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We propose a systematic procedure to study a generalized linear sigma model which can give a physical picture of possible mixing between $q\bar{q}$ and $qq\bar{q}\bar{q}$ low lying spin zero states. In the limit of zero quark masses, we derive the model independent results for the properties of the Nambu Goldstone pseudoscalar particles. For getting information on the scalars it is necessary to make a specific choice of terms. We impose two plausible physical criteria - the modeling of the axial anomaly and the suppression of effective vertices representing too many fermion lines - for limiting the large number of terms which are allowed on general grounds. We calculate the tree-level spectrum based on the leading terms in our approach and find that it prominently exhibits a very low mass isosinglet scalar state. Finally we point out that the low energy result for scattering of pions continues to hold in the general version of the model.

Evidence has been accumulating [1] for a very light mass scalar-isoscalar particle, $f_0(600)$ as well as a possible similar scalar-isospinor particle, $K_0^*(800)$. As has been widely discussed, these may be joined with the well established scalars $a_0(980)$ and $f_0(980)$ to make a putative light scalar nonet. The upside down mass ordering of such a nonet suggests a four quark rather than a two quark structure; both qq-$\bar{q}\bar{q}$ [2] and $q\bar{q}$-$q\bar{q}$ [3] forms have been proposed. Either alternative would be of great importance for a full understanding of QCD in its nonperturbative low energy regime. The relation to the usual $q\bar{q}$ scalar mesons is of clear relevance in a such a picture. It has been suggested [4] that a mixing between the two quark and four quark nonets may help to better understand certain anomalies of the two quark nonet spectrum. The resulting picture is complicated and one may wonder, for example, whether the ordinary pions (believed to be of $q\bar{q}$ type) are chiral partners of the lighter four quark scalars or heavier two quark scalars. Historically, it has been of great value to study such questions in the framework of simple linear sigma models. Such a generalized linear sigma model was proposed in [5] and studied further in [6] and [7]. These papers have suggested the plausibility of a situation in which the lightest, approximate Nambu-Goldstone boson pseudoscalars are followed in ascending mass by scalars with relatively large four quark content. However, the model of interest may have many more terms than previously considered; for example if the interaction terms are restricted to be renormalizable, there are [7] 21 chiral invariant terms and 21 additional terms with the chiral transformation property of the QCD mass terms. In the present note we attempt to understand the essential structure more clearly, to differentiate between model dependent and model independent results as well as to suggest physical ways to choose the most important terms. As an aid we first simplify the analysis by setting the light quark masses to zero. It is accepted that this is a reasonable qualitative approximation since the largest parts of the masses of all particles made of light quarks, other than the lightest 0^- octet, are expected to arise from spontaneous breakdown of chiral symmetry.

The fields of our "toy" model consist of a 3×3 matrix chiral nonet field M, which represents $q\bar{q}$ type states as well as a 3×3 matrix chiral nonet field M', which represents four quark type states. They have the decompositions into scalar and pseudoscalar pieces: $M = S + i\phi$, $M' = S' + i\phi'$ and behave under $\mathrm{SU}(3)_\mathrm{L}\times\mathrm{SU}(3)_\mathrm{R}$ transformations as $M \rightarrow U_L M U_R^\dagger$ and $M' \rightarrow U_L M' U_R^\dagger$. However, the $\mathrm{U}(1)_\mathrm{A}$ transformation which acts at the quark level as $q_{aL}\rightarrow e^{i\nu}q_{aL}$, $q_{aR}\rightarrow e^{-i\nu}q_{aR}$ distinguishes the two fields [5] according to

$$M \rightarrow e^{2i\nu}M, \qquad M' \rightarrow e^{-4i\nu}M'. \tag{1}$$

Note that our treatment is based only on the symmetry structure and hence applies when M' is any linear combination of qq-$\bar{q}\bar{q}$ and $q\bar{q}$-$q\bar{q}$ type fields. We will be interested in the situation where non-zero vacuum values of S and S' may exist: $\langle S_a^b\rangle = \alpha\delta_a^b$, $\langle S_a'^b\rangle = \beta\delta_a^b$, corresponding to an assumed $\mathrm{SU}(3)_\mathrm{V}$ invariant vacuum. The Lagrangian density which defines our model is

$$\begin{aligned}\mathcal{L} = &- \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M\partial_\mu M^\dagger\right) - \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M'\partial_\mu M'^\dagger\right)\\ &- V_0\left(M, M'\right) - V_{SB},\end{aligned} \tag{2}$$

where $V_0(M, M')$ stands for a general function made from $\mathrm{SU}(3)_\mathrm{L}\times\mathrm{SU}(3)_\mathrm{R}$ (but not necessarily $\mathrm{U}(1)_\mathrm{A}$) invariants formed out of M and M'. The quantity V_{SB} which represents the effective chiral symmetry breaking light quark mass terms will be set to zero here.

The $\mathrm{U}(1)_\mathrm{A}$ transformation, which plays a special role in this model, suggests another useful simplification. In QCD there is a special instanton induced term- the "'t Hooft determinant" [8]- which breaks the $\mathrm{U}(1)_\mathrm{A}$ symmetry and can be modeled as $\det(M) + \det(M^\dagger)$. It thus may be natural to require all the terms to satisfy $\mathrm{U}(1)_\mathrm{A}$ invariance except for a particular subset which could model the $\mathrm{U(1)_A}$ anomaly. If one demands that, as in QCD, the $\mathrm{U(1)_A}$ variation of the effective Lagrangian be proportional to the gluonic axial anomaly then a similar effective term $\mathcal{L}_\eta = -c_3[\ln(\det(M)) - \ln(\det(M^\dagger))]^2$, where c_3 is a numerical parameter, seems appropriate [9].

A lot of information concerning especially the pseudoscalar particles in the model may be obtained in general without even specifying the terms in the potential. This may be achieved by studying "generating" equations which arise from the demand that the infinitesimal symmetry transformations in the model hold. For the nine axial transformations one finds [7]:

$$[\phi, \frac{\partial V_0}{\partial S}]_+ - [S, \frac{\partial V_0}{\partial \phi}]_+ + (\phi, S) \to (\phi', S') =$$
$$1[2\mathrm{Tr}(\phi' \frac{\partial V_0}{\partial S'} - S' \frac{\partial V_0}{\partial \phi'}) - 8c_3 i \ln(\frac{\det M}{\det M^\dagger})]. \quad (3)$$

To get constraints on the particle masses we will differentiate these equations once with respect to each of the two matrix fields ϕ and ϕ' and evaluate the equations in the ground state, taking into account the "minimum" conditions, $\langle \frac{\partial V_0}{\partial S} \rangle = 0$ and $\langle \frac{\partial V_0}{\partial S'} \rangle = 0$. Further differentiations with respect to all four matrix fields will similarly yield "model independent" information on 3 and 4 point vertices. We also require the Noether currents, $(J_\mu^{axial})_a^b = \alpha \partial_\mu \phi_a^b + \beta \partial_\mu {\phi'}_a^b + \cdots$, where the dots stand for terms bilinear in the fields. Using Eq.(3) the squared mass matrix which mixes the degenerate two quark and degenerate four quark pseudoscalar octets is:

$$(M_\pi^2) = y_\pi \begin{bmatrix} \beta^2/\alpha^2 & -\beta/\alpha \\ -\beta/\alpha & 1 \end{bmatrix}, \quad (4)$$

where $y_\pi = \langle \frac{\partial^2 V_0}{\partial {\phi'}_1^2 \partial {\phi'}_2^1} \rangle$. Clearly, $\det(M_\pi^2) = 0$ and the zero mass pion octet is a mixture of two quark and four quark fields. The transformation between the diagonal fields π^+ and π'^+ and the original pion fields is defined as:

$$\begin{bmatrix} \pi^+ \\ \pi'^+ \end{bmatrix} = R_\pi^{-1} \begin{bmatrix} \phi_1^2 \\ {\phi'}_1^2 \end{bmatrix} = \begin{bmatrix} \cos\theta_\pi & -\sin\theta_\pi \\ \sin\theta_\pi & \cos\theta_\pi \end{bmatrix} \begin{bmatrix} \phi_1^2 \\ {\phi'}_1^2 \end{bmatrix}, \quad (5)$$

which also defines the transformation matrix, R_π. The explicit diagonalization yields:

$$\tan\theta_\pi = -\frac{\beta}{\alpha}, \quad (6)$$

which may be interpreted as the ratio of the four quark condensate to the two quark condensate in the underlying QCD. We see that the mixing between the two quark pion and the four quark pion would vanish if the four quark condensate were to vanish in this model. Rewriting the Noether current as $(J_\mu^{axial})_1^2 = F_\pi \partial_\mu \pi^+ + F_{\pi'} \partial_\mu \pi'^+ + \cdots$ shows that

$$F_\pi = 2\sqrt{\alpha^2 + \beta^2}, \qquad F_{\pi'} = 0. \quad (7)$$

Note that the physical higher mass pion state decouples from the axial current. Altogether the (eight) zero mass pseudoscalars are characterized by the three parameters α, β and y_π. On the other hand, there are only two experimental inputs: $F_\pi = 131$ MeV and the mass of $\pi(1300)$, the presumed higher mass pion candidate. Thus, the interesting question of the four quark content of the pion has an inevitably model dependent answer.

Next let us consider the "model independent" information available for the two pseudoscalar SU(3) singlet states. This sector is related to the QCD axial anomaly. In the single M model, the anomaly can be modeled by the term $\mathcal{L}_\eta$ mentioned above. In the M-M' model under consideration this form is no longer unique and it is natural to consider a generalization [10] in which $\ln(\det(M)/\det(M^\dagger))$ is replaced by $\gamma_1[\ln(\det(M)/\det(M^\dagger))] + (1 - \gamma_1)[\ln(\mathrm{Tr}(MM'^\dagger)/\mathrm{Tr}(M'M^\dagger))]$, where γ_1 is a dimensionless parameter. Then the squared mass matrix which mixes the two SU(3) pseudoscalar singlet states is obtained as:

$$(M_0^2) = \begin{bmatrix} -\frac{8c_3(2\gamma_1+1)^2}{3\alpha^2} + z_0^2 y_0 & -z_0 y_0 + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} \\ -z_0 y_0 + \frac{8c_3(1-\gamma_1)(2\gamma_1+1)}{3\alpha\beta} & y_0 - \frac{8c_3(1-\gamma_1)^2}{3\beta^2} \end{bmatrix}. \quad (8)$$

Here $z_0 = -2\beta/\alpha$ and $y_0 = \langle \frac{\partial^2 V}{\partial \phi_0' \partial {\phi'}_0} \rangle$. Note that when c_3 is set to zero, making the entire Lagrangian $\mathrm{U(1)_A}$ invariant, $\det(M_0^2) = 0$. Then one of the singlet pseudoscalars becomes, as well known, a Nambu Goldstone particle. This occurs in the large number of colors limit but we will not make that approximation here.

In order to get information about the scalar meson masses and mixings as well as to complete the description of the pseudoscalars it is necessary to make a specific choice of interaction terms. To proceed in a systematic way we define the following quantity for each term,

$$N = 2n + 4n', \quad (9)$$

where n and n' are respectively the number of M fields and the number of M' fields contained in that term. We shall restrict our choice to the lowest non-trivial value of N, which corresponds physically to the total number of quark and antiquark lines at each vertex. In addition to the two special terms which saturate the $\mathrm{U(1)_A}$ anomaly already mentioned, this gives the leading (N=8) potential

$$\begin{aligned} V_0 = & - c_2\,\mathrm{Tr}(MM^\dagger) + c_4^a\,\mathrm{Tr}(MM^\dagger MM^\dagger) \\ & + d_2\,\mathrm{Tr}(M'M'^\dagger) + e_3^a(\epsilon_{abc}\epsilon^{def} M_d^a M_e^b M_f'^c + h.c.) \\ & + \cdots, \end{aligned} \quad (10)$$

where the dots stand for the $\mathrm{U(1)_A}$ non-invariant terms. For simplicity, we have neglected the N=8 term, $c_4^b[\mathrm{Tr}(MM^\dagger)]^2$ which is suppressed, in the single M model by the quark line rule. It may be noted that the quantities $\det(M)$ and $\mathrm{Tr}(MM'^\dagger)$ which enter into those two terms which saturate the $\mathrm{U}(1)_\mathrm{A}$ anomaly have N=6. In this counting scheme, $\mathrm{U}(1)_\mathrm{A}$ invariant terms with N=12 (and higher) might be successively added to improve the approximation. The minimum equations for this potential are:

$$\left\langle \frac{\partial V_0}{\partial S_a^a} \right\rangle = 2\,\alpha\,\left[-c_2 + 2\,c_4^a\,\alpha^2 + 4\,e_3^a\,\beta\right] = 0, \tag{11}$$

$$\left\langle \frac{\partial V_0}{\partial S'^a_a} \right\rangle = 2\left[d_2\,\beta + 2\,e_3^a\,\alpha^2\right] = 0. \tag{12}$$

The $\mathrm{U}(1)_\mathrm{A}$ violating c_3 terms do not contribute to these equations. Notice that α is an overall factor in Eq. (11) so that, in addition to the physical spontaneous breakdown solution where $\alpha \neq 0$ there is a solution with $\alpha = 0$. On the other hand, β is not an overall factor of Eq. (12) and it is easy to see that β, which measures the "4 quark condensate", is necessarily non-zero in the physical situation where α is non-zero. From the specific form in Eq.(10) we find the mixing squared mass matrices for the degenerate octet scalars as well as the SU(3) singlet scalars:

$$(X_a^2) = \begin{bmatrix} 2\,\left[-c_2 + 6\,c_4^a\,\alpha^2 - 2\,e_3^a\,\beta\right] & -4\alpha e_3^a \\ -4\alpha e_3^a & 2d_2 \end{bmatrix} \tag{13}$$

$$(X_0^2) = \begin{bmatrix} 2\,\left[-c_2 + 6\,c_4^a\,\alpha^2 + 4\,e_3^a\,\beta\right] & 8\alpha e_3^a \\ 8\alpha e_3^a & 2d_2 \end{bmatrix} \tag{14}$$

Now let us consider the comparison of the model with experiment. To start with there are 8 parameters (α, β, c_2, d_2, c_4^a, e_3^a, c_3 and γ_1). The last two parameters appear only in the mass matrices of the pseudoscalar SU(3) singlets and are conveniently discussed separately. The other six are effectively reduced to four by using the two minimum equations (11) and (12). As the corresponding four experimental inputs [1] we take the non-strange quantities:

$$\begin{aligned} m(0^+\text{octet}) &= m[a_0(980)] = 984.7 \pm 1.2\,\text{MeV} \\ m(0^+\text{octet}') &= m[a_0(1450)] = 1474 \pm 19\,\text{MeV} \\ m(0^-\text{octet}') &= m[\pi(1300)] = 1300 \pm 100\,\text{MeV} \\ F_\pi &= 131\,\text{MeV} \end{aligned} \tag{15}$$

Evidently, the largest experimental uncertainty appears in the mass of $\pi(1300)$; we shall consider the other masses as fixed at their central values and vary this mass in the indicated range. From studying the scalar SU(3) singlet states we find the consistency condition for positivity of the eigenvalues of their squared mass matrix, Eq.(14):

$$m[\pi(1300)] < 1302\,\text{MeV}. \tag{16}$$

The model predicts, as $m[\pi(1300)]$ varies from 1200 to 1300 MeV,

$$\begin{aligned} m(0^+\text{singlet}) &= 510 \rightarrow 28\ \text{MeV}, \\ m(0^+\text{singlet}') &= 1506 \rightarrow 1555\ \text{MeV}. \end{aligned} \tag{17}$$

Clearly, the most dramatic feature is the very low mass of the lighter SU(3) singlet scalar meson. Of course, one expects the addition of quark mass type terms to modify the details somewhat.

To calculate the masses of the SU(3) singlet pseudoscalars we must diagonalize Eq.(8) with the specific choices of parameters $y_0 = 2d_2$ and $z_0 = 4e_3^a\alpha/d_2$ corresponding to the potential of Eq.(10). This enables us to fit in principle, for any choice of $m[\pi(1300)]$, the two parameters c_3 and γ_1 in terms of the experimental masses of $\eta(958)$ and one of the candidates $\eta(1295)$, $\eta(1405)$, $\eta(1475)$ and $\eta(1760)$. However, it turns out that the positivity of the eigenvalues of the matrix (M_0^2) imposes additional constraints on the choice of $m[\pi(1300)]$ in Eq.(16). Furthermore the first two candidates for the heavier η are also ruled out on grounds of this positivity. For $\eta(1475)$ the allowed range of $m[\pi(1300)]$ is restricted to 1200 to 1230 MeV. On the other hand, there is no additional restriction if $\eta(1760)$ is chosen. If the choice of $\eta(1475)$ is made, the predicted range of $m(0^+\text{singlet})$ is narrowed from that given in Eq.(17) to 510 $\rightarrow$ 410 MeV.

It is very interesting to see what the model has to say about the four quark percentages of the particles it describes. These percentages are displayed in Fig.1 as functions of the precise value of the input parameter $m[\pi(1300)]$. The pion four quark content (equal to $100\,\sin^2\theta_\pi$) is seen to be about 17 percent. Of course the heavier pion would have about an 83 percent four quark content. On the other hand, the octet scalar states present a reversed picture: the $a_0(980)$ has a large four quark content while the $a_0(1450)$ has a smaller four quark content. The very light and the rather heavy 0^+ singlets are about maximally mixed, having roughly equal contributions from the 4 quark and 2 quark components.

The perhaps more plausible scenario in the case of the 0^- singlets takes $\eta(1475)$ as the heavy 0^- singlet state. Fig. 1 shows that there are two solutions for each value of $m[\pi(1300)]$; the dotted line gives a mainly $q\bar{q}$ content while the solid line gives a mainly four quark content. Note that this scenario does not allow $m[\pi(1300)]$ to be higher than about 1230 MeV. The choice (not shown) of $\eta(1760)$ as the partner of $\eta(958)$ also leads to two solutions with small and large two quark content.

There are two reasons for next briefly discussing the pi-pi scattering in this model. First, since the iso-singlet scalar resonances above are being considered at tree level, one expects, as can be seen in the single M model also discussed in [5] and at the two flavor level in [11], that unitarity corrections for the scattering amplitudes will alter their masses and widths. Second, since the pion looks unconventional in this model (having a non-negligible four

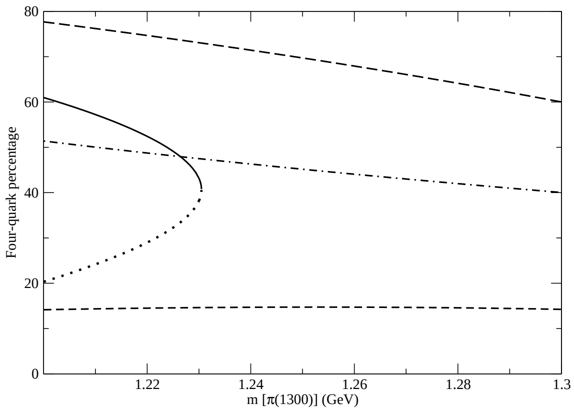

FIG. 1: Four quark percentages of the pion (dashed line), the $a_0(980)$ (top long-dashed line), the very light 0^+singlet (dotted-dashed line) and the $\eta(958)$ in the scenario where the higher state is identified as the $\eta(1475)$ (curve containing both solid and dotted pieces) as functions of the undetermined input parameter, $m[\pi(1300)]$. Note that there are two solutions for the $\eta(958)$: the dotted curve choice gives it a predominant two quark structure and the solid curve choice, a larger four quark content.

quark component) one might worry that the fairly precise "current algebra" formula for the near to threshold scattering amplitude might acquire unacceptably large corrections. In the present massless pion model, this formula [12] should read,

$$A(s,t,u) = 2s/F_\pi^2, \tag{18}$$

where $A(s,t,u)$ is the conventional amplitude term expressed in terms of the Mandelstam variables s, t and u. To obtain $A(s,t,u)$ in the present model, one needs the four point vertices involving the pseudoscalar octet fields as well as the three point vertices involving two pseudoscalar octet fields and one scalar field. It turns out [13] that the result Eq.(18) follows in a "model independent" way just by using the generating Eq.(3): the four point vertices can be related to the three point vertices, which can in turn be related to the two point vertices (masses). For example, the three point vertices involving the SU(3) singlet scalars can be related to the scalar and pseudoscalar squared mass matrices as:

$$\frac{\sqrt{3}F_\pi}{2}\sum_B (R_\pi^{-1})_{1B}\langle\frac{\partial^3 V_0}{\partial(\phi_1^2)_A\partial(\phi_2^1)_B\partial(S_0)_H}\rangle = (X_0^2)_{AH} - (M_\pi^2)_{AH}. \tag{19}$$

Here the capital Latin subscripts refer to summation over the unprimed and primed fields, (M_π^2) is given in Eq.(4) and (X_0^2) is the model independent version of Eq.(14). It is interesting to note that the current algebra theorem will "tolerate" any amount of four quark component in the massless pion. The present model clearly shows that while the (lighter) pion is mainly two quark, the lighter scalars have very large four quark components. This is perhaps the opposite of what one might initially think and is related to the characteristic mixing pattern emerging in a transparent form here.

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