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Tadeusz Iwaniec
Syracuse University and University of Helsinki

Leonid V. Kovalev
Syracuse University

Jani Onninen
Syracuse University

DYNAMICS OF QUASICONFORMAL FIELDS

TADEUSZ IWANIEC, LEONID V. KOVALEV, AND JANI ONNINEN

Abstract. A uniqueness theorem is established for autonomous systems of ODEs, $\dot{x} = f(x)$, where f is a Sobolev vector field with additional geometric structure, such as delta-monoticity or reduced quasiconformality. Specifically, through every non-critical point of f there passes a unique integral curve.

Contents

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1. Introduction and Overview

Let $f\colon \Omega \to \mathbb{R}^n$ be a continuous vector field defined in a domain $\Omega \subset \mathbb{R}^n$. We shall consider the associated autonomous system of ordinary differential equations with given initial data

$$\dot{x}(t) = f\big(x(t)\big)\,, \qquad x(t_0) = x_0 \in \Omega \tag{1.1}$$

By virtue of Peano's Existence Theorem, the system admits a local solution; that is, a solution defined in an open interval containing t_0, in which we have $x(t) \in \Omega$. However, uniqueness of the local solution is not always guaranteed. Every local solution $x = x(t)$ can be extended (as a solution in Ω) to its maximal interval of existence, say for $t \in (\alpha, \beta)$ where $-\infty \leqslant \alpha < \beta \leqslant \infty$. Such an interval will, of course, depend upon the choice of extension of the local solution. The limits $\lim_{t \searrow \alpha} x(t)$ and $\lim_{t \nearrow \beta} x(t)$, if exist in Ω, are the critical points of f; that is, zeros of f. The classical theory of ODEs tells us that Lipschitz vector fields admit unique local solutions; for less regular fields the solutions are seldom unique, see [8, Ch. I Corollary 6.2] for related results. In the present paper, we address the uniqueness question under significantly weaker regularity hypothesis on f. We work with fields f that are locally in Sobolev class $W^{1,p}$ for some $n < p < \infty$. The DiPerna-Lions theory (see [7], [3] and references therein) establishes the existence and uniqueness of suitably generalized flow for Sobolev fields under certain restrictions on their divergence. Our results are different in that we obtain the uniqueness of solutions in the classical sense, for all initial values except for critical points. In order to achieve this, the geometry of f (e.g., quasiconformality or monotonicity) must come into play. It should be noted that the fruitful connection between the theory of ODEs and geometric function theory has a long history [1, 5, 14, 15, 16].

It is easily seen that monotonicity of f yields backwards uniqueness [8, Ch. III Theorem 6.2].

Definition 1.1. A continuous vector field $f\colon \Omega \to \mathbb{R}^n$ is said to be *monotone* if

$$\langle f(a) - f(b)\,,\, a - b\rangle \geqslant 0 \quad \text{ for every } a, b \in \Omega \tag{1.2}$$

It is *strictly monotone* if equality occurs only for $a = b$.

Proposition 1.2. (Backward Uniqueness) *Suppose $f\colon \Omega \to \mathbb{R}^n$ is monotone and $x = x(t)$ and $y = y(t)$ are solutions to the system (1.1) in Ω. Then the distance between them, $t \to |x(t) - y(t)|$, is nondecreasing. In particular, if $x(t_0) = y(t_0)$, then $x(t) = y(t)$ for all values $t \leqslant t_0$ in the range of existence of $x(t)$ and $y(t)$.*

We include a short proof of this proposition, mainly to keep the exposition as self contained as possible.

Proof. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|x(t) - y(t)|^2 = 2\,\langle \dot{x} - \dot{y}\,,\, x - y\rangle = 2\,\langle f(x) - f(y)\,,\, x - y\rangle \geqslant 0$$

Hence, for $t \leqslant t_0$ we obtain $|x(t) - y(t)| \leqslant |x(t_0) - y(t_0)| = 0$. □

In general, forward uniqueness fails for monotone fields (Example 15.1), although it holds for almost every initial value [6]. However, we shall prove that forward uniqueness for δ-monotone fields, holds for every noncritial initial value.

Definition 1.3. A vector field $f\colon \Omega \to \mathbb{R}^n$ is called δ-*monotone*, $0 < \delta \leqslant 1$, if for every $a, b \in \Omega$

$$\langle f(a) - f(b)\,,\, a - b\rangle \geqslant \delta\, |a-b|\, |f(a) - f(b)| \tag{1.3}$$

Note that there is no supposition of continuity here. In fact, a nonconstant δ-monotone mapping is not only continuous but also a K-quasiconformal homeomorphism [12], see Section 2.1 for the definition of K-quasiconformality.

Theorem 1.4. *Let $f\colon \Omega \to \mathbb{R}^n$ be nonconstant δ-monotone. Then the initial value problem*

$$\dot{x}(t) = f\big(x(t)\big)\,, \qquad x(0) = x_0 \in \Omega \tag{1.4}$$

admits unique local solution, provided $f(x_0) \neq 0$.

The condition $f(x_0) \neq 0$ turns out to be necessary, though it is redundant for Lipschitz vector fields, see Example 15.1.

It is also not difficult to see that if f is merely Hölder continuous, $f \in C^\alpha(\Omega)$, with $0 < \alpha < 1$, then the assumption $f(x_0) \neq 0$ does not guarantee uniqueness. However, the uniqueness of integral curves, even for only Hölder regular vector fields, is still possible under additional geometric conditions, like δ-monotonicity in Theorem 1.4. We shall work with homeomorphisms $f\colon \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ normalized by $f(0) = 0$, so the origin of $\mathbb{R}^n$ is the only critical point of the field. In the complex plane there is a close relationship between monotone vector fields and the so-called reduced quasiconformal mappings. In Section 2.3 we take a close look at the reduced distortion inequality

$$|f_{\bar z}| \leqslant k\, |\mathrm{Re}\, f_z|\;, \quad 0 \leqslant k < 1 \quad \text{ for } f \in W^{1,1}_{\mathrm{loc}}(\mathbb{C}) \tag{1.5}$$

This concept and relevant results can be found in [4] and the recent work by the authors [10]. One unusual feature of the homeomorphic solutions to the reduced distortion inequality should be pointed out. The measurable function $\mathrm{Re}\, f_z$ does not change sign in $\mathbb{C}$, see [4, Theorem 6.3.2]. Precisely, we have

$$\text{either} \qquad \mathrm{Re}\, f_z \geqslant 0 \quad \text{a.e. in } \mathbb{C} \tag{1.6}$$

$$\text{or} \qquad \mathrm{Re}\, f_z \leqslant 0 \quad \text{a.e. in } \mathbb{C} \tag{1.7}$$

What is more, though we do not exploit it here, is that (1.6) or (1.7) actually hold with strict inequalities, which is rather deep analytic fact recently established by Alessandrini and Nesi [2] in connection with the question of G-compactness of the Beltrami equation [9, 4]. The property (1.6)-(1.7), does not hold for noninjective solutions of (1.5). It also fails for homeomorphic solutions in proper subdomains $\Omega \subset \mathbb{C}$. Since we confine ourselves to injective vector fields in the entire complex plane, we can certainly assume that $\mathrm{Re}\, f_z \geqslant 0$. Thus,

$$|f_{\bar z}| \leqslant k\, \mathrm{Re}\, f_z \tag{1.8}$$

For if not, replace f by $-f$, which affects only the direction of the integral curves. Homeomorphic solution to (1.8) will be referred to as reduced K-quasiconformal mappings, $K = \frac{1+k}{1-k}$. In Section 3 we shall show that

Proposition 1.5. *Every nonconstant solution to the reduced distortion inequality*

$$|f_{\bar z}| \leqslant k\, \mathrm{Re}\, f_z\,, \quad 0 \leqslant k < 1 \quad \text{ for } f \in W^{1,1}_{\mathrm{loc}}(\mathbb{C}), \quad f(0) = 0$$

is strictly monotone, unless $f(z) = i\omega z$, where ω is a (nonzero) real number, in which case $\langle f(a) - f(b), a - b\rangle \equiv 0$.

We refer to this latter case as degenerate reduced quasiconformal map. Dynamics of the vector field $f(z) = i\omega z$ is rather simple; its integral curves are circles centered at the origin, $z(t) = z_0 e^{i\omega t}$. From now on let us restrict ourselves to discussing nondegenerate reduced K-quasiconformal fields. Since f is strictly monotone, it follows that $\operatorname{Re} f(1) = \langle f(1) - f(0), 1-0\rangle > 0$. Thus it involves no loss of generality in assuming that $\operatorname{Re} f(1) = 1$, by rescaling time parameter if necessary. This yields $|f(1)| \geqslant 1$. We then introduce the following class of the reduced K-quasiconformal fields.

Definition 1.6. Given $K \geqslant 1$ and $d \geqslant 1$, we consider the family

$$\mathcal{F}_K(d) = \big\{ f \colon \mathbb{C} \to \mathbb{C} \colon f(0) = 0 \text{ and } 1 = \operatorname{Re} f(1) \leqslant |f(1)| \leqslant d \big\}$$

where the mappings in consideration are solutions to the differential inequality

$$|f_{\bar z}| \leqslant \frac{K-1}{K+1} \operatorname{Re} f_z \qquad f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$$

Such solutions are automatically K-quasiconformal homeomorphisms. It is perhaps worth noting that $\mathcal{F}_K(d)$ is a convex family; that is, given $f, g \in \mathcal{F}_K(d)$ their convex combination $(1-\lambda)f + \lambda g$, $0 \leqslant \lambda \leqslant 1$, also belongs to $\mathcal{F}_K(d)$.

Let us summarize the above discussion by the following chain of inclusions

$$\begin{aligned} &\begin{Bmatrix} \delta\text{-monotone} \\ \text{mappings} \end{Bmatrix} \subsetneq \begin{Bmatrix} \text{reduced } K\text{-quasiconformal} \\ \text{mappings} \end{Bmatrix} \\ &\subsetneq \begin{Bmatrix} \text{monotone } K\text{-quasiconformal} \\ \text{mappings} \end{Bmatrix} \subsetneq \begin{Bmatrix} K\text{-quasiconformal} \\ \text{mappings} \end{Bmatrix} \end{aligned} \tag{1.9}$$

Here all the inclusions are strict and $K = \frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}}$.

We succeeded in extending Theorem 1.4 to mappings in the second family of this chain.

Theorem 1.7. (Uniqueness) *Given a reduced quasiconformal field $f \in \mathcal{F}_K(d)$. Through every $x_0 \neq 0$ there passes exactly one integral curve $x = x(t)$,*

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = x_0 \tag{1.10}$$

defined in its maximal interval (α, β), where $-\infty \leqslant \alpha < 0 < \beta \leqslant \infty$. We have $x(t) \in \mathbb{C}_0 = \mathbb{C} \setminus \{0\}$ for $t \in (\alpha, \beta)$, and

$$\lim_{t \searrow \alpha} x(t) = 0 \quad \text{and} \quad \lim_{t \nearrow \beta} x(t) = \infty \tag{1.11}$$

Moreover,

$$|x(s)| < |x(t)|, \qquad \text{for } \alpha < s < t < \beta \tag{1.12}$$

In other words, as the point $x(t)$ travels along the curve in the direction of the increasing parameter t, its distance to the critical point of f strictly increases. To accommodate explicit uniform estimates we restrict the integral curves to the annulus

$$\mathbb{C}_r^R = \{ z \colon r \leqslant |z| \leqslant R \}, \qquad 0 < r < R < \infty$$

Consider two integral curves

$$\dot{x}(t) = f\big(x(t)\big)\,, \qquad x(t_0) = x_0 \in \mathbb{C}_r^R$$
$$\dot{y}(s) = f\big(y(s)\big)\,, \qquad y(t_0) = y_0 \in \mathbb{C}_r^R,$$

where the time parameters t and s are restricted to the intervals in which

$$r \leqslant |x(t)| \leqslant R \quad \text{and} \quad r \leqslant |y(s)| \leqslant R$$

respectively. We then have the following Lipschitz dependence of the solutions on both the time parameter and initial data.

Theorem 1.8. (Lipschitz Continuity) *There exist constants $A_r^R = A_r^R(K,d)$ and $B_r^R = B_r^R(K,d)$ such that*

$$|x(t) - y(s)| \leqslant A_r^R\,|t-s| + B_r^R\,|x_0 - y_0| \tag{1.13}$$

In particular,

$$|x(t) - y(t)| \leqslant B_r^R\,|x(s) - y(s)| \tag{1.14}$$

as long as $x(t), y(t), x(s)$ and $y(s)$ lie in the annulus $\mathbb{C}_r^R$.

There is a convenient and geometrically pleasing parametrization of the integral curves for a given field $f \in \mathcal{F}_K(d)$. Every integral curve Γ intersects the unit circle $\mathbb{S}^1$ at exactly one point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$. Denote such curve by Γ_θ and call θ a quasipolar angle of the curve. We have already mentioned that if a point z moves along Γ_θ its distance to the origin strictly increases in time. Thus $\rho = |x|$ can be used as a new parameter in Γ, $0 \leqslant \rho \leqslant \infty$. In this way every point $z \in \mathbb{C}_0$ is uniquely prescribed by its quasipolar coordinates associated with the vector field $f \in \mathcal{F}_K(d)$. This is a pair $(\rho, e^{i\theta}) \in \mathbb{R}_+ \times \mathbb{S}^1$ with $\rho = |z|$ as the polar distance of z and θ as its quasipolar angle; for example, the identity map $f = id\colon \mathbb{C} \to \mathbb{C}$ gives the usual polar coordinates $(\rho, e^{i\theta})$ of $z = \rho e^{i\theta}$. Quasipolar coordinates give rise to a homeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$ defined by the rule $\Phi(\rho, e^{i\theta}) = \rho \cdot e^{i\theta}$. This homeomorphism turns out to be locally bi-Lipschitz in $\mathbb{C}_0$. Precisely, we have

$$c_r^R \leqslant \left|\frac{\Phi(z_1) - \Phi(z_2)}{z_1 - z_2}\right| \leqslant C_r^R \qquad \text{for } z_1, z_2 \in \mathbb{C}_r^R \tag{1.15}$$

see Theorem 9.1. Moreover, Φ is the identity on $\mathbb{S}^1$ and takes every circle $\mathbb{S}_\rho = \{z\colon |z| = \rho\}$ onto itself. More importantly, Φ rectifies each trajectory Γ_θ by mapping it onto the straight ray

$$\mathcal{R}_\theta = \big\{z\colon\ \arg z = \theta\big\}\,, \quad 0 \leqslant \theta < 2\pi$$

see Figure 1.

Every complex number $z \neq 0$ has infinitely many quasipolar angles which differ from each other by multiple of 2π. Let us denote by $\Theta = \Theta(z) = \Theta_f(z)$ the multivalued function that assigns to $z \in \mathbb{C}_0$ all its quasipolar angles. A monodromy principle tells us that Θ has a continuous branch on every simply connected domain $\Omega \subset \mathbb{C}_0$. Two such branches differ by a constant. Therefore, it makes sense to speak of the gradient of Θ, defined by

$$\nabla\Theta = \left(\frac{\partial\Theta}{\partial x}\,, \frac{\partial\Theta}{\partial y}\right)\,, \qquad \text{almost everywhere in } \mathbb{C} \tag{1.16}$$

With the aid of the function $\Theta = \Theta(z)$ we shall factor the vector field $if(z)$ into a product of a gradient field and a scalar function.

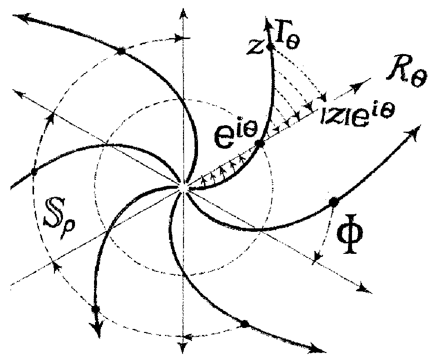

FIGURE 1. Bi-Lipschitz rectification of trajectories.

Theorem 1.9. *The orthogonal vector field $V(x, y) = if(x+iy)$, $f \in \mathcal{F}_k(d)$, admits a factorization*

$$V(x, y) = \lambda(x, y)\nabla\Theta \tag{1.17}$$

The integrating factor is bounded from below and from above,

$$m(|z|) \leqslant \lambda(z) \leqslant M(|z|), \quad \lambda(z) = \frac{|f(z)|}{|\nabla\Theta(z)|} \tag{1.18}$$

Here the lower and upper bounds $m, M : (0, \infty) \to \mathbb{R}_+$ are continuous functions. These functions depend only on the parameters K and d of the family $\mathcal{F}_K(d)$.

It is not difficult to construct a vector field $f \in \mathcal{F}_K(d)$ for which no factorization of the form (1.17) together with (1.18) allows the integrating factor λ to be continuous, see Example 15.2 for details. Curiously, the curvature (in somewhat generalized sense) of the orthogonal trajectories is nonnegative, see Remark 12.2 for an explanation.

To conclude the introduction, let us mention some of the ingredients of our proofs.

Theorem 1.10. *Let $f : \Omega \to \mathbb{R}^n$ be nonconstant δ-monotone. Then the image $f(\Gamma)$ of any C^1-curve $\Gamma \subset \Omega$ is locally rectifiable.*

Due to this property Theorem 1.4 is a corollary of the following more general result.

Theorem 1.11. *Let $f : \Omega \to \mathbb{R}^n$ be a K-quasiconformal field of bounded variation on C^1-curves. Suppose we are given two local solutions of the Cauchy problem*

$$\dot{x}(t) = f\big(x(t)\big), \qquad x(0) = a \in \Omega$$
$$\dot{y}(t) = f\big(y(t)\big), \qquad y(0) = a \in \Omega$$

where $f(a) \neq 0$. Then there exist $\epsilon > 0$ such that $x(t) = y(t)$ for $-\epsilon < t < \epsilon$.

The conclusion of Theorem 1.10 fails for general quasiconformal maps [17], even for reduced ones [11]. This is where the elementary but very useful concept of the modulus of monotonicity

$$\Delta_f(a, b) = \left\langle f(a) - f(b), \frac{a - b}{|a - b|} \right\rangle \tag{1.19}$$

comes into play. We show that for reduced quasiconformal maps Δ_f has the same quasisymmetric behavior as the modulus of continuity for general quasiconformal maps. We exploit this property by computing Δ_f at suitably chosen points on integral curves. Due to a cancellation property of the modulus of monotonicity the sum of Δ_f over such partition points is controlled by the quadratic variation of f. On the other hand, for any planar quasiconformal map f the quadratic variation over a C^1-arc is controlled by the diameter of its image. From this we deduce the uniqueness of solutions.

Our results raise the following

Question 1.12. Let $f\colon \Omega \to \mathbb{R}^n$ be quasiconformal and $f(x_0) \neq 0$. Does the system (1.1) admit a unique local solution?

2. Background

Let us introduce the notation and briefly review basic concepts.

2.1. **Quasisymmetry.** Let Ω be a domain in $\mathbb{R}^n$. A sense preserving homeomorphism $f\colon \Omega \to \mathbb{R}^n$ is said to be K-*quasiconformal*, $1 \leqslant K < \infty$, if

$$\limsup_{\epsilon \to 0} \frac{\max\limits_{|x-a|=\epsilon} |f(x)-f(a)|}{\min\limits_{|y-a|=\epsilon} |f(y)-f(a)|} \leqslant K\,, \qquad \text{for every } a \in \Omega \tag{2.1}$$

It is well known that such mappings belong to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and are Hölder continuous with exponent $\alpha = \frac{1}{K}$. An analytic description of (2.1) can be formulated (equivalently) via the so-called distortion inequality

$$|Df(x)|^n \leqslant \mathcal{K} \cdot J(x,f) \qquad \text{a.e.} \quad f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \tag{2.2}$$

for some $1 \leqslant \mathcal{K} < \infty$. Here $|Df(x)|$ stands for the norm of the differential matrix and $J(x,f)$ for the Jacobian determinant. In the complex plane it reads as

$$|f_{\bar z}(z)| \leqslant k\,|f_z(z)|\,, \quad k = \frac{K-1}{K+1} \quad \text{a.e.} \tag{2.3}$$

The $W^{1,n}$-solutions to the distortion inequality (2.2) or (2.3) (not necessarily injective) are referred to as K-quasiregular mappings. Quasiregular mappings are also locally Hölder continuous. A purely geometric description of quasiconformality, which proves useful in our study here, is the concept of quasisymmetry, also called three point condition.

Theorem 2.1. (Three Points Condition) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiconformal. Then*

$$m_K\left(\frac{|x-a|}{|y-a|}\right) \leqslant \frac{|f(x)-f(a)|}{|f(y)-f(a)|} \leqslant M_K\left(\frac{|x-a|}{|y-a|}\right) \tag{2.4}$$

for $a, x, y \in \mathbb{R}^n$, $a \neq y$, where

$$M_K(t) = C_K \max\left(t^K, t^{\frac{1}{K}}\right)\,, \qquad 0 \leqslant t < \infty$$

and

$$m_K(t) = \left[M_K\left(t^{-1}\right)\right]^{-1} = C_K^{-1} \min\left(t^K, t^{\frac{1}{K}}\right)$$

If f keeps the origin fixed, $f(0)=0$, then

$$m_K(|x|)\ |f(1)| \leqslant |f(x)| \leqslant M_K(|x|)\ |f(1)| \tag{2.5}$$

More generally,

$$m_K\left(\frac{|x|}{|y|}\right) \leqslant \frac{|f(x)|}{|f(y)|} \leqslant M_K\left(\frac{|x|}{|y|}\right) \tag{2.6}$$

2.2. **Modulus of monotonicity.** Let $f\colon \Omega \to \mathbb{R}^n$ be continuous and monotone. The modulus of monotonicity $\Delta_f\colon \Omega\times\Omega \to [0,\infty)$ is defined by the rule

$$0 \leqslant \Delta_f(a,b) = \begin{cases} \left\langle f(a)-f(b), \frac{a-b}{|a-b|}\right\rangle & \text{if } a\neq b \\ 0 & \text{if } a=b \end{cases} \leqslant\ |f(a)-f(b)|$$

We shall also work with $\delta_f\colon \Omega\times\Omega \to [0,1]$ given by

$$\delta_f(a,b) = \left\langle \frac{f(a)-f(b)}{|f(a)-f(b)|}, \frac{a-b}{|a-b|}\right\rangle \qquad \text{for } a\neq b \tag{2.7}$$

Thus f is δ-monotone if and only if

$$\delta_f(a,b) \geqslant \delta\,, \qquad 0\leqslant \delta < 1 \tag{2.8}$$

or, equivalently,

$$\Delta_f(a,b) \geqslant \delta\,|f(a)-f(b)|\ , \qquad \text{for all } a,b\in\Omega \tag{2.9}$$

2.3. **Reduced K-quasiconformal fields.** We will be dealing with the reduced distortion inequality

$$|f_{\bar z}| \leqslant k \operatorname{Re} f_z\,, \quad 0\leqslant k = \frac{K-1}{K+1} < 1\,, \quad f(0)=0 \tag{2.10}$$

for $f:\mathbb{C}\to\mathbb{C}$ in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Such solutions form a convex cone in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. Precisely, if f_1 and f_2 solve (2.10) and $\lambda_1,\lambda_2\geqslant 0$, then so does the mapping $\lambda_1 f_1+\lambda_2 f_2$. As an example, consider the linear map $f_\circ(z)=az+b\bar z$ in which $|b|\leqslant k\operatorname{Re} a$. Adding $f_\circ$ to a solution of (2.10) gives another solution $F(z)=f(z)+az+b\bar z$. We recall rather unexpected topological fact that every nonconstant quasiregular mapping $f\colon\mathbb{C}\to\mathbb{C}$, with $\operatorname{Re} f_z\geqslant 0$ almost everywhere, is a homeomorphisms of $\mathbb{C}$ onto $\mathbb{C}$ [10]. Actually such mapping satisfies strict inequality $\operatorname{Re} f_z>0$, almost everywhere, except for the case of the degenerate monotone mapping

$$f(z)\equiv i\omega\, z\,, \qquad \text{with some } \omega\in\mathbb{R}\setminus\{0\} \tag{2.11}$$

The integral curves $\dot z = i\omega\, z$ are circles centered at the origin, $z(t)=z_0 e^{i\omega\, t}$. As this case is completely clear, we shall focus on nondegenerate reduced K-quasiconformal fields; that is mappings $f\colon\mathbb{C}\to\mathbb{C}$ of Sobolev class $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ such that

$$|f_{\bar z}| \leqslant k\operatorname{Re} f_z\,, \quad f(0)=0\,, \quad f(z)\not\equiv i\omega z\,, \quad 0\leqslant k<1 \tag{2.12}$$

The simple case is the complex linear vector field $f(z)=(\lambda+i\omega)z$, $\lambda>0$. Its trajectories are spirals $z(t)=z_0e^{\lambda t}e^{i\omega t}$, $-\infty<t<\infty$, except for $\omega=0$. In this latter case

$$f(z)=\lambda z\,, \qquad \lambda>0 \tag{2.13}$$

for which the trajectories are straight rays $z(t)=z_0\,e^{\lambda t}$, $-\infty<t<\infty$. We shall see latter that the trajectories of every (nondegenerate) reduced K-quasiconformal field are images of straight rays via a homeomorphism $\Psi\colon\mathbb{C}\xrightarrow{\text{onto}}\mathbb{C}$, $\Psi(0)=0$. This homeomorphism turns out to be locally bi-Lipschitz on $\mathbb{C}_0$, see section 9.

3. Estimates of reduced K-quasiconformal fields

Let $\lambda + i\omega$ be a complex number in the right half plane, $\lambda \geqslant 0$. Given any (nondegenerate) reduced K-quasiconformal map $f\colon \mathbb{C} \to \mathbb{C}$, we consider a map $F(z) = f(z) + (\lambda + i\omega)z$. This is a nonconstant solution to the same distortion inequality as f. Indeed,

$$|F_{\bar z}| = |f_{\bar z}| \leqslant k \operatorname{Re} f_z \leqslant k \operatorname{Re} F_z$$

Thus F is K-quasiregular. By virtue of Corollary 1.5 [10] F is a homeomorphism. In particular, the three point condition applies to F to yield the inequalities

$$m_K\left(\left|\frac{x-a}{y-a}\right|\right) \leqslant \frac{|x-a|\left|\frac{f(x)-f(a)}{x-a} + \lambda + i\omega\right|}{|y-a|\left|\frac{f(y)-f(a)}{y-a} + \lambda + i\omega\right|} \leqslant M_K\left(\left|\frac{x-a}{y-a}\right|\right) \tag{3.1}$$

for every x and $y \neq a$. Therefore, $\frac{f(x)-f(a)}{x-a} + \lambda + i\omega \neq 0$, whenever $x \neq a$. Putting $\omega = -\operatorname{Im}\frac{f(x)-f(a)}{x-a}$ we arrive at the inequality

$$\lambda + \operatorname{Re}\frac{f(x)-f(a)}{x-a} \neq 0 \qquad \text{for all } \lambda \geqslant 0 \quad \text{and} \quad x \neq a$$

This gives

$$\Delta_f(x,a) = |x-a| \operatorname{Re}\frac{f(x)-f(a)}{x-a} > 0 \tag{3.2}$$

We just proved that every (nondegenerate) reduced quasiconformal map is strictly monotone, as stated in Proposition 1.5.

Setting $a = 0$ and $x = 1$, we obtain

$$\operatorname{Re} f(1) = \Delta_f(1,0) > 0 \tag{3.3}$$

This inequality makes it legitimate to normalize the (nondegenerate) reduced K-quasiconformal fields by the condition $\operatorname{Re} f(1) = 1$. We indeed have used such normalization in the definition of the class $\mathcal{F}_K(d)$. Let us record this normalization once again as

$$\Delta_f(1,0) = \operatorname{Re} f(1) = 1 \tag{3.4}$$

We now return to (3.1), and set $\lambda = 0$ and $\omega = -\operatorname{Im}\frac{f(x)-f(a)}{x-a}$. We then arrive at the same three point condition for Δ_f as for a general K-quasiconformal mapping in (2.4),

$$\frac{\Delta_f(x,a)}{\Delta_f(y,a)} = \frac{\left\langle f(x)-f(a), \frac{x-a}{|x-a|}\right\rangle}{\left\langle f(y)-f(a), \frac{y-a}{|y-a|}\right\rangle} \geqslant m_K\left(\left|\frac{x-a}{y-a}\right|\right) = M_K^{-1}\left(\left|\frac{y-a}{x-a}\right|\right) \tag{3.5}$$

In particular, setting $a = 0$, we obtain

$$m_K\left(\left|\frac{x}{y}\right|\right) \leqslant \frac{\Delta_f(x,0)}{\Delta_f(y,0)} \leqslant M_K\left(\left|\frac{x}{y}\right|\right) \tag{3.6}$$

Then, letting $y = 1$, this simplifies to:

$$m_K(|x|) \leqslant |x| \operatorname{Re}\frac{f(x)}{x} = \Delta_f(x,0) \leqslant M_K\big(|x|\big), \qquad \text{for } f \in \mathcal{F}_K(d) \tag{3.7}$$

As for the estimate of $|f(x)|$, we may use the three point condition and the assumption that $1 \leqslant |f(1)| \leqslant d$, to infer that

$$m_K\,(|x|) \leqslant \left| \frac{f(x) - f(0)}{f(1) - f(0)} \right| \leqslant M_K\,(|x|) \tag{3.8}$$

Hence

$$m_K\,(|x|) \leqslant |f(x)| \leqslant d\, M_K\,(|x|) \tag{3.9}$$

This combined with (3.7) gives a lower bound of $\delta_f(x,0)$,

$$\delta_f(x,0) = \frac{\Delta_f(x,0)}{|f(x)|} \geqslant \frac{1}{d}\frac{m_K(|x|)}{M_K(|x|)} \tag{3.10}$$

4. Estimates along integral curves

Let $\Gamma \subset \mathbb{C}_0$ be any integral curve of $f \in \mathcal{F}_K(d)$ parametriced by a solution of the differential equation $\dot{x}(t) = f\big(x(t)\big)$, $\alpha \leqslant t \leqslant \beta$. It follows from the previous computation that

$$\frac{\mathrm{d}|x|}{\mathrm{d}t} = |x|\frac{\mathrm{d}\ln|x|}{\mathrm{d}t} = |x|\,\mathrm{Re}\,\frac{\dot{x}}{x} = \Delta_f(x,0) \geqslant m_K(|x|) \tag{4.1}$$

This shows that the function $t \to |x(t)|$ is strictly increasing in time, whenever $x(t)$ stays away from the critical point of f. Moreover, we have

$$\frac{\mathrm{d}|x|}{\mathrm{d}t} = \delta_f(x,0)\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|$$

Let us now assume that $\Gamma \subset \mathbb{C}_r^R$. This means that $r \leqslant |x(t)| \leqslant R$ for all $\alpha \leqslant t \leqslant \beta$, so $\delta_f(x,0) \geqslant \frac{m_K(r)}{d\,M_K(R)}$ for $x \in \Gamma$. Hence

$$\begin{aligned} |x(\beta) - y(\alpha)| &\leqslant \int_\alpha^\beta |\dot{x}(t)|\,\mathrm{d}t \leqslant \int_\alpha^\beta \frac{1}{\delta_f(x,0)}\frac{\mathrm{d}|x|}{\mathrm{d}t}\mathrm{d}t \\ &= d\,\frac{M_K(R)}{m_K(r)}\big(|x(\beta)| - |x(\alpha)|\big) \end{aligned} \tag{4.2}$$

We just proved a reverse type triangle inequality along Γ.

Lemma 4.1. *Let x_1 and x_2 be points in an integral curve of f such that $r \leqslant |x_1|, |x_2| \leqslant R$. Then*

$$|x_1 - x_2| = C_r^R \cdot \big||x_1| - |x_2|\big|\,, \qquad x_1, x_2 \in \Gamma \tag{4.3}$$ [1]

Another point of significance is that the time difference between points in $\Gamma \subset \mathbb{C}_r^R$ is finite. Indeed, the time between $x(\beta)$ and $x(\alpha)$ can be estimated as follows.

$$|x(\beta)| - |x(\alpha)| = \int_\alpha^\beta \frac{\mathrm{d}|x|}{\mathrm{d}t}\mathrm{d}t \geqslant \int_\alpha^\beta m_K(|x|)\,\mathrm{d}t \geqslant m_K(r)(\beta - \alpha)$$

Hence, whenever $r \leqslant |x(\alpha)| \leqslant |x(\beta)| \leqslant R$ we have

$$\beta - \alpha \leqslant \frac{|x(\beta)| - |x(\alpha)|}{m_K(r)} \leqslant \frac{R - r}{m_k(r)} \tag{4.4}$$

[1]Hereafter, C_r^R stands for a constant depending on the annulus $\mathbb{C}_r^R$. It also depends on the family $\mathcal{F}_K(d)$, though we shall not indicate the dependence on the parameters K and d, as the need will not arise. However, for the sake of notational simplicity we shall allow C_r^R to vary from line to line.

On the basis of these inequalities we may now prove that the endpoints of the maximal extension of the local integral curves are the critical points of f.

Corollary 4.2. *Let $x = x(t)$ be a solution to the system $\dot{x} = f(x)$, $f \in \mathcal{F}_K(d)$ for $t \in (\alpha, \beta)$. Here $-\infty \leqslant \alpha < \beta \leqslant +\infty$ are the endpoints of the maximal interval of existence. Then*

$$\lim_{t \searrow \alpha} x(t) = 0 \quad \text{and} \quad \lim_{t \nearrow \beta} x(t) = \infty \tag{4.5}$$

Proof. As the function $t \to |x(t)|$ is increasing, it follows that

$$0 \leqslant r = \lim_{t \searrow \alpha} |x(t)| < \lim_{t \nearrow \beta} |x(t)| = R \leqslant \infty$$

Suppose to the contrary that $r > 0$. Then the reverse triangle inequality (4.3) shows that we also have a limit $\lim\limits_{t \searrow \alpha} x(t) = a \neq 0$. By (4.4) this limit is attained in finite time. But then, by virtue of Peano's Existence Theorem, the solution $x = x(t)$ admits an extension beyond α (for some $t < \alpha$) contradicting maximality of the interval (α, β). The same contradiction follows if one assumes that $R < \infty$ □

Lemma 4.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a (nondegenerate) reduced K-quasiconformal field. Consider the integral arcs of the same time-length*

$$\begin{aligned} \mathfrak{X} &= \left\{ x(t) \colon \dot{x} = f(x)\,, \quad \alpha \leqslant t \leqslant \beta \right\} \subset \mathbb{C}_r^R \\ \Upsilon &= \left\{ y(t) \colon \dot{y} = f(y)\,, \quad \alpha \leqslant t \leqslant \beta \right\} \subset \mathbb{C}_r^R \end{aligned}$$

We assume that time-length equals the distance between these arcs,

$$\operatorname{dist}(\mathfrak{X}, \Upsilon) = \beta - \alpha$$

Denote by $x_\alpha = x(\alpha)$, $x_\beta = x(\beta)$ the endpoints of $\mathfrak{X}$ and $y_\alpha = y(\alpha)$, $y_\beta = y(\beta)$ the endpoints of Υ. Then, for all $x \in \mathfrak{X}$ and $y \in \Upsilon$ we have

$$\Delta_f(x, y) \leqslant C_r^R \Delta_f(x_\beta, x_\alpha) \tag{4.6}$$

Moreover,

$$\Delta_f(x_\beta, x_\alpha) \leqslant |f(x_\beta)| - |f(x_\alpha)| + \frac{\operatorname{diam}^{\,2} f(\mathfrak{X})}{2\, m_K(r)} \tag{4.7}$$

$$\log \frac{|x_\beta - y_\beta|}{|x_\alpha - y_\alpha|} \leqslant C_r^R \left[|f(x_\beta)| - |f(x_\alpha)| + \operatorname{diam}^{\,2} f(\mathfrak{X}) \right] \tag{4.8}$$

Proof. By the three point condition in (3.5), we have

$$\begin{aligned} \Delta_f(x, y) &= M_K\left(\left| \frac{x - y}{x_\beta - y} \right| \right) \Delta_f(x_\beta, y) \\ &\leqslant M_K\left(\left| \frac{x - y}{x_\beta - y} \right| \right) M_K\left(\left| \frac{x_\beta - y}{x_\beta - x_\alpha} \right| \right) \Delta_f(x_\beta, x_\alpha) \end{aligned} \tag{4.9}$$

We need to estimate the fractions under the function M_K; the numerators from above and the denominators from below. For this, choose and fix the time parameters $t, s \in [\alpha, \beta]$ such that

$$|x(t) - x(s)| = \operatorname{dist}(\mathfrak{X}, \Upsilon) = \beta - \alpha$$

By Mean Value Theorem,

$$\begin{aligned} |x-y| &\leqslant |x-x(t)|+|x(t)-y(s)|+|y(s)-y| \\ &\leqslant |\dot{x}(\xi)|\,(\beta-\alpha)+(\beta-\alpha)+|\dot{y}(\zeta)|\,(\beta-\alpha) \end{aligned}$$

for some $\alpha \leqslant \xi, \zeta \leqslant \beta$. On the other hand $|\dot{x}| = |f(x)| \leqslant d\, M_K(|x|) \leqslant d\, M_K(R)$. Similarly, $|\dot{y}| \leqslant d\, M_K(R)$. Thus, we have

$$|x-y| \leqslant [1+2dM_K(R)]\,(\beta-\alpha)$$

and, in particular,

$$|x_\beta - y| \leqslant [1+2dM_K(R)]\,(\beta-\alpha)$$

As regards the denominators, we have

$$|x_\beta - y| \geqslant \operatorname{dist}(\mathfrak{X}, \Upsilon) = \beta - \alpha$$

and, again by the Mean Value Theorem,

$$|x_\beta - x_\alpha| = |\dot{x}(\zeta)|\,(\beta-\alpha) = |f(x_\zeta)|\,(\beta-\alpha) \geqslant m_K(r)(\beta-\alpha)$$

The inequality (4.6) is now immediate from (4.9), simply set

$$C_r^R = M_K\big(1+2dM_K(R)\big)\cdot M_K\left(\frac{1+2dM_K(R)}{m_K(r)}\right)$$

To prove (4.7) we appeal to the following

Lemma 4.4. *Let A, B, Z be vectors of an inner product space, $|Z| = 1$. Then*

$$\langle A-B, Z\rangle \leqslant |A| - |B| + \frac{|B-\lambda Z|^2}{2\lambda}$$

for all $\lambda > 0$.

Proof.

$$\langle A-B, Z\rangle \leqslant |A| - \langle B, Z\rangle \leqslant |A| - |B| + \frac{|B-\lambda Z|^2 - (|B|-\lambda)^2}{2\lambda}$$

□

We take $X = f(x_\beta)$, $Y = f(x_\alpha)$, $Z = \frac{x_\beta - x_\alpha}{|x_\beta - x_\alpha|}$ and $\lambda = \frac{|x_\beta - x_\alpha|}{\beta-\alpha}$. This gives us the estimate

$$\Delta_f(x_\beta, x_\alpha) \leqslant |f(x_\beta)| - |f(x_\alpha)| + \frac{\left|f(x_\beta) - \frac{x_\beta - x_\alpha}{\beta-\alpha}\right|^2}{2\left|\frac{x_\beta - x_\alpha}{\beta-\alpha}\right|} \tag{4.10}$$

The letter term is handled with the aid of the Mean Value Theorem. Precisely, there is $\xi \in [\alpha, \beta]$ such that

$$\frac{\left|f(x_\beta) - \frac{x_\beta - x_\alpha}{\beta-\alpha}\right|}{2\left|\frac{x_\beta - x_\alpha}{\beta-\alpha}\right|} = \frac{|f(x_\beta) - \dot{x}(\xi)|^2}{2\,|\dot{x}(\xi)|} = \frac{|f(x_\beta) - f(x_\xi)|^2}{2\,|f(x_\xi)|} \leqslant \frac{\operatorname{diam}^2 f(\mathfrak{X})}{2\, m_K\,(r)}$$

as desired. The proof of (4.8) proceeds as follows

$$\begin{aligned}
\log \frac{|x_\beta - y_\beta|}{|x_\alpha - y_\alpha|} &= \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t} \log|x(t)-y(t)|\,\mathrm{d}t \\
&= \int_\alpha^\beta \left\langle \dot{x}(t) - \dot{y}(t),\, \frac{x(t)-y(t)}{|x(t)-y(t)|^2} \right\rangle \mathrm{d}t \\
&= \int_\alpha^\beta \frac{\Delta_f\big(x(t),y(t)\big)}{|x(t)-y(t)|}\,\mathrm{d}t \leqslant C_r^R \Delta_f(x_\beta, x_\alpha) \\
&\leqslant C_r^R \left[|f(x_\beta)| - |f(x_\alpha)| + \frac{\operatorname{diam}^2 f(\mathfrak{X})}{2m_K(r)} \right]
\end{aligned}$$

Here, for the inequality before the last, we estimated the numerator in the integrand by (4.6) while for the denominator we observed

$$|x(t) - y(t)| \geqslant \operatorname{dist}(\mathfrak{X}, \Upsilon) = \beta - \alpha$$

Then (4.8) follows from (4.7). The proof of Lemma 4.3 is complete. □

5. Quadratic variation along C^1-arcs

A parametric curve in $\mathbb{R}^n$ is a continuous function $x = x(t)$ defined in an interval I (bounded or unbounded) with values in $\mathbb{R}^n$. The orientation of a parametric curve is given in the direction of increasing parameter. If $x : I \to \mathbb{R}^n$ is one-to-one, then $x = x(t)$ is called a simple parametric curve; it is called an arc if $I = [\alpha, \beta]$ is closed and bounded, in which case $x(\alpha)$ and $x(\beta)$ are called the left and the right endpoints. Let $\Gamma = \{x(t) \colon \alpha \leqslant t \leqslant \beta\}$. A partition of parameters $\alpha = t_0 < t_1 < \cdots < t_N = \beta$ gives rise to a partition of the curve Γ, with $x_j = x(t_j)$, $j = 0, 1, ..., N$, called partition points of Γ. Furthermore, to every interval $[t_{j-1}, t_j]$ there corresponds a subarc $\gamma_j = x[t_{j-1}, t_j]$ in Γ. The arc length of Γ is denoted by $|\Gamma|$.

Recall that p-variation, $p \geqslant 1$, of a continuous map $f \colon \Omega \to \mathbb{R}^n$ along a compact C^1-arc $\Gamma \subset \Omega$ is defined by

$$|f(\Gamma)|_p = \sup \left(\sum_{\nu=1}^N |\operatorname{diam} f(\gamma_\nu)|^p \right)^{\frac{1}{p}} < \infty \tag{5.1}$$

where the supremum runs over all finite partitions of Γ into subarcs $\gamma_1, \gamma_2, \cdots, \gamma_N$. Note that

$$|f(\Gamma)|_p \leqslant |f(\Gamma)|_q\,, \qquad \text{when } 1 \leqslant q \leqslant p$$

When $p = 1$ we recover the classical concept of bounded variation, and denote $|f(\Gamma)|_1 = |f(\Gamma)|$ for simplicity.

The quadratic variation along C^1-arcs of any homeomorphism $f \colon \mathbb{C} \to \mathbb{C}$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ is finite, see [13, Theorem 4.3]. We shall demonstrate this property, together with specific bounds, for planar K-quasiconformal mappings.

Theorem 5.1. *Let $f \colon \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping and Γ a C^1-arc in $\mathbb{C}$. Then $|f(\Gamma)|_2 < \infty$. If, moreover, $f \in \mathcal{F}_K(d)$ and Γ lies in the annulus $\mathbb{C}_r^R$, then*

$$|f(\Gamma)|_2 \leqslant C_r^R \operatorname{diam} f(\Gamma) \tag{5.2}$$

Proof. Let $z = z(\tau)$, $\alpha \leqslant \tau \leqslant \beta$, be the arc-length parametrization of Γ; that is, $|\dot{z}(\tau)| \equiv 1$ and $|\Gamma| = \beta - \alpha$. The C^1-modulus of regularity of Γ is defined by

$$\Lambda(\tau) = \sup \{ |\dot{z}(t) - \dot{z}(s)| \colon \alpha \leqslant t,\, s \leqslant \beta,\, |t - s| \leqslant \tau \} \tag{5.3}$$

Clearly, the function $\Lambda\colon [0,|\Gamma|] \to [0,2]$ is continuously nondecreasing and $\Lambda(0)=0$. By the definition, we have

$$|\dot z(t)-\dot z(s)| \leqslant \Lambda(t-s)\,, \qquad \text{for } \alpha\leqslant s\leqslant t\leqslant \beta \tag{5.4}$$

We first consider short arcs, assuming that $\Lambda(|\Gamma|)\leqslant \frac{\sqrt2}{2}$, or equivalently,

$$|\dot z(t)-\dot z(s)| \leqslant \frac{\sqrt2}{2} \qquad \text{for all } \alpha\leqslant s\leqslant t\leqslant \beta \tag{5.5}$$

Claim. *Under the assumption (5.5) we have*

$$|f(\Gamma)|_2 \leqslant C_K \operatorname{diam} f(\Gamma) \tag{5.6}$$

where C_K depends only on the distortion K of the mapping f.

Proof of Claim. With the aid of a rigid motion we place Γ into a position in which its endpoints are real numbers, say the left endpoint is the origin and the right endpoint is a positive number L. By the Mean Value Theorem, there exists a middle point $\zeta\in[\alpha,\beta]$ such that

$$1=\frac{z(\beta)-z(\alpha)}{|z(\beta)-z(\alpha)|}=\dot z(\zeta)$$

Then, in view of Condition (5.5),

$$|1-\dot z(\xi)| = |\dot z(\zeta)-\dot z(\xi)| \leqslant \frac{\sqrt2}{2} \qquad \text{for every } \xi\in[\alpha,\beta] \tag{5.7}$$

This, by the Mean Value Theorem again, yields

$$\left|1-\frac{z(t)-z(s)}{|z(s)-z(t)|}\right| \leqslant \frac{\sqrt2}{2} \qquad \text{for } \alpha\leqslant s< t\leqslant \beta \tag{5.8}$$

Then (5.7) combined with the identity $|\dot z(\xi)|=1$ gives

$$|\operatorname{Im}\dot z(\xi)| \leqslant \frac{\sqrt7}{3}\operatorname{Re}\dot z(\xi) \leqslant \operatorname{Re}\dot z(\xi) \tag{5.9}$$

for every $\alpha\leqslant\xi\leqslant\beta$. In other words, the function $t\to\operatorname{Re} z(t)$ is strictly increasing from 0 to L. In particular, Γ becomes a graph of a function over the interval $[0,L]$. Given any parameters $\alpha\leqslant t,s\leqslant\beta$, by Cauchy's Mean-Value Theorem, we have

$$\frac{\operatorname{Im} z(t)-\operatorname{Im} z(s)}{\operatorname{Re} z(t)-\operatorname{Re} z(s)}=\frac{\operatorname{Im}\dot z(\xi)}{\operatorname{Re}\dot z(\xi)}$$

for some $\xi\in[s,t]$. If we combine this with (5.9), we obtain

$$|\operatorname{Im} z(t)-\operatorname{Im} z(s)| \leqslant \operatorname{Re} z(t)-\operatorname{Re} z(s) \tag{5.10}$$

In particular, letting $s=\alpha$, we see that

$$|\operatorname{Im} z(t)| \leqslant \operatorname{Re} z(t) \leqslant L \tag{5.11}$$

Next, we choose and fix an arbitrary partition points of Γ. Denote them by $0=z_0,z_1,\cdots,z_{N-1},z_N=L$. We consider the rectangles

$$\mathcal R_j=\big\{z\colon \operatorname{Re} z_{j-1}\leqslant \operatorname{Re} z\leqslant \operatorname{Re} z_j\,,\ -L\leqslant \operatorname{Im} z\leqslant L\big\}$$

and the subarcs of Γ,

$$\gamma_j=\mathcal R_j\cap\Gamma\,,\quad j=1,2,...,N$$

Inequality (5.10) shows that

$$\max_{z\in\gamma_j}\{\operatorname{Im} z\} \;-\; \min_{z\in\gamma_j}\{\operatorname{Im} z\} \leqslant \operatorname{Re} z_j - \operatorname{Re} z_{j-1}$$

This in turn allows us to confine each arc γ_j in a square $Q_j \subset \mathcal{R}_j$. Such squares are mutually disjoint and lie in a square Q centered at 0 and of side length $2L$

$$Q = \big\{ z\colon \; -L \leqslant \operatorname{Re} z \leqslant L\,, \quad -L \leqslant \operatorname{Im} z \leqslant L \big\}$$

We note that Γ joins ∂Q with the center of Q. Therefore

$$\min_{z\in\partial Q} |f(z) - f(0)| \leqslant \operatorname{diam} f(\Gamma)$$

On the other hand, since f is K-quasiconformal, we have

$$|f(Q)| \leqslant \pi \max_{z\in\partial Q} |f(z) - f(0)|^2 \leqslant C_K \min_{z\in\partial Q} |f(z) - f(0)|^2 \leqslant C_K \operatorname{diam}^2 f(\Gamma)$$

We also have the reverse type estimates for the squares Q_j, namely

$$\operatorname{diam}^2 f(Q_j) \leqslant C_K\, |f(Q_j)|$$

This is a consequence of K-quasiconformality of f, as well.

Now we are ready to estimate the quadratic variation of f along Γ. As cubes Q_j are mutually disjoint, we have

$$\begin{aligned} \sum_{j=1}^{N} \operatorname{diam}^2 f(\gamma_j) \;&\leqslant\; \sum_{j=1}^{N} \operatorname{diam}^2 f(Q_j) \leqslant C_K \sum_{j=1}^{N} |f(Q_j)| \\ &=\; C_K\, |f(\cup Q_j)| \leqslant C_K\, |f(Q)| \leqslant C_K \operatorname{diam}^2 f(\Gamma) \end{aligned} \tag{5.12}$$

This completes the proof of our Claim.

To estimate $|f(\Gamma)|_2$ for long C^1-arcs, we partition Γ into ℓ disjoint subarcs, say $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell$, where ℓ is large enough to ensure condition (5.5) on each subarc. We fix this partition. Now, let $\gamma_1, \gamma_2, ..., \gamma_N$ be any partition of Γ into subarcs γ_j, $1 \leqslant j \leqslant N$, to be used for computing the quadratic variation of f along Γ. There are two kinds of subarcs in this partition. The first kind of the subarcs, denoted by γ_j', are those which lay entirely in one of $\Gamma_1, ..., \Gamma_\ell$. Certainly, using (5.6), we have

$$\sum \operatorname{diam}^2 f(\gamma_j') \leqslant \sum_{\nu=1}^{\ell} |f(\Gamma_\nu)|_2^2 \leqslant \ell\, C_K \operatorname{diam}^2 f(\Gamma)$$

Then, there are at most ℓ remaining subarcs, denoted by γ_j''. Each of them contains at least one endpoint of the partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell$. For these subarcs we have trivial estimate

$$\sum \operatorname{diam}^2 f(\gamma_j'') \leqslant \sum \operatorname{diam}^2 f(\Gamma) \leqslant \ell \operatorname{diam}^2 f(\Gamma)$$

In summary

$$\sum_{j=1}^{N} \operatorname{diam}^2 f(\gamma_j) \leqslant \ell(1 + C_K) \operatorname{diam}^2 f(\Gamma) < \infty$$

Since the partition $\gamma_1, ..., \gamma_N$ of Γ was chosen arbitrarily, it follows that

$$|f(\Gamma)|_2^2 \leqslant \ell(1 + C_K) \operatorname{diam}^2 f(\Gamma) < \infty \tag{5.13}$$

as desired.

When $\Gamma \subset \mathbb{C}_r^R$ is an integral curve of a reduced K-quasiconformal field this estimate lets us deduce specific bound of the quadratic variation.

Proof of (5.2). Let $z = z(\tau)$ be the arc-length parametrization of Γ; that is, $\dot{z}(\tau) = \frac{f(z(\tau))}{|f(z(\tau))|}$. We aim to partition Γ into subarcs $\Gamma_1, ..., \Gamma_\ell$ so that

$$\left|\frac{f(a)}{|f(a)|} - \frac{f(b)}{|f(b)|}\right| \leqslant \frac{\sqrt{2}}{2}\,, \quad \text{whenever } a, b \in \Gamma_\nu \tag{5.14}$$

For this, we observe that

$$\begin{aligned} \left|\frac{f(a)}{|f(a)|} - \frac{f(b)}{|f(b)|}\right| &= \left|\frac{f(a)-f(b)}{|f(a)|} - \frac{f(b)}{|f(b)|}\frac{|f(a)|-|f(b)|}{|f(a)|}\right| \\ &\leqslant \frac{2\,|f(a)-f(b)|}{|f(a))|} \leqslant 2M_K\left(\left|\frac{a-b}{a}\right|\right) \\ &\leqslant 2M_K\left(\frac{a-b}{r}\right) \leqslant \frac{\sqrt{2}}{2} \end{aligned}$$

provided $|a-b| \leqslant \epsilon_K r$, where ϵ_K is determined from the equation $M_K(\epsilon_K) = \frac{\sqrt{2}}{4}$. In view of the reverse triangle inequality (4.3) it suffices to make the partition of Γ fine enough to satisfy

$$\big||b| - |a|\big| \leqslant \frac{\epsilon_K \cdot r}{C_r^R} \quad \text{for } a, b \in \Gamma_\nu,\ \nu = 1, 2, ..., \ell \tag{5.15}$$

To this effect we divide the annulus $\mathbb{C}_r^R$ into ℓ annuli $\mathbb{C}_{r_1}^{R_1}$, ..., $\mathbb{C}_{r_\ell}^{R_\ell}$, each of width $R_\nu - r_\nu = \frac{1}{\ell}(R-r)$. Inequality (5.15) yields a sufficient lower bound for ℓ.

$$\ell \geqslant \frac{(R-r)C_r^R}{r\epsilon_K}$$

Finally, we notice that the intersections $\Gamma_\nu = \Gamma \cap \mathbb{C}_{r_\nu}^{R_\nu}$ are subarcs of Γ, because the function $t \to |x(t)|$ is strictly increasing along Γ. Inequality (5.14) now holds for all $a, b \in \Gamma_\nu$, $\nu = 1, 2, ..., \ell$. Substitute this integer value $\ell = \ell(r, R, K, d)$ into (5.13) to conclude with (5.2). □

6. A partition of two curves

Lemma 6.1. (Partition Lemma) *Suppose we are given two continuous functions $x, y\colon [-\infty, t_0] \to \mathbb{R}^n$ such that $x(-\infty) = y(-\infty)$, whereas $x(t_0) \neq y(t_0)$. Then there exists (unique) sequence $t_0 > t_1 > \cdots > t_k > t_{k+1} \cdots \to t_\infty \geqslant -\infty$, such that*

$$\inf_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - x(s)| = t_k - t_{k+1}\,, \quad x(t_\infty) = y(t_\infty) \tag{6.1}$$

If, in addition, x and y are continuously differentiable, then for every $\tau \in [t_{k+1}, t_k]$ we have

$$|x(t_k) - y(t_k)| \leqslant (1 + C)\,|x(\tau) - y(\tau)| \tag{6.2}$$

where

$$C = \sup_{\tau \leqslant t \leqslant t_k} \left(|\dot{x}(t)| + |\dot{y}(t)|\right)\ {}^{2} \tag{6.3}$$

[2] It will be important for the use of (6.2) that the supremum in (6.3) runs over the interval $[\tau, t_k]$, not over the entire interval $[t_{k+1}, t_k]$.

Proof. We construct such sequence $\{t_k\}$ by induction. Suppose we are given $t_0 > t_1 \cdots > t_k > -\infty$, $x(t_k) \neq y(t_k)$. Consider the following function

$$\varphi_k(\tau) = \tau + \inf_{\tau \leqslant t, s \leqslant t_k} |x(t) - y(s)|\,, \quad \tau \in [-\infty, t_k]$$

Clearly, φ_k is continuous and strictly increasing. Before we make the induction step, let us think of k to be equal zero. Since $\varphi_k(t_k) > t_k$ and $\varphi_k(-\infty) = -\infty$, we find (unique) parameter $t_{k+1} < t_k$ such that $\varphi_k(t_{k+1}) = t_k$. This means that

$$\inf_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - y(s)| = t_k - t_{k+1}$$

In particular, $x(t_{k+1}) \neq y(t_{k+1})$. Now, the same reasoning provides for the induction step. We then obtain the desired decreasing sequence

$$t_0 > t_1 > \cdots > t_k > t_{k+1} \cdots \to t_\infty \geqslant -\infty$$

Finally, if $t_\infty = -\infty$, then $x(t_\infty) = x(-\infty) = y(-\infty) = y(t_\infty)$ If, however, the sequence $\{t_k\}$ is converging to some finite number t_∞, then by (6.1) we conclude that $x(t_\infty) = y(t_\infty)$.

The proof of (6.2) is a matter of triangle inequality combined with (6.1).

$$\begin{aligned}
|x(t_k) - y(t_k)| &\leqslant |x(\tau) - y(\tau)| + |x(t_k) - x(\tau) + y(\tau) - y(t_k)| \\
&= |x(\tau) - y(\tau)| + \left| \int_\tau^{t_k} [\dot{x}(t) - \dot{y}(t)]\, \mathrm{d}t \right| \\
&\leqslant |x(\tau) - y(\tau)| + C\ |t_k - \tau| \\
&\leqslant |x(\tau) - y(\tau)| + C|t_k - t_{k+1}| \\
&= |x(\tau) - y(\tau)| + C \inf_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - y(s)| \\
&\leqslant (1 + C)\ |x(\tau) - y(\tau)|
\end{aligned}$$

as claimed. □

7. Uniqueness, proof of Theorem 1.7

We have already established (1.11) and (1.12) by Corollary 4.2 and Inequality 4.1. To complete the proof of Theorem 1.7 it remains to establish uniqueness of the local solutions. Let us state this task explicitly:

Proposition 7.1. *Suppose we are given two local solutions of the differential system*

$$\dot{x}(t) = f\big(x(t)\big) \quad \text{and} \quad \dot{y}(t) = f\big(y(t)\big) \quad \textit{for } t \in (\alpha, \beta)$$

where $f \in \mathcal{F}_K(d)$ and $x(t') = y(t') \neq 0$ for some $t' \in (\alpha, \beta)$. Then $x(t) = y(t)$ for all $t \in (\alpha, \beta)$.

Proof. The equality $x(t) = y(t)$ for $x \in (\alpha, t']$ is immediate since the function $t \to |x(t) - y(t)|$ is nondecreasing. Suppose, to the contrary, that $|x(t) - y(t)| \not\equiv 0$. Thus, there exists $s \in [t', \beta)$ such that $x(s) = y(s)$ and $|x(t) - y(t)| > 0$ for all $t \in (s, \beta)$. For notational convenience we can certainly assume that $s = 0$. Therefore, $x(t) \neq y(t)$ for all $0 < t < \beta$ and the common value $x(0) = y(0)$ is not the critical point of f. Choose and fix $t_0 \in (0, \beta)$. Thus we have

$$x(t), y(t) \in \mathbb{C}_r^R \quad \text{for } 0 \leqslant t \leqslant t_0$$

where we define

$$r = |x(0)| = |y(0)| \quad \text{and} \quad R = \max\big\{|x(t_0)|, |y(t_0)|\big\}$$

We shall make use of the partition

$$t_0 > t_1 > \cdots t_k > t_{k+1} \to 0$$

as in Lemma 6.1. Accordingly,

$$\operatorname{dist}\{x[t_{k+1}, t_k],\, y[t_{k+1}, t_k]\} = t_k - t_{k+1}\,, \quad k = 0, 1, 2, ...$$

Denote by x_k and y_k the values of x and y at time t_k, respectively. We also denote by γ_k the arc $\gamma_k = \{x(t)\colon t_{k+1} \leqslant t < t_k\}$. Then, in view of Inequality (4.8) in Lemma 4.3, we obtain

$$\log \frac{|x_k - y_k|}{|x_{k+1} - y_{k+1}|} \leqslant C_r^R \left[|f(x_k)| - |f(x_{k+1})| + \operatorname{diam}^2 f(\gamma_k) \right]$$

The telescoping structure of the terms in this inequality helps us to sum them up, with substantial cancellations. Summing with respect to $k = 0, 1, 2, ..., m-1$, the surviving terms are:

$$\log \left| \frac{x_0 - y_0}{x_m - y_m} \right| \leqslant C_r^R \left[|f(x_0)| - |f(x_m)| + |f(\Gamma)|_2^2 \right] \leqslant C_r^R \tag{7.1}$$

where $|f(\Gamma)|_2$ stands for the quadratic variation of f along $\Gamma = \{x(t)\colon 0 \leqslant t \leqslant t_0\}$. By Theorem 5.1, the right hand side of (7.1) is bounded by a constant C_r^R independent of m. However, the left hand side increases to $+\infty$ as $m \to -\infty$, because $x_m - y_m = x(t_m) - y(t_m) \to x(0) - y(0) = 0$. This contradiction proves Theorem 7.1 □

8. Proof of Theorem 1.8

First we prove Inequality (1.14). Let (α_1, β_1) denote the maximal interval of existence of the solution $x = x(t)$ of the system $\dot{x} = f(x)$ in $\mathbb{C}_0$, as in Theorem 1.7. It will be convenient to view x as a solution in $\mathbb{C}$ defined in the interval $[-\infty, \beta_1)$, by setting $x(t) = 0$ for $-\infty \leqslant t \leqslant \alpha_1$. The extended solution is a continuous function $x\colon [-\infty, \beta_1) \to \mathbb{C}$. Now consider another extended solution $y\colon [-\infty, \beta_2) \to \mathbb{C}$. Suppose that at some time $t_0 < \min\{\beta_1, \beta_2\}$ we have $x(t_0) \neq y(t_0)$. In particular, $t_0 \neq -\infty$. We make use of a decreasing sequence

$$t_0 > t_1 > \cdots t_k > t_{k+1} \to t_\infty$$

as in Lemma 6.1. With the same arguments as were used in (7.1) we obtain $|x(t_0) - y(t_0)| \leqslant C_r^R |x(t_k) - y(t_k)|$. Then the complementary inequality (6.2) yields

$$|x(t_0) - y(t_0)| \leqslant C_r^R\, |x(t_k) - y(t_k)| \leqslant C_r^R\, |x(\tau) - y(\tau)|$$

provided

$$r \leqslant |x(\tau)| \leqslant |x(t_0)| \leqslant R$$

and

$$r \leqslant |y(\tau)| \leqslant |y(t_0)| \leqslant R$$

We just proved Inequality (1.14). Now (1.13) is immediate.

$$\begin{aligned} |x(t) - y(s)| &\leqslant |x(t) - x(s)| + |x(s) - y(s)| \\ &\leqslant |\dot{x}(\xi)|\, |t - s| + C_r^R |x(0) - y(0)| \\ &\leqslant M_K(R)|t - s| + C_r^R |x(0) - y(0)| \end{aligned}$$

9. Bi-Lipschitz continuity of Φ

The purpose of this section is to prove the inequality (1.15). For this we represent Φ in quasipolar coordinates. Then (1.15) is the same as

Theorem 9.1. *Given two points $z_1 = (\rho_1, e^{i\theta_1})$ and $z_2 = (\rho_2, e^{i\theta_2})$ in $\mathbb{C}_r^R$, we have*

$$c_r^R\,|z_1 - z_2| \leqslant \left|\rho_1 e^{i\theta_1} - \rho_2 e^{i\theta_2}\right| \leqslant C_r^R\,|z_1 - z_2| \tag{9.1}$$

Proof. First we prove the following

Lemma 9.2. *Given two integral arcs $\mathfrak{X}, \Upsilon \subset \mathbb{C}_r^R$ and points $a \in \mathfrak{X}$, $b \in \Upsilon$ such that $|a| = |b| = \rho$. Then*

$$|a - b| \leqslant C_r^R \operatorname{dist}(\mathfrak{X}, \Upsilon)$$

In other words,

$$|a - b| \leqslant C_r^R\,|x_0 - y_0|\,, \qquad \text{whenever } x_0 \in \mathfrak{X} \text{ and } y_0 \in \Upsilon \tag{9.2}$$

Proof. **Case 1.** Suppose x_0 and y_0 lie in the opposite side of the circle $\mathbb{S}_\rho = \{z : |z| = \rho\}$. For example, $|x_0| \leqslant \rho \leqslant |y_0|$. Then,

$$\begin{aligned}
|a - b| &\leqslant |a - x_0| + |x_0 - y_0| + |y_0 - b| \\
&\leqslant C_r^R\big(|a| - |x_0|\big) + |x_0 - y_0| + C_r^R\big(|y_0| - |b|\big) \\
&= C_r^R\big(|y_0| - |x_0|\big) + |y_0 - x_0| \\
&\leqslant (1 + C_r^R)|y_0 - x_0|
\end{aligned}$$

Here in the second line we have used the reverse triangle inequality 4.3.

Case 2. Suppose both x_0 and y_0 lie inside $\mathbb{S}_\rho$. We use time parametrization for $\mathfrak{X}$ and Υ, with $t = 0$ as starting time for $x_0 = x(0)$ and $y_0 = y(0)$. Therefore $a = x(t)$ and $b = y(s)$ for some parameters t and s. We have $|x(t)| = |y(s)|$. Since the functions $t \to |x(t)|$ and $s \to |x(s)|$ are increasing, it follows that $t \geqslant 0$ and $s \geqslant 0$. We may assume without loss of generality that $0 \leqslant s \leqslant t$. Thus the point $x(s)$ lies in $\mathfrak{X}$. Clearly,

$$r \leqslant |x_0| = |x(0)| \leqslant |x(s)| \leqslant |x(t)| = |a| \leqslant \rho$$

Hence, $x(s) \in \mathfrak{X} \cap \mathbb{C}_r^R$. Now using the reverse triangle inequality (4.3) we obtain

$$\begin{aligned}
|a - b| &= |x(t) - y(s)| \leqslant |x(t) - x(s)| + |x(s) - y(s)| \\
&\leqslant C_r^R\big(|x(t)| - |x(s)|\big) + |x(s) - y(s)| \\
&= C_r^R\left(|y(s)| - |x(s)|\right) + |x(s) - y(s)| \\
&\leqslant (1 + C_r^R)\,|x(s) - y(s)| \\
&\leqslant (1 + C_r^R)\,C_r^R\,|x(0) - y(0)|
\end{aligned}$$

where in the last step we have appealed to (1.14).

In much the same way we prove (9.2) when both $x_0 = x(0)$ and $y_0 = y(0)$ lie outside $\mathbb{S}_\rho$. The only difference is that the parameters t and s will be negative. $\square$

Proof of Theorem 9.1. Obviously, we have

$$|\rho_1 - \rho_2| = \big||z_1| - |z_2|\big| \leqslant |z_1 - z_2|$$

Denote by $\mathfrak{X}, \Upsilon \subset \mathbb{C}_r^R$ the integral arcs which intersect the unit circle at the points $a = e^{i\theta_1}$ and $b = e^{i\theta_2}$, respectively. Thus $z_1 \in \mathfrak{X}$ and $z_2 \in \Upsilon$. By Lemma 9.2, we have

$$\left|e^{i\theta_1} - e^{i\theta_2}\right| = |a - b| \leqslant C_r^R \operatorname{dist}(\mathfrak{X}, \Upsilon) \leqslant C_r^R|z_1 - z_2|$$

These two inequalities prove the estimate in the right hand side of (9.1). For the opposite estimate we choose two points $z_1 = (\rho_1, e^{i\theta_1}) \in \mathfrak{X}$ and $z_2 = (\rho_2, e^{i\theta_2}) \in \Upsilon$, where $r \leqslant \rho_1, \rho_2 \leqslant R$. Define $a = z_1 = (\rho_1, e^{i\theta_1}) \in \mathfrak{X}$ and $b = (\rho_1, e^{i\theta_2}) \in \Upsilon$. These are points of the same distance from the origin, $|a| = |b| = \rho_1$. By Lemma 9.2, we have

$$|z_1 - b| = |a - b| \leqslant C_r^R \operatorname{dist}(\mathfrak{X}, \Upsilon) \leqslant C_r^R \left| e^{i\theta_1} - e^{i\theta_2} \right|$$

On the other hand, b and z_2 belong to the same integral arc in $\mathbb{C}_r^R$, so by the reverse triangle inequality (4.3)

$$|b - z_2| \leqslant C_r^R \big| |b| - |z_2| \big| = C_r^R \, |\rho_1 - \rho_2|$$

Summing up the above inequalities we obtain

$$\begin{aligned} |z_1 - z_2| &\leqslant |z_1 - b| + |b - z_2| \leqslant C_r^R \left(|\rho_1 - \rho_2| + \left| e^{i\theta_1} - e^{i\theta_2} \right| \right) \\ &\leqslant C_r^R \left| \rho_1 e^{i\theta_1} - \rho_2 e^{i\theta_2} \right| \end{aligned}$$

as claimed. Theorem 9.1 is proved. □

Inequality 1.15 tells us that Φ and its inverse, denoted by $\Psi = \Phi^{-1} \colon \mathbb{C} \to \mathbb{C}$, are locally Lipschitz on $\mathbb{C}_0$. Therefore, both Φ and Ψ are differentiable almost everywhere.

10. Polar equation for integral curve Γ_θ

The points $z \in \Gamma_\theta$ can be represented by the equation

$$z = \rho e^{i\varphi(\rho)}\,, \qquad 0 < \rho < \infty \tag{10.1}$$

where φ solves the initial value problem

$$\begin{cases} \dot{\varphi}(\rho) = F(\rho, e^{i\varphi}) = \dfrac{1}{\rho} \dfrac{\operatorname{Im} e^{-i\varphi} f\left(\rho e^{i\varphi}\right)}{\operatorname{Re} e^{-i\varphi} f\left(\rho e^{i\varphi}\right)} \\ \varphi(1) = \theta \end{cases} \tag{10.2}$$

The scalar function $F : \mathbb{R}_+ \times \mathbb{S}^1 \to \mathbb{R}$ can be found as follows

$$\begin{aligned} f(z) &= \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\rho} \cdot \frac{\mathrm{d}\rho}{\mathrm{d}t} = e^{i\varphi} \left(1 + i\rho\dot{\varphi}\right) \cdot \frac{\mathrm{d}|z(t)|}{\mathrm{d}t} \\ &= \left(1 + i\rho\dot{\varphi}\right) z \cdot \operatorname{Re} \frac{f(z)}{z} \end{aligned}$$

Hence,

$$\rho\, \dot{\varphi}(\rho) = \frac{\operatorname{Im} \frac{f(z)}{z}}{\operatorname{Re} \frac{f(z)}{z}}$$

and

$$F(\rho, e^{i\varphi}) = \frac{1}{\rho} \frac{\operatorname{Im} e^{-i\varphi} f\left(\rho e^{i\varphi}\right)}{\operatorname{Re} e^{-i\varphi} f\left(\rho e^{i\varphi}\right)} \tag{10.3}$$

The single equation just established for φ is no longer autonomous. But it can be useful for a discussion of geometric properties of the integral curves.

11. Integrating factor, Proof of Theorem 1.9

Every complex number $z \neq 0$ has infinitely many quasipolar angles which differ from each other by multiple of 2π. These are real numbers $\theta \in \mathbb{R}$ such that the integral curve through the point $e^{i\theta} \in \mathbb{S}^1$ contains z. We denote by $\Theta = \Theta(z)$ the multivalent function that assigns to z its quasipolar angles. It is worth pointing out that $\Theta(z)$ has a continuous branch on every simply connected domain $\Omega \subset \mathbb{C}_0$. Two such branches differ by a constant. In other words, it makes sense to speak of the gradient of Θ,

$$\nabla\Theta(z) = \left(\frac{\partial\Theta}{\partial x}\,,\,\frac{\partial\Theta}{\partial y}\right)$$

where we have chosen a continuous branch of Θ near the given point $z \in \mathbb{C}_0$. By the definition of the map $\Phi\colon \mathbb{C}_0 \to \mathbb{C}_0$, we have the identity

$$e^{i\Theta} = \frac{\Phi(z)}{|\Phi(z)|}$$

Inequality (1.15) shows that for almost every $z \in \mathbb{C}_0$

$$0 < \mathfrak{m}\big(|z|\big) \leqslant |\nabla\Theta(z)| \leqslant \mathfrak{M}\big(|z|\big) < \infty \tag{11.1}$$

Here the lower and upper bounds are given by continuous functions. Of course these functions blow up at the critical point $z = 0$ and at $z = \infty$. Any continuous branch of Θ along an integral curve Γ_θ is constant, namely $\Theta \equiv \theta + 2\pi k$. We differentiate $\Theta(z)$ along Γ_θ to obtain

$$f(z)\,\Theta_z + \overline{f(z)}\,\Theta_{\bar z} = 0 \qquad \text{where } \Theta_{\bar z} = \overline{\Theta_z}$$

Hence

$$if(z) = \pm\frac{|f|}{|\Theta_z|}\Theta_{\bar z} \tag{11.2}$$

We then see that the vector field

$$V(x,y) = if(x+iy) \tag{11.3}$$

takes the form

$$V(x,y) = \lambda\nabla\Theta \qquad \text{with } \lambda(z) = \pm\frac{|f(z)|}{2|\Theta_z(z)|} \tag{11.4}$$

It is not difficult to see that the sign is plus. Indeed, since Θ is increasing in the direction counterclockwise on every circle $\mathbb{S}_\rho = \{\rho e^{i\theta}\colon 0 \leqslant \theta \leqslant 2\pi\}$, we have

$$0 \leqslant \frac{\partial\Theta(z)}{\partial\theta} = 2|z|^2\,\mathrm{Im}\,\frac{\Theta_{\bar z}}{z}$$

On the other hand

$$0 \leqslant \mathrm{Re}\,\frac{f(z)}{z} = \mathrm{Im}\,\frac{if(z)}{z} = \pm\frac{|f|}{|\Theta_z|}\,\mathrm{Im}\,\frac{\Theta_{\bar z}}{z}$$

Thus $\lambda(z) > 0$, almost everywhere. From (11.2) we deduce that the real valued function λ, called the integrating factor, is uniformly bounded from below and from above

$$m\big(|z|\big) \leqslant \lambda(z) \leqslant M\big(|z|\big) \tag{11.5}$$

where $m(t)$ and $M(t)$ are positive continuous function in $\mathbb{R}_+$. Equation (11.4) simply means that f is orthogonal to the gradient field $\nabla\Theta$.

12. Remarks

Remark 12.1. As one moves along a trajectory of a field $f \in \mathcal{F}_K(d)$ in the positive direction its distance $|x(t)|$ to the critical point strictly increases to ∞. It is a simple matter to see that the length of the tangent vector $|\dot{x}(t)|$ also increases. Indeed, for sufficiently small h we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|x(t+h)-x(t)|^2 = 2\langle \dot{x}(t+h)-\dot{x}(t), x(t+h)-x(t)\rangle \geqslant 0$$

Hence

$$|x(t+h)-x(t)| \geqslant |x(s+h)-x(s)|\,, \quad \text{for } t \geqslant 0$$

We divide by h and let it go to zero to conclude that

$$|\dot{x}(t)| \geqslant |\dot{x}(s)|$$

or, equivalently

$$|f(x(t))| \geqslant |f(x(s))|$$

Remark 12.2. It is interesting to note that the orthogonal trajectories, locally defined by the autonomous system

$$\dot{w}(t) = i f\big(w(t)\big)\,, \qquad \alpha < t < \beta \tag{12.1}$$

have well defined curvature at almost every $t \in (\alpha, \beta)$, and it is nonnegative. To carry out the details of this observation we call on the classical formula for the curvature of C^2-simple arc $w = w(t)$

$$k = \operatorname{Im} \frac{\ddot{w}\,\overline{\dot{w}}}{|\dot{w}|^3} = \frac{1}{|\dot{w}|} \operatorname{Im} \frac{\ddot{w}}{\dot{w}} = \frac{1}{|\dot{w}|} \frac{\mathrm{d}}{\mathrm{d}t}\,[\arg \dot{w}] \tag{12.2}$$

On the other hand, when $w = w(t)$ is the orthogonal trajectory of f, we may perform the following computation

$$\operatorname{Re} \frac{f\big(w(t+h)\big) - f\big(w(t)\big)}{w(t+h)-w(t)} \geqslant 0$$

by monotonicity of f. This translates into the inequality

$$\operatorname{Im} \frac{\dot{w}(t+h)-\dot{w}(t)}{w(t+h)-w(t)} \geqslant 0$$

or, equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} \arg\,[w(t+h)-w(t)] \geqslant 0$$

This means that the function $t \to \arg[w(t+h)-w(t)]$ is increasing, so we may write

$$\arg \frac{w(t+h)-w(t)}{h} \geqslant \arg \frac{w(s+h)-w(s)}{h}$$

for $t \geqslant s$ and h sufficiently small. Letting h go to zero we arrive at the inequality

$$\arg \dot{z}(t) \geqslant \arg \dot{z}(s)\,, \qquad t \geqslant s$$

This shows that the function $t \to \arg \dot{z}(t)$ is increasing and, as such, has a nonnegative derivative at almost every point $t \in (\alpha, \beta)$. Now we can define the curvature by the rule

$$k = \frac{1}{|\dot{w}(t)|} \frac{\mathrm{d}}{\mathrm{d}t}\,[\arg \dot{w}(t)] \geqslant 0 \tag{12.3}$$

or

$$k = \frac{1}{|f(w)|} \frac{\mathrm{d}}{\mathrm{d}t} \left[\arg f(w)\right] \tag{12.4}$$

13. Variation of δ-monotone mappings along C^1-arcs, proof of Theorem 1.10

Proof. Let $\Gamma = \{x(t) \colon -\epsilon < t < \epsilon\}$, where $x \colon (-\epsilon, \epsilon) \to \Omega$ is a C^1-parametrization of Γ. Here we assume that $\dot{x}(0) \neq 0$. In particular, $x(t) \neq x(s)$ whenever the parameters $t \neq s$ are close to 0; that is, we assume that ϵ is sufficiently small. We shall construct a strictly increasing homeomorphism $\tau \colon (-\eta, \eta) \xrightarrow{\text{into}} (-\epsilon, \epsilon)$, $\tau(0) = 0$, such that the function $s \to f\big(x(\tau(s))\big)$ becomes Lipschitz continuous. Obviously, this is enough to claim that $f(\Gamma)$ is rectifiable near the given point $f\big(x(0)\big)$. By means of translation of Ω and $f(\Omega)$ we may (and do) assume that $x(0) = 0$ and $f(0) = 0$. Consider two parameters in $(-\epsilon, \epsilon)$, say $-\epsilon < s < t < \epsilon$. In view of δ-monotonicity, we have

$$\left\langle f\big(x(t)\big) - f\big(x(s)\big), \frac{x(t) - x(s)}{|x(t) - x(s)|} \right\rangle \geqslant \delta \left| f\big(x(t)\big) - f\big(x(s)\big) \right| \tag{13.1}$$

On the other hand, since $x \in C^1(-\epsilon, \epsilon)$ we also have

$$\lim_{t,s \to 0} \left| \frac{x(t) - x(s)}{|x(t) - x(s)|} - \frac{\dot{x}(0)}{|\dot{x}(0)|} \right| = 0 \tag{13.2}$$

In particular, we find an interval $(-\epsilon', \epsilon') \subset (-\epsilon, \epsilon)$ such that

$$\left| \frac{x(t) - x(s)}{|x(t) - x(s)|} - \frac{\dot{x}(0)}{|\dot{x}(0)|} \right| \leqslant \frac{\delta}{2} \tag{13.3}$$

for all $s, t \in (-\epsilon', \epsilon')$. This condition together with (13.1) yields

$$\left\langle f\big(x(t)\big) - f\big(x(s)\big), \frac{\dot{x}(0)}{|\dot{x}(0)|} \right\rangle \geqslant \frac{\delta}{2} \left| f\big(x(t)\big) - f\big(x(s)\big) \right| > 0 \tag{13.4}$$

It shows that the function $\varphi(t) = \left\langle f\big(x(t)\big), \frac{\dot{x}(0)}{|\dot{x}(0)|} \right\rangle$ is strictly increasing for $\epsilon' < t < \epsilon'$. It vanishes at $t = 0$. The image of the interval $(-\epsilon', \epsilon')$ under φ covers an interval $(-\eta, \eta)$. Let $\tau = \tau(s)$ denote the inverse of φ, defined for $-\eta < s < \eta$. By the definition

$$\left\langle f\big(x(\tau(s))\big), \frac{\dot{x}(0)}{|\dot{x}(0)|} \right\rangle = s\,, \qquad -\eta < s < \eta \tag{13.5}$$

It is this function $\tau(s)$ that gives us a Lipschitz parametrization of $f(\Gamma)$. We set

$$y(s) = f\big(x(\tau(s))\big) \in f(\Gamma)\,, \qquad -\eta < s < \eta \tag{13.6}$$

Then y satisfies a Lipschitz condition with constant $\frac{2}{\delta}$. Indeed, for $s_1 > s_2$, in view of (13.4), we have

$$\begin{aligned}
|y(s_1) - y(s_2)| &= \left| f\big(x(\tau(s_1))\big) - f\big(x(\tau(s_2))\big) \right| \\
&\leqslant \frac{2}{\delta} \left\langle f\big(x(\tau(s_1))\big) - f\big(x(\tau(s_2))\big), \frac{\dot{x}(0)}{|\dot{x}(0)|} \right\rangle \\
&= \frac{2}{\delta}(s_1 - s_2)
\end{aligned}$$

as derired. $\square$

14. Quasiconformal fields of bounded variation on C^1-curves, proof of Theorem 1.11

Proof. It suffices to prove forward uniqueness. The backward uniqueness follows by considering the field $-f$. We choose $\epsilon > 0$ small enough to satisfy

$$\big|f\big(x(t)\big) - f(a)\big| + \big|f\big(y(t)\big) - f(a)\big| \leqslant \frac{1}{2}\min\{1, |f(a)|\} \tag{14.1}$$

for all $0 \leqslant t \leqslant \epsilon$. Suppose, to the contrary, that for some $0 < t_0 < \epsilon$ we have $x(t_0) \neq y(t_0)$. Consider the sequence $t_0 > t_1 > \cdots t_k > t_{k+1} \cdots t_\infty \geqslant 0$ constructed in Partition Lemma 6.1. Accordingly, we have

$$\inf_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - x(s)| = t_k - t_{k+1}\,, \quad x(t_\infty) = y(t_\infty) \tag{14.2}$$

Let us denote by $x_k = x(t_k)$ and $y_k = y(t_k)$. For each k we compute

$$\begin{aligned}
\log \frac{|x_k - y_k|}{|x_{k+1} - y_{k+1}|} &= \int_{t_{k+1}}^{t_k} \frac{\mathrm{d}}{\mathrm{d}t} \log |x(t) - y(t)|\, \mathrm{d}t \\
&\leqslant \int_{t_{k+1}}^{t_k} \frac{|\dot{x}(t) - \dot{y}(t)|}{|x(t) - y(t)|}\, \mathrm{d}t = \int_{t_{k+1}}^{t_k} \frac{\big|f\big(x(t)\big) - f\big(y(t)\big)\big|}{|x(t) - y(t)|}\, \mathrm{d}t \\
&\leqslant \frac{1}{t_k - t_{k+1}} \int_{t_{k+1}}^{t_k} \big|f\big(x(t)\big) - f\big(y(t)\big)\big|\, \mathrm{d}t
\end{aligned} \tag{14.3}$$

We claim that

$$\big|f\big(x(t)\big) - f\big(y(t)\big)\big| \leqslant C\,|f(x_k) - f(x_{k+1})|\,, \quad t_{k+1} \leqslant t \leqslant t_k \tag{14.4}$$

with a constant independent of k. To this end, observe that the expressions

$$|\dot{x}(t)| = |f(x(t))| \quad \text{and} \quad |\dot{y}(t)| = |f(y(t))|$$

are bounded by $\frac{3}{2}|f(a)|$. Hence

$$\begin{aligned}
\sup_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - y(s)| &\leqslant \inf_{t_{k+1} \leqslant t, s \leqslant t_k} |x(t) - y(s)| + 3|f(a)||t_k - t_{k+1}| \\
&= (1 + 3|f(a)|)\,(t_k - t_{k+1})
\end{aligned}$$

Also $|x_k - y(t)| \geqslant t_k - t_{k+1}$, by (14.2).

Next we see that

$$x_{k+1} - x_k = \int_{t_{k+1}}^{t_k} \big[f\big(x(\tau)\big) - f(a)\big]\, \mathrm{d}\tau + (t_k - t_{k+1}) f(a)$$

Hence

$$\begin{aligned}
|x_{k+1} - x_k| &\geqslant (t_k - t_{k+1})|f(a)| - \frac{1}{2}|f(a)|(t_k - t_{k+1}) \\
&= \frac{1}{2}(t_k - t_{k+1})|f(a)|
\end{aligned}$$

Now, the three point condition yields

$$\begin{aligned}
\frac{\big|f\big(x(t)\big)-f\big(y(t)\big)\big|}{|f(x_k)-f(x_{k+1})|} &\leqslant \frac{\big|f\big(x(t)\big)-f\big(y(t)\big)\big|}{\big|f(x_k)-f\big(y(t)\big)\big|}\cdot\frac{\big|f(x_k)-f\big(y(t)\big)\big|}{|f(x_k)-f(x_{k+1})|}\\
&\leqslant M_K\left(\frac{|x(t)-y(t)|}{|x_k-y(t)|}\right)\cdot M_K\left(\frac{|x_k-y(t)|}{|x_k-x_{k+1})|}\right)\\
&\leqslant M_K\left(1+3|f(a)|\right)\cdot M_K\left(\frac{2}{|f(a)|}+6\right)\\
&= C
\end{aligned}$$

This proves (14.4).

We now substitute (14.4) into (14.3) to obtain

$$\log\frac{|x_k-y_k|}{|x_{k+1}-y_{k+1}|}\leqslant C\;|f(x_{k+1})-f(x_k)| \tag{14.5}$$

Using telescoping structure on the left hand side we compute

$$\log\frac{|x_0-y_0|}{|x_\ell-y_\ell|}\leqslant C\sum_{k=0}^{\ell}|f(x_{k+1})-f(x_k)| \tag{14.6}$$

Finally, letting ℓ go to infinity we see that the left hand side approaches ∞, because

$$x_\ell-y_\ell\to x(t_\infty)-y(t_\infty)=0$$

However, the right hand side is bounded by the total variation of f along the C^1-curve $x=x(t)$. This contradiction completes the proof of Theorem 1.11. □

15. Examples

Example 15.1. Consider the complex function

$$f(z)=\frac{10z}{\sqrt{|z|}}\,,\quad z=x_1+ix_2 \tag{15.1}$$

It satisfies the reduced Beltrami equation

$$f_{\bar z}=\mu(z)\,\mathrm{Re}\,f_z\,,\quad \mu(z)=-\frac{1}{3}\,\frac{z}{\bar z} \tag{15.2}$$

and is δ-monotone. Nevertheless, there are two integral curves passing through the origin

$$z^{\pm}(t)=\begin{cases}(24\pm7i)\,t^2 & \text{if}\quad t\geqslant 0\\ 0 & \text{if}\quad t\leqslant 0\end{cases} \tag{15.3}$$

Note that f is Hölder continuous with exponent $\alpha=\frac{1}{2}$.

Example 15.2. It is not difficult to construct a (nondegenerate) reduced K-quasiconformal field f for which any factorization of the form

$$i\,f(z)=\lambda(z)\nabla U(z)\,,\quad \lambda(z)\in\mathbb{R}\quad U\text{-locally Lipschitz in } C_0$$

does not allow λ to be continuous, equivalently U to be C^1-smooth. Set

$$f(z)=\begin{cases}2z & \mathrm{Im}\,z\geqslant 0\\ 3z-\bar z & \mathrm{Im}\,z\leqslant 0\end{cases} \tag{15.4}$$

Indeed, U must be constant on every integral curve of f, among which are half lines

$$y = cx \quad c \geqslant 0 \quad x \geqslant 0$$

and half-parabolas

$$y = cx^2 \quad c \leqslant 0 \quad x \geqslant 0$$

This forces U to be of the form $U(x, y) = A(y/x)$ in the first quadrant and $U(x, y) = B(y/x^2)$ in the fourth quadrant. It follows that $\lim\limits_{y\to 0+} U_y(x, y) = a/x$ and $\lim\limits_{y\to 0-} U_y(x, y) = b/x^2$ where a and b are nonzero constants because $U_x(x, 0) \equiv 0$. This contradicts the smoothness of U.

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Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: tiwaniec@syr.edu

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: lvkovale@syr.edu

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: jkonnine@syr.edu