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Quasisymmetric Graphs and Zygmund Functions

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QUASISYMMETRIC GRAPHS AND ZYGMUND FUNCTIONS

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Abstract. A quasisymmetric graph is a curve whose projection onto a line is a quasisymmetric map. We show that this class of curves is related to solutions of the reduced Beltrami equation and to a generalization of the Zygmund class Λ_*. This relation makes it possible to use the tools of harmonic analysis to construct nontrivial examples of quasisymmetric graphs and of quasiconformal maps.

1. Introduction

Let X and Y be subsets of a Euclidean space $\mathbb{R}^n$. An embedding $f\colon X \to Y$ is quasisymmetric if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that for any triple of distinct points $a, b, x \in X$

$$|f(x) - f(a)| \leqslant \eta(t)|f(x) - f(b)| \qquad \text{where} \qquad t = \frac{|x-a|}{|x-b|}. \tag{1.1}$$

We call a set $\Gamma \subset \mathbb{C}$ a *quasisymmetric graph* if the orthogonal projection of Γ onto $\mathbb{R}$ is a quasisymmetric homeomorphism between Γ (with the metric induced from $\mathbb{C}$) and $\mathbb{R}$. This should be compared to Lipschitz graphs, which can be defined by requiring the projection to be bi-Lipschitz, a stronger property than quasisymmetry. For instance, we shall see that the graph of any function in the Zygmund class Λ_* is quasisymmetric.

This paper has three main goals.

(I) Parametrize quasisymmetric graphs by homeomorphic solutions of the reduced Beltrami equation;

(II) Use a generalization of the Zygmund class Λ_* to construct quasisymmetric graphs;

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(III) Use (I) and (II) to solve a problem from [25] concerning the variation of reduced quasiconformal maps.

Our success in (I) is partial in that we can parametrize only quasisymmetric graphs with small distortion. This is made precise with the concept of an s-quasisymmetric map introduced by Tukia and Väisälä [34]. Namely, the map f in (1.1) is called s-quasisymmetric (where $s > 0$ is a constant) if η can be chosen so that $\eta(t) \leqslant t + s$ for $0 \leqslant t \leqslant 1/s$. Observe that any quasisymmetric map is s-quasisymmetric for large enough s. The term *s-quasisymmetric graph* should be self-explanatory.

Definition 1.1. A $W^{1,2}_{\mathrm{loc}}$-homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is quasiconformal if there exists a constant $k \in [0, 1)$ such that

$$|f_{\bar z}| \leqslant k|f_z| \qquad \text{a.e. in } \mathbb{C}. \tag{1.2}$$

We sometimes refer to the constant k in (1.2) by writing that f is k-quasiconformal. The images of circles and lines under a quasiconformal map are called quasicircles and quasilines, respectively. These curves are ubiquitous in geometric function theory and still pose challenging problems [16, 27, 28, 30].

Inequality (1.2) is a form of the Beltrami equation $f_{\bar z} = \nu(z) f_z$ where $\|\nu\|_{L^\infty} < 1$. A closely related equation with f_z replaced by $\operatorname{Re} f_z$ (or $\operatorname{Im} f_z$) arises from consideration of elliptic PDE in the plane and generated considerable interest recently [4, 7, 8, 17, 22, 23, 25]. We state this *reduced Beltrami equation* as an inequality, without an explicit coefficient ν.

Definition 1.2. A nonconstant continuous $W^{1,2}_{\mathrm{loc}}$-mapping $f\colon \mathbb{C} \to \mathbb{C}$ is *reduced quasiconformal* if there exists a constant $k \in [0, 1)$ such that

$$|f_{\bar z}| \leqslant k \operatorname{Re} f_z, \qquad \text{a.e. in } \mathbb{C}. \tag{1.3}$$

Definition 1.2 does not explicitly require f to be a homeomorphism, but the injectivity of f is a consequence of inequality (1.3) [21, Corollary 1.5]. In addition, f maps every horizontal line onto a graph over $\mathbb{R}$ [22, Proposition 1.5] except for the degenerate case

$$f(z) = i\lambda z + b, \qquad \lambda \in \mathbb{R}, \quad b \in \mathbb{C}, \tag{1.4}$$

when both sides of (1.3) vanish identically.

We are now ready to state the result that achieves Goal (I) for graphs of small distortion.

Theorem 1.3. *There exists a constant $s_0 > 0$ such that any s-quasisymmetric graph $\Gamma \subset \mathbb{C}$ with $s < s_0$ is the image of $\mathbb{R}$ under a reduced quasiconformal mapping $f\colon \mathbb{C} \to \mathbb{C}$. Moreover, the constant k in* (1.3) *depends only on s and $k \to 0$ as $s \to 0$.*

It should be mentioned that even though Γ has a natural quasisymmetric parametrization by $\mathbb{R}$ (the inverse of projection), this parametrization cannot in general be extended to a reduced quasiconformal mapping of $\mathbb{C}$. Instead we use the parametrization that comes from the conformal map of upper half-plane onto the domain above Γ.

Our Goal (II) is achieved by means of Theorem 1.4. It employs the generalized Zygmund class Λ_μ which is introduced in Definition 2.6.

Theorem 1.4. *Let μ be a doubling measure on $\mathbb{R}$. Let u and v be real functions on $\mathbb{R}$ such that $u' = \mu$ and $v \in \Lambda_\mu$. Then the image of $\mathbb{R}$ under the map $\Gamma(t) = u(t) + iv(t)$ is a quasisymmetric graph.*

Furthermore, if the doubling constant of μ and the Λ_μ-seminorm of v are sufficiently small, then $\Gamma(\mathbb{R})$ is an s-quasisymmetric graph where s is small.

Theorems 1.3 and 1.4 from the basis for the proof of our third main result. To state it, let $\Phi_q\colon [0,\infty) \to [0,\infty)$ be any convex increasing function such that

$$\Phi_q(t) = \frac{t}{(\log 1/t)^q} \qquad \text{for small } t. \tag{1.5}$$

We refer to Definition 2.7 for the notion of Φ-variation.

Theorem 1.5. *There exists a reduced quasiconformal mapping $f\colon \mathbb{C} \to \mathbb{C}$ whose restriction to the line segment $[0,1]$ has infinite Φ_q-variation for every $0 < q < 1$.*

This result was previously known only for $q < 1/2$ [25, Remark 4.1]. On the other hand, for $q > 1$ every reduced quasiconformal map has finite Φ_q-variation on line segments [25, Theorem 1.7]. The borderline case $q = 1$ remains open. Using the additivity of reduced quasiconformal maps, one can strengthen the conclusion of Theorem 1.5 by replacing one line segment with an arbitrary countable set of lines. See [25] for details. The size of such exceptional sets for Sobolev and quasiconformal maps was recently studied in [9].

We do not know if the restriction $s < s_0$ is necessary in Theorem 1.3. The converse statement holds without such restrictions.

Proposition 1.6. *If $f\colon \mathbb{C} \to \mathbb{C}$ is a reduced quasiconformal map which is not of the form* (1.4)*, then $f(\mathbb{R})$ is an s-quasisymmetric graph with $s = s(k) \to 0$ as $k \to 0$. Here k is the constant in* (1.3)*. In addition,*

$$\operatorname{Im} f\big|_{\mathbb{R}} \in \Lambda_\mu \qquad \text{where } \mu = \frac{d}{dx} \operatorname{Re} f(x). \tag{1.6}$$

This leads to a conjecture.

Conjecture 1.7. *The images of $\mathbb{R}$ under reduced quasiconformal maps $\mathbb{C} \to \mathbb{C}$ are precisely quasisymmetric graphs and vertical lines.*

Parametrization of Lipschitz graphs is much easier to achieve. They corresponds to delta-monotone maps, which are defined as follows. A map $f\colon \mathbb{C} \to \mathbb{C}$ is delta-monotone if there exists a constant $\delta > 0$ such that

$$\operatorname{Re} \frac{f(z) - f(\zeta)}{z - \zeta} \geqslant \delta \frac{|f(z) - f(\zeta)|}{|z - \zeta|} \qquad \text{for all distinct } z, \zeta \in \mathbb{C}.$$

This is a proper subclass of reduced quasiconformal maps [22].

Proposition 1.8. *The images of $\mathbb{R}$ under nonconstant delta-monotone maps $\mathbb{C} \to \mathbb{C}$ are precisely Lipschitz graphs.*

Remark 1.9. The concept of a quasisymmetric graph also makes sense for k-hypersurfaces in $\mathbb{R}^n$, although it reduces to Lipschitz graphs when $2k > n$. It would be interesting to investigate, e.g., 2-dimensional quasisymmetric graphs in $\mathbb{R}^4$, but we do not pursue this direction here.

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2. Preliminaries

By an *embedding* we understand a map that is a homeomorphism onto its image. An embedding $\Gamma\colon \mathbb{R} \to \mathbb{C}$ satisfies the Ahlfors condition if there exists a constant K such that

$$\operatorname{diam}\Gamma([a,b]) \leqslant K|\Gamma(a)-\Gamma(b)| \qquad \text{whenever } a<b. \tag{2.1}$$

By a classical theorem of Ahlfors [1], the condition (2.1) characterizes quasilines, i.e., images of lines under quasiconformal maps. Tukia [32] proved that every quasisymmetric embedding $\mathbb{R}\to\mathbb{C}$ extends to a quasiconformal map $\mathbb{C}\to\mathbb{C}$. It immediately follows that every quasisymmetric graph is a quasiline. However, a quasiline may be a graph without being a quasisymmetric graph. Such examples are easy to find, e.g., the graphs $y=\sqrt{x^+}$ and $y=e^x$.

The foundational results on s-quasisymmetric maps were obtained by Tukia and Väisälä in 1980s. We will use three of them. For simplicity, the theorems are stated here in the planar case.

Theorem 2.1. [34, Theorem 5.4] *There is a number $s_0>0$ such that for $0\leqslant s\leqslant s_0$ any s-quasisymmetric embedding of $\mathbb{R}$ into $\mathbb{C}$ extends to a s_1-quasisymmetric mapping $\mathbb{C}\to\mathbb{C}$. Here $s_1=s_1(s)\to 0$ as $s\to 0$.*

Theorem 2.2. [34, Theorem 2.6] *Any s-quasisymmetric homeomorphism $f\colon\mathbb{C}\to\mathbb{C}$ is k-quasiconformal with $k=k(s)\to 0$ as $s\to 0$. Conversely, any k-quasiconformal homeomorphism $f\colon\mathbb{C}\to\mathbb{C}$ is s-quasisymmetric with $s=s(k)\to 0$ as $k\to 0$.*

Theorem 2.3. [35, Theorem 3.9] *Let $0<\varkappa\leqslant\frac{1}{25}$, and let $f\colon\mathbb{R}\to\mathbb{C}$ be a map such that for any $a<b$ there is an affine map $h\colon[a,b]\to\mathbb{C}$ with*

$$\sup_{[a,b]}|h-f|\leqslant\varkappa|h(a)-h(b)|. \tag{2.2}$$

Then f is s-quasisymmetric, where $s=s(\varkappa)\to 0$ as $\varkappa\to 0$.

Definition 2.4. A positive Radon measure μ on $\mathbb{R}$ is doubling if there exists $\delta>0$ such that

$$\mu(I)\leqslant(1+\delta)\mu(J) \tag{2.3}$$

for any adjacent intervals I,J of equal length.

Definition 2.5. A continuous function $g\colon\mathbb{R}\to\mathbb{R}$ belongs to the Zygmund class Λ_* if there exists a constant $M>0$ such that

$$|g(x+h)-2g(x)+g(x-h)|\leqslant 2Mh \qquad \text{for all } x\in\mathbb{R},\ h>0 \tag{2.4}$$

The smallest such M is the Zygmund seminorm of g.

It is often said that (2.4) is an additive form of (2.3). One can interpret (2.4) by saying that the nonlinearity of g on any interval is controlled by the length of the interval. The relevance of the class Λ_* to geometric function theory is evident by now [14, 29, 11]. But our subject required a wider class of functions, in which the length is replaced by a general nonatomic Radon measure on $\mathbb{R}$. A measure is nonatomic if it gives zero mass to every singleton. All our measures are positive.

Definition 2.6. Let μ be a nonatomic Radon measure on $\mathbb{R}$. A continuous function $g\colon \mathbb{R} \to \mathbb{R}$ belongs to the generalized Zygmund class Λ_μ if there exists a constant $M > 0$ such that

$$|g(x+h) - 2g(x) + g(x-h)| \leqslant M\mu([x-h, x+h]) \tag{2.5}$$

for all $x \in \mathbb{R}$, $h \geqslant 0$. The smallest such M is the seminorm of g in Λ_μ.

We should make precise the remark about the controlled nonlinearity of g. Given distinct points $a, b \in \mathbb{R}$, let

$$g_{ab}(x) = \frac{b-x}{b-a}\, g(a) + \frac{x-a}{b-a}\, g(b) \tag{2.6}$$

denote the affine function that agrees with g at a and b. If g satisfies (2.5), then

$$\sup_{[a,b]} |g - g_{ab}| \leqslant M\mu([a,b]) \qquad \text{whenever } a < b. \tag{2.7}$$

Indeed, we lose no generality in assuming that $g(a) = g(b) = 0$ and $|g|$ attains its maximum on $[a, b]$ at a point $\xi \leqslant \frac{a+b}{2}$. Applying (2.5) with $x = \xi$ and $h = \xi - a$, we find

$$|g(2\xi - a) - 2g(\xi)| \leqslant M\mu([a,b]), \qquad \text{hence} \qquad |g(\xi)| \leqslant M\mu([a,b]).$$

Conversely, (2.7) yields (2.5) with $2M$ in place of M.

Definition 2.7. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a convex increasing function. A function $v\colon [a,b] \to \mathbb{R}$ has finite Φ-variation if

$$\sup \sum_{j=1}^{N} \Phi(|v(x_j) - v(x_{j-1})|) < \infty, \tag{2.8}$$

where the supremum is taken over all partitions $a = x_0 < \cdots < x_N = b$ and over all $N \geqslant 1$. If v is defined on $\mathbb{R}$, we say that it has locally finite Φ-variation if (2.8) holds for every bounded interval.

In the sequel, the constants C and c in estimates may be different from one line to another.

3. Proof of Propositions 1.6 and 1.8

Two of the results stated in the introduction admit simple proofs.

Proof of Proposition 1.6. To a reduced quasiconformal map f we associate the one-parameter family $f_\lambda(z) = f(z) + i\lambda z$, $\lambda \in \mathbb{R}$. Unless f is of the form (1.4), each f_λ is also reduced quasiconformal, as it is nonconstant and satisfies (1.3) with the same constant as f. Therefore, f_λ is η-quasisymmetric with η independent of λ. In particular, for any triple of distinct points $a, b, x \in \mathbb{R}$ we have

$$|f_\lambda(x) - f_\lambda(a)| \leqslant \eta(|\tau|)|f_\lambda(x) - f_\lambda(b)|, \qquad \tau = \frac{x-a}{x-b}. \tag{3.1}$$

Setting $\lambda = -\operatorname{Im}\frac{f(x)-f(b)}{x-b}$ results in

$$|f(x) - f(a) - i\tau \operatorname{Im}(f(x) - f(b))| \leqslant \eta(|\tau|)\,|\operatorname{Re}(f(x) - f(b))|. \tag{3.2}$$

There are two ways to use (3.2). First, we can take the real part and obtain

$$|\operatorname{Re}(f(x) - f(a))| \leqslant \eta(|\tau|)\ |\operatorname{Re}(f(x) - f(b))| \tag{3.3}$$

which simply says that $\operatorname{Re} f$ is a quasisymmetric map from $\mathbb{R}$ onto $\mathbb{R}$. Combining (3.3) with the quasisymmetry of f, we conclude that the projection $w \mapsto \operatorname{Re} w$ is a quasisymmetric map from Γ to $\mathbb{R}$.

Let μ denote the distributional derivative of $\operatorname{Re} f(x)$ with respect to x. Since $\operatorname{Re} f$ is quasisymmetric, μ is a doubling measure on $\mathbb{R}$ [20, Remark 13.20b]. Taking the imaginary part in (3.2) yields

$$|\operatorname{Im}(f(x) - f(a)) - \tau \operatorname{Im}(f(x) - f(b))| \leqslant \eta(|\tau|)\ |\operatorname{Re}(f(x) - f(b))|\,. \tag{3.4}$$

Choosing $x = \frac{a+b}{2}$, we conclude that $\operatorname{Im} f \in \Lambda_\mu$. □

Remark 3.1. Every quasisymmetric graph $y = g(x)$ admits a natural quasisymmetric parametrization by $\mathbb{R}$, namely $f(x) = x + ig(x)$. In general, this function f does not satisfy (1.6) and therefore cannot be extended to a reduced quasiconformal map of the plane. For a concrete example, take the graph $y = x^{1/3}$.

Proof of Proposition 1.8. It is obvious that $f(\mathbb{R})$ is a Lipschitz graph for every delta-monotone map $f\colon \mathbb{C} \to \mathbb{C}$. Conversely, for any L-Lipschitz real function g the mapping

$$f(z) := \operatorname{Re} z + iL^2 \operatorname{Im} z + ig(\operatorname{Re} z)$$

satisfies

$$\operatorname{Re} f_z = \frac{L^2+1}{2}, \qquad |\operatorname{Im} f_z| \leqslant \frac{L}{2} \leqslant \operatorname{Re} f_z, \tag{3.5}$$

and

$$|f_{\bar z}| \leqslant \sqrt{\frac{(L^2-1)^2}{4} + \frac{L^2}{4}} \leqslant k \operatorname{Re} f_z \tag{3.6}$$

with $k = k(L) < 1$. The combination of (3.5) and (3.6) implies that f is delta-monotone, see [24, Lemma 12]. □

4. Proof of Theorem 1.3

Let $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$ denote the upper half-plane. By Theorems 2.1 and 2.2 the curve Γ is a k-quasiline where k is small if s is.

The curve Γ divides the plane into two domains; let Ω denote the upper one. Let $f\colon \mathbb{H} \to \Omega$ be a conformal mapping such that $f(\infty) = \infty$ in the sense of boundary correspondence. Since Γ is a k-quasiline, f extends to Γ by continuity. It then extends to the entire plane by quasiconformal reflection, and the extended mapping is $\frac{2k}{1+k^2}$-quasiconformal [1]. By Theorem 2.2 the correspondence $x \mapsto \operatorname{Re} f(x)$ is s_1-quasisymmetric where s_1 is small if k is.

We claim that there exists $\tilde k \in [0,1)$ such that $\tilde k \to 0$ as $k \to 0$ and

$$2 \operatorname{Im} z\, |f''(z)| \leqslant \tilde k \operatorname{Re} f'(z) \qquad \text{for all } z \in \mathbb{H}. \tag{4.1}$$

Assume (4.1) for now and complete the proof of the theorem.

The Koebe $1/4$-theorem [26, (I.6.7)] yields

$$\operatorname{Im} z\, |f'(z)| \leqslant 2 \operatorname{dist}(f(z), \mathbb{C} \setminus f(\mathbb{H})) \qquad \text{for all } z \in \mathbb{H}. \tag{4.2}$$

Hence

$$\lim_{z\to\zeta} \operatorname{Im} z\, |f'(z)| = 0 \qquad \text{for any } \zeta \in \mathbb{R}. \tag{4.3}$$

We extend f to $\mathbb{C}$ following the method that goes back to Ahlfors and Weill [3] and was further developed in [2, 5, 19]. Namely, we define $F\colon \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \operatorname{Im} z \geqslant 0; \\ f(\bar{z}) + (z - \bar{z}) f'(\bar{z}) & \operatorname{Im} z < 0. \end{cases} \tag{4.4}$$

By virtue of (4.3) the mapping F is continuous in $\mathbb{C}$. For $z \in \mathbb{C} \setminus \overline{\mathbb{H}}$ we have

$$F_z = f'(\bar{z}) \qquad \text{and} \qquad F_{\bar{z}} = (z - \bar{z}) f''(\bar{z}). \tag{4.5}$$

The comparison of (4.1) and (4.5) shows that F is reduced quasiconformal. The theorem is proved, modulo (4.1). ☐

Proof of (4.1). The first step is to observe that $\operatorname{Re} f' > 0$ in $\mathbb{H}$. To this end, introduce the function

$$u_h(z) := \arg(f(z+h) - f(z)) \qquad \text{for a fixed } h > 0.$$

Here we choose the branch of arg so that $|u_h| < \pi/2$ on $\partial\mathbb{H}$: this is possible because f extends to a homeomorphism $f\colon \overline{\mathbb{H}} \to \overline{\Omega}$ and $\partial\Omega$ is a graph. The maximum principle implies $|u_h| < \pi/2$ in $\mathbb{H}$, and letting $h \to 0$ we obtain the desired conclusion $\operatorname{Re} f' > 0$.

The harmonic function $u = \operatorname{Re} f'$, being positive in $\mathbb{H}$, admits the Herglotz representation [15, Theorem I.3.5]

$$u(z) = \beta \operatorname{Im} z + \frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Im} \frac{1}{t - z} \, d\mu(t) \tag{4.6}$$

where $\beta \geqslant 0$ and μ is a positive measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1+t^2} \, d\mu(t) < \infty. \tag{4.7}$$

Integration of (4.6) yields $\mu([a,b]) = \operatorname{Re}(f(b) - f(a))$ for any finite interval $[a,b] \subset \mathbb{R}$. Recall that the map $x \mapsto \operatorname{Re} f(x)$ is s_1-quasisymmetric where $s_1 \to 0$ as $k \to 0$. Therefore, the measure μ satisfies the doubling condition (2.3) where $\delta \to 0$ as $k \to 0$.

To proceed further, we must establish that $\beta = 0$ in (4.6). To this end, we need the following growth estimate for univalent functions $F\colon \mathbb{H} \to \mathbb{C}$:

$$|F(x+iy)| \leqslant |F(i)| + \frac{(y+1)^4}{y^2} |F'(i)| \qquad \text{for } y \geqslant 1, \quad |x| \leqslant y + 1. \tag{4.8}$$

To prove (4.8), introduce

$$G(\zeta)=\frac{-i}{2F'(i)}\left\{F\left(i\frac{1+\zeta}{1-\zeta}\right)-F(i)\right\},\qquad |\zeta|<1, \tag{4.9}$$

and observe that $G(0)=G'(0)-1=0$. The growth theorem for class S [13, Theorem 2.6] asserts that

$$|G(\zeta)|\leqslant\frac{|\zeta|}{(1-|\zeta|)^2}=\frac{|\zeta|(1+|\zeta|)^2}{(1-|\zeta|^2)^2}\leqslant\frac{4}{(1-|\zeta|^2)^2}. \tag{4.10}$$

We set $x+iy=i\frac{1+\zeta}{1-\zeta}$ and observe that $|\zeta|^2\leqslant\frac{y^2+1}{(y+1)^2}$. Combinng this with (4.10) and (4.9) the inequality (4.8) follows.

We may assume $0\in\partial\Omega$. For $r>0$ let Γ be the connected component of the set $\{z\in\mathbb{C}\setminus\Omega\colon |z|=r\}$ that contains the point $-ir$. By virtue of the Ahlfors condition (2.1) the length of Γ is bounded from below by cr, with $c>0$ independent of r. Therefore the mapping $z\mapsto z^p$, where $p=\frac{2\pi}{2\pi-c}>1$, is univalent in Ω. This allows us to apply (4.8) with $F=f^p$ and conclude that $|f(x+iy)|=O(y^{2/p})$ as $y\to\infty$, $|x|\leqslant y$. The Cauchy inequality for f' yields $|f'(iy)|=O(y^{\frac{2}{p}-1})$ as $y\to\infty$. Since the exponent of y is strictly less than 1, the coefficient β in (4.6) must vanish.

Returning to (4.6), we compute

$$f''(z)=2\frac{\partial u}{\partial z}(z)=\frac{1}{\pi i}\int_{\mathbb{R}}\frac{1}{(t-z)^2}\,d\mu(t) \tag{4.11}$$

and

$$\operatorname{Re} f'(z)=u(z)=\frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}}\frac{1}{|t-z|^2}\,d\mu(t) \tag{4.12}$$

Thus, the desired inequality (4.1) takes the form

$$\left|\int_{\mathbb{R}}\frac{1}{(t-z)^2}\,d\mu(t)\right|\leqslant\frac{\tilde{k}}{2}\int_{\mathbb{R}}\frac{1}{|t-z|^2}\,d\mu(t) \tag{4.13}$$

The following lemma yields (4.13). It is not particularly new; one can find a similar, but less precise, statement in [12, p. 157]. □

Lemma 4.1. *For any $\varepsilon>0$ there exists $\delta>0$ such that the following holds. If μ satisfies the doubling condition* (2.3) *then*

$$\left|\int_{\mathbb{R}}\frac{1}{(t-z)^2}\,d\mu(t)\right|\leqslant\varepsilon\int_{\mathbb{R}}\frac{1}{|t-z|^2}\,d\mu(t)<\infty\qquad \textit{for all } z\in\mathbb{H}. \tag{4.14}$$

Proof. We write $|I|$ for the length of an interval I. Repeated application of the doubling property yields the growth/decay estimate

$$(1-\gamma)\min(\tau,\tau^{-1})^{\gamma} \leqslant \frac{\mu(I)}{\tau\mu(J)} \leqslant (1+\gamma)\max(\tau,\tau^{-1})^{\gamma}, \qquad \tau = \frac{|I|}{|J|} \tag{4.15}$$

for any two intervals I and J with a common point. Here $\gamma \in (0,1)$ depends only on δ, and $\gamma \to 0$ as $\delta \to 0$.

Using shift, scaling, and normalization, we reduce (4.14) to the case $z = i$ and $\mu([-1,1]) = 1$. By virtue of (4.15), for all $t > 0$ we have

$$(1-\gamma)t\min(t,t^{-1})^{\gamma} \leqslant \mu([-t,t]) \leqslant (1+\gamma)t\max(t,t^{-1})^{\gamma}. \tag{4.16}$$

For small γ the estimates (4.15) yield the following uniform bounds in t,

$$|\mu([-t,t]) - t| = \begin{cases} O(\gamma) & \text{if } 0 < t < 1 \\ O(\gamma t^{1+\gamma}(1+\log t)) & \text{if } t > 1 \end{cases} \tag{4.17}$$

We proceed to estimate both sides of (4.14) via integration by parts followed by (4.17).

$$\begin{aligned} \int_{\mathbb{R}} \frac{1}{t^2+1}\,d\mu(t) &= \int_0^\infty \frac{2t}{(t^2+1)^2}\mu([-t,t])\,dt \\ &= \pi + \int_0^\infty \frac{2t}{(t^2+1)^2}(\mu([-t,t]) - t)\,dt \end{aligned} \tag{4.18}$$

which in view of (4.17) implies

$$\int_{\mathbb{R}} \frac{1}{t^2+1}\,d\mu(t) = \pi + O(\gamma) \qquad \text{as } \gamma \to 0. \tag{4.19}$$

Next,

$$\begin{aligned} \operatorname{Re}\int_{\mathbb{R}} \frac{1}{(t-i)^2}\,d\mu(t) &= \int_{\mathbb{R}} \frac{t^2-1}{(t^2+1)^2}\,d\mu(t) \\ &= \int_0^\infty \frac{2t(t^2-3)}{(t^2+1)^3}\mu([-t,t])\,dt \\ &= \int_0^\infty \frac{2t(t^2-3)}{(t^2+1)^3}(\mu([-t,t]) - t)\,dt \\ &= O(\gamma) \end{aligned} \tag{4.20}$$

Finally,

$$
\begin{aligned}
\operatorname{Im} \int_{\mathbb{R}} \frac{1}{(t-i)^2}\, d\mu(t) &= \int_{\mathbb{R}} \frac{2t}{(t^2+1)^2}\, d\mu(t) \\
&= \int_0^\infty \frac{2(3t^2-1)}{(t^2+1)^3} (\mu([0,t]) - \mu([-t,0]))\, dt \\
&\leqslant \delta \int_0^\infty \frac{2(3t^2-1)}{(t^2+1)^3} \mu([0,t])\, dt \\
&= O(\delta)
\end{aligned}
\tag{4.21}
$$

The combination of (4.19)–(4.21) proves (4.14). □

5. Proof of Theorem 1.4

By virtie of the doubling condition, the map $u\colon \mathbb{R} \to \mathbb{R}$ is s-quasisymmetric where s is small if δ is small. Thus we may consider the map $t \mapsto \Gamma(t)$ instead of the projection $\Gamma(t) \mapsto u(t)$.

Fix $a, b \in \mathbb{R}$, $a < b$. The growth estimate for μ, (4.15), yields

$$\sup_{[a,b]} |u - u_{a,b}| \leqslant C(u(b) - u(a)) \tag{5.1}$$

where $C = C(\delta) \to 0$ as $\delta \to 0$. On the other hand, the definition of Λ_μ implies

$$\sup_{[a,b]} |v - v_{a,b}| \leqslant \|v\|_{\Lambda_\mu} (u(b) - u(a)). \tag{5.2}$$

When δ and $\|v\|_{\Lambda_\mu}$ are small, Theorem 2.3 implies that Γ is s-quasisymmetric with small s.

Without the smallness condition, we can still conclude from (5.1)–(5.2) that

$$|\Gamma(x) - \Gamma_{a,b}(x)| \leqslant K(u(b) - u(a)), \qquad x \in [a,b], \tag{5.3}$$

with K independent of a, b. We shall demonstrate the existence of a constant H such that

$$|\Gamma(x) - \Gamma(a)| \leqslant H|\Gamma(x) - \Gamma(b)| \qquad \text{whenever } |x-a| \leqslant |x-b|. \tag{5.4}$$

The property (5.4) implies the quasisymmetry of Γ [20, Theorem 10.19]. We split the proof of (5.4) in two cases. If $a \leqslant x \leqslant b$, then (5.3) yields

$$|\Gamma(x) - \Gamma(a)| \leqslant \frac{x-a}{b-x} |\Gamma(x) - \Gamma(b)| + K(u(b) - u(a)).$$

Since $|x-a| \leqslant |x-b|$, the doubling condition implies

$$u(b)-u(a) \leqslant (2+\delta)\big(u(b)-u(x)\big),$$

hence

$$|\Gamma(x)-\Gamma(a)| \leqslant [1+K(2+\delta)]|\Gamma(x)-\Gamma(b)|.$$

The other case to consider is $x<a<b$. Now

$$|\Gamma(a)-\Gamma_{x,b}(a)| \leqslant K(u(x)-u(b)) \leqslant K|\Gamma(x)-\Gamma(b)|$$

and

$$|\Gamma(x)-\Gamma_{x,b}(a)| \leqslant |\Gamma(x)-\Gamma(b)|.$$

Hence

$$|\Gamma(x)-\Gamma(a)| \leqslant (K+1)|\Gamma(x)-\Gamma(b)|$$

from which (5.4) follows. □

6. Generalized variation of Zygmund functions

Any function in the Zygmund class Λ_* has a modulus of continuity of the form $C\delta\log(1/\delta)$ on every finite interval [36, Theorem II.3.4]. The example $g(x)=x\log x$ demonstrates that this modulus of continuity is best possible. However, at most points the local modulus of continuity can be improved to $C\delta\sqrt{\log(1/\delta)\log\log(1/\delta)}$, see [6, Theorem 1]. Such an improvement is also possible on the average, i.e., in terms of generalized variation. This fact may be known, but being unable to find a reference, we give a proof.

Proposition 6.1. *Any function of class Λ_* has locally finite Φ_q variation for every $q>1/2$. Here Φ_q is the gauge function from* (1.5).

We need a lemma.

Lemma 6.2. [25, Lemma 3.4]. *If a function $g\colon [a,b]\to\mathbb{R}$ satisfies*

$$\sum_{j=1}^{N}|g(x_j)-g(x_{j-1})| \leqslant C\log^p(N+1)$$

for any partition $a=x_0<\cdots<x_N=b$, then g has finite Φ_q variation for every $q>p$.

Proof of Proposition 6.1. Let $g \in \Lambda_*$. We claim that there exists a constant C such that for any triple $a < x < b$

$$\frac{(g(x)-g(a))^2}{x-a} + \frac{(g(x)-g(b))^2}{b-x} \leqslant \frac{(g(b)-g(a))^2}{b-a} + C(b-a). \tag{6.1}$$

Using the linear interpolant (2.6) we rewrite the left-hand side of (6.1) in terms of the difference $\delta := g(x) - g_{ab}(x)$:

$$\begin{aligned} &\frac{(g(x)-g(a))^2}{x-a} + \frac{(g(x)-g(b))^2}{b-x} \\ &\qquad = \frac{\delta^2}{x-a} + \frac{\delta^2}{b-x} + \frac{(g_{ab}(x)-g(a))^2}{x-a} + \frac{(g_{ab}(x)-g(b))^2}{b-x} \\ &\qquad = \frac{\delta^2}{x-a} + \frac{\delta^2}{b-x} + \frac{(g(b)-g(a))^2}{b-a} \end{aligned}$$

It remains to prove that

$$\frac{\delta^2}{\min(x-a, b-x)} \leqslant C(b-a). \tag{6.2}$$

Recall that $\delta \leqslant C(b-a)$ by (2.7). This immediately implies (6.2) when $(x-a)$ is comparable to $(b-x)$. If x is very close to, say, a, then we use the log-Lipschitz estimate $\delta \leqslant C(x-a)|\log(x-a)|$, see [10, Proposition 1]. Thus (6.2) holds in either case.

Repeated application of (6.1) shows that for any partition $x_0, \dots, x_N$ of the interval $[a, b]$ we have

$$\sum_{j=1}^{N} \frac{|g(x_j) - g(x_{j-1})|^2}{x_j - x_{j-1}} \leqslant C \log(N+1).$$

where C is independent of N. The Cauchy-Schwarz inequality yields

$$\sum_{j=1}^{N} |g(x_j) - g(x_{j-1})| \leqslant C \log^{1/2}(N+1),$$

and Lemma 6.2 completes the proof. □

Turning to the generalized Zygmund class Λ_μ, we immediately find that the modulus of continuity is not log-Lipschitz in general. Indeed, Λ_μ always contains an antiderivative of μ. On the other hand, a version of Proposition 6.1 holds in this generality, albeit with a worse exponent.

Proposition 6.3. *Let μ be a nonatomic Radon measure on $\mathbb{R}$. Any function of class Λ_μ has locally finite Φ_q variation for every $q > 1$.*

Proof. Let $g \in \Lambda_\mu$. We claim that there exists a constant C such that for any triple $a < x < b$

$$|g(x) - g(a)| + |g(x) - g(b)| \leqslant |g(a) - g(b)| + C\mu([a, b]). \tag{6.3}$$

Indeed, in terms of the linear interpolant (2.6) we have

$$\begin{aligned} &|g(x) - g(a)| + |g(x) - g(b)| \\ &\qquad \leqslant |g_{ab}(x) - g(a)| + |g_{ab}(x) - g(b)| + 2|g(x) - g_{ab}(x)| \\ &\qquad = |g(a) - g(b)| + 2|g(x) - g_{ab}(x)| \end{aligned}$$

where the last term is controlled by $\mu([a, b])$ by the definition of Λ_μ.

Consider a partition $a = x_0 < \cdots < x_N = b$ where $N = 2^m$. Applying (6.3) to the triples like x_0, x_1, x_2, we obtain

$$\sum_{j=1}^{2^m} |g(x_j) - g(x_{j-1})| \leqslant C\mu([a, b]) + \sum_{j=1}^{2^{m-1}} |g(x_j) - g(x_{j-1})|$$

After m iterations of this process the estimate becomes

$$\sum_{j=1}^{2^m} |g(x_j) - g(x_{j-1})| \leqslant Cm\mu([a, b]) + |g(a) - g(b)|.$$

Thus, for any N point partition of $[a, b]$ we have the estimate

$$\sum_{j=1}^{N} |g(x_j) - g(x_{j-1})| \leqslant C \log(N + 1) \tag{6.4}$$

where C is independent of N. An application of Lemma 6.2 completes the proof. □

In the next section we prove that Proposition 6.3 is essentially sharp, even if the measure μ is assumed to be doubling with a small constant.

7. Infinite generalized variation

The principal result of this section concerns the class Λ_μ for singular measures μ.

Theorem 7.1. *Let $\delta > 0$. There exists a Radon measure μ on $\mathbb{R}$ with the doubling property* (2.3) *such that the class Λ_μ contains a function which has infinite Φ_q-variation on $[a, b]$ for any $0 < q < 1$ and any $a < b$.*

Together with previous results this quickly yields Theorem 1.5.

Proof of Theorem 1.5. We use the function $v \in \Lambda_\mu$ provided by Theorem 7.1, scaling it down to make the Λ_μ seminorm of v as small as needed for Theorem 1.4. Then use Theorem 1.3 to produce the desired reduced quasiconformal map. □

Proof of Theorem 7.1. Consider 4-adic intervals

$$I_{n,j} = \{x\colon 0 \leqslant 4^n x - j < 1\} = \left[\frac{j}{4^n}, \frac{j+1}{4^n}\right), \qquad n = 1, 2, \dots,\ j \in \mathbb{Z},$$

and define, for $n \geqslant 1$, the Rademacher-type functions

$$\rho_n(x) = \begin{cases} 0, & x \in I_{n,j}, \quad j \equiv 0, 3 \mod 4 \\ 1, & x \in I_{n,j}, \quad j \equiv 1 \mod 4 \\ -1, & x \in I_{n,j}, \quad j \equiv 2 \mod 4 \end{cases}$$

For future references we record several properties of the family $\{\rho_n\}$.

(i) ρ_n is constant on $I_{m,j}$ when $m \geqslant n$;
(ii) ρ_n has zero mean on $I_{m,j}$ when $m < n$.
(iii) the set of discontinuities of ρ_n is $\{j\, 4^{-n}\colon n \geqslant 1,\ 4 \nmid j\}$;
(iv) if ρ_n is discontinuous at x, then $\rho_m(y) = 0$ whenever $m > n$ and $|x - y| < 4^{-m}$;
(v) the antiderivative $R_n(x) := \int_0^x \rho_n(t)\, dt$ is 4^{1-n}-periodic and $|R_n| \leqslant 4^{-n}$;
(vi) the product $R_n \rho_m$ is continuous on $\mathbb{R}$ provided that $m < n$;
(vii) if Ψ is a function of $\rho_1, \dots, \rho_{n-1}, \rho_{n+1}, \dots \rho_m$, then

$$\int_0^1 \Psi(x)\, dx = 4 \int_{[0,1] \cap \{\rho_n = 1\}} \Psi(x)\, dx.$$

(viii) Under the assumptions of (vii), $\int_0^1 \rho_n(x) \Psi(x)\, dx = 0$.

Fix a number $\gamma \in (0, 1)$ and define for $n \geqslant 1$

$$v_n(x) = \prod_{k=1}^{n} (1 + \gamma \rho_{2k-1}(x))$$

The measures $v_n(x)\, dx$ have a weak* limit, denoted μ. It is routine to check that μ satisfies the doubling condition (2.3) where $\delta \to 0$ as $\gamma \to 0$. Indeed, the weights v_n are doubling with a uniformly controlled constant, and $\mu(I)$ can be compared to $\int_I v_n$ as long as the length of I is comparable to 4^{-2n}. See [33].

Let us introduce

$$g(x) = \sum_{n=1}^{\infty} R_{2n}(x) v_n(x) \tag{7.1}$$

where R_{2n} is the antiderivative of ρ_{2n}. Each summand is continuous by virtue of (vi). The property (v) ensures that the series converges uniformly and at an exponential rate.

Step 1: $g \in \Lambda_\mu$. For this we will show that (2.7) holds for all $a, b \in \mathbb{R}$ such that $a < b$. Since g is bounded, it suffices to consider the case $b - a < 1/16$. Let m be the greatest integer such that

$$b - a < 4^{-2m}. \tag{7.2}$$

By virtue of (v) the difference between g and the partial sum

$$g_m(x) = \sum_{n=1}^{m} R_{2n}(x) v_n(x)$$

on the interval $[a, b]$ does not exceed

$$\left(\sup_{[a,b]} v_m\right) \sum_{n>m} 4^{-2n}(1+\gamma)^{n-m} \leqslant C\, 4^{-2m} \sup_{[a,b]} v_m \leqslant C\mu([a,b]).$$

Therefore, it suffices to prove the desired property (2.7) for g_m. Differentiation of g_m yields

$$g_m'(x) = \sum_{n=1}^{m} \rho_{2n}(x) v_n(x) \tag{7.3}$$

because v_n is locally constant on the support of R_{2n}. If g_m' is constant on $[a, b]$ then we are done. Suppose otherwise. By virtue of (iii) the set of discontinuities of g_m' is a subset of $\{j\, 4^{-2n} : 1 \leqslant n \leqslant m,\ 4 \nmid j\}$. Therefore g_m' has exactly one point of discontinuity on $[a, b]$, say $\theta = \ell \cdot 4^{-2r}$, $4 \nmid \ell$. The oscillation of g_m' at this point is at most $2v_r(\theta)$. The property (iv) implies that $v_m(x) \equiv v_r(\theta)$ for $x \in [a, b]$. Hence, the deviation of g_m from an affine function on the interval $[a, b]$ does not exceed

$$2v_r(\theta)(b-a) = 2\int_a^b v_m(x) \leqslant C\mu([a,b])$$

as desired.

Step 2: the variation of g. Fix $0 < q < 1$. We must show that g has infinite Φ_q-variation on every 4-adic interval. It suffices to consider the interval $[0,1]$. Note that g coincides with the partial sum g_m at all points of the form $j\,4^{-2m}$, $j \in \mathbb{Z}$. Hence

$$\sum_{j=1}^{4^{2m}} |g(j\,4^{-2m}) - g((j-1)\,4^{-2m})| \geqslant \int_0^1 |g_m'(x)|\,dx. \tag{7.4}$$

Let $v_m^* = \max(v_1, \dots, v_m)$. For $\lambda > 0$ and $k = 1, \dots, m$ define

$$E_k(\lambda) = \{x \in [0,1] \colon v_k(x) = v_m^*(x) = \lambda,\ v_n(x) < \lambda \text{ for } n < k\}.$$

By definition, the sets $E_k(\lambda)$ from a finite partition of the interval $[0,1]$. We claim that

$$\int_{E_k(\lambda)} |g_m'(x)|\,dx \geqslant \frac{\lambda}{4} |E_k(\lambda)|, \tag{7.5}$$

where $|\cdot|$ denotes the Lebesgue measure. To this end, restrict the set of integration to $E' = E_k(\lambda) \cap \{\rho_{2k} = 1\}$. The property (vii) implies $|E'| = {}^1\!/\!_4 |E_k(\lambda)|$. According to (viii),

$$\int_{E'} \rho_{2n} v_n = \begin{cases} \lambda |E'| & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

From (7.3) we obtain

$$\int_{E'} |g_m'(x)|\,dx \geqslant \int_{E'} g_m'(x)\,dx = \lambda |E'| = \frac{\lambda}{4} |E_k(\lambda)|$$

which proves (7.5).

Summing (7.5) over all $k = 1, \dots, m$ and all $\lambda > 0$ yields

$$\int_0^1 |g_m'(x)|\,dx \geqslant \frac{1}{4} \int_0^1 v_m^*(x)\,dx. \tag{7.6}$$

We need a lemma, the proof of which is postponed to the end of this section.

Lemma 7.2. *There exists a positive constant $c > 0$ such that*

$$\int_0^1 v_m^*(x)\,dx \geqslant cm, \qquad m = 1, 2, \dots \tag{7.7}$$

From (7.4), (7.6) and (7.7) it follows that

$$\sum_{j=1}^{4^{2m}} \left| g(j\,4^{-2m}) - g((j-1)\,4^{-2m}) \right| \geqslant cm, \qquad m = 1, 2, \dots$$

Jensen's inequality yields

$$\sum_{j=1}^{4^{2m}} \Phi_q \left(\left| g(j\,4^{-2m}) - g((j-1)\,4^{-2m}) \right| \right) \geqslant 4^{2m} \Phi_q \Big(\frac{cm}{4^{2m}} \Big) \sim m^{1-q} \to \infty$$

as $m \to \infty$. □

Proof of Lemma 7.2. Introduce the random variables

$$X_k = \log(1+\gamma\rho_k) - \frac{1}{4}\log(1-\gamma^2)$$

with $[0,1]$ being the probability space. Since X_k are independent, identically distributed, and have zero mean, the large deviation bound (Bernstein's inequality [18, Theorem 5.11.4]) yields

$$\mathbf{P}\left\{\sum_{k=1}^{m} X_{2k-1} > \log\frac{1}{4} - \frac{m}{4}\log(1-\gamma^2)\right\} \leqslant e^{-cm} \tag{7.8}$$

where $c > 0$ depends only on γ. An equivalent form of (7.8) is

$$|\{x \in [0,1] \colon v_m \geqslant 1/4\}| \leqslant e^{-cm}. \tag{7.9}$$

For $\lambda \geqslant 1$ let $A(\lambda) = \{x \in [0,1] \colon v_m(x) \geqslant \lambda\}$. The estimate (7.9) yields

$$\int_{[0,1]\setminus A(\lambda)} v_m \leqslant \frac{1}{4} + \lambda e^{-cm}. \tag{7.10}$$

The right-hand side of (7.10) is less than $1/2$ provided that $\lambda \leqslant \frac{1}{4}e^{cm}$. Hence

$$\int_{A(\lambda)} v_m \geqslant \frac{1}{2}, \qquad 1 \leqslant \lambda \leqslant \frac{1}{4}e^{cm}. \tag{7.11}$$

Recall a lower bound for maximal function [31, p. 32]

$$|\{x \in [0,1] \colon v_m^*(x) \geqslant c_1\lambda\}| \geqslant \frac{c_2}{\lambda}\int_{A(\lambda)} v_m \tag{7.12}$$

with universal constants $c_1, c_2 > 0$. Integrating (7.12) with respect to λ and using (7.11), we arrive at (7.7). □

References

1. L.V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301.
2. L.V. Ahlfors, *Sufficient conditions for quasiconformal extension.* In "Discontinuous groups and Riemann surfaces" (Proc. Conf., Univ. Maryland, College Park, Md., 1973), pp. 23–29. Ann. of Math. Studies, No. 79, Princeton Univ. Press, Princeton, N.J., 1974.
3. L.V. Ahlfors and G. Weill, *A uniqueness theorem for Beltrami equations*, Proc. Amer. Math. Soc. **13** (1962), no. 6, 975–978.
4. G. Alessandrini and V. Nesi, *Beltrami operators, non-symmetric elliptic equations and quantitative Jacobian bounds*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 1, 47–67.
5. J.M. Anderson and A. Hinkkanen, *A univalency criterion*, Michigan Math. J. **32** (1985), no. 1, 33–40.
6. J.M. Anderson and L.D. Pitt, *Probabilistic behaviour of functions in the Zygmund spaces Λ^* and λ^**, Proc. London Math. Soc. (3) **59** (1989), no. 3, 558–592.
7. K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton University Press, Princeton, 2009.
8. K. Astala and J. Jääskeläinen, *Homeomorphic solutions to reduced Beltrami equations*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 2, 607–613.

9. Z. Balogh, R. Monti and J. Tyson, *Frequency of Sobolev and quasiconformal dimension distortion*, preprint, 2010.
10. J.J. Carmona and J.J. Donaire, *On removable singularities for the analytic Zygmund class*, Michigan Math. J. **43** (1996), no. 1, 51–65.
11. Z. Chen, J. Chen and C. He, *Functions with unbounded $\bar{\partial}$-derivative and their boundary functions*, J. Austral. Math. Soc. (Series A) **63** (1997), 100–109.
12. E. Doubtsov and A. Nicolau, *Symmetric and Zygmund measures in several variables*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 1, 153–177.
13. P. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.
14. F.P. Gardiner and D.P. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), no. 4, 683–736.
15. J.B. Garnett, *Bounded analytic functions*. Revised 1st ed. Springer, New York, 2007.
16. F.W. Gehring, *Characterizations of quasidisks*. In "Quasiconformal geometry and dynamics (Lublin, 1996)", 11–41, Banach Center Publ., 48, Polish Acad. Sci., Warsaw, 1999.
17. F. Giannetti, T. Iwaniec, L. Kovalev, G. Moscariello and C. Sbordone, *On G-compactness of the Beltrami operators*. In "Nonlinear homogenization and its applications to composites, polycrystals and smart materials", 107–138, NATO Sci. Ser. II Math. Phys. Chem., 170, Kluwer Acad. Publ., Dordrecht, 2004.
18. G.R. Grimmett and D.R. Stirzaker, *Probability and random processes*, 3rd ed., Oxford University Press, New York, 2001.
19. R. Harmelin, *A univalency criterion*, Complex Variables Theory Appl. **10** (1988), no. 4, 327–331.
20. J. Heinonen, *Lectures on analysis on metric spaces*, Springer-Verlag, New York, 2001.
21. T. Iwaniec, L. V. Kovalev and J. Onninen, *On injectivity of quasiregular mappings*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1783–1791.
22. T. Iwaniec, L. V. Kovalev and J. Onninen, *Dynamics of quasiconformal fields*, J. Dynam. Differential Equations **23** (2011), no. 1, 185–212.
23. J. Jääskeläinen, *On reduced Beltrami equations and linear families of quasiregular mappings*, preprint, arXiv:1009.2614.
24. L.V. Kovalev, *Quasiconformal geometry of monotone mappings*, J. London Math. Soc. **75** (2007), no. 2, 391–408.
25. L.V. Kovalev and J. Onninen, *Variation of quasiconformal mappings on lines*, Studia Math. **195** (2009), no. 3, 257–274.
26. O. Lehto, *Univalent functions and Teichmüller spaces*, Springer-Verlag, New York, 1987.
27. D. Meyer, *Bounded turning circles are weak-quasicircles*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1751–1761.
28. I. Prause, *A remark on quasiconformal dimension distortion on the line*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 341–352.
29. E. Reich and J. Chen, *Extensions with bounded $\bar{\partial}$-derivative*, Ann. Acad Sci. Fenn. Ser. A. I Math. **16** (1991), 377–389.
30. S. Smirnov, *Dimension of quasicircles*, Acta Math. **205** (2010), no. 1, 189–197.
31. E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1971.
32. P. Tukia, *Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane*, Ann. Acad. Sci. Fenn. Ser. A I Math. **86** (1981), 89–94.
33. P. Tukia, *Hausdorff dimension and quasisymmetric mappings*, Math. Scand. **65** (1989), no. 1, 152–160.
34. P. Tukia and J. Väisälä, *Extension of embeddings close to isometries or similarities*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 153–175.
35. J. Väisälä, *Bi-Lipschitz and quasisymmetric extension properties*, Ann. Acad. Sci. Fenn. Ser. A I Math. **11** (1986), no. 2, 239–274.

36. A. Zygmund, *Trigonometric series*, 3rd ed. Cambridge University Press, Cambridge, 2002.

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