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Resolutions of Subsets of Finite Sets of Points in Projective Space

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RESOLUTIONS OF SUBSETS OF FINITE SETS OF POINTS IN PROJECTIVE SPACE

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Abstract. Given a finite set, X, of points in projective space for which the Hilbert function is known, a standard result says that there exists a subset of this finite set whose Hilbert function is "as big as possible" inside X. Given a finite set of points in projective space for which the minimal free resolution of its homogeneous ideal is known, what can be said about possible resolutions of ideals of subsets of this finite set? We first give a maximal rank type description of the most generic possible resolution of a subset. Then we show that this generic resolution is not always achieved, by incorporating an example of Eisenbud and Popescu. However, we show that it *is* achieved for sets of points in projective two space: given any finite set of points in projective two space for which the minimal free resolution is known, there must exist a subset having the predicted resolution.

1. Introduction

We work over an algebraically closed field, k. Let $X = \{P_1, \dots, P_d\}$ be a finite set of d distinct points in projective n-space over k, $\mathbb{P}^n$. Associated to X we have its homogeneous ideal $I(X) \subset k[x_0, \dots, x_n] = S$ and its homogeneous coordinate ring $S(X) = S/I(X)$. A fundamental invariant of X is its Hilbert function, h_X, defined to be the Hilbert function of $S(X)$:

$$h_X(t) = \dim_k S(X)_t.$$

Lacking some uniformity property such as the Uniform Position Property (UPP), the subsets of X of fixed cardinality may have different Hilbert functions. Given this, it is somewhat surprising at first glance that there is always at least one subset with a predetermined Hilbert function. Indeed, one of the fundamental results about Hilbert functions of subsets of X is the following:

Lemma 1.1. *Fix an integer e, $1 \le e < d$. Then there exists a subset Y of X of exactly e points such that*

$$h_Y(t) = \min\{h_X(t), e\}.$$

Proof. This is very well known. See for instance [GMR], Lemma 2.5 (c). See also Remark 4.5. □

The Hilbert function is a very coarse measure of the properties of X. Related finer measures that are often studied are the graded Betti numbers and twists of the minimal graded free resolution of $S(X)$ or equivalently $I(X)$.

$$(1.1) \qquad 0 \to F_{n-1} \to \cdots \to F_1 \to F_0 \to I(X) \to 0$$

where $F_i = \oplus_{j=1}^{r_i}(S(-\gamma_{ij}))^{\alpha_{ij}}$. The γ_{ij} are the twists and the α_{ij} are the graded Betti numbers. Since Lemma 1.1 is so useful for Hilbert functions, one may wonder whether there is a corresponding result for resolutions.

In section 2 we state a natural first guess at a possible generalization of Lemma 1.1 to resolutions. The guess is stated in terms of Koszul cohomology. The basic idea of the guess is that at least one subset of X of each cardinality should behave as generically as possible subject to some obvious constraints imposed by being a subset of X. The guess is very similar to the Minimal Resolution Conjecture of Lorenzini [L2] except that the Minimal Resolution Conjecture does not deal with subsets.

In section 3 we show that the guess of section 2 is incorrect. In fact a counterexample to the Minimal Resolution Conjecture provided in [EP] is used to construct a counterexample to the guess. While the Minimal Resolution Conjecture did not turn out to be true in full generality, it is true in many cases and is perhaps a good first start at understanding the true situation. (The end of the introduction to [HS] contains a good list of references to results about the Minimal Resolution Conjecture.) One might still hope that the guess of section 2 would behave similarly. We make this hope more precise with some questions at the end of section 3.

In sections 4 and 5 we answer these questions for $\mathbb{P}^2$ by showing that the guess is true for sets of points in $\mathbb{P}^2$ (a place where the Minimal Resolution Conjecture of Lorenzini is also known to be true). A variety of tools are used to carry this out. We divide the problem into four cases, depending on the number of minimal generators of $I(X)$ in the maximum possible degree. The three easiest of these cases are treated in section 4. The most difficult is the case where $I(X)$ has two minimal generators in this degree, and this case is treated in section 5. Here we combine liaison theory with a careful study of certain sections of a certain twist of $\Omega^1_{\mathbb{P}^2}$, the sheaf of differential one-forms on $\mathbb{P}^2$.

2. A First Guess

We first recall briefly how the graded Betti numbers of an ideal may be computed using Koszul cohomology; see [G] section 1 for more details. One makes a complex

(2.1)

$$\cdots \to \bigwedge^{p+1} S_1 \otimes I(X)_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} S_1 \otimes I(X)_{q} \xrightarrow{d_{p,q}} \bigwedge^{p-1} S_1 \otimes I(X)_{q+1} \xrightarrow{d_{p-1,q+1}} \bigwedge^{p-2} S_1 \otimes I(X)_{q+2} \to \dots$$

where $d_{p,q}(l_1 \wedge l_2 \wedge \cdots \wedge l_p \otimes f) = \sum_{i=1}^{p}(-1)^{p-i} l_1 \wedge \cdots \wedge l_{i-1} \wedge l_{i+1} \wedge \cdots \wedge l_p \otimes l_i f$.

In the resolution (1.1) the exponent of $S(-(p+q))$ in F_p is the dimension, as a vector space over k, of the cohomology group

$$\frac{\ker d_{p,q}}{\text{im } d_{p+1,q-1}}.$$

Of course an exponent of 0 means that $S(-(p+q))$ does not appear. One certainly knows the dimensions of the vector spaces $\bigwedge^i S_1$. If one also knew the Hilbert function of X and thus the dimensions of the vector spaces $I(X)_j$, then to compute all the graded Betti numbers and twists for $I(X)$ it would be sufficient to know the ranks of all the maps $d_{i,j}$. Thus, our guess will combine Lemma 1.1 with a guess about these ranks.

As before $X = \{P_1, \dots, P_d\}$ consists of d distinct points. We assume that the Hilbert function and resolution of X are known. We fix an integer $1 \le e < d$. We guess that there should exist a subset Y of X of exactly e points such that the graded Betti numbers and twists of the graded minimal free resolution of $I(Y)$ are determined as follows in (a) and (b).

(a) The Hilbert function of Y is as in Lemma 1.1. Since $Y \subset X$, for each i, $I(X)_i \subset I(Y)_i$ and we may compare the complex (2.1) for $I(X)$ and the corresponding one for $I(Y)$ by the following commutative diagram.

$$\begin{array}{ccccc} \cdots \to & \bigwedge^p S_1 \otimes I(X)_q & \xrightarrow{d_{p,q}} & \bigwedge^{p-1} S_1 \otimes I(X)_{q+1} & \to \dots \\ & \bigcap & & \bigcap & \\ \cdots \to & \bigwedge^p S_1 \otimes I(Y)_q & \xrightarrow{e_{p,q}} & \bigwedge^{p-1} S_1 \otimes I(Y)_{q+1} & \to \dots \end{array}$$

Assuming (a), we know the dimensions of all the $I(Y)_j$. We then guess that the ranks of the $e_{i,j}$ will be as follows

(b) Of course $e_{0,p+q}$ is the zero map. Having determined the rank of $e_{i,p+q-i}$, the rank of $e_{i+1,p+q-i-1}$ is as large as possible subject to the two constraints:
   (i) $\ker e_{i+1,p+q-i-1}$ must contain $\ker d_{i+1,p+q-i-1}$
   (ii) im $e_{i+1,p+q-i-1}$ must be contained in $\ker e_{i,p+q-i}$.
   In other words rank $e_{i+1,p+q-i-1}$ is the smaller of
   (i′) $\dim \bigwedge^{i+1} S_1 \otimes I(Y)_{p+q-i-1} - \dim \ker d_{i+1,p+q-i-1}$ and
   (ii′) $\dim \ker e_{i,p+q-i}$.

3. A Counter-Example to the First Guess

As mentioned in the introduction this guess is similar to the Minimal Resolution Conjecture of Lorenzini. They are both Maximal Rank Conjectures in that they conjecture that certain vector space maps have ranks as large as possible. It is not surprising, therefore, that one can construct a counterexample to the guess out of a counterexample to the Minimal Resolution Conjecture.

In the introduction to [EP] they point out that for 11 general points in $\mathbb{P}^6$ the Minimal Resolution Conjecture predicts the resolution to be

$$0 \to S(-8)^4 \to S(-7)^{18} \to S(-6)^{25} \oplus S(-5)^4 \to S(-4)^{45} \to S(-3)^{46} \to S(-2)^{17} \to I \to 0$$

whereas the actual resolution is

$$0 \to S(-8)^4 \to S(-7)^{18} \to S(-6)^{25} \oplus S(-5)^5 \to S(-5)^1 \oplus S(-4)^{45} \to S(-3)^{46} \to S(-2)^{17} \to I \to 0.$$

From [L1] section 3 or [L2] section 3 we know that the Minimal Resolution Conjecture is true for 22 general points in $\mathbb{P}^6$. Thus one can work out that the resolution of 22 general points in $\mathbb{P}^6$ is

$$0 \to S(-8)^{15} \to S(-7)^{84} \to S(-6)^{190} \to S(-5)^{216} \to S(-4)^{120} \to S(-2)^6 \oplus S(-3)^{20} \to I \to 0.$$

Any subset of 11 points of 22 general points is a set of 11 general points. Let us see what the guess predicts as the resolution of 11 points contained in 22 general points. The crucial term is $S(-5)$ so we compute only that. The relevant Koszul complex to look at is

$$(3.1)\quad 0 \to \bigwedge^3 S_1 \otimes I_2 \xrightarrow{d_{3,2}} \bigwedge^2 S_1 \otimes I_3 \xrightarrow{d_{2,3}} \bigwedge^1 S_1 \otimes I_4 \xrightarrow{d_{1,4}} \bigwedge^0 S_1 \otimes I_5 \to 0.$$

When I is the ideal of 22 general points, using that these points have generic Hilbert function $1, 7, 22, 22, \ldots$, one easily computes the dimensions in (3.1) as

$$0 \to 210 \xrightarrow{d_{3,2}} 1302 \xrightarrow{d_{2,3}} 1316 \xrightarrow{d_{1,4}} 440 \to 0.$$

To get the known resolution we must then have

$$\begin{aligned} \operatorname{rank} d_{1,4} &= 440 \qquad & \dim\ker d_{1,4} &= 1316 - 440 = 876 \\ \operatorname{rank} d_{2,3} &= 876 \qquad & \dim\ker d_{2,3} &= 1302 - 876 = 426 \\ \operatorname{rank} d_{3,2} &= 210 \qquad & \dim\ker d_{3,2} &= \ \ 210 - 210 = 0 \end{aligned}$$

so that $\dim \frac{\ker d_{2,3}}{\text{im } d_{3,2}} = 426 - 210 = 216$.

When I is the ideal of 11 general points, one again easily computes the dimensions in (3.1) as

$$0 \to 595 \xrightarrow{e_{3,2}} 1533 \xrightarrow{e_{2,3}} 1393 \xrightarrow{e_{1,4}} 451 \to 0$$

Applying the guess we get

$$\begin{aligned} \operatorname{rank} e_{1,4} &= 451 \qquad & \dim\ker e_{1,4} &= 1393 - 451 = 942 \\ \operatorname{rank} e_{2,3} &= 942 \qquad & \dim\ker e_{2,3} &= 1533 - 942 = 591 \\ \operatorname{rank} e_{3,2} &= 591 \qquad & \dim\ker e_{3,2} &= \ \ 595 - 591 = 4 \end{aligned}$$

so that $\dim \frac{\ker e_{2,3}}{\text{im } e_{3,2}} = 591 - 591 = 0$ and $\dim \frac{\ker e_{3,2}}{\text{im } e_{4,1}} = 4 - 0 = 4$. That is, the guess predicts the same resolution for 11 general points in $\mathbb{P}^6$ as the Minimal Resolution Conjecture, which is wrong.

The guess does give some restrictions on what resolutions of subsets of X can be. Conditions (i) and (ii) must always be satisfied, but the rank could be smaller than this upper bound. Also, because of the inductive nature of the upper bounds, once one $e_{i,j}$ fails to achieve the upper bound, the upper bounds on $e_{i+s,j-s}$ for $s \geq 1$ can change.

The above calculations give some preliminary evidence that the following question may have an affirmative answer. See also Remark 5.6.

Question 3.1. *When X is a general set of d points in projective space, does the guess for a subset of $e < d$ points of X always give the same graded Betti numbers as those given by the Minimal Resolution Conjecture for a general set of e points?*

Since the known counter-examples to the Minimal Resolution conjecture go wrong in the "middle" of the resolution, it may well be that parts of the Minimal Resolution Conjecture are always true. In particular, it is known that the Cohen-Macaulay Type Conjecture is true ([TV], [La]), and one naturally wonders if the Ideal Generation Conjecture is true. This leads to the second question:

Question 3.2. *Is the guess true at least at the ends of the resolution? In particular, given any finite set of points in $\mathbb{P}^n$, is there always a subset with the predicted minimal generators and the predicted Cohen-Macaulay type?*

Note that the guess does not assume that we have a general set of points, or even that we have some sort of uniformity! The next two sections show that Question 3.2 has an affirmative answer for subsets of $\mathbb{P}^2$.

4. The Subset Resolution Theorem for points in $\mathbb{P}^2$

We now restrict our attention to points in $\mathbb{P}^2$. Let $X = \{P_1, \dots, P_d\}$ be a set of d distinct points, with homogeneous ideal $I = I(X) \subset k[X_0, X_1, X_2] = S$. At first glance the Koszul complex would seem to involve sequences of the form

$$0 \to \bigwedge^3 S_1 \otimes I_{s-2} \xrightarrow{d_{3,s-2}} \bigwedge^2 S_1 \otimes I_{s-1} \xrightarrow{d_{2,s-1}} \bigwedge^1 S_1 \otimes I_s \xrightarrow{d_{1,s}} \bigwedge^0 S_1 \otimes I_{s+1} \to 0$$

However, we know that the graded minimal free resolution for I has only two terms, so we must have that $d_{3,s-2}$ is injective and $\ker d_{2,s-1} = \text{im } d_{3,s-2}$. The only thing in question is the rank of the map $d_{1,s}$. This is just the multiplication map

$$\mu_s : S_1 \otimes I_s \to I_{s+1}$$
$$L \otimes F \mapsto LF.$$

Definition 4.1. If $Y \subset X$ has the Hilbert function given in Lemma 1.1, we will say that it has *truncated Hilbert function*.

For any subset $Z \subset X$ and any positive integer s, we have a commutative diagram

$$\begin{array}{ccc} S_1 \otimes I(X)_s & \xrightarrow{\mu_{s,X}} & I(X)_{s+1} \\ \bigcap & & \bigcap \\ S_1 \otimes I(Z)_s & \xrightarrow{\mu_{s,Z}} & I(Z)_{s+1} \end{array} \tag{4.1}$$

The subset resolution guess for points in $\mathbb{P}^2$ then becomes the following.

Theorem 4.2. *Let X be a reduced set of d points in $\mathbb{P}^2$. Fix an integer m, $1 \leq m < d$. Then there exists a subset $Z \subset X$ of cardinality m and with truncated Hilbert function, as given in Lemma 1.1, and such that for all positive integers s*

$$\text{rank } \mu_{s,Z} = \min\{\dim I(Z)_{s+1}, \text{rank } \mu_{s,X} + \dim S_1 \otimes I(Z)_s - \dim S_1 \otimes I(X)_s\}.$$

Proof. First observe that if X is contained in a line then X and all its subsets are complete intersections. The resolution of a complete intersection is well known. We let the reader check the theorem in this case. Thus we may assume that X is not contained in a line.

Now we show that it is enough to prove the theorem for $m = d - 1$. To do this, it is enough to show the following. Let $Z \subset X$ be a subset consisting of $m = m_0$ points, such that Z has truncated Hilbert function and $\mu_{s,Z}$ has the predicted rank, for any s. Assume that there is a subset $Z_1 \subset Z$ consisting of $m_0 - 1$ points such that Z_1 has truncated Hilbert function, and such that for all s, the rank of μ_{s,Z_1} is what is predicted in the theorem if we take $X = Z$ and $d = m_0$. Then we have to show that this is the same rank that is predicted by the theorem if we had taken $X = X$ and $Z = Z_1$.

The fact that Z has the predicted rank says that for all s, either $\mu_{s,Z}$ is surjective or $\ker \mu_{s,Z} = \ker \mu_{s,X}$. By our assumption on Z_1, we get that for all s either μ_{s,Z_1} is surjective or $\ker \mu_{s,Z_1} = \ker \mu_{s,Z}$. For those s with μ_{s,Z_1} surjective we are done. For those s with $\ker \mu_{s,Z_1} = \ker \mu_{s,Z} = \ker \mu_{s,X}$ we are done. This leaves only those s for which $\ker \mu_{s,Z_1} = \ker \mu_{s,Z}$ but $\ker \mu_{s,Z} \neq \ker \mu_{s,X}$. But if $\ker \mu_{s,Z} \neq \ker \mu_{s,X}$ then $\mu_{s,Z}$ is surjective. Furthermore, the Hilbert function of X in degree s is different from that of Z in degree s. Since Z has truncated Hilbert function, this says that Z imposes m_0 independent conditions on forms of degree s, and Z_1 imposes $m_0 - 1$ independent conditions on forms of degree s. Hence μ_{s,Z_1} is surjective. Thus it is enough to prove the theorem for the case $m = d - 1$.

Part of the proof will be by induction on d. The reader can easily get this induction argument started by checking directly that the theorem is true for small values of d.

Let l be the smallest positive integer such that X imposes d conditions on forms of degree l. Then $I(X)$ is generated in degrees less than or equal to $l + 1$, [DGM] Prop. 3.7. Let $Z \subset X$ be any subset with $d - 1$ points. Then $I(Z)$ is also generated in degrees less than or equal to $l + 1$. Thus for $s \geq l + 1$ the multiplication map $\mu_{s,Z}$ is surjective and thus satisfies the conclusion of the proposition.

Next consider $s \leq l - 1$. *From now on, unless specified otherwise, we assume that our subset Z has truncated Hilbert function.* Then X imposes at most $d - 1$ conditions on forms of degree s. By Lemma 1.1, Z imposes the smaller of $d - 1$ and the number of conditions imposed by X on forms of degree s. Thus, $I(X)_s = I(Z)_s$. This says that $\mu_{s,X}$ and $\mu_{s,Z}$ are the same map. Certainly the conclusion of the proposition follows in this case. We are only left to consider the case $s = l$. We have to show that *among subsets with cardinality $d - 1$ and with truncated Hilbert function*, we can find one with the right number of minimal generators in degree $l + 1$.

Consider the diagram (4.1) with $s = l$. Regardless of whether or not Z has truncated Hilbert function, $I(X)_l$ has codimension one in $I(Z)_l$, and similarly for $l + 1$. Let $F_1, \ldots, F_t$ be a basis for $I(X)_l$ and let G be a form of degree l in $I(Z)_l - I(X)_l$, so that $F_1, \ldots, F_t$, G is a basis for $I(Z)_l$. The proof breaks down into four cases according to the codimension of the image of $\mu_{l,X}$ in $I(X)_{l+1}$, in other words, the number of generators $I(X)$ needs in degree $l + 1$. Note that if $I(X)_l$ is zero dimensional then $I(Z)_l$ is one dimensional, so $\mu_{l,Z}$ is injective. We may assume that $I(X)_l$ has positive dimension.

Case 1. $I(X)$ is generated in degrees $\leq l$. This says that $\mu_{l,X}$ is surjective. We wish to show that $\mu_{l,Z}$ is also surjective, for any Z (hence in particular one with truncated Hilbert function). Since $I(X)_{l+1}$ has codimension one in $I(Z)_{l+1}$ we simply need to find a single form in the image of $\mu_{l,Z}$ not in the image of $\mu_{l,X}$. Let L be a linear form not vanishing on the single point of $X - Z$. Note that G also does not vanish on the single point of $X - Z$. LG is certainly in the image of $\mu_{l,Z}$, but not in the image of $\mu_{l,X}$ because LG does not vanish on all of X.

Case 2. The image of $\mu_{l,X}$ has codimension one in $I(X)_{l+1}$. We need to show that there is at least one subset $\mathbb{Z}$, with cardinality $d - 1$ and truncated Hilbert function, so that $\mu_{l,\mathbb{Z}}$ is surjective. For this case we consider all subsets of X of cardinality $d - 1$. Set $Z_i = X - \{P_i\}$, $i = 1, \ldots, d$. Let G_i be a form of degree l in $I(Z_i)_l - I(X)_l$. Note that G_i is well defined up to elements of $I(X)_l$. One can see that $F_1, \ldots, F_t, G_1, \ldots, G_d$ form a basis for S_l. Indeed, since $I(X)_l$ has codimension d in S_l there are the right number of them to be a basis, and any linear relation $a_1 F_1 + \cdots + a_t F_t + a_{t+1} G_1 + \cdots + a_{t+d} G_d = 0$ would need to have $a_{t+i} = 0$, $i = 1, \ldots, d$, since $G_i(P_i) \neq 0$ but all the other G's and F's vanish at P_i.

This would give a linear relation among the F's which is impossible because they form a basis for $I(X)_l$.

We only need to find one Z_i such that μ_{l,Z_i} is surjective, and such that Z_i has truncated Hilbert function. We will first argue that in this situation, if μ_{l,Z_i} is surjective then Z_i *must* have truncated Hilbert function.

Let L be a general linear form and let

$$J = \frac{I(X)+(L)}{(L)} \qquad\qquad J_i = \frac{I(Z_i)+(L)}{(L)}$$

be the corresponding ideals in $R = S/(L) \cong k[x,y]$. Note that L is not a zero divisor on $S/I(X)$ or $S/I(Z_i)$. By slight abuse of notation, we will call the rings $A = R/J$ and $A_i = R/J_i$ the *Artinian reductions* of X and Z_i, respectively.

Claim 4.3. *$I(X)$ and J (resp. $I(Z_i)$ and J_i) have the same number of minimal generators, occurring in the same degrees.*

Proof. This is standard. See for instance [M], p. 28. ☐

Claim 4.4. *J_i is* equal *to J in degrees greater than or equal to l if and only if Z_i does* not *have truncated Hilbert function.*

Proof. Notice that $J \subset J_i$ for all i, and notice that $J_i = J$ in degrees $\geq l+1$, so it is enough to prove that $\dim J_i = \dim J$ in degree l if and only if Z_i does not have truncated Hilbert function.

Consider the Hilbert functions of A and of A_i:

$$\begin{array}{llllllll} h_A: & 1 & a_1 & a_2 & \dots & a_{l-1} & a_l & 0 \\ h_{A_i}: & 1 & b_1 & b_2 & \dots & b_{l-1} & b_l & 0 \end{array}$$

Since $J \subset J_i$ we have $a_j \geq b_j \geq 0$ for all j. We also have $\sum a_j = d$ and $\sum b_j = d-1$. It follows that for *one* value of j, say j_0, we have $a_{j_0} = b_{j_0}+1$, and for all other j we have $a_j = b_j$. Since Z_i has truncated Hilbert function if and only if $j_0 = l$, this completes the proof of the claim. ☐

It follows from Claims 4.3 and 4.4 that if Z_i does not have truncated Hilbert function then it is impossible that X has a minimal generator in degree $l+1$ but Z_i does not have a minimal generator in degree $l+1$. So if we prove the existence of a Z_i with no minimal generator in degree $l+1$ then the truncated Hilbert function will follow automatically.

Suppose that μ_{l,Z_i} is never surjective. Since $\dim S_1 \otimes I(Z_i)_l = \dim S_1 \otimes I(X)_l + 3$, $\dim I(Z_i)_{l+1} = \dim I(X)_{l+1} + 1$, and by assumption

$$\dim I(X)_{l+1} = \dim \mu_{l,X}(S_1 \otimes I(X)_l) + 1,$$

we see that for every i the kernel of μ_{l,Z_i} must have dimension at least two larger than the dimension of the kernel of $\mu_{l,X}$. That is, there must be two degree one relations of the form

$$L_{i,1}F_1 + \cdots + L_{i,t}F_t + L_{i,t+1}G_i = 0$$
$$M_{i,1}F_1 + \cdots + M_{i,t}F_t + M_{i,t+1}G_i = 0.$$

These relations must be linearly independent of each other and no linear combination of the two of them can involve only F's and not G_i. From this one can see that all $2d$ of these relations (as you vary i) are linearly independent elements of the kernel of the multiplication map $S_1 \otimes S_l \to S_{l+1}$ which remain independent modulo the kernel of $\mu_{l,X}$.

Using our assumption on the codimension of the image of $\mu_{l,X}$ in $I(X)_{l+1}$ we conclude that this image has codimension $d+1$ in S_{l+1}. Comparing $\mu_{l,X}$ with the multiplication map $S_1 \otimes S_l \to S_{l+1}$ we see that $\dim S_1 \otimes S_l = \dim S_1 \otimes I(X)_l + 3d$. However, from the previous paragraph we know that the dimension of the kernel of $S_1 \otimes S_l \to S_{l+1}$ is at least $2d$ larger than the dimension of the kernel of $\mu_{l,X}$. Counting dimensions we get that $S_1 \otimes S_l \to S_{l+1}$ is not surjective. But, it is a well known triviality that $S_1 \otimes S_l \to S_{l+1}$ is surjective. This contradiction finishes case 2.

Case 3. The image of $\mu_{l,X}$ has codimension two in $I(X)_{l+1}$. This will involve quite a bit more work and will be done in section 5.

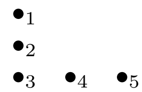
Case 4. The image of $\mu_{l,X}$ has codimension $c \geq 3$ in $I(X)_{l+1}$. Let Z_i, F_i, G_i, J and J_i be as in case 2. The codimension of the image of $\mu_{l,X}$ in $I(Z_i)_{l+1}$ is $c+1 \geq 4$. We want to show that there is a Z_i with truncated Hilbert function, such that $I(Z_i)$ has $c-3$ minimal generators in degree $l+1$. By Claim 4.4, if Z_i does not have truncated Hilbert function then $J_i = J$ in degrees $\geq l$. Hence J and J_i have the same number of minimal generators in degree $l+1$, and by Claim 4.3, the same is true of Z_i and X. So just as in case 2, it is enough to prove the existence of a Z_i with the right number of minimal generators, and it will automatically have truncated Hilbert function.

The proof will be by induction on d. Hence we can assume that the theorem is true for all the Z_i, but suppose that it fails for X. In this case we conclude that for each $i = 1, \ldots, d$ we have at least one degree one relation of the form $L_{i,1}F_1 + \cdots + L_{i,t}F_t + L_{i,t+1}G_i = 0$. If there were always two or more such relations we could arrive at a contradiction as in case 2, so assume for $i = 1$ there is only one such relation.

As indicated above, we may assume that the theorem holds for Z_1. Thus we can find P_j, $j \in \{2, \ldots, d\}$ such that $Z_{1,j} = Z_1 - \{P_j\}$ satisfies the conclusion of the theorem with respect to Z_1. A basis for $I(Z_1)_l$ consists of $F_1, \ldots, F_t, G_1$ and a basis for $I(Z_{1,j})_l$ consists of $F_1, \ldots, F_t, G_1, G_j$. The relations $L_{i,1}F_1 + \cdots + L_{i,t}F_t + L_{i,t+1}G_i = 0$ for $i = 1, j$ say that the codimension of the image of μ_{l,Z_1} in $I(Z_1)_{l+1}$ is exactly $c-1 \geq 2$ (because we assumed only one such relation) and the codimension of the image of $\mu_{l,Z_{1,j}}$ in $I(Z_{1,j})_{l+1}$ is at least $c-2 \geq 1$. But the assumption that $Z_{1,j} \subset Z_1$ satisfies the theorem says that the codimension of the image of $\mu_{l,Z_{1,j}}$ in $I(Z_{1,j})_{l+1}$ is $c-3$. This contradiction finishes case 4. □

Remark 4.5. The proof of Lemma 1.1 is surprisingly simple. The idea is to start with a subset Y' of X (beginning with any single point) and add one point of X at a time in such a way that at each step, the new subset Y has the predicted Hilbert function. This is done by considering the linear system of hypersurfaces of any degree t containing Y'. If the general element of this linear system vanishes on all of X then consider degree $t+1$. If not, there is some point P of X not in the base locus of this linear system, and we take $Y = Y' \cup P$.

One would naturally wonder if the same approach, building up to X point by point rather than taking point after point away from X, would similarly be an easier approach to Theorem 4.2. In fact this seems to not work. Consider, for instance, a set of points with the following configuration:



(i.e. points 1, 2, and 3 are collinear and points 3, 4 and 5 are collinear). If we build up X starting with point 3, it is of course possible to do so in such a way that at each step the subset obtained has the right (truncated) Hilbert function. For example, the sequence 3, 1, 4, 2, 5 works. However, it is *impossible* to find a sequence beginning with 3 such that at each step the subset has the right minimal free resolution according to Theorem 4.2. Indeed, the only subset of X consisting of four points and having the right number of minimal generators is the set of points labeled 1, 2, 4 and 5.

5. The Final Case

This section is devoted to proving case 3 of the proof of Theorem 4.2. We are thus assuming that X has two minimal generators in degree $l+1$, and we are trying to show that there is a subset Z_i of cardinality $d-1$ having truncated Hilbert function and no minimal generator in degree $l+1$. With this assumption on X, the following fact is proved exactly as in case 2 in the preceding section: *If a subset Z_i of $d-1$ points exists with no minimal generator in degree $l+1$ then it must have truncated Hilbert function.*

We begin by taking care of a special subcase. For the following lemma we will actually need to use the truncated Hilbert function to find the desired subset, so we have to be a little careful.

Lemma 5.1. *Assume that X satisfies case 3, i.e. the image of $\mu_{l,X}$ has codimension two in $I(X)_{l+1}$. If the base locus of $I(X)_l$ is one-dimensional then X contains a subset, Z, of cardinality $d-1$ which satisfies the rank condition asserted in Theorem 4.2, namely $I(Z)$ has no minimal generators in degree $l+1$.*

Proof. By Lemma 1.1, there is at least one subset Z_i whose Hilbert function is the truncation of that of X. We will find our desired Z from among these subsets, so from now on we will assume that this is the Hilbert function of Z. Then we have that the ideal of X agrees with that of Z in degrees $\leq l-1$ and Z and X both impose independent conditions on curves of degree l. It follows that

$$\begin{aligned} \dim I(X)_l + 1 &= \dim I(Z)_l, \\ \dim I(X)_{l+1} + 1 &= \dim I(Z)_{l+1} \end{aligned}$$

We are assuming, furthermore, that X has precisely two minimal generators in degree $l+1$. We need to show that Z can be chosen with *no* minimal generator in degree $l+1$.

The assumption about the dimension of the zero locus means that $I(X)_l$ has a GCD, F. Let k be the degree of F. By abuse of notation we will use F both for the curve in $\mathbb{P}^2$ and for the polynomial.

Claim 5.2. $k \leq 2$.

Proof. We will use ideas from [BGM] Proposition 2.3 (closely related to work of Davis [D]). Let X_1 be the subset of X lying on F and let X_2 be the subset not lying on F. We have $I(X_1) = [I(X) + (F)]^{\rm sat}$ and $I(X_2) = [I(X) : F]$, which is already saturated. For $k \le t \le l$ we have

$$\Delta h_{X_2}(t-k) = \Delta h_X(t) - k. \tag{5.1}$$

Notice that $I(X)_l = F \cdot I(X_2)_{l-k}$. From this we deduce two things. First, X_2 imposes independent conditions on forms of degree $l-k$ since X does on forms of degree l. Second, $\mathrm{rk}\ \mu_{l,X} = \mathrm{rk}\ \mu_{l-k,X_2}$.

Let J be the ideal generated by $I(X)_{\le l}$. We have just seen that $\dim J_l = \dim I(X_2)_{l-k}$. In degree $l+1$ we have the inequality $\dim J_{l+1} \le \dim I(X_2)_{l-k+1}$, where the failure to be an equality is measured by the number of minimal generators of $I(X_2)$ in degree $l-k+1$. Let $h(S/J,t)$ be the corresponding Hilbert function and consider $\Delta h(S/J, l+1)$. From the above considerations, one can check that

$$\begin{array}{rcl} \Delta h(S/J, l+1) & \ge & k + \Delta h_{X_2}(l-k+1) \\ & = & k. \end{array}$$

On the other hand, since $I(X)$ has two minimal generators in degree $l+1$, we have $2 = \dim I(X)_{l+1} - \dim J_{l+1}$. This gives

$$\begin{array}{rcl} k & \le & \Delta h(S/J, l+1) \\ & = & l+2 - \dim J_{l+1} + \dim J_l \\ & = & l+2 - [\dim I(X)_{l+1} - 2] + \dim I(X)_l \\ & = & \Delta h_X(l+1) + 2 \\ & = & 2 \end{array}$$

and this proves the claim. □

Claim 5.3. *If $k = 2$ then X_1 consists of exactly $2l+1$ points on F. If $k = 1$ then X_1 consists of exactly $l+1$ points on F.*

Proof. Let us collect the following facts.

1. The initial degree of $I(X)$ is ≥ 2.
2. $\Delta h_{X_2}(l-k+1) = 0$ since X_2 imposes independent conditions on forms of degree $l-k$ ($k = 1, 2$).
3. $\Delta h_X(l+1) = 0$.
4. $\Delta h_X(l) \ge 2$. This follows because we are assuming that X has two minimal generators in degree $l+1$. It can be seen, for example, by applying [C] Theorem 2.1 (d), since in our situation certainly X is contained in a complete intersection of type (α, β) with $\alpha < \beta = l+1$.
5. $\deg X = \deg X_1 + \deg X_2$.
6. $\sum_t \Delta h_{X_2}(t) = \deg X_2$.
7. $\sum_t \Delta h_X(t) = \deg X$.

If one now considers the Hilbert function of the Artinian reduction of $S/I(X)$ (i.e. the function given by $\Delta h_X(t)$) and applies the equation (5.1), in the case $k = 2$ (resp. $k = 1$) one gets from the above facts that $\deg X_1 = 2l+1$ (resp. $\deg X_1 = l+1$) as claimed. □

We consider the cases $k = 2$ and $k = 1$ separately. Suppose that X_1 consists of $2l+1$ points on either a smooth conic or else a union of two lines. In the latter case, either one point lies at the intersection of the two lines or else there are l points on one line and $l+1$ points on the other. (Otherwise X fails to impose independent conditions on forms of degree l.) In any of these cases, the removal of a suitable point P leaves $2l$ points which form the complete intersection of F and some curve G of degree l. Let Z be the subset of X obtained by removing this point.

We know that there exists an element of $I(Z)_l$ which is not in $I(X)_l$. We first claim that such an element must meet F in finitely many points. Certainly the base locus of the linear system $|I(Z)_l|$ cannot contain all of F since then it contains the deleted point P, contradicting the fact that X imposes independent conditions on forms of degree l. Hence the assertion is clear if F is irreducible. Suppose that $F = L_1 L_2$ is reducible and suppose (without loss of generality) that L_1 is in the base locus of $|I(Z)_l|$. Then any element of this linear system consists of the product of L_1 with a homogeneous polynomial of degree $l-1$ containing the remaining points of X. By construction, the remaining points include l points on L_2, so in fact all of F is in the base locus, a contradiction.

Hence without loss of generality we may assume that the subset of Z lying on F is the complete intersection of F and a form $G \in I(Z)_l$ (i.e. G contains all of Z). Now, if $(F_1, \dots, F_m)$ form a basis for $I(X)_l$ and $(F_1, \dots, F_m, G)$ form a basis for $I(Z)_l$, then any linear relation

$$L_1 F_1 + \cdots + L_m F_m + LG = 0$$

implies $L = 0$ since no factor of F is a factor of G, and $\deg F = 2$. The conclusion follows from this fact.

We now turn to the case $k = 1$. We have that X is the union $X = X_1 \cup X_2$, where X_1 consists of $l+1$ points on the line F. Since $|I(X)_{l+1}|$ has a zero-dimensional base locus, a general element, G, of this linear system does not contain F as a component. Hence in particular X_1 is the complete intersection of F and G. Also, in particular we have that $G \in I(X_2)$. We now note that $X = X_2 \cup X_1$ is a liaison addition (cf. [GM], [S])! Hence its ideal is of the form

$$I(X) = F \cdot I(X_2) + (G).$$

Since $I(X)$ has two minimal generators of degree $l+1$, clearly G must be one of these and $I(X_2)$ must have exactly one minimal generator in degree l (which is the maximum possible degree).

Let Z be the subset of X obtained by removing a point, P, of X_1. Let Z_1 be the subset of X_1 obtained by removing P. Exactly as above, $Z = Z_1 \cup X_2$ is a liaison addition, since the linear system $|I(Z)_l|$ does not have all of F in its base locus. Its ideal is of the form

$$I(Z) = F \cdot I(X_2) + (G')$$

where $\deg G' = l$, $G' \in I(X_2)$ and (F, G') form a complete intersection.

We want to show that P can be removed in such a way that $I(Z)$ has no minimal generator in degree $l+1$.

Claim 5.4. *The following are equivalent:*

(1) *$I(Z)$ has a minimal generator in degree $l+1$.*
(2) *$I(X_2)$ has a minimal generator in degree l other than G'.*
(3) *G' is not a minimal generator for $I(X_2)$.*

Proof. The equivalence of (2) and (3) is clear since we have already observed that $I(X_2)$ has exactly one minimal generator in degree l. The fact that (1) implies

(2) follows from the equation $I(Z) = F \cdot I(X_2) + (G')$. Now assume (2), and let H be the minimal generator in $I(X_2)$ other than G'. We want to show that FH is a minimal generator for $I(Z)$. Suppose not, and let $F_1, \dots, F_k$ be a basis for $I(X_2)_{l-1}$. Then we have

$$FH = LG' + \sum_{i=1}^{k} L_i F F_i,$$

or equivalently

$$F\left(H - \sum_{i=1}^{k} L_i F_i\right) = LG'.$$

Since F does not divide G', we get that $H = G' + \sum L_i F_i$ up to scalar multiples, contradicting the assumption that H is a minimal generator for $I(X_2)$ other than G'. □

As a result of Claim 5.4 we have in particular that $I(Z)$ *has a minimal generator in degree* $l+1$ *if and only if* G' *is* not *a minimal generator of* $I(X_2)$. We want to show that we can find the subset Z with no minimal generator in degree $l+1$. We thus have to show that among the $l+1$ collinear points of $G \cap F$, there is at least one point P whose removal leads to a $G' \in I(X_2)_l$ as above which is not in the image of μ_{l-1,X_2}. We will do this by contradiction. Let $P_1, \dots, P_{l+1}$ be the points of $G \cap F$. For each i, $1 \le i \le l+1$, let $G_i \in I(X_2 \cup P_1 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_{l+1})_l \subset I(X_2)_l$ such that G_i does not contain F as a factor, as was done with G' above. Note that $G_1, \dots, G_{l+1}$ are linearly independent in S_l, as was done in Case 2 in the previous section.

We want to show that it is impossible for $G_1, \dots, G_{l+1}$ to all be in the image of μ_{l-1,X_2}. Suppose otherwise. Note that F is a non zero-divisor of $S/I(X_2)$, and let

$$J = \frac{I(X_2) + (F)}{(F)} \qquad \text{and} \qquad R = S/(F).$$

We may view R/J as the Artinian reduction of $S/I(X_2)$, and in particular J has a minimal generator in degree l since we saw that $I(X_2)$ must have a minimal generator in degree l. On the other hand, if we let $\bar{G}_i$ be the image of G_i in R, the same argument as above gives that the $\bar{G}_i$ are linearly independent in $J_l \subset R_l$. Note that $\dim J_l = l+1$, so the $\bar{G}_i$ form a basis. If the G_i are all in the image of μ_{l-1,X_2} then none of the $\bar{G}_i$ is a minimal generator of J, so J has no minimal generator in degree l, a contradiction. This concludes the proof of Lemma 5.1. □

Let Z_i, F_i, and G_i be as in case 2. As noted above, the truncated Hilbert function will follow immediately once we find the Z_i with no minimal generator in degree $l+1$. Since we are assuming that the proposition fails for X, we conclude that for each $i = 1, \dots, d$ we have at least one degree one relation of the form

$$L_{i,1}F_1 + \dots + L_{i,t}F_t + L_{i,t+1}G_i = 0 \tag{5.2}$$

where $L_{i,t+1} \neq 0$. If there were always two or more such relations we could arrive at a contradiction as in case 2. So for some i's there must be exactly one such relation.

As a result of Lemma 5.1, we may assume that the base locus of $|I(X)_l|$ is zero-dimensional. Consequently we may choose homogeneous polynomials $H, K \in I(X)$ *each of degree* $\le l$ which form a regular sequence, hence link X to a zeroscheme D.

Note that we may choose H and K to be minimal generators of $I(X)$. Without loss of generality, say $\deg K \leq \deg H \leq l$.

Claim 5.5. *The ideal $I(D)$ of D contains a form of degree less than* $\deg(K)$.

Proof. We use the Cayley-Bacharach Theorem [DGO] Thm. 3(b). The first difference function for the Hilbert function of the complete intersection $H \cap K$ is:

$$1, 2, 3, \ldots, \deg K, \deg K, \ldots, \deg K, \deg K - 1, \deg K - 2, \ldots, 3, 2, 1,$$

with $\deg K$ repeated $\deg H - \deg K + 1$ times. The first difference function for the Hilbert function of X does not reach 0 until degree $l + 1$. Since $\deg H \leq l$, when we subtract these and read backwards to get the first difference function for D we see its maximum is less than $\deg K$. ☐

Now we wish to use liaison theory to compare the resolutions of $I(X)$ and $I(D)$. We follow the presentation in [CGO]. Suppose the resolution for $I(X)$ is

$$0 \to \bigoplus_{i=1}^{e+1} S(-m_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{e+2} S(-d_i) \to I(X) \to 0$$

with $d_1 \geq d_2 \geq \cdots \geq d_{e+2}$, $m_1 \geq m_2 \geq \cdots \geq m_{e+1}$. By our assumptions on generators of $I(X)$ in degree $l+1$ we have that $d_1 = d_2 = l+1$, $d_3 \leq l$. The map φ is given by a matrix of forms

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,e+1} & a_{1,e+2} \\ a_{21} & a_{22} & \cdots & & \\ \vdots & \vdots & & \vdots & \vdots \\ a_{e+1,1} & a_{e+1,2} & \cdots & a_{e+1,e+1} & a_{e+1,e+2} \end{bmatrix}$$

As in [CGO] we use $\partial\mathcal{A}$ to denote the integer matrix whose (i,j) entry is $\deg a_{ij} = u_{ij} = \max\{0, m_i - d_j\}$.

$$\partial\mathcal{A} = \begin{bmatrix} u_{11} & \cdots & u_{1,e+2} \\ \vdots & \vdots & \\ u_{e+1,1} & \cdots & u_{e+1,e+2} \end{bmatrix}.$$

The fact that $I(X)$ has generators in degree $l+1$ says that X has defining equations of high degree as defined in [CGO] after Prop. 1.2, so that Cor. 2.5 of [CGO] says $u_{1,1} = 1$. This together with inequalities found in Remark 2.2 of [CGO] and the previously mentioned facts about d_1, d_2, d_3 says that $\partial\mathcal{A}$ has the form

$$\left[\begin{array}{cc|c} 1 & 1 & \\ 1 & 1 & \\ \vdots & & A \\ 1 & 1 & \\ \hline 0 & 0 & \\ \vdots & & B \\ 0 & 0 & \end{array}\right]$$

where all entries of A are greater than one. Applying the description of liaison of [CGO] section 3 we get the corresponding matrix for $I(D)$ by eliminating from $\partial\mathcal{A}$ the columns corresponding to H and K, taking the transpose, and reversing the order of columns and rows to get something that looks like

$$\left[\begin{array}{ccc|ccc} & B' & & & A' & \\ \hline 0 & & 0 & 1 & & 1 \\ & \cdots & & & \cdots & \\ 0 & & 0 & 1 & & 1 \end{array}\right]$$

where all the entries of A' are greater than 1. One way to look at this is to say that the two minimal generators of $I(X)$ of maximal degree $l+1$ induce two degree 1 relations among the forms of lowest degree in $I(D)$.

We can also use H and K to link Z_i and $D \cup \{P_i\}$ for each $i = 1, 2, \ldots, d$. (Here and subsequently, we abuse notation somewhat and write $D \cup \{P_i\}$ for the scheme residual to Z_i, even when P_i is a common component of X and D.) Unfortunately, H and K may not be minimal generators for Z_i, so we have to argue slightly differently to get the desired matrix for $I(D \cup \{P_i\})$. We have seen that if there is a Z_i with no minimal generator in degree $l+1$ then it has truncated Hilbert function, and it is the subset that we are seeking. Hence without loss of generality we may assume that Z_i has at least one minimal generator in degree $l+1$, and, as above, it has a degree matrix of the form

$$\left[\begin{array}{c|c} 1 & \\ \vdots & A_i \\ 1 & \\ \hline 0 & \\ \vdots & B_i \\ 0 & \end{array}\right]$$

where all entries of A_i are greater than or equal to one. As a result, the minimal free resolution for $I(Z_i)$ has the form

$$0 \to \bigoplus_{i=1}^{f+1} S(-n_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{f+2} S(-e_i) \to I(Z_i) \to 0$$

with $l+1 = e_1 \geq e_2 \geq \cdots \geq e_{f+2}$, $l+2 = n_1 \geq n_2 \geq \cdots \geq n_{f+1}$. (This can also be deduced directly by considering the regularity of $I(Z_i)$.) Now we link using K and H with $k = \deg K \leq h = \deg H < l+1$. Let C be the complete intersection of H

and K. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigoplus_{i=1}^{f+1} S(-n_i) & \xrightarrow{\varphi} & \bigoplus_{i=1}^{f+2} S(-e_i) & \to & I(Z_i) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & S(-h-k) & \to & S(-h)\oplus S(-k) & \to & I(C) & \to & 0 \end{array}$$

and by the usual mapping cone trick (cf. [M] Proposition 5.2.10) we get a resolution for $I(D \cup \{P_i\})$ of the form

$$0 \to \bigoplus_{i=1}^{f+2} S(e_i - h - k) \to \bigoplus_{i=1}^{f+1} S(n_i - h - k) \oplus S(-h) \oplus S(-k) \to I(D \cup \{P_i\}) \to 0.$$

This is not necessarily minimal. It may be that zero, one or two terms can be split off, depending on whether neither, one or both of H and K are minimal generators of $I(Z_i)$, respectively. However, by the assumption that $k \le h < l+1$, we see that in any case the smallest term $S(e_1 - h - k)$ is *not* split off. It follows that the degree matrix for $I(D \cup \{P_i\})$ is of the form

$$\left[\begin{array}{c|c} B_i' & A_i' \\ \hline 0 \cdots 0 & 1 \cdots 1 \end{array}\right]$$

where all the entries of A_i' are greater than or equal to one. In this case we can say that the minimal generator of $I(Z_i)$ of maximal degree $l+1$ induces a degree 1 relation among the forms of lowest degree in $I(D \cup \{P_i\})$.

Putting these facts together we get the following. Let v be the lowest degree of forms in $I(D)$. We have two independent degree 1 relations among $I(D)_v$. For each $i = 1, 2, \ldots, d$ there exists a linear combination (depending on i) of these two relations that becomes a degree 1 relation on $I(D \cup \{P_i\})_v$.

We will convert this to a question about sections of twisted bundles of differential forms on $\mathbb{P}^2$. The basic idea is fairly standard; see for instance [HS] section 9 or [B] section 1. Let $\Omega^1_{\mathbb{P}^2}$ represent the sheaf of differential one-forms on $\mathbb{P}^2$. Consider the dual of the Euler sequence twisted by $v+1$ (cf. [H] Theorem II.8.13):

$$(5.3) \qquad 0 \to \Omega^1_{\mathbb{P}^2}(v+1) \to \mathcal{O}_{\mathbb{P}^2}(v)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(v+1) \to 0$$

Taking cohomology, we obtain

$$0 \to H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(v+1)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(v)^{\oplus 3}) \xrightarrow{\alpha} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(v+1)) \to \cdots.$$

The map α is given by $\alpha(f_1, f_2, f_3) = x_0 f_1 + x_1 f_2 + x_2 f_3$. A degree 1 relation among elements of $I(D)_v$ can be written as $x_0 g_1 + x_2 g_2 + x_2 g_3 = 0$ where $g_i \in I(D)_v$ and therefore represents an element of the kernel of α vanishing on D. By exactness it is the image of an element of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(v+1))$ vanishing on D. The same argument applies to relations on $I(D \cup \{P_i\})_v$. We deduce that the two degree one relations among $I(D)_v$ correspond to two linearly independent sections, call them s_1 and s_2, in $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(v+1))$ that vanish on D, and the degree one relation on $I(D \cup \{P_i\})_v$ corresponds to a linear combination of s_1 and s_2 that also vanishes at P_i. (When $P_i \in \text{Support}(D)$ this must be interpreted appropriately in terms of ideals.) The locus of points where sections of vector bundles fail to be linearly independent can be analyzed using Chern classes.

Set $Y = \{\ P \in \mathbb{P}^2 \mid s_1(P) \text{ and } s_2(P) \text{ are linearly dependent }\}$. Note that since s_1 and s_2 are linearly independent as elements of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(v+1))$, $Y \neq \mathbb{P}^2$. From [F] Example 14.3.2 we see that the class of Y with an appropriate scheme structure is the first Chern class of $\Omega^1_{\mathbb{P}^2}(v+1)$. Another way to see this is to observe that $s_1 \wedge s_2$ is a nonzero section of the line bundle $\bigwedge^2(\Omega^1_{\mathbb{P}^2}(v+1))$. From the Whitney sum formula [F] Theorem 3.2(e) applied to the Euler sequence (5.3) we get that the first Chern class of $\Omega^1_{\mathbb{P}^2}(v+1)$ is $2v-1$ times a line. In other words, Y is a curve of degree $2v-1$. Certainly Y passes through D because s_1 and s_2 vanish at D, and Y passes through $X = \{P_1, \dots, P_d\}$ because for each i some linear combination of s_1 and s_2 vanishes at P_i.

We need to know more about the local nature of Y near the points of the complete intersection $H \cap K$, which we denote by C. By abuse of notation we also let Y represent a polynomial generating the homogeneous ideal of Y. We consider the ideals $I(X)$, $I(D)$ and $I(C)$. Let Q be a point of C. The subscript Q on an ideal or polynomial means we are considering the corresponding ideal or function in the local ring of $\mathbb{P}^2$ at Q.

(a) Suppose $Q \in X$, $Q \notin \text{Support}(D)$. Then $Y_Q \in I(X)_Q = I(C)_Q$.

(b) Suppose $Q \in \text{Support}(D)$, $Q \notin X$. Choose local coordinates x, y on $\mathbb{P}^2$ centered at Q and trivialize $\Omega^1_{\mathbb{P}^2}(v+1)$ locally near Q. Then each s_i is given locally by a pair of functions $s_i(x,y) = (s_{i,1}(x,y), s_{i,2}(x,y))$, $i = 1, 2$. Saying that s_i vanishes on D is saying that $s_{i,j}(x,y) \in I(D)_Q$ for both $j = 1$ and $j = 2$. The local equation of Y near Q is the determinant

$$\begin{vmatrix} s_{1,1}(x,y) & s_{1,2}(x,y) \\ s_{2,1}(x,y) & s_{2,2}(x,y) \end{vmatrix} = s_{11}s_{22} - s_{12}s_{21}.$$

Thus $Y_Q \in I(D)^2_Q \subset I(C)_Q$ since $I(D)_Q = I(C)_Q$.

(c) Suppose $Q \in X \cap \text{Support}(D)$. We continue with the notation of (b). Some linear combination of s_1 and s_2 vanishes on the residual scheme to $X - Q$ in C, which we called $D \cup \{Q\}$ by abuse of notation. In the determinant replacing s_1 by this linear combination and s_2 by some other independent linear combination we get that $Y_Q \in I(C)_Q \cdot I(D)_Q \subset I(C)_Q$.

Putting the three local calculations (a), (b) and (c) together, we get that globally $Y \in I(C)$, so we may write $Y = SH + TK$. Again consider Q a point of $\text{Support}(D)$. Whether we are in case (b) or (c), we have $Y_Q \in I(C)_Q \cdot I(D)_Q$. Because $I(C)_Q = \langle H_Q, K_Q \rangle$, we may write $Y_Q = aH_Q + bK_Q$, where $a, b \in I(D)_Q$. On the other hand, of course we also have $Y_Q = S_Q H_Q + T_Q K_Q$. This gives $(S_Q - a)H_Q = (b - T_Q)K_Q$. The local ring of $\mathbb{P}^2$ at Q is a unique factorization domain, and H_Q and K_Q have no common factors. We conclude that K_Q divides $S_Q - a$, so that $S_Q - a \in I(C)_Q \subset I(D)_Q$, giving $S_Q \in I(D)_Q$. Similarly, $T_Q \in I(D)_Q$. Since this is true locally for all $Q \in \text{Support}(D)$, we conclude that globally $S, T \in I(D)$. Furthermore, since $Y \neq \mathbb{P}^2$, S and T cannot both be zero.

From Claim 5.5, we see that H and K both have degree greater than v = smallest degree of a form in $I(D)$. Since Y has degree $2v-1$, both S and T have degree less than $v-1$, meaning that they could not be in $I(D)$. This final contradiction completes the proof of Theorem 4.2. ∎

Remark 5.6. It should be noted that Theorem 4.2 gives another proof of the Minimal Resolution conjecture for general sets of points in $\mathbb{P}^2$. (Other solutions can be found in [GGR], [GMa], [GM]. See also [L2].)

To see this first note that by [GMa] Prop. 1.4, any set, $\mathbb{X}$, of $\binom{d+2}{2}$ points in $\mathbb{P}^2$ not lying on a curve of degree d, always has resolution

$$0 \to S(-(d+2))^{d+1} \longrightarrow S(-(d+1))^{d+2} \longrightarrow I(\mathbb{X}) \to 0.$$

Thus, if $\mathbb{Z}$ is a general set of $t = \binom{d+1}{2} + r$ points in $\mathbb{P}^2$, $0 < r < d+1$, then $\mathbb{Z}$ satisfies the minimal resolution conjecture if and only if the multiplication map

$$\mu_{d,\mathbb{Z}} : S_1 \otimes I(\mathbb{Z})_d \longrightarrow I(\mathbb{Z})_{d+1}$$

has maximal rank (see e.g. [GMa] section 2), i.e.

$$\text{rank } \mu_{d,\mathbb{Z}} = \min\{ \dim I(\mathbb{Z})_{d+1}, 3 \dim I(\mathbb{Z})_d \}. \tag{5.4}$$

Now, (5.4) follows immediately from Theorem 4.2 once we note that since $I(\mathbb{X})_d = 0$ then both rank $\mu_{d,\mathbb{X}} = 0$ and $\dim(S_1 \otimes I(\mathbb{X})_d) = 0$.

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