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PRESENTATIONS OF RINGS WITH NON-TRIVIAL SEMIDUALIZING MODULES

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Abstract. Let R be a commutative noetherian local ring. A finitely generated R-module C is *semidualizing* if it is self-orthogonal and satisfies the condition $\mathrm{Hom}_R(C,C) \cong R$. We prove that a Cohen-Macaulay ring R with dualizing module D admits a semidualizing module C satisfying $R \ncong C \ncong D$ if and only if it is a homomorphic image of a Gorenstein ring in which the defining ideal decomposes in a cohomologically independent way. This expands on a well-known result of Foxby, Reiten and Sharp saying that R admits a dualizing module if and only if R is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring.

1. Introduction

Throughout this paper $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring.

A finitely generated R-module C is *self-orthogonal* if $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. Examples of self-orthogonal R-modules include the finitely generated free R-modules and the dualizing module of Grothendieck. (See Section 2 for definitions and background information.) Results of Foxby [10], Reiten [17] and Sharp [21] precisely characterize the local rings which possess a dualizing module: the ring R admits a dualizing module if and only if R is Cohen–Macaulay and there exist a Gorenstein local ring Q and an ideal $I \subset Q$ such that $R \cong Q/I$.

The point of this paper is to similarly characterize the local Cohen–Macaulay rings with a dualizing module which admit certain other self-orthogonal modules. The specific self-orthogonal modules of interest are the *semidualizing* R-modules, that is, those self-orthogonal R-modules satisfying $\mathrm{Hom}_R(C,C) \cong R$. A free R-module of rank 1 is semidualizing, as is a dualizing R-module, when one exists. We say that a semidualizing is *non-trivial* if it is neither free nor dualizing.

Our main theorem is the following expansion of the aforementioned result of Foxby, Reiten and Sharp; we prove it in Section 3. It shows, assuming the existence of a dualizing module, that R has a non-trivial semidualizing module if and only if R is Cohen-Macaulay and $R \cong Q/(I_1 + I_2)$ where Q is Gorenstein and the rings Q/I_1 and Q/I_2 enjoy considerable cohomological vanishing over Q. Thus, it addresses both of the following questions: what conditions guarantee that R admits a non-trivial semidualizing module, and what are the ramifications of the existence of such a module?

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Theorem 1.1. *Let R be a local Cohen–Macaulay ring with a dualizing module. Then R admits a semidualizing module that is neither dualizing nor free if and only if there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ satisfying the following conditions:*

(1) *There is a ring isomorphism $R \cong Q/(I_1 + I_2)$;*
(2) *For $j = 1, 2$ the quotient ring Q/I_j is Cohen–Macaulay and not Gorenstein;*
(3) *For all $i \in \mathbb{Z}$, we have the following vanishing of Tate cohomology modules: $\widehat{\operatorname{Tor}}_i^Q(Q/I_1, Q/I_2) = 0 = \widehat{\operatorname{Ext}}_Q^i(Q/I_1, Q/I_2)$;*
(4) *There exists an integer c such that $\operatorname{Ext}_Q^c(Q/I_1, Q/I_2)$ is not cyclic; and*
(5) *For all $i \geqslant 1$, we have $\operatorname{Tor}_i^Q(Q/I_1, Q/I_2) = 0$; in particular, there is an equality $I_1 \cap I_2 = I_1 I_2$.*

A prototypical example of a ring admitting non-trivial semidualizing modules is the following.

Example 1.2. Let k be a field and set $Q = k[\![X, Y, S, T]\!]$. The ring

$$R = Q/(X^2, XY, Y^2, S^2, ST, T^2) = Q/[(X^2, XY, Y^2) + (S^2, ST, T^2)]$$

is local with maximal ideal $(X, Y, S, T)R$. It is artinian of socle dimension 4, hence Cohen–Macaulay and non-Gorenstein. With $R_1 = Q/(X^2, XY, Y^2)$ it follows that the R-module $\operatorname{Ext}_{R_1}^2(R, R_1)$ is semidualizing and neither dualizing nor free; see [22, p. 92, Example].

Proposition 4.1 shows how Theorem 1.1 can be used to construct numerous rings admitting non-trivial semidualizing modules. To complement this, the following example shows that rings that do not admit non-trivial semidualizing modules are easy to come by.

Example 1.3. Let k be a field. The ring $R = k[X, Y]/(X^2, XY, Y^2)$ is local with maximal ideal $\mathfrak{m} = (X, Y)R$. It is artinian of socle dimension 2, hence Cohen–Macaulay and non-Gorenstein. From the equality $\mathfrak{m}^2 = 0$, it is straightforward to deduce that the only semidualizing R-modules, up to isomorphism, are the ring itself and the dualizing module; see [22, Prop. (4.9)].

2. Background on Semidualizing Modules

We begin with relevant definitions. The following notions were introduced independently (with different terminology) by Foxby [10], Golod [12], Grothendieck [13, 14], Vasconcelos [22] and Wakamatsu [23].

Definition 2.1. Let C be an R-module. The *homothety homomorphism* is the map $\chi_C^R \colon R \to \operatorname{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$.

The R-module C is *semidualizing* if it satisfies the following conditions:

(1) The R-module C is finitely generated;
(2) The homothety map $\chi_C^R \colon R \to \operatorname{Hom}_R(C, C)$, is an isomorphism; and
(3) For all $i \geqslant 1$, we have $\operatorname{Ext}_R^i(C, C) = 0$.

An R-module D is *dualizing* if it is semidualizing and has finite injective dimension.

Note that the R-module R is semidualizing, so that every local ring admits a semidualizing module.

Fact 2.2. Let C be a semidualizing R-module. It is straightforward to show that a sequence $\mathbf{x} = x_1, \ldots, x_n \in \mathfrak{m}$ is C-regular if and only if it is R-regular. In particular, we have $\operatorname{depth}_R(C) = \operatorname{depth}(R)$; see, e.g., [18, (1.4)]. Thus, when R is Cohen–Macaulay, every semidualizing R-module is a maximal Cohen–Macaulay module. On the other hand, if R admits a dualizing module, then R is Cohen–Macaulay by [20, (8.9)]. As R is local, if it admits a dualizing module, then its dualizing module is unique up to isomorphism; see, e.g. [5, (3.3.4(b))].

The following definition and fact justify the term "dualizing".

Definition 2.3. Let C and B be R-modules. The natural *biduality homomorphism* $\delta_C^B \colon C \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, B), B)$ is given by $\delta_C^B(c)(\phi) = \phi(c)$. When D is a dualizing R-module, we set $C^\dagger = \operatorname{Hom}_R(C, D)$.

Fact 2.4. Assume that R is Cohen–Macaulay with dualizing module D. Let C be a semidualizing R-module. Fact 2.2 says that C is a maximal Cohen–Macaulay R-module. From standard duality theory, for all $i \neq 0$ we have

$$\operatorname{Ext}_R^i(C, D) = 0 = \operatorname{Ext}_R^i(C^\dagger, D)$$

and the natural biduality homomorphism $\delta_C^D \colon C \to \operatorname{Hom}_R(C^\dagger, D)$ is an isomorphism; see, e.g., [5, (3.3.10)]. The R-module $C^\dagger$ is semidualizing by [7, (2.12)]. Also, the evaluation map $C \otimes_R C^\dagger \to D$ given by $c \otimes \phi \mapsto \phi(c)$ is an isomorphism, and one has $\operatorname{Tor}_i^R(C, C^\dagger) = 0$ for all $i \geqslant 1$ by [11, (3.1)].

The following construction is also known as the "idealization" of M. It was popularized by Nagata, but goes back at least to Hochschild [15], and the idea behind the construction appears in work of Dorroh [8]. It is the key idea for the proof of the converse of Sharp's result [21] given by Foxby [10] and Reiten [17].

Definition 2.5. Let M be an R-module. The *trivial extension* of R by M is the ring $R \ltimes M$, described as follows. As an additive abelian group, we have $R \ltimes M = R \oplus M$. The multiplication in $R \ltimes M$ is given by the formula

$$(r, m)(r', m') = (rr', rm' + r'm).$$

The multiplicative identity on $R \ltimes M$ is $(1, 0)$. We let $\epsilon_M \colon R \to R \ltimes M$ and $\tau_M \colon R \ltimes M \to R$ denote the natural injection and surjection, respectively.

The next assertions are straightforward to verify.

Fact 2.6. Let M be an R-module. The trivial extension $R \ltimes M$ is a commutative ring with identity. The maps ϵ_M and τ_M are ring homomorphisms, and $\operatorname{Ker}(\tau_M) = 0 \oplus M$. We have $(0 \oplus M)^2 = 0$, and so $\operatorname{Spec}(R \ltimes M)$ is in order-preserving bijection with $\operatorname{Spec}(R)$. It follows that $R \ltimes M$ is quasilocal and $\dim(R \ltimes M) = \dim(R)$. If M is finitely generated, then $R \ltimes M$ is also noetherian and

$$\operatorname{depth}(R \ltimes M) = \operatorname{depth}_R(R \ltimes M) = \min\{\operatorname{depth}(R), \operatorname{depth}_R(M)\}.$$

In particular, if R is Cohen–Macaulay and M is a maximal Cohen–Macaulay R-module, then $R \ltimes M$ is Cohen–Macaulay as well.

Next, we discuss the correspondence between dualizing modules and Gorenstein presentations given by the results of Foxby, Reiten and Sharp.

Fact 2.7. Sharp [21, (3.1)] showed that if R is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring Q, then R admits a dualizing module. The proof proceeds as follows. If $g = \operatorname{depth}(Q) - \operatorname{depth}(R) = \dim(Q) - \dim(R)$, then $\operatorname{Ext}^i_Q(R, Q) = 0$ for $i \neq g$ and the module $\operatorname{Ext}^g_Q(R, Q)$ is dualizing for R.

The same idea gives the following. Let A be a local Cohen–Macaulay ring with a dualizing module D, and assume that R is Cohen–Macaulay and a module-finite A-algebra. If $h = \operatorname{depth}(A) - \operatorname{depth}(R) = \dim(A) - \dim(R)$, then $\operatorname{Ext}^i_A(R, D) = 0$ for $i \neq h$ and the module $\operatorname{Ext}^h_A(R, D)$ is dualizing for R.

Fact 2.8. Independently, Foxby [10, (4.1)] and Reiten [17, (3)] proved the converse of Sharp's result from Fact 2.7. Namely, they showed that if R admits a dualizing module, then it is Cohen–Macaulay and a homomorphic image of a local Gorenstein ring Q. We sketch the proof here, as the main idea forms the basis of our proof of Theorem 1.1. See also, e.g., [5, (3.3.6)].

Let D be a dualizing R-module. It follows from [20, (8.9)] that R is Cohen–Macaulay. Set $Q = R \ltimes D$, which is Gorenstein with $\dim(Q) = \dim(R)$. The natural surjection $\tau_D \colon Q \to R$ yields an presentation of R as a homomorphic image of the local Gorenstein ring Q.

The next notion we need is Auslander and Bridger's G-dimension [1, 2]. See also Christensen [6].

Definition 2.9. A complex of R-modules

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

is *totally acyclic* if it satisfies the following conditions:

(1) Each R-module X_i is finitely generated and free; and
(2) The complexes X and $\operatorname{Hom}_R(X, R)$ are exact.

An R-module G is *totally reflexive* if there exists a totally acyclic complex of R-modules such that $G \cong \operatorname{Coker}(\partial^X_1)$; in this event, the complex X is a *complete resolution* of G.

Fact 2.10. An R-module G is totally reflexive if and only if it satisfies the following:

(1) The R-module G is finitely generated;
(2) The biduality map $\delta^R_G \colon G \to \operatorname{Hom}_R(\operatorname{Hom}_R(G, R), R)$, is an isomorphism; and
(3) For all $i \geqslant 1$, we have $\operatorname{Ext}^i_R(G, R) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(G, R), R)$.

See, e.g., [6, (4.1.4)].

Definition 2.11. Let M be a finitely generated R-module. Then M has *finite G-dimension* if it has a finite resolution by totally reflexive R-modules, that is, if there is an exact sequence

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$$

such that each G_i is a totally reflexive R-module. The *G-dimension* of M, when it is finite, is the length of the shortest finite resolution by totally reflexive R-modules:

$$\operatorname{G-dim}_R(M) = \inf \left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} \text{there is an exact sequence of } R\text{-modules} \\ 0 \to G_n \to \cdots \to G_0 \to M \to 0 \\ \text{such that each } G_i \text{ is totally reflexive} \end{array} \right\}.$$

Fact 2.12. The ring R is Gorenstein if and only if every finitely generated R-module has finite G-dimension; see [6, (1.4.9)]. Also, the AB formula [6, (1.4.8)] says that if M is a finitely generated R-module of finite G-dimension, then

$$\text{G-dim}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

Fact 2.13. Let S be a Cohen–Macaulay local ring equipped with a module-finite local ring homomorphism $\tau\colon S \to R$ such that R is Cohen–Macaulay. Then $\text{G-dim}_S(R) < \infty$ if and only if there exists an integer $g \geqslant 0$ such that $\operatorname{Ext}^i_S(R,S) = 0$ for all $i \neq g$ and $\operatorname{Ext}^g_S(R,S)$ is a semidualizing R-module; when these conditions hold, one has $g = \text{G-dim}_S(R)$. See [7, (6.1)].

Assume that S has a dualizing module D. If $\text{G-dim}_S(R) < \infty$, then $R \otimes_S D$ is a semidualizing R-module and $\operatorname{Tor}^S_i(R,D) = 0$ for all $i \geqslant 1$; see [7, (4.7),(5.1)].

Our final background topic is Avramov and Martsinkovsky's notion of Tate cohomology [4].

Definition 2.14. Let M be a finitely generated R-module. Considering M as a complex concentrated in degree zero, a *Tate resolution* of M is a diagram of degree zero chain maps of R-complexes $T \xrightarrow{\alpha} P \xrightarrow{\beta} M$ satisfying the following conditions:

1. The complex T is totally acyclic, and the map α_i is an isomorphism for $i \gg 0$;
2. The complex P is a resolution of M by finitely generated free R-modules, and β is the augmentation map

Remark 2.15. In [4], Tate resolutions are called "complete resolutions". We call them Tate resolutions in order to avoid confusion with the terminology from Definition 2.9. This is consistent with [19].

Fact 2.16. By [4, (3.1)], a finitely generated R-module M has finite G-dimension if and only if it admits a Tate resolution.

Definition 2.17. Let M be a finitely generated R-module of finite G-dimension, and let $T \xrightarrow{\alpha} P \xrightarrow{\beta} M$ be a Tate resolution of M. For each integer i and each R-module N, the ith *Tate homology* and *Tate cohomology* modules are

$$\widehat{\operatorname{Tor}}^R_i(M,N) = \operatorname{H}_i(T \otimes_R N) \qquad \widehat{\operatorname{Ext}}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(T,N)).$$

Fact 2.18. Let M be a finitely generated R-module of finite G-dimension. For each integer i and each R-module N, the modules $\widehat{\operatorname{Tor}}^R_i(M,N)$ and $\widehat{\operatorname{Ext}}^i_R(M,N)$ are independent of the choice of Tate resolution of M, and they are appropriately functorial in each variable by [4, (5.1)]. If M has finite projective dimension, then we have $\widehat{\operatorname{Tor}}^R_i(M,-) = 0 = \widehat{\operatorname{Ext}}^i_R(M,-)$ and $\widehat{\operatorname{Tor}}^R_i(-,M) = 0 = \widehat{\operatorname{Ext}}^i_R(-,M)$ for each integer i; see [4, (5.9) and (7.4)].

3. Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into two pieces. The first piece is the following result which covers one implication. Note that, if $\operatorname{pd}_Q(Q/I_1)$ or $\operatorname{pd}_Q(Q/I_2)$ is finite, then condition (3) holds automatically by Fact 2.18.

Theorem 3.1 (Sufficiency of conditions (1)–(5) of Theorem 1.1)**.** *Let R be a local Cohen–Macaulay ring with dualizing module. Assume that there exist a Gorenstein local ring Q and ideals $I_1, I_2 \subset Q$ satisfying the following conditions:*

(1) *There is a ring isomorphism* $R \cong Q/(I_1 + I_2)$*;*
(2) *For* $j = 1, 2$ *the quotient ring* Q/I_j *is Cohen–Macaulay, and* Q/I_2 *is not Gorenstein;*
(3) *For all* $i \in \mathbb{Z}$*, we have* $\widehat{\operatorname{Tor}}_i^Q(Q/I_1, Q/I_2) = 0 = \widehat{\operatorname{Ext}}_Q^i(Q/I_1, Q/I_2)$*;*
(4) *There exists an integer* c *such that* $\operatorname{Ext}_Q^c(Q/I_1, Q/I_2)$ *is not cyclic; and*
(5) *For all* $i \geqslant 1$*, we have* $\operatorname{Tor}_i^Q(Q/I_1, Q/I_2) = 0$*; in particular, there is an equality* $I_1 \cap I_2 = I_1 I_2$*.*

Then R *admits a semidualizing module that is neither dualizing nor free.*

Proof. For $j = 1, 2$ set $R_j = Q/I_j$. Since Q is Gorenstein, we have $\text{G-dim}_Q(R_1) < \infty$ by Fact 2.12, so R_1 admits a Tate resolution $T \xrightarrow{\alpha} P \xrightarrow{\beta} R_1$ over Q; see Fact 2.16.

We claim that the induced diagram $T \otimes_Q R_2 \xrightarrow{\alpha \otimes_Q R_2} P \otimes_Q R_2 \xrightarrow{\beta \otimes_Q R_2} R_1 \otimes_Q R_2$ is a Tate resolution of $R_1 \otimes_Q R_2 \cong R$ over R_2. The condition (5) implies that $P \otimes_Q R_2$ is a free resolution of $R_1 \otimes_Q R_2 \cong R$ over R_2, and it follows that $\beta \otimes_Q R_2$ is a quasi-isormorphism. Of course, the complex $T \otimes_Q R_2$ consists of finitely generated free R_2-modules, and the map $\alpha^i \otimes_Q R_2$ is an isomorphism for $i \gg 0$. The condition $\widehat{\operatorname{Tor}}_i^Q(R_1, R_2) = 0$ from (3) implies that the complex $T \otimes_Q R_2$ is exact. Hence, to prove the claim, it remains to show that the first complex in the following sequence of isomorphisms is exact:

$$\operatorname{Hom}_{R_2}(T \otimes_Q R_2, R_2) \cong \operatorname{Hom}_Q(T, \operatorname{Hom}_{R_2}(R_2, R_2)) \cong \operatorname{Hom}_Q(T, R_2).$$

The isomorphisms here are given by Hom-tensor adjointness and Hom cancellation. This explains the first step in the next sequence of isomorphisms:

$$\mathrm{H}_i(\operatorname{Hom}_{R_2}(T \otimes_Q R_2, R_2)) \cong \mathrm{H}_i(\operatorname{Hom}_Q(T, R_2)) \cong \widehat{\operatorname{Ext}}_Q^{-i}(R_1, R_2) = 0.$$

The second step is by definition, and the third step is by assumption (3). This establishes the claim.

From the claim, we conclude that $g = \text{G-dim}_{R_2}(R)$ is finite; see Fact 2.16. It follows from Fact 2.13 that $\operatorname{Ext}_{R_2}^g(R, R_2) \neq 0$, and that the R-module $C = \operatorname{Ext}_{R_2}^g(R, R_2)$ is semidualizing.

To complete the proof, we need only show that C is not free and not dualizing. By assumption (4), the fact that $\operatorname{Ext}_{R_2}^i(R, R_2) = 0$ for all $i \neq g$ implies that $C = \operatorname{Ext}_{R_2}^g(R, R_2)$ is not cyclic, so $C \ncong R$.

There is an equality of Bass series $I_{R_2}^{R_2}(t) = t^e I_R^C(t)$ for some integer e. (For instance, the vanishing $\operatorname{Ext}_{R_2}^i(R, R_2) = 0$ for all $i \neq g$ implies that there is an isomorphism $C \simeq \mathsf{\Sigma}^g \mathbf{R}\operatorname{Hom}_{R_2}(R, R_2)$ in $\mathsf{D}(R)$, so we can apply, e.g., [7, (1.7.8)].) By assumption (2), the ring R_2 is not Gorenstein. Hence, the Bass series $I_{R_2}^{R_2}(t) = t^e I_R^C(t)$ is not a monomial. It follows that the Bass series $I_R^C(t)$ is not a monomial, so C is not dualizing for R. □

The remainder of this section is devoted to the proof of the following.

Theorem 3.2 (Necessity of conditions (1)–(5) of Theorem 1.1)**.** *Let* R *be a local Cohen–Macaulay ring with dualizing module* D*. Assume that* R *admits a semidualizing module* C *that is neither dualizing nor free. Then there exist a Gorenstein local ring* Q *and ideals* $I_1, I_2 \subset Q$ *satisfying the following conditions:*

(1) *There is a ring isomorphism* $R \cong Q/(I_1 + I_2)$*;*

(2) *For $j = 1,2$ the quotient ring Q/I_j is Cohen–Macaulay with a dualizing module D_j and is not Gorenstein;*
(3) *For all $i \in \mathbb{Z}$, we have $\widehat{\mathrm{Tor}}_i^Q(Q/I_1, Q/I_2) = 0 = \widehat{\mathrm{Ext}}_Q^i(Q/I_1, Q/I_2)$ and $\widehat{\mathrm{Tor}}_i^Q(Q/I_2, Q/I_1) = 0 = \widehat{\mathrm{Ext}}_Q^i(Q/I_2, Q/I_1)$;*
(4) *The modules $\mathrm{Hom}_Q(Q/I_1, Q/I_2)$ and $\mathrm{Hom}_Q(Q/I_2, Q/I_1)$ are not cyclic;*
(5) *For all $i \geqslant 1$, we have $\mathrm{Ext}_Q^i(Q/I_1, Q/I_2) = 0 = \mathrm{Ext}_Q^i(Q/I_2, Q/I_1)$ and $\mathrm{Tor}_i^Q(Q/I_1, Q/I_2) = 0$; in particular, there is an equality $I_1 \cap I_2 = I_1 I_2$;*
(6) *For $j = 1,2$ we have $\text{G-dim}_{Q/I_j}(R) < \infty$; and*
(7) *There exists an R-module isomorphism $D_1 \otimes_Q D_2 \cong D$, and for all $i \geqslant 1$ we have $\mathrm{Tor}_i^Q(D_1, D_2) = 0$.*

Proof. For the sake of readability, we include the following roadmap of the proof.

Outline 3.3. The ring Q is constructed as an iterated trivial extension of R. As an R-module, it has the form $Q = R \oplus C \oplus C^\dagger \oplus D$ where $C^\dagger = \mathrm{Hom}_R(C, D)$. The ideals I_j are then given as $I_1 = 0 \oplus 0 \oplus C^\dagger \oplus D$ and $I_2 = 0 \oplus C \oplus 0 \oplus D$. The details for these constructions are contained in Steps 3.4 and 3.5. Conditions (1), (2) and (6) are then verified in Lemmas 3.6–3.8. The verification of conditions (4) and (5) requires more work; it is proved in Lemma 3.12, with the help of Lemmas 3.9–3.11. Lemma 3.13 contains the verification of condition (7). The proof concludes with Lemma 3.14 which contains the verification of condition (3).

The following two steps contain notation and facts for use through the rest of the proof.

Step 3.4. Set $R_1 = R \ltimes C$, which is Cohen–Macaulay with $\dim(R_1) = \dim(R)$; see Facts 2.2 and 2.6. The natural injection $\epsilon_C \colon R \to R_1$ makes R_1 into a module-finite R-algebra, so Fact 2.7 implies that the module $D_1 = \mathrm{Hom}_R(R_1, D)$ is dualizing for R_1. There is a sequence of R-module isomorphisms

$$D_1 = \mathrm{Hom}_R(R_1, D) \cong \mathrm{Hom}_R(R \oplus C, D) \cong \mathrm{Hom}_R(C, D) \oplus \mathrm{Hom}_R(R, D) \cong C^\dagger \oplus D.$$

It is straightforward to show that the resulting R_1-module structure on $C^\dagger \oplus D$ is given by the following formula:

$$(r, c)(\phi, d) = (r\phi, \phi(c) + rd).$$

The kernel of the natural epimorphism $\tau_C \colon R_1 \to R$ is the ideal $\mathrm{Ker}(\tau_C) \cong 0 \oplus C$.

Fact 2.8 implies that the ring $Q = R_1 \ltimes D_1$ is local and Gorenstein. The R-module isomorphism in the next display is by definition:

$$Q = R_1 \ltimes D_1 \cong R \oplus C \oplus C^\dagger \oplus D.$$

It is straightforward to show that the resulting ring structure on Q is given by

$$(r, c, \phi, d)(r', c', \phi', d') = (rr', rc' + r'c, r\phi' + r'\phi, \phi'(c) + \phi(c') + rd' + r'd).$$

The kernel of the epimorphism $\tau_{D_1} \colon Q \to R_1$ is the ideal

$$I_1 = \mathrm{Ker}(\tau_{D_1}) \cong 0 \oplus 0 \oplus C^\dagger \oplus D.$$

As a Q-module, this is isomorphic to the R_1-dualizing module D_1. The kernel of the composition $\tau_C \circ \tau_{D_1} \colon Q \to R$ is the ideal $\mathrm{Ker}(\tau_C \tau_{D_1}) \cong 0 \oplus C \oplus C^\dagger \oplus D$.

Since Q is Gorenstein and $\mathrm{depth}(R_1) = \mathrm{depth}(Q)$, Fact 2.12 implies that R_1 is totally reflexive as a Q-module. Using the the natural isomorphism $\mathrm{Hom}_Q(R_1, Q) \xrightarrow{\cong} (0 :_Q I_1)$ given by $\psi \mapsto \psi(1)$, one shows that the map $\mathrm{Hom}_Q(R_1, Q) \to I_1$ given by

$\psi \mapsto \psi(1)$ is a well-defined Q-module isomorphism. Thus I_1 is totally reflexive over Q, and it follows that $\mathrm{Hom}_Q(I_1, Q) \cong R_1$.

Step 3.5. Set $R_2 = R \ltimes C^\dagger$, which is Cohen–Macaulay with $\dim(R_2) = \dim(R)$. The injection $\epsilon_{C^\dagger} \colon R \to R_2$ makes R_2 into a module-finite R-algebra, so the module $D_2 = \mathrm{Hom}_R(R_2, D)$ is dualizing for R_2. There is a sequence of R-module isomorphisms

$$D_2 = \mathrm{Hom}_R(R_2, D) \cong \mathrm{Hom}_R(R \oplus C^\dagger, D) \cong \mathrm{Hom}_R(C^\dagger, D) \oplus \mathrm{Hom}_R(R, D) \cong C \oplus D.$$

The last isomorphism is from Fact 2.4. The resulting R_2-module structure on $C \oplus D$ is given by the following formula:

$$(r, \phi)(c, d) = (r\phi, \phi(c) + rd).$$

The kernel of the natural epimorphism $\tau_{C^\dagger} \colon R_2 \to R$ is the ideal $\mathrm{Ker}(\tau_{C^\dagger}) \cong 0 \oplus C^\dagger$.

The ring $Q' = R_2 \ltimes D_2$ is local and Gorenstein. There is a sequence of R-module isomorphisms

$$Q' = R_2 \ltimes D_2 \cong R \oplus C \oplus C^\dagger \oplus D$$

and the resulting ring structure on $R \oplus C \oplus C^\dagger \oplus D$ is given by

$$(r, c, \phi, d)(r', c', \phi', d') = (rr', rc' + r'c, r\phi' + r'\phi, \phi'(c) + \phi(c') + rd' + r'd).$$

That is, we have an isomorphism of rings $Q' \cong Q$. The kernel of the epimorphism $\tau_{D_2} \colon Q \to R_2$ is the ideal

$$I_2 = \mathrm{Ker}(\tau_{D_2}) \cong 0 \oplus C \oplus 0 \oplus D.$$

This is isomorphic, as a Q-module, to the dualizing module D_2. The kernel of the composition $\tau_{C^\dagger} \circ \tau_{D_2} \colon Q \to R$ is the ideal $\mathrm{Ker}(\tau_{C^\dagger}\tau_{D_2}) \cong 0 \oplus C \oplus C^\dagger \oplus D$.

As in Step 3.4, the Q-modules R_2 and $\mathrm{Hom}_Q(R_2, Q) \cong I_2$ are totally reflexive, and $\mathrm{Hom}_Q(I_2, Q) \cong R_2$.

Lemma 3.6 (Verification of condition (1) from Theorem 3.2)**.** *With the notation of Steps 3.4–3.5, there is a ring isomorphism $R \cong Q/(I_1 + I_2)$.*

Proof. Consider the following sequence of R-module isomorphisms:

$$\begin{aligned} Q/(I_1 + I_2) &\cong (R \oplus C \oplus C^\dagger \oplus D)/((0 \oplus 0 \oplus C^\dagger \oplus D) + (0 \oplus C \oplus 0 \oplus D)) \\ &\cong (R \oplus C \oplus C^\dagger \oplus D)/(0 \oplus C \oplus C^\dagger \oplus D)) \\ &\cong R. \end{aligned}$$

It is straightforward to check that these are ring isomorphisms. □

Lemma 3.7 (Verification of condition (2) from Theorem 3.2)**.** *With the notation of Steps 3.4 and 3.5, each ring $R_j \cong Q/I_j$ is Cohen–Macaulay with a dualizing module D_j and is not Gorenstein.*

Proof. It remains only to show that each ring R_j is not Gorenstein, that is, that D_j is not isomorphic to R_j as an R_j-module.

For R_1, suppose by way of contradiction that there is an R_1-module isomorphism $D_1 \cong R_1$. It follows that this is an R-module isomorphism via the natural injection $\epsilon_C \colon R \to R_1$. Thus, we have R-module isomorphisms

$$C^\dagger \oplus D \cong D_1 \cong R_1 \cong R \oplus C.$$

Computing minimal numbers of generators, we have

$$\begin{aligned}\mu_R(C^\dagger)+\mu_R(D)=\mu_R(C^\dagger\oplus D)&=\mu_R(R\oplus C)=\mu_R(R)+\mu_R(C)\\&=1+\mu_R(C)\leqslant 1+\mu_R(C)\mu_R(C^\dagger)=1+\mu_R(D).\end{aligned}$$

The last step in this sequence follows from Fact 2.4. It follows that $\mu_R(C^\dagger)=1$, that is, that $C^\dagger$ is cyclic. From the isomorphism $R\cong\operatorname{Hom}_R(C,C)$, one concludes that $\operatorname{Ann}_R(C)=0$, and hence $C^\dagger\cong R/\operatorname{Ann}_R(C^\dagger)\cong R$. It follows that

$$C\cong\operatorname{Hom}_R(C^\dagger,D)\cong\operatorname{Hom}_R(R,D)\cong D$$

contradicting the assumption that C is not dualizing for R. (Note that this uses the uniqueness statement from Fact 2.2.)

Next, observe that $C^\dagger$ is not free and is not dualizing for R; this follows from the isomorphism $C\cong\operatorname{Hom}_R(C^\dagger,D)$ contained in Fact 2.4, using the assumption that C is not free and not dualizing. Hence, the proof that R_2 is not Gorenstein follows as in the previous paragraph. □

Lemma 3.8 (Verification of condition (6) from Theorem 3.2)**.** *With the notation of Steps 3.4–3.5, we have* $\operatorname{G-dim}_{R_j}(R)=0$ *for* $j=1,2$.

Proof. To show that $\operatorname{G-dim}_{R_1}(R)=0$, it suffices to show that $\operatorname{Ext}^i_{R_1}(R,R_1)=0$ for all $i\geqslant 1$ and that $\operatorname{Hom}_{R_1}(R,R_1)\cong C$; see Fact 2.13. To this end, we note that there are isomorphisms of R-modules

$$\operatorname{Hom}_R(R_1,C)\cong\operatorname{Hom}_R(R\oplus C,C)\cong\operatorname{Hom}_R(C,C)\oplus\operatorname{Hom}_R(R,C)\cong R\oplus C\cong R_1$$

and it is straightforward to check that the composition $\operatorname{Hom}_R(R_1,C)\cong R_1$ is an R_1-module isomorphism. Furthermore, for $i\geqslant 1$ we have

$$\operatorname{Ext}^i_R(R_1,C)\cong\operatorname{Ext}^i_R(R\oplus C,C)\cong\operatorname{Ext}^i_R(C,C)\oplus\operatorname{Ext}^i_R(R,C)=0.$$

Let I be an injective resolution of C as an R-module. The previous two displays imply that $\operatorname{Hom}_R(R_1,I)$ is an injective resolution of R_1 as an R_1-module. Using the fact that the composition $R\xrightarrow{\epsilon_C}R_1\xrightarrow{\tau_C}R$ is the identity id_R, we conclude that

$$\operatorname{Hom}_{R_1}(R,\operatorname{Hom}_R(R_1,I))\cong\operatorname{Hom}_R(R\otimes_{R_1}R_1,I)\cong\operatorname{Hom}_R(R,I)\cong I$$

and hence

$$\operatorname{Ext}^i_{R_1}(R,R_1)\cong\mathrm{H}^i(\operatorname{Hom}_{R_1}(R,\operatorname{Hom}_R(R_1,I)))\cong\mathrm{H}^i(I)\cong\begin{cases}0&\text{if } i\geqslant 1\\ C&\text{if } i=0\end{cases}$$

as desired.[1]

The proof for R_2 is similar. □

The next three results are for the proof of Lemma 3.12.

Lemma 3.9. *With the notation of Steps 3.4 and 3.5, one has* $\operatorname{Tor}^R_i(R_1,R_2)=0$ *for all* $i\geqslant 1$*, and there is an* R_1*-algebra isomorphism* $R_1\otimes_R R_2\cong Q$.

[1]Note that the finiteness of $\operatorname{G-dim}_{R_1}(R)$ can also be deduced from [16, (2.16)].

Proof. The Tor-vanishing comes from the following sequence of R-module isomorphisms

$$\begin{aligned}\operatorname{Tor}_i^R(R_1,R_2) &\cong \operatorname{Tor}_i^R(R\oplus C, R\oplus C^\dagger)\\ &\cong \operatorname{Tor}_i^R(R,R)\oplus\operatorname{Tor}_i^R(C,R)\oplus\operatorname{Tor}_i^R(R,C^\dagger)\oplus\operatorname{Tor}_i^R(C,C^\dagger)\\ &\cong \begin{cases} R\oplus C\oplus C^\dagger\oplus D & \text{if } i=0\\ 0 & \text{if } i\neq 0.\end{cases}\end{aligned}$$

The first isomorphism is by definition; the second isomorphism is elementary; and the third isomorphism is from Fact 2.4.

Moreover, it is straightforward to verify that in the case $i=0$ the isomorphism $R_1\otimes_R R_2\cong Q$ has the form $\alpha\colon R_1\otimes_R R_2\xrightarrow{\cong} Q$ given by

$$(r,c)\otimes(r',\phi')\mapsto(rr',r'c,r\phi',\phi'(c)).$$

It is routine to check that this is a ring homomorphism, that is, a ring isomorphism. Let $\xi\colon R_1\to R_1\otimes_R R_2$ be given by $(r,c)\mapsto(r,c)\otimes(1,0)$. Then one has $\alpha\xi=\epsilon_{D_1}\colon R_1\to Q$. It follows that $R_1\otimes_R R_2\cong Q$ as an R_1-algebra. $\square$

Lemma 3.10. *Continue with the notation of Steps 3.4 and 3.5. In the tensor product $R\otimes_{R_1} Q$ we have $1\otimes(0,c,0,d)=0$ for all $c\in C$ and all $d\in D$.*

Proof. Recall that Fact 2.4 implies that the evaluation map $C\otimes_R C^\dagger\to D$ given by $c'\otimes\phi\mapsto\phi(c')$ is an isomorphism. Hence, there exist $c'\in C$ and $\phi\in C^\dagger$ such that $d=\phi(c')$. This explains the first equality in the sequence

$$\begin{aligned}1\otimes(0,0,0,d) &= 1\otimes(0,0,0,\phi(c')) = 1\otimes[(0,c')(0,0,\phi,0)]\\ &= [1(0,c')]\otimes(0,0,\phi,0) = 0\otimes(0,0,\phi,0)=0.\end{aligned}\tag{3.10.1}$$

The second equality is by definition of the R_1-module structure on Q; the third equality is from the fact that we are tensoring over R_1; the fourth equality is from the fact that the R_1-module structure on R comes from the natural surjection $R_1\to R$, with the fact that $(0,c)\in 0\oplus C$ which is the kernel of this surjection.

On the other hand, using similar reasoning, we have

$$\begin{aligned}1\otimes(0,c,0,0) &= 1\otimes[(0,c)(1,0,0,0)] = [1(0,c)]\otimes(1,0,0,0)\\ &= 0\otimes(1,0,0,0)=0.\end{aligned}\tag{3.10.2}$$

Combining (3.10.1) and (3.10.2) we have

$$1\otimes(0,c,0,d)=[1\otimes(0,0,0,d)]+[1\otimes(0,c,0,0)]=0$$

as claimed. $\square$

Lemma 3.11. *With the notation of Steps 3.4 and 3.5, one has $\operatorname{Tor}_i^{R_1}(R,Q)=0$ for all $i\geqslant 1$, and there is a Q-module isomorphism $R\otimes_{R_1}Q\cong R_2$.*

Proof. Let P be an R-projective resolution of R_2. Lemma 3.9 implies that $R_1\otimes_R P$ is a projective resolution of $R_1\otimes_R R_2\cong Q$ as an R_1-module. From the following sequence of isomorphisms

$$R\otimes_{R_1}(R_1\otimes_R P)\cong(R\otimes_{R_1}R_1)\otimes_R P\cong R\otimes_R P\cong P$$

it follows that, for $i\geqslant 1$, we have

$$\operatorname{Tor}_i^{R_1}(R,Q)\cong \mathrm{H}_i(R\otimes_{R_1}(R_1\otimes_R P))\cong\mathrm{H}_i(P)=0$$

where the final vanishing comes from the assumption that P is a resolution of a module and $i \geqslant 1$.

This reasoning shows that there is an R-module isomorphism $\beta\colon R_2 \xrightarrow{\cong} R \otimes_{R_1} Q$. This isomorphism is equal to the composition

$$R_2 \xrightarrow{\cong} R \otimes_R R_2 \xrightarrow{\cong} R \otimes_{R_1} (R_1 \otimes_R R_2) \xrightarrow[R \otimes_{R_1} \alpha]{\cong} R \otimes_{R_1} Q$$

and is therefore given by

$$(r, \phi) \mapsto 1 \otimes (r, \phi) \mapsto 1 \otimes [(1, 0) \otimes (r, \phi)] \mapsto 1 \otimes (r, 0, \phi, 0). \tag{3.11.1}$$

We claim that β is a Q-module isomorphism. Recall that the Q-module structure on R_2 is given via the natural surjection $Q \to R_2$, and so is described as

$$(r, c, \phi, d)(r', \phi') = (r, \phi)(r', \phi') = (rr', r\phi' + r'\phi).$$

This explains the first equality in the following sequence

$$\beta((r, c, \phi, d)(r', \phi')) = \beta(rr', r\phi' + r'\phi) = 1 \otimes (rr', 0, r\phi' + r'\phi, 0).$$

The second equality is by (3.11.1). On the other hand, the definition of β explains the first equality in the sequence

$$\begin{aligned}(r, c, \phi, d)\beta(r', \phi') &= (r, c, \phi, d)[1 \otimes (r', 0, \phi', 0)]\\ &= 1 \otimes [(r, c, \phi, d)(r', 0, \phi', 0)]\\ &= 1 \otimes (rr', r'c, r\phi' + r'\phi, r'd + \phi'(c))\\ &= [1 \otimes (rr', 0, r\phi' + r'\phi, 0)] + [1 \otimes (0, r'c, 0, r'd + \phi'(c))]\\ &= 1 \otimes (rr', 0, r\phi' + r'\phi, 0).\end{aligned}$$

The second equality is from the definition of the Q-modules structure on $R \otimes_{R_1} Q$; the third equality is from the definition of the multiplication in Q; the fourth equality is by bilinearity; and the fifth equality is by Lemma 3.10. Combining these two sequences, we conclude that β is a Q-module isomorphism, as claimed. □

Lemma 3.12 (Verification of conditions (4)–(5) from Theorem 3.2)**.** *With the notation of Steps 3.4 and 3.5, the modules* $\operatorname{Hom}_Q(R_1, R_2)$ *and* $\operatorname{Hom}_Q(R_2, R_1)$ *are not cyclic. Also, one has* $\operatorname{Ext}^i_Q(R_1, R_2) = 0 = \operatorname{Ext}^i_Q(R_2, R_1)$ *and* $\operatorname{Tor}^Q_i(R_1, R_2) = 0$ *for all* $i \geqslant 1$*; in particular, there is an equality* $I_1 \cap I_2 = I_1 I_2$*.*

Proof. Let L be a projective resolution of R over R_1. Lemma 3.11 implies that the complex $L \otimes_{R_1} Q$ is a projective resolution of $R \otimes_{R_1} Q \cong R_2$ over Q. We have isomorphisms

$$(L \otimes_{R_1} Q) \otimes_Q R_1 \cong L \otimes_{R_1} (Q \otimes_Q R_1) \cong L \otimes_{R_1} R_1 \cong L$$

and it follows that, for $i \geqslant 1$, we have

$$\operatorname{Tor}^Q_i(R_2, R_1) \cong \mathrm{H}_i((L \otimes_{R_1} Q) \otimes_Q R_1) \cong \mathrm{H}_i(L) = 0$$

since L is a projective resolution.

The equality $I_1 \cap I_2 = I_1 I_2$ follows from the direct computation

$$I_1 \cap I_2 = (0 \oplus 0 \oplus C^\dagger \oplus D) \cap (0 \oplus C \oplus 0 \oplus D) = 0 \oplus 0 \oplus 0 \oplus D = I_1 I_2$$

or from the sequence $(I_1 \cap I_2)/(I_1 I_2) \cong \operatorname{Tor}^Q_1(Q/I_1, Q/I_2) = 0$.

Let P be a projective resolution of R_1 over Q. From the fact that $\operatorname{Tor}^Q_i(R_2, R_1) = 0$ for all $i \geq 1$ we get that $P \otimes_Q R_2$ is a projective resolution of R over R_2. Since the complexes $\mathrm{Hom}_Q(P, R_2)$ and $\mathrm{Hom}_{R_2}(P \otimes_Q R_2, R_2)$ are isomorphic, we therefore have the isomorphisms

$$\mathrm{Ext}^i_Q(R_1, R_2) \cong \mathrm{Ext}^i_{R_2}(R, R_2)$$

for all $i \geq 0$. By the fact that $\text{G-dim}_{R_2}(R) = 0$, we conclude that

$$\mathrm{Ext}^i_Q(R_1, R_2) \cong \begin{cases} C^\dagger & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Since C is not dualizing, the module $\mathrm{Hom}_Q(R_1, R_2) \cong \mathrm{Ext}^0_Q(R_1, R_2) \cong C^\dagger$ is not cyclic.

The verification for $\mathrm{Hom}_Q(R_2, R_1)$ and $\mathrm{Ext}^i_Q(R_2, R_1)$ is similar. □

Lemma 3.13 (Verification of condition (7) from Theorem 3.2)**.** *With the notation of Steps 3.4 and 3.5, there is an R-module isomorphism $D_1 \otimes_Q D_2 \cong D$, and for all $i \geqslant 1$ we have $\mathrm{Tor}^Q_i(D_1, D_2) = 0$.*

Proof. There is a short exact sequence of Q-module homomorphisms

$$0 \to D_1 \to Q \xrightarrow{\tau_{D_1}} R_1 \to 0.$$

For all $i \geqslant 1$, we have $\mathrm{Tor}^Q_i(Q, R_2) = 0 = \mathrm{Tor}^Q_i(R_1, R_2)$, so the long exact sequence in $\mathrm{Tor}^Q_i(-, R_2)$ associated to the displayed sequence implies that $\mathrm{Tor}^Q_i(D_1, R_2) = 0$ for all $i \geqslant 1$. Consider the next short exact sequence of Q-module homomorphisms

$$0 \to D_2 \to Q \xrightarrow{\tau_{D_2}} R_2 \to 0.$$

The associated long exact sequence in $\mathrm{Tor}^Q_i(D_1, -)$ implies that $\mathrm{Tor}^Q_i(D_1, D_2) = 0$ for all $i \geqslant 1$.

It is straightforward to verify the following sequence of Q-module isomorphisms

$$R \otimes_{R_1} D_1 \cong \left(\frac{R \ltimes C}{0 \oplus C}\right) \otimes_{R \ltimes C} (C^\dagger \oplus D) \cong \frac{C^\dagger \oplus D}{(0 \oplus C)(C^\dagger \oplus D)} \cong \frac{C^\dagger \oplus D}{0 \oplus D} \cong C^\dagger$$

and similarly

$$R \otimes_{R_2} D_2 \cong C.$$

These combine to explain the third isomorphism in the following sequence:

$$D_1 \otimes_Q D_2 \cong R \otimes_Q (D_1 \otimes_Q D_2) \cong (R \otimes_Q D_1) \otimes_R (R \otimes_Q D_2) \cong C^\dagger \otimes_R C \cong D.$$

For the first isomorphism, use the fact that D_j is annihilated by $D_j = I_j$ for $j = 1, 2$ to conclude that $D_1 \otimes_Q D_2$ is annihilated by $I_1 + I_2$; it follows that $D_1 \otimes_Q D_2$ is naturally a module over the quotient $Q/(I_1 + I_2) \cong R$. The second isomorphism is standard, and the fourth one is from Fact 2.4. □

Lemma 3.14 (Verification of condition (3) from Theorem 3.2)**.** *With the notation of Steps 3.4–3.5, we have $\widehat{\mathrm{Tor}}^Q_i(R_1, R_2) = 0 = \widehat{\mathrm{Ext}}^i_Q(R_1, R_2)$ and $\widehat{\mathrm{Tor}}^Q_i(R_2, R_1) = 0 = \widehat{\mathrm{Ext}}^i_Q(R_2, R_1)$ for all $i \in \mathbb{Z}$.*

Proof. We verify that $\widehat{\mathrm{Tor}}^Q_i(R_1, R_2) = 0 = \widehat{\mathrm{Ext}}^i_Q(R_1, R_2)$. The proof of the other vanishing is similar.

Recall from Step 3.4 that R_1 is totally reflexive as a Q-module. We construct a complete resolution of R_1 over Q by splicing a minimal Q-free resolution P of R_1 with its dual $P^* = \mathrm{Hom}_Q(P, Q)$. Using the fact that R_1^* is isomorphic to I_1, the first syzygy of R_1 in P, we conclude that $X^* \cong X$. This explains the second isomorphism in the next sequence wherein i is an arbitrary integer:

$$
\begin{aligned}
\widehat{\mathrm{Tor}}_i^Q(R_1, R_2) &\cong \mathrm{H}_i(X \otimes_Q R_2) \cong \mathrm{H}_i(X^* \otimes_Q R_2) \\
&\cong \mathrm{H}_i(\mathrm{Hom}_Q(X, R_2)) \cong \widehat{\mathrm{Ext}}_Q^{-i}(R_1, R_2).
\end{aligned}
\tag{3.14.1}
$$

The third isomorphism is standard, since each Q-module X_i is finitely generated and free, and the other isomorphisms are by definition.

For $i \geqslant 1$, the complex X provides the second steps in the next displays:

$$
\begin{aligned}
\widehat{\mathrm{Ext}}_Q^{-i}(R_1, R_2) &\cong \widehat{\mathrm{Tor}}_i^Q(R_1, R_2) \cong \mathrm{Tor}_i^Q(R_1, R_2) = 0 \\
\widehat{\mathrm{Tor}}_{-i}^Q(R_1, R_2) &\cong \widehat{\mathrm{Ext}}_Q^{i}(R_1, R_2) \cong \mathrm{Ext}_Q^i(R_1, R_2) = 0.
\end{aligned}
$$

The first steps are from (3.14.1), and the third steps are from Lemma 3.12.

To complete the proof it suffices by (3.14.1) to show that $\widehat{\mathrm{Ext}}_Q^{0}(R_1, R_2) = 0$. For this, we recall the exact sequence

$$
0 \to \mathrm{Hom}_Q(R_1, Q) \otimes_Q R_2 \xrightarrow{\nu} \mathrm{Hom}_Q(R_1, R_2) \to \widehat{\mathrm{Ext}}_Q^{0}(R_1, R_2) \to 0
$$

from [4, (5.8(3))]. Note that this uses the fact that R_1 is totally reflexive as a Q-module, with the condition $\widehat{\mathrm{Ext}}_Q^{-1}(R_1, R_2) = 0$ which we have already verified. Also, the map ν is given by the formula $\nu(\psi \otimes r_2) = \psi_{r_2} \colon R_1 \to R_2$ where $\psi_{r_2}(r_1) = \psi(r_1) r_2$. Thus, to complete the proof, we need only show that the map ν is surjective.

As with the isomorphism $\alpha \colon \mathrm{Hom}_Q(R_1, Q) \xrightarrow{\cong} I_1$, it is straightforward to show that the map $\beta \colon \mathrm{Hom}_Q(R_1, R_2) \to C^\dagger$ given by $\phi \mapsto \phi(1)$ is a well-defined Q-module isomorphism. Also, from Lemma 3.12 we have that $I_1 I_2 = 0 \oplus 0 \oplus 0 \oplus D$, considered as a subset of $I_1 = 0 \oplus 0 \oplus C^\dagger \oplus D \subset R \oplus C \oplus C^\dagger \oplus D = Q$. In particular, the map $\sigma \colon I_1/I_1 I_2 \to C^\dagger$ given by $\overline{(0,0,f,d)} \mapsto f$ is a well-defined Q-module isomorphism.

Finally, it is straightforward to show that the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Hom}_Q(R_1, Q) \otimes_Q R_2 \arrow[rrr, "\nu"] \arrow[d, "\alpha \otimes_Q R_2"', "\cong"] & & & \mathrm{Hom}_Q(R_1, R_2) \arrow[d, "\beta"', "\cong"] \\
I_1 \otimes_Q R_2 \arrow[r, "="] & I_1 \otimes_Q Q/I_2 \arrow[r, "\delta", "\cong"'] & I_1/I_1 I_2 \arrow[r, "\sigma", "\cong"'] & C^\dagger.
\end{tikzcd}
$$

From this, it follows that ν is surjective, as desired. $\square$

This completes the proof of Theorem 3.2. $\square$

4. Constructing Rings with Non-trivial Semidualizing Modules

We begin this section with the following application of Theorem 3.1.

Proposition 4.1. *Let R_1 be a local Cohen–Macaulay ring with dualizing module $D_1 \not\cong R_1$ and $\dim(R_1) \geqslant 2$. Let $\mathbf{x} = x_1, \ldots, x_n \in R_1$ be an R_1-regular sequence with $n \geqslant 2$, and fix an integer $t \geqslant 2$. Then the ring $R = R_1/(\mathbf{x})^t$ has a semidualizing module C that is neither dualizing no free.*

Proof. We verify the conditions (1)–(5) from Theorem 3.1.

(1) Set $Q = R_1 \ltimes D_1$ and $I_1 = 0 \oplus D_1 \subset Q$. Consider the elements $y_i = (x_i, 0) \in Q$ for $i = 1, \ldots, n$. It is straightforward to show that the sequence $\mathbf{y} = y_1, \ldots, y_n$ is Q-regular. With $R_2 = Q/(\mathbf{y})^t$, we have $R \cong R_1 \otimes_Q R_2$. That is, with $I_2 = (\mathbf{y})^t$, condition (1) from Theorem 3.1 is satisfied.

(2) The assumption $D_1 \not\cong R_1$ implies that R_1 is not Gorenstein. It is well-known that $\operatorname{type}(R_2) = \binom{t+n-2}{n-1} > 1$, so R_2 is not Gorenstein.

(3) By Fact 2.18, it suffices to show that $\operatorname{pd}_Q(R_2) < \infty$. Since $\mathbf{y}$ is a Q-regular sequence, the associated graded ring $\oplus_{i=0}^{\infty}(\mathbf{y})^i/(\mathbf{y})^{i+1}$ is isomorphic as a Q-algebra to the polynomial ring $Q/(\mathbf{y})[Y_1, \ldots, Y_n]$. It follows that the Q-module $R_2 \cong Q/(\mathbf{y})^t$ has a finite filtration $0 = N_r \subset N_{r-1} \subset \cdots \subset N_0 = R_2$ such that $N_{i-1}/N_i \cong Q/(\mathbf{y})$ for $i = 1, \ldots, r$. Since each quotient $N_{i-1}/N_i \cong Q/(\mathbf{y})$ has finite projective dimension over Q, the same is true for R_2.

(4) The following isomorphisms are straightforward to verify:

$$R_2 = Q/(\mathbf{y})^t \cong [R_1/(\mathbf{x})^t] \ltimes [D_1/(\mathbf{x})^t D_1] \cong R \ltimes [D_1/(\mathbf{x})^t D_1].$$

Since $\mathbf{x}$ is R_1-regular, it is also D_1-regular. Using this, one checks readily that

$$\operatorname{Hom}_Q(R_1, R_2) \cong \{z \in R_2 \mid I_1 z = 0\} = 0 \oplus [D_1/(\mathbf{x})^t D_1].$$

Since D_1 is not cyclic and $\mathbf{x}$ is contained in the maximal ideal of R_1, we conclude that $\operatorname{Hom}_Q(R_1, R_2) \cong D_1/(\mathbf{x})^t D_1$ is not cyclic.

(5) The Q-module R_1 is totally reflexive; see Facts 2.12–2.13. It follows from [6, (2.4.2(b))] that $\operatorname{Tor}_i^Q(R_1, N) = 0$ for all $i \geqslant 1$ and for all Q-modules N of finite flat dimension; see also [2, (4.13)]. Thus, we have $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geqslant 1$. □

Remark 4.2. One can use the results of [3] directly to show that the ring R in Proposition 4.1 has a non-trivial semidualizing module. (Specifically, the relative dualizing module of the natural surjection $R_1 \to R$ works.) However, our proof illustrates the concrete criteria of Theorem 3.1.

We conclude by showing that there exists a Cohen–Macaulay local ring R that does not admit a dualizing module and does admit a semidualizing module C such that $C \not\cong R$. The construction is essentially from [22, p. 92, Example].

Example 4.3. Let A be a local Cohen–Macaulay ring that does not admit a dualizing module. (Such rings are known to exist by a result of Ferrand and Raynaud [9].) Set $R = A[X,Y]/(X,Y)^2 \cong A \ltimes A^2$ and consider the R-module $C = \operatorname{Hom}_A(R, A)$. Since R is finitely generated and free as an A-module, Fact 2.13 shows that C is a semidualizing R-module. The composition of the natural inclusion $A \to R$ and the natural surjection $R \to A$ is the identity on A.

If R admitted a dualizing module D, then the module $\operatorname{Hom}_R(A, D)$ would be a dualizing A-module by Fact 2.7, contradicting our assumption on A. (Alternately, since A is not a homomorphic image of a Gorenstein ring, we conclude from the surjection $R \to A$ that R is not a homomorphic image of a Gorenstein ring.)

We show that $C \not\cong R$. It suffices to show that $\operatorname{Hom}_R(A, C) \not\cong \operatorname{Hom}_R(A, R)$. We compute:

$$\operatorname{Hom}_R(A, C) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_A(R, A)) \cong \operatorname{Hom}_A(R \otimes_R A, A) \cong \operatorname{Hom}_A(A, A) \cong A$$
$$\operatorname{Hom}_R(A, R) \cong \{r \in R \mid (0 \oplus A^2) r = 0\} = 0 \oplus A^2 \cong A^2$$

which gives the desired conclusion.

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