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Probabilistic Analysis of the Median Rule: Asymptotics and Applications

Anil Ravindran Menon
Syracuse University

Kishan Mehrotra
Syracuse University, mehrotra@syr.edu

Chilukuri K. Mohan
Syracuse University, ckmohan@syr.edu

Sanjay Ranka
Syracuse University

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Chilukuri K. Mohan, and Sanjay Ranka

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School of Computer and Information Science
Syracuse University
Suite 4-116, Center for Science and Technology
Syracuse, New York 13244-4100

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Anil Menon
armenon@top.cis.syr.edu

Kishan Mehrotra
kishan@top.cis.syr.edu

Chilukuri K. Mohan
mohan@top.cis.syr.edu

Sanjay Ranka
ranka@cis.ufl.edu

1 Introduction

The solution of integer optimization problems by relaxation methods consists of three parts. First, the discrete problem is converted into a continuous optimization problem, which is generally more tractable. Second, the relaxed problem is solved efficiently, yielding a optimal solution in the continuous space. Finally, an assignment procedure is used to map this solution to a "suitable" discrete solution.

One heuristic — we call it the *relaxation* heuristic — that often guides the choice and design of assignment algorithms is: "given a continuous optimal solution, the corresponding integer optimal solution is likely to be nearby " (with respect to some well defined metric). Intuitively, this heuristic is reasonable for objective functions that are, say, Lipschitz functions. For such functions, an assignment algorithm might map the continuous optimal solution to the *nearest* feasible[1] solution in the discrete space, in the hope that the discrete solution will be optimal as well.

In this paper, we consider properties of a particular assignment algorithm known as the median rule. Define a binary vector to be *balanced* when the numbers of its 1's and 0's differ at most by one. The median rule used to assign n-dimensional real vectors to n-dimensional *balanced* binary vectors, may be loosely described as follows: map the ith component of a real vector to a 0 or 1, depending on whether that component is smaller or greater than the median value of the vector components.

We address two aspects of the median rule. The first result is that given a real vector, the median rule produces the closest balanced binary vector, with respect to *any* Schur-convex distance criteria. This includes several Minkowski norms, entropy measures, gauge functions etc. In this sense, the median rule optimally implements the relaxation heuristic.

The second result addresses the issue of *relaxation error*. Though the median rule produces the nearest balanced integer solution to a given real vector, it is possible that this solution is sub-optimal, and the actual optimal solution is located elsewhere. The difference between the actual optimal cost and the cost of the solution obtained by the median rule is called the relaxation error.

We consider the optimization of real valued, parametrized, multivariable Lipschitz functions where domains are the set of balanced binary vectors. Varying the parameters over the range of their values, we obtain an ensemble of such problems. Each problem instance in the ensemble has an

[1]By a "feasible" solution we mean a solution that satisfies the constraints of the problem.

optimal real cost, an integer cost, and an associated relaxation error. We establish upper bounds on the probability that the relaxation error is greater than a given threshold t. In general, these bounds depend on the random model being considered.

These results have an immediate bearing on the important graph bisection width problem, which involves the minimization of a certain semidefinite quadratic cost function over balanced binary domains. This important problem arises in a variety of areas including load balancing [11, 16], storage management [22], distributed directories [15], and VLSI design [10]. The results obtained indicate that the median rule in a certain precise sense, is an optimal assignment procedure for this problem.

The rest of the paper is organized as follows: In section 3, we prove the shortest distance properties of the median rule. In section 4, we introduce the concept of relaxation error and the Lipschtiz bisection problem. Upper bounds on the relaxation error are obtained in Section 5. A discussion on these results is given in Section 6.

2 Preliminaries

We will use the following conventions:

- $\mathbb{R}^n$ denotes n-dimensional real space.
- $\mathbb{R}^n_+ = \{(x_1, \ldots x_n) : x_i > 0, \text{ for } i = 1, \ldots, n\}$
- $\mathbb{D}^n = \{(x_1, \ldots, x_n) : 0 \leq x_i \leq 1, \text{ for } i = 1, \ldots, n\} \equiv [0,1]^n$
- $\mathbb{A}^n_2 = \{(b_1, \ldots, b_n) : b_i \in \{0,1\}, \mid \sum_i b_i - \lfloor n/2 \rfloor \mid \leq 1 \text{ for } i = 1, \ldots, n\}$. $\mathbb{A}^n_2$ is referred to as the set of balanced binary vectors.

If $\boldsymbol{x}$ is a vector, then $\boldsymbol{x}^t$ stands for its transpose. Also,

- x_m = median value of the vector $\boldsymbol{x}^t = (x_1, \ldots x_n)$
- $\mathbf{0}^t = (0, 0, \cdots, 0)$
- $\boldsymbol{e}^t = (1, 1, \cdots, 1)$

For any $\boldsymbol{x}^t = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

- $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ denote the components of $\boldsymbol{x}$ arranged in decreasing order. $\boldsymbol{x}_\downarrow = (x_{[1]}, \ldots, x_{[n]})$ is the decreasing rearrangement of $\boldsymbol{x}$.
- Similarly, $\boldsymbol{x}_\uparrow = (x_{(1)}\, x_{(2)} \cdots x_{(n)})$ denotes the increasing rearrangement of $\boldsymbol{x}$.

The following definition is central to what follows:

Definition 2.1 (Majorization, Schur-Convexity) [12, pp. 7,54] If $\boldsymbol{x}, \boldsymbol{y} \in, \mathbb{R}^n$ then, $\boldsymbol{x}$ is said to *majorize* $\boldsymbol{y}$, denoted $\boldsymbol{y} \preceq \boldsymbol{x}$ (equivalently, $\boldsymbol{x} \succeq \boldsymbol{y}$) if the following conditions are satisfied:

1. $\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]} \quad \forall k = 1, \ldots, n-1$
2. $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$

Definition 2.2 (Schur-Convexity) A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex, if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{x} \preceq \boldsymbol{y}$ implies that $F(\boldsymbol{x}) \leq F(\boldsymbol{y})$. If the inequalities listed above are strict then F is said to be *strictly* Schur-convex. ∎

3 Median Rule

Consider the following optimization problem:

> Given a vector $\boldsymbol{x} \in \mathbb{R}^n$, find a vector $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ that minimizes $\sum_i (x_i - \mu_i)^2$, subject to the condition that $\mu_1 = \mu_2 = \cdots, = \mu_n$.

Let $\bar{x}$ denote the average of the components x_i. Then it is well known that the vector $\mu = (\bar{x}, \bar{x}, \ldots, \bar{x})$ is the unique solution to the above problem. In this section we consider a similar problem:

> **Problem A:** Given a vector $\boldsymbol{x} \in \mathbb{R}^n$, find a vector $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbf{A}_2^n$ that minimizes $F(\boldsymbol{x} - \boldsymbol{b})$, where $F : \mathbb{R}^n \to \mathbb{R}$ is any Schur-convex function.

Thus we seek a balanced vector $\boldsymbol{b}$ closest to the given vector $\boldsymbol{x}$ with respect to a distance measure based on the class of Schur-convex functions. Schur-convex functions occupy an enormous amount of mathematical real estate. Examples of such functions include the Minkowski l_p norms[2] for $p \geq 1$, Shannon entropy, symmetric quasi-convex functions, star shaped functions, elementary symmetric functions, and many "inequality measures" such as the Gini coefficient, Simpson's measure, and the Schutz coefficient [12, pp. 54-99,407-411]. We shall show that the following algorithm, known as the *median rule*, produces such a balanced vector. Basically, the procedure involves (1) finding the median of the components of the vector $\boldsymbol{x}$, and (2) assigning the i^{th} component of $\boldsymbol{b}$ to 0, if the i^{th} component of $\boldsymbol{x}$ is less than the median, and to 1 otherwise. Components of $\boldsymbol{x}$ equal to median are only technically troublesome. Formally, the rule may be defined as follows:

Definition 3.1 Let $\boldsymbol{x} \in \mathbb{R}^n$, and x_m denote the associated median. The median rule is a function $M : \mathbb{R}^n \to \mathbf{A}_2^n$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$M((x_1, x_2, \cdots, x_n)) = (b_1, b_2, \cdots, b_n) \quad \text{where} \quad b_i = \begin{cases} 0 & \text{if } x_i < x_m \\ 1 & \text{if } x_i > x_m \end{cases} \tag{3.1}$$

Those components of $\boldsymbol{x}$ equal in value to x_m are assigned in the following manner. Let $k \leq n$ be the the number of components of $\boldsymbol{x}$ equal in value to x_m. Assign $\lfloor k/2 \rfloor$ of these components to 0, and the remaining $\lceil k/2 \rceil$ components to 1. We shall write $b_i = M(x_i)$, to indicate that the i-th real co-ordinate is mapped to b_i by the median assignment rule. ∎

The following theorem is the solution to Problem A.

Theorem 3.1 (Optimality) Let $F : \mathbb{R}^n \to \mathbb{R}$ be *any* Schur-convex function. Define:

$$\begin{aligned} &G_F : \mathbb{R}^n \times \mathbf{A}_2^n \to \mathbb{R}^n \\ &G_F(\boldsymbol{x}, \boldsymbol{b}) = F(\boldsymbol{x} - \boldsymbol{b}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbf{A}_2^n \end{aligned} \tag{3.2}$$

If $M : \mathbb{R}^n \to \mathbf{A}_2^n$ is the median rule and π is *any* $n \times n$ permutation matrix, then $G_F(\boldsymbol{x}, M(\boldsymbol{x})) \leq G_F(\boldsymbol{x}, \pi(M(\boldsymbol{x})))$. ∎

[2] If $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$, then $\|\boldsymbol{x}\|_p = \{\sum_{i=1}^n |x_i|^p\}^{(1/p)}$.

Proof: Let $\boldsymbol{c} = \pi(M(\boldsymbol{x}))$; in other words assignment of x_i's to binary values in $\boldsymbol{c}$ differs from the median assignment and is some permutation of $M(\boldsymbol{x})$). In view of Schur-convexity of F, it suffices to show that the vector $\boldsymbol{x} - M(\boldsymbol{x}) \preceq \boldsymbol{x} - \boldsymbol{c}$ for all π. Without loss of generality, assume that the components of $\boldsymbol{x}$ are in decreasing order; $M(\boldsymbol{x})$ will therefore be of the form: $(1, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$. From a standard result in majorization, to prove $\boldsymbol{x} - M(\boldsymbol{x}) \preceq \boldsymbol{x} - \boldsymbol{c}$, it suffices to establish the result for the case when $\boldsymbol{x} - M(\boldsymbol{x})$ and $\boldsymbol{x} - \boldsymbol{c}$ differ in any two components only [12, pp. 58, note A.5]. Since $F(\cdot)$ is Schur-convex it is necessarily symmetric, hence the order of the components do not matter, and without loss of generality we may assume $\boldsymbol{x} - M(\boldsymbol{x})$ and $\boldsymbol{x} - \boldsymbol{c}$ differ in the first two components, i.e. let,

$$\begin{aligned} \boldsymbol{x} - M(\boldsymbol{x}) &= (x_1 - M(x_1), x_2 - M(x_2), y_3, y_4, \ldots, y_n), \\ \boldsymbol{x} - \boldsymbol{c} &= (x_1 - c_1, x_2 - c_2, y_3, y_4, \ldots, y_n), \end{aligned}$$

and $x_1 - M(x_1) \neq x_1 - c_1, x_2 - M(x_2) \neq x_2 - c_2$. Since, $\boldsymbol{c}$ is a permutation of $M(\boldsymbol{x})$, it follows that $c_1 = M(x_2)$. Similarly, $c_2 = M(x_1)$. Also, $x_1 \geq x_2$ implies that $M(x_1) \geq M(x_2)$ (and therefore $c_1 \leq c_2$). Now,

$$\begin{aligned} x_1 - M(x_1) &\leq x_1 - M(x_2) = x_1 - c_1 \\ x_1 - M(x_1) + x_2 - M(x_2) &= x_1 - M(x_2) + x_2 - M(x_1) = x_1 - c_1 + x_2 - c_2 \end{aligned}$$

Other components of $\boldsymbol{x} - M(\boldsymbol{x})$ and $\boldsymbol{x} - \boldsymbol{c}$ being the same it follows that $\boldsymbol{x} - M(\boldsymbol{x}) \preceq \boldsymbol{x} - \boldsymbol{c}$. ∎

G_F is a really a functional, but we have chosen to finesse over this point, writing it as a function.

Remark 3.1 Theorem 3.1 is a significant extension of the work of Chan et. al. [7], the starting point for this paper. They had proved the above result with respect to the Minkowski l_p norms with $p \geq 1$. Since all symmetric gauge functions — of which the l_s norms with $s \geq 1$ constitute a special sub-class — are Schur-convex functions [12, pp. 96], the result of Chan et. al. [7] is an immediate consequence of Theorem 3.1. Many such results could be generated by suitably choosing the Schur-convex function, and (perhaps) restricting the domains of the real vectors $\boldsymbol{x}$.

The proof of Theorem 3.1 reveals that:

- Specific aspects of the median rule enter the arguments in a relatively minor way, and rules based on other positional means could well be constructed that give similar Schur-convex optimality results.
- Theorem 3.1 can be extended to other finite discrete spaces, as follows. Let $\mathbf{A}_p^n \subseteq \{0, 1, 2 \cdots, p-1\}^n$ such that if $\boldsymbol{c} \in \mathbf{A}_p^n$, then $n_0 = n_1 = \cdots = n_{p-1}$, where n_i is the number of i's in $\boldsymbol{c}$. Then what is the vector in A_p^n nearest to a vector $\boldsymbol{x} \in \mathbb{R}^n$? The proof of Theorem 3.1 can be extended to show that the discrete vector obtained by first sorting the elements of $\boldsymbol{x}$, and then assigning the first k elements to 0, the next k elements to 1 etc., and finally assigning the last k elements to $p-1$, is the closest vector to $\boldsymbol{x}$ (with respect to the same distance measures as in Theorem 3.1). In other words assign a real component x_i to l if x_i lies between the $(l/p)^{\text{th}}$ quantile and the $((l+1)/p)^{\text{th}}$ quantile. The case when n is not a multiple of p can be handled by suitably defining the assignment rule, in a manner similar to Definition 3.1.

- The restriction of the median rule's range to $\{0,1\}$ vectors is not crucial for proving Theorem 3.1. If the median rule maps the real components greater than the median to say, α, and elements less than the median to β, with ties being decided in a manner similar to that outlined in Definition 3.1, then as long as $\alpha > \beta$, the theorem is essentially unchanged. For example, we could use $\{-1,1\}^n$ as the range of $M(\cdot)$, rather than $\{0,1\}^n$.

By choosing appropriate Schur-convex functions F and using Theorem 3.1, many interesting special cases may be proved, as shown by the next example.

Example 3.1 If $\boldsymbol{p} \in \mathbb{D}^n$ such that $\sum_i p_i = 1$, then the (Shannon) entropy of $\boldsymbol{p}$ is given by:

$$H(\boldsymbol{p}) = H(p_1, p_2, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i. \tag{3.3}$$

It can be shown that $-H(\boldsymbol{p})$ is strictly Schur-convex [12, pp. 71]. Let $U \subset \mathbb{R}_+^n$ such that if $\boldsymbol{x} \in U$ then $\forall i : x_i > 1$. Set $K(\boldsymbol{x}) = (\sum_{i=1}^n x_i) - (n/2)$ so that $(\boldsymbol{x} - \boldsymbol{b})/K(\boldsymbol{x}) \in \mathbb{D}^n$ and $\sum_i((x_i - b_i)/K(\boldsymbol{x}) = 1$. Therefore, $(\boldsymbol{x} - M(\boldsymbol{x}))/K(\boldsymbol{x})$ represents the normalized "deviation vector" of $\boldsymbol{x}$ with respect to its median mapping $M(\boldsymbol{x})$. The entropy $H((\boldsymbol{x} - \boldsymbol{b})/K(\boldsymbol{x}))$ is well defined for all $\boldsymbol{x} \in U$ and $\boldsymbol{b} \in \mathbf{A}_2^n$. From Theorem 3.1, it follows that

$$H\left(\frac{\boldsymbol{x} - M(\boldsymbol{x})}{K(\boldsymbol{x})}\right) \geq H\left(\frac{\boldsymbol{x} - \pi(M(\boldsymbol{x}))}{K(\boldsymbol{x})}\right). \tag{3.4}$$

Thus the Shannon entropy of $(\boldsymbol{x} - M(\boldsymbol{x}))/K(\boldsymbol{x})$ exceeds that obtained with any other permutation of $M(\boldsymbol{x})$. ∎

The following lemma generalizes Theorem 3.1 to the class of *arrangement decreasing* functions. See [?, pp. 375-378], [12, pp. 158-164] for definitions and many examples of this class of functions.

Lemma 3.1 Let $G : \mathbf{A}_2^n \times \mathbb{R}^n \to \mathbb{R}$ be any arrangement decreasing function. Let $\boldsymbol{x}_\uparrow = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})$, $\boldsymbol{b}_\uparrow = (b_{(1)}, b_{(2)}, \ldots, b_{(n)})$. Let π and π' be two $n \times n$ permutation matrices such that π is *better arranged* than π'. Then, $G(\boldsymbol{b}_\uparrow, \pi(\boldsymbol{x}_\uparrow)) \leq G(\boldsymbol{b}_\uparrow, \pi'(\boldsymbol{x}_\uparrow))$. ∎

Proof: The function G_F defined in Theorem 3.1 can be shown to be arrangement decreasing [?, pp. 377]. From the definition of arrangement decreasing functions, it follows that other arrangement decreasing functions must also decrease upon any rearrangement that disturb the natural order specified by $\boldsymbol{x}_\uparrow$ and $\boldsymbol{b}_\uparrow$. ∎

Thus, with respect to a very wide family of functions, the median rule is optimal. One may think of the balanced vectors as Voronoi *seed* points; they partition real space into regions (Voronoi cells) such that every point in a cell is closer (with respect to some distance metric) to one seed than to another. Points equidistant from two or more seed points lie on the boundaries of the Voronoi regions. The construction of Voronoi cells is dependent on the choice of the seed points, and on the choice of a distance measure. The same set of seed points with different distant measures can give rise to different Voronoi cells. Theorem 3.1 and Lemma ?? attest that with the balanced vectors as seed points, and with respect to a wide variety of distance measures, the same Voronoi cells are obtained.

4 Relaxation error and the Median rule

We are interested in studying the efficacy of the median rule for the graph bisection problem with respect to a criterion we call the relaxation error. We study the relaxation error in a more general setting, viz. the class of Lipschitz continuous functions. We refer to this general class of problems as *Lipschitz bisectioning*. We first define the graph bisection problem, and then the Lipschitz bisection problem.

4.1 Graph Bisection

Let $G = (V, E)$ be an undirected graph, with node set V, edge set E, and no self loops. Let the total number of nodes be $n = |V|$. The nodes are assumed be numbered 1 through n. To each edge from node i to node j is assigned a positive real value, the "edge-weight" w_{ij}. Define the weighted adjacency matrix $A(G) = (a_{ij})_{n\times n}$, and the associated row-sum diagonal matrix $D(G) = (d_{ij})_{n\times n}$ by,

$$a_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin E \\ w_{ij} = w_{ji} & \text{otherwise} \end{cases} \qquad d_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{k=1}^{n} a_{ik} & \text{otherwise} \end{cases} \tag{4.1}$$

Then the Laplacian (Fiedler matrix, Kirchoff matrix, admittance matrix) of the graph is given by, $L(G) = D(G) - A(G)$. The Laplacian enjoys several useful and remarkable properties; positive semidefiniteness is one of them [14]. The graph bisection problem, formally stated below:

$$\text{Determine } \boldsymbol{b} \in \mathbf{A}_2^n \text{ that minimizes } \boldsymbol{b}^t L(G)\, \boldsymbol{b}. \tag{4.2}$$

is NP-complete [8, pp. 209].

The graph bisection problem is of particular relevance to parallel computing where it translates to minimizing the net communication cost (cut-size) between processors given that each processor is to be subject to the same computational load. Other applications include storage management [22], distributed directories [15], and VLSI design [10].

To obtain a (suboptimal) solution of the graph bisection problem one method is to relax the constraint on the domain of $\boldsymbol{b}$ from $\mathbf{A}_2^n$ to $\mathbb{R}_+^n$, and solve:

$$\text{Minimize } : \boldsymbol{x}^t L(G)\, \boldsymbol{x} \tag{4.3}$$

$$\text{such that}: \|\boldsymbol{x}\|^2 = \frac{n}{2} \text{ and } \boldsymbol{x} \in \mathbb{R}_+^n. \tag{4.4}$$

The quadratic function $\boldsymbol{b}^t L(G)\, \boldsymbol{b}$ for $\boldsymbol{b} \in \mathbf{A}_2^n$, identified as the net communication cost, is also known as the bisection width of the graph. Note that the cost of the optimal real solution is always less than or equal to that of the optimal discrete solution since $\mathbf{A}_2^n \subseteq \mathbb{R}_+^n$.

For any graph, irrespective of the Laplacian, the relaxed graph bisection problem has a globally minimizing solution $\boldsymbol{x}^* \propto \boldsymbol{e}$, with a corresponding cost of 0 (i.e., $(\boldsymbol{x}^*)^t L(G) \boldsymbol{x}^* = 0$), and carries no information about the integer solution. We therefore impose an additional constraint:

$$\boldsymbol{x} \neq s\boldsymbol{e} \qquad \text{for any } s \in \mathbb{R}. \tag{4.5}$$

In fact the solution $\boldsymbol{e}$ is an eigenvector corresponding to $\lambda_1 = 0$, the smallest eigenvalue of the Laplacian $L(G)$, and Equation ?? excludes solutions that correspond to the smallest eigenvalue. An eigenvector corresponding to the *second* smallest eigenvalue $\lambda_2(L)$ of $L(G)$ is the desired continuous optimal solution [14]. The median rule is then applied to map the continuous solution to a discrete one, hopefully also optimal. We generalize bisection problems to cost functions that are Lipschitz continuous.

4.2 Lipschitz bisection

Let $U \subseteq \mathbb{R}_+^k$ and $C : \mathbb{R}_+^n \times U \rightarrow \mathbb{R}$ be a multivariable Lipschitz function with constant $\alpha > 0$. We shall refer to variables in U as *parameters* of the function C. In short, C is a Lipschitz function with n variables and k parameters (the reason for introducing parameters is to set up an "ensemble" later on in this section). To emphasize the special role of the parameters, we shall use the notation $C_{\boldsymbol{u}}(\boldsymbol{x})$ to denote $C(\boldsymbol{x}, \boldsymbol{u})$, where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in U$. $C_{\boldsymbol{u}}$ is a Lipschitz function with constant $\alpha > 0$ if,

$$|C_{\boldsymbol{u}}(\boldsymbol{x}) - C_{\boldsymbol{u}}(\boldsymbol{y})| \leq \alpha \|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n, \ \boldsymbol{u} \in U, \tag{4.6}$$

where $\| \ \|$ is the l_2 norm, i.e., $\|\boldsymbol{x}\| = (\sum_i x_i^2)^{1/2}$. $C_{\boldsymbol{u}}$ is to be thought of as an *instance* of a family of similar problems with the same Lipschitz constant, differing only in the values of certain parameters. We are interested in the following minimization problem:

$$\text{For a fixed } \boldsymbol{u} \in U, \text{ minimize } C_{\boldsymbol{u}}(\boldsymbol{c}) \text{ where } \boldsymbol{c} \in \mathbf{A}_2^n$$

Denote the (unknown) optimum solution to the above problem by $\boldsymbol{b}^*$, i.e., $C_{\boldsymbol{u}}(\boldsymbol{b}^*) = \min_{\boldsymbol{c} \in \mathbf{A}_2^n} C_{\boldsymbol{u}}(\boldsymbol{c})$. Assume that there exists an efficient method to solve the following relaxed version,

$$\text{Minimize } C_{\boldsymbol{u}}(\boldsymbol{x}) \text{ such that } \|\boldsymbol{x}\|^2 = \frac{n}{2} \quad \text{and} \quad \boldsymbol{x} \neq s\boldsymbol{e}, \quad \forall\, s \in \mathbb{R} \text{ and } \boldsymbol{x} \in \mathbb{R}_+^n \tag{4.7}$$

Thus in the relaxed version of the optimization problem, we place a norm constraint ($\|\boldsymbol{x}\|^2 = n/2$), a domain constraint ($\boldsymbol{x} \in \mathbb{R}_+^n$), and one structural constraint, viz., all components of $\boldsymbol{x}^*$ are not the same ($\boldsymbol{x} \neq s\boldsymbol{e}$). If $\boldsymbol{x}^*$ denotes the optimal solution to the above problem, i.e., $C_{\boldsymbol{u}}(\boldsymbol{x}^*) = \min_{\boldsymbol{x} \in \mathbb{R}^n} C_{\boldsymbol{u}}(\boldsymbol{x})$ then, due to $\mathbf{A}_n^2 \subseteq \mathbb{R}_+^n$,

$$C_{\boldsymbol{u}}(\boldsymbol{x}^*) \leq C_{\boldsymbol{u}}(\boldsymbol{b}^*). \tag{4.8}$$

Given a continuous optimal solution $\boldsymbol{x}^*$, the median rule can be used to find the nearest (in l_2 norm) feasible solution $\hat{\boldsymbol{b}} = M(\boldsymbol{x}^*) \in A_n^2$. It is possible that $\hat{\boldsymbol{b}}$ is not the optimal integer solution, i.e., $\hat{\boldsymbol{b}} \neq \boldsymbol{b}^*$ and,

$$C_{\boldsymbol{u}}(\boldsymbol{b}^*) = \min_{\boldsymbol{b} \in \mathbf{A}_2^n} C_{\boldsymbol{u}}(\boldsymbol{b}) \ < \ C_{\boldsymbol{u}}(\hat{\boldsymbol{b}}) = C_{\boldsymbol{u}}(M(\boldsymbol{x}^*)). \tag{4.9}$$

Let $\boldsymbol{b}$ denote any feasible solution. The relaxation error δ is defined as the absolute difference between the two costs associated with the solutions $\boldsymbol{b}^*$ and $\boldsymbol{b}$,

$$\delta = |C_{\boldsymbol{u}}(\boldsymbol{b}^*) - C_{\boldsymbol{u}}(\boldsymbol{b})| \tag{4.10}$$

Now, it is possible that $C_{\boldsymbol{u}}(\boldsymbol{b}^*)$ will be equal to its lowest possible value $C_{\boldsymbol{u}}(\boldsymbol{x}^*)$, so that $\delta \leq |C_{\boldsymbol{u}}(\boldsymbol{x}^*) - C_{\boldsymbol{u}}(\boldsymbol{b})|$. Since $C_{\boldsymbol{u}}(\cdot)$ is Lipschitz it follows that,

$$\delta \leq |C_{\boldsymbol{u}}(\boldsymbol{x}^*) - C_{\boldsymbol{u}}(\boldsymbol{b})| \leq \alpha \|\boldsymbol{x}^* - \boldsymbol{b}\| \tag{4.11}$$

Therefore, smaller the value of $\|\boldsymbol{x}^* - \boldsymbol{b}\|$, the smaller the worst case relaxation error is expected to be. Since $\|\boldsymbol{x}^* - \boldsymbol{b}\|$ has the least value when $\boldsymbol{b} = \hat{\boldsymbol{b}} = M(\boldsymbol{x}^*)$ (Theorem 3.1), the relaxation error is likely to be smallest when the median rule is used. Our goal is to study the expected behavior of δ in terms of its maximum possible value $\alpha \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\|$. In other words we consider the problem of bounding the relaxation error in a probabilistic setting.

5 Relaxation Error: Probabilistic Analysis

A probabilistic analysis can be initiated by constructing an ensemble of problems, assign the ensemble a suitable measure and then study the relaxation error with respect to this measure. Thus, assign the uniform measure to U. To each random selected vector $\boldsymbol{u} \in U$, there corresponds a *unique* minimization problem:

$$\text{Minimize } C_{\boldsymbol{u}}(\boldsymbol{c}) \text{ where } \boldsymbol{c} \in \mathbf{A}_2^n.$$

Associated with this problem is its continuous relaxation, the quantities $\boldsymbol{b}$, $\boldsymbol{b}^*$, $\boldsymbol{x}^*$ and relaxation error δ. Since $\boldsymbol{u}$ is a random variable, so are δ, $\boldsymbol{b}$, $\boldsymbol{b}^*$, and $\boldsymbol{x}^*$.

Endowing U with the uniform measure leads to a sample space of random minimization problems, where each problem is uniquely characterized by its parameters. We refer to this sample space as the ensemble $\mathcal{C}(n, \alpha)$. With respect to the assigned measure, it is valid to ask for the probability that the relaxation error is greater than some arbitrary positive value. Lemma 5.1 addresses this issue.

Note: Though we should distinguish between the optimal solution to a *randomly* chosen problem from the ensemble $\mathcal{C}(n, \alpha)$, and the optimal solution $\boldsymbol{x}^*$ to a fixed, *particular* instance C, we shall be denote both by $\boldsymbol{x}^*$. This will enable us to avoid writing $\mathbf{X}^*$ instead of $\boldsymbol{x}^*$, or $\mathbf{B}^*$ instead of $\boldsymbol{b}^*$ and so on. Thus $\boldsymbol{x}^*$, $\boldsymbol{b}^*$, $\boldsymbol{b}$ and δ are all understood to be random variables, and not solutions to a given problem. ∎

The following lemma establishes bounds on the relaxation error for Lipschitz bisectioning when the median rule is used.

Lemma 5.1 Let $\mathcal{C}(n, \alpha)$ denote the ensemble of random minimization problems C with Lipschitz constant α. Let the random variables $\boldsymbol{x}^*$, $\hat{\boldsymbol{b}}$, and δ be as defined in Section 4.2. Then,

$$\text{Prob}(\,\delta > t\,) \leq \text{Prob}\left\{ \sum_{i=1}^{n/2} x^*_{[i]} \leq \frac{n - (t/\alpha)^2}{2} \right\} \tag{5.1}$$

∎

Proof: By the definition of Lipschitz functions and arguments presented before Equation (4.13),

$$\delta \leq \alpha \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\| \tag{5.2}$$

In other words for any $t > 0$, the event $\{\delta > t\}$ is a subset of the event $\{\alpha \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\| > t\}$. Hence:

$$\begin{aligned} \text{Prob}(\delta > t) &\leq \text{Prob}\left\{ \alpha \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\| > t \right\} \\ &= \text{Prob}\left\{ \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\|^2 > (t/\alpha)^2 \right\} \end{aligned} \tag{5.3}$$

But $\|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\|^2 = n - \sum_{i=1}^{n} x_i^* \hat{b}_i$. Since $\hat{b}_i = 1$ for the largest half of $\boldsymbol{x}^*$ and 0 otherwise, it follows that,

$$\begin{aligned} \text{Prob}(\delta > t) &\leq \text{Prob}\left\{ \sum_{i=1}^{n} x_i^* \hat{b}_i \leq \frac{n - (t/\alpha)^2}{2} \right\} \\ &\leq \text{Prob}\left\{ \sum_{i=1}^{n/2} x_{[i]}^* \leq \frac{n - (t/\alpha)^2}{2} \right\} \quad \blacksquare \end{aligned} \tag{5.4}$$

Equation (5.1) has important consequences. Since we have shown that the random variable δ is stochastically smaller[3] than $\alpha \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\|$, it follows that

$$E[\delta] \leq \alpha E[\, \|\boldsymbol{x}^* - \hat{\boldsymbol{b}}\| \,]. \tag{5.5}$$

We use this result to obtain bounds on the expected error in relaxation.

The following corollary, a consequence of the arithmetic mean-geometric mean inequality, provides another bound on the relaxation error in an alternative form.

Corollary 5.1 Let $\boldsymbol{x}^*$, δ, n, α be as defined in Section 4.2, and let $\tau = t/\alpha$. Then,

$$\text{Prob}(\delta > t) \leq \text{Prob}\left\{ \prod_{i=1}^{n/2} x_{[i]}^* \leq \left(1 - \frac{\tau^2}{n}\right)^{n/2} \right\} \tag{5.6}$$

$$\leq \text{Prob}\left\{ \prod_{i=1}^{n/2} x_{[i]}^* \leq \exp(-\tau^2/2) \right\} \quad \blacksquare \tag{5.7}$$

The median rule is optimal in the sense that if $\boldsymbol{x}^*$ was mapped to an integer solution $\boldsymbol{b} \neq \hat{\boldsymbol{b}}$ then,

$$\text{Prob}(\delta > t) \leq \text{Prob}\left\{ \sum_{i=1}^{n} b_i x_i^* \leq \frac{n - (t/\alpha)^2}{2} \,\middle|\, \sum_{j=1}^{n} b_j = \frac{n}{2} \right\}. \tag{5.8}$$

In other words, the tightest bounds on the relaxation error are obtained when the median rule is used. Any other rule leaves greater room for the relaxation error[4].

[3] A random variable X is stochastically smaller than Y iff $\text{Prob}(X > t) \leq \text{Prob}(Y > t)$ for all $t \in \mathbb{R}$ [12, pp. 481].

[4] On the other hand, the "clocking paradox" complicates matters. A random variable L_1 can be stochastically less than random variable L_2, i.e., $\text{Prob}(L_1 > t) < \text{Prob}(L_2 > t)$ for all t, but $\text{Prob}(L_1 > L_2)$ can be arbitrarily close to 1 [2].

The geometry of the problem may be exploited to give rough upper bounds on the maximum possible relaxation error. To see this, let θ denote the angle between $\boldsymbol{x}^*$ and $\hat{\boldsymbol{b}}$ so that,

$$\cos(\theta) = \frac{\sum_{i=1}^{n} x_i^* \hat{b}_i}{\|\boldsymbol{x}^*\| \, \|\hat{\boldsymbol{b}}\|} = \frac{\sum_{i=1}^{n/2} x_{[i]}^*}{\|\boldsymbol{x}^*\| \, \|\hat{\boldsymbol{b}}\|} \tag{5.9}$$

Since $\|\boldsymbol{x}^*\| = \|\boldsymbol{b}\| = \sqrt{n/2}$, and $\boldsymbol{x}^*$ is most distant from $\boldsymbol{b}$ when $\boldsymbol{x}^* = (1/\sqrt{2}, 1/\sqrt{2}, \ldots, 1/\sqrt{2})$ equidistant from all the balanced vectors, it follows that

$$1 \geq \cos(\theta) \geq \frac{1}{\sqrt{2}} \quad \text{or} \quad \frac{n}{2} \geq \sum_{i=1}^{n/2} x_{[i]}^* \geq \frac{n}{2\sqrt{2}} \tag{5.10}$$

If θ were greater than $\pi/4$, then the median rule would map x^* to some other balanced vector. Equation (5.9) is simple but useful. For instance if $[n - (t/\alpha)^2]/2 \leq n/(2\sqrt{2})$, i.e. $t \geq 0.5412\,\alpha\sqrt{n}$ then $\sum_{i=1}^{n/2} x_{[i]}^* \leq [n - (t/\alpha)^2]/2$ is impossible, and hence

$$\delta \leq 0.5412\,\alpha\sqrt{n} \tag{5.11}$$

Thus, the relaxation error δ is $O(\alpha\sqrt{n})$. If a little more information is available, such as the mean and variance of the components of $\boldsymbol{x}^*$, then the classical bounds in order statistics may be used [1, 5]. To illustrate this approach, we present two examples. In the first, we assume the existence and knowledge of the mean and variance of the components x_i^*, and apply the Arnold-Groeneveld bound on the median to the relaxation error. In the second, we assume that the distributions of the x_i's are available, and apply a recent bound due to Caraux and Gascuel [4].

5.1 Arnold-Groeneveld Bounds

In what follows, assume[5] that the expectations and variances of x_i^*, μ_i and σ_i^2 respectively, exist and are all equal, i.e., $\mu_1 = \cdots = \mu_n = \mu$, and similarly $\sigma_1^2 = \cdots = \sigma_n^2 = \sigma^2$. Then the Arnold-Groeneveld bound [17] on the expected value of the median $E[x_m^*]$ reduces to:

$$\mu - \sigma \leq E[x_m^*] \leq \mu + \sigma \tag{5.12}$$

In Equation (5.13) σ is the common standard deviation of the components x_i^*. We establish two bounds: the first is tighter but requires an assumption, while the second is looser but requires no such assumption:

$$E[\delta] \leq \alpha\sqrt{n}\sqrt{1 + \sigma - \mu} \quad \text{provided} \quad \mu \leq 1 + \sigma, \tag{5.13}$$

and,

$$E[\delta] \leq \alpha\sqrt{n}(1 + (\sigma - \mu)/2). \tag{5.14}$$

In particular, since $\delta \geq 0$, $E[\delta] \geq 0$ and the following non-obvious inequality is implied,

$$1 + (\sigma - \mu)/2 \geq 0 \tag{5.15}$$

$$\Rightarrow \quad \mu \leq \sigma + 2 \tag{5.16}$$

[5]The assumptions are not essential, but greatly simplify the algebra.

The remainder of this section will establish (5.14) and (5.15). Our starting point is Lemma 5.1.

$$\begin{aligned}
\text{Prob}\,(\delta > t) &\leq \text{Prob}\left\{\sum_{i=1}^{n} x_i^* b_i \leq \frac{n - (t/\alpha)^2}{2}\right\} && (5.17)\\
&\leq \text{Prob}\left\{\frac{n}{2} x_m^* \leq \frac{n - \tau^2}{2}\right\}\\
&\leq \text{Prob}\,(x_m^* \leq 1 - \tau^2/n)\\
&\leq \text{Prob}\,(1 - x_m^* \geq \tau^2/n)\\
&\leq \text{Prob}\,(\alpha\sqrt{n}\,(1 - x_m^*) \geq t) && (5.18)\\
\Rightarrow \quad E[\delta] &\leq \alpha\sqrt{n}\,E[\sqrt{1 - x_m^*}] && (5.19)
\end{aligned}$$

Equation (5.19) asserts that δ is stochastically smaller than $\alpha\sqrt{n}\sqrt{1 - x_m^*}$, and provided the expectations of δ and x_m^* exist, we get Equation (5.20) [12, pp. 481]. Since $\sqrt{1 - x_m^*}$ is a concave function, from Jensen's inequality yields,

$$E[\sqrt{1 - x_m^*}] \leq \sqrt{1 - E[x_m^*]}. \tag{5.20}$$

We digress briefly to mention that in Equation (5.21) the term $\sqrt{1 - E[x_m^*]}$ is always well-defined, since Equation (5.9) and $x_m^* \leq x_{[i]}$ for $0 \leq i \leq n/2$, imply that

$$\begin{aligned}
\frac{n}{2} &\geq \sum_{i=1}^{n/2} x_{[i]}^* \geq \frac{n}{2} x_m^* && (5.21)\\
\Rightarrow \quad 1 &\geq x_m^* && (5.22)\\
1 &\geq E[x_m^*] && (5.23)
\end{aligned}$$

Also, since $\boldsymbol{x}^* \in \mathbb{R}_+^n$

$$0 \leq E[x_m^*] \leq 1 \tag{5.24}$$

From the Arnold-Groeneveld bound,

$$\begin{aligned}
1 + \sigma - \mu &\geq 1 - E[x_m^*] \geq 1 - (\sigma + \mu) && (5.25)\\
\Rightarrow \quad \sqrt{1 + \sigma - \mu} &\geq \sqrt{1 - E[x_m^*]} \quad \text{if} \quad 1 + \sigma - \mu \geq 0 && (5.26)
\end{aligned}$$

Hence, provided $\mu \leq 1 + \sigma$,

$$E[\delta] \leq \alpha\sqrt{n}\sqrt{1 - E[x_m^*]} \leq \alpha\sqrt{n}\sqrt{1 + \sigma - \mu} \tag{5.27}$$

Alternatively, from Equation (5.25),

$$E[\sqrt{1 - x_m^*}] \leq \sqrt{1 - E[x_m]} \leq 1 - (E[x_m^*]/2) \tag{5.28}$$

Applying the Arnold-Groeneveld bound to $1 - (E[x_m^*]/2)$ we obtain,

$$E[\delta] \leq \alpha\sqrt{n}\,(1 - E[x_m^*]/2) \leq \alpha\sqrt{n}\,(1 + (\sigma - \mu)/2) \tag{5.29}$$

Equation (5.28) and Equation (5.30) are useful provided they give tighter bounds than Equation (5.12). The main thing to notice about these bounds is that they are relatively distribution-free, i.e., they do not assume any knowledge about the specific form of the distribution, only its mean and variance. Also, note that the mean and variances of the components x_i^* cannot be arbitrary. By virtue of Lemma 5.1 and the non-negativity of the relaxation error, the difference between the mean (μ) and the standard deviation (σ) is always less than 2.

5.2 Distribution Bounds

A different set of bounds can be obtained if the cumulative distribution functions (c.d.f.) of x_i^* are known. In particular, a recent result due to Caraux and Gascuel is applicable. We reproduce their result here for convenience.

Proposition 5.1 [4, Proposition 1]: Let $Y_1, \ldots, Y_n$ be a set of n random variables which are identically distributed with c.d.f. F_Y. Then for the c.d.f. of the rth increasing statistic $F_{(r)}$,

$$F_{(r)}(t) \leq \inf\left(\frac{n}{r}F_Y(t), 1\right) \tag{5.30}$$

∎

Assume for convenience that the components x_i^* are identically distributed according to a common distribution $F_{x^*}(t)$, i.e.,

$$F_{x_i^*}(t) = \text{Prob}\,(x_i^* \leq t) = F_{x^*}(t) \tag{5.31}$$

Consequently, from Proposition 5.1;

$$\text{Prob}(\delta > t) \leq \text{Prob}(x_m^* \leq 1 - \frac{t^2}{n\alpha^2}) \leq \inf\left\{1,\ 2\,F(1 - \frac{t^2}{n\alpha^2})\right\} \tag{5.32}$$

So, for some known distribution functions of x_i^*'s, the relaxation error may be bounded for any threshold t.

An alternate approach to distribution-based bounds is to make specific assumptions about the distribution of $\boldsymbol{x}^*$ itself, i.e., consider what happens when the $\boldsymbol{x}^*$ in the ensemble are distributed in different ways. We consider three random models; in the first, $\boldsymbol{x}^*$ is assumed to be uniformly distributed on the hypersphere of radius $\sqrt{n/2}$, in the second the angle between $\boldsymbol{x}^*$ and $\boldsymbol{b}$ is assumed to be uniformly distributed and in the third, the sine of the angle is uniformly distributed.

Model 1: It will be shown that the relaxation error goes to zero asymptotically if $\boldsymbol{x}^*$ is assumed to be uniformly distributed on the surface of the non-negative orthant of an n-dimensional hypersphere of radius $\sqrt{n/2}$. In this section we show that asymptotically, the expected relaxation error goes to zero, at a rate given by $O(1/\sqrt{n})$. We begin with an intuitive explanation of why this is to be expected, given the above uniform distribution of $\boldsymbol{x}^*$.

In Lemma 5.1, to place an upper bound on δ, one needs to place a bound on the probability that $\|\boldsymbol{x}^* - \boldsymbol{b}\| > t/\alpha$, for a given t. Though $\boldsymbol{x}^*$ may be uniformly distributed on the hypersphere, the fact that $\boldsymbol{b} = M(\boldsymbol{x}^*)$ means that to obtain an upper bound on this probability, we need only consider those points that are closer to $\boldsymbol{b}$ (in l_2 norm) than to any other balanced vector. One may fix a balanced vector say, $\boldsymbol{b}_o = (0, 0, \ldots, 0, 1, 1, 1, \ldots, 1)$ and try to obtain an upper bound on the probability that $\|\boldsymbol{x}^* - \boldsymbol{b}_o\| > t/\alpha$ given that $\boldsymbol{x}^*$ is closer to $\boldsymbol{b}_o$ than any other discrete solution. However, if we compute the measure of the set of points that are closer to $\boldsymbol{b}_o$ than to any other discrete balanced vector (of which there are $\binom{n}{n/2}$), it turns out that the measure tends to zero asymptotically. The following simple calculation shows why.

Assume without loss of generality that n is even, i.e. $n = 2k$. Each of the $\binom{n}{n/2}$ discrete solutions also lie on this hypersphere. Consider the surface area of the first quadrant *per* balanced vector in $\mathbf{A}_2^n$. Now the surface area of an n-dimensional hypersphere of radius r is given by,

$$S^n(r) = \frac{n\,\pi^{n/2}\,r^{n-1}}{\Gamma(\frac{n}{2}+1)} \tag{5.33}$$

where $\Gamma(x)$ is the Euler Gamma function. From Equation (5.34), it follows that the surface area of the first quadrant for $n = 2k$ and $r = \sqrt{n/2} = \sqrt{k}$ is

$$\begin{aligned} S_+^n(\sqrt{k}) &= S^n(\sqrt{n/2})/2^n && (5.34)\\ &= \frac{(2k)\,\pi^k\,(\sqrt{n/2})^{2k-1}}{2^{2k}\,\Gamma(k+1)} = \frac{2\,\pi^k\,k^{k+1/2}}{4^k\,\Gamma(k+1)} && (5.35)\\ &= \frac{2\,\pi^k\,k^{k+1/2}}{4^k\,(k!)} && (5.36) \end{aligned}$$

where we have used the fact that for integer $k > 0$, $\Gamma(k+1) = (k!)$. $S_+^n(\sqrt{k})$ is monotonically increasing with respect to n. But if $\boldsymbol{x}^*$ is to be mapped to $\boldsymbol{b}$, it must lie in a certain area of the hypersphere. In this area, all points are closer to $\boldsymbol{b}$ than to any other balanced vector. Since there are $\binom{n}{n/2}$ balanced vectors in n-dimensions, the surface area *per* discrete balanced vector is given by,

$$\begin{aligned} \frac{S_+^n(\sqrt{k})}{\binom{n}{n/2}} &= \frac{2\,k^{k+1/2}}{\binom{2k}{k}\,(k!)}\,\frac{\pi^k}{4^k} && (5.37)\\ &= 2\sqrt{k}\,\frac{k^k\,k!}{(2k)!}\,(\pi/4)^k && (5.38)\\ &= 2\sqrt{k}\,(\pi/4)^k\,\frac{k\times k\times\cdots\times k}{(k+1)\times(k+2)\times\cdots\times(k+k)} && (5.39) \end{aligned}$$

From Equation (5.40) we see that,

$$\lim_{k\to\infty}\frac{S_+^n(\sqrt{k})}{\binom{n}{n/2}} \to 0 \tag{5.40}$$

One may conclude that the measure of the set of points that are closer to $\boldsymbol{b}_o$ than to any other balanced vector is zero (asymptotically). In other words, for this model, if we find an optimal continuous solution, then with probability one, the associated relaxation error is zero. The next section gives an estimate for the rate of convergence. For simplicity we have relied heavily on symmetry arguments, using the 2-dimensional example to illustrate the calculations for the general case.

5.2.1 Asymptotics for Model 1

To clarify what will follow, consider the case when $n = 2$, shown in Figure 1. There are two possible discrete solutions, viz., $\boldsymbol{b}_o = (0,1)$ and $\boldsymbol{b}_1 = (1,0)$. Real solutions $\boldsymbol{x}^*$ are closer to $\boldsymbol{b}_o$ provided

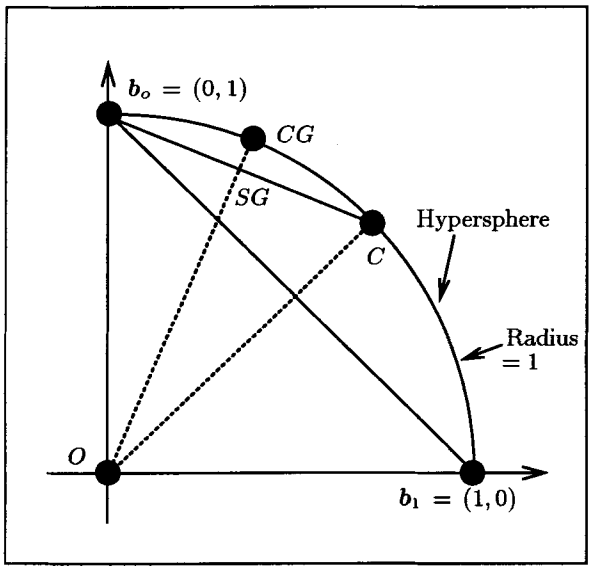

Figure 1: Geometry of the relaxation error for $n = 2$. There are only two discrete solutions, viz. $\boldsymbol{b}_o = (0,1)$ and $\boldsymbol{b}_1 = (1,0)$. The random continuous solution $\boldsymbol{x}^*$ is mapped to $\boldsymbol{b}_o$ if it lies in that section of the circle (two dimensional "hypersphere") between $\boldsymbol{b}_o$ and C, else to $\boldsymbol{b}_1$. The *expected* distance $E[||\boldsymbol{x}^* - \boldsymbol{b}_o||]$ is the distance between CG and $\boldsymbol{b}_o$, where CG is the center of mass of the distribution of $\boldsymbol{x}^*$ in the section $\boldsymbol{b}_o$–CG–C.

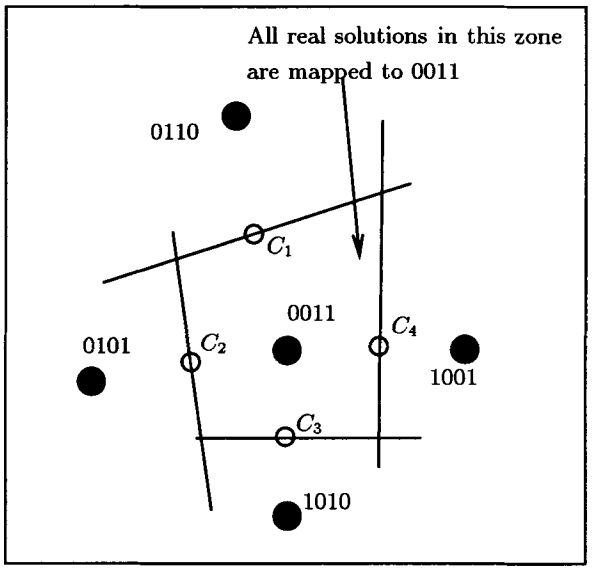

Figure 2: For $n = 4$, there are six discrete solutions, from 0011 ($\boldsymbol{b}_o$) to 1100. The "zone of attraction" for 0011 is delimited by "edges" (hyperplanes), where each edge represents a region in which points are equidistant between 0011 and a neighbor. Knowing the co-ordinates of $C1, \ldots, C4$ enables the calculation of the center of mass of 0011's zone of attraction.

they lie on the circular arc $\boldsymbol{b}_o$–CG–C. If the distribution of $\boldsymbol{x}^*$ is known, then one can calculate the *expected* distance between such a solution and $\boldsymbol{b}_o$. Irrespective of whether $\boldsymbol{b}_o$ is at the actual optimal distance or not, the expected relaxation error is necessarily less than this expected distance. To illustrate this approach, we perform the explicit computations for the two dimensional case, under the assumption that the $\boldsymbol{x}^*$ is *uniformly* distributed along the circle in Figure 1. From elementary geometry, the co-ordinates of C are given by $(1/\sqrt{2}, 1/\sqrt{2})$. So if a solution $\boldsymbol{x}^* = (x_1^*, x_2^*)$ is to be mapped to $\boldsymbol{b}_o$, we must have,

$$x_1^* \leq \frac{1}{\sqrt{2}} \qquad x_2^* \geq \frac{1}{\sqrt{2}} \tag{5.41}$$

If $\boldsymbol{x}^*$ were uniformly distributed along the line $\boldsymbol{b}_o$–C, then the expected distance between $\boldsymbol{x}^*$ and $\boldsymbol{b}^*$ would be the distance between the centroid SG and $\boldsymbol{b}_o$. The co-ordinates of SG are $(\frac{1}{2\sqrt{2}}, \frac{1+\sqrt{2}}{2\sqrt{2}})$. However the solutions are distributed uniformly along the circular arc. The expected distance then becomes the distance between $\boldsymbol{b}_o$ and CG. The co-ordinates of CG are computable from those of SG, using the fact that CG is the intersection of the line 0–SG and the circular arc. This gives,

$$CG = (\frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2}}}{2}) \tag{5.42}$$

Then the Euclidean distance between CG and $\boldsymbol{b}_o$ is equal to

$$d(\boldsymbol{b}_o, CG)^2 = \left(0 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 + \left(1 - \frac{\sqrt{2+\sqrt{2}}}{2}\right)^2 \tag{5.43}$$

$$\approx 0.1521$$

$$d(\boldsymbol{b}_o, CG) \approx 0.39 \tag{5.44}$$

Then,

$$\text{Prob}(\delta > t) \leq \text{Prob}\{\|\boldsymbol{x}^* - \boldsymbol{b}\| > t/\alpha\} \tag{5.45}$$

$$\Rightarrow \quad E[\delta] \leq \alpha E[\|\boldsymbol{x}^* - \boldsymbol{b}\|]$$

$$\leq \alpha\, d(\boldsymbol{b}_o, CG)$$

$$\leq 0.39\,\alpha \tag{5.46}$$

This analysis can be extended to the multi-dimensional case as well. Figure 2 shows a schematic illustration of the four dimensional case. There are six discrete solutions. As before, we fix $\boldsymbol{b}_o = (0,0,1,1)$. $\boldsymbol{b}_o$ has *four* nearest neighbors, i.e discrete solutions at a hamming distance of 2; $(1,0,0,1)$, $(1,0,1,0)$, $(0,1,0,1)$ and $(0,1,1,0)$. The edges in Figure 2 represent regions containing points equidistant from $\boldsymbol{b}_o$ and a neighbor. These edges delimit a Voronoi polytope, such that if a continuous solution lies in this "zone", then it is mapped to $\boldsymbol{b}_o$, rather than any other discrete solution. Geometrically, all solutions lie on a hypersphere of radius $\sqrt{n/2} = \sqrt{2}$, and origin at $O = (0,0,\ldots,0,0)$. The points C_1 through C_4 are analogous to the point C in Figure 1. For example, C_1 lies on the midpoint of the geodesic (on the hypersphere) joining the point $(0,1,1,0)$ and $(0,0,1,1)$. Therefore, the co-ordinates of the C_i's are given by,

$$C_1 = (0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}) \qquad C_2 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1) \tag{5.47}$$

$$C_3 = (\frac{1}{\sqrt{2}}, 0, 1, \frac{1}{\sqrt{2}}) \qquad C_4 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1) \tag{5.48}$$

If the $\boldsymbol{x}^*$ were uniformly distributed on the polytope formed by $\boldsymbol{b}_o$ and the points C_i's, then the expected distance of $E[\|\boldsymbol{x}^* - \boldsymbol{b}_o\|]$ would be given by the Euclidean distance between $\boldsymbol{b}_o$ and the polytope's centroid SG, where the co-ordinates of SG are given by,

$$SG = (\boldsymbol{b}_o + C_1 + C_2 + C_3 + C_4)/5 \tag{5.49}$$

$$= \frac{1}{5}\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, 2 + \frac{2}{\sqrt{2}}, 2 + \frac{2}{\sqrt{2}}\right) \tag{5.50}$$

$$= \left(\frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}, \frac{2+\sqrt{2}}{5}, \frac{2+\sqrt{2}}{5}\right) \tag{5.51}$$

SG is then used to compute the co-ordinates of CG, by using the fact that CG lies at the intersection of the line 0–SG and the hypersphere. Finally, knowing CG's co-ordinates enables one to compute an upper bound on the expected value of δ, analogous to that for the two dimensional case. When

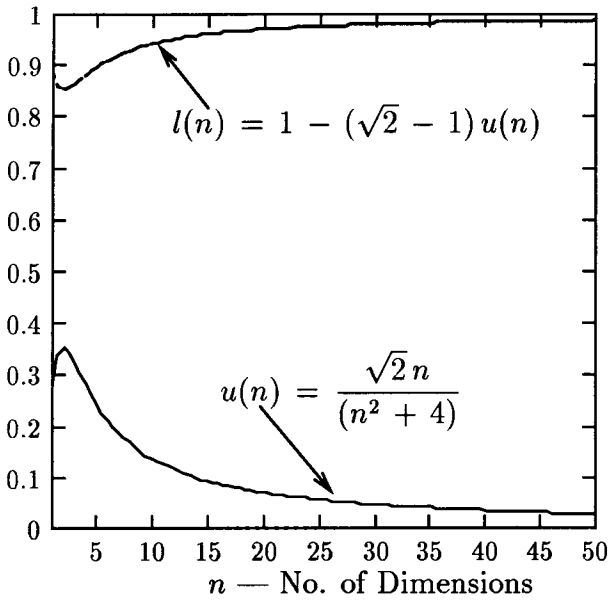

Figure 3: Plot of $u(n), l(n)$ versus n

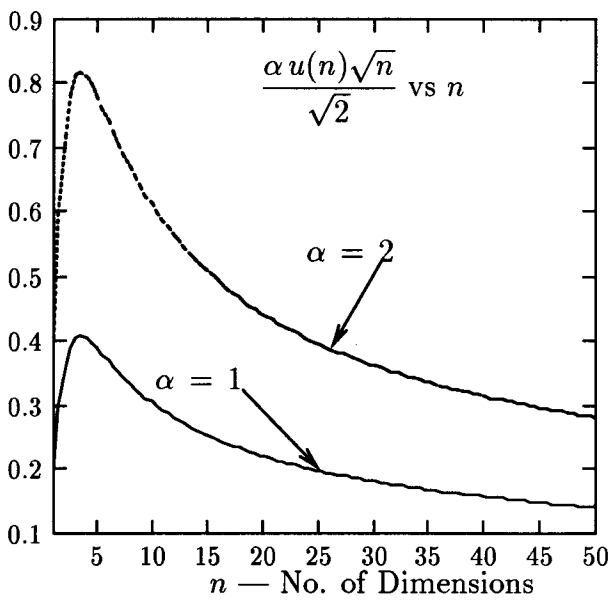

Figure 4: The expected relaxation error as a function of problem size, and Lipschitz constant of cost surface

generalized to n dimensions, we obtain the following:

$$SG = \frac{1}{(n/2)^2 + 1}\,(\boldsymbol{b}_o + \sum_{i=1}^{(n/2)^2} C_i) \tag{5.52}$$

$$= (\underbrace{u(n), \ldots, u(n)}_{\text{first } n/2 \text{ terms}}, \underbrace{1 - (\sqrt{2} - 1)\,u(n), \ldots, l(n), \ldots, l(n)}_{\text{last } n/2 \text{ terms}}) \tag{5.53}$$

where

$$u(n) = \frac{\sqrt{2}\,n}{(n^2 + 4)} \tag{5.54}$$

$$l(n) = 1 - (\sqrt{2} - 1)\,u(n) \tag{5.55}$$

Figure 3 is a graph of $u(n)$ and $l(n)$ versus n. Clearly, as the number of dimensions increase, $u(n)$ rapidly goes to zero (at a rate of $O(1/n)$). This fact is significant in controlling the asymptotic behavior of the relaxation error.

From Equation (5.54) it is now possible to compute the centroid CG; tedious but simple calculations give,

$$CG = (\underbrace{\frac{u(n)}{\sqrt{u(n)^2 + l(n)^2}}, \ldots, \frac{u(n)}{\sqrt{u(n)^2 + l(n)^2}}}_{\text{first } n/2 \text{ terms}}, \underbrace{\frac{l(n)}{\sqrt{u(n)^2 + l(n)^2}}, \ldots, \frac{l(n)}{\sqrt{u(n)^2 + l(n)^2}}}_{\text{last } n/2 \text{ terms}})$$

Since $u(n) \to 0$ as $n \to \infty$, it follows that $l(n) \to 1$ as $n \to \infty$, i.e., asymptotically CG tends to $\boldsymbol{b}_o = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$, and $d(\boldsymbol{b}_o, CG) \to 0$ for increasing n, where $d(\boldsymbol{b}_o, CG)$ is the distance

between $\boldsymbol{b}$ and CG, or the expected value of $\|\boldsymbol{x}^* - \boldsymbol{b}_o\|$. Precisely, $d(\boldsymbol{b}_o, CG)$ is given by,

$$d(\boldsymbol{b}_o, CG) = \sqrt{n}\sqrt{u(n)^2 + l(n)^2 - l(n)\sqrt{u(n)^2 + l(n)^2}} \tag{5.56}$$

where $u(n)$ and $l(n)$ are as given earlier. Equation (5.57) implies and Equation (5.47) imply that

$$E[\delta] \leq \alpha\sqrt{n}\sqrt{u(n)^2 + l(n)^2 - l(n)\sqrt{u(n)^2 + l(n)^2}} \tag{5.57}$$

Therefore,

$$\lim_{n\to\infty} E[\delta] \rightarrow \frac{\alpha\, u(n)\sqrt{n}}{\sqrt{2}} \tag{5.58}$$

$$\rightarrow \frac{\alpha\, n\sqrt{n}}{n^2 + 4} \rightarrow \frac{\alpha}{\sqrt{n}} \tag{5.59}$$

Thus $E[\delta] \rightarrow 0$ as $n \rightarrow \infty$, and does so at a rate given by $O(1/\sqrt{n})$ (see Figure 4). Asymptotically, if the random optimal continuous solutions are uniformly distributed over the hypersphere, the relaxation error goes to zero.

The worst case occurs when $\boldsymbol{x}^*$ is *not* uniformly distributed over the hypersphere but the distribution is instead concentrated around a point, which while still in the zone of attraction of $\boldsymbol{b}_o$, is fartherest from it. The co-ordinates of this point are easily seen to be given by $(1/\sqrt{2}, 1/\sqrt{2}, \ldots, 1/\sqrt{2})$ which is the unique point on the hypersphere equidistant from *all* the discrete balanced solutions. The relaxation error is then bounded by

$$E[\delta] \leq \alpha\sqrt{n}[1 - \frac{1}{\sqrt{2}}]^{1/2} \approx 0.541\,\alpha\sqrt{n} \tag{5.60}$$

which is $O(\sqrt{n})$ as given by Equation (5.12).

Model 2: From Equation (5.7), viz.,

$$\cos(\theta) = \frac{\sum_{i=1}^{n} x_i^* b_i}{\|\boldsymbol{x}^*\|\,\|\boldsymbol{b}\|} = \frac{\sum_{i=1}^{n/2} x_{[i]}^*}{\|\boldsymbol{x}^*\|\,\|\boldsymbol{b}\|} \tag{5.61}$$

one can rewrite Equation (5.2) as follows:

$$\text{Prob}(\,\delta > t\,) \leq \text{Prob}\left\{\sum_{i=1}^{n} x_i^* b_i \leq \frac{n - (t/\alpha)^2}{2}\right\} \tag{5.62}$$

$$= \text{Prob}\left\{\cos(\theta) \leq 1 - \tau^2/n\right\} \tag{5.63}$$

$$= \text{Prob}\left\{\sin(\theta/2) \geq \tau/\sqrt{2\,n}\right\} \tag{5.64}$$

$$= \text{Prob}\left\{\Theta \geq \tau/\sqrt{2\,n}\right\} \tag{5.65}$$

where $0 \leq \theta \leq \pi/4$ (see Equation (5.9)), and we have introduced the variable $\Theta = \sin(\theta/2)$.

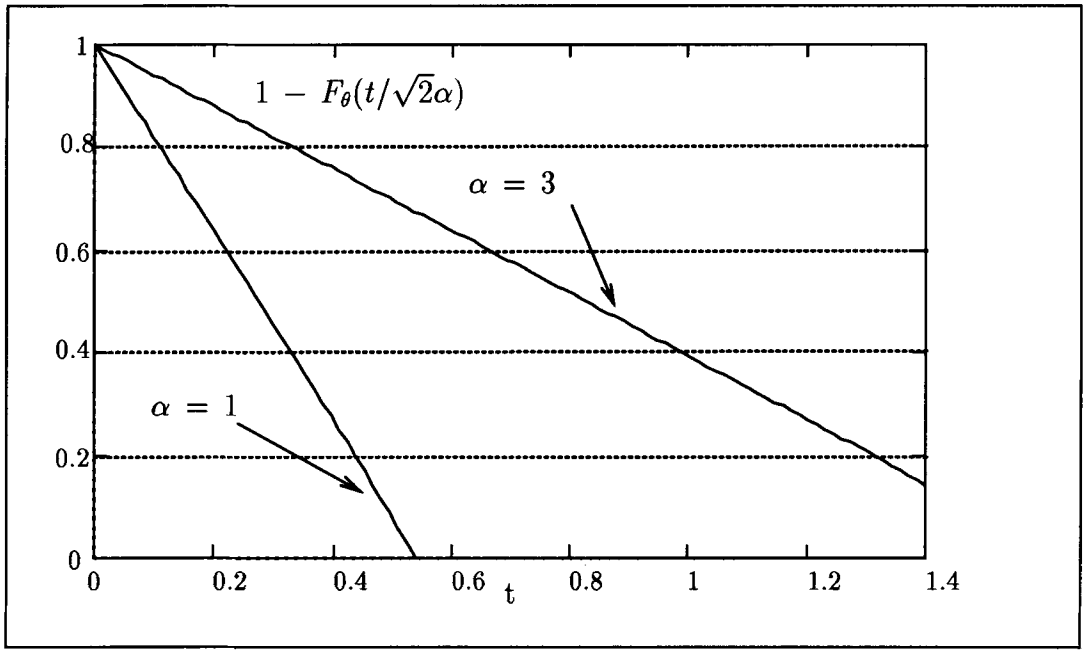

Figure 5: Plot of $1 - F_\Theta(t/\sqrt{2}\alpha)$ with respect to t for two different values of α. The number of dimensions n is fixed at 20. Notice that for increasing α ("rougher") surfaces the upper bound on the relaxation error increases (for a given t).

Consider the random model in which θ is uniformly distributed in the interval $[0, \pi/4]$. Then it is possible to explicitly evaluate $F_\Theta(t)$, the c.d.f. of the random variable $\Theta = \sin(\theta/2)$. From routine transform operations, we get

$$\text{Prob}(\delta > t) \leq 1 - \frac{8}{\pi}\sin^{-1}(\tau/\sqrt{2\,n}) = 1 - F_\Theta(\tau/\sqrt{2\,n}), \quad \forall t \leq \alpha\sqrt{n}\sqrt{1 - 1/\sqrt{2}} \tag{5.66}$$

Or equivalently (renaming t appropriately),

$$\text{Prob}(\delta > t\sqrt{n}) \leq 1 - \frac{8}{\pi}\sin^{-1}\left(\frac{t}{\sqrt{2}\,\alpha}\right) = 1 - F_\Theta(t/\sqrt{2}\,\alpha), \quad \forall t \leq \alpha\sqrt{1 - 1/\sqrt{2}} \tag{5.67}$$

The right hand side of Equation (5.67) is the c.d.f of the arc-sin density distribution, and is a function of three variables, n, t and α. Figure 5 plots the right hand side of Equation (5.68) with respect to t for two different values of the Lipschitz constant. Equation (5.68) shows that for any fixed multiple of $\sqrt{n}$, the probability that δ is greater than it is a constant, irrespective of problem size. In other words, the *relative* relaxation error $\delta_r = \delta/\sqrt{n}$ is independent of problem size in this model.

Model 3: Rather than letting θ be uniform in $[0, \pi/4]$, an alternative model could require θ to be so distributed that $\Theta = \sin(\theta/2)$ is uniform in $[0, 1/\sqrt{2}]$ (for example, this is the case if θ has an arc-sin c.d.f). It then follows that,

$$\text{Prob}(\delta > t) \leq 1 - \sqrt{2}\frac{t}{\sqrt{2n}\,\alpha} = 1 - \frac{t}{\sqrt{n}\,\alpha} \tag{5.68}$$

Or equivalently (renaming t appropriately),

$$\text{Prob}(\delta > t\sqrt{n}) \leq 1 - \frac{t}{\alpha} \tag{5.69}$$

Therefore, the asymptotic behavior of δ in both models is essentially the same.

6 Discussion

The usefulness of the upper bounds on the relaxation error (developed in Section 5.2), depends on the available information regarding the distribution of $\boldsymbol{x}^*$ in the ensemble $\mathcal{C}(n, \alpha)$. Typically, one would expect the solutions $(\boldsymbol{x}^*)$ to be tightly concentrated around the discrete solutions, i.e., x_i^* is either close to 0 to 1. Such is the case, for instance, if $\boldsymbol{x}^*$ was distributed according to the arc-sine density distribution (Beta distribution with both parameters set to $1/2$). Such a concentration of measure can only reduce the relaxation error. Typically, the situation is usually even much better than that afforded by assuming the distribution to be uniform; certainly, there is no reason to suppose that the continuous optimal solutions are concentrated around the point $(1/\sqrt{2}, 1/\sqrt{2}, \ldots, 1/\sqrt{2})$ (worst case scenario).

Second, the results make explicit the relationship between δ and the "smoothness" constant of the surface, α. The expected relaxation error is *linearly* dependent on the smoothness of the cost surface. With respect to relaxation, regularization techniques which smooth a cost surface by some kind of averaging process (say) thus have only a limited benefit, since they only reduce the relaxation error by a linear factor.

We have studied the relaxation error associated with a particular relaxation. Other problem relaxations may yield different conclusions. For example, rather than relaxing as in Equation (4.8),

$$\text{Minimize } C(\boldsymbol{x}) \text{ such that } \|\boldsymbol{x}\|^2 = \frac{n}{2} \text{ and } \boldsymbol{x} \in \mathbb{R}_+^n \tag{6.1}$$

one could have relaxed it as follows,

$$\text{Minimize } C(\boldsymbol{x}) \text{ such that } \sum_{i=}^{n} x_i = \frac{n}{2} \text{ and } \boldsymbol{x} \in \mathbb{R}_+^n \tag{6.2}$$

Now $\boldsymbol{x}$ is constrained to lie on a simplex rather than a hypersphere. The conclusions obtained in this paper remain essentially unchanged, for this particular relaxation, especially the asymptotic results.

It is interesting to compare the bounds obtained on the relaxation error with related work in the literature. In mathematical programming, seminal work by Baum, Chandrashekaran, Giles, Orlin, Shirali, Trotter and several others have led to deterministic results that bound the difference between the continuous and optimum integer solutions for constrained separable convex programming problems [18, pp. 237-242], [23]. Typically, their results state that for this class of problems, the distance (usually with respect to l_∞ norm) between the integer and continuous solutions, is bounded by some value, which depends on the number of variables and constraint matrix. These bounds are often the *best* possible, in the sense that one can always construct examples achieving the stated bounds.

But it also implies that such results refer to *worst-case* scenarios, since they have to account for all possibilities. Fundamentally, this class of results is pessimistic in nature. Probabilistic analyses seem to have been few, though in recent years there has been an upsurge in new techniques and promising results, such as [6, 13, 20, 21].

In the area of graph bisectioning, the results are of a mixed nature. On the one hand it is possible to construct graphs on which "standard" techniques (such as spectral methods) are guaranteed to produce poor bisection [9]. In fact, there are no known approximation algorithms for the graph bisection problem. On the other hand, there *is* a catalogue of positive results; for example the performance of spectral methods (or alternative techniques) on "regular" graphs such as finite element meshes is asymptotically optimal [19, 3]. There is also a great deal of folklore that seems to suggest that many heuristic algorithms are quite efficacious "in practice." Intuitively , the graph bisection problem is "easier" than say, the Hamiltonian path problem. Our results on the relaxation error suggest that the Lipschitz continuity of the cost function is the single most important factor underlying the ease of obtaining good solutions for the graph bisection problem. It is likely that similar results hold for Lipschitz continuous cost functions subject to general convex constraints.

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