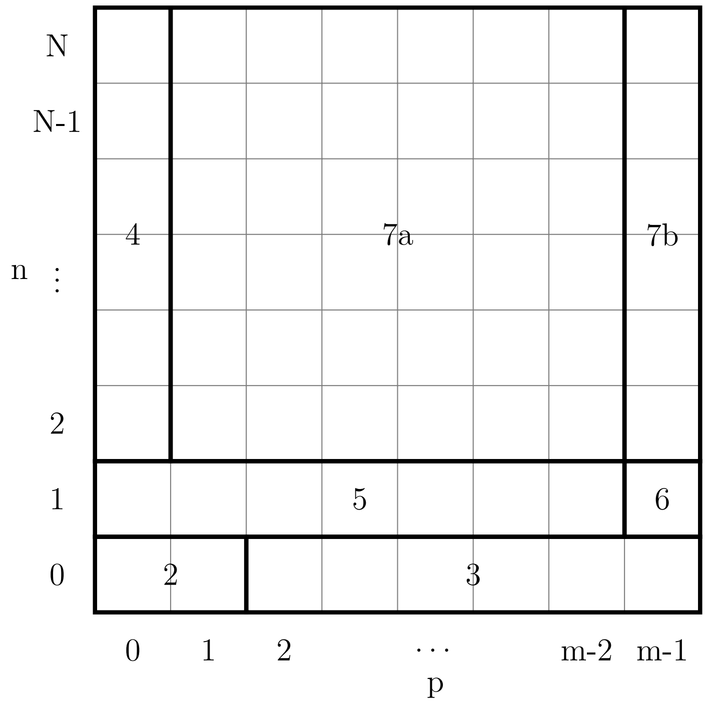


Figure 6.2: Blocks of computation steps from Algorithm 6.3.

Step 2. Compute $F_{0,0}(x)$ *and* $F_{1,0}(x)$ *for each* x *using Equations* (6.8) *and* (6.9).

Step 3. Compute $F_{0,1}(x) = \sqrt{2}F_{1,0}(x)$ *for each* x.

Step 4. Compute $F_{1,1}(x) = \dfrac{-x\omega(x)}{\sqrt{2}\sqrt[4]{\pi}} + \dfrac{1}{\sqrt{2}}F_{0,0}(x)$ *for each* x.

Step 5. Compute $F_{0,n+1}(x) = h_n(0) - h_n(x)\omega(x)$.

Step 6. Compute $F_{1,n+1}(x) = \dfrac{1}{\sqrt{2(n+1)}}\left(-xh_n(x)\omega(x) + F_{0,n}(x)\right)$.

Theorem 6.5. *Let* $\mathcal{N}_{N,m,x}$ *be equal to the computational cost of completing Algorithm 6.3 for fixed values* N, m *and* x. *Then*

$$\mathcal{N}_{N,m,x} = 2N + mN - m + 6.$$

Proof. For each fixed x, computing $\omega(x)$ and each $h_n(x)$ for $n = 0, 1, \ldots, N$ via the three term recurrence in *Step 1* of Algorithm 6.3 requires a total of $2(N+1)+1$ multiplications.

Computing *Step 2* requires only a single multiplication. Since multiplication and division by 2 can be achieved by only shifting digits in binary, computing $F_{p,0}(x)$ for each x and each p via Equation (6.10) requires two multiplications. Therefore *Step 3* of Algorithm 6.3 requires $2(m-3)$ multiplications. Computing Equation (6.11) for each x and n can be accomplished in one multiplication, so *Step 4* requires $N-3$ multiplications. Since Equation (6.13) is multiplication by 2, *Step 5* is free for each x. Equation (6.14) requires one multiplication and Equation (6.15) requires four multiplications for each x, p, and n. Therefore *Step 6* requires exactly 4 multiplications and *Step 7* requires $(N-3)(m-1)+4$.

Therefore, altogether the computational cost of completing all steps associated with Algorithm 6.3 in one dimension for fixed values N, m and x is the total computation cost from each step: $2N + mN - m + 6$. □

For each fixed value $j \in \mathbb{S}^1_M$, to compute the collection

$\{A_{j,r}(n) : \ r \in W_j, \ n = 0 : N-1\}$ via Equation (6.7) and Algorithm 6.3, requires evaluating each $F_{p,n}$ at $2^j + 1$ values of x. Namely at the points in the collection $\{I_{s,j} : s = 0 : 2^j\}$.

Theorem 6.6. *Fix positive integers N and m. To fully compute the collection*

$$\{F_{p,n}(I_{i,j}) \ : \ p \in \mathbb{Z}_m, \ n \in \mathbb{Z}_N, \ j \in \mathbb{Z}_M, \ i \in \mathbb{Z}_{2^j}\} \tag{6.16}$$

requires $(2N + mN - m + 6)(2^{M+1} - 1)$ *multiplications.*

Proof. From the previous Theorem, we know that for each $I_{i,j}$, computing the collection $\{F_{p,n}(I_{i,j})\}$ for all $p \in \mathbb{Z}_m$ and all $n \in \mathbb{Z}_N$ requires $2N + mN - m + 6$ multiplications. The cardinality of the set of all $I_{i,j}$ of interest is

$$|\{j \in \mathbb{Z}_M,\ i \in \mathbb{Z}_j\}| = \sum_{j=0}^{M} 2^j = 2^{M+1} - 1.$$

$\square$

Corollary 6.7. *Let N, m be fixed positive integers. Choose $M = \lceil \log_2 \log_2^k N \rceil$, where k is chosen as in Theorem 5.5. Then the computational complexity of computing the collection*

$$\left\{A_{j,r}(n)\ :\ j \in \mathbb{S}_M^1,\ r \in W_j,\ n \in \mathbb{Z}_N\right\} \tag{6.17}$$

using Equation (6.7) *is* $\mathcal{O}(N \log_2^k N)$.

In several dimensions, the values $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$ are defined using products as in Equation (6.5). If they are computed naively, this will require d multiplications for each $(\mathbf{j}, \mathbf{r}, \mathbf{n})$.

Algorithm 6.8. *Let N, m be fixed positive integers. Choose $M = \lceil \log_2 \log 2^k N \rceil$, where k is chosen as in Theorem 5.5. Compute the collection*

$$\left\{A_{\mathbf{j},\mathbf{r}}(\mathbf{n})\ :\ \mathbf{n} \in \mathbb{L}_N^d,\ \mathbf{j} \in \mathbb{S}_M^d,\ \mathbf{r} \in \mathbb{W}_{\mathbf{j}}\right\}$$

as follows.

Step 1. Compute the single dimension collection

$$\{A_{j,r}(n)\ :\ j \in \mathbb{Z}_M,\ r \in \mathbb{W}_j,\ n \in \mathbb{Z}_N\}$$

using Algorithm 6.3.

Step 2. For each $\mathbf{n} \in \mathbb{L}_N^d$*, compute the sum*

$$\check{c}_{\mathbf{n}} = \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in \mathbb{W}_{\mathbf{j}}} \prod_{k=1}^{d} A_{j_k, r_k}(n_k).$$

Theorem 6.9. *Let* N, m *be fixed positive integers. Choose* $M = \lceil \log_2 \log 2^k N \rceil$ *where* k *is chosen as in Theorem 5.5. Computing the collection*

$$\left\{ A_{\mathbf{j},\mathbf{r}}(\mathbf{n}) \ : \ \mathbf{n} \in \mathbb{L}_N^d, \ \mathbf{j} \in \mathbb{S}_M^d, \ \mathbf{r} \in \mathbb{W}_{\mathbf{j}} \right\}$$

via Algorithm 6.8 requires $\mathcal{O}(N \log_2^{2(d-1)+k} N)$ *multiplications.*

Proof. From Corollary 6.7, computing the collection

$$\{A_{j,r}(n) \ : \ n \in \mathbb{Z}_N, \ j \in \mathbb{Z}_M, \ r \in \mathbb{W}_j\}$$

as in Step 1 of Algorithm 6.8 requires $\mathcal{O}(N \log_2^k N)$ multiplications. These values should be stored in memory for future use.

The cardinality of the set $\mathbb{L}_N^d$ is $\mathcal{O}(N \log_2^{d-1} N)$, and the cardinality of the set

$$\left\{ \mathbf{j} \in \mathbb{S}_M^d, \ \mathbf{r} \in \mathbb{W}_{\mathbf{j}} \right\}$$

is $M^{d-1} 2^M \le c k^{d-1} \log_2^{d-1}(\log_2 N) \log_2^k N$ for some constant c. Therefore the cardinality of the index set

$$\left\{\mathbf{n} \in \mathbb{L}_N^d,\ \mathbf{j} \in \mathbb{S}_M^d,\ \mathbf{r} \in \mathbb{W}_{\mathbf{j}}\right\}$$

is $\mathcal{O}(Nk^{d-1}\log_2^{d-1}(\log_2 N)\log_2^{d-1+k} N) = \mathcal{O}(N\log_2^{2(d-1)+k} N)$, using the estimate of k given in Equation (5.5).

For each triplet $(\mathbf{j}, \mathbf{r}, \mathbf{n})$ in this index set, we require d multiplications of the univariate quantities stored in memory to compute $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$.

Therefore the cost of computing $A_{\mathbf{j},\mathbf{r}}(\mathbf{n})$ for all $\mathbf{j} \in \mathbb{S}_M^d$, $\mathbf{r} \in \mathbb{W}_{\mathbf{j}}$ and $\mathbf{n} \in \mathbb{L}_N^d$ is $\mathcal{O}(N\log_2^{2(d-1)+k} N)$. The sum of all of these as in Step 2 of Algorithm 6.8 does not contribute any additional multiplications. □

The values $\left\{\eta_{j,r}(f) : j \in \mathbb{S}_M^d, r \in \mathbb{W}_j\right\}$ defined in Equation (4.8) are computed using the Algorithm written by Xu and Jiang appearing in [12]. It is proven therein that the complexity to compute the set using their algorithm is $\mathcal{O}(N\log_2^{d-1} N)$.

Now we come to the primary result concerning the computational cost of the sparse discrete Fourier Hermite approximation scheme proposed in this dissertation.

Theorem 6.10. *Let N, m be fixed positive integers. Let $M = \lceil \log_2 \log_2^k N \rceil$ where k is chosen as in Theorem 5.5. Then the cost of computing the inner product*

$$\langle \mathcal{S}_M^d(f), H_{\mathbf{n}} \rangle_\omega$$

for all $\mathbf{n} \in \mathbb{L}_N^d$ is $\mathcal{O}(N\log_2^{2(d-1)+k} N)$.

Proof. Recall the inner products are decomposed via the formula

$$\langle \mathcal{S}_M^d(f), H_{\mathbf{n}} \rangle_\omega = \sum_{\mathbf{j} \in \mathbb{S}_M^d} \sum_{\mathbf{r} \in \mathbb{W}_{\mathbf{j}}} \eta_{\mathbf{j},\mathbf{r}}(f) A_{\mathbf{j},\mathbf{r}}(\mathbf{n}).$$

We compute $\{A_{\mathbf{j},\mathbf{r}}(\mathbf{n})\}$ in $\mathcal{O}(N \log_2^{2(d-1)+k} N)$ multiplications. We compute $\{\eta_{\mathbf{j},\mathbf{r}}(f)\}$ in $\mathcal{O}(N \log_2^{d-1} N)$ multiplications.

To compute the sum, we require one multiplication for each pair $(\mathbf{j}, \mathbf{r})$ in the index set $\{\mathbf{j} \in \mathbb{S}_M^d,\ \mathbf{r} \in \mathbb{W}_{\mathbf{j}}\}$, a set of cardinality $\mathcal{O}(\log_2^{d-1+k} N)$.

Therefore the entire collection can be computed using $\mathcal{O}(N \log_2^{2(d-1)+k} N)$ multiplications.

□

Chapter 7

Numerical Results

In this chapter, numerical examples are presented to demonstrate the accuracy and computational efficiency of Algorithm 6.3, which numerically computes $\check{c}_{n,M}$, the coefficients in the discrete sparse Fourier Hermite approximation of a given function f. The computer program is run on an HP ENVY laptop with an Intel i5 2.67 GHz CPU and 4GB RAM. The machine precision of the computer used is $2.2204e-016$.

In the numerical examples, use 'Err' to denote the relative error, which is computed as

$$\text{Err} \ = \frac{||f - \check{f}_{M,N}||_\omega}{\sqrt{||f||^2_{\kappa^s} + ||f||^2_\omega + |f|^2_{X^m_M(I^d)}}}.$$

The presented results verify the theoretical error estimate from Theorem 5.6 if Err $< N^{-s}$. The computing time in seconds spend calculating the coefficients $\check{c}_{M,\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{L}^d_N$ is denoted by 'CT'. The compression ratio CR denotes the value $|\mathbb{L}^d_N|/N^d$. The value $\mathcal{M}_N$ is the actual number of multiplications used to compute the Fourier Hermite coefficients. Since the compression ratio depends only on N and d, it needs to be investigated in only one example.

As discussed in Chapter 5, the parameter k may be fixed independently of N. In the following examples, the value of k will be specified in a manner that does not depend on N, however, we will still achieve, or even beat, the desired approximation accuracy.

The first two examples use multivariate polynomials that are separable in each variable. These examples are designed to demonstrate the accuracy of the proposed algorithm. Since the example functions are polynomials, it is trivial to express them each as a finite Fourier Hermite series, and the error of the proposed discrete sparse Fourier Hermite approximation is easy to compute. Note that although the two example functions admit a finite Fourier Hermite series expression, the proposed method does not guarantee the approximation will equal the exact expansion. This is because of the quadrature method used to numerically compute the coefficients. It will compute the coefficients within an acceptable tolerance, but it will not produce a termwise exact result.

The theoretical results presented in this paper guarantee a global approximation relative to the weighted L^2 norm. However, as the reader will see from the Figures appearing in this section, the computed approximations in fact give very good pointwise approximations on the interval I^d.

Example 7.1. *Consider in this example the function* $f_{1,d}(\mathbf{x}) := \prod_{j=1}^{d} x_j^2$*, for* $\mathbf{x} \in I^d$ *and* $d = 2, 3, 4$*. The parameters will be set as* $m = 3$ *and* $s = 2.99$*. The value* k *is fixed at* 3 *and is not chosen to depend on* N*.*

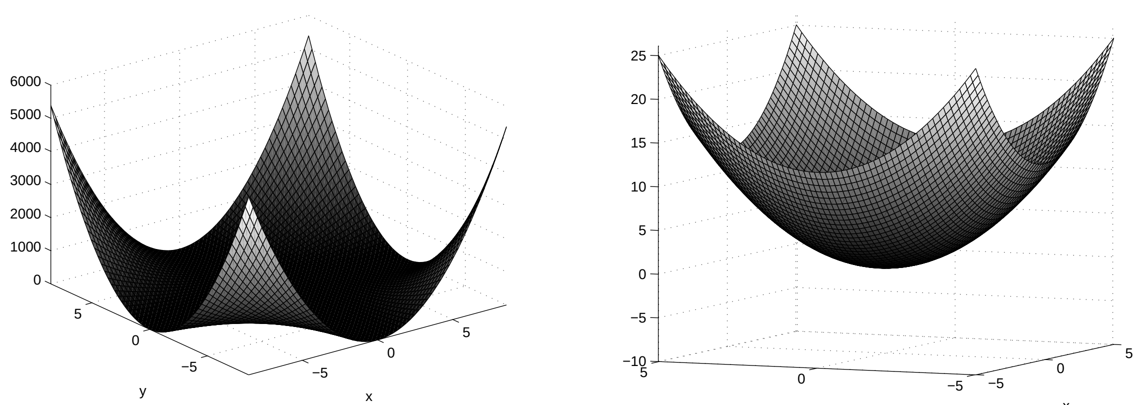
To compute the accuracy of the algorithm in this example, first the exact Fourier Hermite series of f must be computed. Using the standard definition of the Hermite polynomials, as well as the normalized definition, the exact expansions are

$$
\begin{aligned}
f_{1,2} &= \frac{H_{2,2}}{16} + \frac{H_{2,0}+H_{0,2}}{8} + \frac{H_{0,0}}{4} \\
&= \frac{\sqrt{\pi}}{2} h_{2,2} + \frac{\sqrt{\pi}}{2\sqrt{2}} (h_{2,0}+h_{0,2}) + \frac{\sqrt{\pi}}{2} h_{0,0},
\end{aligned}
$$

and

$$
\begin{aligned}
f_{1,3} &= \frac{H_{2,2,2}}{64} + \frac{H_{0,2,2}+H_{2,0,2}+H_{2,2,0}}{32} + \frac{H_{2,0,0}+H_{0,2,0}+H_{0,0,2}}{16} + \frac{H_{0,0,0}}{8}. \\
&= \pi^{3/4}\left(\frac{h_{2,2,2}}{\sqrt{8}} + \frac{h_{2,2,0}+h_{2,0,2}+h_{0,2,2}}{4} + \frac{h_{2,0,0}+h_{0,2,0}+h_{0,0,2}}{2\sqrt{8}} + \frac{h_{0,0,0}}{8}\right).
\end{aligned}
$$

The results are displayed in Table 7.1 and graphically in the Figures 7.1 and 7.2. The large jump in 'Err' from $N = 5$ to $N = 10$ for $d = 2$ is due to the fact that $(2,2) \notin \mathbb{L}_5^2$, but the term $h_{2,2}$ is in the exact expansion of f. This discrepancy is also shown in Figure 7.1. However, for all $N \geq 10$, and for each term $h_{\mathbf{n}}$ in the exact expansion of $f_{1,2}$, the index $\mathbf{n}$ is an element of the sparse index set $\mathbb{L}_N^d$, and therefore the numerical results in a very small error, and visually the plots of the approximations are indistinguishable from the exact plot.



(a) Approximation with $N = 30$.　　(b) Approximation with $N = 5$.

Figure 7.1: Plots of Fourier-Hermite approximations from Example 1 with $m = 3, d = 2$.

	N	Err	CT	CR	Mn	N^{-s}
$d=2$	5	1.3841e-003	0.001543	4.0000e-001	12960	8.1298e-003
	10	6.5795e-008	0.018664	2.7000e-001	186624	1.0233e-003
	15	6.5953e-010	0.030042	2.0000e-001	311040	3.0443e-004
	20	2.2772e-012	0.094269	1.6500e-001	1026432	1.2880e-004
	25	1.7673e-014	0.125249	1.3920e-001	1353024	6.6094e-005
	30	3.0603e-015	0.160367	1.2333e-001	1726272	3.8318e-005
$d=3$	5	7.1779e-005	0.125678	1.2800e-001	179712	8.1298e-003
	10	8.6133e-005	0.311694	5.3000e-002	3938112	1.0233e-003
	15	4.8937e-005	0.562010	2.8148e-002	7058880	3.0443e-004
	20	2.3254e-005	2.438147	1.9000e-002	27841536	1.2880e-004
	25	1.1657e-006	3.344474	1.3376e-002	38282112	6.6094e-005
	30	9.6284e-007	4.447853	1.0222e-002	50554368	3.8318e-005
$d=4$	5	1.5978e-006	0.124670	3.6800e-002	1788480	8.1298e-003
	10	3.4845e-006	4.472589	8.9000e-003	54788400	1.0233e-003
	15	6.4826e-006	8.525389	3.3383e-003	104036400	3.0443e-004
	20	5.7423e-007	41.372797	1.8250e-003	478716480	1.2880e-004
	25	1.2881e-007	57.721236	1.0701e-003	685285920	6.6094e-005
	30	7.5379e-009	77.918464	6.9383e-004	921365280	3.8318e-005

Table 7.1: Numerical Results from Example 1.

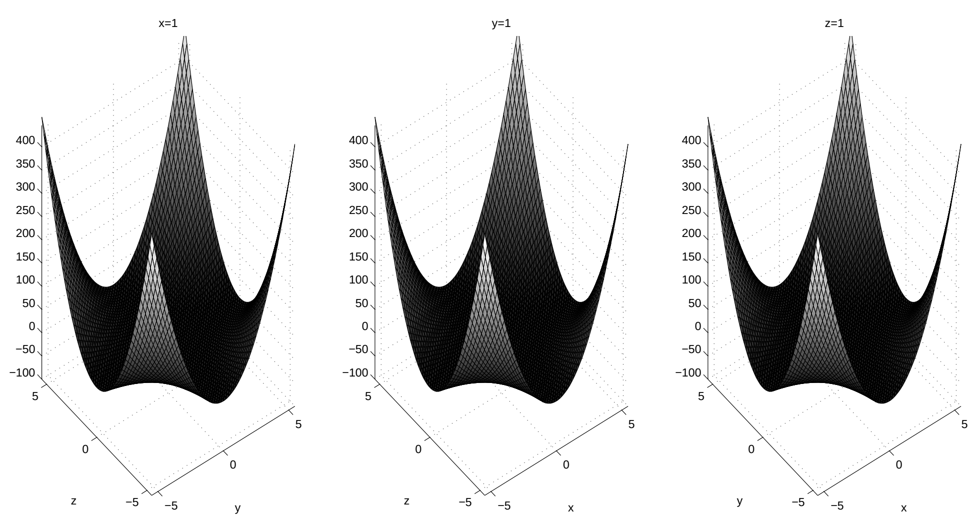


Figure 7.2: Cross sections from the Fourier Hermite Approximation of Example 1 with $d=3$.

Example 7.2. *Consider in this example the functions* $f_{2,2}(x,y) = xy^3$ *and* $f_{2,3}(x,y,z) = xy^3z^2$*. Let* $m = 3$ *and* $s = 2.99$*. Again, fix the parameter* $k = 3$*.*

The functions from Example 7.2 can be written exactly using a finite number of Hermite polynomials or the normalized Hermite polynomials as

$$\begin{aligned} f_{2,2} &= \frac{1}{16}H_{1,3} + \frac{3}{8}H_{1,1} \\ &= \frac{\sqrt{6\pi}}{4}h_{1,3} + \frac{3\sqrt{\pi}}{4}h_{1,1} \end{aligned}$$

and

$$\begin{aligned} f_{2,3} &= \frac{1}{64}H_{1,3,2} + \frac{3}{32}H_{1,1,2} + \frac{1}{32}H_{1,3,0} + \frac{3}{16}H_{1,1,0} \\ &= \frac{2\sqrt{3}\pi^{3/4}}{8}h_{1,3,2} + \frac{3\sqrt{2}\pi^{3/4}}{8}h_{1,1,2} + \frac{\sqrt{6}\pi^{3/4}}{8}h_{1,3,0} + \frac{3\pi^{3/4}}{8}h_{1,1,0}. \end{aligned}$$

The three norms used in the evaluation of the order of accuracy of the approximation are computed exactly as

$$||f_{2,2}||_\omega = \tfrac{\sqrt{15\pi}}{4}, \qquad ||f_{2,3}||_\omega = \tfrac{3\sqrt{5}}{8}\pi^{3/4}$$

$$||f_{2,2}||_{\kappa^s} = \tfrac{\sqrt{\pi}}{4}\left(6\cdot 8^{2s} + 9\cdot 2^{4s}\right)^{1/2}, \qquad ||f_{2,3}||_{\kappa^s} = \tfrac{2^{2s}\sqrt{3}\pi^{3/4}}{8}\left(4\cdot 6^{2s} + 6\cdot 3^{2s} + 2\cdot 2^{2s} + 3\right)^{1/2}$$

$$|f_{2,2}|_{X_M^m(I^d)} = A^4, \qquad |f_{2,3}|_{X_M^m(I^d)} = A^6.$$

For small values of N, the numerical results, presented in Table 7.2, suffer slightly since not enough terms are used to fully describe each function $f_{2,d}$. However, for larger N, the results not only confirm the theoretical results from Theorem 5.6, but are much better than the theoretical results. In particular, notice for $N = 30$ that Err is several orders of

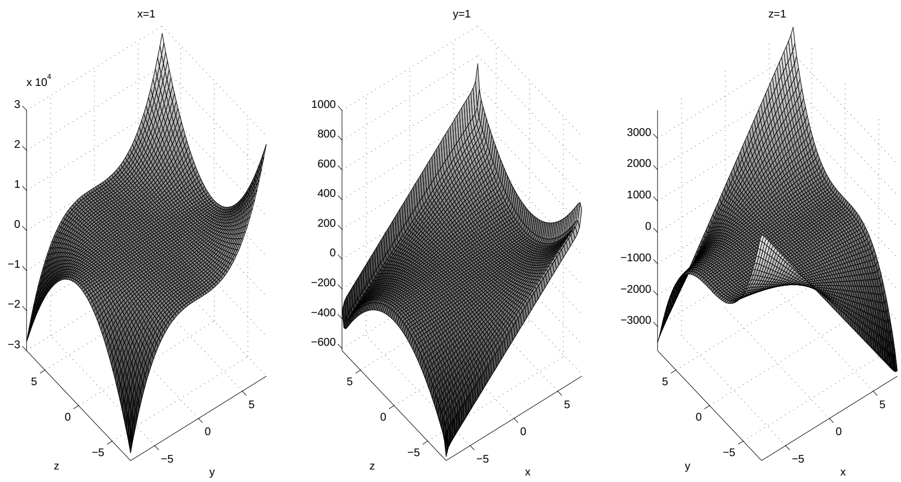


Figure 7.3: Three cross sections of the Fourier Hermite approximation of the function from Example 2 with $N = 30$, $d = 3$.

magnitude smaller than N^{-s} for both $d = 2, 3$.

Plots of the resulting approximation for $d = 3$ and $N = 30$ are presented in Figure 7.3. The three plots give cross sections of the approximation for $x = 1$, $y = 1$ and $z = 1$ respectively. Notice the three plots are visually indistinguishable from the cross sections of the exact function $f_{2,3}$, which are defined as $c_x(y,z) = y^3z^2$, $c_y(x,z) = xz^2$ and $c_z(x,y) = xy^3$.

Example 7.3. *Consider in this example the functions* $f_{3,2}(x,y) = xe^y$ *and* $f_{3,3}(x,y,z) = xz^2e^y$. *Take* $m = 2$ *and* $s = 1.99$. *For* $d = 2$ *use* $k = 3$ *and for* $d = 3$ *use* $k = 2$.

Since $f_{3,2}$ and $f_{3,3}$ are not polynomials, they do not possess finite exact Fourier Hermite expansions as in the previous two examples. To compute the norms $||\cdot||_\omega$ and $||\cdot||_{\kappa^s}$, first the functions are written as a truncation of a Taylor series, then converted to Hermite functions.

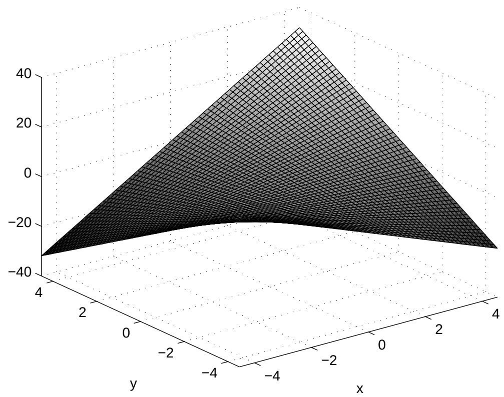
The following norms are computed up to the precision of the PC used.

$$||f_{3,2}||_\omega \approx 2.0664, \qquad ||f_{3,3}||_\omega \approx 3.3208$$

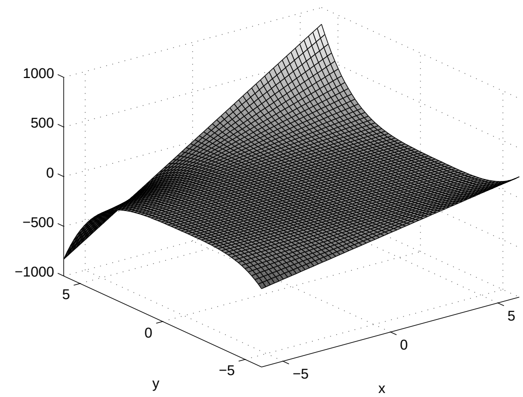
$$||f_{3,2}||_{\kappa^{1.99}} \approx 30.6512, \qquad ||f_{3,3}||_{\kappa^{1.99}} \approx 3.1113e+002$$

$$|f_{3,2}|_{X^2_M(I^2)} = A * exp(A), \qquad |f_{3,3}|_{X^2_M(I^3)} = A^3 * exp(A).$$

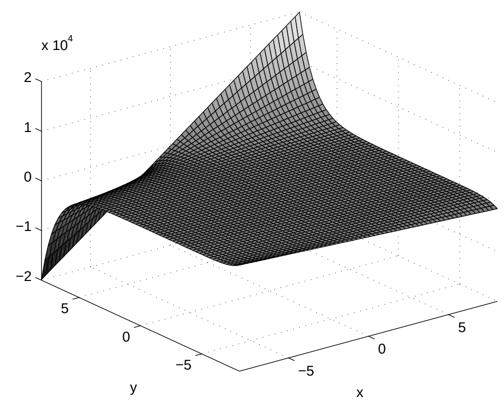
In the previous two examples, once N was large enough so that the set $\{h_\mathbf{n} | \mathbf{n} \in \mathbb{L}^d_N\}$ contained all of the terms in the exact finite Fourier Hermite series of the sample function, the approximation error results were very good. In this example, the exact Fourier Hermite series contains an infinite number of terms, so the results are not as good. However, they still confirm the theoretical error results of Theorem 5.6. A few plots of the two dimensional approximation are given in Figure 7.4. Notice that as N increases, the 'curling' feature near the boundary of the plot is diminished.



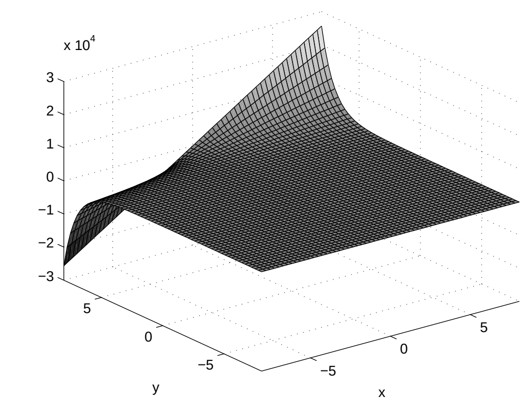
(a) $N = 5$



(b) $N = 10$



(c) $N = 20$



(d) Exact

Figure 7.4: Plots of the function and its Fourier-Hermite approximation from Example 3.

Example 7.4. *Consider in this example the functions* $f_{4,d}(\mathbf{x}) = \sqrt{\sum_{j=1}^{d}(10 + x_j)}$. *Take* $m = 2, s = 1.5$ *and* $k = 2$.

This example is designed to confirm the accuracy results for a function that is not separable in each variable. The purpose of the factor of 10 is to ensure $f_{4,d}$ is continuously differentiable on the set $I = [-A, A]^d$ for all values of N considered. Notice from Figure 7.5, that even though $f_{4,2}$ is not a polynomial or very polynomial-like, the plot of the approximation is close to the exact function on the interval I^2.

	N	Err	CT	$\mathcal{M}_N$	N^{-s}
$d=2$	5	1.9392e-003	0.001912	12960	8.1298e-003
	10	1.8200e-007	0.018205	186624	1.0233e-003
	15	2.1562e-007	0.029993	311040	3.0443e-004
	20	2.5875e-008	0.094582	1026432	1.2880e-004
	25	2.3903e-008	0.125978	1353024	6.6094e-005
	30	2.2007e-008	0.162680	1726272	3.8318e-005
$d=3$	5	1.2745e-004	0.014539	179712	8.1298e-003
	10	1.0101e-004	0.312031	3938112	1.0233e-003
	15	3.3673e-005	0.550564	7058880	3.0443e-004
	20	1.5604e-005	2.437823	27841536	1.2880e-004
	25	5.6098e-010	3.397974	38282112	6.6094e-005
	30	4.2848e-010	4.527295	50554368	3.8318e-005

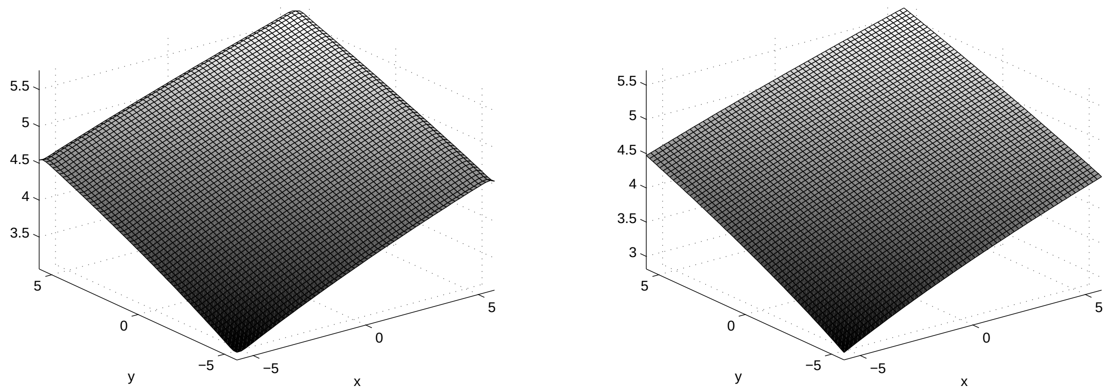
Table 7.2: Numerical Results from Example 2.

	N	Err	CT	$\mathcal{M}_N$	N^{-s}
$d=2$	5	1.5184e-003	0.001550	5760	4.0600e-002
	10	3.2744e-004	0.009040	82944	1.0233e-002
	15	9.8112e-005	0.015834	138240	4.5664e-003
	20	3.6438e-005	0.045008	456192	2.5760e-003
	25	1.5449e-005	0.061540	601344	1.6523e-003
	30	7.1808e-006	0.073578	767232	1.1496e-003
$d=3$	5	9.5993e-004	0.002681	19456	4.0649e-002
	10	1.5370e-004	0.017275	176384	1.0233e-002
	15	3.4559e-005	0.031582	316160	4.5664e-003
	20	9.8577e-006	0.116372	1323008	2.5760e-003
	25	3.3580e-006	0.160051	1819136	1.6523e-003
	30	1.3016e-006	0.208652	2402304	1.1496e-003

Table 7.3: Numerical Results for Example 3.

	N	Err	CT	$\mathcal{M}_N$	N^{-s}
$d=2$	5	7.7113e-003	0.001054	2400	8.9443e-002
	10	5.2524e-003	0.002961	15552	3.1623e-002
	15	4.2081e-003	0.004563	25920	1.7213e-002
	20	3.5441e-003	0.011196	88704	1.1180e-002
	25	2.7385e-003	0.014464	116928	8.0000e-003
$d=2$	5	3.1208e-002	0.002751	19456	8.9443e-002
	10	1.5203e-002	0.017136	176384	3.1623e-002
	15	1.0963e-002	0.030415	316160	1.7213e-002
	20	1.0891e-002	0.114381	1323008	1.1180e-002

Table 7.4: Numerical Results for Example 4.



(a) $N = 20$ (b) Exact

Figure 7.5: Plots of the function and its Fourier-Hermite approximation from Example 4.

Example 7.5. *Consider the function*

$$f_{5,2}(x,y) = \begin{cases} x^2 y, & \text{for } -A \le x < -A/3 \\ 0, & \text{for } -A/3 \le x < A/3 \\ (x-1) * y, & \text{for } A/3 \le x \le A \end{cases} \cdot$$

Take $m = 2$*,* $s = 1.5$ *and* $k = 3$*.*

This example is designed to confirm the results for functions that are not continuous. In order for any candidate function to be in the space $X_M^m(I^d)$, the points of discontinuity can be located only at points in the refinable set V_M. However, as N increases, so does the value A, which in turn adjusts the values in the refinable sets. So for this example, the value $A(N)$ will be fixed at $A(30)$. This will affect the complexity of the numerically computed coefficients, especially for smaller values of N, but enables a meaningful comparison of results.

	N	Err	CT	$\mathcal{M}_N$	N^{-s}
$d = 2$	5	1.9252e-002	0.002324	5760	8.9443e-002
	10	4.6874e-003	0.009723	82944	3.1623e-002
	15	1.9433e-003	0.014964	138240	1.7213e-002
	20	1.5688e-003	0.045885	456192	1.1180e-002
	25	7.9197e-004	0.060433	601344	8.0000e-003

Table 7.5: Numerical Results for Example 5.

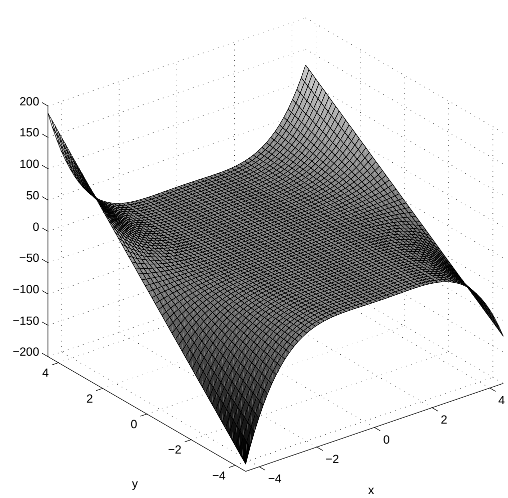


Figure 7.6: Plot of the Fourier-Hermite approximation from Example 5 with $N = 10$.

Appendices

Appendix A

Various Results

Theorem A.1. *For a positive fixed value s, let*

$$\langle \cdot, \cdot \rangle_{\kappa^s} : L^2_\omega(\mathbb{R}^d) \times L^2_\omega(\mathbb{R}^d) \to \mathbb{R}$$

be defined via

$$\langle f, g \rangle_{\kappa^s} := \sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{\langle f, H_\mathbf{n} \rangle_\omega \langle g, H_\mathbf{n} \rangle_\omega}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}} \prod_{j=1}^d (1 + n_j)^{2s},$$

for any functions $f, g \in L^2_\omega(\mathbb{R}^d)$. Then $\langle \cdot, \cdot \rangle_{\kappa^s}$ defines an inner product.

Proof. Notice the map is symmetric since for any $f, g \in L^2_\omega(\mathbb{R}^d)$,

$$\begin{aligned} \langle f, g \rangle_{\kappa^s} &= \sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{\langle f, H_\mathbf{n} \rangle_\omega \langle g, H_\mathbf{n} \rangle_\omega}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}} \prod_{j=1}^d (1 + n_j)^{2s} \\ &= \sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{\langle g, H_\mathbf{n} \rangle_\omega \langle f, H_\mathbf{n} \rangle_\omega}{\mathbf{n}! 2^\mathbf{n} \pi^{d/2}} \prod_{j=1}^d (1 + n_j)^{2s} \\ &= \langle g, f \rangle_{\kappa^s}. \end{aligned}$$

The map is linear in the first argument since

$$\begin{aligned}\langle \alpha f, g\rangle_{\kappa^s} &= \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle \alpha f, H_{\mathbf{n}}\rangle_\omega \langle g, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (1+n_j)^{2s} \\ &= \alpha \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle f, H_{\mathbf{n}}\rangle_\omega \langle g, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (1+n_j)^{2s} \\ &= \alpha \langle f, g\rangle_{\kappa^s},\end{aligned}$$

and

$$\begin{aligned}\langle f+g, h\rangle_{\kappa^s} &= \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle f+g, H_{\mathbf{n}}\rangle_\omega \langle h, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (1+n_j)^{2s} \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle f, H_{\mathbf{n}}\rangle_\omega \langle h, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (1+n_j)^{2s} + \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle g, H_{\mathbf{n}}\rangle_\omega \langle h, H_{\mathbf{n}}\rangle_\omega}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (1+n_j)^{2s} \\ &= \langle f, h\rangle_{\kappa^s} + \langle g, h\rangle_{\kappa^s},\end{aligned}$$

for any $f, g, h \in L^2_\omega(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$.

Since each term in the sum

$$\langle f, f\rangle_{\kappa^s} = \sum_{\mathbf{n}\in\mathbb{N}_0^d} \frac{\langle f, H_{\mathbf{n}}\rangle_\omega^2}{\mathbf{n}!2^{\mathbf{n}}\pi^{d/2}} \prod_{j=1}^d (a+n_j)^{2s}$$

is non-negative, $\langle f, f\rangle_{\kappa^s} \geq 0$ for every $f \in L^2_\omega(\mathbb{R}^d)$. Additionally, if $\langle f, f\rangle_{\kappa^s} = 0$, then it follows that $\langle f, H_{\mathbf{n}}\rangle_\omega$ must equal 0 for every $\mathbf{n} \in L^2_\omega(\mathbb{R}^d)$. Since $\{H_{\mathbf{n}}\}$ is a complete orthogonal collection in $L^2_\omega(\mathbb{R}^d)$, this guarantees $f \equiv 0$. Therefore the map is positive definite.

Since the map $\langle \cdot, \cdot\rangle_{\kappa^s}$ is symmetric, linear, and positive definite, it defines an inner product on $L^2_\omega(\mathbb{R}^d)$. □

Theorem A.2. *Let* $|\cdot|_{\infty,I^d} : L^2_\omega(\mathbb{R}^d) \to \mathbb{R}$ *be defined via*

$$|f|_{\infty,I^d} := \sup_{x\in I^d} |f(x)|, \textit{ for any } f \in L^2_\omega(\mathbb{R}^d).$$

Then $|\cdot|_{\infty,I^d}$ *is a seminorm.*

Proof. Note that for any $f, g \in L^2_\omega(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}|\alpha f|_{\infty,I^d} &= \sup_{x\in I^d} |\alpha f(x)| \\ &= |\alpha| \sup_{x\in I^d} |f(x)| \\ &= |\alpha||f|_{\infty,I^d},\end{aligned}$$

$$\begin{aligned}|f+g|_{\infty,I^d} &= \sup_{x\in I^d} |f(x)+g(x)| \\ &\le \sup_{x\in I^d} |f(x)| + \sup_{y\in I^d} |g(y)| \\ &= |f|_{\infty,I^d} + |g|_{\infty,I^d},\end{aligned}$$

and clearly $|f|_{\infty,I^d} \ge 0$. However, if $|f|_{\infty,I^d} = 0$, this does not mean $f \equiv 0$, but only that $f|_{I^d} \equiv 0$.

Therefore $|\cdot|_{\infty,I^d}$ defines a seminorm. □

Theorem A.3. *Let* $H_n(x)$ *be the univariate Hermite polynomial of degree* n*, for* $n \in \mathbb{N}_0$*. If*

n is odd, then H_n contains only odd powers of x. If n is even, then H_n has only even powers of x.

Proof. We will proceed by induction on n.

The theorem is clearly true for $n = 0$, since $H_n(x) \equiv 1$.

Suppose it is true for some integer $n - 1$. Then, by Theorem 2.5,

$$H_n'(x) = 2nH_{n-1}(x).$$

If n is an even integer, then $n-1$ is an odd integer, and therefore, by the assumption, H_{n-1} is polynomial with all odd-degree powers of x. Thus H_n' is an odd polynomial, and so one can conclude that H_n is an even polynomial.

If n is an odd integer, then $n - 1$ is an even integer, and then H_{n-1} and H_n' are both even polynomials. Thus H_n is a polynomial, but this does not yet show H_n is an odd polynomial. However, one can conclude H_{n-2} is an odd polynomial. Therefore, by the three term recurrence,

$$H_n(x) = 2(0)H_{n-1}(0) - 2nH_{n-2}(x) = 0,$$

for any integer $n \geq 2$, and $H_1(0) = 0$ since $H_1(x) = 2x$. Thus we have shown H_n is an odd polynomial.

Therefore the theorem is true for any non-negative integer n. □

In a single dimension, some formulas for the direct computation of integrals involving the Gaussian weight are included below.

1. $\displaystyle\int_a^b e^{-x^2}\,dx = erf(x)|_a^b$

2. $\displaystyle\int_a^b xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_a^b$

3. For $m > 2$ even, $\displaystyle\int_a^b x^m e^{-x^2}\,dx =$

$$= \left[\frac{-x^{m-1}}{2}e^{-x^2} - \sum_{j=1}^{m/2-1}\frac{\prod_{i=1}^{j}(m-2i+1)}{2^{j+1}}x^{m-2j-1}e^{-x^2} + \frac{(m-1)!!}{2^{m/2}}erf(x)\right|_a^b$$

4. For $m > 1$ odd, $\displaystyle\int_a^b x^m e^{-x^2}\,dx =$

$$= \left[\frac{-1}{2}x^{m-1}e^{-x^2} - \sum_{j=1}^{\frac{m-1}{2}}\frac{\prod_{i=1}^{j}(m-2i+1)}{2^{j+1}}x^{m-2j-1}e^{-x^2}\right|_a^b.$$

Lemma A.4. *For any $m, n \in \mathbb{N}_0$, the product $H_m H_n$ can be written as the sum*

$$H_m H_n = \sum_{i=0}^{\min\{m,n\}} \frac{2^i}{i!}\frac{m!}{(m-i)!}\frac{n!}{(n-i)!}H_{m+n-2i}.$$

Proof. If $n = 0$, then the summation reduces to the single term H_m. If $n = 1$ and $m \geq 1$, then the summation is $H_{m+1} + 2mH_{m-1}$. By the three term recurrence of the Hermite polynomials, this is equal to $2xH_m$ which is $H_n H_m$. Since the statement is symmetric in m and n, we have shown that it is true for any pair (m, n) with either $m \leq 1$ or $n \leq 1$.

We now proceed by induction. Consider a pair $(m, n+1)$ with $n \geq 1$ and $n + 1 \leq m$. Suppose the statement of the lemma is true for the pairs (m, n) and $(m, n-1)$. For notational convenience, denote $\frac{2^i}{i!}\frac{m!}{(m-i)!}\frac{n!}{(n-i)!}$ by $\gamma(m, n, i)$. Then using the three term recurrence and the induction step gives

$$\begin{aligned}
H_m H_{n+1} &= H_m \left(2xH_n - 2nH_{n-1}\right) \\
&= 2x \left(H_m H_n\right) - 2n \left(H_m H_{n-1}\right) \\
&= 2x \left(\sum_{i=0}^{n} \gamma(m,n,i) H_{m+n-2i} \right) - 2n \left(\sum_{i=0}^{n-1} \gamma(m,n-1,i) H_{m+n-1-2i} \right).
\end{aligned}$$

Using the three term recurrence, the first summation in the previous line can be rewritten in the form

$$\begin{aligned}
&\sum_{i=0}^{n} \gamma(m,n,i) 2x H_{m+n-2i} = \\
&\quad = \sum_{i=0}^{n} \gamma(m,n,i) \left[H_{m+n-2i+1} + 2(m+n-2i) H_{m+n-2i-1} \right] \\
&\quad = \left(\sum_{i=0}^{n} \gamma(m,n,i) H_{m+n-2i+1} \right) + \left(\sum_{i=0}^{n} \gamma(m,n,i) 2(m+n-2i) H_{m+n-2i-1} \right) \\
&\quad = \left(H_{m+n+1} + \sum_{i=1}^{n} \gamma(m,n,i) H_{m+n-2i+1} \right) + \\
&\qquad\quad + \left(\frac{2^{n+1} m!}{(m-n-1)!} H_{m-n-1} + \sum_{i=0}^{n-1} \gamma(m,n,i) 2(m+n-2i) H_{m+n-2i-1} \right) \\
&= H_{m+n+1} + \frac{2^{n+1} m!}{(m-n-1)!} H_{m+n-1} + \left(\sum_{i=0}^{n-1} \gamma(m,n,i+1) H_{m+n-2i-1} \right) + \\
&\qquad\quad + \left(\sum_{i=0}^{n-1} \gamma(m,n,i) 2(m+n-2i) H_{m+n-2i-1} \right),
\end{aligned}$$

after re-indexing to obtain the last equality.

Now consider a combination of the γ terms that will arise when the results of the last two strings of equalities are combined.

$$\begin{aligned}
&\gamma(m,n,i+1)+2(m+n-2i)\gamma(m,n,i)-2(n-i)\gamma(m,n,i)= \\
&\quad=\gamma(m,n,i+1)+2(m-i)\gamma(m,n,i) \\
&\quad=\left(\frac{2^{i+1}}{(i+1)!}\frac{m!}{(m-i-1)!}\frac{n!}{(n-i-1)!}\right)+\left(2(m-i)\frac{2^i}{i!}\frac{m!}{(m-i)!}\frac{n!}{(n-i)!}\right) \\
&\quad=\left(\frac{2^{i+1}}{(i+1)!}\frac{m!}{(m-i-1)!}\frac{n!}{(n-i-1)!}\right)+\left(\frac{2^{i+1}}{i!}\frac{m!}{(m-i-1)!}\frac{n!}{(n-i)!}\right) \\
&\quad=\left(\frac{2^{i+1}}{i!}\frac{m!}{(m-i-1)!}\frac{n!}{(n-i-1)!}\right)\left(\frac{1}{i+1}+\frac{1}{n-i}\right) \\
&\quad=\left(\frac{2^{i+1}}{i!}\frac{m!}{(m-i-1)!}\frac{n!}{(n-i-1)!}\right)\left(\frac{n+1}{(i+1)(n-i)}\right) \\
&\quad=\frac{2^{i+1}}{(i+1)!}\frac{m!}{(m-i-1)!}\frac{(n+1)!}{(n-i)!} \\
&\quad=\gamma(m,n+1,i+1).
\end{aligned}$$

Combining all the previous results gives

$$\begin{aligned}
H_mH_{n+1}&=H_{m+n+1}+\frac{2^{n+1}m!}{(m-n-1)!}H_{m+n-1}+ \\
&+\left(\sum_{i=0}^{n-1}\left(\gamma(m,n,i+1)+\gamma(m,n,i)2[(m+n-2i)-(n-i)]\right)H_{m+n-2i-1}\right) \\
&=H_{m+n+1}+\frac{2^{n+1}m!}{(m-n-1)!}H_{m+n-1}+\left(\sum_{i=0}^{n-1}\gamma(m,n+1,i+1)H_{m+n-2i-1}\right) \\
&=H_{m+n+1}+\frac{2^{n+1}m!}{(m-n-1)!}H_{m+n-1}+\left(\sum_{i=1}^{n}\gamma(m,n+1,i)H_{m+n-2i+1}\right) \\
&=\sum_{i=0}^{n+1}\gamma(m,n+1,i)H_{m+n+1-2i}.
\end{aligned}$$

So the statement of the lemma is true for $(m,n+1)$ with $m\geq n+1\geq 2$. Therefore, by symmetry, the lemma is true for all pairs of nonnegative integers m and n. □

Parseval's identity [14] generalizes to collections of complete orthogonal systems.

Theorem A.5. *Let* $\{H_n\}_{n=0}^{\infty}$ *be the collection of Hermite polynomials. Let* $f \in L_{\omega}^2(\mathbb{R})$. *If the Fourier Hermite expansion of* f *is* $\sum_{n=0}^{\infty} c_n H_n$, *then*

$$||f||_{\omega}^2 = \sum_{n=0}^{\infty} |c_n|^2 ||H_n||_{\omega}^2.$$

Proof. The collection of Hermite polynomials can be normalized.

Let $h_n := H_n/||H_n||_{\omega}$. Then f can be expressed in terms of the normalized Hermite polynomials:

$$\begin{aligned} f &= \sum_{n=0}^{\infty} c_n H_n \\ &= \sum_{n=0}^{\infty} c_n ||H_n||_{\omega} \frac{H_n}{||H_n||_{\omega}} \\ &= \sum_{n=0}^{\infty} d_n h_n, \end{aligned}$$

where $d_n := c_n ||H_n||_{\omega}$ for each $n \in \mathbb{N}_0$.

Since $\{H_n\}$ is a complete orthogonal system in $L_{\omega}^2(\mathbb{R})$, so is $\{h_n\}$. By the classical

Parseval's identity,

$$
\begin{aligned}
||f||_\omega^2 &= \sum_{n=0}^{\infty} |d_n|^2 \\
&= \sum_{n=0}^{\infty} |c_n|^2 ||H_n||_\omega^2.
\end{aligned}
$$

□

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Vita

The author grew up in Kansas City, Missouri, and earned B.S. and M.S. degrees in Mathematics from Kansas State University in 2002 and 2005 respectively.

She came to Syracuse University in 2006 to study Mathematics with Professor Yuesheng Xu. As a graduate student she had many fantastic opportunities to enrich her studies. In 2008 she visited the Jet Propulsion Laboratory of NASA to help investigate possible future scientific missions. During the summer of 2009, she was awarded the East Asian and Pacific Summer Institutes fellowship from the NSF to study mathematics at the Chinese Academy of Sciences in Beijing, China. This was a fruitful trip, and she visited the Academy again the following December. During summer 2010 she studied for two months at Sun-Yat Sen University in Guangzhou, China with Professors Yuesheng Xu and Lixin Shen.

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Education

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Duties as a Graduate Student: Perform primary, independent research of new mathematics. Review literature, identify intelligence gaps, and develop data and theorems to close gaps. Prepare reports, present findings to diverse audience, and defend activity to PhD advisor and committee.

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Conference and Seminar Presentations

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Programming and Projects

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