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SEQUENCES OF REFLECTION FUNCTORS AND THE PREPROJECTIVE COMPONENT OF A VALUED QUIVER

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Dedicated to the memory of L. Gaunce Lewis, Jr.

Abstract. This paper concerns preprojective representations of a finite connected valued quiver without oriented cycles. For each such representation, an explicit formula in terms of the geometry of the quiver gives a unique, up to a certain equivalence, shortest (+)-admissible sequence such that the corresponding composition of reflection functors annihilates the representation. The set of equivalence classes of the above sequences is a partially ordered set that contains a great deal of information about the preprojective component of the Auslander-Reiten quiver. The results apply to the study of reduced words in the Weyl group associated to an indecomposable symmetrizable generalized Cartan matrix.

Introduction

The motivation for this work comes from two sources. The first is the paper [12], which assigns a canonical (+)-admissible sequence to each indecomposable preprojective module over the path algebra of a finite connected quiver without oriented cycles and then uses the combinatorial structure of the set $\mathfrak{S}$ of (+)-admissible sequences, and reflection functors instead of the Coxeter functor (Auslander-Reiten translation), to give an explicit description of the preprojective component of the Auslander-Reiten quiver [1]. In this connection a question is whether similar results hold in a more general setting of representations of valued quivers studied in [3]. The question is especially relevant in view of [10], which is our second source of motivation. Using combinatorics of the set $\mathfrak{S}$, the latter paper relates properties of reduced words in the Weyl group $\mathcal{W}(A)$ associated to an indecomposable symmetric generalized $n \times n$ Cartan matrix A [9] to properties of preprojective modules over the path algebra of a quiver without oriented cycles whose underlying graph is the graph associated to A. Let $\sigma_1, \dots, \sigma_n$ be the simple reflections, and let c be any Coxeter element, i.e., $c = \sigma_{x_n} \dots \sigma_{x_1}$ where $x_1, \dots, x_n$ is any permutation of the numbers $1, \dots, n$. The authors of [10] proved that $\mathcal{W}(A)$ is infinite if and only if the powers of c are reduced words in the σ_h's, after Andrei Zelevinsky brought to their attention the following two results. Howlett proved that any Coxeter group $\mathcal{W}$ is infinite if and only if c has infinite order [8, Theorem 4.1]. Fomin and Zelevinsky proved the following. Let A be symmetrizable and bipartite, i.e., the set $\{1, \dots, n\}$ is a disjoint union of nonempty subsets I, J and, for $h \neq l$, $a_{hl} = 0$ if either $h, l \in I$ or $h, l \in J$. If $c = \prod_{i \in I} \sigma_i \prod_{j \in J} \sigma_j$, then $\mathcal{W}(A)$ is infinite if and only if the powers of c are reduced words [7, Corollary 9.6]. The aforementioned result of [10] is a strengthening of the indicated results of Howlett and Fomin-Zelevinsky in the case $\mathcal{W} = \mathcal{W}(A)$ where A is symmetric. A goal of [11] is to

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obtain the strengthening for any symmetrizable A, using properties of preprojective modules and of the set $\mathfrak{S}$. Since there is a one-to-one correspondence between valued graphs and symmetrizable Cartan matrices [3, p. 1], one has to replace graphs with valued graphs and representations of quivers with representations of valued quivers. Thus we lay a foundation for [11].

This paper continues the study of combinatorial properties of $\mathfrak{S}$ initiated in [12] and further developed in [10]. These properties allow us to extend the main results of [12] from representations of quivers to representations of valued quivers (Section 2), as well as to give new, more transparent proofs. The rich combinatorics of $\mathfrak{S}$ is not fully understood and is useful for representation theory. Our intention is to study it in the future.

We now recall some facts, definitions, and notation, using freely [1, 2, 3]. A *graph* is a pair $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices and Γ_1 is the set of edges, i.e., of two-element subsets of Γ_0. Any subset $X \subset \Gamma_0$ determines a *full subgraph* of Γ with the set of vertices X and the set of edges consisting of all those two-element subsets $\{i, j\} \in \Gamma_1$ that satisfy $i, j \in X$. A *valuation* $\mathbf{b}$ of a graph Γ is a set of nonnegative integers $\{b_{ij}\}$ for all pairs $i, j \in \Gamma_0$ where $b_{ii} = 0$ and there exist nonzero natural numbers d_i satisfying

$$d_i b_{ij} = d_j b_{ji}, \quad \text{for all } i, j \in \Gamma_0.$$

The pair $(\Gamma, \mathbf{b})$ is a *valued graph*, and the above condition says that the matrix $[b_{ij}]$ is symmetrizable. The valued graph $(\Gamma, \mathbf{b})$ is *connected* if for all vertices $h \neq l$, there is a sequence $h, \ldots, i, j, \ldots, l$ in Γ_0 such that $b_{ij} \neq 0$ for each pair of subsequent vertices i, j. Throughout the paper, $(\Gamma, \mathbf{b})$ is a fixed finite connected valued graph with $|\Gamma_0| > 1$, where $|X|$ stands for the cardinality of a set X.

An orientation, Λ, on Γ consists of two functions $s : \Gamma_1 \to \Gamma_0$ and $e : \Gamma_1 \to \Gamma_0$. For an edge $a \in \Gamma_1$, $s(a)$ and $e(a)$ are the vertices incident with a, and they are called the starting point and the endpoint of a, respectively; one writes $a : s(a) \to e(a)$. The ordered triple $(\Gamma, \mathbf{b}, \Lambda)$ is called a *valued quiver* and a is then called an arrow of the quiver. Any subset $X \subset \Gamma_0$ determines a *full subquiver* of $(\Gamma, \mathbf{b}, \Lambda)$ by taking the full subgraph of Γ determined by X and preserving the valuation and orientation of each edge. Given a sequence of arrows $a_1, \ldots, a_t,\ t > 0$, satisfying $e(a_i) = s(a_{i+1}),\ 0 < i < t$, one forms a path $p = a_t \ldots a_1$ of length t in $(\Gamma, \mathbf{b}, \Lambda)$. By definition, $s(p) = s(a_1),\ e(p) = e(a_t)$, so one writes $p : s(p) \to e(p)$ and says that p is a path from $s(p)$ to $e(p)$. By definition, for all $x \in \Gamma_0$ there is a unique path of length 0 from x to x, denoted by e_x. A path p of length at least 1 is an oriented cycle if $s(p) = e(p)$. The set of vertices of any valued quiver without oriented cycles (no finiteness assumptions) acquires a structure of a partially ordered set (poset) by putting $x \leq y$ if there is a path from x to y. If $(\Gamma, \mathbf{b}, \Lambda)$ has no oriented cycles, we denote this poset by (Γ_0, Λ). All orientations Λ, Θ, etc., are such that $(\Gamma, \mathbf{b}, \Lambda)$, $(\Gamma, \mathbf{b}, \Theta)$, etc., have no oriented cycles.

To define representations of a valued quiver $(\Gamma, \mathbf{b}, \Lambda)$, one has to choose a *modulation* $\mathfrak{B}$ of the valued graph $(\Gamma, \mathbf{b})$, which by definition is a set of division rings $\mathbf{k}_i,\ i \in \Gamma_0$, together with a $\mathbf{k}_i - \mathbf{k}_j$-bimodule ${}_iB_j$ and a $\mathbf{k}_j - \mathbf{k}_i$-bimodule ${}_jB_i$ for each edge $\{i, j\} \in \Gamma_1$ such that

(i) there are $\mathbf{k}_j - \mathbf{k}_i$-bimodule isomorphisms

$${}_jB_i \cong \mathrm{Hom}_{\mathbf{k}_i}({}_iB_j, \mathbf{k}_i) \cong \mathrm{Hom}_{\mathbf{k}_j}({}_iB_j, \mathbf{k}_j)$$

and

(ii) $\dim_{\mathbf{k}_i}({}_iB_j) = b_{ij}$.

For the rest of the paper we denote by Γ a valued graph with a fixed valuation $\mathbf{b}$ and modulation $\mathfrak{B}$, denote by (Γ, Λ) the corresponding valued quiver with orientation Λ, and assume that the division rings $\mathbf{k}_i$ are finite dimensional vector spaces over a common central subfield k acting centrally on all bimodules ${}_iB_j$. The latter assumption is sufficient for the applications that we have in mind. However, the results of [5] imply that most of our considerations hold without this assumption. Under the assumption, each ${}_iB_j$ is a finite dimensional k-space, so setting $d_i = \dim_k \mathbf{k}_i$, we get $d_i b_{ij} = \dim_k({}_iB_j) = \dim_k({}_jB_i) = d_j b_{ji}$.

A (left) representation (V, f) of (Γ, Λ) is a set of finite dimensional left $\mathbf{k}_i$-spaces V_i, $i \in \Gamma_0$, together with $\mathbf{k}_j$-linear maps

$$f_a : {}_jB_i \otimes_{\mathbf{k}_i} V_i \to V_j$$

for all arrows $a : i \to j$, and morphisms of representations are defined in a natural way. We obtain the category $\mathrm{Rep}(\Gamma, \Lambda)$ of representations of the valued quiver (Γ, Λ).

Putting $\mathbf{k} = \prod_{i\in\Gamma_0} \mathbf{k}_i$ and viewing $B = \bigoplus_{i\to j} {}_jB_i$ as a $\mathbf{k}$-$\mathbf{k}$-bimodule where $\mathbf{k}$ acts on ${}_jB_i$ from the left via the projection $\mathbf{k} \to \mathbf{k}_j$ and from the right via the projection $\mathbf{k} \to \mathbf{k}_i$, one forms the tensor ring $\mathrm{T}(\mathbf{k}, B) = \bigoplus_{n=0}^{\infty} B^{(n)}$ where $B^{(n)} = B \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} B$ is the n-fold tensor product, and the multiplication is given by the isomorphisms $B^{(n)} \otimes B^{(m)} \to B^{(n+m)}$ [4, p. 386]. Since (Γ, Λ) has no oriented cycles, $\mathrm{T}(\mathbf{k}, B)$ is a finite dimensional k-algebra and we denote it by $k(\Gamma, \Lambda)$. Let $e_i \in \mathbf{k}$ be the n-tuple that has $1 \in \mathbf{k}_i$ in the ith place and 0 elsewhere. A left $k(\Gamma, \Lambda)$-module M is *finite dimensional* if $\dim_{\mathbf{k}_i} e_i M < \infty$ for all i, which is equivalent to $\dim_k M < \infty$. We let f.d. $k(\Gamma, \Lambda)$ denote the category of finite dimensional left $k(\Gamma, \Lambda)$-modules. The categories $\mathrm{Rep}(\Gamma, \Lambda)$ and f.d. $k(\Gamma, \Lambda)$ are equivalent [4, Proposition 10.1] and we view the equivalence as an identification. In this paper all $k(\Gamma, \Lambda)$-modules are finite dimensional.

Given a valued quiver (Γ, Λ) and a vertex $x \in \Gamma_0$, let $\sigma_x\Lambda$ be the orientation on Γ obtained by reversing the direction of each arrow incident with x and preserving the directions of the remaining arrows. There results a new valued quiver $(\Gamma, \sigma_x\Lambda)$ (remember, the valuation $\mathbf{b}$ and modulation $\mathfrak{B}$ of the valued graph Γ are fixed). A vertex x is a *sink* if no arrow starts at x. For each sink x, the *reflection* functor $F_x^+ : \mathrm{Rep}(\Gamma, \Lambda) \to \mathrm{Rep}(\Gamma, \sigma_x\Lambda)$ is defined [3, pp. 15-16], and we recall the definition for the convenience of the reader.

Let $(V, f) \in \mathrm{Rep}(\Gamma, \Lambda)$ and let $(W, g) = F_x^+(V, f)$. Then $W_y = V_y$ for all $y \neq x$, and $g_b = f_b$ for all those arrows b of $(\Gamma, \sigma_x\Lambda)$ that do not start at x. Let $a_i : y_i \to x,\ i = 1, \ldots, l$, be the arrows of (Γ, Λ) ending at x. Then the reversed arrows $a_i' : x \to y_i,\ i = 1, \ldots, l$, are all the arrows of $(\Gamma, \sigma_x\Lambda)$ starting at x. Consider the exact sequence

$$0 \to \operatorname{Ker} h \xrightarrow{j} \bigoplus_{i=1}^{l} {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \xrightarrow{h} V_x$$

of $\mathbf{k}_x$-spaces, where the map h is induced by the maps $f_{a_i} : {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \to V_x$. Then $W_x = \operatorname{Ker} h$ and each map $g_{a_i'} : {}_{y_i}B_x \otimes_{\mathbf{k}_x} W_x \to W_{y_i} = V_{y_i}$ is obtained from the map $W_x \to {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}$ induced by j using the following chain of isomorphisms of k-spaces [3, pp. 14-15].

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{k}_x}(W_x, {}_xB_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}) &\cong \mathrm{Hom}_{\mathbf{k}_x}(W_x, \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x, \mathbf{k}_{y_i}) \otimes_{\mathbf{k}_{y_i}} W_{y_i}) \\
&\cong \mathrm{Hom}_{\mathbf{k}_x}(W_x, \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x, W_{y_i})) \\
&\cong \mathrm{Hom}_{\mathbf{k}_{y_i}}({}_{y_i}B_x \otimes_{\mathbf{k}_x} W_x, W_{y_i})
\end{aligned}$$

A sequence of vertices $S = x_1, x_2, \ldots, x_s,\ s \geq 0$, is called *(+)-admissible* on (Γ, Λ) if it either is empty, or satisfies the following conditions: x_1 is a sink with respect to Λ, x_2 is a sink with respect to $\sigma_{x_1}\Lambda$, and so on; sometimes we write $x_1x_2\ldots x_s$ instead of $x_1, x_2, \ldots, x_s$. Recall that we denote by $\mathfrak{S}$ the set of (+)-admissible sequences on (Γ, Λ). If $S = x_1, \ldots, x_s$ is in $\mathfrak{S}$, we put $\Lambda^S = \sigma_{x_s} \ldots \sigma_{x_1}\Lambda$ and $F(S) = F_{x_s}^+ \ldots F_{x_1}^+ : \mathrm{Rep}(\Gamma, \Lambda) \to \mathrm{Rep}(\Gamma, \Lambda^S)$. If the sequence S consists of distinct vertices and contains each vertex of the quiver, then $F(S) = \Phi^+$ does not depend on the choice of S and is called the *Coxeter* functor [3, p. 19]. For $S \in \mathfrak{S}$ we say that S *annihilates* a module $M \in$ f.d. $k(\Gamma, \Lambda)$ if $F(S)(V, f) = 0$, where (V, f) is the representation of (Γ, Λ) identified with M. In light of this identification, we often write $F(S)M$ or Φ^+M.

A *source* is a vertex of a quiver at which no arrow ends. Replacing sinks with sources, one gets similar definitions of a *reflection* functor F_x^-, a $(-)$-*admissible* sequence, and the *Coxeter* functor Φ^- [3].

In [3, p. 22], the authors make the following definition.

Definition 0.1. A representation (V, f) of (Γ, Λ) is *preprojective* if $(\Phi^+)^m(V, f) = 0$ for some integer $m > 0$.

Definition 0.1 is equivalent to the following.

Definition 0.2. A module $M \in$ f.d. $k(\Gamma, \Lambda)$ is *preprojective* if there exists an $S \in \mathfrak{S}$ that annihilates it.

We describe all $S \in \mathfrak{S}$ that annihilate a preprojective $k(\Gamma, \Lambda)$-module M by proving that, up to a certain equivalence $\sim$, there exists a unique *shortest* $(+)$-admissible sequence S_M that annihilates M (Theorem 2.2(a)), where an $S \in \mathfrak{S}$ is a shortest sequence that annihilates M if S annihilates M but no proper subsequence of S does. Suppose now that M is indecomposable. Then $S_M \in \mathfrak{P}$ (Theorem 2.6) where $\mathfrak{P}$ is the subset of $\mathfrak{S}$ consisting of the *principal* $(+)$-admissible sequences defined below in terms of the poset (Γ_0, Λ) and geometry of Γ, and S_M determines M uniquely up to isomorphism (Theorem 2.2(d)). If m is the smallest positive integer satisfying $(\Phi^+)^m M = 0$, then $m = \nu + 1$ where ν is a unique nonnegative integer for which $(\Phi^+)^\nu M = P$ is indecomposable projective; $P = P_x$ is determined up to isomorphism by a unique $x \in \Gamma_0$; and $M \cong (\Phi^-)^\nu P_x \cong (\mathrm{TrD})^\nu P_x$. It easy to compute S_M from $(\nu, x) = (\nu(M), x(M))$ and vice versa, and it is more efficient to compute M from S_M than from (ν, x) (Corollary 2.7). If (Γ, Λ) is of infinite representation type, then $\mathfrak{P} = \{S_M \mid M \text{ indecomposable preprojective}\}$ (Corollary 2.9(c)). If $M_1, \ldots, M_t$ are the nonisomorphic indecomposable summands of a preprojective module M, it is easy to compute S_M in terms of $S_{M_1}, \ldots, S_{M_t}$ (Theorem 2.2(c)).

The preprojective (connected) component of the Auslander-Reiten quiver of $k(\Gamma, \Lambda)$ is closely related to the translation quiver $\mathbb{N} \times (\Gamma, \Lambda^{op})$, and if (Γ, Λ) is of infinite representation type, the two coincide. Recall that $\mathbb{N} \times (\Gamma, \Lambda^{op})$, with $\mathbb{N}$ being the set of nonnegative integers and Λ^{op} the opposite orientation of Λ, is an infinite connected valued quiver that can be visualized as a disjoint union of countably many copies of the valued quiver (Γ, Λ^{op}) where, for each $i \in \mathbb{N}$, one draws additional arrows starting at vertices of $\{i\} \times (\Gamma, \Lambda^{op})$ and ending at vertices of $\{i+1\} \times (\Gamma, \Lambda^{op})$; here the valuation of new edges is assigned in a natural way and the translation is a left shift. One of the reasons to study $(+)$-admissible sequences is that a significant part of the combinatorial structure of $\mathbb{N} \times (\Gamma, \Lambda^{op})$ can be recovered from a simpler combinatorics of the set $\mathfrak{S}$, which has a natural poset structure (up to the equivalence $\sim$): if $S, T \in \mathfrak{S}$, we set $S \preccurlyeq T$ if $T \sim SS'$ where S' is a $(+)$-admissible sequence on (Γ, Λ^S). Since the translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$ has no oriented cycles, its set of vertices $\mathbb{N} \times \Gamma_0$ is a poset. We prove that this poset is isomorphic to $\mathfrak{P}$ viewed as a subposet of $\mathfrak{S}$ (Theorem 1.11(a)). A large class of valued quivers, which is easy to describe combinatorially, is characterized by the fact that (Γ, Λ) with the valuation ignored coincides with the Hasse diagram of the poset (Γ_0, Λ). For these valued quivers, the Hasse diagram of $\mathfrak{P}$ is the underlying quiver of the valued translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$ (Theorem 1.11(b)), i.e., $(+)$-admissible sequences contain all information about the preprojective component except for the valuation.

We now describe the content of the paper section by section. Section 1 presents the necessary definitions and results of [12, 10] concerning the combinatorics of the sets $\mathfrak{S}$ and $\mathfrak{P}$: the equivalence $\sim$, the partial order $\preccurlyeq$, and a canonical form and the lattice structure on the set $\mathfrak{S}$; the filters of the poset (Γ_0, Λ) play a major role. These considerations do not involve representation theory, valuation, or modulation of (Γ, Λ), so most of the proofs are omitted. Section 2 describes the properties of the shortest sequence S_M associated to a preprojective module M, as well as the connection between the preprojective component of the Auslander-Reiten quiver of (Γ, Λ) and the poset $\mathfrak{P}$.

By duality, one can study $(-)$-admissible sequences and the preinjective component of the valued quiver, using ideals, instead of filters, of the poset (Γ_0, Λ) and the same equivalence $\sim$. We leave this to the reader.

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1. Posets, Admissible Sequences, and Canonical Forms

Throughout this section (Γ, Λ) is a valued quiver without oriented cycles. By definition, a $(+)$-admissible sequence on (Γ, Λ) depends neither on the valuation $\mathbf{b}$ nor on the modulation $\mathfrak{B}$. Therefore the considerations of [12, Sections 1 and 2] and [10, Section 2] apply and we quote, mostly without proofs, those results that are needed in the rest of the paper.

We recall some notions about posets; see [6]. Let $(P, \leq)$ be a poset. A subset $F \subset P$ is called a *filter* if whenever $x \in F$ and $y \geq x$, we have $y \in F$. We say that a filter F is generated by $X \subset P$ and write $F = \langle X \rangle$ if $F = \{y \in P \,|\, y \geq x \text{ for some } x \in X\}$. If F is generated by a single element x, we call F a *principal filter* and write $F = \langle x \rangle$. For $x, y \in P$ we say that y covers x and write $x \lessdot y$ if (i) $x < y$ and (ii) $x < y' \leq y$ implies $y' = y$. The Hasse diagram, $\mathscr{H}(P)$, of P is the quiver with the set of vertices P and the set of arrows that contains a single arrow $x \to y$ if and only if $x \lessdot y$, and has no other arrows. For all $x, y \in \Gamma_0$, we set $x \leq y$ if there is a path from x to y in (Γ, Λ). Since (Γ, Λ) has no oriented cycles, this turns Γ_0 into a poset, which we denote by (Γ_0, Λ).

Definition 1.1. If $S = x_1, \ldots, x_s,\ s \geq 0$, is in $\mathfrak{S}$, we write $\Lambda^S = \sigma_{x_s} \ldots \sigma_{x_1} \Lambda$ and, in particular, $\Lambda^{\emptyset} = \Lambda$. The *support* of S, $\operatorname{Supp} S$, is the set of distinct vertices among x_j, $1 \leq j \leq s$. As in [10, Definition 2.1], the *length* of S is $\ell(S) = s$; the *multiplicity* of $v \in \Gamma_0$ in S, $m_S(v)$, is the (nonnegative) number of subscripts j satisfying $x_j = v$. A sequence $K \in \mathfrak{S}$ is *complete* if $m_K(v) = 1$ for all $v \in \Gamma_0$. If $S = x_1, \ldots, x_s$ and $T = y_1, \ldots, y_t$ are $(+)$-admissible on (Γ, Λ) and (Γ, Λ^S), respectively, the concatenation of S and T is the sequence $ST = x_1, \ldots, x_s, y_1, \ldots, y_t$. If K is complete, $\Lambda^K = \Lambda$ so that if $m > 0$, then K^m denotes the concatenation of m copies of K and $K^m \in \mathfrak{S}$.

The following statement, which is [12, Proposition 1.3], relates the elements of $\mathfrak{S}$ to filters of the poset (Γ_0, Λ). In particular, it tells us precisely when a subset of Γ_0 can be realized as the support of a sequence $S \in \mathfrak{S}$.

Proposition 1.1. *Let $\Omega \subset \Gamma_0$. There exists a sequence $S = x_1, \ldots, x_s,\ s \geq 0$, in $\mathfrak{S}$ satisfying $\operatorname{Supp} S = \Omega$ if and only if Ω is a filter of (Γ_0, Λ). Moreover, if $\Omega \neq \emptyset$ is a filter, the sequence $S = x_1, \ldots, x_s$ can be chosen so that $x_1, \ldots, x_s$ are distinct.*

The following is [12, Definition 1.2].

Definition 1.2. If a sequence $S = x_1, \ldots, x_i, x_{i+1}, \ldots, x_s,\ 0 < i < s$, in $\mathfrak{S}$ has the property that no edge of Γ connects x_i with x_{i+1}, then $T = x_1, \ldots, x_{i+1}, x_i, \ldots, x_s$ is in $\mathfrak{S}$, and we set SrT. We denote by $\sim$ the equivalence relation that is a reflexive and transitive closure of the symmetric binary relation r.

The above definition is motivated by the fact that if distinct vertices x and y are both sinks in (Γ, Λ), then $F_x^+ F_y^+ = F_y^+ F_x^+$, as follows from the analog of [2, Lemma 1.2, proof of part 3)] for representations of valued quivers. Hence $S \sim T$ implies $F(S) = F(T)$.

The following is [12, Proposition 1.6].

Proposition 1.2. *If $S, T \in \mathfrak{S}$ are nonempty and consist of distinct vertices, the following are equivalent.*

(a) $S \sim T$.
(b) $\operatorname{Supp} S = \operatorname{Supp} T$.
(c) $\Lambda^S = \Lambda^T$.

The next result, which is [12, Proposition 1.9], produces a canonical form in $\mathfrak{S}$.

Proposition 1.3. *Let $S \in \mathfrak{S}$ be nonempty.*

(a) *We have $S \sim S_1 S_2 \dots S_r$ where, for all i, S_i consists of distinct vertices, and $\operatorname{Supp} S_i = \operatorname{Supp} S_i S_{i+1} \dots S_r$. Further, if $\operatorname{Supp} S_i \neq \Gamma_0$ then $\operatorname{Supp} S_{i+1} \subsetneq \operatorname{Supp} S_i$.*
(b) *Let $T \sim T_1 T_2 \dots T_q$ be a nonempty sequence in $\mathfrak{S}$ where, for all j, T_j consists of distinct vertices, and $\operatorname{Supp} T_j = \operatorname{Supp} T_j T_{j+1} \dots T_q$. Then $S \sim T$ if and only if $r = q$ and $S_i \sim T_i$ on $(\Gamma, \Lambda^{S_1 \dots S_{i-1}})$, $i = 1, \dots, r$.*

Definition 1.3. If $S \sim S_1 S_2 \dots S_r$ is a nonempty sequence in $\mathfrak{S}$ and the S_i satisfy the conditions of Proposition 1.3(a), we say that $S_1 S_2 \dots S_r$ is the *canonical form*, and r is the *size*, of S. If $S = S_1 S_2 \dots S_r$, we say that S is in the canonical form. The size of the empty sequence is zero.

By Proposition 1.3(b), the size of a nonempty sequence is uniquely determined and each S_i is unique up to equivalence.

We quote [10, Remark 2.1].

Remark 1.1. In the setting of Proposition 1.3(a), if $v \in \Gamma_0$ then $v \in \operatorname{Supp} S_i$ if and only if $m_S(v) \geq i$.

According to [12, Definition 1.5], for each filter F of (Γ_0, Λ), the *hull* of F is the smallest filter of (Γ_0, Λ) containing F, as well as each vertex of $\Gamma_0 \setminus F$ that is connected by an edge to a vertex in F. The hull of F is denoted by $H_\Lambda(F)$.

We quote [10, Remark 2.2].

Remark 1.2. If F is a filter of (Γ_0, Λ) and the full subgraph of Γ determined by $\operatorname{Supp} F$ is connected (for example, if F is a principal filter), then the full subgraph of Γ determined by $\operatorname{Supp} H_\Lambda(F)$ is connected.

An effective way of constructing all possible (+)-admissible sequences is given by the next statement, which is [12, Proposition 1.11].

Proposition 1.4. (a) *If $S = S_1 S_2 \dots S_r \in \mathfrak{S}$ is a nonempty sequence in the canonical form then, for all i, $\operatorname{Supp} S_i$ is a filter of (Γ_0, Λ) and, for $0 < i < r$, $H_\Lambda(\operatorname{Supp} S_{i+1}) \subset \operatorname{Supp} S_i$.*

(b) *If $F_1 \supset \dots \supset F_{r-1} \supset F_r$ is a sequence of nonempty filters of (Γ_0, Λ) satisfying $H_\Lambda(F_{i+1}) \subset F_i$ for $0 < i < r$, then there exists a unique up to equivalence sequence $S_1 S_2 \dots S_r \in \mathfrak{S}$ in the canonical form satisfying $\operatorname{Supp} S_i = F_i$ for all i.*

We now introduce a partial order on the set of equivalence classes of $\sim$, define the subset $\mathfrak{P}$ of principal (+)-admissible sequences in $\mathfrak{S}$, and relate the poset structure of $\mathfrak{P}$ to the combinatorial structure of the translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$. We quote [12, Definition 2.1].

Definition 1.4. If $S, T \in \mathfrak{S}$, we say that S is a *subsequence* of T and write $S \preccurlyeq T$ if $T \sim SU$ for some (+)-admissible sequence U.

Proposition 1.5. (a) *The relation $\preccurlyeq$ is a preorder.*

Let $S, T \in \mathfrak{S}$.

(b) *We have $S \preccurlyeq T$ and $T \preccurlyeq S$ if and only if $S \sim T$.*
(c) *If S, T are nonempty and if $S_1 \dots S_r$, $T_1 \dots T_q$ are their canonical forms, respectively, then $S \preccurlyeq T$ if and only if $r \leq q$ and $S_i \preccurlyeq T_i$ for $0 < i \leq r$.*
(d) *$S \preccurlyeq T$ if and only if for all $v \in \Gamma_0$, $m_S(v) \leq m_T(v)$.*

Proof. (a), (b), and (c) are proved in [12, Proposition 2.1].
(d) This is a direct consequence of (c), Proposition 1.3(a), and Remark 1.1. □

We quote [10, Corollary 2.4 and Proposition 2.5].

Corollary 1.6. *If $S, T \in \mathfrak{S}$, then $S \sim T$ if and only if for all $v \in \Gamma_0$, $m_S(v) = m_T(v)$.*

Proposition 1.7. *Let $S \in \mathfrak{S}$ and let U, V be (+)-admissible sequences on (Γ, Λ^S).*
(a) *$SU \preccurlyeq SV$ if and only if $U \preccurlyeq V$.*
(b) *$SU \sim SV$ if and only if $U \sim V$.*

Proof. Part (a) is an immediate consequence of Proposition 1.5(d), and (b) follows directly from Corollary 1.6. □

By Proposition 1.5(b), the preorder $\preccurlyeq$ induces a partial order, which we denote by the same symbol, on the set of equivalence classes of $\sim$ in $\mathfrak{S}$; when no confusion arises we identify a sequence with its equivalence class. The poset $\mathfrak{S}$ is a lattice [10], as is demonstrated below.
The following is [10, Definition 2.4].

Definition 1.5. Let $S, T \in \mathfrak{S}$ be nonempty and let $S_1S_2\dots S_r$, $T_1T_2\dots T_q$ be their canonical forms, respectively, where without loss of generality we assume that $r \leq q$. We set:
(a) $S \wedge T$ to be the empty sequence if $\operatorname{Supp} S \cap \operatorname{Supp} T = \emptyset$; and if $\operatorname{Supp} S \cap \operatorname{Supp} T \neq \emptyset$, then $S \wedge T$ is a (+)-admissible sequence with the canonical form $R_1R_2\dots R_s$, where $s \leq q, r$ is the largest integer satisfying $\operatorname{Supp} R_i = \operatorname{Supp} S_i \cap \operatorname{Supp} T_i \neq \emptyset$ for $0 < i \leq s$.
(b) $S \vee T$ to be a (+)-admissible sequence with the canonical form $R_1R_2\dots R_q$, where $\operatorname{Supp} R_i = \operatorname{Supp} S_i \cup \operatorname{Supp} T_i$ for $0 < i \leq r$, and $\operatorname{Supp} R_i = \operatorname{Supp} T_i$ for $r < i \leq q$.

If $\emptyset$ is the empty sequence in $\mathfrak{S}$, then for all $S \in \mathfrak{S}$, we set $S \wedge \emptyset = \emptyset$ and $S \vee \emptyset = S$.

That $S \wedge T$ and $S \vee T$ are in fact (+)-admissible sequences is contained in the proof of the following statement, which is [10, Proposition 2.6].

Proposition 1.8. *The poset of equivalence classes of $\sim$ in $\mathfrak{S}$ with the partial order $\preccurlyeq$ is a lattice where the operations of the greatest lower bound and the least upper bound are $\wedge$ and $\vee$, respectively.*

Proof. The intersection or union of two filters is always a filter. If F_1, F_2 are filters of (Γ_0, Λ), then it is straight forward that $H_\Lambda(F_1 \cap F_2) \subset H_\Lambda(F_1) \cap H_\Lambda(F_2)$ and $H_\Lambda(F_1 \cup F_2) = H_\Lambda(F_1) \cup H_\Lambda(F_2)$. Therefore, in view of Proposition 1.4, we conclude that if $S, T \in \mathfrak{S}$, then $S \wedge T$ and $S \vee T$ are in $\mathfrak{S}$. It follows from Proposition 1.5, parts (c) and (d), that $S \wedge T$ and $S \vee T$ are the greatest lower bound and the least upper bound, respectively, of S and T. □

We quote [10, Theorem 2.7].

Theorem 1.9. *Let $S, T \in \mathfrak{S}$.*
(a) *$S \sim (S \wedge T)\, S'$, $T \sim (S \wedge T)\, T'$ where S', T' are (+)-admissible sequences on $(\Gamma, \Lambda^{S \wedge T})$ that are unique up to equivalence.*
(b) $\operatorname{Supp} S' \cap \operatorname{Supp} T' = \emptyset$.

Proof. (a) This is a direct consequence of Propositions 1.8 and 1.7(b).
(b) By (a), we have $(S \wedge T)(S' \wedge T') \preccurlyeq S, T$, so Proposition 1.8 implies $(S \wedge T)(S' \wedge T') \preccurlyeq S \wedge T$ whence $S' \wedge T' = \emptyset$. By Definition 1.5(a) and Proposition 1.3(a), $\operatorname{Supp} S' \cap \operatorname{Supp} T' = \emptyset$. □

The following is [12, Definition 2.2].

Definition 1.6. A sequence $S \in \mathfrak{S}$ is *tight* if it is nonempty and its canonical form $S_1 S_2 \dots S_r$ satisfies $\operatorname{Supp} S_i = H_\Lambda(\operatorname{Supp} S_{i+1})$ for $0 < i < r$, and S is *principal* if it is tight and $\operatorname{Supp} S_r$ is a principal filter. We denote by $\mathfrak{T}$ ($\mathfrak{P}$) the set of tight (principal) sequences in $\mathfrak{S}$; clearly, $\mathfrak{P} \subset \mathfrak{T} \subset \mathfrak{S}$. By Proposition 1.2, a tight sequence is uniquely determined by its size and the set $\operatorname{Supp} S_r$, so we let $S_{r,x}$ denote the principal sequence of size r with $\operatorname{Supp} S_r = \langle x \rangle,\ x \in \Gamma_0$. Thus $\mathfrak{P} = \{S_{r,x} \mid r \in \mathbb{Z}^+, x \in \Gamma_0\}$ where $\mathbb{Z}^+$ is the set of positive integers.

We quote [10, Remark 3.1].

Remark 1.3. It follows from Remark 1.2 that if $S \in \mathfrak{P}$, the full subgraph of Γ determined by $\operatorname{Supp} S$ is connected.

The next statement, which is [12, Corollary 2.3], explains how to compare an arbitrary sequence in $\mathfrak{S}$ with a tight one, and shows that the last vertex of a principal sequence is uniquely determined.

Proposition 1.10. *Let $S \in \mathfrak{S}$ be nonempty and $T \in \mathfrak{T}$ with canonical forms $S_1 \dots S_r$ and $T_1 \dots T_q$, respectively.*

(a) *We have $T \preccurlyeq S$ if and only if $q \le r$ and $\operatorname{Supp} T_q \subset \operatorname{Supp} S_q$. If $T = S_{q,x} \in \mathfrak{P}$ then $T \preccurlyeq S$ if and only if $q \le r$ and $x \in \operatorname{Supp} S_q$.*

(b) *If $T = S_{q,x} = x_1, x_2, \dots, x_t$ then $x_t = x$.*

We now recall the notion of a translation quiver [13, p. 47]. If $\Delta = (\Delta_0, \Delta_1)$ is a locally finite graph with an orientation Θ, the quiver (Δ, Θ) is a *translation quiver* if it is equipped with a partially defined injective map $\tau : \Delta_0 \to \Delta_0$, called the *translation* of (Δ, Θ), such that for all $z \in \Delta_0$ in the domain of τ and all $y \in \Delta_0$ there is an arrow from y to z if and only if there is an arrow from τz to y (remember, in this paper no graph has multiple edges). In particular, the translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$ of the opposite quiver of (Γ, Λ) is defined as follows. The set of vertices of $\mathbb{N}(\Gamma, \Lambda^{op})$ is $\mathbb{N} \times \Gamma_0$, and each arrow $a : u \to v$ of (Γ, Λ), which by definition is the only arrow $u \to v$, gives rise to two series of arrows, $(n, a^\circ) : (n, v) \to (n, u)$ and $(n, a^\circ)' : (n, u) \to (n+1, v)$. The translation is defined by $\tau(n, u) = (n-1, u)$ for all $n > 0$ and $u \in \Gamma_0$. By construction, $\mathbb{N}(\Gamma, \Lambda^{op})$ is a locally finite quiver without oriented cycles, so $\mathbb{N} \times \Gamma_0$ is a poset. We note that since (Γ, Λ) is a valued quiver, $\mathbb{N}(\Gamma, \Lambda^{op})$ is a *valued translation quiver* (see [1, Sections VII.4 and VIII.1]). However, we do not use the valuation on $\mathbb{N}(\Gamma, \Lambda^{op})$ because our method is to obtain information about the latter set using the combinatorics of $\mathfrak{S}$ and $\mathfrak{P}$, which are independent of the valuation or modulation on (Γ, Λ).

We end this section by relating the Hasse diagram of $\mathfrak{P}$ to $\mathbb{N}(\Gamma, \Lambda^{op})$. Recall that if $a : x \to y$ is an arrow in a quiver, then a path $a_t \dots a_1 : x \to y$ of length $t > 1$ in the quiver is called a *bypass* of a. The following is [12, Theorem 2.5].

Theorem 1.11. *Let $\mathfrak{P}$ be the set of principal $(+)$-admissible sequences on (Γ, Λ).*

(a) *The map $\psi : \mathfrak{P} \to \mathbb{N} \times \Gamma_0$ given by $\psi(S_{r,x}) = (r-1, x)$ is an isomorphism of posets.*

(b) *Suppose no arrow in (Γ, Λ) has a bypass. Then ψ induces an isomorphism of quivers $\psi : \mathscr{H}(\mathfrak{P}) \to \mathbb{N}(\Gamma, \Lambda^{op})$, and the map $S_{r,x} \mapsto S_{r-1,x},\ x \in \Gamma_0,\ r > 1$, is a translation on $\mathscr{H}(\mathfrak{P})$ that turns ψ into an isomorphism of translation quivers.*

2. Preprojective Modules

Throughout this section (Γ, Λ) is a quiver without oriented cycles with a fixed valuation $\mathbf{b}$ and modulation $\mathfrak{B}$. We apply the combinatorial results of Section 1 to the preprojective component of the Auslander-Reiten quiver.

Definition 2.1. If $S = x_1, \dots, x_s,\ s > 0$, is in $\mathfrak{S}$, we let $F(S)$ denote the composition of reflection functors $F^+_{x_s} \dots F^+_{x_1}$; when $S = K$ is a complete $(+)$-admissible sequence then $F(S) = \Phi^+$ is the (positive) Coxeter functor [3], and if $S = \emptyset$ then $F(S)$ is the identity functor on f.d. $k(\Gamma, \Lambda)$. We say that S *annihilates* a $k(\Gamma, \Lambda)$-module M if $F(S)M = 0$; if, in addition, no proper subsequence of S annihilates M, then S is a *shortest* sequence annihilating M.

Recall that if $(V, f) \in \mathrm{Rep}(\Gamma, \Lambda)$, the *support* of (V, f) is defined as $\mathrm{Supp}\,(V, f) = \{x \in \Gamma_0 \mid V_x \neq 0\}$. If $M \in$ f.d. $k(\Gamma, \Lambda)$ and (V, f) is the representation identified with M, then, by definition, $\mathrm{Supp}\, M = \mathrm{Supp}\,(V, f)$.

Remark 2.1. Let $M \in$ f.d. $k(\Gamma, \Lambda)$. If $S \in \mathfrak{S}$ annihilates M, then $\mathrm{Supp}\, M \subset \mathrm{Supp}\, S_M$. If M is indecomposable, the full subgraph of Γ determined by $\mathrm{Supp}\, M$ is connected.

For each $x \in \Gamma_0$, let $L_x \in \mathrm{Rep}(\Gamma, \Lambda)$ be defined by $L_x = (V_i, f_a)$, where $V_i = 0$ for $i \neq x$, $V_x = \mathbf{k}_x$, and $f_a = 0$ for all arrows a. That is, the representations L_x are the simple objects of $\mathrm{Rep}(\Gamma, \Lambda)$. The following is an analog of [2, Corollary 1.1].

Proposition 2.1. *Let $x_1, \ldots, x_s$, $s > 0$, be a $(+)$-admissible sequence on the valued quiver (Γ, Λ).*

(a) *For any i $(1 \leq i \leq s)$, $F^-_{x_1} \cdots F^-_{x_{i-1}}(L_{x_i})$ is either 0 or an indecomposable object in $\mathrm{Rep}(\Gamma, \Lambda)$.*

(b) *If $(V, f) \in \mathrm{Rep}(\Gamma, \Lambda)$ is indecomposable and $F^+_{x_s} \cdots F^+_{x_1}(V, f) = 0$, then for some i, $(V, f) \cong F^-_{x_1} \cdots F^-_{x_{i-1}}(L_{x_i})$.*

Proof. The statement follows from [3, Proposition 2.1] in the same way as [2, Corollary 1.1] follows from [2, Theorem 1.1]. □

The following result extends [12, Theorem 3.1] and [10, Theorem 3.4(a)] to representations of valued quivers. For an integer $m > 0$ and $N \in$ f.d. $k(\Gamma, \Lambda)$, we denote by N^m the direct sum of m copies of N.

Theorem 2.2. *Let $M \in$ f.d. $k(\Gamma, \Lambda)$ be preprojective.*

(a) *There exists a unique up to equivalence shortest sequence $S_M \in \mathfrak{S}$ annihilating M.*

(b) *If $M \cong N_1 \oplus \cdots \oplus N_s$ then each N_i is preprojective and $S_M = S_{N_1} \vee \cdots \vee S_{N_s}$. In particular, for all integers $m > 0$, $S_{M^m} = S_M$.*

(c) *If $M \cong M_1^{m_1} \oplus \cdots \oplus M_t^{m_t}$ where the M_i's are nonisomorphic indecomposable $k(\Gamma, \Lambda)$-modules and $m_i > 0$ for all i, then $S_M = S_{M_1} \vee \cdots \vee S_{M_t}$.*

(d) *If M is indecomposable and $N \in$ f.d. $k(\Gamma, \Lambda)$ is indecomposable preprojective, then $S_N \sim S_M$ if and only if $N \cong M$.*

Proof. (a) Let S, T be shortest sequences in $\mathfrak{S}$ annihilating M, where $\ell(S) \leq \ell(T)$. To show that $S \sim T$, proceed by induction on $\ell(S)$. If $\ell(S) = 0$, then $S = \emptyset$ whence $M = 0$ and $T = \emptyset$. Suppose now that $\ell(S) > 0$ and that the statement holds for all preprojective $k(\Gamma, \Theta)$-modules N (for all orientations Θ without oriented cycles) and all pairs S', T' of shortest $(+)$-admissible sequences on (Γ, Θ) annihilating N where $\ell(S') \leq \ell(T')$ and $\ell(S') < \ell(S)$. Since $\ell(S) > 0$, then $M \neq 0$.

By Theorem 1.9, $S \sim (S \wedge T)S'$ and $T \sim (S \wedge T)T'$ where S', T' are $(+)$-admissible sequences on $(\Gamma, \Lambda^{S \wedge T})$ satisfying $\mathrm{Supp}\, S' \cap \mathrm{Supp}\, T' = \emptyset$. It follows that $\ell(S') \leq \ell(T')$. If $S \wedge T = \emptyset$ then $S \sim S'$, $T \sim T'$, and $\mathrm{Supp}\, S \cap \mathrm{Supp}\, T = \emptyset$. Let $(W, h) \in \mathrm{Rep}(\Gamma, \Lambda)$ be identified with M. Since $F(S)M = 0$, if $W_i \neq 0$ for some $i \in \Gamma_0$, then $i \in \mathrm{Supp}\, S$. Since $\mathrm{Supp}\, S \cap \mathrm{Supp}\, T = \emptyset$, then $F(T)$ does not change any of the nonzero $\mathbf{k}_i$-spaces W_i, which exist because $M \neq 0$. We obtained a contradiction with $F(T)M = 0$, so $S \wedge T \neq \emptyset$ whence $\ell(S') < \ell(S)$ and S', T' are shortest $(+)$-admissible sequences on $(\Gamma, \Lambda^{S \wedge T})$ annihilating the preprojective $k(\Gamma, \Lambda^{S \wedge T})$-module $F(S \wedge T)M$. By the induction hypothesis, we have $S' \sim T'$ whence $S \sim (S \wedge T)S' \sim (S \wedge T)T' \sim T$.

(b) Since every reflection functor is additive, each N_i is preprojective. By (a), a sequence $S \in \mathfrak{S}$ annihilates M if and only if $S_{N_i} \preccurlyeq S$ for all i. Since $\mathfrak{S}$ is a lattice by Proposition 1.8, we have $S_M = S_{N_1} \vee \cdots \vee S_{N_s}$.

(c) This is an immediate consequence of (b).

(d) This follows from Proposition 2.1. □

As in [12], we now show that if M is an indecomposable preprojective $k(\Gamma,\Lambda)$-module, then $S_M \in \mathfrak{P}$, and we begin with the case when $M = P$ is projective.

Lemma 2.3. *Let $P_x \in$ f.d. $k(\Gamma,\Lambda)$ be the indecomposable projective module associated with $x \in \Gamma_0$, let (V,f) be the representation of (Γ,Λ) identified with P_x, and let $x \neq z \in \Gamma_0$.*

(a) *If the set $\{a_i : y_i \to z,\ i = 1,\dots,l\}$ of all arrows ending at z is not empty, then the map $h : \oplus_{i=1}^{l}({}_zB_{y_i} \underset{\mathbf{k}_{y_i}}{\otimes} V_{y_i}) \to V_z$ induced by the maps $f_{a_i} : {}_zB_{y_i} \underset{\mathbf{k}_{y_i}}{\otimes} V_{y_i} \to V_z$ is an isomorphism.*

(b) *If z is a sink in (Γ,Λ) and if $Q_x \in$ f.d. $k(\Gamma, \sigma_z \Lambda)$ is the indecomposable projective module associated with x, then $F_z^+(P_x) \cong Q_x$.*

Proof. (a) We recall the structure of (V,f), see [4, Section 10] and [5]. For all $u \in \Gamma_0$, denote by $\mathcal{W}_u^x$ the set of all paths from x to u in (Γ,Λ) and let $p \in \mathcal{W}_u^x$. If $p = b_t \cdots b_1,\ t > 0$, we set $B_p = {}_{e(b_t)}B_{s(b_t)} \underset{\mathbf{k}_{s(b_t)}}{\otimes} \cdots \underset{\mathbf{k}_{e(b_1)}}{\otimes} {}_{e(b_1)}B_{s(b_1)}$, and if $u = x$ and $p = e_x$ is the trivial path, we set $B_p = \mathbf{k}_x$. Then $V_z = \underset{p\in\mathcal{W}_z^x}{\oplus} B_p$; note that $V_z = 0$ if $\mathcal{W}_z^x = \emptyset$. To describe the map associated to an arrow $y \to z$, say, to $a_1 : y_1 \to z$, we note first that

$$V_z = (\underset{p=a_1q}{\oplus} B_p) \oplus (\underset{p\neq a_1q}{\oplus} B_p) = (\underset{q\in\mathcal{W}_{y_1}^x}{\oplus} ({}_zB_{y_1} \underset{\mathbf{k}_{y_1}}{\otimes} B_q)) \oplus (\underset{p\neq a_1q}{\oplus} B_p),$$

while $V_{y_1} = \underset{q\in\mathcal{W}_{y_1}^x}{\oplus} B_q$. The function $f_{a_1} : {}_zB_{y_1} \underset{\mathbf{k}_{y_1}}{\otimes} V_{y_1} \to V_z$ maps its domain onto the first summand of its codomain via the usual isomorphism ${}_zB_{y_1} \underset{\mathbf{k}_{y_1}}{\otimes} (\underset{q\in\mathcal{W}_{y_1}^x}{\oplus} B_q) \to \underset{q\in\mathcal{W}_{y_1}^x}{\oplus} ({}_zB_{y_1} \underset{\mathbf{k}_{y_1}}{\otimes} B_q)$. It is now clear that h is an isomorphism.

(b) Let (U,j) and (W,g) be the representations of $(\Gamma, \sigma_z \Lambda)$ identified with Q_x and $F_z^+(P_x)$, respectively. Let $z \neq y \in \Gamma_0$. Since z is a sink in (Γ,Λ) and a source in $(\Gamma,\sigma_z\Lambda)$, a path from x to y in $(\Gamma,\sigma_z\Lambda)$ is a path from x to y in (Γ,Λ), and vice versa. Thus, $U_y = V_y$ for all $y \neq z$, and $j_a = f_a$ for all arrows a not ending at z. Since F_z^+ affects only the space at z and the maps into this space, $U_y = W_y$ for all $y \neq z$, and $j_a = g_a$ for all arrows a not ending at z. Since z is a source in $(\Gamma,\sigma_z\Lambda)$, there is no path from x to z in that quiver, whence $U_z = 0$. It remains to show that $W_z = 0$. Since z is a sink in (Γ,Λ), $x \neq z$, and the graph Γ is connected, the set of arrows stopping at z is not empty. Hence the map h of part (a) is an isomorphism, and $W_z = \operatorname{Ker} h = 0$. □

Proposition 2.4. *If P_x is the indecomposable projective $k(\Gamma,\Lambda)$-module associated with $x \in \Gamma_0$, then $S_{P_x} \sim S_{1,x}$.*

Proof. By Propositions 1.1 and 1.2, there exists a unique up to equivalence $(+)$-admissible sequence $S = x_1,\dots,x_s$ that consists of distinct vertices and satisfies $\{x_1,\dots,x_s\} = \langle x\rangle$. We first show by induction on s that S annihilates P_x. When $s = 1$ this follows from Proposition 2.1. Suppose $s > 1$ and the statement holds for all orientations Θ on Γ without oriented cycles and all indecomposable projective $k(\Gamma,\Theta)$-modules associated with vertices w satisfying $|\langle w\rangle| < s$. Since $s > 1$, then $x < x_1$ in (Γ_0,Λ), and in $(\Gamma_0,\sigma_{x_1}\Lambda)$ we have $\langle x\rangle = \{x_2,\dots,x_s\}$. By the induction hypothesis, the sequence $x_2,\dots,x_s$ annihilates Q_x, the indecomposable projective $k(\Gamma,\sigma_{x_1}\Lambda)$-module associated with x. Since $x < x_1$ and x_1 is a sink in (Γ,Λ), Lemma 2.3(b) says that $F_{x_1}^+(P_x) \cong Q_x$, so S annihilates P_x. To show that no proper subsequence of S annihilates P_x, let $(V,f) \in \operatorname{Rep}(\Gamma,\Lambda)$ be identified with P_x and note that if $y \geq x$ in (Γ_0,Λ), then $V_y \neq 0$. Since V_y may be changed by F_z^+ only if $z = y$, any sequence annihilating P_x must contain y. □

We need the following combinatorial statement whose necessity is [10, Proposition 3.6].

Proposition 2.5. *Let* $S = x_1, \dots, x_s,\ s > 1$, *be in* $\mathfrak{S}$ *and set* $T = x_2, \dots, x_s$. *Then* $S \in \mathfrak{P}$ *if and only if the full subgraph of* Γ *determined by* $\operatorname{Supp} S$ *is connected and* T *is a principal (+)-admissible sequence on* $(\Gamma, \sigma_{x_1} \Lambda)$.

Proof. We only have to prove the sufficiency. Parts of the proof are similar to the proofs of [10, Proposition 3.6 and Theorem 4.5].

For a given graph Γ and orientation Λ, the sets $\mathfrak{P}$ and $\mathfrak{S}$ depend neither on the valuation $\mathbf{b}$ nor on the modulation $\mathfrak{B}$. Hence, without loss of generality, we may assume for the rest of this proof that (Γ, Λ) is an ordinary, not valued, quiver in which at least one of the arrows has multiplicity greater than 1: since Γ is a connected graph with more than one vertex, (Γ, Λ) has at least one arrow. Then the finite dimensional path algebras $k(\Gamma, \Lambda)$ and $k(\Gamma, \sigma_{x_1} \Lambda)$ are of infinite representation type (see [2]), and the results of [12] apply.

Since $k(\Gamma, \sigma_{x_1} \Lambda)$ is of infinite representation type, [12, Corollary 3.8, parts (a) and (c)] says that $T \sim S_N$ for some indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module N, and [1, VIII Proposition 1.14] says that N is not a preinjective module, hence, not a simple injective module. By [2, Theorem 1.1, part (2)], $M = F^-_{x_1} N$ is an indecomposable $k(\Gamma, \Lambda)$-module and $N \cong F^+_{x_1} M$, so that S annihilates M. Therefore M is preprojective, $S_M \preccurlyeq S$, and [12, Theorem 3.5] says that $S_M \in \mathfrak{P}$. To show that $S \in \mathfrak{P}$, it suffices to prove that $x_1 \in \operatorname{Supp} S_M$. For if the latter is true, then $S_M \sim y_1, \dots, y_t$ where $y_1 = x_1$ and

$$F^+_{y_t} \dots F^+_{y_2}(N) \cong F^+_{y_t} \dots F^+_{y_2} F^+_{y_1}(F^-_{x_1} N) = F(S_M) M = 0,$$

whence $y_2, \dots, y_t$ is a (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$ that annihilates N, so that $\ell(S_M) - 1 \geq \ell(T) = \ell(S) - 1$ and $\ell(S_M) \geq \ell(S)$. Since $S_M \preccurlyeq S$, then $S_M \sim S$.

If $x_1 \notin \operatorname{Supp} S_M$ then $x_1 \in \operatorname{Supp} U$, where $S \sim S_M U$, and x_1 is a sink in (Γ, Λ^{S_M}) because $\operatorname{Supp} S_M$, being a filter of (Γ_0, Λ), contains no $v \in \Gamma_0$ satisfying $v \leq x_1$. By [12, Lemma 1.7], for all $v \in \operatorname{Supp} S_M$, no arrow connects v and x_1 whence $S_M x_1 \sim x_1 S_M$ on (Γ, Λ). Therefore S_M is a (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$ and we have $0 = F^+_{x_1}(F(S_M) M) = F(S_M)(F^+_{x_1} M) \cong F(S_M) N$. Hence $S_N \preccurlyeq S_M$ so that $s - 1 \leq \ell(S_M)$, which implies $s - 1 = \ell(S_M)$ and $S \sim S_M x_1 \sim x_1 S_M$. Then the full subgraph of Γ determined by $\operatorname{Supp} S$ is disconnected, a contradiction. ☐

Although the statement of Proposition 2.5 does not involve representation theory, our proof uses representations of quivers. We know a purely combinatorial proof, but it is much longer and more technical than the one given above.

The next result is an extension of [12, Theorem 3.5] to representations of valued quivers.

Theorem 2.6. *If* $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ *is indecomposable preprojective,* S_M *is a principal* (+)*-admissible sequence.*

Proof. Since $M \neq 0$, then $S_M = x_1, \dots, x_s,\ s > 0$, and we proceed by induction on s. The case $s = 1$ is trivial, so let $s > 1$ and suppose that the theorem holds for all orientations Θ on Γ without oriented cylces and all indecomposable preprojective $k(\Gamma, \Theta)$-modules N satisfying $\ell(S_N) < s$. Since $s > 1$, $N = F^+_{x_1} M$ is an indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$-module and $S_N = x_2, \dots, x_s$. By the induction hypothesis, S_N is a principal (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$. In view of Proposition 2.5, to prove that $S_M \in \mathfrak{P}$, it suffices to show that the full subgraph of Γ determined by $\operatorname{Supp} S_M$ is connected.

Assume, to the contrary, that the subgraph is disconnected. Since S_N is a principal (+)-admissible sequence, Remark 1.3 says that the full subgraph of Γ determined by $\operatorname{Supp} S_N$ is connected, whence $\operatorname{Supp} S_M = \operatorname{Supp} S_N \cup \{x_1\}$ where $x_1 \notin \operatorname{Supp} S_N$ and, moreover, no edge of Γ connects x_1 to a vertex in $\operatorname{Supp} S_N$. It follows that $S_M = x_1 S_N \sim S_N x_1$ so that $S_N \in \mathfrak{S}$. According to Remark 2.1, the full subgraph of Γ determined by $\operatorname{Supp} M$ is connected and $\operatorname{Supp} M \subset \operatorname{Supp} S_M$. Then either $\operatorname{Supp} M = \{x_1\}$ or $\operatorname{Supp} M \subset \operatorname{Supp} S_N$. In the former case, $M \cong L_{x_1}$ whence $S_M = x_1$, which contradicts $s > 1$. In the latter case, $0 = F(S_M)M = F^+_{x_1}(F(S_N)M)$ implies $F(S_N)M = 0$ because $x_1 \notin \operatorname{Supp} S_N$, which contradicts that S_M is the shortest sequence annihilating M. $\square$

Corollary 2.7. *Let $M \in \text{f.d.}\, k(\Gamma, \Lambda)$ be indecomposable and satisfy $(\Phi^+)^\nu M \cong P_x$, where $\nu \in \mathbb{N}$ and P_x is the indecomposable projective $k(\Gamma, \Lambda)$-module associated with $x \in \Gamma_0$.*

(a) *$S_M \sim S_{\nu+1,x}$.*
(b) *If $S_M = x_1, \ldots, x_s$ then $x_s = x$ and $M \cong F^-_{x_1} \ldots F^-_{x_{s-1}}(L_x)$ where L_x is the simple projective $k(\Gamma, \sigma_{x_{s-1}} \ldots \sigma_{x_1} \Lambda)$-module associated with x.*

Proof. (a) By Theorem 2.6, $S_M \sim S_{r,y}$. We have $S_M \preccurlyeq K^{r-1}S_{1,y} \preccurlyeq K^r$ by Propositions 1.4 and 1.5, so $F(S_{1,y})((\Phi^+)^{r-1}M) = (\Phi^+)^r M = 0$. Since $(\Phi^+)^{r-1}M \neq 0$ by Theorem 2.2(a), then $(\Phi^+)^{r-1}M \cong P_x$ and $\nu = r - 1$ (see [3, Proposition 2.4(i)]). Since $S_{1,y}$ annihilates P_x then $S_{1,x} \preccurlyeq S_{1,y}$ by Proposition 2.4. Since $K^{r-1}S_{1,x}$ annihilates M then $S_M \preccurlyeq K^{r-1}S_{1,x}$, whence $S_{1,y} \preccurlyeq S_{1,x}$ and $S_{1,y} \sim S_{1,x}$ in light of Proposition 1.5. Using Proposition 1.2, we get $x = y$.

(b) This is an easy consequence of (a), Corollary 1.10(b), and Proposition 2.1. $\square$

In order to apply our results to the preprojective component of (Γ, Λ), we recall some definitions and facts from [1, 13]. If $X \in \text{f.d.}\, k(\Gamma, \Lambda)$ is indecomposable, let $[X]$ be the isomorphism class of X. If $Y \in \text{f.d.}\, k(\Gamma, \Lambda)$ is indecomposable, a path of length $m > 0$ from X to Y is a sequence of nonzero nonisomorphisms $X = A_0 \to \cdots \to A_m = Y$, where $A_i \in \text{f.d.}\, k(\Gamma, \Lambda)$ is indecomposable for all i. By definition, there exists a path of length zero from X to X. One writes $[X] \prec [Y]$ if there exists a path of positive length from X to Y.

The preprojective component of (Γ, Λ), $\tilde{\mathscr{P}}(\Gamma, \Lambda)$, is a locally finite connected valued translation quiver whose set of vertices, $\tilde{\mathscr{P}}(\Gamma, \Lambda)_0$, consists of the isomorphism classes of indecomposable preprojective $k(\Gamma, \Lambda)$-modules. If $X, Y \in \text{f.d.}\, k(\Gamma, \Lambda)$ are indecomposable, there is an arrow $[X] \to [Y]$ if and only if there exists an irreducible map $X \to Y$ (remember, we disregard the valuations of arrows). The translation is defined by $[X] \mapsto [\operatorname{DTr} X] = [\Phi^+ X]$ for all nonprojective X. If X, Y are indecomposable, Y is preprojective, and $X = A_0 \to \cdots \to A_m = Y$, $m > 0$, is a path from X to Y, then $[X] \neq [Y]$ and A_i is preprojective for all i. It follows that the reflexive closure $\preccurlyeq$ of the transitive binary relation $\prec$ is a partial order on $\tilde{\mathscr{P}}(\Gamma, \Lambda)_0$. Moreover, $[X] \prec [Y]$ if and only if there is a finite sequence of irreducible morphisms $X = B_0 \to \cdots \to B_n = Y$, where $n > 0$ and B_j is indecomposable preprojective for all j.

We finish the paper by extending [12, Proposition 3.7 and Corollary 3.8] to representations of valued quivers. Consider the map $\phi : \tilde{\mathscr{P}}(\Gamma, \Lambda) \to \mathbb{N}(\Gamma, \Lambda^{op})$ defined on the vertices by $\phi([L]) = (\nu, x) = (\nu(L), x(L))$, where x is the vertex of (Γ, Λ) associated with the indecomposable projective module $(\Phi^+)^\nu L$, and defined on the arrows in a natural way [1, VIII Proposition 1.15].

Proposition 2.8. (a) *The map $\phi : \tilde{\mathscr{P}}(\Gamma, \Lambda) \to \mathbb{N}(\Gamma, \Lambda^{op})$ is a full embedding of translation quivers whose restriction $\phi : \tilde{\mathscr{P}}(\Gamma, \Lambda)_0 \to \mathbb{N} \times \Gamma_0$ is an injective morphism of posets.*
(b) *The map ϕ is an isomorphism when (Γ, Λ) is of infinite representation type.*
(c) *The image of ϕ is an ideal of $\mathbb{N} \times \Gamma_0$, i.e., if $[M] \in \tilde{\mathscr{P}}(\Gamma, \Lambda)_0$ and $(l, u) \leq \phi([M])$, then there exists an indecomposable preprojective $k(\Gamma, \Lambda)$-module L with $\phi([L]) = (l, u)$.*
(d) *Given an $[M] \in \tilde{\mathscr{P}}(\Gamma, \Lambda)_0$, the map ϕ induces a bijection between the set of paths in $\tilde{\mathscr{P}}(\Gamma, \Lambda)$ ending at $[M]$ and the set of paths in $\mathbb{N}(\Gamma, \Lambda^{op})$ ending at $\phi([M])$.*

Proof. (a) and (b) These are [1, VIII Propositions 1.15 and 1.16].

(c) This is an easy consequence of the following obvious statement. If $0 \to A \to B \to C \to 0$ is an almost split sequence of finitely generated modules over a hereditary artin algebra where B has a nonzero injective direct summand, then C is injective.

(d) This is an immediate consequence of (a) and (c). $\square$

We now obtain a module-theoretic version of Theorem 1.11.

Corollary 2.9. (a) *The map* $\chi : \tilde{\mathscr{P}}(\Gamma, \Lambda)_0 \to \mathfrak{P}$ *given by* $[L] \mapsto S_L$ *is an injective morphism of posets.*

(b) *If each arrow* $x \to y$ *is the only path from* x *to* y *in* (Γ, Λ)*, then the map* χ *induces a full embedding* $\chi : \tilde{\mathscr{P}}(\Gamma, \Lambda) \to \mathscr{H}(\mathfrak{P})$ *of translation quivers, where* $S_{r,x} \mapsto S_{r-1,x}$*,* $x \in \Gamma_0,\ r > 1$*, is the translation on* $\mathscr{H}(\mathfrak{P})$*.*

(c) *If* (Γ, Λ) *is of infinite representation type, the map* χ *in (a) and in (b) is an isomorphism.*

Proof. This is an immediate consequence of Theorems 2.2, 2.6, and 1.11, together with Proposition 2.8. ☐

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