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DIFFEOMORPHIC APPROXIMATION OF SOBOLEV HOMEOMORPHISMS

TADEUSZ IWANIEC, LEONID V. KOVALEV, AND JANI ONNINEN

Abstract. Every homeomorphism $h\colon \mathbb{X} \to \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$, $1 < p < \infty$, can be approximated in the Sobolev norm by $\mathscr{C}^\infty$-smooth diffeomorphisms.

1. Introduction

By the very definition, the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{R})$, $1 \leqslant p < \infty$, in a domain $\mathbb{X} \subset \mathbb{R}^n$, is the completion of $\mathscr{C}^\infty$-smooth real functions having finite Sobolev norm

$$\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathscr{L}^p(\mathbb{X})} + \|\nabla u\|_{\mathscr{L}^p(\mathbb{X})} < \infty.$$

The question of smooth approximation becomes more intricate for Sobolev mappings, whose target is not a linear space, say a smooth manifold [11, 19, 20, 21] or even for mappings between open subsets $\mathbb{X},\mathbb{Y}$ of the Euclidean space $\mathbb{R}^n$. If a given homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ it is not obvious at all as to whether one can preserve injectivity property of the $\mathscr{C}^\infty$-smooth approximating mappings. It is rather surprising that this question remained unanswered after the global invertibility of Sobolev mappings became an issue in nonlinear elasticity [4, 17, 31, 35]. It was formulated and promoted by John M. Ball in the following form.

Question. [6, 7] If $h \in \mathscr{W}^{1,p}(\mathbb{X},\mathbb{R}^n)$ is invertible, can h be approximated in $\mathscr{W}^{1,p}$ by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans and points out its relevance to the regularity of minimizers of neohookean energy functionals [5, 9, 14, 16, 34]. Partial results toward the Ball-Evans problem were obtained in [30] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [10] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [6, 33] illustrate the difficulty of preserving invertibility in the approximation process. In [24] we provided an affirmative answer to the Ball-Evans question in the planar case when $p = 2$. In the present

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paper we extend the result of [24] to all Sobolev classes $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ with $1<p<\infty$. The case $p=1$ still remains open.

Let $\mathbb{X}$ be a nonempty open set in $\mathbb{R}^2$. We study complex-valued functions $h=u+iv\colon \mathbb{X}\to\mathbb{C}\simeq\mathbb{R}^2$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{C})$, $1<p<\infty$. Their real and imaginary part have well defined gradient in $\mathscr{L}^p(\mathbb{X},\mathbb{R}^2)$

$$\nabla u\colon \mathbb{X}\to\mathbb{R}^2 \quad\text{and}\quad \nabla v\colon \mathbb{X}\to\mathbb{R}^2.$$

Then we introduce the gradient mapping of h, by setting

$$\nabla h = (\nabla u,\nabla v)\colon \mathbb{X}\to\mathbb{R}^2\times\mathbb{R}^2. \tag{1.1}$$

The $\mathscr{L}^p$-norm of the gradient mapping and the p-energy of h are defined by

$$\|\nabla h\|_{\mathscr{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} \left(|\nabla u|^p+|\nabla v|^p\right)\right]^{\frac{1}{p}}, \quad \mathsf{E}_{\mathbb{X}}[h]=\mathsf{E}^p_{\mathbb{X}}[h]=\|\nabla h\|^p_{\mathscr{L}^p(\mathbb{X})}. \tag{1.2}$$

The reader may wish to notice that this norm is slightly different from what can be found in other texts in which the authors use the differential matrix of h instead of the gradient mapping, so

$$\|Dh\|_{\mathscr{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} \left(|\nabla u|^2+|\nabla v|^2\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}. \tag{1.3}$$

Thus our approach involves *coordinate-wise* p-harmonic mappings, which we still call p-harmonic for the sake of brevity. We shall take an advantage of the gradient mapping on numerous occasions, by exploring the associated *uncoupled* system of real p-harmonic equations for mappings with smallest p-energy. Our theorem reads as follows.

Theorem 1.1. *Let $h\colon \mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X},\mathbb{Y})$, $1<p<\infty$, defined for open sets $\mathbb{X},\mathbb{Y}\subset\mathbb{R}^2$. Then there exist $\mathscr{C}^\infty$-diffeomorphisms $h_\ell\colon \mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$, $\ell=1,2,\dots$ such that*

(i) $h_\ell-h\in\mathscr{W}^{1,p}_\circ(\mathbb{X},\mathbb{R}^2)$, $\ell=1,2,\dots$

(ii) $\lim\limits_{\ell\to\infty}(h_\ell-h)=0$, *uniformly on* $\mathbb{X}$

(iii) $\lim\limits_{\ell\to\infty}\|\nabla h_\ell-\nabla h\|_{\mathscr{L}^p(\mathbb{X})}=0$

(iv) $\|\nabla h_\ell\|_{\mathscr{L}^p(\mathbb{X})}\leqslant\|\nabla h\|_{\mathscr{L}^p(\mathbb{X})}$, *for* $\ell=1,2,\dots$

(v) If h is a $\mathscr{C}^\infty$-diffeomorphism outside of a compact subset of $\mathbb{X}$, then there is a compact subset of $\mathbb{X}$ outside which $h_\ell\equiv h$, for all $\ell=1,2,\dots$

A straightforward triangulation argument yields the following corollary.

Corollary 1.2. *Let $h\colon \mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X},\mathbb{Y})$, $1<p<\infty$, defined for open sets $\mathbb{X},\mathbb{Y}\subset\mathbb{R}^2$. Then there exist piecewise affine homeomorphisms $h_\ell\colon \mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$, $\ell=1,2,\dots$ such that*

(i) $h_\ell-h\in\mathscr{W}^{1,p}_\circ(\mathbb{X},\mathbb{R}^2)$, $\ell=1,2,\dots$

(ii) $\lim_{\ell\to\infty}(h_\ell - h) = 0$, *uniformly on* $\mathbb{X}$

(iii) $\lim_{\ell\to\infty}\|\nabla h_\ell - \nabla h\|_{\mathscr{L}^p(\mathbb{X})} = 0$.

(iv) If h *is affine outside of a compact subset of* $\mathbb{X}$*, then there is a compact subset of* $\mathbb{X}$ *outside which* $h_\ell \equiv h$*, for all* $\ell = 1, 2, \dots$

We conclude this introduction with a sketch of the proof. The construction of an approximating diffeomorphism involves five consecutive modifications of h. Steps 1, 2, and 4 are p-harmonic replacements based on the Alessandrini-Sigalotti extension [3] of the Radó-Kneser-Choquet Theorem. The other steps involve an explicit smoothing procedure along crosscuts. For this, we adopted some lines of arguments used in J. Munkres' work [32].

2. p-harmonic mappings and preliminaries

Let Ω be a bounded domain in the complex plain $\mathbb{C} \simeq \mathbb{R}^2$. A function $u\colon \Omega \to \mathbb{R}$ in the Sobolev class $\mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, is called p-harmonic if

$$\operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \tag{2.1}$$

meaning that

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla \varphi\rangle = 0 \qquad \text{for every } \varphi \in \mathscr{C}^\infty_\circ(\Omega). \tag{2.2}$$

The first observation is that the gradient map $f = \nabla u\colon \Omega \to \mathbb{R}^2$ is K-quasiregular with $1 \leqslant K \leqslant \max\{p-1, 1/(p-1)\}$, see [12]. Consequently $u \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with some $0 < \alpha = \alpha(p) \leqslant 1$. In fact [25] the foremost regularity of a p-harmonic function ($p \neq 2$) is $\mathscr{C}^{k,\alpha}_{\mathrm{loc}}(\Omega)$, where the integer $k \geqslant 1$ and the Hölder exponent $\alpha \in (0,1]$ are determined by the equation

$$k + \alpha = \frac{7p - 6 + \sqrt{p^2 + 12p - 12}}{6p - 6} > 1 + \frac{1}{3}.$$

Thus, regardless of the exponent p, we have $u \in \mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = 1/3$. Clearly, by elliptic regularity theory, outside the singular set

$$\mathcal{S} = \big\{z \in \Omega\colon \nabla u(z) = 0\big\},$$

we have $u \in \mathscr{C}^\infty(\Omega \setminus \mathcal{S})$. The singular set, being the set of zeros of a quasiregular mapping, consists of isolated points; unless $u \equiv \text{const}$. Pertaining to regularity up to the boundary, we consider a domain Ω whose boundary near a point $z_\circ \in \partial\Omega$ is a $\mathscr{C}^\infty$-smooth arc, say $\Gamma \subset \partial\Omega$. Precisely, we assume that there exist a disk $D = D(z_\circ, \epsilon)$ and a $\mathscr{C}^\infty$-smooth diffeomorphism $\varphi\colon D \xrightarrow{\text{onto}} \mathbb{C}$ such that

$$\begin{aligned}
\varphi(D \cap \Omega) &= \mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}\\
\varphi(\Gamma) &= \mathbb{R} = \{z\colon \operatorname{Im} z = 0\}\\
\varphi(D \setminus \overline{\Omega}) &= \mathbb{C}_- = \{z\colon \operatorname{Im} z < 0\}.
\end{aligned}$$

Proposition 2.1 (Boundary Regularity)**.** *Suppose $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ is p-harmonic in Ω and $\mathscr{C}^\infty$-smooth when restricted to Γ. Then u is $\mathscr{C}^{1,\alpha}$-regular up to Γ, meaning that u extends to D as a $\mathscr{C}^{1,\alpha}(D)$-regular function, where α depends only on p.*

2.1. **The Dirichlet problem.** There are two formulations of the Dirichlet boundary value problem for p-harmonic equation; both are essential for our investigation. We begin with the variational formulation.

Lemma 2.2. *Let $u_\circ \in \mathscr{W}^{1,p}(\Omega)$ be a given Dirichlet data. There exists precisely one function $u \in u_\circ + \mathscr{W}_\circ^{1,p}(\Omega)$ which minimizes the p-harmonic energy:*

$$\mathcal{E}_p[u] = \inf \left\{ \int_\Omega |\nabla w|^p \colon w \in u_\circ + \mathscr{W}_\circ^{1,p}(\Omega) \right\}.$$

The solution u is certainly a p-harmonic function, so $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$-regular. However, more efficient to us will be the following classical formulation of the Dirichlet problem.

Problem 2.3. Given $u_\circ \in \mathscr{C}(\partial\Omega)$ find a p-harmonic function u in Ω which extends continuously to $\overline{\Omega}$ such that $u_{|\partial\Omega} = u_\circ$.

It is not difficult to see that such solution (if exists) is unique. However, the existence poses rather delicate conditions on $\partial\Omega$ and the data $u_\circ \in \mathscr{C}(\overline{\Omega})$. We shall confine ourselves to Jordan domains $\Omega \subset \mathbb{C}$ and the Dirichlet data $u_\circ \in \mathscr{C}(\overline{\Omega})$ of finite p-harmonic energy. In this case both formulations are valid and lead to the same solution. Indeed, the variational solution is continuous up to the boundary because each boundary point of a planar Jordan domain is a regular point for the p-Laplace operator Δ_p [18, p.418]. See [22, 6.16] for the discussion of boundary regularity and relevant capacities and [27, Lemma 2] for a capacity estimate that applies to simply connected domains.

Proposition 2.4 (Existence)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain and $u_\circ \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$. There exists, unique, p-harmonic function $u \in \mathscr{W}^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ such that $u_{|\partial\Omega} = u_{\circ|\partial\Omega}$.*

2.2. **Radó-Kneser-Choquet Theorem.** Let $h = u + iv$ be a complex harmonic mapping in a Jordan domain $\mathbb{U}$ that is continuous on $\overline{\mathbb{U}}$. Assume that the boundary mapping $h \colon \partial\mathbb{U} \xrightarrow{\text{onto}} \Gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve. Then h is a $\mathscr{C}^\infty$-smooth diffeomorphism of $\mathbb{U}$ onto the bounded component of $\mathbb{C} \setminus \Gamma$. Thus, in particular, the Jacobian determinant $J(z,h) = |h_z|^2 - |h_{\bar z}|^2$ is strictly positive in $\mathbb{U}$, see [15, p.20]. Suppose, in addition, that $\partial\mathbb{U}$ contains a $\mathscr{C}^\infty$-smooth arc $\gamma \subset \partial\mathbb{U}$, and h takes γ onto a $\mathscr{C}^\infty$-smooth subarc in Γ. Then h is $\mathscr{C}^\infty$-smooth up to γ and its Jacobian determinant is positive on γ as well, see [15, p.116]. Numerous presentations of the proof of Radó-Kneser-Choquet Theorem can be found, [15]. The idea that goes back to Kneser [26] and Choquet [13] is to look at the structure of the level curves of the coordinate functions $u = \operatorname{Re} h$, $v = \operatorname{Im} h$ and their linear combinations. These ideas have been applied to more general linear and nonlinear elliptic systems of PDEs in the complex plane [8], see also [1, 2, 28, 29] for related problems concerning critical points. In the present paper we shall explore a result due to G. Alessandrini and M. Sigalotti [3] for a nonlinear system that consists of two p-harmonic equations

$$\begin{cases} \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \\ \operatorname{div}|\nabla v|^{p-2}\nabla v = 0 \end{cases}, \qquad 1 < p < \infty, \quad h = u + iv.$$

Call it *uncoupled p-harmonic system.* The novelty and key element in [3] is the associated single linear elliptic PDE of divergence type (with variable coefficients) for a linear combination of u and v. Such combination represents a real part of a quasiregular mapping and, therefore, admits only isolated critical points. We shall not go into their arguments in detail, but instead extract the following p-harmonic analogue of the Radó-Kneser-Choquet Theorem.

Theorem 2.5 (G. Alessandrini and M. Sigalotti)**.** *Let $\mathbb{U}$ be a bounded Jordan domain and $h = u + iv \colon \overline{\mathbb{U}} \to \mathbb{C}$ be a continuous mapping whose coordinate functions $u, v \in \mathscr{W}^{1,p}(\mathbb{U})$, $1 < p < \infty$, are p-harmonic. Suppose that $h \colon \partial\mathbb{U} \xrightarrow{\text{onto}} \gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve γ. Then*

(i) h is a $\mathscr{C}^\infty$-diffeomorphism from $\mathbb{U}$ onto the bounded component of $\mathbb{C}\setminus\gamma$. In particular,

$$J(z, h) = |h_z|^2 - |h_{\bar z}|^2 > 0 \quad \text{in } \mathbb{U}.$$

(ii) If, in addition, $\partial\mathbb{U}$ contains a $\mathscr{C}^\infty$-smooth arc $\Gamma \subset \partial\mathbb{U}$ and $h(\Gamma)$ is a $\mathscr{C}^\infty$-smooth subarc in γ, then h is $\mathscr{C}^{1,\alpha}$-regular up to Γ, for some $0 < \alpha = \alpha(p) < 1$ (actually $\mathscr{C}^\infty$). Moreover $J(z, h) > 0$ on Γ as well.

This theorem is a straightforward corollary of Theorem 5.1 in [3]. However, three remarks are in order.

(1) In their Theorem 5.1 the authors of [3] assume that $\mathbb{U}$ satisfies an exterior cone condition. This is needed only insofar as to ensure the existence of a continuous extension of a given homeomorphism $\Phi \colon \partial\mathbb{U} \to \gamma$ into $\mathbb{U}$ whose coordinate functions are p-harmonic in $\mathbb{U}$. Obviously, such an extension is unique, though the p-harmonic energy need not be finite. Once we have such a mapping the exterior cone condition on $\mathbb{U}$ for the conclusion of Theorem 5.1 is redundant, see Remark 3.2 in [3]. This is exactly the case we are dealing with in Theorem 2.5.

(2) In regard to the statement (ii) we point out that in Theorem 5.1 of [3] the authors work with the mappings that are smooth up to the entire boundary of $\mathbb{U}$. Nonetheless their proof that $J(z, h) > 0$ on $\partial\mathbb{U}$ is local, so applies without any change to our case (ii).

(3) Since $J(z,h) > 0$ in $\mathbb{U}$ up to the arc $\Gamma \subset \partial\mathbb{U}$ the coordinate functions of h have nonvanishing gradient. This means that p-harmonic equation is uniformly elliptic up to Γ. Consequently, h is $\mathscr{C}^\infty$-smooth on $\mathbb{U}$ up to Γ.

2.3. **The p-harmonic replacement.** Let Ω be a bounded domain in $\mathbb{R}^2 \simeq \mathbb{C}$. We consider a class $\mathcal{A}(\Omega) = \mathcal{A}^p(\Omega)$, $1 < p < \infty$, of uniformly continuous functions $h = u + iv \colon \Omega \to \mathbb{C}$ having finite p-harmonic energy and furnish it with the norm

$$\|h\|_{\mathcal{A}^p(\Omega)} = \|h\|_{\mathscr{C}(\Omega)} + \|\nabla h\|_{\mathscr{L}^p(\Omega)}.$$

The closure of $\mathscr{C}_\circ^\infty(\Omega)$ in $\mathcal{A}^p(\Omega)$ will be denoted by $\mathcal{A}_\circ^p(\Omega)$.

Proposition 2.6. *Let $\mathbb{U} \Subset \Omega$ be a Jordan subdomain of Ω. There exists a unique operator*

$$\mathbf{R}_\mathbb{U} \colon \mathcal{A}^p(\Omega) \to \mathcal{A}^p(\Omega)$$

(nonlinear if $p \neq 2$) such that for every $h \in \mathcal{A}^p(\Omega)$

$$\begin{aligned} \mathbf{R}_\mathbb{U} h &= h \qquad \textit{in } \Omega \setminus \mathbb{U} \\ \mathbf{R}_\mathbb{U} &\in h + \mathscr{W}_\circ^{1,p}(\mathbb{U}) \\ \Delta_p \mathbf{R}_\mathbb{U} h &= 0 \qquad \textit{in } \mathbb{U} \end{aligned} \tag{2.3}$$

$$\mathsf{E}_\Omega[\mathbf{R}_\mathbb{U} h] \leqslant \mathsf{E}_\Omega[h] \tag{2.4}$$

Equality occurs in (2.4) *if and only if h is p-harmonic in $\mathbb{U}$.*

Proof. For $h = u + iv$ we define

$$\mathbf{R}_\mathbb{U} h = \mathbf{R}_\mathbb{U} u + i\, \mathbf{R}_\mathbb{U} v.$$

It is therefore enough to construct the replacement for real-valued functions. For $u \in \mathcal{A}^p(\Omega)$ real, we define

$$\mathbf{R}_\mathbb{U} u = \begin{cases} u & \text{in } \Omega \setminus \mathbb{U} \\ \tilde{u} & \text{in } \mathbb{U} \end{cases}$$

where $\tilde{u}$ is determined uniquely as a solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} = 0 & \text{in } \mathbb{U} \\ \tilde{u} \in u + \mathscr{W}_\circ^{1,p}(\mathbb{U}) \end{cases}$$

so conditions (2.3) are fulfilled. That $\mathbf{R}_\mathbb{U} u$ is continuous in Ω is guaranteed by Proposition 2.4. The solution $\tilde{u}$ is found as the minimizer of the p-harmonic energy in the class $u + \mathscr{W}_\circ^{1,p}(\mathbb{U})$, so we certainly have

$$\mathsf{E}_\Omega[\mathbf{R}_\mathbb{U} u] \leqslant \mathsf{E}_\Omega[u]$$

The same estimate holds for the imaginary part of h, so adding them up yields

$$\mathsf{E}_\Omega[\mathbf{R}_\mathbb{U} h] \leqslant \mathsf{E}_\Omega[h].$$ □

Remark 2.7. The reader may wish to know that the operator $\mathbf{R}_\mathbb{U} \colon \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is continuous, though we do not appeal to this fact.

2.4. **Smoothing along a crosscut.** Consider a bounded Jordan domain $\mathbb{U}$ and a $\mathscr{C}^\infty$-smooth crosscut $\Gamma \subset \mathbb{U}$ with two distinct end-points in $\partial\mathbb{U}$. By definition, this means that there is a $\mathscr{C}^\infty$-diffeomorphism $\varphi \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{R})$, and its distinct endpoints are given by

$$\lim_{x\to-\infty} \varphi(x) \in \partial\mathbb{U}$$
$$\lim_{x\to\infty} \varphi(x) \in \partial\mathbb{U}$$

Such Γ splits $\mathbb{U}$ into two Jordan subdomains

$$\mathbb{U}_+ = \varphi(\mathbb{C}_+), \quad \mathbb{C}_+ = \{z \colon \operatorname{Im} z > 0\}$$
$$\mathbb{U}_- = \varphi(\mathbb{C}_-), \quad \mathbb{C}_- = \{z \colon \operatorname{Im} z < 0\}.$$

Suppose we are given a homeomorphism $f \colon \overline{\mathbb{U}} \to \mathbb{C}$ such that each of two mappings

$$f \colon \mathbb{U}_+ \to \mathbb{R}^2 \quad \text{and} \quad f \colon \mathbb{U}_- \to \mathbb{R}^2$$

is $\mathscr{C}^\infty$-smooth up to Γ. Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leqslant m \quad \text{and} \quad \det Df(z) \geqslant \frac{1}{m}$$

on $\mathbb{U}_+$ and on $\mathbb{U}_-$. Thus $f \colon \mathbb{U} \to \mathbb{R}^2$ is in fact locally bi-Lipschitz.

Proposition 2.8. *Under the above conditions there is a constant $0 < M < \infty$ such that for every open set $\mathbb{V} \subset \mathbb{U}$ containing Γ one can find a homeomorphism $g \colon \overline{\mathbb{U}} \xrightarrow{\text{onto}} f(\overline{\mathbb{U}})$ which is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{U}$, with the following properties:*

$$g(z) = f(z), \text{ for } z \in (\overline{\mathbb{U}} \setminus \mathbb{V}) \cup \Gamma \tag{2.5}$$

$$|Dg(z)| \leqslant M \quad \text{and} \quad \det Dg(z) > \frac{1}{M} \text{ on } \mathbb{U}. \tag{2.6}$$

The key element of this smoothing device is that the constant M is independent of the neighborhood $\mathbb{V}$ of Γ, see Figure 1. The proof is given in [24] following the ideas of [32].

We shall recall similar smoothing device for cuts along Jordan curves. Let $\mathbb{U}$ be a simply connected domain with $\mathscr{C}^\infty$-regular cut along a Jordan curve $\Gamma \subset \mathbb{U}$. This means there is a diffeomorphism $\varphi \colon \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{S}^1)$, $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$. As before Γ splits $\mathbb{U}$ into

$$\mathbb{U}_+ = \varphi(\mathbb{D}_+), \quad \mathbb{D}_+ = \{z \colon |z| < 1\}$$
$$\mathbb{U}_- = \varphi(\mathbb{D}_-), \quad \mathbb{D}_- = \{z \colon |z| > 1\}.$$

Suppose we are given a homeomorphism $f \colon \mathbb{U} \to \mathbb{R}^2$ such that each of two mappings

$$f \colon \mathbb{U}_+ \to \mathbb{R}^2 \quad \text{and} \quad f \colon \mathbb{U}_- \to \mathbb{R}^2$$

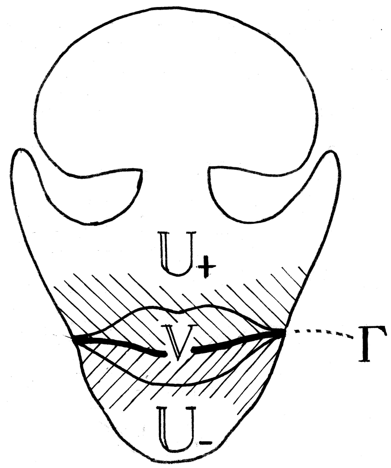

FIGURE 1. Jordan domain with a crosscut Γ and its neighborhood $\mathbb{V}$.

is $\mathscr{C}^\infty$-smooth up to Γ. Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leqslant m \quad \text{ and } \quad \det Df(z) \geqslant \frac{1}{m}$$

on $\mathbb{U}_+$ and $\mathbb{U}_-$.

Proposition 2.9. *Under the above conditions there is a constant* $0 < M < \infty$ *such that for every open set* $\mathbb{V} \subset \mathbb{U}$ *containing* Γ *one can find a* $\mathscr{C}^\infty$-*diffeomorphism* $g \colon \mathbb{U} \xrightarrow{\text{onto}} f(\mathbb{U})$ *with the following properties*

$$g(z) = f(z), \ \textit{for } z \in (\mathbb{U} \setminus \mathbb{V}) \cup \Gamma \tag{2.7}$$

$$|Dg(z)| \leqslant M \quad \textit{and} \quad \det Dg(z) > \frac{1}{M} \ \textit{on } \mathbb{U}. \tag{2.8}$$

Having disposed of the above preliminaries we shall now proceed to the construction of the approximating sequence of diffeomorphisms.

3. The proof

3.1. **Scheme of the proof.** Let us begin with a convention. We will often suppress the explicit dependence on the Sobolev exponent $1 < p < \infty$ in the notation, whenever it becomes selfexplanatory. For every $\epsilon > 0$ we shall construct a $\mathscr{C}^\infty$-diffeomorphism $\hbar \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that

(A) $\hbar - h \in \mathcal{A}_\circ(\mathbb{X})$

(B) $\|\hbar - h\|_{\mathscr{C}(\mathbb{X})} \leqslant \epsilon$

(C) $\|\nabla \hbar - \nabla h\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \epsilon$

(D) $\mathsf{E}_{\mathbb{X}}[\hbar] \leqslant \mathsf{E}_{\mathbb{X}}[h]$

(E) If h is a $\mathscr{C}^\infty$-diffeomorphism outside of a compact subset of $\mathbb{X}$, then there exist a compact subset of $\mathbb{X}$ outside of which we have $\hbar \equiv h$, for all $\epsilon > 0$.

We may and do assume that h is not a $\mathscr{C}^\infty$-diffeomorphism, since otherwise $\hbar = h$ satisfies the desired properties. Let $x_\circ \in \mathbb{X}$ be a point such that h fails to be $\mathscr{C}^\infty$-diffeomorphism in any neighborhood of $x_\circ$.

We shall consider dyadic squares in $\mathbb{Y}$ with respect to a selected rectangular coordinate system in $\mathbb{R}^2$. By choosing the origin of the system we ensure that $h(x_\circ)$ does not lie on the boundary of any dyadic square.

Let us fix $\epsilon > 0$. The construction of $\hbar$ proceeds in 5 steps, each of which gives a homeomorphism $\hbar_k \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $k = 0, 1, \dots, 5$, in the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ such that $\hbar_0 = h$, $\hbar_k \in \hbar_{k-1} + \mathcal{A}_\circ(\mathbb{X})$, $k = 1, \dots, 5$ and $\hbar_5 = \hbar$ is the desired diffeomorphism. For each $k = 1, 2, \dots, 5$ we will secure conditions analogous to (A)-(E). Namely,

(A_k) $\hbar_k - \hbar_{k-1} \in \mathcal{A}_\circ(\mathbb{X})$

(B_k) $\|\hbar_k - \hbar_{k-1}\|_{\mathscr{C}(\mathbb{X})} \leqslant \epsilon/5$

(C_k) $\|\nabla \hbar_k - \nabla \hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \epsilon/5$

(D_k) $\|\nabla \hbar_1\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\nabla \hbar_0\|_{\mathscr{L}^p(\mathbb{X})} - 2\delta$, for some $\delta > 0$;
$\|\nabla \hbar_k\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\nabla \hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})}$, for $k = 2, 4$;
$\|\nabla \hbar_k\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\nabla \hbar_{k-1}\|_{\mathscr{L}^p(\mathbb{X})} + \delta$, for $k = 3, 5$

(E_k) If h_{k-1} is a $\mathscr{C}^\infty$-diffeomorphism outside of a compact subset of $\mathbb{X}$, then there exists a compact subset in $\mathbb{X}$ outside which we have $\hbar_k \equiv \hbar_{k-1}$ for all $\epsilon > 0$.

3.2. **Partition of $\mathbb{X}$ into cells.** Let us distinguish one particular Whitney type partition of $\mathbb{Y}$ and keep it fixed for the rest of our arguments.

$$\mathbb{Y} = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{Y}_\nu},$$

where $\mathbb{Y}_\nu$ are mutually disjoint open dyadic squares such that

$$\operatorname{diam} \mathbb{Y}_\nu \leqslant \operatorname{dist}(\mathbb{Y}_\nu, \partial \mathbb{Y}) \leqslant 3 \operatorname{diam} \mathbb{Y}_\nu \quad \text{for } \nu = 1, 2, \dots$$

unless $\mathbb{Y} = \mathbb{R}^2$, in which case $\mathbb{Y}_\nu$ are unit squares. Thus the cover of $\mathbb{Y}$ by $\overline{\mathbb{Y}_\nu}$ is locally finite. The preimages

$$\mathbb{X}_\nu = h^{-1}(\mathbb{Y}_\nu), \qquad \nu = 1, 2, \dots$$

are Jordan domains which we call *cells* in $\mathbb{X}$. In the forthcoming Step 1 we shall need to further divide each cell into a finite number of *daughter cells* in $\mathbb{X}$. Note that all but finite number of cells $\mathbb{X}_\nu$, $\nu = 1, 2, ...$ lie outside a given compact subset of $\mathbb{X}$.

Step 1

To avoid undue indexing in the forthcoming division of cells, we shall argue in two substeps.

Step 1a. Examine one of the cells in $\mathbb{X}$, say $\mathfrak{X} = \mathbb{X}_\nu$, for some fixed $\nu = 1, 2, \dots$. Call it a *parent cell.* Thus $h(\mathfrak{X}) = \Upsilon$ is the corresponding Whitney square $\Upsilon = \mathbb{Y}_\nu \subset \mathbb{Y}$. To every $n = 1, 2, \dots$, there corresponds a partition of Υ into 4^n-dyadic congruent squares Υ_i, $i = 1, \dots, 4^n$

$$\overline{\Upsilon} = \overline{\Upsilon_1} \cup \dots \cup \overline{\Upsilon_{4^n}}.$$

This gives rise to a division of $\mathfrak{X}$ into daughter cells $\mathfrak{X}_i = h^{-1}(\Upsilon_i)$

$$\overline{\mathfrak{X}} = \overline{\mathfrak{X}_1} \cup \overline{\mathfrak{X}_2} \cup \dots \cup \overline{\mathfrak{X}_{4^n}}.$$

We look at the homeomorphisms

$$h \colon \overline{\mathfrak{X}_i} \xrightarrow{\text{onto}} \overline{\Upsilon_i}, \qquad i = 1, 2, \dots 4^n$$

By virtue of Proposition 2.6 we may replace them with p-harmonic homeomorphisms

$$\widetilde{h}_i = \mathbf{R}_{\mathfrak{X}_i} h \colon \overline{\mathfrak{X}_i} \xrightarrow{\text{onto}} \overline{\Upsilon_i}, \qquad i = 1, 2, \dots, 4^n$$

which coincide with h on $\partial \mathfrak{X}_i$. This procedure may not be necessary if $h \colon \mathfrak{X}_i \to \Upsilon_i$ is already a $\mathscr{C}^\infty$-diffeomorphism. In such cases we always use the *trivial replacement* $\widetilde{h}_i = h$. After all such replacements are made, we arrive at a homeomorphism

$$\widetilde{h} \colon \overline{\mathfrak{X}} \xrightarrow{\text{onto}} \overline{\Upsilon}$$

which is a $\mathscr{C}^\infty$-diffeomorphism in each cell $\mathfrak{X}_i$ and coincides with h on $\partial \mathfrak{X}_i$. Obviously,

$$\widetilde{h} = h + \sum_{i=1}^{4^n} [\widetilde{h}_i - h]_\circ \in h + \mathcal{A}_\circ(\mathfrak{X})$$

where $[\widetilde{h}_i - h]_\circ$ stands for zero extension of $\widetilde{h}_i - h$ outside $\mathfrak{X}_i$ and, therefore, belongs to $\mathcal{A}_\circ(\mathfrak{X}_i)$. Furthermore, by principle of minimal p-harmonic energy, we have

$$\mathsf{E}_{\mathfrak{X}}[\widetilde{h}] = \sum_{i=1}^{4^n} \mathsf{E}_{\mathfrak{X}_i}[\widetilde{h}_i] \leqslant \sum_{i=1}^{4^n} \mathsf{E}_{\mathfrak{X}_i}[h] = \mathsf{E}_{\mathfrak{X}}[h].$$

The eventual aim is to fix the number of daughter cells in $\mathfrak{X}$. For this we vary n and look closely at the resulting homeomorphisms, denoted by f_n. This sequence of mappings is bounded in $\mathcal{A}(\mathfrak{X})$. It actually converges to h uniformly on $\overline{\mathfrak{X}}$. Indeed, given any point $x \in \overline{\mathfrak{X}}$, say $x \in \overline{\mathfrak{X}_i}$, for some $i = 1, 2, \dots, 4^n$, we have

$$|f_n(x) - h(x)| = |\widetilde{h}_i(x) - h(x)| \leqslant \operatorname{diam} \Upsilon_i = 2^{-n} \operatorname{diam} \Upsilon.$$

Thus

$$\lim_{n \to \infty} f_n = h, \quad \text{ uniformly in } \overline{\mathfrak{X}}.$$

On the other hand the mappings f_n are bounded in the Sobolev space $\mathscr{W}^{1,p}(\mathfrak{X})$, so converge to h weakly in $\mathscr{W}^{1,p}(\mathfrak{X})$. The key observation now is that

$$\|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})} \leqslant \liminf_{n \to \infty} \|\nabla f_n\|_{\mathscr{L}^p(\mathfrak{X})} \leqslant \|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})}$$

because of convexity of the energy functional. This gives

$$\lim_{n\to\infty} \|\nabla f_n\|_{\mathscr{L}^p(\mathfrak{X})} = \|\nabla h\|_{\mathscr{L}^p(\mathfrak{X})}$$

Then, the usual application of Clarkson's inequalities in $\mathscr{L}^p$-spaces, $1 < p < \infty$, yields

$$\lim_{n\to\infty} \|\nabla f_n - \nabla h\|_{\mathscr{L}^p(\mathfrak{X})} = 0$$

meaning that $f_n - h \to 0$ in the norm topology of $\mathcal{A}(\mathfrak{X})$. We can now determine the number $n = n_\nu = n(\mathfrak{X})$, simply requiring the division of $\mathfrak{X}$ be fine enough to satisfy two conditions.

$$\begin{cases} \operatorname{diam} \Upsilon_i = 2^{-n} \operatorname{diam} \Upsilon \leqslant \epsilon/5, \quad i = 1, \dots, 4^n \\ \|\nabla f_n - \nabla h\|_{\mathscr{L}^p(\mathfrak{X})} \leqslant \frac{\epsilon}{5\cdot 2^\nu} \end{cases} \tag{3.1}$$

where we recall that $\mathfrak{X}$ stands for $\mathbb{X}_\nu$.

Step 1b. Now, having $n = n_\nu$ fixed for each cell $\mathfrak{X}_\nu$, we construct our first approximating mapping

$$\hbar_1 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

by setting

$$\hbar_1 := h + \sum_{\nu=1}^{\infty} [f_{n_\nu} - h]_\circ \in h + \mathcal{A}_\circ(\mathbb{X})$$

where, as always, $[f_{n_\nu} - h]_\circ$ stands for the zero extension of $f_{n_\nu} - h$ outside $\mathbb{X}_\nu$. This mapping is a $\mathscr{C}^\infty$-diffeomorphism in every daughter cell. Clearly, we have the condition

$$\hbar_1 - h \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_1}$$

Moreover, by the condition in (3.1) imposed on every n_ν,

$$\|\hbar_1 - h\|_{\mathscr{C}(\mathbb{X})} \leqslant \sup_{\nu=1,2,\dots} \{\operatorname{diam} \Upsilon_i \colon \Upsilon_i \subset \mathbb{Y}_\nu, i = 1, \dots, 4^{n_\nu}\} \leqslant \frac{\epsilon}{5} \tag{B_1}$$

and

$$\|\nabla \hbar_1 - \nabla h\|^p_{\mathscr{L}^p(\mathbb{X})} = \sum_{\nu=1}^{\infty} \|\nabla \hbar_1 - \nabla h\|^p_{\mathscr{L}^p(\mathbb{X}_\nu)} \leqslant \left(\frac{\epsilon}{5}\right)^p \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} < \left(\frac{\epsilon}{5}\right)^p. \tag{C_1}$$

Regarding condition (D_1), we observe that summing up the energies over all daughter cells $\mathfrak{X}_i \subset \mathbb{X}_\nu$, $i = 1, 2, \dots 4^{n_\nu}$ and $\nu = 1, 2, \dots$, gives the total energy of $\hbar_1$ not larger than that of h. Even more, since h fails to be a $\mathscr{C}^\infty$-diffeomorphism in at least one of these cells, the p-harmonic replacement takes place in this cell and, consequently, $\hbar_1$ has strictly smaller energy. Hence

$$\|\nabla \hbar_1\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\nabla h\|_{\mathscr{L}^p(\mathbb{X})} - 2\delta, \quad \text{for some } \delta > 0. \tag{D_1}$$

Regarding condition (E_1), we note that under the assumption therein we made only a finite number of nontrivial (p-harmonic) replacements. The same remark will apply to the subsequent steps and will not be mentioned again. The step 1 is complete.

Before proceeding to Step 2, let us put all daughter cells in $\mathbb{X}$ in a single sequence

$$\mathfrak{X}^1, \mathfrak{X}^2, \dots \subset \mathbb{X}.$$

Thus from now on the daughter cells from different parents are indistinguishable as far as the mapping $\hbar_1$ is concerned. The point is that $\hbar_1$ is a $\mathscr{C}^\infty$-diffeomorphism in every such cell, a property that will be pertinent to all new cells coming later either by splitting or merging the existing cells. Note that the images $\Upsilon^\alpha = h(\mathfrak{X}^\alpha)$, $\alpha = 1, 2, \dots$, form a partition of $\mathbb{Y}$ into dyadic squares

$$\mathbb{Y} = \bigcup_{\alpha=1}^{\infty} \overline{\Upsilon^\alpha}, \qquad \text{where} \qquad \operatorname{diam} \Upsilon^\alpha \leqslant \frac{\epsilon}{5}.$$

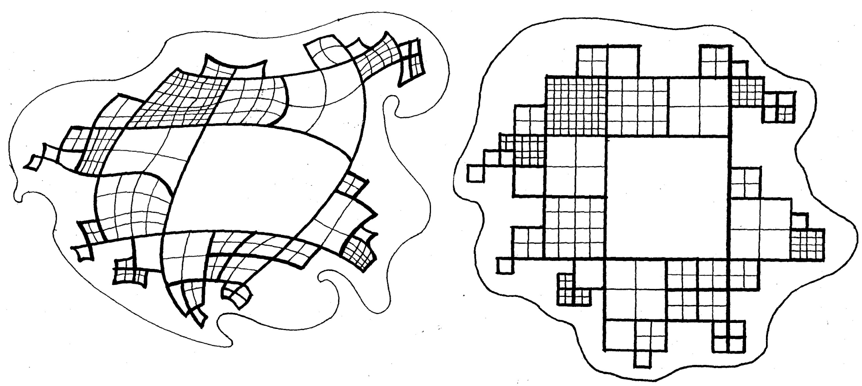

FIGURE 2. $\hbar_1$ is a $\mathscr{C}^\infty$-diffeomorphism in each cell $\mathfrak{X}^\alpha \subset \mathbb{X}$.

STEP 2

Step 2a. (Adjacent cells) Let $\mathcal{C}(\mathbb{Y}) \subset \mathbb{Y}$ be the collection of all corners of dyadic squares Υ^α, $\alpha = 1, 2, \dots$, and $\mathcal{V}(\mathbb{X}) \subset \mathbb{X}$ denote the set of their preimages under h, called *vertices of cells*. Whenever two closed cells $\overline{\mathfrak{X}^\alpha}$ and $\overline{\mathfrak{X}^\beta}$, $\alpha \neq \beta$, intersect, their common part is either a point in $\mathcal{V}(\mathbb{X})$ or an edge, that is, a closed Jordan arc with endpoints in $\mathcal{V}(\mathbb{X})$. In this latter case we say that $\mathfrak{X}^\alpha$ and $\mathfrak{X}^\beta$ are adjacent cells with common edge

$$\overline{C^{\alpha\beta}} = \overline{\mathfrak{X}^\alpha} \cap \overline{\mathfrak{X}^\beta}.$$

This is the closure of a Jordan open arc $C^{\alpha\beta} = \overline{C^{\alpha\beta}} \setminus \mathcal{V}(\mathbb{X})$. The mappings

$$\hbar_1 \colon \mathfrak{X}^\alpha \xrightarrow{\text{onto}} \Upsilon^\alpha \quad \text{and} \quad \hbar_1 \colon \mathfrak{X}^\beta \xrightarrow{\text{onto}} \Upsilon^\beta$$

are $\mathscr{C}^\infty$-diffeomorphisms but they do not necessarily match smoothly along the edges. We shall now produce a new cell $\mathfrak{X}^{\alpha\beta}$, a daughter of the adjacent cells $\mathfrak{X}^\alpha$ and $\mathfrak{X}^\beta$, such that

$$C^{\alpha\beta} \subset \mathfrak{X}^{\alpha\beta} \subset \mathfrak{X}^\alpha \cup C^{\alpha\beta} \cup \mathfrak{X}^\beta.$$

To construct $\mathfrak{X}^{\alpha\beta}$ we look at the adjacent dyadic squares $\overline{\Upsilon^\alpha}$ and $\overline{\Upsilon^\beta}$ in $\mathbb{Y}$. The intersection $\overline{\Upsilon^\alpha} \cap \overline{\Upsilon^\beta} = h(\overline{C^{\alpha\beta}})$ is a closed interval. Let R be a number greater than the length of $h(C^{\alpha\beta})$ to be chosen sufficiently large later on. There exist exactly two open disks of radius R for which $h(C^{\alpha\beta})$ is a chord. Their intersection, denoted by $\mathcal{L}^{\alpha\beta}$, is a symmetric doubly convex lens of curvature R^{-1}. Thus $\mathcal{L}^{\alpha\beta}$ is enclosed between two open circular arcs $\gamma^{\alpha\beta} = \Upsilon^\alpha \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\alpha$ and $\gamma^{\beta\alpha} = \Upsilon^\beta \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\beta$. Note that $\mathcal{L}^{\alpha\beta} = \mathcal{L}^{\beta\alpha}$, but $\gamma^{\alpha\beta} \neq \gamma^{\beta\alpha}$. We call

$$\mathfrak{X}^{\alpha\beta} = \hbar_1^{-1}(\mathcal{L}^{\alpha\beta}), \quad \text{a daughter of the adjacent cells } \mathfrak{X}^\alpha \text{ and } \mathfrak{X}^\beta. \tag{3.2}$$

As the curvature of the lens $\mathcal{L}^{\alpha\beta}$ approaches zero, the area of $\mathfrak{X}^{\alpha\beta}$ tends to 0. This allows us to choose R so that

$$\|\nabla \hbar_1\|_{\mathscr{L}^p(\mathfrak{X}^{\alpha\beta})} \leqslant \frac{\epsilon}{5 \cdot 2^{\alpha+\beta}}. \tag{3.3}$$

The lenses $\mathcal{L}^{\alpha\beta}$ are disjoint because the opening angle of each lens (the angle between arcs at their common endpoints) is at most $\pi/3$ and their long axes are either parallel or orthogonal, see Figure 3. Therefore, the cells $\mathfrak{X}^{\alpha\beta} = \hbar_1^{-1}(\mathcal{L}^{\alpha\beta})$ are also disjoint. However, their closures may have a common point that lies in $\mathcal{V}(\mathbb{X})$. The boundary of $\mathfrak{X}^{\alpha\beta}$ consists of two open arcs

$$\Gamma^{\alpha\beta} = \mathfrak{X}^\alpha \cap \partial\mathfrak{X}^{\alpha\beta} \quad \text{and} \quad \Gamma^{\beta\alpha} = \mathfrak{X}^\beta \cap \partial\mathfrak{X}^{\alpha\beta}$$

plus their endpoints. These open arcs are $\mathscr{C}^\infty$-smooth because they come as images of the circular arcs enclosing the lens $\mathcal{L}^{\alpha\beta}$ under a $\mathscr{C}^\infty$-diffeomorphism.

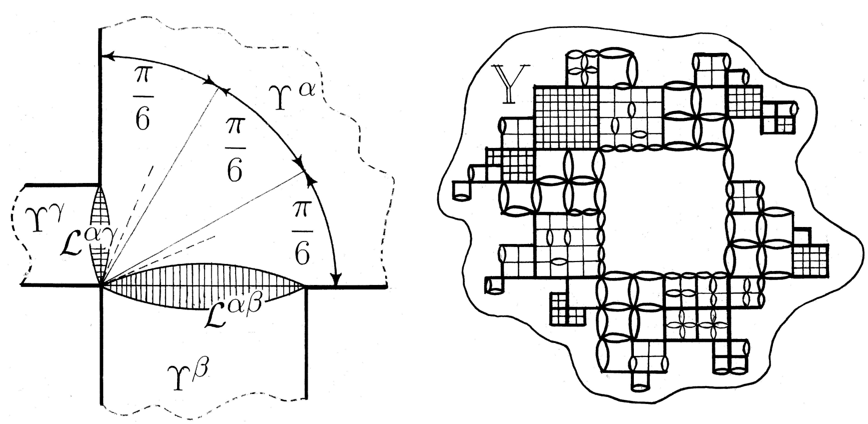

Figure 3. Lenses.

Remark 3.1. In what follows we shall consider only the pairs (α, β) of indices $\alpha = 1, 2, \ldots$ and $\beta = 1, 2, \ldots$ which correspond to adjacent cells. Such pairs will be designated the symbol $\alpha\beta$.

Step 2b. (Replacements in $\mathfrak{X}^{\alpha\,\beta}$) The lenses $\mathcal{L}^{\alpha\,\beta} \subset \mathbb{Y}$ are convex, so with the aid of Proposition 2.6 and Theorem 2.5, we may replace $\hbar_1 \colon \mathfrak{X}^{\alpha\,\beta} \to \mathcal{L}^{\alpha\,\beta}$ with the p-harmonic extension of $\hbar_1 \colon \partial\mathfrak{X}^{\alpha\,\beta} \to \partial\mathcal{L}^{\alpha\,\beta}$. We do this, and denote the result by $\hbar_2^{\alpha\beta} \colon \mathfrak{X}^{\alpha\beta} \to \mathcal{L}^{\alpha\beta}$, only on the cells in which $\hbar_1 \colon \mathfrak{X}^{\alpha} \cup \mathfrak{X}^{\beta} \cup \mathfrak{X}^{\alpha\beta} \to \mathbb{R}^2$ is not a $\mathscr{C}^\infty$-diffeomorphism. In other cells we set $\hbar_2^{\alpha\beta} = \hbar_1$. In either case $\hbar_2^{\alpha\beta} \in \hbar_1 + \mathcal{A}_\circ(\mathfrak{X}^{\alpha\,\beta})$ so we define

$$\hbar_2 = \hbar_1 + \sum_{\alpha\beta} [\hbar_2^{\alpha\,\beta} - \hbar_1]_\circ .$$

Thus we have

$$\hbar_2 - \hbar_1 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_2}$$

The advantage of using $\hbar_2$ in the next step lies in the fact that it is not only a $\mathscr{C}^\infty$-diffeomorphism in every cell, but also is $\mathscr{C}^\infty$-smooth with positive Jacobian determinant, up to each edge of the cells created here. These edges are $\mathscr{C}^\infty$-smooth open arcs. By cells created here we mean not only $\mathfrak{X}^{\alpha\,\beta}$ but also those obtained from the parent cell $\mathfrak{X}^\alpha$ by removing the adjacent daughters; that is,

$$\mathfrak{X}^\alpha \setminus \bigcup_{\alpha\beta} \mathfrak{X}^{\alpha\,\beta}, \qquad \alpha = 1, 2, \dots$$

See Figure 4. The estimates of $\hbar_2$ run as follows. By (3.1) we have,

$$\|\hbar_2 - \hbar_1\|_{\mathscr{C}(\mathbb{X})} \leqslant \sup_{\alpha\beta}\{\operatorname{diam} \mathcal{L}^{\alpha\,\beta}\} \leqslant \sup_{\alpha}\{\operatorname{diam} \mathbb{Y}^\alpha\} \leqslant \frac{\epsilon}{5}. \tag{B_2}$$

In view of the minimum p-harmonic energy principle, we have

$$\begin{aligned} \|\nabla\hbar_2 - \nabla\hbar_1\|_{\mathcal{L}^p(\mathbb{X})} &= \sum_{\alpha\beta} \|\nabla\hbar_2 - \nabla\hbar_1\|_{\mathcal{L}^p(\cup\mathfrak{X}^{\alpha\,\beta})} \\ &\leqslant \sum_{\alpha\beta} \Big[\|\nabla\hbar_2\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\,\beta})} + \|\nabla\hbar_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\,\beta})} \Big] \\ &\leqslant 2\sum_{\alpha\beta} \|\nabla\hbar_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\,\beta})} \leqslant \frac{2\epsilon}{5} \sum_{\alpha\beta} 2^{-\alpha-\beta}. \end{aligned}$$

by (3.3). Hence

$$\|\nabla\hbar_2 - \nabla\hbar_1\|_{\mathcal{L}^p(\mathbb{X})} \leqslant \frac{\epsilon}{5}. \tag{C_2}$$

The minimum energy principle also yields estimate

$$\begin{aligned} \|\nabla\hbar_2\|^p_{\mathcal{L}^p(\mathbb{X})} &= \|\nabla\hbar_2\|^p_{\mathcal{L}^p(\cup\mathfrak{X}^{\alpha\,\beta})} + \|\nabla\hbar_1\|^p_{\mathcal{L}^p(\mathbb{X}\setminus\cup\mathfrak{X}^{\alpha\,\beta})} \\ &\leqslant \|\nabla\hbar_1\|^p_{\mathcal{L}^p(\cup\mathfrak{X}^{\alpha\,\beta})} + \|\nabla\hbar_1\|^p_{\mathcal{L}^p(\mathbb{X}\setminus\cup\mathfrak{X}^{\alpha\,\beta})} = \|\nabla\hbar_1\|^p_{\mathcal{L}^p(\mathbb{X})}. \end{aligned}$$

In particular

$$\|\nabla\hbar_2\|_{\mathcal{L}^p(\mathbb{X})} \leqslant \|\nabla\hbar_1\|_{\mathcal{L}^p(\mathbb{X})}, \tag{D_2}$$

completing the proof of Step 2.

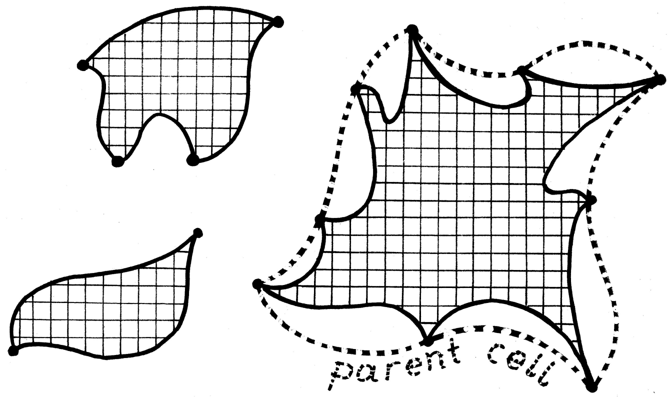

FIGURE 4. Three types of cells.

Note that $\hbar_2$ is locally bi-Lipschitz in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. The exceptional set $\mathcal{V}(\mathbb{X})$ is discrete.

STEP 3

We shall now merge all the adjacent cells together, by smoothing $\hbar_2$ around the edges $\Gamma^{\alpha\beta} \subset \mathfrak{X}^\alpha$. To achieve proper estimates we need to remove small neighborhoods of all vertices, outside which $\hbar_2$ is certainly locally bi-Lipschitz.

Step 3a. First we cover the set $\mathcal{C}(\mathbb{Y})$ of corners of dyadic squares by disks $\mathbb{D}_c$ centered at $c \in \mathcal{C}(\mathbb{Y})$. These disks will be chosen small enough to satisfy all the conditions listed below.

(i) $\operatorname{diam} \mathbb{D}_c < \epsilon/5$ for every $c \in \mathcal{C}(\mathbb{Y})$,

(ii) $\displaystyle\sum_{v \in \mathcal{V}(\mathbb{X})} \int_{\mathbb{F}_v} |\nabla \hbar_2|^p \leqslant \left(\frac{\epsilon}{20}\right)^p$, where $\mathbb{F}_v = \hbar_2^{-1}(\mathbb{D}_c)$, $c = \hbar_2(v) = h(v)$.

Denote by $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup \overline{\mathbb{F}_v}$. We truncate each edge $\Gamma^{\alpha\beta}$ near the endpoints by setting

$$\Gamma^{\alpha\beta}_\circ = \Gamma^{\alpha\beta} \cap \mathbb{X}_\circ. \tag{3.4}$$

These are mutually disjoint open arcs; their closures are isolated continua in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. This means that there are disjoint neighborhoods of them. We are actually interested in neighborhoods $\mathbb{U}^{\alpha\beta} \subset \mathfrak{X}^\alpha$ of $\Gamma^{\alpha\beta}_\circ$ that are Jordan domains in which $\Gamma^{\alpha\beta}_\circ \subset \mathbb{U}^{\alpha\beta}$ are $\mathscr{C}^\infty$-smooth crosscuts with two endpoints in $\partial\mathbb{U}^{\alpha\beta}$, see Section 2. It is geometrically clear that such mutually disjoint neighborhoods exist. Now the stage for next substep is established.

Step 3b. ($\mathscr{C}^\infty$-replacement within $\mathbb{U}^{\alpha\beta}$) It is at this stage that we will improve $\hbar_2$ in $\mathbb{U}^{\alpha\beta}$ to a $\mathscr{C}^\infty$-smooth diffeomorphism with no harm to the previously established estimates for $\hbar_2$. The tool is Proposition 2.8. As always, we shall make no replacement of $\hbar_2 \colon \mathbb{U}^{\alpha\beta} \to \Upsilon^\alpha$ if it is already $\mathscr{C}^\infty$-diffeomorphism. Recall that we have a bi-Lipschitz mapping $\hbar_2 \colon \mathbb{U}^{\alpha\beta} \to \hbar_2(\mathfrak{X}^\alpha) = \Upsilon^\alpha$ that takes the crosscut $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ onto a circular arc. Denote the components $\mathbb{U}_+^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \setminus \overline{\mathfrak{X}^{\alpha\beta}}$ and $\mathbb{U}_-^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \cap \mathfrak{X}^{\alpha\beta}$. Furthermore, we have

$$|D\hbar_2| \leqslant m_{\alpha\beta} \quad \text{ and } \quad \det D\hbar_2 \geqslant \frac{1}{m_{\alpha\beta}}, \quad \text{for some } \; m_{\alpha\beta} > 0$$

on each component. The mappings $\hbar_2 \colon \mathbb{U}_+^{\alpha\beta} \to \Upsilon^\alpha$ and $\hbar_2 \colon \mathbb{U}_-^{\alpha\beta} \to \Upsilon^\alpha$ are $\mathscr{C}^\infty$-diffeomorphisms up to $\Gamma_\circ^{\alpha\beta}$. In accordance with Proposition 2.8 we find a constant $M_{\alpha\beta}$ such that: whenever open set $\mathbb{V}^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ contains the crosscut $\Gamma_\circ^{\alpha\beta}$ there exists a homeomorphism $\hbar_3^{\alpha\beta} \colon \overline{\mathbb{U}^{\alpha\beta}} \xrightarrow{\text{onto}} \hbar_2(\overline{\mathbb{U}^{\alpha\beta}})$ which is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{U}^{\alpha\beta}$, with the following properties

- $\hbar_3^{\alpha\beta} \equiv \hbar_2$ on $(\overline{\mathbb{U}^{\alpha\beta}} \setminus \mathbb{V}^{\alpha\beta}) \cup \Gamma_\circ^{\alpha\beta}$;
- $|\nabla \hbar_3^{\alpha\beta}| \leqslant M_{\alpha\beta}$ and $\det \nabla \hbar_3^{\alpha\beta} \geqslant \frac{1}{M_{\alpha\beta}}$ in $\mathbb{U}^{\alpha\beta}$.

Since $M_{\alpha\beta}$ does not depend on $\mathbb{V}^{\alpha\beta}$ it will be advantageous to take neighborhoods $\mathbb{V}^{\alpha\beta}$ of $\Gamma_\circ^{\alpha\beta}$ thin enough to satisfy

- $\overline{\mathbb{V}^{\alpha\beta}} \subset \mathbb{U}^{\alpha\beta} \cup \overline{\Gamma_\circ^{\alpha\beta}}$;
- $|\mathbb{V}^{\alpha\beta}| \leqslant \frac{1}{5^p \cdot 2^{\alpha+\beta}} \left[\frac{\epsilon}{m_{\alpha\beta} + M_{\alpha\beta}}\right]^p$ and also $|\mathbb{V}^{\alpha\beta}| \leqslant \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}}$.

Note that $\hbar_3^{\alpha\beta}, \hbar_2 \in \mathscr{W}^{1,\infty}(\mathbb{U}^{\alpha\beta}) \subset \mathscr{W}^{1,p}(\mathbb{U}^{\alpha\beta})$ and $\hbar_3^{\alpha\beta} = \hbar_2$ on $\partial \mathbb{U}^{\alpha\beta}$, so we have

$$\hbar_3^{\alpha\beta} - \hbar_2 \in \mathscr{W}_\circ^{1,p}(\mathbb{U}^{\alpha\beta}).$$

Step 3c. We now define a homeomorphism $\hbar_3 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by the rule

$$\hbar_3 = \begin{cases} \hbar_3^{\alpha\beta} & \text{in } \mathbb{U}^{\alpha\beta} \\ \hbar_2 & \text{in } \mathbb{X} \setminus \bigcup\limits_{\alpha\beta} \mathbb{U}^{\alpha\beta}. \end{cases}$$

Obviously, $\hbar_3$ is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{X}_\circ$ and $\hbar_3 - \hbar_2 \in \mathscr{W}_\circ^{1,p}(\mathbb{X}_\circ)$. Since $\hbar_3$ coincides with $\hbar_2$ outside $\mathbb{X}_\circ$ we have $\hbar_3 = \hbar_2 + [\hbar_3 - \hbar_2]_\circ$. Hence

$$(A_3) \qquad \hbar_3 - \hbar_2 \in \mathcal{A}_\circ(\mathbb{X}).$$

Then, for every $x \in \mathbb{X}$,

$$|\hbar_3(x) - \hbar_2(x)| \leqslant \begin{cases} \operatorname{diam} \hbar_2(\mathbb{U}^{\alpha\beta}), & \text{for } x \in \mathbb{U}^{\alpha\beta} \\ 0, & \text{otherwise} \end{cases} \leqslant \operatorname{diam} \Upsilon^\alpha \leqslant \frac{\epsilon}{5}$$

meaning that

$$(B_3) \qquad \|\hbar_3 - \hbar_2\|_{\mathscr{C}(\mathbb{X})} \leqslant \frac{\epsilon}{5}.$$

The computation of p-norms goes as follows

$$\begin{aligned}\|\nabla \hbar_3 - \nabla \hbar_2\|^p_{\mathscr{L}^p(X)} &= \sum_{\alpha\beta} \int_{\mathbb{V}^{\alpha\,\beta}} |\nabla \hbar_3 - \nabla \hbar_2|^p \\ &\leqslant \sum_{\alpha\beta} |\mathbb{V}^{\alpha\,\beta}| \left[\|\nabla \hbar_3\|_{\mathscr{C}(\mathbb{V}^{\alpha\,\beta})} + \|\nabla \hbar_2\|_{\mathscr{C}(\mathbb{V}^{\alpha\,\beta})} \right]^p \\ &\leqslant \sum_{\alpha\beta} |\mathbb{V}^{\alpha\,\beta}| \, (m_{\alpha\beta} + M_{\alpha\beta})^p \leqslant \sum_{\alpha\beta} \frac{\epsilon^p}{5^p\, 2^{\alpha+\beta}} \leqslant \left(\frac{\epsilon}{5}\right)^p .\end{aligned}$$

Hence

$$(C_3) \qquad \|\nabla \hbar_3 - \nabla \hbar_2\|_{\mathscr{L}^p(X)} \leqslant \frac{\epsilon}{5}.$$

In the finite energy case, when $\|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} < \infty$, we observe that

$$\|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X}\setminus\cup\mathbb{V}^{\alpha\,\beta})} = \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X}\setminus\cup\mathbb{V}^{\alpha\,\beta})} \leqslant \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})}.$$

Therefore, by triangle inequality,

$$\begin{aligned}\|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} &\leqslant \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{V}^{\alpha\,\beta})} \\ &\leqslant \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} |\mathbb{V}^{\alpha\,\beta}| \cdot \|\nabla \hbar_3\|_{\mathscr{C}(\mathbb{V}^{\alpha\,\beta})} \\ &\leqslant \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}} \cdot M_{\alpha\beta}\end{aligned}$$

which yields

$$(D_3) \qquad \|\nabla \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\nabla \hbar_2\|_{\mathscr{L}^p(\mathbb{X})} + \delta.$$

The third step is completed.

Step 4

We have already upgraded the mapping h to a homeomorphism $\hbar_3 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ that is a $\mathscr{C}^\infty$-diffeomorphism in $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup_{v\in\mathcal{V}(\mathbb{X})} \overline{\mathbb{F}_v}$, where $\mathbb{F}_v$ are small surroundings of the vertices of cells. Their images $\hbar_3(\mathbb{F}_v) = \hbar_2(\mathbb{F}_v) = \mathbb{D}_c$ are small disks centered at $c = h(v)$. In Step 3a, one of the preconditions on those disks was that $\operatorname{diam} \mathbb{D}_c < \epsilon/5$. Furthermore, the closed disks $\overline{\mathbb{D}_c}$ are isolated continua in $\mathbb{Y}$ for all $c \in \mathcal{C}(\mathbb{Y})$, so are the sets $\overline{\mathbb{F}_v}$ in $\mathbb{X}$. We shall now consider slightly larger concentric open disks $\mathbb{D}'_c \supset \overline{\mathbb{D}_c}$, $c \in \mathcal{C}(\mathbb{Y})$, and their preimages $\mathbb{F}'_v = h_3^{-1}(\mathbb{D}'_c) \subset \mathbb{X}$, $v = h^{-1}(c) \in \mathcal{V}(\mathbb{X})$. The annulus $\mathbb{D}'_c \setminus \overline{\mathbb{D}_c}$ will be thin enough to ensure that $\mathbb{D}'_c$ are still disjoint,

$$\operatorname{diam} \mathbb{D}'_c < \frac{\epsilon}{5} \qquad \text{for all } c \in \mathcal{C}(\mathbb{Y})$$

and

$$\sum_{v\in\mathcal{V}(\mathbb{X})} \|\nabla \hbar_3\|^p_{\mathcal{L}^p(\mathbb{F}'_v\setminus\mathbb{F}_v)} \leqslant \left(\frac{\epsilon}{20}\right)^p .$$

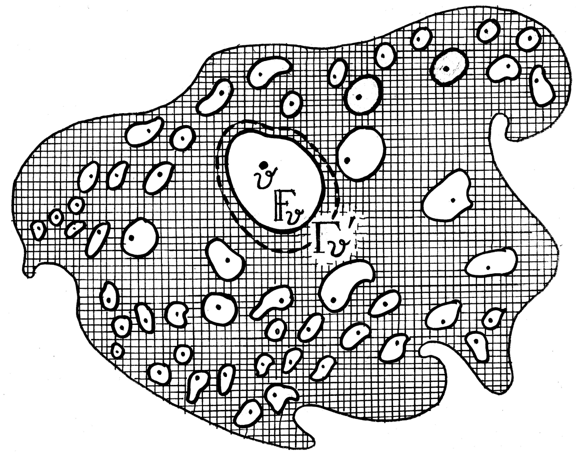

FIGURE 5. Neighborhoods of vertices.

Let Γ'_v, $v \in \mathcal{V}(\mathbb{X})$, denote the boundary of $\mathbb{F}'_v$. These are $\mathscr{C}^\infty$-smooth Jordan curves. We now define a homeomorphism $\hbar_4 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ by performing p-harmonic replacement of mappings $\hbar_3 \colon \mathbb{F}'_v \xrightarrow{\text{onto}} \mathbb{D}'_c$, whenever such a mapping fails to be $\mathscr{C}^\infty$-diffeomorphism. Thus every $\hbar_4 \colon \mathbb{F}'_v \xrightarrow{\text{onto}} \mathbb{D}'_c$ is a $\mathscr{C}^\infty$-diffeomorphism up to Γ'_v. Moreover $\hbar_4 \in \hbar_3 + \mathscr{W}_\circ^{1,p}(\mathbb{F}'_c)$, so

$$\hbar_4 - \hbar_3 \in \mathcal{A}_\circ(\mathbb{X}). \tag{A_4}$$

For every $x \in \mathbb{X}$, we have

$$|\hbar_4(x) - \hbar_3(x)| \leqslant \begin{cases} \operatorname{diam} \mathbb{D}'_c & \text{in } \mathbb{F}'_v,\ c = h(v) \\ 0 & \text{otherwise} \end{cases} \leqslant \frac{\epsilon}{5}.$$

Hence

$$\|\hbar_4 - \hbar_3\|_{\mathscr{C}(\mathbb{X})} \leqslant \frac{\epsilon}{5}. \tag{B_4}$$

By virtue of the minimum energy principle we compute the p-norms

$$\begin{aligned}
\|\hbar_4 - \hbar_3\|^p_{\mathscr{L}^p(\mathbb{X})} &= \sum_{v\in\mathcal{V}(\mathbb{X})} \|\hbar_4 - \hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}'_v)} \\
&\leqslant \sum_{v\in\mathcal{V}(\mathbb{X})} \left[\|\hbar_4\|_{\mathscr{L}^p(\mathbb{F}'_v)} + \|\hbar_3\|_{\mathscr{L}^p(\mathbb{F}'_v)}\right]^p \\
&\leqslant 2^p \sum_{v\in\mathcal{V}(\mathbb{X})} \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}'_v)} \\
&\leqslant 2^{2p-1} \sum_{v\in\mathcal{V}(\mathbb{X})} \left[\|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}'_v\setminus\mathbb{F}_v)} + \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{F}_v)}\right] \\
&\leqslant 2^{2p-1} \left[\left(\frac{\epsilon}{20}\right)^p + \sum_{v\in\mathcal{V}(\mathbb{X})} \|\hbar_2\|^p_{\mathscr{L}^p(\mathbb{F}_v)}\right] \\
&\leqslant 2^{2p} \left(\frac{\epsilon}{20}\right)^p = \left(\frac{\epsilon}{5}\right)^p.
\end{aligned}$$

Hence

$$(C_4) \qquad \|\hbar_4 - \hbar_3\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \frac{\epsilon}{5}.$$

Again by minimum energy principle we find that

$$(D_4) \qquad \|\hbar_4\|^p_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\hbar_3\|^p_{\mathscr{L}^p(\mathbb{X})}.$$

Just as in the previous steps, condition (E_4) remains valid, finishing Step 4.

Step 5

The final step consists of smoothing $\hbar_4$ in a neighborhood of each smooth Jordan curve Γ'_v, $v \in \mathcal{V}(\mathbb{X})$. We argue in much the same way as in Step 3, but this time we appeal to Proposition 2.9 instead of Proposition 2.8. By smoothing $\hbar_4$ in a sufficiently thin neighborhood of each Γ'_v we obtain a $\mathscr{C}^\infty$-diffeomorphism $\hbar_5 \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$,

$$(A_5) \qquad \hbar_5 - \hbar_4 \in \mathcal{A}_\circ(\mathbb{X}).$$

$$(B_5) \qquad \|\hbar_5 - \hbar_4\|_{\mathscr{C}(\mathbb{X})} \leqslant \frac{\epsilon}{5}.$$

$$(C_5) \qquad \|\hbar_5 - \hbar_4\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \frac{\epsilon}{5}.$$

$$(D_5) \qquad \|\hbar_5\|_{\mathscr{L}^p(\mathbb{X})} \leqslant \|\hbar_4\|_{\mathscr{L}^p(\mathbb{X})} + \delta. \quad \square$$

4. Open questions

Question 4.1. Does Theorem 1.1 extend to $n = 3$?

Question 4.2. A bi-Sobolev homeomorphism $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a mapping of class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqslant p < \infty$, whose inverse $h^{-1}\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ belongs to a Sobolev class $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$, $1 \leqslant q < \infty$. Can h be approximated by bi-Sobolev diffeomorphisms $\{h_\ell\}$ so that $h_\ell \to h$ in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ and $h_\ell^{-1} \to h^{-1}$ in $\mathscr{W}^{1,q}(\mathbb{Y}, \mathbb{X})$?

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