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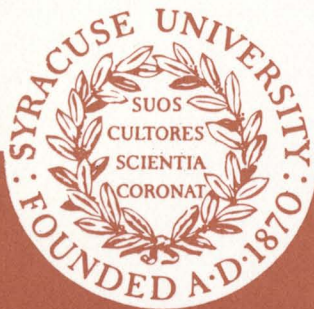
SOME RESULTS ON OPTIMUM
PREMIUM PAYMENT PLANS

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SOME RESULTS ON OPTIMUM PREMIUM

PAYMENT PLANS

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1. INTRODUCTION

Any insurance plan consists of a sequence of payments every year (or some other fixed time interval) in return for certain death benefits. The benefits may take the form of a wide variety of insurances or annuities. For simplicity, we will assume that premiums and benefits are paid annually. In this paper, we investigate the appropriateness of this type of plan. Naturally, appropriateness of any plan cannot be measured without an optimality criteria. Three such criteria, which are statistical in nature, are introduced in this paper. For the principal "safety" criterion which we use, the optimal premium are those which minimize a certain "profit variance" subject to a familiar profit constraint. We also develop a "profitability" criterion and then solve an associated optimality problem.

Our main results state that if the sequence of present values of total benefits is nonincreasing, then the profit variance is minimum when the insured pays a net single premium at once, and if this cannot be done, the insured should pay off the policy as early as possible.

2. NOTATION AND PROBLEM FORMULATION

In this section, we formulate the problem for a simple situation. It is assumed that a client aged x is to receive some benefit at the end of the year of his (or her) death. In return, the client promises to pay annual premiums while he is living. We assume that the mortality

of the population of lives to which the client belongs is known and we put

$${}_k|q_x = P\{\text{subject aged } x \text{ dies between ages } x+k \text{ and } x+k+1\}$$

for $k = 0, 1, 2, \dots, w - x - 1$, where w is an upper limit on the age variable. For convenience and without loss of generality, we take $w = \infty$.

Of course, we assume that ${}_k|q_x$ satisfies $\sum_{k=0}^{\infty} {}_k|q_x = 1$.

Let P_{k-1} = premium payment at the beginning of the k -th year,
 $k = 1, 2, 3, \dots$;

ρ_{k-1} = present value of premium payments up to and including
the k -th payment, $k = 1, 2, 3, \dots$;

and β_{k-1} = present value of death benefit if death occurs between
ages $x+k-1$ and $x+k$, $k = 1, 2, 3, \dots$.

For example, if the annual effective rate of interest is $i > 0$,

and $v = (1+i)^{-1}$, then $\rho_t = \sum_{k=0}^t P_k v^k$, $t = 0, 1, 2, \dots$. If the

benefits constitute a whole life insurance policy, then

$$\beta_t = v^{t+1}, \quad t = 0, 1, 2, \dots$$

For a term insurance policy with term period of n years, $\beta_t = v^{t+1}$,
 $t = 0, 1, \dots, n-1$ and $\beta_t = 0$, $t \geq n$. For an n -year endowment
insurance,

$$\beta_t = v^{t+1}, \quad t = 0, 1, \dots, n-1 \quad \text{and} \quad \beta_t = v^n, \quad t \geq n.$$

We will assume throughout that the ρ 's and β 's satisfy

$$0 \leq \rho_1 \leq \rho_2 \leq \rho_3 \leq \dots$$

$$\beta_t \geq 0 \quad \text{for all } t.$$

These simply mean that there are no negative premium payments or negative death benefits.

In the sequel, we will take the β 's as known and the ρ 's to be determined consistent with our optimality criterion.

Define the random variables X , Y and Z as follows:

X = present value of the benefit payments

Y = present value of the premium payments

and $Z = Y - X$ = the present value of the net profit (to the insurance company).

It is clear that for these random variables

$${}_k|q_x = P[X = \beta_k] = P[Y = \rho_k] = P[Z = \rho_k - \beta_k],$$

$k = 0, 1, 2, \dots$ and that for $k \neq \ell$, $P[Z = \rho_k - \beta_\ell] = 0$.

We propose to minimize the "profit variance"

$$\sum_{k=0}^{\infty} (\rho_k - \beta_k)^2 {}_k|q_x$$

for variation in ρ 's, subject to the constraint that

$$\sum_{k=0}^{\infty} (\rho_k - \beta_k)k|q_x = 0 .$$

This constraint is the familiar one that the expected present value of the premium payments equals that of the benefit payments, i.e., $E(X) = E(Y)$ and therefore $E(Z) = 0$.

Remark 1. The mean criterion is implicitly used to obtain net single or net annual premiums in actuarial science literature. (See Jordan, "Life Contingencies.") For example, for whole life insurance where $\beta_t = v^{t+1}$, the criteria $E(Z) = 0$ gives the net single premium

$$P_0 = \sum_{t=0}^{\infty} v^{t+1} t|q_x = A_x , \quad P_1 = P_2 = \dots = 0 ,$$

i.e., $\rho_0 = \rho_1 = \dots = A_x$.

To obtain the net annual premium P , we have

$$\rho_k = P + Pv + Pv^2 + \dots + Pv^k = P\ddot{a}_{\overline{k+1}|}$$

and $E(Z) = 0$ gives

$$P = \frac{A_x}{\ddot{a}_x} .$$

The concept of variance, however, is not discussed in the standard actuarial literature. The model we have developed which defines present values of payments and benefits in terms of random variables, allows us to naturally introduce the variance of these variables. Minimizing the profit variance subject to keeping the net profit zero (or as we shall see later,

at some fixed amount) puts high probability on profits near zero. Thus, both the insurer and the insured are protected against large deviations in profit. For this reason, we shall refer to this as the "safety" criterion.

Although not dealt with in this paper, one could use this variance concept to construct interval estimates of net profit for various types of policies, make normal approximations to probabilities of loss and so on.

Remark 2. Notice that by taking the point of view that the present values of the payments are known or to be determined, rather than the payments themselves, we need not specify the interest structure in advance. For example, once the optimal ρ 's are found, the actual premiums $\{\rho_t\}$ can be determined year by year according to the current interest rate.

Remark 3. There are two important classes of β 's which will yield us explicit solutions. They are

$$(1) \quad \beta_0 \geq \beta_1 \geq \dots \geq 0 \quad (\beta\text{'s nonincreasing})$$

$$\text{and } (2) \quad 0 \leq \beta_0 \leq \beta_1 \leq \dots \quad (\beta\text{'s nondecreasing}) .$$

3. THE MAIN RESULTS

In this section, we present our main results. Proofs of these results are deferred to the Appendix. We consider monotone values of β 's

as described above in Remark 3.

3.1 Nonincreasing Premium Values

Assume that $\beta_0 \geq \beta_1 \geq \dots \geq 0$. This situation is quite common for whole life and term insurances. For variation of the ρ 's subject to

$0 \leq \rho_0 \leq \rho_1 \leq \dots$ and $\sum_{k=0}^{\infty} \rho_k k |q_x = \sum_{k=0}^{\infty} \beta_k k |q_x$, our concern is to find

what choices of ρ 's minimize $\sum_{k=0}^{\infty} (\rho_k - \beta_k)^2 k |q_x$, the profit variance.

We have

Lemma 1. Under the conditions outlined above,

$$\min_{\{\rho_k\}} \sum_{k=0}^{\infty} (\rho_k - \beta_k)^2 k |q_x = \sum_{k=0}^{\infty} (\rho^* - \beta_k)^2 k |q_x$$

where

$$\rho^* = \sum_{k=0}^{\infty} \beta_k k |q_x .$$

Proof: See the Appendix for all proofs.

Put another way, Lemma 1 states that $\min \text{Var}(Z) = \text{Var}(X)$. Thus, the profit variance is minimized when the insured pays a net single premium at once.

In Lemma 2, we compare profit variances when a single premium is made at age $x + k$ for the entire policy at age x .

Lemma 2. If for some fixed $k \geq 0$, we have $\rho = 0$ for $t < k$ and $0 < \rho_k = \rho_{k+1} = \dots$ and if Z_k is the associated net profit random variable, then

$$\text{Var}(X) = \text{Var}(Z_0) \leq \text{Var}(Z_1) \leq \dots .$$

We are now ready to state our main result.

Theorem 1. If for $0 \leq k$, $0 \leq m$, we have $\rho_t = 0$ when $t < k$ and $0 \leq \rho_k \leq \rho_{k+1} \leq \dots \leq \rho_{k+m} = \rho_{k+m+1} = \rho_{k+m+2} = \dots$ and if $Z_k^{(m)}$ is the associated net profit random variable, then

$$\text{Var}(Z_k) \leq \text{Var}(Z_k^{(m)}) \leq \text{Var}(Z_{k+m})$$

where Z_k and Z_{k+m} are the net profit random variables associated with single premium payments at year k and $k + m$, respectively.

Lemmas 1, 2 and Theorem 1 and the continuity in the ρ_i 's of $\text{Var}(Z)$ indicate that $\text{Var}(Z_k^{(m)}) \downarrow \text{Var}(Z_k)$ as $\rho_j \downarrow \rho_k$, $j = k + 1, k + 2, \dots$ and $\text{Var}(Z_k^{(m)}) \uparrow \text{Var}(Z_{k+m})$ as $\rho_j \uparrow 0$, $j = k, k + 1, \dots, k + m - 1$. Hence, if paying a single premium is unreasonable, then from the point of keeping the profit variance small, it is desirable to make the bulk of the premium payments in the early years of the policy.

A criterion similar to that of minimizing profit variance can be given in terms of median and mean absolute deviation. An associated

result is given in the following theorem.

Theorem 2. Let m be a positive integer such that

$$.5 \leq \sum_{t=m}^{\infty} t|q_x \quad \text{and} \quad .5 \leq \sum_{t=0}^m t|q_x .$$

If we put $\rho_0 = \rho_1 = \dots = \beta_m$, then $\text{median}(Z) = 0$ and $E|Z - \text{median}(Z)|$ is minimum among all policies for which the median is zero. [Of course, $\beta_0 \geq \beta_1 \geq \dots$.]

From the nature of the proof of Theorem 2, we can similarly conclude that if the optimum net single premium solution is not practical, then in view of minimizing the mean absolute deviation, it is best to pay off the policy as soon as possible.

So far our discussion has stressed premium paying plans which are fair [i.e., $E(Z) = 0$] and safe [i.e., $\text{Var}(Z)$ is minimum] both to the client and the insurance company. It is possible, however, that we might want a premium payment plan which, subject to $E(Z) = 0$, maximizes $P[Z \geq 0]$. This is a criterion of profitability from the point of view of the company. This criterion of profitability is not relevant when we have $\text{median}(Z) = 0$ since we always have $P[Z \geq 0] \geq .5$ and $P[Z \leq 0] \geq .5$.

Suppose for definiteness, that we want to maximize $P[Z \geq 0]$ subject to $E(Z) = 0$. We have

Theorem 3. Let C_k be the discounted net single premium associated with Z_k . That is,

$$C_k = \frac{\sum_{t=0}^{\infty} \beta_t t |q_x}{\sum_{t=k}^{\infty} t |q_x} .$$

Let k be such that $C_k \geq \beta_k$ and $C_{k-1} < \beta_{k-1}$. Then, if $E(Z) = 0$, $P[Z \geq 0]$ is maximum when $\rho_0 = \rho_1 = \dots = \rho_{k-1} = 0$ and $\rho_k = \rho_{k+1} = \dots = C_k$.

In other words, Theorem 3 implies that to maximize the probability of making a profit, the insurance company should require a single premium payment at the "turning point" age $x + k$. More generally, if this "best" premium payment plan is not feasible, it follows that the company should seek only nominal premium payments until the "turning point" age $x + k$ is reached. This is in direct contrast to the "safety" criteria which requires that the bulk of the payments be made in the early years. Depending on the interest of the company, suitable adjustments should be made between these two extremes to suit its particular needs.

The special case when $\beta_t = \beta$ for all t results in $\rho_t = \beta$ for all t by the "safety" criteria. Then $\text{Var}(Z) = 0$ and $P[Z = 0] = 1$. It follows that $P[Z \geq 0]$ is also maximized and, hence, both the criteria of "profitability" and "safety" have their optimal solution at the net single premium of β .

3.2 Nondecreasing Premium Values

Suppose now that $\beta_0 \leq \beta_1 \leq \beta_2 \leq \dots$. Clearly this is a common situation when the benefit takes the form of annuities of certain types, or of an insurance whose benefits increase in present value. In this case, the solution for the minimum profit variance is trivial. Namely, the profit variance can be made to equal zero by simply taking $\phi_t = \beta_t$.

4. OTHER EXPENSES

The conclusions of the paper hold also if we change the constraint $E(Z) = 0$ to $E(Z - L) = 0$, where L is a "loading" constant assumed not to depend upon the premium payments. In the context, the "safety" criteria becomes more meaningful in that the company may wish to achieve its fixed margin of profit L with the minimum of "risk."

5. CONTINUOUS CASE

All the results of this paper follow when one or both of the benefit and premium payments are continuous. We generalize our model by defining the random variable T_x = time of death beyond x for a person aged x . For simplicity, we shall drop the subscript in T_x since we are always working with an individual aged x . We assume T is continuous with probability density function $f(t)$.

Remark 4. The standard actuarial notation for probabilities of death and survival can all be expressed in terms of $f(t)$. For instance,

$${}_t p_x = P(x \text{ survives } t \text{ more years}) = 1 - F(t) ,$$

where

$$F(t) = P(T \leq t)$$

$$= \int_0^t f(y) dy .$$

$$\mu_{x+t} = \text{force of mortality}$$

$$= \frac{-\frac{\partial}{\partial t} {}_t p_x}{{}_t p_x}$$

$$= \frac{f(t)}{1 - F(t)}$$

$${}_t q_x = P(x \text{ dies with } t \text{ years})$$

$$= \int_0^t f(y) dy$$

$${}_t | q_x = P(x \text{ survives } t \text{ years and dies in the coming year})$$

$$= \int_t^{t+1} f(y) dy .$$

The benefit (X), payment (Y) and profit (Z) random variables are all functions of T . If, for instance, the payment made is annual and the benefit paid at time of death, then

$$\begin{aligned}
 Y &= Y(t) \\
 &= \rho_k \quad \text{if } k-1 \leq t < k, \quad k = 0, 1, 2, \dots
 \end{aligned}$$

$$\begin{aligned}
 X &= X(t) \\
 &= \beta_t, \quad t \geq 0
 \end{aligned}$$

$$\begin{aligned}
 Z &= Z(t) \\
 &= Y - X.
 \end{aligned}$$

Then

$$\begin{aligned}
 E(X) &= \int_0^{\infty} x(t) f(t) dt \\
 &= \int_0^{\infty} \beta_t f(t) dt
 \end{aligned}$$

$$\begin{aligned}
 E(Y) &= \int_0^{\infty} y(t) f(t) dt \\
 &= \sum_{t=0}^{\infty} \rho_t t |q_x
 \end{aligned}$$

$$\begin{aligned}
 E(Z) &= \int_0^{\infty} [y(t) - x(t)] f(t) dt \\
 &= \sum_{k=0}^{\infty} \int_k^{k+1} (\rho_k - \beta_k) f(t) dt.
 \end{aligned}$$

All the proofs in the Appendix now follow if we replace summation signs by integral signs with the corresponding modifications in notation.

APPENDIX

In this Appendix we establish properties of the variance of the net profit random variable defined in Section 2. Our main result is Theorem 1. For convenience of presentation, we first establish some propositions,

denote $t|q_x$ by w_t , $\sum_{t=k}^{\infty} t|q_x = {}_k p_x$ by W_k and the expected benefit

$\sum_{t=0}^{\infty} \beta_t w_t$ by A .

Proposition A1. If $\beta_0 \geq \beta_1 \geq \dots \geq 0$, $0 \leq \rho_0 \leq \rho_1 \leq \dots$ and w_0, w_1, \dots are probabilities such that

$$\sum_{t=0}^{\infty} \rho_t w_t = \sum_{t=0}^{\infty} \beta_t w_t = A.$$

Then,

$$\sum_{t=0}^{\infty} (\rho_t - A)(\beta_t - A)w_t \leq 0 \tag{A.1}$$

Proof: Observe that $\sum_{t=0}^{\infty} (\rho_t - A)(\beta_t - A)w_t = \sum_{t=0}^{\infty} (\rho_t - A)\beta_t w_t$.

Since ρ_t is nondecreasing, there exists a t_0 such that $\rho_t - A \leq 0$ for $t \leq t_0$ and $\rho_t - A > 0$ for $t > t_0$. Consequently, using the nonincreasing property of β 's

$$\begin{aligned} \sum_{t=0}^{\infty} (\rho_t - A)\beta_t w_t &= \sum_{t=0}^{t_0} (\rho_t - A)\beta_t w_t + \sum_{t=t_0+1}^{\infty} (\rho_t - A)\beta_t w_t \\ &\leq \beta_{t_0} \sum_{t=0}^{t_0} (\rho_t - A)w_t + \beta_{t_0} \sum_{t=t_0+1}^{\infty} (\rho_t - A)w_t = 0. \end{aligned}$$

The inequality follows because $\beta_{t_0} \geq \beta_t$ for all $t \leq t_0$ and $(\rho_t - A)$ is negative in this range of t . A similar argument gives the desired inequality for the second term and the last equality is obtained because

$$A = \sum_{t=0}^{\infty} \rho_t w_t .$$

Proposition A2. Under the conditions of the above proposition,

$$\sum_{t=0}^{\infty} (\rho_t - \beta_t)^2 w_t \geq \sum_{t=0}^{\infty} (\beta_t - A)^2 w_t \quad (\text{A.2})$$

and the equality occurs if and only if $\rho_0 = \rho_1 = \dots = A$.

Proof: By adding and subtracting A in $(\rho_t - \beta_t)$ and then taking the square, we obtain

$$\begin{aligned} \sum_{t=0}^{\infty} (\rho_t - \beta_t)^2 w_t &= \sum_{t=0}^{\infty} (\rho_t - A)^2 w_t + \sum_{t=0}^{\infty} (\beta_t - A)^2 w_t \\ &\quad - 2 \sum_{t=0}^{\infty} (\rho_t - A)(\beta_t - A) w_t . \end{aligned}$$

Using Proposition A1 and $\sum_{t=0}^{\infty} (\rho_t - A)^2 w_t \geq 0$, we get the desired inequality.

Finally, $\sum_{t=0}^{\infty} (\rho_t - A)^2 w_t$ is strictly positive unless $\rho_t = A$ for

all t in which case $\sum_{t=0}^{\infty} (\rho_t - A)(\beta_t - A) w_t = 0$.

In view of the fact that ρ^* and A are the same quantities, the second part of the above proposition establishes Lemma 1.

In Lemma 1, we have already established that $\text{Var}(X) = \text{Var}(Z_0)$. Thus, to prove Lemma 2 it remains to establish that $\text{Var}(Z_k) \leq \text{Var}(Z_{t+1})$ for any positive integer k . Recall that Z_k is the net profit random variable associated with a single payment at year k , $k \geq 0$. But $\text{Var}(Z_k) \leq \text{Var}(Z_{k+1})$ is a special case of our more general result stated as Theorem 1. We now prove this theorem.

Theorem 1. Let k and m be two nonnegative integers and $\rho_t = 0$ for $t < k$ and $0 < \rho_k \leq \rho_{k+1} \leq \dots \leq \rho_{k+m} = \rho_{k+m+1} = \dots$. If $Z_k^{(m)}$ is the net profit random variable associated with the above payment plan then

$$\text{Var}(Z_k) \leq \text{Var}(Z_k^{(m)}) \leq \text{Var}(Z_{k+m}) \quad (\text{A.3})$$

where Z_k is the net profit random variable associated with a single payment at k , $k \geq 0$.

Proof: Let Y_k , $Y_k^{(m)}$ and Y_{k+m} be the present value of the payments random variables corresponding to Z_k , $Z_k^{(m)}$ and Z_{k+m} , respectively. We recall that $Z_k = Y_k - X$, $Z_k^{(m)} = Y_k^{(m)} - X$ and $Z_{k+m} = Y_{k+m} - X$ where X is the random variable denoting the present value of the benefit. Since

$$\begin{aligned}\text{Var}(Z_k^{(m)}) &= \text{Var}(Y_k^{(m)} - X) = \text{Var}(Y_k^{(m)} - Y_k + Z_k) \\ &= \text{Var}(Z_k) + \text{Var}(Y_k^{(m)} - Y_k) + 2 \text{Cov}(Y_k^{(m)} - Y_k, Z_k),\end{aligned}$$

to establish $\text{Var}(Z_k^{(m)}) \geq \text{Var}(Z_k)$, it suffices to show that $\text{Cov}(Y_k^{(m)} - Y_k, Z_k)$ is nonnegative.

Let C_k denote the present value of the single premium payment at year k . Then by the definitions of Y_k and $Y_k^{(m)}$,

$$C_k W_k = \sum_{t=k}^{k+m-1} \rho_t W_t + \rho_{k+m} W_{k+m} = A \quad (\text{A.4})$$

$$\begin{aligned}\text{Cov}(Y_k^{(m)} - Y_k, Z_k) &= \sum_{t=k}^{k+m} (C_k - \beta_t)(\rho_t - C_k)W_t \\ &\quad + \sum_{t=k+m+1}^{\infty} (C_k - \beta_t)(\rho_{k+m} - C_k)W_t\end{aligned}$$

where recall that X takes values β_0, β_1, \dots with probabilities w_0, w_1, \dots , respectively. By (A.4) and the fact that ρ_k is increasing, it follows that $\rho_k \leq C_k \leq \rho_{k+m}$. Hence, there exists a t_0 between k and $k+m$ such that

$$\begin{aligned}\rho_t - C_k &\leq 0 \quad \text{for } t \leq t_0 \\ \text{and} & \\ \rho_t - C_k &\geq 0 \quad \text{for } t > t_0.\end{aligned} \quad (\text{A.5})$$

Due to nonincreasing property of the β 's we have

$$\begin{aligned}
C_k - \beta_t &\leq C_k - \beta_{t_{0+1}} && \text{for } t \leq t_0 \\
&\geq C_k - \beta_{t_{0+1}} && \text{for } t > t_0 .
\end{aligned}
\tag{A.6}$$

Using (A.5) and (A.6), we obtain

$$\begin{aligned}
\text{Cov}(Y_k^{(m)} - Y_k, Z_k) &= \sum_{t=k}^{t_0} (C_k - \beta_t)(\rho_t - C_k)w_t + \sum_{t=t_{0+1}}^{k+m} (C_k - \beta_t)(\rho_t - C_k)w_t \\
&+ \sum_{t=k+m+1}^{\infty} (C_k - \beta_t)(\rho_{k+m} - C_k)w_t \\
&\geq \sum_{t=k}^{t_0} (C_k - \beta_{t_{0+1}})(\rho_t - C_k)w_t + \sum_{t=t_{0+1}}^{k+m} (C_k - \beta_{t_{0+1}})(\rho_t - C_k)w_t \\
&+ \sum_{t=k+m+1}^{\infty} (C_k - \beta_{t_{0+1}})(\rho_{k+m} - C_k)w_t \\
&= (C_k - \beta_{t_{0+1}}) \left[\sum_{t=k}^{k+m} (\rho_t - C_k)w_t + \sum_{t=k+m+1}^{\infty} (\rho_{k+m} - C_k)w_t \right] \\
&= (C_k - \beta_{t_{0+1}}) E(Y_k^{(m)} - Y_k) = 0 .
\end{aligned}$$

This proves the desired inequality that $\text{Var}(Z_k) \leq \text{Var}(Z_k^{(m)})$.

To prove $\text{Var}(Z_{k+m}) \geq \text{Var}(Z_k^{(m)})$ we again proceed in a similar manner.

Since

$$\text{Var}(Z_{k+m}) = \text{Var}(Z_k^{(m)}) + \text{Var}(Y_{k+m} - Y_k^{(m)}) + 2 \text{Cov}(Y_{k+m} - Y_k^{(m)}, Z_k^{(m)})$$

it suffices to show that $\text{Cov}(Y_{k+m} - Y_k^{(m)}, Z_k^{(m)})$ is nonnegative. Let

C_{k+m} denote the present value of the single premium payment at year

$k + m$. Then

$$C_{k+m} W_{k+m} = \sum_{t=k}^{k+m-1} \rho_t w_t + \rho_{k+m} W_{k+m} = A \quad (\text{A.7})$$

$$\begin{aligned} \text{Cov}(Y_{k+m} - Y_k^{(m)}, Z_k^{(m)}) &= \sum_{t=k}^{k+m-1} (\rho_t - \beta_t) (-o_t) w_t \\ &\quad + \sum_{t=k+m}^{\infty} (\rho_{k+m} - \beta_t) (C_{k+m} - \rho_{k+m}) w_t . \end{aligned}$$

By (A.7) and since ρ_t is increasing and β_t is decreasing, $(\beta_t - \rho_t) \geq (\beta_{t+k} - \rho_{t+k})$ and $\rho_{k+m} - \beta_t \geq \rho_{k+m} - \beta_{k+m}$ for all $t \geq k+m$. It follows that

$$\begin{aligned} \text{Cov}(Y_{k+m} - Y_k^{(m)}, Z_k^{(m)}) &\geq \sum_{t=k}^{k+m-1} (\beta_{k+m} - \rho_{k+m}) \rho_t w_t \\ &\quad + \sum_{t=k+m}^{\infty} (\rho_{k+m} - \beta_{k+m}) (C_{k+m} - \rho_{k+m}) w_t \\ &= (\beta_{k+m} - \rho_{k+m}) \left[\sum_{t=k}^{k+m-1} \rho_t w_t - \sum_{t=k+m}^{\infty} (C_{k+m} - \rho_{k+m}) w_t \right] \\ &= (\beta_{k+m} - \rho_{k+m}) \left[E \left(Y_k^{(m)} - Y_{k+m} \right) \right] = 0 . \end{aligned}$$

This completes the proof of the theorem.

In the remainder of this Appendix we establish our results corresponding to the median optimality and the criterion of profitability. First, consider the result corresponding to the median optimality.

Theorem 2. Let m be a positive integer such that $.5 \leq \sum_{t=m}^{\infty} w_t$ and

$.5 \leq \sum_{t=0}^m w_t$. If we put $\rho_0 = \rho_1 = \dots = \beta_m$ then for the corresponding

net profit random variable Z , median $Z = 0$, and $E|Z - \text{median}(Z)|$ is minimum among all policies for which the median is zero.

Proof: Since $\rho_t = \beta_m$ for all t and since $\beta_0 \geq \beta_1 \geq \dots$ we obtain

$$P[Z \leq 0] = \sum_{\{t: \rho_t - \beta_t \leq 0\}} w_t = \sum_{t=0}^m w_t \geq .5$$

and

$$P[Z \geq 0] = \sum_{\{t: \rho_t - \beta_t \geq 0\}} w_t = \sum_{t=m}^{\infty} w_t \geq .5 \quad . \quad (A.8)$$

Clearly, median(Z) = 0 .

Let $\rho_0^*, \rho_1^*, \dots$ be any other sequence of total premium payments such that for the corresponding Z variable, denoted by (Z^*) , median is zero. Then we must have

$$\rho_t^* - \beta_t \leq 0 \quad \text{for } t \leq m \quad \text{and} \quad \rho_t^* - \beta_t \geq 0 \quad \text{for } t \geq m .$$

This, in particular, implies that $\rho_m^* = \beta_m$ and

$$\begin{aligned} E|Z^* - \text{median}(Z^*)| &= \sum_{t=0}^m (\beta_t - \rho_t^*) + \sum_{t=m}^{\infty} (\rho_t^* - \beta_t) w_t \\ &\geq \sum_{t=0}^m (\beta_t - \rho_m^*) w_t + \sum_{t=m}^{\infty} (\rho_m^* - \beta_t) w_t \quad \text{since } \rho_t^* \\ &\quad \text{is increasing and } \rho_m^* = \beta_m = \rho_m \\ &= E|Z - \text{median}(Z)| . \end{aligned}$$

Finally, we consider the criteria of profitability. Suppose we consider all those premium payment plans ρ_0, ρ_1, \dots such that $E(Z) = 0$, i.e.,

$$\sum_{t=0}^{\infty} \rho_t w_t = \sum_{t=0}^{\infty} \beta_t w_t . \quad (\text{A.9})$$

On the other hand,

$$P[Z \geq 0] = \sum_{\{t: \rho_t \geq \beta_t\}} w_t = \sum_{t=m}^{\infty} w_t$$

because β_t is nonincreasing and ρ_t is nondecreasing and m denotes the first time when $\rho_t \geq \beta_t$. Since β_t 's are predetermined, to maximize $P[Z \geq 0]$ we must find a payment sequence such that the corresponding m is the smallest. First, we confine our attention to a single payment plan at k for $k = 0, 1, 2, \dots$. For any such plan (A.9) implies that the dis-

counted net single premium for year k , C_k , must be equal to $\left[\sum_{t=0}^{\infty} \beta_t w_t \right] / w_k$.

For any nonsingle payment plan ρ_0, ρ_1, \dots with $E(Z) = 0$, we obtain $C_k w_k =$

$\sum_{i=0}^{\infty} \rho_i w_i$ and, consequently,

$$\sum_{i=k}^{\infty} (C_k - \rho_i) w_i = \sum_{i=0}^{k-1} \rho_i w_i \geq 0 . \quad (\text{A.10})$$

But ρ_i are nondecreasing therefore, if $C_k - \rho_k < 0$, then $C_k - \rho_i < 0$

for all $i > k$ and, consequently, $\sum_{i=k}^{\infty} (C_k - \rho_i) w_i$ will be negative.

This contradicts (A.10). Hence, for any payment plan ρ_0, ρ_1, \dots , we must have $C_k \geq \rho_k$. Thus, if $\rho_k \geq \beta_k$ for the first time at $k = m$,

then there exists a single payment plan such that the corresponding C_m is also greater than β_m . Since $C_m \geq \rho_m$, the $P[Z \geq 0]$ corresponding to C_m can not be smaller than the $P[Z \geq 0]$ corresponding to ρ_0, ρ_1, \dots . Thus, the following theorem.

Theorem 3. Let C_k be the discounted net single premium such that for the associated Z_k , $E(Z_k) = 0$. Let k be such that $C_k \geq \beta_k$ and $C_{k-1} < \beta_{k-1}$. Then among all possible payment plans satisfying $E(Z) = 0$, the plan which maximizes $P[Z \geq 0]$ is given by the single premium payment at k , i.e., $\rho_0 = \dots = \rho_{k-1}$ and $C_k = \rho_k = \rho_{k+1} = \dots$.

REFERENCES

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