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A Critique of the Link Approach to Exact Lattice Supersymmetry

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We examine the link approach to constructing a lattice theory of $\mathcal{N} = 2$ super Yang Mills theory in two dimensions. The goal of this construction is to provide a discretization of the continuum theory which preserves all supersymmetries at non-zero lattice spacing. We show that this approach suffers from an inconsistency and argue that a maximum of just one of the supersymmetries can be implemented on the lattice.

I. INTRODUCTION

For the study of non-perturbative effects in supersymmetric field theories, important for e.g. supersymmetry breaking, it is very useful to have lattice formulations of these theories. Such formulations allow one to do computer simulations. From a practical point of view the main problem that arises when one tries to write down such lattice theories lies in the failure of the ordinary product rule of differentiation, as explained in e.g. [1]. This generically leads to lattice actions which classically break all the supersymmetries of the continuum theory. This in turn renders recovering a supersymmetric continuum limit problematic – typically the lattice action must be supplemented by the addition of large numbers of relevant supersymmetry violating terms whose couplings must be carefully fine tuned as the lattice spacing is reduced.

In light of this a couple of recent approaches to the problem of lattice supersymmetry attempt to preserve a fraction of the original supersymmetry exactly at non-zero lattice spacing [2, 3, 4]. In one approach a supersymmetric lattice action is constructed by orbifolding a supersymmetric matrix model while the other proceeds by finding combinations of the original supercharges which behave like internal symmetries and hence can be transferred to the lattice. While these two constructions appear quite different, they have recently been shown to be intimately related and to correspond to discretizations of *twisted* formulations of the continuum supersymmetric theories [5].

In [6] an alternative discretization of these twisted theories was described, which aimed at repairing the product rule of differentiation, the Leibniz rule, by introducing a non-commutativity between the bosonic and fermionic coordinates of twisted superspace. The claim is that lattice theories constructed in this way exhibit invariance under the *full* set of continuum supersymmetries. How-

ever, as explained in [7], unfortunately this approach suffers from an inconsistency.

Another approach related to these non-commutative lattice formulations is the link approach, which was introduced by the same authors in [8]. Because the paper [7] never addressed the link approach, a certain amount of confusion and discussion has arisen as to whether this approach is also inconsistent or not. In this paper we describe this construction and show that a similar problem does indeed arise in this case, too.

In the following section a summary of the non-commutativity approach and its inconsistency is given. The third section describes the link approach, and the fourth section highlights an apparent inconsistency inherent also in that construction. The paper ends with conclusions and some discussion.

II. THE NC APPROACH

Instead of having derivative operators, on the lattice one has to deal with difference operators. For example, for the forward and backward difference operator, acting on functions f on a lattice with coordinates x^μ as

$$\Delta_{\pm\mu} f(\mathbf{x}) = \pm \frac{1}{|\mathbf{n}_\mu|} \left(f(\mathbf{x} \pm \mathbf{n}_\mu) - f(\mathbf{x}) \right), \qquad (1)$$

(where $\mathbf{n}_\mu$ corresponds to the shift of one lattice spacing in the μ-direction) the following product rule holds

$$
\begin{aligned}
\Delta_{\pm\mu} \left[f_1(\mathbf{x}) f_2(\mathbf{x}) \right] &= \left[\Delta_{\pm\mu} f_1(\mathbf{x}) \right] f_2(\mathbf{x}) \\
&\quad + f_1(\mathbf{x} \pm \mathbf{n}^\mu) \left[\Delta_{\pm\mu} f_2(\mathbf{x}) \right].
\end{aligned}
\qquad (2)
$$

This differs from the ordinary Leibniz rule by a shift in the argument of one of the functions.

The non-commutativity (NC) approach, as introduced in [6], can now be explained as follows. For writing down supersymmetric lattice theories, the real necessity for having the Leibniz rule lies in having this rule

hold for the supersymmetry variations, not just the difference operator. The main idea is to introduce a non-commutativity between the bosonic and fermionic coordinates in a superspace representation, such that the combinations $\delta_A = \epsilon^A Q_A$ (no sum), where the Q_A are the supercharges and the ϵ^A Grassmannian susy parameters, obey the Leibniz rule.

Ignoring the Lorentz transformations, the general supersymmetry algebra can be written as

$$\{Q_A, Q_B\} = f_{AB}^\mu \Delta_{\pm\mu}, \quad [Q_A, \Delta_{\pm\mu}] = 0. \quad (3)$$

As explained in [6, 7] the supersymmetry variations δ_A will obey the Leibniz rule if the following non-commutativities between the coordinates of superspace and the susy parameters are introduced

$$[\mathbf{x}, \theta^A] = \mathbf{a}_A \theta^A, \quad [\mathbf{x}, \epsilon^A] = \mathbf{a}_A \epsilon^A \quad \text{(no sum)}. \quad (4)$$

This can only be done consistently if the following equations hold:

$$\mathbf{a}_A + \mathbf{a}_B = \pm\mathbf{n}_\mu \text{ for } f_{AB}^\mu \neq 0. \quad (5)$$

The a_A are called the shift parameters.

D'Adda et al. have found a solution for these relations for the twisted $\mathcal{N} = D = 2$ and twisted $\mathcal{N} = D = 4$ supersymmetry algebras. As explained in [7], these relations can also be satisfied for the $\mathcal{N} = 2$ supersymmetry algebra in one dimension, supersymmetric quantum mechanics.

Let's look in more detail at the $\mathcal{N} = D = 2$ case. Reformulating the algebra by twisting [6], it reads

$$\{Q, Q_\mu\} = i\Delta_{+\mu}, \qquad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}, \quad (6)$$

where only the non-vanishing anticommutators are shown. Introducing the supercoordinates $\theta, \tilde{\theta}, \theta^\mu$, the supercharges read

$$\begin{aligned} Q &= \frac{\partial}{\partial\theta} + \frac{i}{2}\theta^\mu \Delta_{+\mu}, \\ \tilde{Q} &= \frac{\partial}{\partial\tilde{\theta}} - \frac{i}{2}\epsilon_{\mu\nu}\theta^\mu \Delta_{-\nu}, \\ Q_\mu &= \frac{\partial}{\partial\theta_\mu} + \frac{i}{2}\theta\Delta_{+\mu} - \frac{i}{2}\epsilon_{\mu\nu}\tilde{\theta}\Delta_{-\nu}. \end{aligned} \quad (7)$$

Demanding that independently ϵQ, $\tilde{\epsilon}\tilde{Q}$, $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ satisfy the Leibniz rule, one is lead to the following non-commutativities

$$[x^\mu, \theta] = a\theta, [x^\mu, \tilde{\theta}] = \tilde{a}\tilde{\theta}, [x^\mu, \theta^\nu] = a^\nu\theta^\nu \text{ (no sum), } (8)$$

and similarly for the ϵ_A. For consistency the shift parameters have to satisfy the following conditions:

$$a + a_\mu = n_\mu, \quad \tilde{a} + a_\mu = -|\epsilon_{\mu\nu}|n_\nu,$$
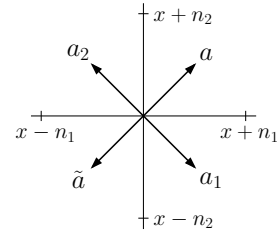
$$a + \tilde{a} + a_1 + a_2 = 0. \quad (9)$$

FIG. 1: Symmetric choice of shift parameter a_A for twisted $\mathcal{N} = D = 2$.

These conditions have a symmetric solution

$$a = -\tilde{a} = (1/2, 1/2), \quad a_1 = -a_2 = (1/2, -1/2), \quad (10)$$

as shown in figure 1. Working with this solution one is thus forced to introduce new lattice points at the half integer lattice sites, effectively doubling the lattice. Actually, one of the parameters is left undetermined by the equations (9), leading to a linear space of solutions. Another interesting solution is given by

$$a = (0,0), \, \tilde{a} = (-1,-1), \, a_1 = (1,0), \, a_2 = (0,1), \quad (11)$$

where one of the shift parameters vanishes and all others point to already existing lattice points.

However, the non-commutativity approach is plagued by an inconsistency. This inconsistency holds in general, as explained in [7]. Since the $\epsilon^A Q_A$ (no sum) obey the Leibniz rule, also the $\epsilon^A s_A$ will obey the Leibniz rule, where the s_A denote the supersymmetry transformations of the component fields. It thus holds that

$$\begin{aligned} &\epsilon^A s_A [f_1(\mathbf{x})f_2(\mathbf{x})] \\ &= [\epsilon^A s_A f_1(\mathbf{x})] f_2(\mathbf{x}) + f_1(\mathbf{x}) [\epsilon^A s_A f_2(\mathbf{x})] , \end{aligned} \quad (12)$$

for any two component fields f_1 and f_2. Changing the order in the product, it just as well holds that

$$\begin{aligned} &\epsilon^A s_A [f_1(\mathbf{x})f_2(\mathbf{x})] \quad (13) \\ &= \epsilon^A s_A \left[(-1)^{|f_1||f_2|} f_2(\mathbf{x})f_1(\mathbf{x}) \right] \\ &= (-1)^{|f_1||f_2|} \left([\epsilon^A s_A f_2(\mathbf{x})] f_1(\mathbf{x}) + f_2(\mathbf{x}) [\epsilon^A s_A f_1(\mathbf{x})] \right) \\ &= f_1(\mathbf{x} - \mathbf{a}_A) [\epsilon^A s_A f_2(\mathbf{x})] + [\epsilon^A s_A f_1(\mathbf{x})] f_2(\mathbf{x} + \mathbf{a}_A), \end{aligned}$$

where the last equality follows due to the non-commutativity. Since the two expressions for the susy transformation of $f_1 f_2$ are not equal, the action of susy transformations on products is not well defined in the NC approach.

III. THE LINK APPROACH

A. General Ideas

Inspired by the NC approach the same authors have proposed a novel link formulation of twisted lattice supersymmetry in [8]. This construction again has the

merit of appearing to retain *all* continuum supersymmetries on the lattice. Instead of thinking in terms of non-commuting supercoordinates x^μ and θ^A, the θ^A are given a link variable interpretation. They now describe constant link variables associated to the links $(x, x+a_A)$, θ^A being their common value:

$$x \,\bullet\!\!\xrightarrow{}\!\!\bullet\, x+a_A \qquad (14)$$
$$\theta^A$$

Objects on the right of θ_A are now forced to be at $x+a_A$, objects on the left at x, thereby *replacing* the non-commutativity relation

$$[\mathbf{x}, \theta^A] = \mathbf{a}_A \theta^A \Leftrightarrow \mathbf{x}\theta^A = \theta^A(\mathbf{x} + \mathbf{a}_A) \quad \text{(no sum). (15)}$$

In a similar way the $\frac{\partial}{\partial \theta^A}$ are associated to constant link variables and can be viewed as the same link variable as θ^A, but with opposite link orientation. The standard algebraic properties of the θ^A and $\frac{\partial}{\partial \theta^A}$ variables are now interpreted as link relations, for instance $\theta^A \theta^B = -\theta^B \theta^A$ reads

$$\theta^A_{x+a_A+a_B, x+a_B} \theta^B_{x+a_B, x} = -\theta^B_{x+a_B+a_A, x+a_A} \theta^A_{x+a_A, x}. (16)$$

Also the difference operator $\Delta_{\pm\mu}$ gets the interpretation of a constant link variable, on the link $(x+n_\mu, x)$. It thus follows that also the supercharges have a link interpretation. For example, in case of the twisted $\mathcal{N} = D = 2$ algebra, the supercharge Q reads

$$Q_{x,x-a} = \frac{\partial}{\partial \theta}_{x,x-a} + \frac{i}{2}(\theta^\mu)_{x,x+a_\mu}(\Delta_{+\mu})_{x+a_\mu, x-a}, \ (17)$$

and the algebra itself reads

$$
\begin{aligned}
Q_{x+n_\mu, x+a_\mu}&(Q_\mu)_{x+a_\mu, x} + (Q_\mu)_{x+n_\mu, x+a} Q_{x+a, x} \\
&= i(\Delta_\mu)_{x+n_\mu, x}, \\
\tilde{Q}_{x, x+\tilde{a}}&(Q_\mu)_{x+\tilde{a}, x+\tilde{a}+a_\mu} + (Q_\mu)_{x, x+a_\mu} \tilde{Q}_{x+a_\mu, x+a_\mu+\tilde{a}} \\
&= -i\epsilon_{\mu\nu}(\Delta_{-\nu})_{x-n_\nu, x}. \quad (18)
\end{aligned}
$$

The consistency conditions for the link approach are the same as for the NC approach, namely equation (9).

Unfortunately, a full consistent set of calculation rules for the link approach has never been presented. For instance, it is not clear how to consistently multiply two superfields. This should be contrasted with the non-commutative constructions where the calculation rules are the usual ones plus a non-commutativity between the bosonic and fermionic coordinates of superspace.

However, working on a component field level we will nevertheless show that it is possible to define a Leibniz rule for taking susy variations of products of fields which allows one to construct lattice actions which are formally invariant under the twisted lattice supersymmetries. The supersymmetry transformations s_A are treated as link variables, like the supercharges Q_A. All component fields $\varphi_{x, x+a_\varphi}$ are treated as link fields, albeit including the degenerate case $a_\varphi = 0$ in which case we speak of a site field. A supersymmetry transformation will thus map a site or a link field onto another site or link field in such a way that the supersymmetry algebra is preserved. The lattice action is then built out of the site and link component fields.

Since the link approach incorporates link fields naturally into the discretization of a theory, it forms an excellent playground for lattice gauge theory where gauge fields A_μ are put on the lattice as link fields $U_{n_\mu} \simeq e^{iA_\mu}$.

In [8] the lattice formulation of twisted $\mathcal{N} = D = 2$ Super Yang-Mills theory is treated precisely along the lines described above. This is discussed in more detail in the next section.

B. Twisted $\mathcal{N} = D = 2$ SYM on the Lattice

In this section the discretization of twisted $\mathcal{N} = D = 2$ SYM following the link approach is summarized.

To gauge the twisted $\mathcal{N} = D = 2$ lattice theory introduced before, the constant link variables $\Delta_\pm$ and Q_A are replaced with corresponding gauge degrees of freedom[1]:

$$(\Delta_{\pm\mu})_{x\pm n_\mu, x} \rightarrow \mp(\mathcal{U}_{\pm\mu})_{x, x\pm n_\mu}, \qquad (19)$$
$$(Q_A)_{x, x-a_A} \rightarrow (\nabla_A)_{x, x+a_A}. \qquad (20)$$

The link fields $(\mathcal{U}_{\pm\mu})$ and ∇_A are x (or better link) dependent elements of the gauge group, just like gauge links in ordinary lattice gauge theory[2]. The gauge transformation of these link variables are given by

$$(\mathcal{U}_{\pm\mu})_{x, x\pm n_\mu} \rightarrow G_x (\mathcal{U}_{\pm\mu})_{x, x\pm n_\mu} G^{-1}_{x\pm n_\mu}, \qquad (21)$$
$$(\nabla_A)_{x, x+a_A} \rightarrow G_x (\nabla_A)_{x, x+a_A} G^{-1}_{x+a_A}, \qquad (22)$$

where G_x denotes the finite gauge transformation at the site x.

The following twisted $\mathcal{N} = D = 2$ susy constraints follow from 'gauging' the susy algebra (6),

$$\{\nabla, \nabla_\mu\}_{x, x+a+a_\mu} = +i(\mathcal{U}_{+\mu})_{x, x+n_\mu}, \qquad (23)$$
$$\{\tilde{\nabla}, \nabla_\mu\}_{x, x+\tilde{a}+a_\mu} = -i\epsilon_{\mu\nu} (\mathcal{U}_{-\nu})_{x, x-n_\nu}, \qquad (24)$$

where the left hand side of (23)–(24) should be understood as link anti-commutators, as in (16). The equations (9) are obviously crucial for consistency.

Playing with Jacobi identities of the link matrices one can see that one may consistently define the following

---

non-vanishing fermionic link fields:

$$[\nabla_\mu, \mathcal{U}_{+\nu}]_{x,x+a_\mu+n_\nu} \equiv -\epsilon_{\mu\nu}(\tilde\rho)_{x,x-\tilde a}, \tag{25}$$

$$[\nabla_\mu, \mathcal{U}_{-\nu}]_{x,x+a_\mu-n_\nu} \equiv -\delta_{\mu\nu}(\rho)_{x,x-a}, \tag{26}$$

$$\epsilon_{\mu\nu}[\nabla, \mathcal{U}_{-\nu}]_{x,x+a-n_\nu} \equiv -\epsilon_{\mu\nu}(\lambda_\nu)_{x,x-a_\nu} \tag{27}$$

$$= -[\tilde\nabla, \mathcal{U}_{+\mu}]_{x,x+\tilde a+n_\mu}, \tag{28}$$

which are then $\mathcal{N} = 2$ twisted fermions. The full twisted $\mathcal{N} = 2$ multiplet on the lattice is given by

$$(\mathcal{U}_\pm, \rho, \tilde\rho, \lambda_\mu, K), \tag{29}$$

where

$$K_{x,x} \equiv \frac{1}{2}\{\nabla_\mu, \lambda_\mu\}_{x,x} \tag{30}$$

is an auxiliary (site) field, needed for the supersymmetry algebra to close off-shell. By construction all the link and site fields in the multiplet are elements of the gauge group and transform in the same way under gauge transformations as the $\mathcal{U}_\pm$ and ∇_A. The fields $\mathcal{U}_\pm$ and K are bosonic, the others are fermionic.

Supersymmetry transformations of link fields are defined by

$$(s_A\varphi)_{x,x+a_\varphi+a_A} = s_A(\varphi)_{x,x+a_\varphi} \equiv [\nabla_A, \varphi\}_{x,x+a_\varphi+a_A} \tag{31}$$

where $(\varphi)_{x,x+a_\varphi}$ denotes one of the component fields in $(\mathcal{U}_{\pm\mu}, \rho, \tilde\rho, \lambda_\mu, K)$, or a product of them. This can be worked out further to

$$(s_A\varphi)_{x,x+a_\varphi+a_A} = (\nabla_A)_{x,a+a_A}\varphi_{x+a_A,x+a_A+a_\varphi} \tag{32}$$
$$-(-)^{|\varphi|}\varphi_{x,x+a_\varphi}(\nabla_A)_{x+a_\varphi,x+a_\varphi+a_A},$$

and represented pictorially as

$$\tag{33}$$

For doing calculations one needs to know how a product of the (matrix valued) link fields transforms in terms of the transformations of single fields, in other words, one needs to know a Leibniz rule. It can easily be derived how something like

$$\tag{34}$$

transforms, where the φ_i are single link fields. Using the definition (31), and by writing things out explicitly and adding and subtracting at the same time

$$\tag{35}$$

it follows that

$$(s_A\varphi_1\varphi_2)_{x,x+a_A+a_{\varphi_1}+a_{\varphi_2}} \tag{36}$$
$$= (s_A\varphi_1)_{x,x+a_A+a_{\varphi_1}}(\varphi_2)_{x+a_A+a_{\varphi_1},x+a_A+a_{\varphi_1}+a_{\varphi_2}}$$
$$+(-1)^{|\varphi_1|}(\varphi_1)_{x,x+a_{\varphi_1}}(s_A\varphi_2)_{x+a_{\varphi_1},x+a_A+a_{\varphi_1}+a_{\varphi_2}}.$$

	s
$\mathcal{U}_{+\nu}$	0
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$
λ_ν	0
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$
$\tilde\rho$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]$
K	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \lambda_\rho]$

TABLE I: SUSY transformation of $\mathcal{N} = 2$ lattice SYM multiplet.

This is a Leibniz rule of the well known general form, but as in the NC construction, modified by shifts in the arguments of the fields - in this case the start and end points of link fields. Pictorially it can be expressed as follows:

$$\tag{37}$$

Using the definition of the supersymmetry transformations (31) and the twisted $\mathcal{N} = D = 2$ constraints (23-24), one can deduce the susy transformations of the different component fields explicitly by using Jacobi identities. The result is shown in table I for the transformation s. The full result can be found in [8]. Arguing in a similar way, it can be seen that the following supersymmetry algebra must hold on the lattice

$$\{s, s_\mu\}(\varphi)_{x,x+a_\varphi} = +i[\mathcal{U}_{+\mu}, \varphi]_{x,x+a_\varphi+n_\mu}, \tag{38}$$

$$\{\tilde s, s_\mu\}(\varphi)_{x,x+a_\varphi} = -i\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{x,x+a_\varphi-n_\nu}, \tag{39}$$

$$s^2(\varphi)_{x,x+a_\varphi} = \tilde s^2(\varphi)_{x,x+a_\varphi} = 0, \tag{40}$$

$$\{s, \tilde s\}(\varphi)_{x,x+a_\varphi} = \{s_\mu, s_\nu\}(\varphi)_{x,x+a_\varphi} = 0, \tag{41}$$

where φ denotes any (product of) component(s) of the multiplet $(\mathcal{U}_{\pm\mu}, \rho, \tilde\rho, \lambda_\mu, K)$.

Finally the action of twisted $\mathcal{N} = D = 2$ SYM on the lattice is given by

$$S \equiv \frac{1}{4}\sum_x \text{Tr } s\tilde s\epsilon_{\mu\nu}s_\mu s_\nu \, \mathcal{U}_{+\mu}\mathcal{U}_{-\mu}, \tag{42}$$

where the summation over x should also cover the additional lattice sites introduced by the shift parameters a_A. For the symmetric choice of these parameters (Figure 1) this means a sum over the integer sites (m_1, m_2) and over the half integer sites $(m_1 + \frac{1}{2}, m_2 + \frac{1}{2})$.

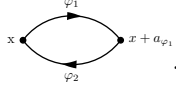The fact that the action is given by the consecutive action of all the susy transformations on $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu}$ suggests that the action is by construction invariant under all the supersymmetries due to the nature of the lattice susy algebra. Furthermore, the closed loop nature of $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu}$

plus the fact that the total shift of $s\tilde{s}\epsilon_{\mu\nu}s_\mu s_\nu$ is given by $a + \tilde{a} + a_1 + a_2 = 0$, makes sure that the action consists of a sum over closed loops, and is therefore manifestly gauge invariant.

IV. THE INCONSISTENCY

It can easily be seen from an expression like (36) that the Leibniz rule is not invariant under flipping the order of φ_1 and φ_2, due to the shifts in the arguments of the fields just as in the NC approach. However due to the link nature of the fields, an expression like $(\varphi_1)_{x,x+a_{\varphi_1}}(\varphi_2)_{x+a_{\varphi_1},x+a_{\varphi_1}+a_{\varphi_2}}$ can not be exchanged for $(\varphi_2)_{x+a_{\varphi_1},x+a_{\varphi_1}+a_{\varphi_2}}(\varphi_1)_{x,x+a_{\varphi_1}}$, since the latter expression is not well defined and in general the two matrices φ_1 and φ_2 do not commute. Thus, in contrast to the NC formulation the Leibniz rule (36) seems to be well defined in this link approach.

However, it is not. Consider the case of a closed loop, $a_{\varphi_1} = -a_{\varphi_2}$. Both $(\varphi_1)_{x,x+a_{\varphi_1}}(\varphi_2)_{x+a_{\varphi_1},x}$ and $(\varphi_2)_{x+a_{\varphi_1},x}(\varphi_1)_{x,x+a_{\varphi_1}}$ are well defined and are graphically represented by the same loop diagram

$$x \bullet \underset{\varphi_2}{\overset{\varphi_1}{\bigcirc}} \bullet \, x+a_{\varphi_1} . \qquad (43)$$
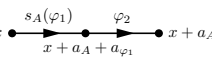
Of particular interest are traces over closed loops since they occur in the action. Upon taking the trace these two expressions are equal up to a factor $(-1)^{|\varphi_1||\varphi_2|}$.
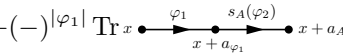
Consider now the susy transformation of the first ordering of the fields. We find

$$s_A \text{Tr}[(\varphi_1)_{x,x+a_{\varphi_1}}(\varphi_2)_{x+a_{\varphi_1},x}] \qquad (44)$$

$$= \text{Tr}[(s_A\varphi_1)_{x,x+a_A+a_{\varphi_1}}(\varphi_2)_{x+a_A+a_{\varphi_1},x+a_A}]$$

$$+ (-1)^{|\varphi_1|}\text{Tr}[(\varphi_1)_{x,x+a_{\varphi_1}}(s_A\varphi_2)_{x+a_{\varphi_1},x+a_A}]$$

$$= \text{Tr} \, x \bullet \overset{s_A(\varphi_1)}{\longrightarrow} \underset{x+a_A+a_{\varphi_1}}{\bullet} \overset{\varphi_2}{\longrightarrow} \bullet \, x+a_A$$
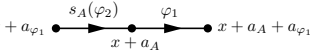
$$+ (-)^{|\varphi_1|} \, \text{Tr} \, x \bullet \overset{\varphi_1}{\longrightarrow} \underset{x+a_{\varphi_1}}{\bullet} \overset{s_A(\varphi_2)}{\longrightarrow} \bullet \, x+a_A.$$

But susy transforming the second ordering leads to the expression (up to the factor $(-1)^{|\varphi_1||\varphi_2|}$)

$$s_A \text{Tr}[(\varphi_2)_{x+a_{\varphi_1},x}(\varphi_1)_{x,x+a_{\varphi_1}}] \qquad (45)$$

$$= \text{Tr}[(s_A\varphi_2)_{x+a_{\varphi_1},x+a_A}(\varphi_1)_{x+a_A,x+a_A+a_{\varphi_1}}]$$

$$+ (-1)^{|\varphi_2|}\text{Tr}[(\varphi_2)_{x+a_{\varphi_1},x}(s_A\varphi_1)_{x,x+a_{\varphi_1}+a_A}]$$

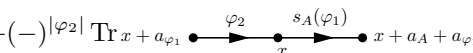$$= \text{Tr} \, x+a_{\varphi_1} \bullet \overset{s_A(\varphi_2)}{\longrightarrow} \underset{x+a_A}{\bullet} \overset{\varphi_1}{\longrightarrow} \bullet \, x+a_A+a_{\varphi_1}$$

$$+ (-)^{|\varphi_2|} \, \text{Tr} \, x+a_{\varphi_1} \bullet \overset{\varphi_2}{\longrightarrow} \underset{x}{\bullet} \overset{s_A(\varphi_1)}{\longrightarrow} \bullet \, x+a_A+a_{\varphi_1}.$$

These expressions are clearly not identical. They do not consist of the same fields. In fact the first ordering leads to a field living on the link $(x, x + a_A)$ while the second gives a field residing on the link $(x + a_{\varphi_1}, x + a_{\varphi_1} + a_A)$. Therefore the supersymmetric transformation of such a gauge invariant loop is not well defined.

The technical reason for the inconsistency comes from the link nature: when transforming the first field in a product like (44) or (45), the second field has to be shifted by the shift of s_A. Hence the supersymmetry transformations treat fields in products according to their order and reversing the order then leads to the shown contradiction. It is easy to see that this inconsistency applies to closed loop products with two or more fields (i.e. closed loops with two or more legs).

A special case of the situation discussed above deserves a little more attention: $a_{\varphi_1} = -a_{\varphi_2} = 0$. We are now considering two matrix valued fields F and G that live on lattice sites. Following the Leibniz rule, the susy transformation of the product $F_{x,x}G_{x,x}$ is given by

$$s_A \text{Tr}[F_{x,x}G_{x,x}] = \text{Tr}[(s_A F)_{x,x+a_A}G_{x+a_A,x+a_A}] \qquad (46)$$

$$+ (-1)^{|F|}\text{Tr}[F_{x,x}(s_A G)_{x,x+a_A}].$$

But since we trace over F and G, we can flip the order of the fields in the product, and the following is obtained (up to the factor $(-1)^{|F||G|}$)

$$s_A \text{Tr}[G_{x,x}F_{x,x}] = \text{Tr}[(s_A G)_{x,x+a_A}F_{x+a_A,x+a_A}] \qquad (47)$$

$$+ (-1)^{|G|}\text{Tr}[G_{x,x}(s_A F)_{x,x+a_A}].$$

This is almost exactly the inconsistency as it was encountered in the NC approach, see [7]. This inconsistency affects all the supercharges associated with the symmetric solution for the shift parameters (10).

Notice, however that there is no inconsistency in the case when the shift parameter $a_A = 0$. This occurs for example for one of the supersymmetries of the solution given by equation (11). In this case the susy transformation s maps a closed loop to a sum of closed loops, and it is easy to see using cyclic permutation of the trace that all orderings of the fields yield exactly the same expression for the susy variation. Such a supercharge has a site character and is hence similar to the conserved supercharges which appear in both [2] and [3]. As in that work, a maximum of one supercharge can be implemented exactly in this lattice model. Furthermore such a charge behaves as a scalar under Lorentz transformations. This can be done for any of the four supersymmetries by working with a suitable choice of the shift parameters. The other three supersymmetries will have non zero shift parameter, and will suffer from the inconsistency.

Of course an interesting question is how this inconsistency affects the supersymmetric transformation of the action. If one derives the action from its definition

(42) by ignoring the inconsistency explained above, i.e. by just taking the fields in the order in which they come, not changing the order in a product using the trace cyclicity, the following form of the action is obtained (taking into account the equations (9))

$$
\begin{aligned}
S \; = \; \sum_x \mathrm{Tr}\Bigg[& \qquad\qquad\qquad\qquad\qquad\qquad (48) \\
& (\frac{i}{2}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}] - K)_{x,x}(-\frac{i}{2}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}] - K)_{x,x} \\
& -\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x+n_\mu+n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x+n_\rho+n_\sigma,x} \\
& -i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x+a}(\rho)_{x+a,x} + \\
& -i(\tilde{\rho})_{x,x-\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x-\tilde{a},x}. \Bigg].
\end{aligned}
$$

Susy transforming this form of the action under s gives the expected result, zero. However, using the the cyclicity of the trace to write the action in the form given in [8],

$$
\begin{aligned}
S \; = \; \sum_x \mathrm{Tr}\Bigg[& \qquad\qquad\qquad\qquad\qquad\qquad (49) \\
& \frac{1}{4}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x,x} + K_{x,x}^2 \\
& -\frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu},\mathcal{U}_{+\nu}]_{x,x+n_\mu+n_\nu}[\mathcal{U}_{-\rho},\mathcal{U}_{-\sigma}]_{x+n_\rho+n_\sigma,x} \\
& -i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x+a}(\rho)_{x+a,x} + \\
& -i(\tilde{\rho})_{x,x-\tilde{a}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu},\lambda_\nu]_{x-\tilde{a},x} \Bigg],
\end{aligned}
$$

and susy transforming this form of the action under s, it leads to the following

$$
\begin{aligned}
sS \; = \; \frac{1}{2}\sum_x \mathrm{Tr}\Bigg[& \frac{1}{2}[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x+a}[\mathcal{U}_{+\nu},\mathcal{U}_{-\nu}]_{x+a,x+a} \quad (50) \\
& -\frac{1}{2}[\mathcal{U}_{+\mu},\mathcal{U}_{-\mu}]_{x,x}[\mathcal{U}_{+\nu},\lambda_\nu]_{x,x+a} + \\
& iK_{x,x}[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x+a} - i[\mathcal{U}_{+\mu},\lambda_\mu]_{x,x+a}K_{x+a,x+a} \Bigg].
\end{aligned}
$$

In the case $a \neq 0$, the situation here is thus very much like that encountered in the NC approach [7], where the action written without interchanging fields in a product was naively invariant, but no longer upon changing the order. In the case $a = 0$, the expression (50) does reduce to zero, and thus, in this situation, there is again no inconsistency associated with the supersymmetry s. However, the other supersymmetries will now have non-zero shift parameters and variation with respect to them will be ill-defined.

For illustration, consider one term in the action in

more detail. The following is part of the action

$$
-i\sum_x \mathrm{Tr}\left[(\tilde{\rho})_{x+\tilde{a},x}(\mathcal{U}_{-1})_{x,x-n_1}(\lambda_2)_{x-n_1,x+\tilde{a}}\right]
$$

$$
= -i\sum_x \mathrm{Tr} \qquad\qquad\qquad\qquad (51)
$$

Transforming the above loop as it stands under s leads to

$$
\begin{aligned}
i\sum_x \mathrm{Tr}\Big[& (\tilde{\rho})_{x+\tilde{a},x}(\lambda_1)_{x,x-a_1}(\lambda_2)_{x-a_1,x+a+\tilde{a}} \quad + \qquad (52) \\
& \frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho},\mathcal{U}_{+\sigma}]_{x+\tilde{a},x+a}(\mathcal{U}_{-1})_{x+a,x-a_1}(\lambda_2)_{x-a_1,x+a+\tilde{a}}\Big],
\end{aligned}
$$

which is a link from $x + \tilde{a}$ to $x + a + \tilde{a}$. Using the trace cyclicity, (51) can equivalently be written as

$$
-i\sum_x \mathrm{Tr}\left[(\mathcal{U}_{-1})_{x,x-n_1}(\lambda_2)_{x-n_1,x+\tilde{a}}(\tilde{\rho})_{x+\tilde{a},x}\right]. \quad (53)
$$

Transforming this expression leads to

$$
\begin{aligned}
i\sum_x \mathrm{Tr}\Big[& (\lambda_1)_{x,x-a_1}(\lambda_2)_{x-a_1,x+a+\tilde{a}}(\tilde{\rho})_{x+a+\tilde{a},x+a} \quad (54) \\
& +(\mathcal{U}_{-1})_{x,x-n_1}(\lambda_2)_{x-n_1,x+\tilde{a}}(\frac{i}{2})\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho},\mathcal{U}_{+\sigma}]_{x+\tilde{a},x+a}\Big],
\end{aligned}
$$

which is a link from x to $x + a$. Choosing $a = 0$, the expressions (52) and (54) are two equivalent ways of writing the same loop, and there is no inconsistency for s. Choosing $a \neq 0$, the two expressions are different links, and the inconsistency is present.

V. DISCUSSION AND CONCLUSION

In a series of recent papers D'Adda et al. [6, 8] have developed novel approaches to discretizing certain supersymmetric theories with the goal of preserving *all* continuum (twisted) supersymmetries.

Two approaches have been constructed both of which modify the supersymmetry variation of products of fields so as to make it compatible with a modified lattice Leibniz rule. In the approach described in [6] this property is ensured by introducing a non-commutativity between superspace coordinates. An inconsistency in this approach was pointed out in [7] and we have summarized this problem again here.

This paper focuses on an analysis of the second approach, termed the link construction, in which the fields live either on links or on sites of a lattice and transform under supersymmetry variation into fields of opposite Grassmann character living on neighboring links or sites. The supersymmetry transformations also either have a link or a site character. The precise link or site nature of the fields and the supersymmetry transformations

depends on the choice of shift parameters, and ensures that a closed lattice supersymmetry algebra can be constructed incorporating lattice difference operators [8]. In this paper we summarize this construction and point to another apparent inconsistency of this approach which is similar to that encountered with the non-commutative construction. While the arguments are quite general we illustrate them by referring to the lattice action for $\mathcal{N} = 2$ super Yang-Mills theory described in [8].

The problem is best exposed by considering the supersymmetry variation of a gauge invariant Wilson loop. If a supersymmetry variation has a link character, we show that the result of supersymmetry variation of this closed loop is not invariant under an initial cyclic permutation of the fields in the loop. Since the action is built out of traces over such loops, this inconsistency is visible at the level of the action – different cyclic permutations of terms in the action yield the same action written in different ways whose supersymmetry variations are not equal: they differ from zero to non-zero.

In the $\mathcal{N} = D = 2$ model the shift parameters are constrained such that at most one of them is zero. Choosing one of the parameters zero, the corresponding susy variation has a site character and can be placed on the lattice consistently. The other three have a link character and their action on gauge invariant loops is not well defined. Choosing none of them zero, all will suffer from this inconsistency.

Furthermore, if a susy transformation has a link character, the variation of the action under this transformation will always be a link variable. Since link variables are not gauge invariant, this means that the expectation value of the supersymmetry variation of the action will be zero, regardless of the way the action is written. This, in turn, means that the partition function will actually be susy invariant. At first glance this reasoning appears to be a way out of the inconsistency, but it is not. Indeed, this observation actually implies that the susy variation of *any* gauge invariant lattice action will have vanishing expectation value – clearly a non-physical result.

As a final remark, the transformations described here and in [8] do not contain an infinitessimal Grassman parameter - it is possible that the introduction of such a parameter with link character may alleviate some of the problems [9].

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