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On the Growth of the Betti Sequence of the Canonical Module

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ON THE GROWTH OF THE BETTI SEQUENCE OF THE CANONICAL MODULE

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Abstract. We study the growth of the Betti sequence of the canonical module of a Cohen–Macaulay local ring. It is an open question whether this sequence grows exponentially whenever the ring is not Gorenstein. We answer the question of exponential growth affirmatively for a large class of rings, and prove that the growth is in general not extremal. As an application of growth, we give criteria for a Cohen–Macaulay ring possessing a canonical module to be Gorenstein.

Introduction

A canonical module ω_R for a Cohen–Macaulay local ring R is a maximal Cohen–Macaulay module having finite injective dimension and such that the natural homomorphism $R \longrightarrow \operatorname{Hom}_R(\omega_R, \omega_R)$ is an isomorphism. If such a module exists, then it is unique up to isomorphism. The ring R is Gorenstein if and only if R itself is a canonical module, that is, if and only if ω_R is free. Although the cohomological behavior of the canonical module, both in algebra and in geometry, is quite well understood, little is known about its homological aspects. In this note we study the growth of the Betti numbers—the ranks of the free modules occurring in a minimal free resolution—of ω_R over R. Specifically, we seek to answer the following question, a version of which we first heard from C. Huneke.

Question. *If R is not Gorenstein, must the Betti numbers of the canonical module grow exponentially?*

By exponential growth of a sequence $\{b_i\}$ we mean that there exist real numbers $1 < \alpha < \beta$ such that $\alpha^i < b_i < \beta^i$ for all $i \gg 0$. Our main result of Section 1 answers this question affirmatively for a large class of local rings.

It is well-known that the growth of the Betti sequence of the residue field k of a local ring R characterizes its regularity: *R is regular if and only if the Betti sequence of k is finite.* This is the foundational Auslander–Buchsbaum–Serre Theorem. Gulliksen ([10], [11]) extends this theorem with a characterization of local complete intersections: *R is a complete intersection if and only if the Betti sequence of k grows polynomially.* By polynomial growth of a sequence $\{b_i\}$ we mean that there is an integer d and a positive constant c such that $b_i \le ci^d$ for all $i \gg 0$. One motivation for the question above is whether there are analogous statements regarding the growth of the Betti sequence of the canonical module ω_R of a local Cohen–Macaulay ring. The Auslander-Buchsbaum formula implies that R is Gorenstein if and only if ω_R has a finite Betti sequence. However, we do not know

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whether there exists a class of Cohen–Macaulay rings for which the Betti sequence of the canonical module grows polynomially. In other words, we do not know if there exists a class of Cohen–Macaulay rings which are near to being Gorenstein in the same sense that complete intersections are near to being regular.

Since the canonical module is maximal Cohen–Macaulay over a Cohen–Macaulay ring R, we may, and often do, reduce both R and ω_R modulo a maximal regular sequence and assume that R has dimension zero. Then ω_R is isomorphic to the injective hull of the residue field. In particular, the Betti numbers of ω_R are equal to the Bass numbers of R, that is, the multiplicities of ω_R in each term of the minimal injective resolution of R. In this case we may rephrase the question above as follows:

Question′. *If the minimal injective resolution of an Artinian local ring R as a module over itself grows sub-exponentially, is R necessarily self-injective?*

By abuse of language, throughout this note we will simply say that a finitely generated module *has exponential growth* (or *polynomial growth*) to mean that its sequence of Betti numbers has exponential growth (or polynomial growth).

We now briefly describe the contents below. In the first section, we identify a broad class of rings for which the canonical module grows exponentially. In some cases, exponential growth follows from more general results about the growth of all free resolutions over the rings considered. In fact, in these cases we can be more precise: the Betti numbers of the canonical module are eventually strictly increasing. This condition is of particular interest, and we return to it in Section 2. We also consider modules having *linear* resolutions with exponential growth, and give a comparison result (Lemma 1.4) for their Betti numbers. As an application, we prove exponential growth of the canonical module for rings defined by certain monomial ideals.

In section 2 we demonstrate an upper bound for the growth of Betti numbers in the presence of certain vanishing Exts or Tors (Lemma 2.1). This allows us to give criteria for a Cohen-Macaulay ring to be Gorenstein, which are in the spirit of the work by Ulrich [22] and Hanes–Huneke [12].

In the final section, we give a family of examples showing that the canonical module need not have *extremal* growth among all R-modules. Based on this, we introduce a notion for a Cohen–Macaulay ring to be 'close to Gorenstein' and compare our notion with other ones in the literature.

Throughout this note, we consider only Noetherian rings, which we usually assume to be Cohen–Macaulay (CM) with a canonical module ω_R, and we consider only finitely generated modules. When only one ring is in play, we often drop the subscript and write ω for its canonical module. Our standard reference for facts about canonical modules is Chapter Three of [9]. We denote the length of a module M by $\lambda(M)$, its minimal number of generators by $\mu(M)$, and its i^{th} Betti number by $b_i(M)$. When M is a maximal Cohen–Macaulay (MCM) module, we write $M^\vee$ for the canonical dual $\operatorname{Hom}_R(M, \omega)$.

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1. Exponential growth

We first prove that there are several situations in which extant literature applies to show that the canonical module grows exponentially. This is due to the fact that in these situations 'most' modules of infinite projective dimension have exponential growth. In fact, in each case, if the Betti sequence grows exponentially, then it is also eventually strictly increasing. (The usefulness of this condition on the Betti sequence will become clear in the next section.) We list these cases, along with references.

(1) R is a Golod ring [19], cf. [20];
(2) R has codimension ≤ 3 [4], [21];
(3) R is one link from a complete intersection [4], [21];
(4) R is radical cube zero [18].

We combine the consequences of assumptions (1)–(4) on the canonical module in the following.

Proposition 1.1. *Let R be a CM ring possessing a canonical module ω and satisfying one of the conditions (1)–(4). If R is not Gorenstein, then the canonical module grows exponentially. Moreover, if this is the case then the Betti sequence $\{b_i(\omega)\}$ is eventually strictly increasing.*

A common way for an R-module M to have polynomial growth is for it to have *finite complete intersection dimension.* Before going through the proof of Proposition 1.1, we observe that this is impossible for the canonical module, as pointed out to us by S. Sather-Wagstaff. We first recall the definition of complete intersection dimension from [7]: a surjection $Q \longrightarrow R$ of local rings is called a *(codimension c) deformation* of R if its kernel is generated by a regular sequence (of length c) contained in the maximal ideal of Q. A diagram of local ring homomorphisms $R \longrightarrow R' \longleftarrow Q$ is said to be a *(codimension c) quasi-deformation of* R if $R \longrightarrow R'$ is flat and $R' \longleftarrow Q$ is a (codimension c) deformation. Finally, an R-module M has *finite complete intersection dimension* if there exists a quasi-deformation $R \longrightarrow R' \longleftarrow Q$ of R such that $M \otimes_R R'$ has finite projective dimension over Q. If this is the case then M necessarily has polynomial growth over R [7, Theorem 5.6].

By [7, Theorem. 1.4], modules of finite complete intersection dimension necessarily have *finite G-dimension.* Recall that an R-module M has *G-dimension zero* if M is reflexive and $\operatorname{Ext}^i_R(M,R) = \operatorname{Ext}^i_R(M^*,R) = 0$ for all positive i, where $(\quad)^*$ denotes the ring dual $\operatorname{Hom}_R(\quad, R)$. The G-dimension of an arbitrary module M is then the minimal length of a resolution of M by modules of G-dimension zero.

Proposition 1.2. *The canonical module of a CM local ring R has finite complete intersection dimension if and only if it has finite G-dimension if and only if R is Gorenstein.*

Proof. It suffices to prove that the G-dimension of ω being finite implies that R is Gorenstein. For this we may assume that $\dim R = 0$. The Auslander-Bridger formula [1] then implies that ω has G-dimension zero. In particular, ω is reflexive and $\operatorname{Ext}^i_R(\omega^*, R) = 0$ for $i > 0$, so dualizing a free resolution of ω^* exhibits ω as a submodule of a free module. Since ω is injective, this embedding splits, and ω is free, that is, R is Gorenstein. □

A stronger notion than finite complete intersection dimension is that of finite *virtual projective dimension* [2]. It suffices for our needs simply to note that a module having finite virtual projective dimension necessarily has finite complete intersection dimension. Now we are ready to prove Proposition 1.1.

Proof of Proposition 1.1. (1). Assume that R is a Golod ring. Then as shown in [19], if R is not a complete intersection then the Betti sequence of every module of infinite projective dimension grows exponentially, and moreover is eventually strictly increasing. Since complete intersections are Gorenstein, we have the desired conclusion.

(2) and (3). It is shown in [4] and [21] that a finitely generated module over a ring satisfying (2) or (3) either has finite virtual projective dimension or grows exponentially and the Betti sequence is eventually strictly increasing. By Proposition 1.2 above, if R is not Gorenstein then the canonical module does not have finite virtual projective dimension.

(4). We deduce the following statement from a theorem of Lescot [18]: *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$. Set $e = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and $s = \dim_k(\mathfrak{m}^2)$. Then a finite non-free R-module M has exponential growth, with strictly increasing Betti sequence, unless* $\operatorname{soc}(R) = \mathfrak{m}^2$*, $s = e - 1 \geq 2$, and, assuming $\mathfrak{m}^2 M = 0$, one has $eb_0(M) = \lambda(M)$. In this case the sequence $\{b_i(M)\}$ is stationary.*

We must show that the canonical module ω does not fall into the special case allowed by Lescot's theorem. Assume that $\operatorname{soc}(R) = \mathfrak{m}^2$ and $s = e - 1 \geq 2$, so that $\mu(\omega) = e - 1$. Let X be the first syzygy of ω in a minimal R-free resolution, so in particular $\mathfrak{m}^2 X = 0$, and assume that $eb_0(X) = \lambda(X)$. Then from the short exact sequence $0 \longrightarrow X \longrightarrow R^{e-1} \longrightarrow \omega \longrightarrow 0$, we have that $\lambda(X) = 2e(e-1) - 2e = 2e^2 - 4e$. Putting the two equations together we get $b_0(X) = 2e - 4$. On the other hand, the short exact sequence above induces an exact sequence $0 \longrightarrow X \longrightarrow \mathfrak{m}R^{e-1} \longrightarrow \mathfrak{m}\omega \longrightarrow 0$, and tensoring this with k we obtain an exact sequence $X/\mathfrak{m}X \longrightarrow \mathfrak{m}R^{e-1}/\mathfrak{m}^2 R^{e-1} \longrightarrow \mathfrak{m}\omega/\mathfrak{m}^2\omega \longrightarrow 0$. From this we see that $b_0(X) \geq e(e-1) - e = e^2 - 2e$. Thus $2e - 4 \geq e^2 - 2e$, and this implies $e = 2$, a contradiction. □

Remark 1.3. The class of rings to which Proposition 1.1 applies is less limited than it first appears, thanks to two elementary yet crucial observations.

(1) Let $R \longrightarrow S$ be a flat local map of local Cohen–Macaulay rings such that the closed fibre $S/\mathfrak{m}S$ is Gorenstein. Then $\omega_R \otimes_R S$ is isomorphic to the canonical module ω_S of S, and $b_i(\omega_S) = b_i(\omega_R)$ for all i. In fact, relaxing the flatness condition still gives a useful implication: by [6], if $\varphi : Q \longrightarrow R$ is a local ring homomorphism of finite flat dimension, then we have $b_i(\omega_Q) \leq b_i(\omega_R)$ for all $i \gg 0$. Thus ω_R grows exponentially if ω_Q does.

Let us say that a class of rings is *closed under flat extensions* if whenever $R \longrightarrow S$ is a flat map of local rings then R is in the class if and only if S is in the class. Let us say that a class of rings is *closed under homomorphisms of finite flat dimension* if whenever $Q \longrightarrow R$ is a local ring homomorphism of finite flat dimension, and Q is in the class, then R is also in the class.

(2) If x is a nonzerodivisor in R, then $\omega_{R/(x)} \cong \omega_R/x\omega_R$, and $b_i(\omega_{R/(x)}) = b_i(\omega_R)$ for all i.

Let us say that a class of rings is *closed under deformations* if whenever x is a nonzerodivisor in R, the class contains R if and only if it contains R/xR.

We next identify a class of monomial algebras whose canonical modules grow exponentially. Our main technical tool is a local analogue of [16, 2.7].

We say that a finitely generated module M over a local ring $(R, \mathfrak{m})$ has a *linear resolution* if there exists a minimal R-free resolution $F_\bullet$ of M such that for all i the induced maps $F_i/\mathfrak{m}F_i \longrightarrow \mathfrak{m}F_{i-1}/\mathfrak{m}^2F_{i-1}$ are injective.

Lemma 1.4. *Let $\pi : Q \longrightarrow R$ be a surjection of local rings $(Q, \mathfrak{n}, k)$ and $(R, \mathfrak{m}, k)$ such that $\ker \pi \subseteq \mathfrak{n}^2$. Let M be a Q-module and N an R-module. Suppose that M has a linear resolution over Q and that $\varphi : M \longrightarrow N$ is a homomorphism of Q-modules such that the induced map $\overline{\varphi} : M/\mathfrak{n}M \longrightarrow N/\mathfrak{m}N$ is injective. Then the induced maps $\operatorname{Tor}_i^\pi(\varphi, k) : \operatorname{Tor}_i^Q(M, k) \longrightarrow \operatorname{Tor}_i^R(N, k)$ are injective for each i.*

Proof. By assumption we have a short exact sequence $0 \longrightarrow K \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ with F_0 a free Q-module and the induced map $K/\mathfrak{n}K \longrightarrow \mathfrak{n}F_0/\mathfrak{n}^2F_0$ injective. Let $f_1, \dots, f_n$ be a basis for F_0. From the injection $M/\mathfrak{n}M \longrightarrow N/\mathfrak{m}N$ we choose a free R-module G_0, and a basis $g_1, \dots, g_m$ of G_0, with $m \geq n$, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r] & G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

commutes, where φ_0 is the map defined by regarding G_0 as a Q-module and extending linearly the assignments $\varphi_0(f_i) = g_i$, $i = 1, \dots, n$. Then by construction we have $\overline{\varphi_0} : F_0/\mathfrak{n}F_0 \longrightarrow G_0/\mathfrak{m}G_0$ injective.

If we can show that the induced map $K/\mathfrak{n}K \longrightarrow L/\mathfrak{m}L$ is injective then we may continue inductively, defining maps $\varphi_i : F_i \longrightarrow G_i$ from a linear minimal Q-free resolution of M to a minimal R-free resolution of N such that the induced maps $\overline{\varphi}_i : F_i/\mathfrak{n}F_i \longrightarrow G_i/\mathfrak{m}G_i$ are injective for all i, and hence prove our claim.

Let x be in K and assume that $\varphi_0(x) \in \mathfrak{m}L \subseteq \mathfrak{m}^2G_0$. Writing $x = a_1f_1 + \cdots + a_nf_n$ for some $a_i \in R$, we have $\varphi_0(x) = \pi(a_1)g_1 + \cdots + \pi_n(a_n)g_n$. Hence $\pi(a_i) \in \mathfrak{m}^2$ for each i. It follows from $\ker \pi \subseteq \mathfrak{n}^2$ that $a_i \in \mathfrak{n}^2$ for each i. Thus $x \in \mathfrak{n}^2F_0$. Now the injection $K/\mathfrak{n}K \longrightarrow \mathfrak{n}F_0/\mathfrak{n}^2F_0$ shows that $x \in \mathfrak{n}K$, as desired. □

Theorem 1.5. *Let $\pi : (S, \mathfrak{n}) \longrightarrow (R, \mathfrak{m})$ be a surjection of local rings with R CM and possessing a canonical module, and suppose that $\ker \pi \subseteq \mathfrak{n}^2$. Assume that for some minimal generator x of ω_R, $\operatorname{ann}_S x$ contains an ideal I such that S/I has a linear resolution and exponential growth. Then ω_R grows exponentially.*

Proof. Apply Lemma 1.4 to the map $\varphi : S/I \longrightarrow \omega_R$ defined by $\varphi(\overline{1}) = x$. □

Our application of Theorem 1.5 is stated in our usual local context, though a graded analogue is easily obtained from it.

Corollary 1.6. *Let $(Q, \mathfrak{n})$ be a regular local ring containing a field and $x_1, \dots, x_d$ a regular system of parameters for $\mathfrak{n}$. Let $I \subseteq Q$ be an $\mathfrak{n}$-primary ideal generated by monomials in the x_i. If I contains x_ix_j and x_ix_l for $j \neq l$, then the canonical module of $R = Q/I$ grows exponentially.*

Proof. We may complete both Q and R, and assume that Q is a power series ring in the variables $x_1, \dots, x_d$ over a field k. As I is a monomial ideal, any socle element α of R has a unique representation $\alpha = x_{i_1} \dots x_{i_l}$ as a monomial in Q. Viewing α as a vector-space basis vector for R over k, the dual element $\alpha^* \in \operatorname{Hom}_k(R, k) = \omega_R$

is a minimal generator of ω_R. Since $x_ix_j, x_ix_l \in I$, either x_i annihilates α^* or both x_j and x_l do. We apply the theorem with $S = Q/(x_ix_j, x_ix_l)Q$, over which S/x_iS, S/x_jS, and S/x_lS all have linear minimal resolutions whose Betti numbers have exponential growth. □

Remark 1.7. Like Proposition 1.1, the usefulness of Corollary 1.6 is greatly enhanced by Remark 1.3. In particular, exponential growth of the canonical module holds for any ring R for which there exists a sequence of local rings $R = R_0, S_1, R_1, \ldots, S_n, R_n$ such that R_n is as in the statement of Corollary 1.6, and for each $i = 1, \ldots, n$ both R_i and R_{i-1} are quotients of S_i by S_i-regular sequences. We give one application of this observation below.

Example 1.8. Let k be a field and define a pair of Artinian local rings $R = k[a,b]/(a^4, a^3b, b^2)$, $R' = k[b,c]/(b^2, bc, c^2)$. Set further $S = k[\![t^3, t^5, t^7]\!]$, a one-dimensional complete domain. Then S has a presentation

$$S \cong k[\![a,b,c]\!]/(ac - b^2, bc - a^4, c^2 - a^3b)\,,$$

so that $R \cong S/(t^7)$, $R' \cong S/(t^3)$. Corollary 1.6 applies to R', so it follows that the canonical modules of both S and R have exponential growth as well.

To summarize the results thus far, we introduce two classes of CM rings.

Definition 1.9. Let $\mathfrak{C}$ be the smallest class of CM rings with canonical module which contains those satisfying one of (1)–(4) in Proposition 1.1, and which is closed under deformations and flat extensions with Gorenstein closed fibre.

Let $\widetilde{\mathfrak{C}}$ be the smallest class of CM rings with canonical module containing $\mathfrak{C}$ and rings satisfying the hypothesis of Corollary 1.6, and which is closed under deformations and homomorphisms of finite flat dimension.

Theorem 1.10. *For each $R \in \mathfrak{C}$, either R is Gorenstein or the Betti sequence of the canonical module grows exponentially and is eventually strictly increasing. For each $R \in \widetilde{\mathfrak{C}}$, either R is Gorenstein or the canonical module grows exponentially.*

2. Bounds on Betti numbers; criteria for the Gorenstein property

This section supplies a variation on a theme of Ulrich [22] and Hanes–Huneke [12] which gives conditions for a ring to be Gorenstein in terms of certain vanishing Exts involving modules with many generators relative to their multiplicity. The advantage of our results relative to those of Ulrich and Huneke–Hanes is that we need not assume the modules involved have positive rank, and this greatly enhances the applicability of the results. The downside is that we sometimes need to assume more Exts or Tors vanish.

We first need a means of bounding Betti numbers. The following is a strengthening of [13, 1.4(1)]. Note that it generalizes the well-known fact that if N is a MCM R-module and $\mathrm{Tor}_i^R(k, N) = 0$ for some $i > 0$ then N is free.

Lemma 2.1. *Let R be a CM local ring, M a CM R-module of dimension d, and N a MCM R-module. Let n be an integer and assume that either*

(1) $\mathrm{Tor}_i^R(M, N) = 0$ *for all i with* $1 \le n - d \le i \le n$, *or*
(2) $\mathrm{Ext}_R^i(M, N^\vee) = 0$ *for all i with* $1 \le n \le i \le n + d$.

Then for any sequence $\boldsymbol{x} = x_1, \ldots, x_d$ *regular on both* M *and* R,

$$b_n(N) \leq \frac{\lambda(\mathfrak{m}M/\boldsymbol{x}M)}{\mu(M)}\, b_{n-1}(N)\,.$$

Moreover, equality holds if and only if both $\mathfrak{m}(M/\boldsymbol{x}M \otimes_R N) = 0$ *and* $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) = 0$.

Proof. We first prove case (1). Replacing N by a syzygy if necessary, we may assume that $n = d+1$, and we proceed by induction on d. When $d = 0$ our hypotheses are therefore that M has finite length and $\operatorname{Tor}_1^R(M,N) = 0$. Applying $-\otimes_R N$ to the short exact sequence $0 \longrightarrow \mathfrak{m}M \longrightarrow M \xrightarrow{\pi} M/\mathfrak{m}M \longrightarrow 0$, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M/\mathfrak{m}M, N) \longrightarrow \mathfrak{m}M \otimes_R N \longrightarrow M \otimes_R N \xrightarrow{\pi\otimes N} M/\mathfrak{m}M \otimes_R N \longrightarrow 0.$$

Since $M/\mathfrak{m}M$ is isomorphic to a sum of $\mu(M)$ copies of the residue field of R, the monomorphism on the left gives $\mu(M)b_1(N) \leq \lambda(\mathfrak{m}M \otimes_R N)$. Equality holds if and only if $\pi \otimes_R N$ is an isomorphism, and this is equivalent to $M \otimes_R N$ being a vector space over k, in other words, $\mathfrak{m}(M \otimes_R N) = 0$.

Next take a short exact sequence $0 \longrightarrow N_1 \longrightarrow R^{b_0(N)} \xrightarrow{\epsilon} N \longrightarrow 0$. Applying $\mathfrak{m}M \otimes_R -$ gives the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(\mathfrak{m}M, N) \longrightarrow \mathfrak{m}M \otimes_R N_1 \longrightarrow (\mathfrak{m}M)^{b_0(N)} \xrightarrow{\mathfrak{m}M\otimes\epsilon} \mathfrak{m}M \otimes N \longrightarrow 0.$$

The surjection $\mathfrak{m}M \otimes \epsilon$ yields $\lambda(\mathfrak{m}M \otimes_R N) \leq \lambda(\mathfrak{m}M)b_0(N)$. Equality holds if and only if $\ker(\mathfrak{m}M \otimes \epsilon) = 0$

Combining these two inequalities yields $\mu(M)b_1(N) \leq \lambda(\mathfrak{m}M)b_0(N)$, so that

$$b_1(N) \leq \frac{\lambda(\mathfrak{m}M)}{\mu(M)}\, b_0(N)\,,$$

and equality holds if and only if both $\mathfrak{m}(M \otimes_R N) = 0$ and $\ker(\mathfrak{m}M \otimes \epsilon) = 0$. This latter condition is equivalent to both $\mathfrak{m}(M \otimes_R N) = 0$ and $\mathfrak{m}^2 M = 0$.

Now suppose that $d > 0$, and let bars denote images modulo x_d, with $\overline{\boldsymbol{x}} = \overline{x_1}, \ldots, \overline{x_{d-1}}$. The long exact sequence of Tor arising from the short exact sequence $0 \longrightarrow M \xrightarrow{x_d} M \longrightarrow \overline{M} \longrightarrow 0$ yields $\operatorname{Tor}_i^R(\overline{M}, N) = 0$ for $2 \leq i \leq d+1$, and a standard isomorphism gives $\operatorname{Tor}_i^{\overline{R}}(\overline{M}, \overline{N}) = 0$ for $2 \leq i \leq d+1$. By induction, since $\overline{M}$ is a CM R-module of dimension $d-1$, we have

$$b_{d+1}^{\overline{R}}(\overline{N}) \leq \frac{\lambda(\overline{\mathfrak{m}}\overline{M}/\overline{\boldsymbol{x}}\overline{M})}{\mu(\overline{M})} b_d^{\overline{R}}(\overline{N})\,.$$

Since $b_n^R(N) = b_n^{\overline{R}}(\overline{N})$ for all n, $\mu(M) = \mu(\overline{M})$, and $\mathfrak{m}M/\boldsymbol{x}M \cong \overline{\mathfrak{m}}\overline{M}/\overline{\boldsymbol{x}}\overline{M}$ we get the same inequality without the bars. Finally, by induction we achieve equality if and only if both $\overline{\mathfrak{m}}(\overline{M}/\overline{\boldsymbol{x}}\overline{M} \otimes_{\overline{R}} \overline{N}) = 0$ and $\overline{\mathfrak{m}}(\overline{\mathfrak{m}}\overline{M}/\overline{\boldsymbol{x}}\overline{M}) = 0$, and this is equivalent to both $\mathfrak{m}(M/\boldsymbol{x}M \otimes_R N) = 0$ and $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) = 0$.

For case (2), when $d = 0$ Matlis duality yields $\operatorname{Tor}_1^R(M,N) = 0$, and we get the inequality by case (1). For $d > 0$, we reduce modulo the nonzerodivisor x_d. Using the fact that $\operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{\omega}) \cong \operatorname{Hom}_R(N, \omega) \otimes \overline{R}$, and the long exact sequence of Ext derived from the short exact sequence $0 \longrightarrow M \xrightarrow{x_d} M \longrightarrow \overline{M} \longrightarrow 0$, we see that the hypothesis passes to $\overline{R}$, and the inequality follows by induction, with the same condition for equality. $\square$

Using Lemma 2.1 we obtain our criteria for the Gorenstein property analogous to those of Ulrich and Hanes–Huneke.

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a CM local ring with canonical module ω, and M be a CM R-module of dimension d such that for some sequence $\boldsymbol{x}$ of length d regular on both M and R,*

1. *$\lambda(\mathfrak{m}M/\boldsymbol{x}M) < \mu(M)$, and*
2. *$\operatorname{Ext}^i_R(M, R) = 0$ for $1 \le i \le d + \mu(\omega)$,*

then R is Gorenstein. The same statement, except allowing equality in (1), holds if either $\mathfrak{m}((M/\boldsymbol{x}M) \otimes_R \omega) \neq 0$ or $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) \neq 0$.

Proof. Lemma 2.1 and the hypotheses imply that $b_n(\omega) < b_{n-1}(\omega)$ for $1 \le n \le \mu(\omega)$. This forces $b_{\mu(\omega)}(\omega) = 0$, so that ω has finite projective dimension. By the Auslander-Buchsbaum formula, ω is free and R is Gorenstein.

The last statement follows immediately from the last statement of Lemma 2.1. □

Though M need not have constant rank in Theorem 2.2, the result can be improved dramatically by assuming that the canonical module has constant rank. Recall that this is equivalent to requiring that R be generically Gorenstein, that is, that all localizations of R at minimal primes are Gorenstein. In this case the rank of ω is 1.

Proposition 2.3. *Let R be a generically Gorenstein CM local ring with canonical module ω. If R is not Gorenstein, then $b_1(\omega) \ge b_0(\omega)$.*

Proof. Let X be the first syzygy of ω in a minimal R-free resolution. Since ω has rank one, $\operatorname{rank} X = \mu(\omega) - 1$. If $\mu(X) = b_1(\omega) \le b_0(\omega) - 1 = \operatorname{rank} X$, then X is free, so that ω has finite projective dimension. By the Auslander-Buchsbaum formula, then, ω is free and R is Gorenstein. □

Theorem 2.4. *Let R be a generically Gorenstein CM local ring with canonical module ω, and M be a CM R-module of dimension d such that for some sequence $\boldsymbol{x}$ of length d regular on both M and R,*

1. *$\lambda(\mathfrak{m}M/\boldsymbol{x}M) < \mu(M)$, and*
2. *$\operatorname{Ext}^i_R(M, R) = 0$ for $1 \le i \le d + 1$,*

then R is Gorenstein. The same statement, except allowing equality in (1), holds if either $\mathfrak{m}((M/\boldsymbol{x}M) \otimes_R \omega) \neq 0$ or $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) \neq 0$.

Proof. By Lemma 2.1, and the hypotheses (1) and (2) we obtain $b_1(\omega) < b_0(\omega)$. By Proposition 2.3, R must be Gorenstein. □

Lemma 2.1 also places restrictions on the module theory of the rings in the class $\mathfrak{C}$ of Definition 1.9.

Theorem 2.5. *Suppose that $R \in \mathfrak{C}$, and that M is a CM R-module of dimension d such that for some sequence $\boldsymbol{x}$ of length d regular on both M and R,*

1. *$\lambda(\mathfrak{m}M/\boldsymbol{x}M) \le \mu(M)$ and*
2. *$\operatorname{Ext}^i_R(M, R) = 0$ for all $i \gg 0$,*

then R is Gorenstein.

Proof. By Lemma 2.1 and the hypotheses (1) and (2) we obtain $b_{i+1}(\omega) \le b_i(\omega)$ for all $i \gg 0$. By Theorem 1.10, R must be Gorenstein. □

We could improve our Theorem 2.2 to $\lambda(\mathfrak{m}M/\boldsymbol{x}M) \leq \mu(M)$ in hypothesis (1) and to only assuming $\operatorname{Ext}_R^i(M,R) = 0$ for $1 \leq i \leq d+1$ in (2) if we knew that $b_1(\omega) > b_0(\omega)$ held whenever R is not Gorenstein. This prompts a very specialized version of our main question:

Question 2.6. *Does $b_1(\omega) \leq b_0(\omega)$ imply that R is Gorenstein?*

An affirmative answer in one case follows from the Hilbert–Burch Theorem.

Proposition 2.7. *Let R be a CM local ring of codimension two which is not Gorenstein. Then $b_1(\omega) > b_0(\omega)$.*

Proof. We may assume that R is complete. Thus $R = Q/I$ where Q is a complete regular local ring and I is an ideal of height two. By the Hilbert–Burch theorem a minimal resolution of R over Q has the form

$$0 \longrightarrow Q^n \xrightarrow{\varphi} Q^{n+1} \longrightarrow Q \longrightarrow R \longrightarrow 0,$$

where the ideal I is generated by the $n \times n$ minors of a matrix φ representing the map $Q^n \longrightarrow Q^{n+1}$ with respect to fixed bases of Q^n and Q^{n+1}. The canonical module $\omega_R \cong \operatorname{Ext}_Q^2(R,Q)$ is presented by the transpose φ^T of the matrix φ. We claim that φ^T gives in fact a minimal presentation of ω_R. Since R is not Gorenstein we see that $n > 1$, and in this case no row or column of φ has entries contained in I. Therefore φ^T is a minimal presentation matrix, and has n rows and $n+1$ columns. That is, $b_1(\omega) = n+1 > n = b_0(\omega)$. □

Next we give some examples which indicate the sharpness of the results of this section.

Example 2.8. Let k be a field and $R = k[x,y,z]/(x^2,xy,y^2,z^2)$. Then R is a codimension-three Artinian local ring and is not Gorenstein. One may check that the canonical module ω of R has Betti numbers $b_0(\omega) = 2$ and $b_i(\omega) = 3 \cdot 2^{i-1}$ for all $i \geq 1$. Set $M = (z)$. Then $\mu(M) = 1$, $\lambda(\mathfrak{m}M) = 2$, and it is not hard to show that $\operatorname{Ext}_R^i(M,R) = \operatorname{Tor}_i^R(M,\omega) = 0$ for all $i > 0$.

This is an example in which equality in Lemma 2.1 is achieved. The example also shows that neither the strict inequality in Theorem 2.2, nor the inequality in Theorem 2.5 can be improved to $\lambda(\mathfrak{m}M/\boldsymbol{x}M) \leq \mu(M) + 1$.

The next two examples show that the two conditions for equality in Lemma 2.1 are independent of one another.

Example 2.9. Let $R = k[x,y]/(x^3,y^3)$, $M = R/(x)$ and $N = R/(y)$. Then R is Artinian and $\operatorname{Tor}_i^R(M,N) = 0$ for all $i > 0$. We have $\mathfrak{m}(M \otimes_R N) = 0$ yet $\mathfrak{m}^2 M \neq 0$ and $\mathfrak{m}^2 N \neq 0$.

Example 2.10. From [13, Example 2.10], $R = k[x_1,x_2,x_3,x_4]/I$ where I is a specific ideal generated by seven homogeneous quadratics, and R has Hilbert series $1+4t+3t^2$. Then for M defined as the cokernel of the 2×2 matrix with rows (x_3,x_1) and (x_4,x_2), $\operatorname{Tor}_i^R(M,\omega) = 0$ for all $i > 0$. One has $\mathfrak{m}^2 M = 0$ yet $\mathfrak{m}(M \otimes_R \omega) \neq 0$.

Hanes and Huneke prove a criterion for the Gorenstein property which is like our Theorem 2.4, except that they assume M has positive rank and then allow $\lambda(\mathfrak{m}M/\boldsymbol{x}M) \leq \mu(M)$ in hypothesis (1). The next example shows that one cannot in general improve their theorem to assume only that $\lambda(\mathfrak{m}M/\boldsymbol{x}M) \leq \mu(M) + 2$. We do not know if the assumption can be weakened to $\lambda(\mathfrak{m}M/\boldsymbol{x}M) \leq \mu(M) + 1$.

Example 2.11. Let R be the quotient of the polynomial ring in nine variables $k[x_{ij}, x, y, z]$, by the ideal I generated by the 2×2 minors of the 3×2 generic matrix (x_{ij}), and by $xz - y^2$. Then R is a CM domain of dimension six whose canonical module has Betti numbers $b_0(\omega) = 2$, $b_i(\omega) = 3 \cdot 2^{i-1}$ for all $i \geq 1$. Set $M = (x, y)$. Then one can check that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$. The minimum $\lambda(\mathfrak{m}M/\boldsymbol{x}M)$ after reduction by a system of parameters $\boldsymbol{x}$ is 4. Thus $\lambda(\mathfrak{m}M/\boldsymbol{x}M) = \mu(M) + 2$.

We end this section with an application of Theorem 2.2 to a commutative version of a conjecture of Tachikawa (cf. [13], [5]) as follows.

Corollary 2.12. *Let R be an Artinian local ring, and suppose that $2 \dim \operatorname{soc}(R) > \lambda(R)$. If $\operatorname{Ext}^i_R(\omega, R) = 0$ for $1 \leq i \leq d + \mu(\omega)$, then R is Gorenstein.*

Proof. Our hypothesis is equivalent to $2\mu(\omega) > \lambda(R)$, and reformulating gives the inequality $\lambda(\mathfrak{m}\omega) < \mu(\omega)$. Now apply Theorem 2.2. □

3. Non-extremality

Encouraged by the positive results of the first section, one might go so far as to expect that the canonical module is *extremal*, that is, that the minimal free resolution of ω has maximal growth among R-modules. To make this notion precise, recall that the *curvature* $\operatorname{curv}_R(M)$ of a finitely generated R-module M is the exponential rate of growth of the Betti sequence $\{b_i(M)\}$, defined as the reciprocal of the radius of convergence of the Poincaré series $P^R_M(t)$:

$$\operatorname{curv}_R(M) = \limsup_{n \longrightarrow \infty} \sqrt[n]{b_n(M)}\,.$$

It is known [3, Prop. 4.2.4] that the residue field k has extremal growth, so that $\operatorname{curv}_R(M) \leq \operatorname{curv}_R(k)$ for all M. One might thus ask: For a CM local ring $(R, \mathfrak{m}, k)$ with canonical module $\omega \not\cong R$, is $\operatorname{curv}_R(\omega) = \operatorname{curv}_R(k)$?

Here we show by example that this question is overly optimistic. We obtain Artinian local rings $(R, \mathfrak{m}, k)$ so that $\operatorname{curv}_R(\omega) < \operatorname{curv}_R(k)$, and even so that the quotient $\operatorname{curv}_R(\omega)/\operatorname{curv}_R(k)$ can be made as small as desired. The examples are obtained as *local tensors* of Artinian local rings R_1 and R_2. Recall the definition from [15].

Definition 3.1. Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be local rings essentially of finite type over the same field k, with k also being the common residue field of R_1 and R_2. The *local tensor* R of R_1 and R_2 is the localization of $R_1 \otimes_k R_2$ at the maximal ideal $\mathfrak{m} := \mathfrak{m}_1 \otimes_k R_2 + R_1 \otimes_k \mathfrak{m}_2$.

We need three basic facts about local tensors, which are collected below. See [15] for proofs.

Proposition 3.2. *Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be as in Definition 3.1, and let $(R, \mathfrak{m})$ be the local tensor.*

(1) *If R_1 and R_2 are Cohen–Macaulay with canonical modules ω_1, ω_2, respectively, then R is Cohen–Macaulay with canonical module $\omega := (\omega_1 \otimes_k \omega_2)_\mathfrak{m}$.*
(2) *For modules M_1 and M_2 over R_1 and R_2, put $M = (M_1 \otimes_k M_2)_\mathfrak{m}$. Then we have an equality of Poincaré series*

$$P^R_M(t) = P^{R_1}_{M_1}(t) P^{R_2}_{M_2}(t)\,.$$

(3) *For $M = (M_1 \otimes_k M_2)_{\mathfrak{m}}$ as above, we have*

$$\operatorname{curv}_R(M) = \max\{\operatorname{curv}_{R_1}(M_1), \operatorname{curv}_{R_2}(M_2)\}\,.$$

The ingredients of our examples are as follows. We take a pair of Artinian local rings A, B with B Gorenstein and $\operatorname{curv}_B(k)$ large, and with A non-Gorenstein and both $\operatorname{curv}_A(k)$ and $\operatorname{curv}_A(\omega_A)$ small.

Example 3.3. Let k be a field and set $A = k[a,b]/(a^2, ab, b^2)$. Then the curvature of every nonfree A-module is equal to 2. Indeed, any syzygy in a minimal A-free resolution is killed by the maximal ideal of A, so it suffices to observe that $\operatorname{curv}_A(k) = 2$.

Next fix $e \geq 3$ and put

$$B = k[x_1, \dots, x_e]/(x_i^2 - x_{i+1}^2, x_j x_l \mid i = 1, \dots, e-1; j \neq l)\,.$$

We claim that B is a Gorenstein ring with $\operatorname{curv}_B(k) = \frac{2}{e-\sqrt{e^2-4}}$. That B has one-dimensional socle is not hard to see, *cf.* [9, 3.2.11]. By Result 5 of [17] we see that B is Koszul, with Hilbert series $H_R(t) = 1 + et + t^2$. The Poincaré series of k over B is thus

$$P_k^B(t) = \frac{1}{1 - et + t^2}\,,$$

for which one computes the radius of convergence $\frac{1}{2}(e - \sqrt{e^2-4})$.

Let now $(R, \mathfrak{m})$ be the local tensor of A and B. Then the canonical module of R is

$$\omega_R = \omega_A \otimes_k \omega_B = \omega_A \otimes_k B$$

and we have

$$\operatorname{curv}_R(\omega_R) = 2 < \frac{2}{e - \sqrt{e^2-4}} = \operatorname{curv}_R(k)\,.$$

Note that since $\frac{2}{e-\sqrt{e^2-4}} \longrightarrow \infty$ as $e \longrightarrow \infty$, the disparity in curvatures may be made as large as desired by choosing B with $e \gg 0$.

Remark 3.4. One may introduce the quotient $\mathfrak{g}(R) = \operatorname{curv}_R(\omega)/\operatorname{curv}_R(k)$ as a measure of a local ring's deviation from the Gorenstein property. One sees immediately that $0 \leq \mathfrak{g}(R) \leq 1$ for all non-regular R, and that R is Gorenstein if and only if $\mathfrak{g}(R) = 0$. The ring A above illustrates that quite often $\mathfrak{g}(R) = 1$. However, it follows from Example 3.3 that $\mathfrak{g}(R)$ can also be made arbitrarily close to 0 for non-Gorenstein R.

We end by showing that the above notion of 'close' to Gorenstein is different from others in the literature.

In [8] Barucci and Fröberg describe a notion for a one-dimensional ring to be 'almost' Gorenstein, and give $R = k[X,Y,Z]/(XY, XZ, YZ)$ as an example of an almost Gorenstein ring in their sense. However, it is not hard to show that $\operatorname{curv}_R \omega = \operatorname{curv}_R k$, in other words, $\mathfrak{g}(R) = 1$ (R is in fact a Golod ring). Thus R is furthest from being Gorenstein in our sense.

In [14] Huneke and Vraciu also define a notion of a ring R being 'almost' Gorenstein. They show that any Artinian Gorenstein ring modulo its socle is almost Gorenstein in their sense, for example, $R = k[x,y]/(x^2, xy, y^2)$. But this is again a Golod ring, and therefore $\mathfrak{g}(R) = 1$, so again their notion of almost Gorenstein is incomparable to ours.

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