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Notes on (twisted) lattice supersymmetry

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Abstract

We describe a new approach to the problem of putting supersymmetric theories on the lattice. The basic idea is to discretize a *twisted* formulation of the supersymmetric theory. For certain theories with extended supersymmetry these twisted formulations contain only integer spin fields. The twisting exposes a scalar nilpotent supercharge which generates an exact lattice symmetry. We give examples from quantum mechanics, sigma models and Yang-Mills theories.

Introduction

The difficulties of discretizing supersymmetric theories are well known. Generic naive discretizations of continuum supersymmetric theories do not preserve supersymmetry. Quantum corrections then generate a large number of relevant supersymmetry violating interactions whose couplings must be tuned to zero as the lattice spacing is reduced. This is both unnatural and in many cases (especially for models with extended supersymmetry) prohibitively difficult. Various attempts have been made over the last twenty five years to overcome these problems see [1] and [2] and the recent reviews [3, 4]. However most of this work was confined to low dimensional models or Hamiltonian formulations.

Quite recently a series of new approaches have been developed which share the common feature of preserving a sub-algebra of the full supersymmetry algebra *exactly* at finite lattice spacing¹ [6, 7, 8, 9]. The hope is that this exact symmetry will protect the lattice theory against at least some of these dangerous radiative corrections and thus reduce fine tuning.

Theories where these ideas may be applied all possess *extended* supersymmetry. While these theories are not of immediate phenomenological interest they exhibit fascinating connections to string and gravitational theories as exhibited by the well known correspondence between $\mathcal{N} = 4$ super Yang-Mills in four dimensions theory and type IIB string theory on AdS space. Actually, the latter forms perhaps the best known example of a more general conjectured duality between $p + 1$ -dimensional super Yang-Mills and black p-brane solutions in supergravity.

This review will concentrate on just one of these new approaches to formulating lattice supersymmetry – discretization of a twisted version of the supersymmetric theory [7, 10, 11]. The construction applies only to cases where the number of continuum supercharges is a multiple of 2^D in D dimensions. We first start with a toy model, supersymmetric quantum mechanics realized as a $(0 + 1)$ dimensional field theory and show how to realize an exact nilpotent lattice supersymmetry in that model. We then go on to show how to lift this model to two dimensions to construct a lattice action for the two dimensional sigma model which retains an exact supersymmetry at finite lattice spacing. Numerical results deriving from full dynamical fermion simulations and confirming exact lattice supersymmetry in these models are presented. The general twisting procedure is then described in two and four dimensions and the twisted actions of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang-Mills in two and four dimensions are written down. The discretization of these gauge systems is then described in some detail.

¹Very recently a lattice construction of $\mathcal{N} = 2$ super Yang-Mills in $D = 2$ has been proposed which preserves *all* the supercharges [5]

1 Toy model

Consider a toy model consisting of a set of commuting fields $\phi(t)$ and $B(t)$ depending on a single continuous (Euclidean) time parameter t , together with anticommuting fields $\chi(t)$ and $\psi(t)$. Let us also postulate the fermionic symmetry

$$\begin{aligned} Q\phi &= \psi \\ Q\psi &= 0 \\ Q\chi &= B \\ QB &= 0 \end{aligned} \tag{1}$$

Notice that this symmetry is nilpotent off-shell and is reminiscent of a BRST symmetry. Using this structure we can write down a action which resembles a gauge fixing term

$$S = Q \int dt \chi \left(N(\phi) + \frac{1}{2}B \right) \tag{2}$$

Carrying out the variation we find

$$S = \int dt \left(BN + \frac{1}{2}B^2 + \chi \left(\frac{\partial N}{\partial \phi} \right) \psi \right) \tag{3}$$

After integrating over B we are led to the on-shell action

$$S = \int dt \frac{1}{2} N(\phi)^2 + \chi \frac{\partial N}{\partial \phi} \psi \tag{4}$$

To recover a physical theory from this construction it is necessary to choose a specific function $N(\phi)$. If we choose $N(\phi) = \partial_t \phi + P(\phi)$ our action can be recognized as nothing more than Witten's supersymmetric quantum mechanics [13].

$$N^2 \rightarrow (\partial\phi + P^2(\phi))^2 \tag{5}$$

$$\frac{\partial N}{\partial \phi} \rightarrow (\partial + P'(\phi)) \tag{6}$$

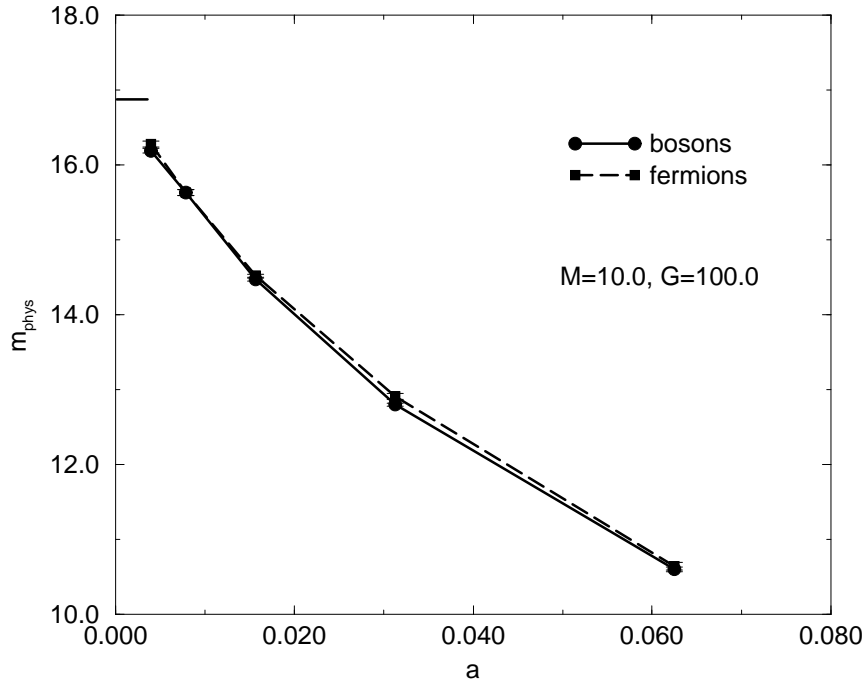
Notice the presence of cross terms in the bosonic action which in the continuum are total derivatives and hence can be neglected. When we latticeize the theory their presence will be necessary to ensure exact supersymmetry. This correspondence requires that we associate the “ghost” and “anti-ghost” field ψ and χ with physical fermion fields. This relationship between twisted and conventional fermion fields will become more complicated in higher dimensions. Unlike usual BRST gauge fixing we must however *not* impose the physical state conditions $Q|\text{phys}\rangle = 0$.

Notice that invariance of this action under Q depends only on its nilpotent property – not the form of the function $N(\phi)$. Indeed the twisted supersymmetry transformation eqn. 1 involves no derivatives in time and hence can be trivially transferred to the lattice. The resulting lattice action is

$$S_L = Q \sum \chi_t \left(D_{tt'}^+ \phi_{t'} + P(\phi_t) + B_t \right) \tag{7}$$

where

$$\Delta_{tt'}^+ \phi_{t'} = (\delta_{t',t+a} - \delta_{t',t}) \phi_{t'} \tag{8}$$



Notice that the use of a forward difference operator ensures no bosonic doublers. Exact SUSY then implies no fermion doublers. We will see later that the use of forward and backward difference operators is natural in higher dimensions also and follows from regarding the fermion fields as components of a Dirac-Kähler field. Well-defined discretizations of the Dirac-Kähler equation necessarily introduce such operators. Of course supersymmetric quantum mechanics possesses two supercharges. In this formulation this second symmetry can be gotten by exchanging $\chi \rightarrow \psi$. Notice that this second symmetry is classically broken by a term $O(a^2)$. However, an absence of divergences ensures this symmetry is automatically restored without fine tuning as $a \rightarrow 0$ [14]. In figure 1 we show some numerical results for the boson and fermion massgaps deriving from a dynamical fermion simulation of this model using $P = m\phi + g\phi^3$ and $m = 10$, $g = 100$ and lattice sizes $L = 16$, $L = 32$, $L = 64$, $L = 128$ and $L = 256$. Even at the largest lattice spacings a clear boson/fermion degeneracy can be seen. Furthermore, the lattice massgap appears to flow to the correct continuum value (calculated using Hamiltonian techniques) without fine tuning. We have also tested a number of Ward identities following from this symmetry. The simplest of these is the expectation of the bosonic action which turns out to be

$$\langle S_B \rangle = \frac{N_{\text{dof}}}{2\beta} \quad (9)$$

For results on this quantity and other simple Ward identities we refer the reader to [6, 14] in which extensive numerical results are provided to support the existence of exact lattice supersymmetry and the claim that no fine tuning is needed in this model to take the continuum limit.

2 Relation to topological quantum field theory

This BRST invariance reflects an underlying local shift symmetry [15]. Consider a model with a finite number of fields ϕ_i and $S_{\text{cl}}(\phi) = 0$. This is trivially invariant under the topological symmetry

$$\phi_t \rightarrow \phi_t + \epsilon_t \quad (10)$$

To quantize this theory requires picking a gauge. Choosing $N_t(\phi) = D_{tt'}^+ \phi_t' + P_t'(\phi) = 0$. leads to

$$Z = \int \prod d\phi_t \delta(N_t) \det \left(\frac{\partial N_t}{\partial \phi_{t'}} \right) \quad (11)$$

If we represent determinant using anticommuting ghosts and introduce a multiplier field for the δ -function we recover our SUSY model in Landau gauge! Notice though that this topological theory requires a projection to states annihilated by Q . This is equivalent to a projection to the vacuum state and *is not* what we do here. Here, the twisted or topological field theory form is simply to be viewed as a change of variables in the underlying supersymmetric theory. In flat space the regular and twisted formulations are completely equivalent.

3 Sigma model

A possible generalization of this quantum mechanical model consists of equipping the scalar fields with an additional index $\phi \rightarrow \phi^i$ and regarding these fields as coordinates on some non-trivial target space with metric $g_{ij}(\phi)$ [16]. The appropriate nilpotent symmetry is now

$$\begin{aligned} Q\phi^i &= \psi^i \\ Q\psi^i &= 0 \\ Q\chi_i &= \left(B_i - \chi_j \Gamma_{ik}^j \psi^k \right) \\ QB_i &= \left(B_j \Gamma_{ik}^j \psi^k - \frac{1}{2} \chi_j R_{ilk}^j \psi^l \psi^k \right) \end{aligned}$$

and the appropriate gauge fermion looks like

$$\Psi = \int_{\sigma} \eta_i \left(N^i(\phi) - \frac{1}{2} g^{ij} B_j \right) \quad (12)$$

with action $S = \beta Q\Psi$. Carrying out the variation and integrating out B as before we find a twisted form of the usual supersymmetric sigma model action

$$S = \alpha \int_{\sigma} \left(\frac{1}{2} g_{ij} N^i N^j - \chi_i \nabla_k N^i \psi^k + \frac{1}{4} R_{jlmk} \chi^j \chi^l \psi^m \psi^k \right) \quad (13)$$

which is then invariant under the scalar supersymmetry

$$\begin{aligned} Q\phi^i &= \psi^i \\ Q\psi^i &= 0 \\ Q\chi_i &= \left(g_{ij} N^j - \chi_j \Gamma_{ik}^j \psi^k \right) \end{aligned}$$

We still must specify the “gauge fixing” function N^i . For a one dimensional base space with coordinate σ we can just take $N^i = \frac{d\phi^i}{d\sigma}$ and the resulting action is

$$S = \beta \int d\sigma \left(\frac{1}{2} g_{ij} \frac{d\phi^i}{d\sigma} \frac{d\phi^j}{d\sigma} - \chi_i \frac{D}{D\sigma} \psi^i + \frac{1}{4} R_{jlmk} \chi^j \chi^l \psi^m \psi^k \right) \quad (14)$$

where the covariant derivative is the pullback of its target space cousin

$$\frac{D}{D\sigma} \psi^i = \frac{d}{d\sigma} \psi^i + \Gamma_{kj}^i \frac{d\phi^k}{d\sigma} \psi^j \quad (15)$$

Discretization of this action is just the same as for quantum mechanics and proceeds by replacing a continuum derivative by a forward difference operator.

The situation becomes more interesting when we take the base space to be two dimensional. The natural gauge fixing term now becomes

$$N^{i\alpha} = \partial^\alpha \phi^i \quad (16)$$

and implies that the anti-ghost χ and multiplier B also acquire an additional base space vector index. Actually this choice will not do. It is clear that if we are to arrive at a supersymmetric model the number of degrees of freedom carried by the anti-ghost must match that of the ghost field (in the end they will turn out to correspond to different chiral components of the physical fermions). Thus we must require the anti-ghost $\chi^{i\alpha}$ and multiplier field $B^{i\alpha}$ satisfy some condition which halves their number of degrees of freedom [15]. The natural way to do this is to introduce projection operators $P^{(-)}$ and $P^{(+)}$ and require that $\chi^{i\alpha}$ and $B^{i\alpha}$ satisfy certain self-duality conditions

$$\begin{aligned} P^{(-)} \chi &= 0 \\ P^{(+)} \chi &= \chi \end{aligned} \quad (17)$$

One choice for these projectors is

$$P_{j\beta}^{i\alpha(\pm)} = \frac{1}{2} \left(\delta_j^i \delta_\beta^\alpha \pm J_j^i \epsilon_\beta^\alpha \right) \quad (18)$$

Here, J_j^i must be a globally defined tensor field on the target space which squares to minus the identity and ϵ_β^α is the usual antisymmetric matrix with constant coefficients. Manifolds possessing such a structure are called *almost complex* and have even dimension. At this point we must be careful to make sure that the BRST transformations we introduced earlier are compatible with these self-duality conditions. This constraint forces the almost complex structure to be covariantly constant $\nabla_k J_j^i = 0$ and the manifold is termed Kähler. The final action in complex coordinates takes the form

$$\begin{aligned} S &= \beta \int d^2\sigma \left(2h^{+-} g_{I\bar{J}} \partial_+ \phi^I \partial_- \phi^{\bar{J}} \right. \\ &\quad \left. - h^{+-} g_{I\bar{J}} \chi_+^I D_- \psi^{\bar{J}} - h^{+-} g_{I\bar{J}} \chi_-^{\bar{I}} D_+ \psi^J + \frac{1}{2} h^{+-} R_{I\bar{I}J\bar{J}} \chi_+^I \chi_-^{\bar{I}} \psi^J \psi^{\bar{J}} \right) \end{aligned} \quad (19)$$

It should be clear by inspection that this model is indeed the $\mathcal{N} = 2$ supersymmetric sigma model with χ_+^I and $\psi^{\bar{I}}$ corresponding to the Weyl components of a Dirac spinor λ^I in chiral basis [17, 18, 19].

To date discretization of this action has been effected by replacing the continuum derivative by a *symmetric* difference operator and adding an additional Wilson term in the form of a holomorphic Killing vector to preserve the Q -exactness of the action. For details we refer to [16]. It should also be possible to proceed by rewriting the fermionic action in Dirac-Kähler form and utilizing the same discretization prescription we will advocate later for Yang-Mills theories. This has yet to be done.

One other interesting limit occurs for these two dimensional theories if I take a flat two dimensional target space. In this case I can deform the model *without losing the Q -exactness of the action* by addition of a holomorphic potential and obtain the two dimensional complex Wess-Zumino model. We refer the interested reader to [20] for details and extensive numerical simulations.

To summarize we have shown that it is possible to find lattice formulations of one and two dimensional supersymmetric theories with extended ($\mathcal{N} = 2$) supersymmetry which are exactly invariant under a single scalar fermionic symmetry. Furthermore this fermionic symmetry corresponds to a particular combination of the usual supercharges and emerges naturally in the context of twisted or topological field theory formulations of the supersymmetric theory. These twisted formulation naturally contain scalar and vector fermions.

There are two problems in what we have said so far; firstly we have not given a general method for constructing the twisted variables in terms of the usual fields. This we will rectify in the next section. Second and more important is the fact that we have not considered theories with a gauge symmetry. Twisted formulations of gauge theories exist in the continuum (indeed the very first topological field theory constructed by Witten corresponded to twisted $\mathcal{N} = 2$ super Yang-Mills in four dimensions) but the twin requirements of exact gauge symmetry and exact supersymmetry render a simple translation of the continuum constructions to the lattice problematic. However, progress was made when in [23], Sugino managed to generalize the twisted supersymmetry transformations to the lattice. Unfortunately, his construction generically yields additional states in the lattice theory with no counterpart in the continuum. In low dimensions it is possible to circumvent these problems by careful choice of the gauge fermion but this approach fails in four dimensions. In [24] we proposed an alternative discretization of these twisted models which does not suffer from these problems. We will spend the last part of this article reviewing this approach which we will see includes the interesting case of $\mathcal{N} = 4$ super Yang-Mills.

4 Twisting as a change of variables

The twist required to expose a scalar supercharge in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ super Yang-Mills theory in $D = 2$ and $D = 4$ dimensions respectively is gotten by replacing the usual rotation group $SO(D)$ by the diagonal subgroup [12, 25, 24, 28]

$$SO(D)' = \text{diagonal subgroup}(SO(D) \times SO(D)_R) \quad (20)$$

where the second factor reflects the additional R-symmetry present in these theories and corresponds to the possible internal rotations of the D Majorana supercharges into each other. The supercharges now transform as matrices under this twisted rotation group and can hence be expanded on a basis of products of gamma matrices

$$q = QI + Q_\mu \gamma^\mu + Q_{\mu\nu} \gamma^{\mu\nu} + \dots \quad (21)$$

where the coefficients are the twisted supercharges. The original SUSY algebra

$$\{q_\alpha^I, \bar{q}_\beta^J\} = 2\delta^{IJ}\gamma_{\alpha\beta}^\mu P^\mu \quad (22)$$

now implies a twisted algebra

$$\{q, q\}_{\alpha\beta} = 4\gamma_{\alpha\beta}^\mu P^\mu \quad (23)$$

which naturally includes the nilpotent scalar supercharge Q . Actually the twisted algebra also implies that the momentum is now Q -exact

$$\{Q, Q_\mu\} = P_\mu \quad (24)$$

This property makes it plausible that the entire energy-momentum tensor and hence action of the theory may be Q -exact. This is in agreement with the BRST form we have exhibited in the previous examples. Finally we should point out that we can match the four supercharges of original SUSY theory by taking the twisted supercharges to be real. This will imply a reality condition on the supercharge matrix

$$q^\dagger = Cq^T C^{-1} \quad (25)$$

where C will be the charge conjugation matrix. If the supercharges form a matrix so do the fermions which hence can be written in terms of anticommuting, antisymmetric tensor fields $\eta, \psi_\mu, \chi_{\mu\nu}$ etc. We can abstract these p-form components and consider the fermions as represented by a real Kähler-Dirac field [24, 28, 12, 25] The original Dirac equation can then be shown to be equivalent to the tensor Dirac-Kähler equation

$$(d - d^\dagger)\Psi = 0 \quad (26)$$

where d and d^\dagger are the usual exterior derivative and its adjoint. This corresponds to a fermion kinetic term (or Dirac-Kähler action)

$$S_F = \Psi^\dagger \cdot (d - d^\dagger)\Psi \quad (27)$$

This equivalence of the D -dimensional Dirac-Kähler equation to the Dirac equation for D fermions remains true when the model is gauged. In the continuum this equivalence has been remarked on many times – see for example [26]. Hamiltonian lattice theories using Dirac-Kähler fermions were first proposed in [27].

In this way we have exhibited the general change of variables implied by the twist, exhibited the nilpotent supercharge explicitly, and shown that such actions can be written in a Q -exact form. What remains is to write down the nilpotent symmetry and gauge fermion in a gauge theory and then describe the prescription used to discretize the theory.

5 Twisted $\mathcal{N} = 2$ SYM in two dimensions

The two dimensional Dirac-Kähler field representing the fermions contains 4 grassman components $(\eta, \psi_\mu, \chi_{12})$. Their corresponding commuting Q -superpartners are labeled $(\bar{\phi}, A_\mu, B_{12})$. All these fields take values in the adjoint of a gauge group with continuum twisted action [23, 24]

$$\begin{aligned} S = & \beta Q \text{Tr} \int d^2x \left(\frac{1}{4} \eta [\phi, \bar{\phi}] + 2\chi_{12} F_{12} + \right. \\ & \left. + \chi_{12} B_{12} + \psi_\mu D_\mu \bar{\phi} \right) \end{aligned} \quad (28)$$

where the scalar symmetry is given by

$$\begin{aligned}
QA_\mu &= \psi_\mu \\
Q\psi_\mu &= -D_\mu\phi \\
Q\phi &= 0 \\
Q\chi_{12} &= B_{12} \\
QB_{12} &= [\phi, \chi_{12}] \\
Q\bar{\phi} &= \eta \\
Q\eta &= [\phi, \bar{\phi}]
\end{aligned} \tag{29}$$

The square of Q is now an infinitesimal gauge transformation given by the field ϕ . Carrying out the Q -variation and integrating out B_{12} yields the on-shell action

$$\begin{aligned}
S &= \beta\text{Tr} \int d^2x \left(\frac{1}{4}[\phi, \bar{\phi}]^2 - \frac{1}{4}\eta[\phi, \eta] - F_{12}^2 \right. \\
&\quad - D_\mu\phi D_\mu\bar{\phi} - \chi_{12}[\phi, \chi_{12}] \\
&\quad - 2\chi_{12}(D_1\psi_2 - D_2\psi_1) - 2\psi_\mu D_\mu\eta/2 \\
&\quad \left. + \psi_\mu[\bar{\phi}, \psi_\mu] \right)
\end{aligned} \tag{30}$$

Notice that the scalar plus gauge part is positive definite along the contour $\bar{\phi}^a = (\phi^a)^*$ (we use AH group generators) and clearly corresponds to the bosonic sector of 2D $\mathcal{N} = 2$ super Yang-Mills. The fermionic piece is nothing more than the Dirac-Kähler action described earlier.

6 Twisted $\mathcal{N} = 4$ SYM in four dimensions

The four dimensional Dirac-Kähler field representing the fermions contains the sixteen grassman components $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\lambda}, \kappa_{1234})$ [25, 28]. Their corresponding Q -superpartners are labeled $(\bar{\phi}, A_\mu, B_{\mu\nu}, W_{\mu\nu\lambda}, C_{1234})$. The corresponding Q -transformations are generalizations of the two dimensional ones:

$$\begin{aligned}
Q\bar{\phi} &= \eta \quad Q\eta = [\phi, \bar{\phi}] \\
QA_\mu &= \psi_\mu \quad Q\psi_\mu = -D_\mu\phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] \quad Q\chi_{\mu\nu} = B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \quad Q\theta_{\mu\nu\lambda} = [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] \quad Q\kappa_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \\
Q\phi &= 0
\end{aligned} \tag{31}$$

Clearly B and C will be multiplier fields which are integrated out to yield the on-shell supersymmetric action. The four fields of W , together with ϕ and $\bar{\phi}$ correspond to the usual six scalars of $\mathcal{N} = 4$ super Yang-Mills. The appropriate gauge fermion is given by $S = \beta Q\Lambda$ with

$$\Lambda = \int d^4x \text{Tr} \left[\chi_{\mu\nu} \left(F_{\mu\nu} + \frac{1}{2}B_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}] \right) \right]$$

$$\begin{aligned}
& + D_\lambda W_{\lambda\mu\nu}) \\
& + \psi_\mu D_\mu \bar{\phi} + \frac{1}{4}\eta[\phi, \bar{\phi}] + \frac{1}{3!}\theta_{\mu\nu\lambda}[W_{\mu\nu\lambda}, \bar{\phi}] \\
& + \frac{1}{4!}\kappa_{\mu\nu\lambda\rho} \left(\sqrt{2}D_{[\mu} W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho} \right) \Big] \tag{32}
\end{aligned}$$

Carrying out the Q -variation and subsequently integrating out $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$ leads to

$$S = \beta (S_B + S_F + S_Y) \tag{33}$$

where

$$S_F = \int d^4x \text{Tr} \left[-\chi_{\mu\nu} D_{[\mu} \psi_{\nu]} - \chi_{\mu\nu} D_\lambda \theta_{\lambda\mu\nu} \right. \tag{34}$$

$$\left. - \eta D_\mu \psi_\mu - \frac{\sqrt{2}}{4!} \kappa_{\mu\nu\lambda\rho} D_{[\mu} \theta_{\nu\lambda\rho]} \right] \tag{35}$$

$$S_B = \int d^4x \text{Tr} \left[-\frac{1}{2} \left(F_{\mu\nu} - \frac{1}{2} [W_{\mu\lambda\rho}, W_{\nu\lambda\rho}] \right)^2 \right. \tag{36}$$

$$\begin{aligned}
& + (D_\lambda W_{\lambda\mu\nu})^2 + \frac{2}{4!} (D_{[\mu} W_{\nu\lambda\rho]})^2 \\
& \left. - D_\mu \phi D_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 - \frac{1}{3!} [\phi, W_{\mu\nu\lambda}] [\bar{\phi}, W_{\mu\nu\lambda}] \right] \tag{37}
\end{aligned}$$

We omit the Yukawas for simplicity. Again, a simple rescaling of the fields renders the Dirac-Kähler nature of the fermionic action manifest while the bosonic sector is nothing more than the Marcus twist of $\mathcal{N} = 4$ super Yang-Mills after replacing the W -field by its dual [29]. Another lattice formulation of $\mathcal{N} = 4$ super Yang-Mills obtained from the orbifold method was recently written down by Kaplan and Unsal [30].

7 Lattice prescription

These twisted gauge actions may be discretized in a natural way. We place 0-forms on sites, 1-forms on links, 2 forms on plaquettes etc. For each orientation of the underlying p-cube we associate a field f and its complex conjugate f^\dagger . Notice this complexification doubles the degrees of freedom in the lattice theory with respect to its continuum cousin. Furthermore, we choose the lattice fields to have the following gauge transformation properties

$$f_{\mu_1 \dots \mu_p}(x) \rightarrow G(x) f_{\mu_1 \dots \mu_p}(x) G^{-1}(x + e_{\mu_1 \dots \mu_p}) \tag{38}$$

where the vector $e_{\mu_1 \dots \mu_p} = \sum_{j=1}^p \mu_j$. A covariant forward difference operator is also defined by [31]

$$D_\mu^+ f_{\mu_1 \dots \mu_p}(x) = U_\mu(x) f_{\mu_1 \dots \mu_p}(x + \mu) - f_{\mu_1 \dots \mu_p}(x) U_\mu(x + e_{\mu_1 \dots \mu_p}) \tag{39}$$

and its adjoint a covariant difference operator via

$$D_\mu^- f_{\mu_1 \dots \mu_p}(x) = f_{\mu_1 \dots \mu_p}(x) U_\mu^\dagger(x + e_{\mu_1 \dots \mu_p} - \mu) - U_\mu^\dagger(x - \mu) f_{\mu_1 \dots \mu_p}(x - \mu) \tag{40}$$

These reduce to continuum derivatives as $a \rightarrow 0$ and ensure that derivatives transform correctly under gauge transformations. We also replace the continuum vector potential A_μ by the Wilson gauge link U_μ which is to be treated as a non-unitary matrix at this stage of the construction.

It was proved in [32] that theories formulated in these geometrical terms can be discretized without encountering spectrum doubling if

$$\begin{aligned} \partial_\mu &\rightarrow D^+ \text{ if acts like } d \\ \partial_\mu &\rightarrow D^- \text{ if acts like } d^\dagger \end{aligned} \tag{41}$$

We also use the following definition of the Yang-Mills field strength

$$F_{\mu\nu}(x) = D_\mu^+ U_\nu(x) \rightarrow F_{\mu\nu}^{\text{cont}} \text{ as } a \rightarrow 0 \tag{42}$$

Using these ingredients we can straightforwardly construct the lattice theory for both continuum twisted theories. The Q -transformations are almost unchanged – the only subtlety is that the explicit covariant derivative appearing on the right hand side of the variation of ψ_μ must be a *forward* difference operator and various commutators are point split in such a way as to transform correctly under lattice gauge transformations. We refer the reader to [24, 28] for details. The discretization of the gauge fermion is straightforward; we replace any term of the form

$$\int A_{\mu_1 \dots \mu_p} B_{\mu_1 \dots \mu_p} \tag{43}$$

by the lattice expression

$$\sum A_{\mu_1 \dots \mu_p}^\dagger B_{\mu_1 \dots \mu_p} + \text{h.c} \tag{44}$$

This lattice theory is formulated in terms of complex fields. In [28] we give arguments that the theory can be truncated to the real line and the twisted supersymmetric Ward identities recovered in the continuum limit without additional fine tuning.

8 Conclusions

It is possible to find formulations of a variety of supersymmetric theory which can be written in the language of differential forms and exterior derivatives. The fermion content of such theories may be embedded in one (or more) Dirac-Kähler fields. Such a theory has a Q -exact action and a scalar nilpotent supercharge. The latter generates a fermionic symmetry which may be implemented exactly on the lattice. We have illustrated this with examples drawn from quantum mechanics, two dimensional sigma models and Yang-Mills theories. The use of Dirac-Kähler fermions evades the standard doubling problems and allows local, Q -symmetric lattice actions to be written down. In the case of gauge theories the requirements of gauge invariance force a complexification of the degrees of freedom. Significant numerical work has already been done in the non-gauge models and is currently starting in the Yang-Mills case. We hope that these studies, complemented by perturbative calculations will help establish these lattice theories as good non-perturbative regulators of the corresponding continuum theories. If this proves correct, then they may be used to explore the strong coupling physics of models such as large N Yang-Mills, which would give us a new non-perturbative handle on various string and supergravity theories.

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