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Low energy scattering with a nontrivial pion

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An earlier calculation in a generalized linear sigma model showed that the well-known current algebra formula for low energy pion pion scattering held even though the massless Nambu Goldstone pion contained a small admixture of a two-quark two-antiquark field. Here we turn on the pion mass and note that the current algebra formula no longer holds exactly. We discuss this small deviation and also study the effects of an SU(3) symmetric quark mass type term on the masses and mixings of the eight SU(3) multiplets in the model. We calculate the s wave scattering lengths, including the beyond current algebra theorem corrections due to the scalar mesons, and observe that the model can fit the data well. In the process, we uncover the way in which linear sigma models give controlled corrections (due to the presence of scalar mesons) to the current algebra scattering formula. Such a feature is commonly thought to exist only in the non-linear sigma model approach.

I. INTRODUCTION

A linear sigma model with both quark-antiquark type fields and fields containing (in an unspecified configuration) two quarks and two antiquarks, seems useful for understanding the light scalar spectrum of QCD. In a previous treatment [1] we considered a usual simplification in which the three light quark masses were taken to be zero. The model was seen to give a neat intuitive explanation of how "four quark" scalar states could be naturally much lighter than the conventional p-wave quark-antiquark scalars. We also verified in detail that, as long as the potential of the model satisfied $SU(3)_L \times SU(3)_R$ invariance, the massless version of the famous current algebra theorem [2] on low energy pion pion scattering was correct.

In the present paper we introduce a common mass for the three light quarks in such a way that the pion gets its correct mass. SU(3) flavor invariance continues to hold, which is a desirable simplification. First we reexamine the masses and mixings of the particles in the model. It is seen that the natural explanation for the lightness of a "four quark" scalar remains unchanged. Then we reexamine the pion pion scattering amplitude to try to see if the low energy theorem continues to hold. Curiously we find that it does not exactly hold. What goes wrong? The algebra of the Noether currents should be good in this model so that is not the cause. It turns out that the partially conserved axial vector current, which is also required for the theorem, does not hold, unlike for the massless case. The axial vector current has a single particle contribution from the "heavy pion" in this chiral model as well as from the ordinary pion. The ordinary pion does not therefore completely saturate the axial current. Actually this is a small effect but is of conceptual interest and may be of more importance for the kaon scattering case.

A more important quantitative effect arises from the contributions of the scalar isosinglet mesons in the model. It is shown that these contributions can explain the experimental s wave isosinglet scattering length.

The notation is reviewed in Section II. The corrections to the masses and mixings, due to the quark mass term, are studied in Section III. An approximate analytic treatment of the scattering, for a general potential, is contained in Section IV. The exact numerical treatment for a "leading order" potential is presented in Section V. Some discussion and conclusions are given in Section VI. The Appendix explains the method of parameter determination from experiment and a listing of typical values for all the parameters.

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II.   NOTATION

We introduce the 3×3 matrix chiral nonet fields;

$$M = S + i\phi, \qquad\qquad M' = S' + i\phi'. \tag{1}$$

Here M represents scalar, S and pseudoscalar, ϕ quark-antiquark type states, while M' represents states which are made of two quarks and two antiquarks. The transformation properties under $\mathrm{SU(3)_L}\times \mathrm{SU(3)_R}\times \mathrm{U(1)_A}$ are

$$M \to e^{2i\nu}\, U_\mathrm{L} M U_\mathrm{R}^\dagger, \qquad\qquad M' \to e^{-4i\nu}\, U_\mathrm{L} M' U_\mathrm{R}^\dagger, \tag{2}$$

where U_L and U_R are unitary unimodular matrices, and the phase ν is associated with the $\mathrm{U(1)_A}$ transformation. The general Lagrangian density which defines our model is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(\partial_\mu M \partial_\mu M^\dagger\right) - \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M' \partial_\mu M'^\dagger\right) - V_0\left(M, M'\right) - V_{SB}, \tag{3}$$

where $V_0(M, M')$ stands for a function made from $\mathrm{SU(3)_L}\times \mathrm{SU(3)_R}$ (but not necessarily $\mathrm{U(1)_A}$) invariants formed out of M and M'. The quantity V_{SB} stands for chiral symmetry breaking terms which transform in the same way as the quark mass terms in the fundamental QCD Lagrangian. In our previous paper [1], we focused on general properties which continued to hold when V_{SB} was set to zero. Here, we include the SU(3) symmetric mass term:

$$V_{SB} = -2\, A\, \mathrm{Tr}(S) \tag{4}$$

where A is a real parameter. A characteristic feature of the model is the presence of "two-quark" and "four-quark" condensates:

$$\left\langle S_a^b \right\rangle = \alpha_a \delta_a^b, \qquad \left\langle S_a'^b \right\rangle = \beta_a \delta_a^b. \tag{5}$$

We shall assume the vacuum to be $\mathrm{SU(3)_V}$ invariant, which implies

$$\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha, \qquad\qquad \beta_1 = \beta_2 = \beta_3 \equiv \beta. \tag{6}$$

The SU(3) particle content of the model consists of two pseudoscalar octets, two pseudoscalar singlets, two scalar octets and two scalar singlets. This gives us eight different masses and four mixing angles. We next give the notations for resolving the nonets into SU(3) octets and singlets. Note the matrix convention $\phi_a^b \to \phi_{ab}$. The properly normalized singlet states are:

$$\phi_0 = \frac{1}{\sqrt{3}}\mathrm{Tr}(\phi), \qquad \phi_0' = \frac{1}{\sqrt{3}}\mathrm{Tr}(\phi'),$$
$$S_0 = \frac{1}{\sqrt{3}}\mathrm{Tr}(S), \qquad S_0' = \frac{1}{\sqrt{3}}\mathrm{Tr}(S'). \tag{7}$$

Then we have the matrix decompositions:

$$\phi = \hat{\phi} + \frac{1}{\sqrt{3}}\phi_0 1, \qquad \phi' = \hat{\phi}' + \frac{1}{\sqrt{3}}\phi_0' 1,$$
$$S = \hat{S} + \frac{1}{\sqrt{3}}S_0 1, \qquad S' = \hat{S}' + \frac{1}{\sqrt{3}}S_0' 1, \tag{8}$$

wherein $\hat{\phi}$, $\hat{\phi}'$, $\hat{S}$ and $\hat{S}'$ are all 3×3 traceless matrices. The singlet scalar fields may be further decomposed as:

$$S_0 = \sqrt{3}\alpha + \tilde{S}_0, \qquad S_0' = \sqrt{3}\beta + \tilde{S}_0'. \tag{9}$$

Here $\tilde{S}_0$ and $\tilde{S}_0'$ are the fluctuation fields around the true ground state of the model. The breaking of SU(3) to the isospin group SU(2) will be examined in the future. In that case there are 16 different masses, four 2×2 mixing matrices and two 4×4 mixing matrices. To fully characterize the system we will also require some knowledge of the axial vector and vector currents obtained by Noether's method:

$$\begin{aligned}
(J_\mu^{axial})_a^b &= (\alpha_a + \alpha_b)\partial_\mu \phi_a^b + (\beta_a + \beta_b)\partial_\mu \phi_a'^b + \cdots, \\
(J_\mu^{vector})_a^b &= i(\alpha_a - \alpha_b)\partial_\mu S_a^b + i(\beta_a - \beta_b)\partial_\mu S_a'^b + \cdots,
\end{aligned} \tag{10}$$

where the dots stand for terms bilinear in the fields.

In our model we use a previously discussed scheme to select the most important terms in the potential, $V_0(M, M')$. The favored terms which are $SU(3)_L \times SU(3)_R$ invariant but violate $U(1)_A$ are:

$$V_\eta = c_3 [F_\eta(M, M')]^2, \tag{11}$$

in which c_3 is a coupling constant and

$$F_\eta(M, M') = \gamma_1 \ln\left(\frac{\det(M)}{\det(M^\dagger)}\right) + (1 - \gamma_1)\ln\left(\frac{\text{Tr}(MM'^\dagger)}{\text{Tr}(M'M^\dagger)}\right), \tag{12}$$

where γ_1 is a dimensionless parameter. This form exactly mocks up the $U(1)_A$ anomaly of QCD. Information about the pseudoscalar particles which is independent of the choice of the $U(1)_A$ invariant terms in V_0 may be obtained by differentiating the following matrix equation representing the response of the potential to an infinitesimal axial transformation:

$$\left[\phi, \frac{\partial V_0}{\partial S}\right]_+ - \left[S, \frac{\partial V_0}{\partial \phi}\right]_+ + (\phi, S) \rightarrow (\phi', S') = 1\left[2\,\text{Tr}\left(\phi'\frac{\partial V_0}{\partial S'} - S'\frac{\partial V_0}{\partial \phi'}\right) - 8\,c_3\,i\,F_\eta(M, M')\right]. \tag{13}$$

To get general constraints on the pseudoscalar particle masses we differentiate this equation once with respect to each of the two matrix fields: ϕ, ϕ' and evaluate the equation in the ground state. Thus we also need the "minimum" condition,

$$\left\langle\frac{\partial V_0}{\partial S}\right\rangle + \left\langle\frac{\partial V_{SB}}{\partial S}\right\rangle = 0, \qquad \left\langle\frac{\partial V_0}{\partial S'}\right\rangle + \left\langle\frac{\partial V_{SB}}{\partial S'}\right\rangle = 0. \tag{14}$$

III.   MASSES AND MIXINGS

As we previously discussed, the leading choice of terms corresponding to eight or fewer quark plus antiquark lines at each effective vertex reads:

$$\begin{aligned} V_0 = & - c_2\,\text{Tr}(MM^\dagger) + c_4^a\,\text{Tr}(MM^\dagger MM^\dagger) \\ & + d_2\,\text{Tr}(M'M'^\dagger) + e_3^a(\epsilon_{abc}\epsilon^{def}M_d^a M_e^b M_f'^c + h.c.) \\ & + c_3\left[\gamma_1 \ln(\frac{\det M}{\det M^\dagger}) + (1 - \gamma_1)\frac{\text{Tr}(MM'^\dagger)}{\text{Tr}(M'M^\dagger)}\right]^2. \end{aligned} \tag{15}$$

All the terms except the last two have been chosen to also possess the $U(1)_A$ invariance.

The minimum equations for this potential are:

$$\left\langle\frac{\partial V_0}{\partial S_a^a}\right\rangle = 2\alpha\left(-c_2 + 2\,c_4^a\,\alpha^2 + 4\,e_3^a\,\beta\right) = 2A, \tag{16}$$

$$\left\langle\frac{\partial V_0}{\partial S'^a_a}\right\rangle = 2\left(d_2\,\beta + 2\,e_3^a\,\alpha^2\right) = 0. \tag{17}$$

Differentiating the potential in Eq.(15) twice will yield four 2×2 mass matrices denoted as (M_π^2), (M_0^2), (X_a^2) and (X_0^2) respectively for the pseudoscalar octets, the pseudoscalar singlets, the scalar octets and the scalar singlets. These may be brought to diagonal (hatted) form by the following 2×2 orthogonal transformations:

$$\begin{aligned} \sum_{B,C}(R_\pi^{-1})_{AB}(M_\pi^2)_{BC}(R_\pi)_{CD} &= (\hat{M}_\pi^2)_{AD}, & \sum_{B,C}(R_0^{-1})_{AB}(M_0^2)_{BC}(R_0)_{CD} = (\hat{M}_0^2)_{AD}, \\ \sum_{B,C}(L_a^{-1})_{AB}(X_a^2)_{BC}(L_a)_{CD} &= (\hat{X}_a^2)_{AD}, & \sum_{B,C}(L_0^{-1})_{AB}(X_0^2)_{BC}(L_0)_{CD} = (\hat{X}_0^2)_{AD}, \end{aligned} \tag{18}$$

Notice that the four mass matrices are identical to those given in section IV of [1] for the zero quark mass case. The numerical values of the entries will however differ because the relations among the coefficients are different due to the presence of 2A rather than zero on the right hand side of Eq.(16).
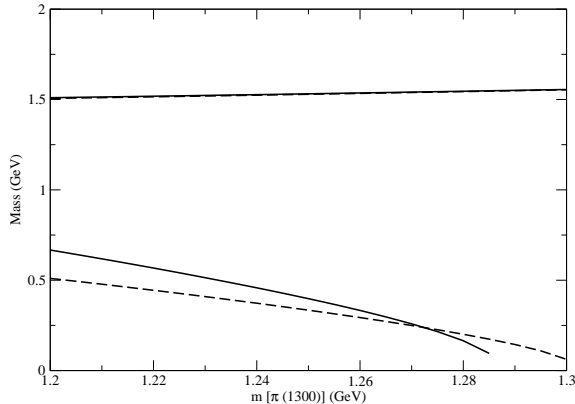
FIG. 1: The predictions for the masses of the two SU(3) singlet scalars vs. $m[\pi(1300)]$. The solid lines correspond to the massive pion case while the dashed lines correspond to the massless pion case previously considered.

In the massive case there are 9 parameters (A, α, β, c_2, d_2, c_4^a, e_3^a, c_3 and γ_1). These can be reduced to seven by use of the two minimum equations just given. We note that the parameters c_3 and γ_1, associated with modeling the $U(1)_A$ anomaly, do not contribute to either the minimum equations or to the mass matrices of the particles which are not 0^- singlets. Thus it is convenient to first determine the other five independent parameters. As the corresponding experimental inputs [3] we take the non-strange quantities:

$$
\begin{aligned}
m(0^+\text{octet}) &= m[a_0(980)] = 984.7 \pm 1.2 \,\text{MeV} \\
m(0^+\text{octet}') &= m[a_0(1450)] = 1474 \pm 19 \,\text{MeV} \\
m(0^-\text{octet}') &= m[\pi(1300)] = 1300 \pm 100 \,\text{MeV} \\
m(0^-\text{octet}) &= m_\pi = 137 \,\text{MeV} \\
F_\pi &= 131 \,\text{MeV}
\end{aligned}
\tag{19}
$$

Evidently, a large experimental uncertainty appears in the mass of $\pi(1300)$; we shall initially take the other masses as fixed at their central values and vary this mass in the indicated range. Essentially $m[\pi(1300)]$ is being treated as an arbitrary parameter of our model. As shown in Eq.(A1) in Appendix A, it is straightforward to determine the five independent parameters in terms of these five inputs. This determination is a generalization of the one in the previous zero mass pion case in which four parameters were determined from four inputs.

The effects of adding a non zero quark mass term on the masses of the two predicted scalar singlets are displayed in Fig.1. It is clear that the small mass term has a negligible effect on the mass of the heavier scalar singlet. On the other hand, there is a larger effect on the mass of the lighter scalar singlet. Still this singlet is exceptionally light so there is no qualitative difference in the result.

In Fig. 2 we display a comparison of the four quark percentages of the π meson, the lighter a_0 meson and the lighter scalar singlet with the corresponding values in the model with zero quark masses. (These are, of course, equal to the two quark percentages of the heavier particles with the same quantum numbers). It is clear that there is not much change compared to the zero quark mass case.

It remains to discuss the four quark percentages of the two SU(3) singlet pseudoscalars. The lightest is the $\eta(958)$ while candidates for the heavier one include $\eta(1295)$, $\eta(1405)$, $\eta(1475)$ and $\eta(1760)$. As in the zero mass case, the first two candidates are ruled out because they do not lead to positive eigenvalues of the prediagonal squared mass matrix (M_0^2). For the other two scenarios we may find the numerical values of the remaining parameters c_3 and γ_1 by using Eqs. (A2), (A3), (A4) and (A5) given in the Appendix. The four quark contents for the $\eta(958)$ are being compared between the massive and massless quark cases in Fig. 3. Note that there are two solutions for each scenario corresponding to Eq. (A4) being of quadratic type. The lower four quark percentage curves seem the most plausible. Again there seems to be little difference between the zero and non-zero quark mass cases. This is understandable by comparing the values of the Lagrangian parameters found in Appendix A with those found in Appendix B of [1].
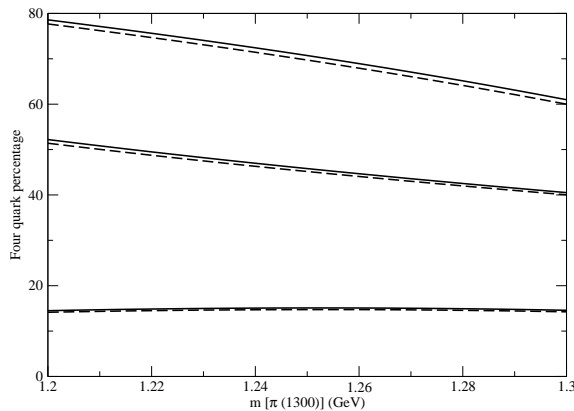
FIG. 2: Plot of the four quark percentages of various particles in the model as functions of the undetermined input parameter, $m[\pi(1300)]$. Starting from the bottom and going up, the curves respectively show the four quark percentages of the pion, the lighter 0^+ singlet, and the $a_0(980)$. Solid lines apply for the massive pion case and dashed lines for the massless pion case.
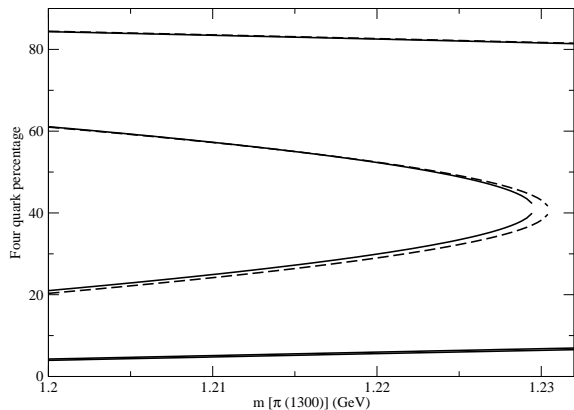

FIG. 3: Plot of the four quark percentages of the $\eta(958)$ as functions of the undetermined input parameter, $m[\pi(1300)]$ for two scenarios. The top and bottom curves correspond to choosing the $\eta(1760)$ as the heavier 0^- SU(3) singlet while the middle two curves correspond to choosing the $\eta(1475)$ as the heavier 0^- SU(3) singlet. Note that for each scenario, the two curves are associated with different solutions of the quadratic equation (A4) for γ_1. The solid lines apply to the massive pion case while the dashed lines apply to the previous massless pion case.

IV. PION SCATTERING: APPROXIMATE ANALYTIC TREATMENT FOR GENERAL $\mathbf{V}_0$

We have seen that there are not very big changes when using an SU(3) symmetric quark mass term of proper strength to give the experimental mass value to the pion. However, for the discussion of the pion pion scattering amplitude near threshold, which is one of the most important applications of the chiral approach, the correct value of the pion mass is important. In the zero quark mass case we showed that for any choice of terms in the potential V_0, the current algebra expression for the threshold pion pion amplitude held exactly. This was understandable since the algebra of chiral currents held by construction and furthermore the pion completely saturated the axial current. In the present case, the pion does not, as we shall see, completely saturate the axial current. Thus it is interesting to study this case in more detail.

Note that the transformation between the diagonal fields (π^+ and π'^+) and the original pion fields is given as:

$$
\begin{bmatrix} \pi^+ \\ \pi'^+ \end{bmatrix} = R_\pi^{-1} \begin{bmatrix} \phi_1^2 \\ \phi_1'^2 \end{bmatrix} = \begin{bmatrix} \cos\theta_\pi & -\sin\theta_\pi \\ \sin\theta_\pi & \cos\theta_\pi \end{bmatrix} \begin{bmatrix} \phi_1^2 \\ \phi_1'^2 \end{bmatrix}. \tag{20}
$$

The value of the mixing angle, θ_π was previously written for an arbitrary potential V_0 and with the symmetry breaker,

Eq.(4) as:

$$\tan(2\theta_\pi) = \frac{-2y_\pi z_\pi}{y_\pi(1 - z_\pi^2) - x_\pi},\tag{21}$$

wherein,

$$x_\pi = \frac{2A}{\alpha}, \quad y_\pi = \left\langle \frac{\partial^2 V}{\partial\phi_2'^1 \partial\phi_1'^2} \right\rangle, \quad z_\pi = \frac{\beta}{\alpha}.\tag{22}$$

The specific values of x_π, y_π and z_π will depend on the particular potential. In the case of no symmetry breaking, there was a big simplification and $\tan\theta_\pi$ was given by just $-\beta/\alpha$. Furthermore, x_π and y_π would be respectively the squared pion mass and the squared $\pi(1300)$ mass in the absence of mixing. Clearly, the ratio x_π/y_π is a very small number in the general case. We now make use of this fact to solve for the first correction to the mixing angle obtained from Eq.(21):

$$\begin{aligned}
\sin\theta_\pi &= \frac{-\beta}{\sqrt{\alpha^2 + \beta^2}} \left[1 + \frac{\alpha^4}{(\alpha^2 + \beta^2)^2} \frac{x_\pi}{y_\pi} \right] + \cdots, \\
\cos\theta_\pi &= \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \left[1 - \frac{\alpha^2\beta^2}{(\alpha^2 + \beta^2)^2} \frac{x_\pi}{y_\pi} \right] + \cdots.
\end{aligned}\tag{23}$$

In these equations the first terms correspond to the massless pion case and the second terms to the leading x_π/y_π corrections when the pion mass is turned on.

An important application of Eq.(23) is to the axial vector current with the pion's quantum numbers. After taking account of the mixing in Eq.(20), the right hand side of the first Eq.(10) may be rewritten as:

$$(J_\mu^{axial})_1^2 = F_\pi \partial_\mu \pi^+ + F_{\pi'} \partial_\mu \pi'^+ + \cdots,\tag{24}$$

where,

$$\begin{aligned}
F_\pi &= 2\,\alpha\cos\theta_\pi - 2\,\beta\sin\theta_\pi, \\
F_{\pi'} &= 2\,\alpha\sin\theta_\pi + 2\,\beta\cos\theta_\pi.
\end{aligned}\tag{25}$$

In the zero pion mass case, $F_{\pi'}$ is seen to vanish. For the non zero pion mass case, we find, using Eq.(23),

$$\begin{aligned}
F_\pi &= 2\sqrt{\alpha^2 + \beta^2} + \mathcal{O}\left(\frac{x_\pi}{y_\pi}\right)^2, \\
F_{\pi'} &= \frac{-2\alpha^3\beta}{(\alpha^2 + \beta^2)^{3/2}} \left(\frac{x_\pi}{y_\pi}\right) + \mathcal{O}\left(\frac{x_\pi}{y_\pi}\right)^2.
\end{aligned}\tag{26}$$

It is seen that F_π does not change much while $F_{\pi'}$ picks up a non zero value. Thus the π' does not decouple from the axial vector current in the massive pion case. This means that PCAC does not strictly hold in the massive case and hence there is no reason to expect that the current algebra threshold theorem should be exactly correct in the present model.

As a check of the accuracy of the approximate formula (26) for $F_{\pi'}$ we made an exact numerical calculation using the specific potential in Eq.(15) and found $F_{\pi'} = -6.937 \times 10^{-4}$ GeV while, for the same parameters, the approximate formula gave $F_{\pi'} = -6.864 \times 10^{-4}$ GeV.

Now let us discuss the pion pion scattering in the threshold region. Our initial goal will be to see what results may be obtained for a general choice of chiral invariant potential, V_0 in Eq.(3). The general pattern of this discussion and the notation is given in sections V, VI and VII of [1] for the massless pion case. We start with the conventional $\pi - \pi$ scattering amplitude at tree level:

$$A(s,t,u) = -\frac{g}{2} + \sum_D \left(\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD} - s} + \frac{g_{0D}^2}{(\hat{X}_0^2)_{DD} - s} \right).\tag{27}$$

In this equation, g denotes the coefficient of the four point contact interaction among the physical (mass diagonal) pions. Furthermore g_{0D} denotes the three point coupling constants connecting the physical pions to the two physical

SU(3) singlet scalar mesons. Similarly g_{8D} stands for the coupling constants connecting the physical pions to the two physical scalar mesons which transform as the eighth component of an SU(3) octet. The usual Mandelstam variables, s, t, u are being employed. It is straightforward to numerically calculate the coupling constants just mentioned and the amplitude if the form of V_0 is specified [e.g. Eq. (15)]; this will be discussed in the next section. To proceed with the general case we note that, for example, the coupling constant g_{0D} may be written as:

$$g_{0D} = \left\langle \frac{\partial^3 V}{\partial \pi^+ \partial \pi^- \partial (S_{0p})_D} \right\rangle = \sum_{A,B,C} (R_\pi)_{A1} (R_\pi)_{B1} (L_0)_{CD} \left\langle \frac{\partial^3 V}{\partial (\phi_1^2)_A \partial (\phi_2^1)_B \partial (S_0)_C} \right\rangle. \tag{28}$$

As discussed in [1], there is a relation between the three point coupling constants on the right hand side and two point elements of the squared mass matrices. Such a relation follows from differentiation of Eq. (13) together with the use of Eq.(14). In fact it is similar in form to the relation obtained for the zero mass pion case:

$$g_{0D} = \frac{2}{\sqrt{3} F_\pi} (R_\pi)_{A1} (L_0)_{HD} \left[(X_0^2)_{AH} - (M_\pi^2)_{AH} \right], \tag{29}$$

wherein we have now adopted the convention of summing over repeated indices. The elements of the pion transformation matrix, $(R_\pi)_{A1}$ are the angles $\sin\theta_\pi$ and $\cos\theta_\pi$ given in Eq. (23) for the present non zero pion mass case. In the zero pion mass case, $\sin\theta_\pi$ and $\cos\theta_\pi$ may be rewritten in terms of just α and β as we see by setting the second and higher order terms on the right hand sides of Eq. (23) to zero. That is the form in which Eq. (29) exactly holds also for the massive pion case. Since Eq. (29) and its analog for the four point vertices play an important role in the proof of the current algebra theorem, we can only prove the theorem in the massive case when we make the (not too bad) approximation that x_π/y_π is zero.

With this approximation we can show, by generalizing the treatment given in [1], that the usual "current algebra" formula holds for the massive pion case. In the previous treatment, the second term on the right hand side of Eq.(29) made no contribution. Now we must take this term's contribution into account. To get the same delicate cancellation due to chiral symmetry we must expand the amplitude in powers of $s - m_\pi^2$ instead of simply powers of s. Explicitly,

$$
\begin{aligned}
A(s,t,u) = & -\frac{g}{2} + \left(\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD} - (\hat{M}_\pi^2)_{11}} + \frac{g_{0D}^2}{(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11}} \right) \\
& + \left(s - (\hat{M}_\pi^2)_{11} \right) \left(\frac{g_{8D}^2}{\left[(\hat{X}_a^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} + \frac{g_{0D}^2}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} \right) + \cdots.
\end{aligned}
\tag{30}
$$

Note that $(\hat{X}_a^2)_{DD}$ is a single number indexed by D. There is a huge simplification of the coefficients of the $(s - m_\pi^2)$ term:

$$
\begin{aligned}
& \frac{g_{0D}^2}{[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11}]^2} \\
& = \frac{4}{3F_\pi^2} (R_\pi)_{A1} \left[(X_0^2)_{AH} - (M_\pi^2)_{AH} \right] (L_0)_{HD} \frac{1}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} (R_\pi)_{C1} [(X_0^2)_{CK} - (M_\pi^2)_{CK}] (L_0)_{KD} \\
& = \frac{4}{3F_\pi^2} (R_\pi^{-1})_{1G} (L_0)_{GE} (L_0^{-1})_{EA} (X_0^2)_{AH} (L_0)_{HD} \frac{1}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} (L_0^{-1})_{DK} (X_0^2)_{KC} (L_0)_{CF} (L_0^{-1})_{FJ} (R_\pi)_{J1} - \\
& \frac{8}{3F_\pi^2} (R_\pi)_{A1} (M_\pi^2)_{AM} (R_\pi)_{MP} (R_\pi^{-1})_{PH} (L_0)_{HD} \frac{1}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} (L_0^{-1})_{DK} (X_0^2)_{KC} (L_0)_{CF} (L_0^{-1})_{FJ} (R_\pi)_{J1} + \\
& \frac{4}{3F_\pi^2} (R_\pi)_{A1} (M_\pi^2)_{AM} (R_\pi)_{MP} (R_\pi^{-1})_{PH} (L_0)_{HD} \frac{1}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} (R_\pi^{-1})_{1C} (M_\pi^2)_{CN} (R_{pi})_{NF} (R_\pi^{-1})_{FK} (L_0)_{KD} \\
& = \frac{4}{3F_\pi^2} (R_\pi^{-1})_{1G} (L_0)_{GE} \frac{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2}{\left[(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11} \right]^2} (L_0^{-1})_{EJ} (R_\pi)_{J1} = \frac{4}{3F_\pi^2}.
\end{aligned}
\tag{31}
$$

Similarly,

$$\frac{g_{8D}^2}{\left[(\hat{X}_a^2)_{DD} - (\hat{M}_\pi^2)_{11}\right]^2} = \frac{2}{3F_\pi^2}. \tag{32}$$

Next we must consider the contribution of the terms independent of $(s - m_\pi^2)$. The four point coupling constant g is approximately related to the matrix elements of the squared mass matrices (again with the proviso that the pion transformation matrices be similarly approximated) as:

$$g = \frac{8}{F_\pi^2}\left[\frac{1}{3}(R_\pi^{-1})_{1D}\left[(X_0^2)_{DJ} - (M_\pi^2)_{DJ}\right](R_\pi)_{J1} + \frac{1}{6}(R_\pi^{-1})_{1D}\left[(X_0^2)_{DJ} - (M_\pi^2)_{DJ}\right](R_\pi)_{J1}\right]. \tag{33}$$

(The analogous calculation for the zero mass pion case is given in section VII of [1].) Using calculations similar to Eq.(31) we also get:

$$\frac{g_{0D}^2}{(\hat{X}_0^2)_{DD} - (\hat{M}_\pi^2)_{11}} = \frac{4}{3F_\pi^2}(R_\pi^{-1})_{1D}\left[(X_0^2)_{DJ} - (M_\pi^2)_{DJ}\right](R_\pi)_{J1} \tag{34}$$

$$\frac{g_{8D}^2}{(\hat{X}_a^2)_{DD} - (\hat{M}_\pi^2)_{11}} = \frac{4}{6F_\pi^2}(R_\pi^{-1})_{1D}\left[(X_a^2)_{DJ} - (M_\pi^2)_{DJ}\right](R_\pi)_{J1} \tag{35}$$

Putting the last three equations into Eq.(30) we see that the sum of the terms independent of $(s - m_\pi^2)$ vanishes in the given approximation. The usual formula,

$$A(s,t,u) = \frac{2}{F_\pi^2}(s - m_\pi^2) + \cdots, \tag{36}$$

is thus obtained as an approximation.

It should be remarked that Eq. (36) corresponds to keeping only terms up to linear order in the expansion for $A(s,t,u)$. That means there are corrections, even at threshold, due to the masses of the scalars not being infinite. To see this and to summarize in a simple way, the preceding steps let us expand to one higher order, adopting a condensed notation in which m_i stands for the mass of any of the four scalars while g_i stands for the corresponding trilinear coupling constant of that scalar with two pions. The first four terms in the expansion of $A(s,t,u)$, as exactly obtained from Eq.(27), are

$$A(s,t,u) = -\frac{g}{2} + \sum_i \frac{g_i^2}{m_i^2 - m_\pi^2}\left[1 + \frac{s - m_\pi^2}{m_i^2 - m_\pi^2} + (\frac{s - m_\pi^2}{m_i^2 - m_\pi^2})^2 + \cdots\right]$$

$$\approx (s - m_\pi^2)\left[\frac{2}{F_\pi^2} + (s - m_\pi^2)\sum_i \frac{g_i^2}{(m_i^2 - m_\pi^2)^3} + \cdots\right]. \tag{37}$$

The exact first equation contains, for each m_i, a geometrical expansion in the quantity $(s - m_\pi^2)/(m_i^2 - m_\pi^2)$. Thus the radius of convergence in s for this expression is the squared mass of the lightest scalar singlet. To apply this expression in the resonance region we must, of course, unitarize the formula in some way. Here we will be content to look at the threshold region. In going from the first to the second equation we used the facts established above that (in the approximation where $F_{\pi'} = 0$): 1) the sum of the first two terms of the first equation vanishes and 2) the third term of the first equation simplifies to becomes the first, current algebra, term of the second equation. The third term of the second equation represents the leading correction to the usual current algebra formula. It depends on the masses of the scalar mesons and would vanish in a hypothetical limit (often used) in which the scalar meson masses are taken to infinity. Note that every term in the approximate amplitude vanishes for $s = m_\pi^2$, an unphysical point called the Adler zero [4]. Our derivation shows that the Adler zero follows from the generating equation (13), which in turn expresses the chiral invariance of the potential, V_0 and from the saturation of the axial vector current by the pion field (so-called partial conservation of the axial current). The second equation in (37) is an approximation, though a numerically good one, because the saturation of the axial current has been seen to be not strictly accurate in the present model.

The situation in the case of zero pion mass [1] is slightly different. There the amplitude is proportional to s so the Adler zero occurs at $s = 0$, which is also the threshold. Thus the current algebra amplitude as well as the corrections due to non-infinite mass scalar mesons vanish at threshold in the zero pion mass case.

In the above we found that the current algebra theorem for a general potential does not seem to be exactly correct. This small deviation and in addition the more important effect of the scalar mesons will next be calculated exactly, by numerical means, for the scattering amplitude using the leading choice of V_0 discussed in section III.

V. PION SCATTERING: EXACT NUMERICAL TREATMENT

In the exact numerical treatment we do not need to make use of the relations between two point and three point functions and between three point and four point functions since we adopt the specific potential V_0 given in Eq. (15). The needed quantities for calculating the scattering amplitude are displayed in Eq. (27): the four physical scalar singlet masses, the four three-point coupling constants conecting these scalars to two pions and the four-point pion physical coupling constant, g. These are obtained by, in turn, differentiating the potential twice with respect to two scalar fields ($\frac{\partial^2}{\partial S \partial S}$), two pseudoscalar fields as well as one scalar field ($\frac{\partial^3}{\partial S \partial \phi \partial \phi}$) and four pseudoscalar fields ($\frac{\partial^4}{\partial \phi \partial \phi \partial \phi \partial \phi}$). Furthermore we must use equations like Eq. (28) to relate the "bare" amplitudes obtained by such differentiations to the physical ones (ie, in mass diagonal bases for the fields). The matrices transforming the fields to mass diagonal bases are defined in Eq. (18) and are obtained by diagonalizing the relevant squared mass matrices.

For our purpose we *define* the current algebra result in terms of the expansion of the tree level amplitude $A(s,t,u)$ in powers of $(s - m_\pi^2)$, as displayed in Eq. (37). Specifically,

$$A(s,t,u) = C_0 + C_1(s - m_\pi^2) + \cdots, \tag{38}$$

where,

$$
\begin{aligned}
C_0 &= -\frac{g}{2} + \sum_i \frac{g_i^2}{m_i^2 - m_\pi^2}, \\
C_1 &= \sum_i \frac{g_i^2}{(m_i^2 - m_\pi^2)^2}.
\end{aligned}
\tag{39}
$$

The current algebra result requires C_0 to vanish and $C_1 = 2/F_\pi^2$.

Plots of C_0 and C_1 as functions of the model parameter $m[\pi(1300)]$ are shown in Figs. 4 and 5 respectively. Even though C_0 is small it is clearly non vanishing. Also C_1 deviates by a few percent from the current algebra prediction. To estimate the numerical accuracy of this calculation it was repeated for the case of zero pion mass. There it was found that $C_0 = \mathcal{O}(10^{-8})$ whereas it should be exactly zero. Thus the accuracy of the calculation method is several orders of magnitude more sensitive than the indicated effect. In this model, the Adler zero is shifted (by about $-C_0/C_1$) very slightly to the left of m_π^2.
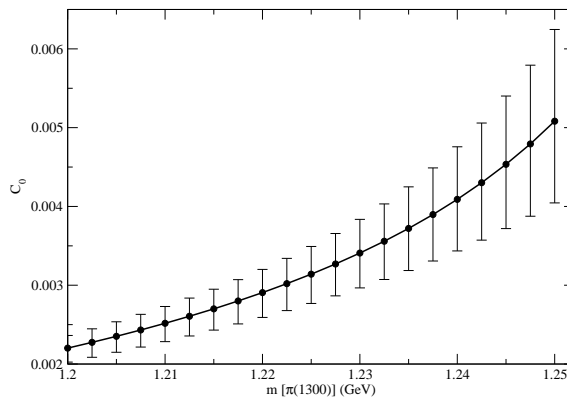

FIG. 4: Constant term, C_0 in expansion of invariant amplitude as a function of the model parameter $m[\pi(1300)]$. The error bars reflect the uncertainty in the mass of $a_0(1450)$. Current algebra would predict $C_0=0$.

The small deviations from the current algebra result just discussed seem to be beyond present experimental accuracy. On the other hand, the "beyond" current algebra contributions to $A(s,t,u)$ due to the higher than linear terms in the expansion shown in Eq. (37) seem to be highly relevant for comparison with present day experiments [5, 6, 7, 8]. These corrections would vanish in a limit where the scalar masses all go to infinity, which essentially corresponds to the use of a non-linear rather than the present linear type of sigma model.
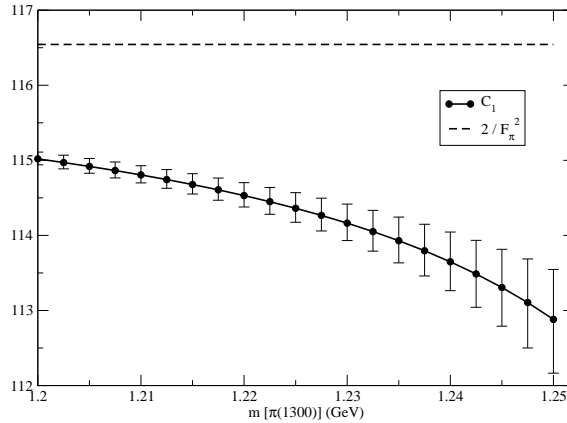
FIG. 5: Coefficient of linear term C_1 (in units of GeV^{-2}) in expansion of invariant amplitude compared with the current algebra result (dashed line) as a function of the model parameter $m[\pi(1300)]$. The error bars reflect the uncertainty in the mass of $a_0(1450)$.

It is usual to discuss the amplitudes near threshold in terms of their partial wave scattering lengths. The $J = 0$ scattering lengths are of course especially affected by the presence of light scalar mesons. Using the compact notation in Eq. (37), the (dimensionless) partial wave scattering lengths may be calculated to be:

$$m_\pi a_0^0 = \frac{1}{32\pi}\left[-\frac{5g}{2} + \sum_i g_i^2\left(\frac{3}{m_i^2 - 4m_\pi^2} + \frac{2}{m_i^2}\right)\right],$$

$$m_\pi a_0^2 = \frac{1}{32\pi}\left[-g + 2\sum_i \frac{g_i^2}{m_i^2}\right]. \tag{40}$$

These formulas are expressed in terms of the physical masses and coupling constants which are being computed exactly by numerical means. Note that the isospin label, I and the angular momentum label, J appear as a_J^I. For comparison, we may give the usual current algebra results [2]:

$$m_\pi a_0^0 = \frac{7m_\pi^2}{16\pi F_\pi^2},$$

$$m_\pi a_0^2 = \frac{-2m_\pi^2}{16\pi F_\pi^2}. \tag{41}$$

The results of our numerical calculation are shown in Fig. 6.

It is seen that the numerical calculation for the scattering length in the non resonant $I = 2$ channel gives about the same value, -0.04 as the current algebra result. In the resonant $I = 0$ s-wave channel the current algebra result of 0.15 is smaller than the result of the exact calculation for the range shown of the model parameter $m[\pi(1300)]$. The exact calculation result in this resonant channel varies strongly with $m[\pi(1300)]$ in contrast to the case in the non resonant channel. To understand what this means we should ask what is the significance of varying $m[\pi(1300)]$. Such a variation might be associated with variations in the masses of the four iso-singlet scalars. But the set up of the model as shown in Eq. (19) fixes the physical masses of the two octet isosinglets (at 985 MeV and 1474 MeV) . Furthermore we see in Fig. 1 that varying $m[\pi(1300)]$ leaves the mass of the heavier SU(3) singlet scalar essentially unchanged (at about 1500 MeV) while it changes the mass of the lightest SU(3) singlet scalar. Thus it seems unavoidable to interpret the variation of $m[\pi(1300)]$ as being associated with the variation of the mass of the lightest scalar. This is confirmed by noticing that the largest change in a_0^0 occurs in the region of $m[\pi(1300)]$ where the mass of the lightest scalar is changing most rapidly.

The correction to the current algebra result for a_0^0 due to the finite masses of light scalars was already discussed and noted to be positive in [9], some years ago. The contribution of a light scalar meson to the scattering length was recently calculated in [10].
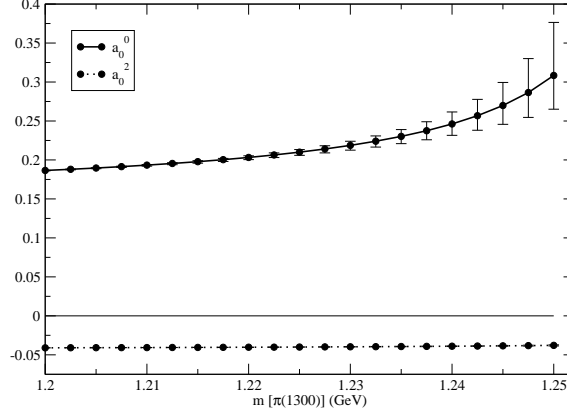
FIG. 6: Top curve: $I = J = 0$ scattering length, $m_\pi a_0^0$ vs. $m[\pi(1300)]$. Bottom curve: $I = 2$, $J = 0$ scattering length, $m_\pi a_0^2$ vs. $m[\pi(1300)]$. The error bars reflect the uncertainty of $m[a_0(1450)]$.

It is very interesting to examine the recent experimental data on the s-wave scattering lengths a_0^0 and a_0^2; these include the following.

NA48/2 collaboration [5]:

$$m_{\pi^+}(a_0^0 - a_0^2) = 0.264 \pm 0.015 \tag{42}$$

$$m_{\pi^+} a_0^0 = 0.256 \pm 0.011 \tag{43}$$

E865 Collaboration [6]:

$$m_{\pi^+} a_0^0 = 0.216 \pm 0.015 \tag{44}$$

DIRAC Collaboration [7]

$$m_{\pi^+} a_0^0 = 0.264^{+0.038}_{-0.020} \tag{45}$$

A general discussion of these experiments is given in [8].

Comparison of experiment with theory shows that the larger values of a_0^0 predicted by the numerical calculation when $m[\pi(1300)]$ is greater than about 1215 MeV, give good agreement. (This corresponds to the lightest scalar singlet lighter than about 460 MeV). In contrast, the current algebra prediction for a_0^0 is clearly too low. The nonresonant channel with ($I = 2$, $J = 0$) is not so well determined from experiment but seems to be consistent with the common prediction of current algebra or the numerical calculation.

The indicated value of the lightest scalar mass in our model is consistent with recent results [11] obtained by using Roy dispersion relation sum rules. The typical values obtained for the mass and width of the lightest scalar are M = 441 MeV and Γ = 544 MeV. These are also similar to what is obtained [12], M = 457 MeV and Γ = 632 MeV, by using a K-matrix unitarized three flavor linear sigma model (with just one chiral nonet). A unitarized two flavor linear sigma model was earlier given in [13]. The s wave pion pion interaction has recently [14] been discussed using the Adler sum rule. Actually, a long time ago the Adler sum rule was used [15] to suggest a light scalar with a similar mass to the above.

The encouraging result for a_0^0 with inclusion of scalar meson corrections corresponds to a tree level treatment of this linear sigma model. One may justifiably wonder whether the agreement would be spoiled by inclusion of loops, i.e. by a unitarization of the model. While this is a more complicated matter we may note that the simplest unitarization, the K matrix approach, does not change the result at all. In this approach the corrected scattering length would be,

$$m_\pi a_0^0(\text{corrected}) = \frac{m_\pi a_0^0}{1 - im_\pi a_0^0 \sqrt{1 - 4m_\pi^2/s}}, \tag{46}$$

where the kinematical square root is to be evaluated at threshold, $s = 4m_\pi^2$. Of course this is just an indication rather than a proof that the effect of unitarization is expected to be small for the scattering lengths.
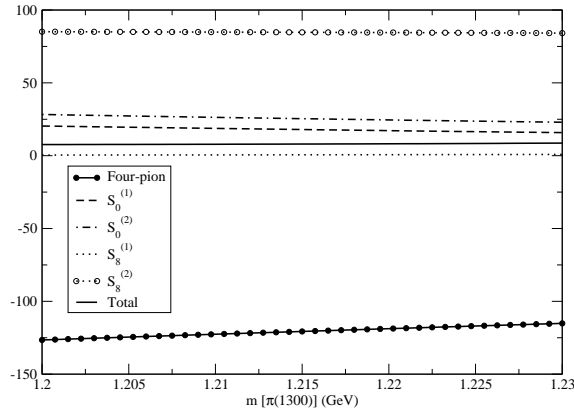
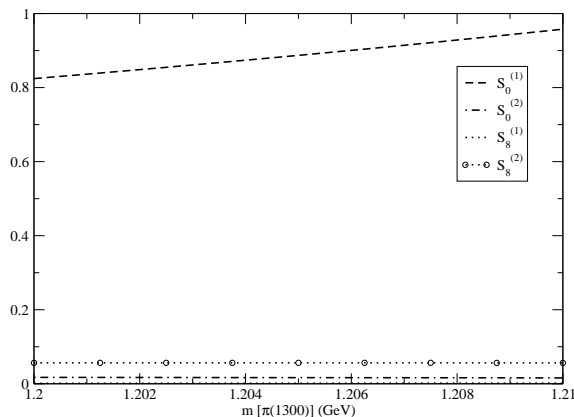FIG. 7: Individual contributions in Eq.(27) to $A(s,t,u)$ at threshold.

FIG. 8: Individual contributions to the second of Eq.(37) to $A(s,t,u)$ at threshold.

VI. DISCUSSION AND CONCLUSIONS

We have seen that the intuitive explanation for the existence of a very light scalar meson as well as light scalar mesons with large "four quark" content given in [1] (which used the simplifying assumption of zero pion mass) still is good when the physical pion mass is used. The main physical input is (almost) spontaneously broken chiral symmetry. Our model treats the chiral symmetry in its linear realization. This is normally considered inconvenient for the treatment of the scattering problem since it is known that there are very large cancellations between the four pion contact and scalar meson pole terms due to the symmetry. However it has the nice feature that it retains the light scalar mesons which give crucial corrections to the current algebra results. The inconvenience due to the large cancellations can be removed by using the Taylor expansion in Eq. (37). The third term in the second line of that equation shows that the beyond current algebra corrections can be neatly displayed in a physical way; they are seen to be of order $(m_\pi/m_i)^2$ compared to the current algebra term. Here m_i is a scalar meson mass. Still higher order corrections are suppressed by additional orders of $(m_\pi/m_i)^2$.

It is very interesting to illustrate how the use of the expanded form of the amplitude given in Eq. (37) leads to a nice picture. In Fig. 7 the individual contributions to the starting form of the amplitude, Eq. (27), associated with each scalar exchange as well as with the four point contact term are displayed. Evidently the contact term is the largest in magnitude and cancels off most of the other contributions. In addition the contribution from the lightest

scalar $S_0^{(1)}$ exchange, using this starting equation, has a very small magnitude, although its effect is expected to be the largest. Evidently, the intricate cancellations completely distort the underlying physics when expressed in this form. On the other hand, the use of the second of Eq.(37) clears things up, as may be seen from Fig. 8. There the correction to the current algebra result is seen to be completely dominated by the lightest scalar.

Clearly the s-wave pion pion scattering is rather nicely described by a linear sigma model. At the present stage it does not seem to matter which one is chosen for this aspect. The virtue of the present toy model is that it accomodates both 2 quark and 4 quark scalar nonets in a consistent way and may help in understanding the relation to QCD as earlier discussed [1]. Another interesting consequence of this linear formulation is the presence of heavy, largely four quark pseudoscalars, of which the $\pi(1300)$ is a possible candidate.

An amusing feature of the present model with a massive pion is that, due to a small deviation from exact PCAC, the current algebra result for near threshold pion pion scattering no longer holds exactly. Associated with this is the feature that the amplitude no longer vanishes at the unphysical point (Adler zero), $s = m_\pi^2$. The zero is shifted somewhat. The measure of this effect is the deviation of the coefficient, $F_{\pi'}$ from zero. We have seen that this is quite small since $F_{\pi'}$ is of the order $x_\pi/y_\pi \approx m_\pi^2/m_{\pi'}^2 \approx 0.01$, as may be seen from Eq. (26). On the other hand the similar effect should be more noticeable for the $K - K'$ system and for $K - \bar{K}$ scattering. In [16] it is shown that this system is formally analogous to the $\pi - \pi'$ system with the substitutions $x_\pi \to x_K$ and $y_\pi \to y_K$. But in this case $F_{K'}$ is expected to be of the order $m_K^2/m_{K'}^2 \approx 0.1$, ten times greater than for the pion case. A full calculation would require setting up the model with the inclusion of SU(3) symmetry breaking quark mass terms. This will be studied elsewhere.

A possible general question about the present model is that it introduces both states made of a quark and an antiquark as well as states with two quarks and two antiquarks. According to the usual 't Hooft large N_c extrapolation [17] of QCD the "four quark" states are expected to be suppressed. However, it was recently pointed out [18] that the alternative, mathematically allowed, Corrigan Ramond [19] extrapolation does not suppress the multiquark states. This kind of extrapolation may be relevant for understanding the physics of the light scalar mesons.

APPENDIX A: PARAMETER DETERMINATION

Given the inputs: the pion decay constant, F_π; the mass of the pion, m_π; the mass of the $a_0(980)$, m_a; the mass of the $a_0(1450)$, $m_{a'}$; the mass of the $\pi(1300)$, $m_{\pi'}$, the independent model parameters which don't involve the $U(1)_A$ violating terms can be successively determined (in the order given) by the equations:

$$2d_2 = \frac{m_a^2 m_{a'}^2 - m_\pi^2 m_{\pi'}^2}{m_a^2 + m_{a'}^2 - m_\pi^2 - m_{\pi'}^2}$$

$$\frac{A}{\alpha} = \frac{m_\pi^2 m_{\pi'}^2}{4 d_2}$$

$$(\alpha e_3^a)^2 = \frac{1}{64}\left((m_a^2 - m_{a'}^2)^2 - [4d_2 - (m_a^2 + m_{a'}^2)]^2\right)$$

$$4c_2 = m_a^2 + m_{a'}^2 - 2d_2 - \frac{56(\alpha e_3^a)^2}{d_2} - \frac{3}{2}m_\pi^2 m_{\pi'}^2$$

$$\frac{\beta}{\alpha} = \frac{-2(\alpha e_3^a)}{d_2}$$

$$\cos 2\theta_\pi = \frac{2 d_2 - 2 d_2 \left(\frac{\beta}{\alpha}\right)^2 - 2\left(\frac{A}{\alpha}\right)}{\sqrt{16 d_2^2 \left(\frac{\beta}{\alpha}\right)^2 + \left[2 d_2 - 2 d_2 \left(\frac{\beta}{\alpha}\right)^2 - 2\left(\frac{A}{\alpha}\right)\right]^2}}$$

$$\alpha = \frac{1}{2}\frac{F_\pi}{\cos\theta_\pi - \left(\frac{\beta}{\alpha}\right)\sin\theta_\pi}$$

$$c_4^a = \frac{1}{2\alpha^2}\left[c_2 + \frac{8(\alpha e_3^a)^2}{d_2} + \left(\frac{A}{\alpha}\right)\right] \tag{A1}$$

Once the above parameters are determined, the parameters γ_1 and c_3 of the $U(1)_A$ violating sector are obtained in terms of the mass of the $\eta(958)$, $m_{\eta 1}$ and the mass of a suitable heavier 0^- isosinglet, $m_{\eta 2}$ using the following procedure. The 2×2 prediagonal mass-squared matrix of the two SU(3) singlet pseudoscalars is written in the form:

$$(M_0^2) = \begin{bmatrix} -\frac{8c_3(2\gamma_1+1)^2}{3\alpha^2} + K_{11} & \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} + K_{12} \\ \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} + K_{12} & -\frac{8c_3(1-\gamma_1)^2}{3\beta^2} + K_{22} \end{bmatrix}, \tag{A2}$$

where K_{ij} is a real symmetric matrix involving the coefficients of the terms in V_0 which are $U(1)_A$ invariant . With the choice of invariant terms in Eq.(15) we have:

$$\begin{aligned} K_{11} &= -2\left(c_2 - 2c_4^a\alpha^2 + 4e_3^a\beta\right) \\ K_{12} &= -8e_3^a\alpha \\ K_{22} &= 2d_2 \end{aligned} \tag{A3}$$

Then, γ_1 is found as a solution of the quadratic equation:

$$\begin{aligned} 0 &= S\gamma_1^2 + T\gamma_1 + U, \\ S &= \frac{R}{\alpha^2}\left(4 + \frac{\alpha^2}{\beta^2}\right) - \frac{K_{11}}{\beta^2} + \frac{4K_{12}}{\alpha\beta} - \frac{4K_{22}}{\alpha^2}, \\ T &= \frac{R}{\alpha^2}\left(4 - 2\frac{\alpha^2}{\beta^2}\right) + \frac{2K_{11}}{\beta^2} - \frac{2K_{12}}{\alpha\beta} - \frac{4K_{22}}{\alpha^2}, \\ U &= \frac{R}{\alpha^2}\left(1 + \frac{\alpha^2}{\beta^2}\right) - \frac{K_{11}}{\beta^2} - \frac{2K_{12}}{\alpha\beta} - \frac{K_{22}}{\alpha^2}, \\ R &= \frac{4m_{\eta 1}^2 m_{\eta 2}^2 - \det(K)}{m_{\eta 1}^2 + m_{\eta 2}^2 - \mathrm{Tr}(K)}. \end{aligned} \tag{A4}$$

In addition,

$$c_3 = \frac{\frac{3}{8}\left(-m_{\eta 1}^2 m_{\eta 2}^2 + \det(K)\right)}{K_{11}\left(\frac{1-\gamma_1}{\beta}\right)^2 + 2K_{12}\left(\frac{1-\gamma_1}{\beta}\right)\left(\frac{1+2\gamma_1}{\alpha}\right) + K_{22}\left(\frac{1+2\gamma_1}{\alpha}\right)^2} \tag{A5}$$

Next we give the numerical values of the parameters for the central values of all the listed input masses except for $m[\pi(1300)]$ which instead will take the typical value allowed by both the data and by the model, 1215 MeV. Table I shows the results for the parameters which are not associated with the $U(1)_A$ violating part of the Lagrangian.

$c_2(\mathrm{GeV}^2)$	8.79×10^{-2}
$d_2(\mathrm{GeV}^2)$	6.30×10^{-1}
$e_3^a(\mathrm{GeV})$	-2.13
c_4^a	42.4
$\alpha(\mathrm{GeV})$	6.06×10^{-2}
$\beta(\mathrm{GeV})$	2.49×10^{-2}
$A(\mathrm{GeV}^3)$	6.66×10^{-4}

TABLE I: Calculated Lagrangian parameters: c_2, d_2, e_3^a, c_4^a and vacuum values: α, β.

Table II shows the calculated Lagrangian parameters associated with the $U(1)_A$ violating terms. Two "scenarios" associated with different identifications of the heavy η which is the partner of the $\eta(958)$ are shown (I assumes $\eta(1475)$ to be chosen while II assumes $\eta(1760)$ to be chosen.) For each scenario, the two solutions (labeled 1 and 2) are shown.

Using these parameters we next list the mixing matrices for, respectively, the two 0^- octet states, the two 0^+ octet states and the two 0^+ singlet states:

	I1	I2	II1	II2
$c_3(\text{GeV}^4)$	-2.39×10^{-4}	-2.38×10^{-4}	-3.42×10^{-4}	-3.37×10^{-4}
γ_1	5.33×10^{-1}	2.52×10^{-1}	8.68×10^{-1}	-8.65×10^{-2}

TABLE II: Calculated parameters: c_3 and γ_1.

$$(R_\pi^{-1}) = \begin{bmatrix} 0.923 & 0.385 \\ -0.385 & 0.923 \end{bmatrix}, \quad (L_a^{-1}) = \begin{bmatrix} -0.486 & 0.874 \\ 0.874 & 0.486 \end{bmatrix}, \quad (L_0^{-1}) = \begin{bmatrix} 0.706 & 0.708 \\ 0.708 & -0.706 \end{bmatrix}. \tag{A6}$$

Similarly, the mixing matrices for the two solutions for scenario I of the 0^- singlet states are:

$$I\,1 : (R_0^{-1}) = \begin{bmatrix} -0.671 & 0.742 \\ 0.742 & 0.671 \end{bmatrix}, \quad I\,2 : (R_0^{-1}) = \begin{bmatrix} 0.853 & -0.522 \\ 0.522 & 0.853 \end{bmatrix}. \tag{A7}$$

Finally, the mixing matrices for the two solutions for scenario II of the 0^- singlet states are:

$$II\,1 : (R_0^{-1}) = \begin{bmatrix} -0.411 & 0.912 \\ 0.912 & 0.411 \end{bmatrix}, \quad II\,2 : (R_0^{-1}) = \begin{bmatrix} 0.972 & -0.235 \\ 0.235 & 0.972 \end{bmatrix}. \tag{A8}$$

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