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WARPED PRODUCT RIGIDITY

CHENXU HE, PETER PETERSEN, AND WILLIAM WYLIE

In memory of Barrett O'Neill

ABSTRACT. In this paper we study the space of solutions to an overdetermined linear system involving the Hessian of functions. We show that if the solution space has dimension greater than one, then the underlying manifold has a very rigid warped product structure. This warped product structure will be used to study warped product Einstein structures in [HPW3].

INTRODUCTION

Let q be a quadratic form on a Riemannian manifold (M, g) and Q be the corresponding symmetric linear operator on M . We shall study the space of solutions

$$W(M; q) = \{w \in C^\infty(M, \mathbb{R}) : \text{Hess}w = wq\}.$$

Solving $\text{Hess}w = wq$ for a fixed q is generally impossible, but it comes up in many places. Perhaps the most well known example is due to Obata [Ob] which we will discuss in section 2. A more complicated example is warped product Einstein structures which are of this type with

$$q = \frac{1}{m}(\text{Ric} - \lambda g)$$

see [HPW2] and [HPW3].

Note that any positive function w has the property that

$$\text{Hess}w = wq$$

defines a quadratic form q . However, if a real valued function w satisfies such an equation, then its zero set is a totally geodesic codimension one submanifold, which is a rather special condition. We shall enhance this by showing that, if such an equation has linearly independent solutions, then the underlying space is a warped product.

Note that when $\dim M = 1$ the equation

$$\text{Hess}w = wq$$

is a scalar equation

$$w'' = Qw$$

with $q = Qdx^2$. Clearly there is a two-dimensional space of solutions unless $M = \mathbb{S}^1$. So this is not a case where we can say much about (M, g, q) . When $M = \mathbb{S}^1$ this equation is also known as Hill's equation. The issue of finding one or two solutions to that equation has a long history (see [MW].) In either case the underlying space

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does have the desired underlying structure of a warped product, albeit in a very trivial fashion with the base being a point and the fiber the space itself. This example shows that one cannot expect q to be determined by the geometry unless there are three or more linearly independent solutions.

The building blocks for all examples consist of base spaces and fiber spaces:

Definition. A *base space* (B, g_B, u) consists of a Riemannian manifold and a smooth function $u : B \rightarrow [0, \infty)$ such that $u^{-1}(0) = \partial B$. We define

$$q_B = \frac{1}{u} \text{Hess}u$$

and when $\partial B \neq \emptyset$ assume that this defines a smooth tensor on B , and that $|\nabla u| = 1$ on ∂B . Moreover, when the functions in the solution space $W(B; q_B)$, that vanish on ∂B if it is not empty, are constant multiples of u we call it a *base manifold* (see [HPW3]).

Definition. A *fiber space* (F, g_F, τ) consists of a space form (F, g_F) and a *characteristic function* $\tau : F \rightarrow \mathbb{R}$ such that $\dim W(F; -\tau g_F) = \dim F + 1$. In case F is a sphere we shall further assume that (F, g_F) is the unit sphere.

Remark. As we shall see τ will almost always be a constant. Only when $\dim F = 1$ is it possible for τ to be a function. We shall also see that F must be simply connected unless it is a circle.

Our main result is that if $W(M; q)$ has dimension larger than one then (M, g) must be isometric to a warped product of a particular sort.

Theorem A. *Let (M, g) be complete and simply connected. If q is a quadratic form such that $\dim W(M; q) = k + 1$ where $k \geq 1$, then there is a simply connected base space (B, g_B, u) and a fiber space (F, g_F, τ) such that*

$$(M, g) = (B \times F, g_B + u^2 g_F).$$

Moreover when $\partial B \neq \emptyset$ or $k > 1$, then the characteristic function is constant.

Remark. In general the base space doesn't have to be a base manifold, see Example 5.2. This is in sharp contrast to what happens when q is more directly related to the geometry (see [HPW3]).

From the warped product structure $M = B \times_u F$ constructed in Theorem A, we obtain two natural projections π_1 and π_2 from M to B and F respectively. The special structure of the manifold M yields the following decomposition of the vector space $W(M; q)$.

Theorem B. *Let (M, g) be complete and simply-connected with $\dim W(M; q) \geq 2$. Suppose $M = B \times_u F$ as in Theorem A. Then we have*

$$W(M; q) = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\tau g_F)\}.$$

Remark. We actually show slightly more general results than Theorems A and B. Namely any subspace $W \subset W(M; q)$ with $\dim W > 1$, not necessarily the whole solution space, induces a warped product structure on M as in Theorem A and such W has the decomposition as in Theorem B, see Theorems 3.7, 5.3 and 5.4. This generalization will be useful when we study the manifold which might not be simply-connected. By lifting the quadratic form q to the universal cover we consider the subspace of solutions which is also invariant under the deck transformation.

Remark. The special type of warped products obtained in Theorem A with constant curvature fiber is also referred as *generalized Robertson-Walker* space in general relativity, see the recent survey [Ze] and references therein.

In Section 1 we show some basic properties about the solution space $W(M; q)$ and what it looks like in some simple cases. In Section 2 we establish some properties for $W(M; q)$ when we know that M is a warped product. Section 3 is devoted to the proof of Theorem A. In Section 4 we use Theorem A to place restrictions on what the quadratic form can look like. This in turn is used in Section 5 to prove Theorem B. Knowing that M is a warped product then allows us to determine what the quadratic form q looks like in terms of the geometry of M . In Section 6 we collect some miscellaneous results: more detailed description of the base space that appears in the warped product structure and the case where M is not simply-connected.

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1. BASIC PROPERTIES AND EXAMPLES

We start by establishing two simple but fundamental properties.

Proposition 1.1. *The evaluation map*

$$\begin{aligned} W(M; q) &\rightarrow \mathbb{R} \times T_p M, \\ w &\mapsto (w(p), \nabla w|_p) \end{aligned}$$

is injective.

Proof. We use a proof adapted from [Co] and which was also used in [HPW1]. Let $w \in W = W(M; q)$ and let γ be a unit speed geodesic emanating from $p \in M$. Define $h(t) = w(\gamma(t))$ and $\Theta(t) = q(\gamma'(t), \gamma'(t))$. Then we have a linear second order o.d.e. along γ for h given by

$$\begin{aligned} h''(t) &= \text{Hess}w(\gamma'(t), \gamma'(t)) \\ &= \frac{\Theta(t)}{m} \cdot h(t). \end{aligned}$$

Thus h is uniquely determined by its initial values

$$\begin{aligned} h(0) &= w(\gamma(0)), \\ h'(0) &= g(\nabla w, \gamma'(0)). \end{aligned}$$

In particular h must vanish if w and its gradient vanish at p .

This shows that the set $A = \{p \in M : w(p) = 0, \nabla w|_p = 0\}$ is open. As it is clearly also closed it follows that w must vanish everywhere if M is connected and A is nonempty. \square

Next we prove a basic fact about the zero set of a $w \in W(M; q)$.

Proposition 1.2. *Let $L = \{p \in M : w(p) = 0\} \neq \emptyset$ for some $w \in W(M; q)$. Then L is a totally geodesic hypersurface.*

Proof. We already know that ∇w can't vanish on L . This shows that 0 is a regular value for w and hence that L is hypersurface. We know in addition from $\nabla_X \nabla w = wQ(X)$ that $\text{Hess}w$ vanishes on L . This shows that L is totally geodesic. \square

We now turn our attention to an enhancement of a well-known result by [Ob].

Theorem 1.3. *Let (M, g) be a complete simply connected Riemannian n -manifold with $n > 1$. If there exists $\tau \in \mathbb{R}$ such that*

$$\dim W(M; -\tau g) = n + 1,$$

then (M, g) is a simply connected space form of constant curvature τ .

Proof. Obata considered the case where $\tau > 0$ and in that case it suffices to assume that $\dim W(M; -\tau g) \geq 1$. In case $\tau \leq 0$ we do however need the stronger assumption.

When $\tau = 0$ we see that constant functions are in $W(M; 0)$. But there will also be an n -dimensional subspace of non-constant functions whose Hessians vanish. This shows that $(M, g) = \mathbb{R}^n$ with the Euclidean flat metric.

When $\tau < 0$ note that any $w \in W(M; -\tau g)$ has the property that $\bar{\mu}(w) = \tau w^2 + |\nabla w|^2$ is constant and thus defines a nondegenerate quadratic form on $W(M; -\tau g)$. By Proposition 1.1 it follows that some w will have $\bar{\mu}(w) < 0$. By scaling we can then assume that some $w \in W(M; -\tau g)$ will satisfy $\tau w^2 + |\nabla w|^2 = \tau$ or

$$-\tau = \frac{|\nabla w|^2}{-1 + w^2} = |\nabla \operatorname{arccosh}(w)|^2.$$

We can argue as in [Ob] that $w = \cosh(\sqrt{-\tau}r)$, where $r : M \rightarrow \mathbb{R}$ is the distance to a point in M . And that the metric has constant curvature τ as it is the warped product metric

$$dr^2 + \frac{\sinh(\sqrt{-\tau}r)}{\sqrt{-\tau}} g_{\mathbb{S}^{n-1}}$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric on the unit sphere. \square

This result can be further extended as follows:

Lemma 1.4. *Let (M, g) be a complete simply connected Riemannian n -manifold with $n \geq 1$. If there exists $\tau : M \rightarrow \mathbb{R}$ such that*

$$\dim W(M; -\tau g) = n + 1,$$

then either $n = 1$ or τ is constant and (M, g) is a simply connected space form of constant curvature τ .

Proof. When $n = 1$ it is clear that $\dim W(M; -\tau g) = 2$ for any function τ . In general having a solution to

$$\operatorname{Hess} w = -\tau w g$$

shows that

$$d(|\nabla w|^2) = -\tau d(w^2)$$

In particular, $d\tau \wedge d(w^2) = 0$ for all $w \in W(M; -\tau g)$. Now use $\dim W(M; -\tau g) = n+1$ together with Proposition 1.1 to find n functions $w_i \in W(M; -\tau g)$, $i = 1, \dots, n$ such that $d(w_i^2)$ form a basis at x . Then $d\tau \wedge d(w_i^2) = 0$ implies that $d\tau$ vanishes at x . \square

In case M has constant curvature we also have the following converse.

Lemma 1.5. *Let (M, g) be a complete simply connected space form and $\tau \in \mathbb{R}$. Either $\dim W(M; -\tau g) = \dim M + 1$ or all functions in $W(M; -\tau g)$ are constant.*

Proof. When $n = 1$ this is obvious. Otherwise we have to show that if w is a non-constant solution to the equation $\text{Hess}w = -\tau w g$ when τ is the curvature of M . However, the fact that $\text{Hess}w = -\tau w g$ together with knowing that $\tau w^2 + |\nabla w|^2$ is constant shows that $\sec(\nabla w, X) = \tau$. \square

Remark 1.6. There is a related, local, rigidity characterization of spaces where $\text{Hess}w = \phi g$ for some $\phi \in C^\infty(M, \mathbb{R})$ as warped products over one dimensional bases. This was first proven by Brinkmann in [Br], also see [CC] and [Be, 9.117].

In Theorem A, even though the manifolds M do not have boundary, manifolds with boundary do arise as base spaces. The warped product structure of M yields a decomposition of the space $W(M; q)$ involving functions in the space $W(B; q|_B)$, see Proposition 2.3. When B has boundary, the boundary conditions satisfied by these functions in $W(B; q|_B)$ also yields some further restrictions. We will encounter both Dirichlet and Neumann boundary conditions, for which we will use the following notation.

Definition 1.7. Let M be a Riemannian manifold with boundary $\partial M \neq \emptyset$ and ν is a normal vector to ∂M , then we define the spaces with Dirichlet and Neumann boundary conditions as

$$\begin{aligned} W(M; q) &= \{w \in C^\infty(M) : \text{Hess}w = wq\}, \\ W(M; q)_D &= \{w \in W(M; q) : w|_{\partial M} = 0\}, \\ W(M; q)_N &= \left\{w \in W(M; q) : \frac{\partial w}{\partial \nu}|_{\partial M} = 0\right\}. \end{aligned}$$

We have the following general fact when $\partial M \neq \emptyset$.

Proposition 1.8. *If $\partial M \neq \emptyset$, then $\dim W(M; q)_D \leq 1$. Moreover if $\dim W(M; q) = 1$, then*

$$W(M; q) = W(M; q)_D \oplus W(M; q)_N.$$

Proof. Let $x \in \partial M$. If $\nabla w(x) = 0$, then Proposition 1.1 implies that w is the zero function. Thus any non-zero element $w \in W(M; q)_D$ satisfies

$$w(x) = 0 \quad \nabla w(x) \neq 0 \quad \nabla w(x) \perp \partial B$$

which, applying Proposition 1.1 again shows that $\dim W(M; q)_D \leq 1$.

In the case when $\dim W(M; q)_D = 1$ Proposition 1.1 also shows that the intersection is zero,

$$W(M; q)_D \cap W(M; q)_N = \{0\},$$

i.e., the decomposition of $W(M; q)$ is a direct sum. \square

Remark 1.9. Note that the condition $\dim W(M; q)_D = 1$ implies that M has totally geodesic boundary.

In the following we describe some simple examples, the manifold M is one dimension. We consider the quadratic form q which is defined by the warped product Einstein equation, i.e.,

$$q = -\frac{\lambda}{m}g.$$

Note that $\text{Ric} = 0$ in this case. In [HPW1, Example 1] we had the classification when the solution $w \in W(M; q)$ is non-negative and only vanishes on the boundary. Here we extend that classification to allow any sign of w .

Example 1.10. Let $(M, g) = (\mathbb{R}, dr^2)$ and then $w \in W(M; q)$ if and only if

$$w'' = -\kappa w \quad \text{with} \quad \kappa = \frac{\lambda}{m}.$$

So we have three different cases depending on the sign of λ .

- (1) When $\lambda > 0$, we have $w = C_1 \cos(\sqrt{\kappa}r) + C_2 \sin(\sqrt{\kappa}r)$.
- (2) When $\lambda = 0$, we have $w = C_1 r + C_2$.
- (3) When $\lambda < 0$, we have $w = C_1 \exp(\sqrt{-\kappa}r) + C_2 \exp(-\sqrt{-\kappa}r)$.

In particular we have $\dim W(M; q) = 2$ for all three cases.

Example 1.11. Let $(M, g) = (\mathbb{S}_a^1, dr^2)$ be the circle with radius a . Then $W(\mathbb{S}_a^1; q)$ corresponds to the elements in $W(\mathbb{R}; q)$ which have period $2\pi a$. This gives us the following

$$\dim W(\mathbb{S}_a^1; q) = \begin{cases} 2 & \text{if } \lambda > 0 \text{ and } a\sqrt{\kappa} \text{ is an integer,} \\ 1 & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now look at one dimensional examples with boundary.

Example 1.12. Let $(M, g) = ([0, \infty), dr^2)$ and then again we have three different cases.

- (1) When $\lambda > 0$, we have

$$\begin{aligned} W(M; q)_D &= \{C \sin(\sqrt{\kappa}r) : C \in \mathbb{R}\} \\ W(M; q)_N &= \{C \cos(\sqrt{\kappa}r) : C \in \mathbb{R}\}. \end{aligned}$$

- (2) When $\lambda = 0$, we have

$$\begin{aligned} W(M; q)_D &= \{Cr : C \in \mathbb{R}\} \\ W(M; q)_N &= \{C : C \in \mathbb{R}\}. \end{aligned}$$

- (3) When $\lambda < 0$, we have

$$\begin{aligned} W(M; q)_D &= \{C \sinh(\sqrt{-\kappa}r) : C \in \mathbb{R}\} \\ W(M; q)_N &= \{C \cosh(\sqrt{-\kappa}r) : C \in \mathbb{R}\}. \end{aligned}$$

Finally we consider the closed interval which is similar to the circle case.

Example 1.13. Let $(M, g) = ([0, 2\pi a], dr^2)$. We have

- (1) If $\lambda > 0$ and $a\sqrt{\kappa}$ is an integer, then

$$\begin{aligned} W(M; q)_D &= \{C \sin(\sqrt{\kappa}r) : C \in \mathbb{R}\} \\ W(M; q)_N &= \{C \cos(\sqrt{\kappa}r) : C \in \mathbb{R}\}. \end{aligned}$$

- (2) If $\lambda = 0$, then

$$\begin{aligned} W(M; q)_D &= \{0\} \\ W(M; q)_N &= \{C : C \in \mathbb{R}\}. \end{aligned}$$

- (3) Otherwise

$$W(M; q) = W(M; q)_D = W(M; q)_N = \{0\}.$$

2. WARPED PRODUCT EXTENSIONS

In this section we create a fairly general class of examples using warped product extensions. The goal is to start with a base space (B, g_B, u) and then construct (M, g) as a warped product over (B, g_B) with fiber (F, g_F) and metric given by

$$g = g_B + u^2 g_F.$$

When $\partial B \neq \emptyset$ there are further conditions in order to obtain a smooth metric on M . The fiber has to be a round sphere which we can assume to be the unit sphere and ∇u a unit normal field to $\partial B \subset B$. There are further conditions on the higher derivatives of u and we also need $\partial B \subset B$ to be totally geodesic. These conditions, however, are automatically satisfied as we assume that

$$uq_B = \text{Hess}_B u$$

for some smooth symmetric tensor q_B on B .

The warped product structure defines two distributions on M , the horizontal one given by TB and the vertical one by TF . We denote these two distributions by \mathcal{B} and \mathcal{F} respectively. The projection onto B is denoted by $\pi_1 : M \rightarrow B$ and the projection onto F by $\pi_2 : M \rightarrow F$. We use X, Y, \dots and U, V, \dots to denote the horizontal and vertical vector fields respectively.

Next we need to define q on M as an extension of q_B on B . We assume that q preserves the horizontal and vertical distributions and that on the horizontal distribution $q(X, Y) = q_B(X, Y)$.

As q preserves the horizontal and vertical distributions it follows that any $w \in W(M; q)$ has the property that its Hessian also preserves these distributions. It follows from that the function w has a special form.

Lemma 2.1. *If $M = B \times_u F$ and $w : M \rightarrow \mathbb{R}$ satisfies*

$$(\text{Hess}_M w)(X, U) = 0$$

for all $X \in TB$ and $U \in TF$, then

$$w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v)$$

where $z : B \rightarrow \mathbb{R}$ and $v : F \rightarrow \mathbb{R}$.

Moreover, if

$$\pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v) = 0$$

then v must be constant and z a multiple of u .

Proof. The second fundamental form for a warped product is particularly simple: if X is a vector field on B and U a vector field on F , then

$$\nabla_X^M U = \nabla_U^M X = \frac{D_X u}{u} U.$$

With that in mind we have

$$\begin{aligned} D_X \frac{1}{u} D_U w &= -\frac{D_X u}{u^2} D_U w + \frac{1}{u} D_X D_U w \\ &= \frac{1}{u} \left(-D_{\nabla_X^M U} w + D_X D_U w \right) \\ &= \frac{1}{u} (\text{Hess}_M w)(X, U) \\ &= 0. \end{aligned}$$

Thus $D_U \frac{w}{u}$ is constant on B . This shows that if we restrict $\frac{w}{u}$ to the fibers $F_i = \{b_i\} \times F$ over points $b_1, b_2 \in B$ then the difference

$$\frac{w}{u}|_{F_1} - \frac{w}{u}|_{F_2}$$

is constant. This shows the claim.

For the uniqueness statement just note that if

$$\pi_1^*(z) = -\pi_1^*(u) \cdot \pi_2^*(v)$$

then the right hand side defines a function on B and thus v must be constant. \square

Remark 2.2. When B has boundary and we insist that both $\pi_1^*(z)$ and $\pi_1^*(u) \cdot \pi_2^*(v)$ be smooth on M , then there are extra conditions. The function $\pi_1^*(u) \cdot \pi_2^*(v)$ is smooth at the singular set only if v is odd $-v(y) = v(-y)$, $y \in \mathbb{S}^k$. On the other hand $\pi_1^*(z)$ can only be smooth if ∇z is tangent to the boundary of B , i.e., it satisfies the Neumann boundary condition on B .

Next we study how $W(M; q)$ relates to u and the fiber F .

Proposition 2.3. *Let $M = B \times_u F$ and assume that $uq_B = \text{Hess}_B u$. Then $w \in W(M; q)$ if and only if there exist $z \in C^\infty(B)$ and $v \in C^\infty(F)$ such that*

- (1) $w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v)$,
- (2) $z \in W(B; q_B)$, and
- (3) $\text{Hess}_F v + v \left(-q|_{\mathcal{F}} + |\nabla u|_B^2 g_F \right) = - \left(-\frac{z}{u} q|_{\mathcal{F}} + g_B(\nabla u, \nabla z) g_F \right)$.

Proof. From Lemma 2.1 we know that any function $w \in W(M; q)$ has the form

$$w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v).$$

On the horizontal distribution we have

$$\begin{aligned} wq|_B &= zq_B + uvq_B, \\ (\text{Hess}_M w)|_B &= \text{Hess}_B z + v\text{Hess}_B u. \end{aligned}$$

Since $u \in W(B; q_B)$ we see that $(\text{Hess}_M w)|_B = wq|_B$ if and only if $z \in W(B; q_B)$.

On the vertical distribution we have

$$\begin{aligned} wq|_{\mathcal{F}} &= zq|_{\mathcal{F}} + uvq|_{\mathcal{F}} \\ (\text{Hess}_M w)|_{\mathcal{F}} &= u\text{Hess}_F v + uv|\nabla u|_B^2 g_F + ug_B(\nabla u, \nabla z) g_F. \end{aligned}$$

Thus $wq|_{\mathcal{F}} = (\text{Hess}_M w)|_{\mathcal{F}}$ is equivalent to condition 3. \square

Note that if $\dim W(B; q_B) = 1$ then all $w \in W(M; q)$ are of the form $w = \pi_1^*(u) \cdot \pi_2^*(v)$. This motivates the following

Corollary 2.4. *Let $M = B \times_u F$ and assume that $uq_B = \text{Hess}_B u$, $W(M; q) \neq 0$, and that some nontrivial $w \in W(M; q)$ is of the form $w = \pi_1^*(u) \cdot \pi_2^*(v)$, then there is a symmetric tensor q_F on F such that*

$$q_F = q|_{\mathcal{F}} - |\nabla u|_B^2 g_F.$$

Proof. As we can write $w \in W(M; q)$ in the form $w = \pi_1^*(u) \cdot \pi_2^*(v)$ it follows from condition 3. in Proposition 2.3 that

$$\text{Hess}_F v = v \left(q|_{\mathcal{F}} - |\nabla u|_B^2 g_F \right).$$

This implies that $q|_{\mathcal{F}} - |\nabla u|_B^2 g_F$ can only depend on F at points where $v \neq 0$. Now w and hence also v can only vanish on a totally geodesic hypersurface so by continuity q_F defines a symmetric tensor on F . \square

Remark 2.5. As we shall see the most important examples of such constructions always have the property that $q|_{\mathcal{F}} = -\kappa u^2 g_F$ for some function $\kappa : M \rightarrow \mathbb{R}$. The previous corollary then shows that $\kappa u^2 + |\nabla u|^2$ is constant on the horizontal leaves and thus defines a function on F . Combining this with Proposition 2.3 and Lemma 1.5 establishes the next result.

Corollary 2.6. *Let $M = B \times_u F$ and assume that there is a function $\kappa : M \rightarrow \mathbb{R}$ such that $q|_{\mathcal{F}} = -\kappa u^2 g_F$ and that (F, g_F) is a simply connected space form. If some nontrivial $w \in W(M; q)$ is of the form $w = \pi_1^*(u) \cdot \pi_2^*(v_1)$, then*

$$\left\{ v \in C^\infty(F) : \text{Hess}_F v = -v \left(\kappa u^2 + |\nabla u|^2 \right) g_F \right\}$$

has dimension $\dim F + 1$ and

$$\left\{ \pi_1^*(u) \cdot \pi_2^*(v) : \text{Hess}_F v = -v \left(\kappa u^2 + |\nabla u|^2 \right) g_F \right\} \subset W(M; q).$$

3. THE WARPED PRODUCT STRUCTURE

In this section we prove Theorem A, i.e., manifolds with $\dim W(M; q) > 1$ are warped products, see Theorem 3.7.

We start with a simple lemma that shows how we construct Killing fields.

Lemma 3.1. *Let $v, w \in C^\infty(M)$ then $v\nabla w - w\nabla v$ is a Killing field if and only if $v\text{Hess}w = w\text{Hess}v$.*

Proof. We prove this by a simple direct calculation:

$$\begin{aligned} \nabla_X (v\nabla w - w\nabla v) &= (D_X v)\nabla w - (D_X w)\nabla v + v\nabla_X \nabla w - w\nabla_X \nabla v \\ &= g(\nabla v, X)\nabla w - g(\nabla w, X)\nabla v + v\nabla_X \nabla w - w\nabla_X \nabla v \\ &= (\nabla w \wedge \nabla v)(X) + v\nabla_X \nabla w - w\nabla_X \nabla v. \end{aligned}$$

This shows that $v\nabla w - w\nabla v$ is a Killing field precisely when

$$v\nabla_X \nabla w = w\nabla_X \nabla v$$

which finishes the proof. \square

In particular, we see that if a manifold has $\dim W(M; q) > 1$, then we get Killing fields.

For the remainder of this section we fix a Riemannian manifold (M, g) and a quadratic form q on M . Furthermore, we select a subspace $W \subset W(M; q)$. For each such subspace we define

$$W_p = \{w \in W : w(p) = 0\}$$

A point p is said to be *regular* if the codimension of $W_p \subset W$ is one. Otherwise a point is called *singular*. The set of singular points is denoted S .

Proposition 3.2. *The singular set S is a totally geodesic submanifold of codimension $\dim W$.*

Proof. It follows by induction from Proposition 1.2 that S has the stated properties. \square

At regular points $p \in M - S$ we define

$$\mathcal{F}_p = \{\nabla w : w \in W_p\}$$

and let \mathcal{B} be the orthogonal distribution on $M - S$. At a regular point there is a unique $u_p \in W$ with

$$\begin{aligned} u_p(p) &= 1 \\ \nabla u_p|_p &\perp \mathcal{F}_p. \end{aligned}$$

This orthogonal distribution has the following properties.

Proposition 3.3. *Suppose $\dim W \geq 1$ and let $k = \dim W - 1$. Then the foliation \mathcal{B} on the regular set $M - S$ is totally geodesic of dimension $n - k$. Let B_p be the leaf of the foliation \mathcal{B} through $p \in M - S$, then u_p is positive on B_p . Finally q preserves the two distributions.*

Proof. Recall that \mathcal{B} is the orthogonal distribution to \mathcal{F} and

$$\mathcal{F}_p = \{\nabla w : w \in W_p\}.$$

If two vector fields are perpendicular to the gradient of a function, then their Lie bracket is clearly also perpendicular to the gradient. This shows that \mathcal{B} is integrable. Moreover the leaf through $p \in M - S$ is the connected component B_p in

$$\{x \in M - S : w(x) = 0 \text{ for all } w \in W_p\}$$

that contains p . This is clearly a totally geodesic submanifold. If u_p vanishes at $x \in B_p$, then $u_p \in W_q$ and consequently also lies in W_p , a contradiction.

Note that on $T_p M$ we have

$$q(X, V) = g(\nabla_X \nabla u_p, V)$$

as ∇u_p is tangent to B_p and B_p is totally geodesic it follows that $q(X, V) = 0$ if $X \in T_p B_p$ and $V \in \mathcal{F}_p$. \square

Remark 3.4. Note that when $W = \{0\}$ we have $S = M$. In the next case where $\dim W = 1$ the regular set $M - S$ has two components. Each of these components is a leaf in the totally geodesic foliation \mathcal{B} .

Remark 3.5. Note that B_p need not be complete even if M is. It can however be completed by adding components of S as boundary pieces. Thus the closure \overline{B}_p is naturally a manifold with boundary when $S \neq \emptyset$.

Next we investigate the $(\dim W - 1)$ -dimensional distribution \mathcal{F} as well as its extension

$$\widehat{\mathcal{F}}_p = \{\nabla w|_p : w \in W\}.$$

Proposition 3.6. *Suppose $\dim W = k + 1 \geq 2$. The distribution \mathcal{F} on $M - S$ is integrable and is generated by a set of Killing fields on M of dimension $\frac{1}{2}k(k + 1)$. Moreover, for any vector fields $Z \in \mathcal{F}$ and $X \in TM$, we have*

$$\nabla_X Z \in \widehat{\mathcal{F}}.$$

Proof. For a fixed point $p \in M - S$, the space W_p is spanned by the following functions

$$v(p)w - w(p)v, \quad \text{for } v, w \in W.$$

It follows that the following vectors form a spanning set of the subspace $\mathcal{F}_p \subset T_pM$:

$$v(p)\nabla w|_p - w(p)\nabla v|_p, \quad \text{for } v, w \in W.$$

So we can write the distribution \mathcal{F} as

$$\mathcal{F} = \{v\nabla w - w\nabla v : v, w \in W\}.$$

Note that $\widehat{\mathcal{F}}$ might not be a distribution on $M - S$ as the dimension of $\widehat{\mathcal{F}}_p$ can be either $k + 1$ or k . It agrees with \mathcal{F}_p for those p where its dimension is k . At the points where its dimension is $k + 1$ the complementary subspace of $\mathcal{F}_p \subset \widehat{\mathcal{F}}_p$ is one-dimensional and spanned by $\nabla u_p|_p$.

Using $\mathcal{F} = \{v\nabla w - w\nabla v : v, w \in W\}$ we see that when $X \in \mathcal{B}$ Proposition 3.3 implies

$$g([\nabla v, \nabla w], X) = -g(v\nabla w - w\nabla v, X) = 0$$

In particular, both $\widehat{\mathcal{F}}$ and \mathcal{F} are integrable where they are distributions. Moreover, we know from Lemma 3.1 that \mathcal{F} is spanned by Killing fields.

Finally we calculate the dimension of this set of Killing fields on M . First note that it can't exceed $\frac{1}{2}k(k + 1)$ as the fields are all tangent to a k -dimensional distribution. Next note that at $p \in M - S$ we have two types of Killing fields

$$v\nabla w - w\nabla v, \quad v, w \in W_p$$

and

$$u_p\nabla w - w\nabla u_p, \quad w \in W_p.$$

The first type of Killing field vanishes at p and has covariant derivative $\nabla w|_p \wedge \nabla v|_p$ which defines a skew symmetric transformation that leaves \mathcal{F}_p invariant. Moreover, as the skew symmetric transformations on \mathcal{F}_p are generated by such transformations these Killing fields generate a subspace of dimension at least

$$\frac{1}{2}k(k - 1).$$

The second type of Killing field has value $\nabla w|_p$ at p . Thus these Killing fields will generate a complementary subspace of dimension at least k . This shows that the Killing fields $\{v\nabla w - w\nabla v : v, w \in W\}$ generate a space of Killing fields of dimension at least $\frac{1}{2}k(k + 1)$. \square

We can now prove our Theorem A from the introduction.

Theorem 3.7. *Let (M^n, g) be a complete simply connected Riemannian manifold with a symmetric tensor q and W be a subspace of $W(M; q)$. If $\dim W = k + 1 \geq 2$, then*

$$M = B \times_u F$$

where u vanishes on the boundary of B and F is either the k -dimensional unit sphere $\mathbb{S}^k(1) \subset \mathbb{R}^{k+1}$, k -dimensional Euclidean space \mathbb{R}^k , or the k -dimensional hyperbolic space \mathbb{H}^k . In the first two cases $k \geq 1$ while in the last $k > 1$.

Proof. Proposition 3.6 shows that the set of Killing fields on M that are tangent to the foliation \mathcal{F} is a subalgebra of the space of all Killing fields on M of dimension $\frac{1}{2}k(k+1)$. As M is complete this means that there'll be a corresponding connected subgroup $G \subset \text{Iso}(M, g)$. First observe that as the Killing fields $v\nabla w - w\nabla v$ vanish on S , the group G fixes S . Next note that G forces the leaves of the foliation \mathcal{F} to be maximally symmetric. In particular, they are complete connected space forms, which are either simply connected or possibly circles or real projective spaces see e.g., [Pe, page 190]. From what we show below it'll be clear that the case of real projective spaces will not occur here as M is simply connected.

Note that we have the group G , we would like to show that the quotient map $\pi_1 : M \rightarrow M/G$ is a Riemannian submersion on $M - S$. When there is no singular set, this follows from [BH, Theorem A]. In fact, due to the group action G the proof of [BH, Theorem A] is somewhat simpler in our case and can be adapted to work in case $S \neq \emptyset$ and $k > 1$. That is, the case where $M - S$ is connected and simply connected (see also [O'N, p. 203] for a similar construction in the context of covering spaces).

First note that, when at least one fiber F_p is compact, G itself is compact and so the action is proper. In particular, if $S \neq \emptyset$, then the fibers F_p for p near $x \in S$ can be identified with the space of unit normal vectors to $x \in S$ and so the fibers are compact.

For each $p \in M - S$ there is a neighborhood U_p and a uniquely defined Riemannian submersion $U_p \rightarrow B_p$ which projects along the leaves of \mathcal{F} . Next note that any two vertical leaves can be connected by a horizontal geodesic in $M - S$. This shows that G acts transitively on the leaves B_p , $p \in M - S$. Now fix a specific horizontal leaf B . By using elements in G we can then construct Riemannian submersions $f_p : U_p \rightarrow B$ with the properties that: If $U_{p_1} \cap U_{p_2} \neq \emptyset$, then there exists $h \in G$ such that $h(B) = B$ and $h \circ f_{p_1} = f_{p_2}$. Since $M - S$ is connected and simply connected a standard monodromy argument then shows that we obtain a global Riemannian submersion $f : M - S \rightarrow B$. Moreover, $\bar{B} = M/G$ so the natural projection $\pi_1 : M \rightarrow M/G$ is a Riemannian submersion when restricted to $M - S$.

This leaves us with the situation where $k = 1$ and $S \neq \emptyset$. In particular, all fibers are circles. In this case G is Abelian. We start by observing in general that if some $h \in G$ fixes all points in a fiber F_p and $S \neq \emptyset$, then h acts trivially. Let $x \in S$ be the closest point to p . Then h must fix the unique shortest geodesic from x to p in B_p . Note that it is unique as it is normal to S in \bar{B}_p . Next observe that we can move this geodesic by isometries from G to get minimal connections from x to all other points in the orbit F_p . Since h fixes all of F_p we see that h not only fixes S but also all normal directions $\nu_x S$. Thus h acts trivially. In case G is Abelian this implies that all principal isotropy groups are trivial. In particular, $\pi_1 : M \rightarrow M/G$ is a Riemannian submersion when restricted to $M - S$.

In all cases we now have that the quotient map $\pi_1 : M \rightarrow M/G$ is a Riemannian submersion on $M - S$. Since $G \subset \text{Iso}(M, g)$ the leaves of \mathcal{F} have the property that their second fundamental forms are also invariant under G . This implies that the leaves are totally umbilic with a mean curvature vector that is invariant under the group action. As the orthogonal foliation is totally geodesic it follows that the mean curvature vector is basic. It then follows from [Be, Chapter 9.J] that these two foliations yield a local warped product structure on $M - S$.

Since G fixes S , to obtain a global warped product structure we need only show that $B_p \cap F_p = \{p\}$ on $M - S$. When there is no singular set we can again appeal to [BH, Theorem A] which says that in this case M is diffeomorphic to $B_p \times F_p$.

When $S \neq \emptyset$ note that the quotient map $\pi_1 : M \rightarrow M/G$ forces M/G to be a Riemannian manifold with totally geodesic boundary S . In particular, M/G is homotopy equivalent to its interior. In this situation we know initially only that π_1 is a Riemannian covering map when restricted to horizontal leaves. However, let $\gamma : [0, 1] \rightarrow \text{int}(M/G)$ be a loop and consider a horizontal lift $\bar{\gamma} : [0, 1] \rightarrow M - S$. As M is simply connected $\bar{\gamma}$ is homotopic to a path in the fiber

$$\pi_1^{-1}(\gamma(0)) = \pi_1^{-1}(\gamma(1))$$

through a homotopy that keeps the endpoints fixed. This in turn shows that γ is homotopic to a point in M/G . Thus $\text{int}(M/G)$ is simply connected and we see that π_1 is an isometry when restricted to horizontal leaves. In particular, for all $p \in M - S$ we have $B_p \cap F_p = \{p\}$. \square

Corollary 3.8. *When $\mathcal{F} = \widehat{\mathcal{F}}$, i.e., the foliation \mathcal{F} is totally geodesic, the manifold M is isometric to a product.*

4. PROPERTIES OF THE QUADRATIC FORM

Assume below that we have a complete simply connected Riemannian n -manifold with $\dim W(M; q) \geq 2$ and a fixed subspace $W \subset \dim W(M; q)$ with $\dim W = k+1$ and $k \geq 1$. Theorem 3.7 then tells us that

$$(M, g) = (B \times F, g_B + u^2 g_F)$$

for some function $u : B \rightarrow [0, \infty)$ that vanishes only on ∂B . In this section we give the details of how to show that the base is a base space and the fiber a fiber space.

Note that we shall not distinguish between fields on B and their corresponding horizontal lifts to M . However, we will be careful with notation in regards to derivatives of such fields. We'll use A_1, A_2 as vector fields on TM , X, Y as horizontal fields and U, V as vertical fields. Also we shall for convenience use u for its pullback to M .

The vertical isometries from G act as isometries on M and so there is a function $\rho : B \rightarrow \mathbb{R}$ such that

$$\text{Ric}_M(V) = \rho V, \quad V \in \mathcal{F}.$$

From [Be, Chapter 9] we obtain the following facts for warped products: the vertical Ricci curvature ρ is related to the Einstein constant ρ^F for F by

$$\rho u^2 + u \Delta_B u + (k-1)|\nabla u|^2 = \rho^F.$$

The horizontal Ricci curvatures satisfy

$$\text{Ric}_M(X, Y) = \text{Ric}_B(X, Y) - \frac{k}{u}(\text{Hess}_B u)(X, Y).$$

The extrinsic geometry of the leaves of \mathcal{F} are governed by

$$g(\nabla_V V, X) = -\frac{1}{u}g(X, \nabla u)g(V, V).$$

In particular,

$$\nabla_V \nabla u = \frac{|\nabla u|^2}{u}V.$$

The goal here is to show that q depends only on $\text{tr}Q$, where $q(A_1, A_2) = g(Q(A_1), A_2)$, and the Ricci curvatures of B and M .

We start by relating the elements in W to the warping function u .

Lemma 4.1. *For any $w \in W$ we have*

$$g(\nabla w, \nabla u) = \frac{|\nabla u|^2}{u} w.$$

Moreover on B_p , the horizontal leaf through p , we have $u_p = \frac{u}{u(p)}$.

Proof. First note that ∇u is basic and invariant under the group action G , and thus commutes with the Killing fields $v\nabla w - w\nabla v$. This shows

$$\begin{aligned} \nabla_{\nabla u}(v\nabla w - w\nabla v) &= \nabla_{v\nabla w - w\nabla v} \nabla u \\ &= \frac{|\nabla u|^2}{u} (v\nabla w - w\nabla v) \end{aligned}$$

but the left hand side is also

$$\nabla_{\nabla u}(v\nabla w - w\nabla v) = g(\nabla v, \nabla u) \nabla w - g(\nabla w, \nabla u) \nabla v.$$

As long as ∇v and ∇w are linearly independent this shows

$$g(\nabla w, \nabla u) = \frac{|\nabla u|^2}{u} w.$$

As $v, w \in W$ are arbitrary we have shown that this holds for all $w \in W$.

Next we claim that ∇u_p stays tangent to B_p . Let $w \in W_p$ then w vanishes on B_p . So for $X \in TB_p$ we have

$$\begin{aligned} D_X g(\nabla u_p, \nabla w) &= \text{Hess} u_p(X, \nabla w) + \text{Hess} w(X, \nabla u_p) \\ &= u_p q(X, \nabla w) + w q(\nabla u_p, X) \\ &= 0. \end{aligned}$$

As $g(\nabla u_p, \nabla w) = 0$ at p , this shows that $g(\nabla u_p, \nabla w) = 0$ on all of B_p . Next recall from Proposition 3.6 that

$$\nabla_V V \in \widehat{\mathcal{F}}.$$

In particular, it follows that $\nabla u \in \widehat{\mathcal{F}} \cap \mathcal{B}$. We clearly also have $\nabla u_p \in \widehat{\mathcal{F}} \cap \mathcal{B}$ so it follows that

$$\begin{aligned} \nabla u_p &= g(\nabla u, \nabla u_p) \frac{\nabla u}{|\nabla u|^2} \\ &= \frac{1}{u} u_p \nabla u. \end{aligned}$$

From which we get the last claim. \square

This lemma allows us to completely determine the horizontal structure of q .

Theorem 4.2. *On \mathcal{B} we have*

$$q|_{\mathcal{B}} = \frac{1}{k} (\text{Ric}_B - \text{Ric}_M) = \frac{1}{u} \text{Hess}_B u.$$

On the base space B , the quadratic form is given by

$$q_B = \frac{1}{u} \text{Hess}_B u.$$

Proof. We calculate on B_p and use the linear operator Q corresponding to q

$$\begin{aligned} Q(X) &= \frac{1}{u_p} \nabla_X \nabla u_p \\ &= \frac{1}{u} \nabla_X \nabla u \\ &= \frac{1}{k} (\text{Ric}_B - \text{Ric}_M)(X). \end{aligned}$$

The second statement follows as (B, g_B) is totally geodesic in M . \square

Next we turn our attention to the vertical structure of q .

Theorem 4.3. *Restricting q to the vertical fibers we have*

$$q|_{\mathcal{F}} = (\rho + \text{tr}Q) g|_{\mathcal{F}} = (\rho + \text{tr}Q) u^2 g_F.$$

Remark 4.4. Note that we cannot expect any more information given what happens when $\dim M = 1$ as $F = M$ in that case.

Proof. Start with $w \in W$, i.e.,

$$\nabla \nabla w = wQ.$$

The Weitzenböck formula for a gradient field ∇w states

$$\text{div} \nabla \nabla w = \nabla \Delta w + \text{Ric}(\nabla w)$$

which for our specific field reduces to

$$\text{div}(wQ) = \nabla(w \text{tr}Q) + \text{Ric}(\nabla w).$$

This implies

$$Q(\nabla w) + w \text{div}(Q) = \text{tr}Q \nabla w + w \nabla(\text{tr}Q) + \text{Ric}(\nabla w).$$

So

$$Q(\nabla w) = \text{Ric}(\nabla w) + \text{tr}Q \nabla w + w(-\text{div}(Q) + \nabla(\text{tr}Q)).$$

This tells us that Q is essentially determined by Ric , $\text{tr}Q$ and $\text{div}(Q)$ on $\widehat{\mathcal{F}}$. On \mathcal{F} we can be more specific. Let $p \in M$ and $w \in W_p$ then

$$Q((\nabla w)|_p) = \text{Ric}((\nabla w)|_p) + (\text{tr}Q)(\nabla w)|_p$$

showing that

$$Q|_{\mathcal{F}} = (\rho + \text{tr}Q) I|_{\mathcal{F}}. \quad \square$$

Finally we note that when $k > 1$ then q is completely determined by the vertical Ricci curvatures and the warping function.

Corollary 4.5. *When $k = 1$ we have*

$$\text{tr}(Q_B) = \frac{\Delta_B u}{u} = -\rho,$$

while if $k > 1$

$$\text{tr}Q = -\frac{1}{k-1} (k\rho + \text{tr}(Q_B)).$$

In particular, q is invariant under the action of G if $k > 1$.

Proof. Our formulas for $q|_{\mathcal{B}}$ and $q|_{\mathcal{F}}$ imply that

$$\mathrm{tr}Q = k(\rho + \mathrm{tr}Q) + \mathrm{tr}(Q_B)$$

and by definition

$$\mathrm{tr}(Q_B) = \frac{\Delta_B u}{u}.$$

Both statements follow immediately from this.

For the last statement note that both ρ and $\frac{\Delta_B u}{u}$ are invariant under G . \square

5. THE STRUCTURE OF W

For a given warped product structure coming from a specific subspace $W \subset W(M; q)$ define

$$\kappa = -\rho - \mathrm{tr}Q$$

and

$$(5.1) \quad \begin{aligned} \bar{\mu}(u) &= \kappa u^2 + |\nabla u|^2, \\ \bar{\mu}(u, z) &= \kappa uz + g(\nabla u, \nabla z). \end{aligned}$$

In this section we prove Theorem B for the subspace W . The argument is split into two cases. In Theorem 5.3 we prove the result when the singular set S is nonempty and in Theorem 5.4 we address $S = \emptyset$ case. In both cases we will see that the characteristic function τ of the fiber space F is equal to $\bar{\mu}(u)$.

We first simplify Proposition 2.3 by using Theorem 4.3.

Proposition 5.1. *Let $M = B \times_u F$ and assume that $uq_B = \mathrm{Hess}_B u$. Then $w \in W(M; q)$ if and only if there exist $z \in C^\infty(B)$ and $v \in C^\infty(F)$ such that*

- (1) $w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v)$,
- (2) $z \in W(B; q_B)$, and
- (3) $\mathrm{Hess}_F v + v\bar{\mu}(u)g_F = -\bar{\mu}(u, z)g_F$.

From just the first two conditions we can see that, if $W(B, q_B)$ is spanned by u , then the conclusion to Theorem B holds. However, this is not always the case as the following example shows.

Example 5.2. Let $B = (\mathbb{R}, dx^2)$ be the real line. Select $u : \mathbb{R} \rightarrow (0, \infty)$ and define $q_B = \frac{u''}{u} dx^2$ where we use $'$ for derivatives on the base. In this case we have $\dim W\left(B; \frac{u''}{u} dx^2\right) = 2$. Next choose a simply connected k -dimensional fiber space $(F, g_F, -\tau g_F)$, where τ is a constant when $k > 1$ or a merely a function on $F = \mathbb{R}$. The warped product

$$(M, g, q) = \left(\mathbb{R} \times F, dx^2 + u^2 g_F, \frac{u''}{u} dx^2 + \left((u')^2 - \tau \right) g_F \right)$$

has the property that

$$W(M; q) \supset \{ \pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\tau g_F) \}$$

and therefore has dimension $k+1$ or $k+2$. In the latter case the metric $dx^2 + u^2 g_F$ is forced to have constant curvature. So, as long as u , (F, g_F) , and τ are selected in such a way that the total space M doesn't have constant curvature we obtain

examples where $\dim W(M; q) = \dim M = k + 1$. In this case the precise condition for $\dim W(M; q) = k + 2$ is

$$\frac{\tau - (u')^2}{u^2} = \kappa = \kappa_B = -\frac{u''}{u}$$

or

$$\frac{\tau}{u^2} = -\left(\frac{u'}{u}\right)'.$$

Note, in particular, that this can never happen if τ is a non-constant function on $F = \mathbb{R}$.

Now we prove Theorem B in the case where there is a singular set.

Theorem 5.3. *Let (M, g) be complete and simply connected and $W \subset W(M; q)$ a subspace of dimension $k + 1 \geq 2$. When $S \neq \emptyset$, then we have*

$$W = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u)g_F)\}$$

and $\bar{\mu}(u)$ is constant on M .

Proof. We start by showing that any $w \in W$ is of the form $w = \pi_1^*(u) \cdot \pi_2^*(v)$. We know from Lemma 2.1 that

$$w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v).$$

So the goal is simply to show that z must be a multiple of u . Recall from Proposition 2.3 that $z \in W(B; q_B)$.

When $S \neq \emptyset$ we know that $w|_S = 0$ so it immediately follows that $z|_{\partial B} = 0$. Then Proposition 1.1 shows that $z = Cu$ for some constant C . We can now use Theorem 4.3 and argue as in Corollary 2.4 that

$$\begin{aligned} q_F &= q|_{\mathcal{F}} - |\nabla u|_B^2 g_F \\ &= -\left(\kappa u^2 + |\nabla u|^2\right) g_F \\ &= -\bar{\mu}(u) g_F \end{aligned}$$

defines a quadratic form on F . In particular $\bar{\mu}(u)$ is a function on F . This gives us the desired structure

$$W = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u)g_F)\}.$$

Finally we show that $\bar{\mu}(u)$ is constant on M . We can think of $p \in M - S$ as a pair of points $p = (x, y) \in \text{int}B \times F$. Thus $\bar{\mu}(u)(x, y)$ is constant in x for a fixed y . Letting $x \rightarrow x_0 \in \partial B = S$ and using that $\kappa = -\rho - \text{tr}Q$ is continuous on M then show that

$$\bar{\mu}(u)(x, y) = \kappa(x_0)u^2(x_0) + |\nabla u|^2|_{x_0} = |\nabla u|^2|_{x_0}.$$

Here the right hand side is clearly independent of y and so the left hand side must be as well. This shows that $\bar{\mu}(u)$ is constant on M . \square

When there is no singular set we have to work a little harder to prove the same result.

Theorem 5.4. *Let (M, g) be complete and simply connected. If $W \subset W(M; q)$ has dimension $k + 1 \geq 2$ and $S = \emptyset$, then $\bar{\mu}(u)$ is a function on F and a constant when $k > 1$. Moreover,*

$$W = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u)g_F)\}.$$

Proof. We start by showing that $\bar{\mu}(u)$ is constant when $k > 1$. The vertical part of the Ricci curvatures for a warped product implies in our case that

$$u^2 \rho = \rho_F - (u \Delta_B u + (k-1) |\nabla u|^2)$$

which can be reduced to

$$(\rho + \text{tr}(Q_B)) u^2 + (k-1) |\nabla u|^2 = \rho_F.$$

When $k > 1$ the relationship developed in Corollary 4.5 implies

$$\kappa = -\rho - \text{tr}Q = \frac{\rho + \text{tr}(Q_B)}{k-1}.$$

Thus

$$\bar{\mu}(u) = \kappa u^2 + |\nabla u|^2 = \frac{\rho_F}{k-1}$$

is constant.

We know from Lemma 2.1 that

$$(5.2) \quad w = \pi_1^*(z) + \pi_1^*(u) \cdot \pi_2^*(v) = z + u \cdot v$$

If we take the gradient of this equation at some point $p = (x, y) \in B \times F$ then

$$\nabla w = (\nabla z)|_x + v(y) (\nabla u)|_x + u(x) (\nabla v)|_y.$$

In this decomposition

$$(\nabla v)|_y \in \mathcal{F}$$

and

$$(\nabla z)|_x + v(y) (\nabla u)|_x \in \mathcal{B} \cap \widehat{\mathcal{F}}.$$

Thus it follows that

$$(\nabla z)|_x \in \mathcal{B} \cap \widehat{\mathcal{F}}$$

which in turn implies that $(\nabla z)|_x$ and $(\nabla u)|_x$ are linearly dependent for all $b \in B$.

Let $W_B \subset W(B; q_B)$ be the subspace spanned by u and all z that appear in equation (5.2). As these functions all have proportional gradients the foliation $\widehat{\mathcal{F}}_B$ on B defined by W_B has dimension at most 1 and consequently $\dim W_B \leq 2$.

If $\dim W_B = 1$, then $W_B = \text{span}\{u\}$ and so we always have:

$$w = \pi_1^*(u) \cdot \pi_2^*(v).$$

We can then use Proposition 5.1 and Corollary 2.4 to see that

$$\begin{aligned} q_F &= q|_{\mathcal{F}} - |\nabla u|_B^2 g_F \\ &= -\left(\kappa u^2 + |\nabla u|^2\right) g_F \\ &= -\bar{\mu}(u) g_F \end{aligned}$$

defines a quadratic form on F . In particular, $\bar{\mu}(u)$ is constant on the horizontal leaves and $v \in W(F; -\bar{\mu}(u) g_F)$. Moreover, when $k = 1$ we clearly have that $\dim W(F; -\bar{\mu}(u) g_F) = k + 1$ as $F = \mathbb{R}$, while if $k > 1$ $\bar{\mu}(u)$ is constant so Lemma 1.5 implies that $\dim W(F; -\bar{\mu}(u) g_F) = k + 1$. Thus

$$W = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u) g_F)\}.$$

If $\dim W_B = 2$, then Corollary 3.8 applied to the space $W_B \subset W(B; q_B)$ shows that

$$(B, g_B) = (H \times \mathbb{R}, g_H + dt^2).$$

Moreover the functions in W_B are constant on $H \times \{t_0\}$ for all $t_0 \in \mathbb{R}$. This shows that $u = u(t)$ and

$$W_B = \left\{ z \in C^\infty(\mathbb{R}) : z'' = z \frac{u''}{u} \right\}.$$

The functions without a z component

$$W^F = \{w \in W : w = \pi_1^*(u) \cdot \pi_2^*(v)\}$$

will then form a subspace of dimension k . In particular, it is nonempty and thus Proposition 5.1 shows that

$$\text{Hess}_F v = -v \bar{\mu}(u) g_F$$

for some v . This shows via Corollary 2.4 that $\bar{\mu}(u)$ defines a function on F and that

$$\begin{aligned} q_F &= q|_{\mathcal{F}} - |\nabla u|_B^2 g_F \\ &= -\left(\kappa u^2 + |\nabla u|^2\right) g_F \\ &= -\bar{\mu}(u) g_F \end{aligned}$$

defines a quadratic form on F . We can then argue as before that $\dim W(F; -\bar{\mu}(u) g_F) = k + 1$.

To reach a contradiction in this case select $w_1 = z + uv_1 \in W$ and $w_2 = uv_2 \in W^F - \{0\}$. Then

$$\begin{aligned} TF &\ni w_1 \nabla w_2 - w_2 \nabla w_1 \\ &= (z + uv_1)(u \nabla v_2 + v_2 \nabla u) - uv_2(\nabla z + u \nabla v_1 + v_1 \nabla u) \\ &= v_2(z \nabla u - u \nabla z) + u^2(v_1 \nabla v_2 - v_2 \nabla v_1) + zu \nabla v_2. \end{aligned}$$

Since $\nabla v_1, \nabla v_2 \in TF$ it follows that $z \nabla u - u \nabla z = 0$ as v_2 is non-trivial. But this shows that z is a multiple of u contradicting that $\dim W_B = 2$. \square

Finally we show that we cannot expect $\bar{\mu}(u)$ to be constant unless we are in the situations covered by Theorems 5.3 and 5.4.

Example 5.5. Since M is assumed to be simply connected, Theorems 5.3 and 5.4 show that the only case where $\bar{\mu}(u)$ might not be constant is when $F = \mathbb{R}$. We can construct examples of this type as follows. Fix a base manifold (B, g_B) with a positive function $u : B \rightarrow (0, \infty)$ such that $\dim W(B; \frac{1}{u} \text{Hess}u) = 1$. Let

$$M = B \times_u \mathbb{R}, g = g_B + u^2 dt^2$$

and define

$$q = \frac{1}{u} \text{Hess}_B u + \left(|\nabla u|^2 - \tau\right) dt^2$$

where $\tau : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth function. This gives us a complete collection of examples where $F = \mathbb{R}$ and $\dim W(M; q) = 2$.

6. MISCELLANEOUS RESULTS

In this section we present some related results. In subsection 6.1 we study the base space in more detail. In subsection 6.2 we consider the situation where there is a group of isometries that leaves the quadratic form invariant. This allows us to extend our results to the case where M might not be simply-connected.

6.1. Base Space Structure. We start by extending the definition of $\bar{\mu}$ in (5.1) to all of W :

$$\bar{\mu}(w) = \kappa w^2 + |\nabla w|^2.$$

Proposition 6.1. *If $w = \pi_1^*(u) \cdot \pi_2^*(v) \in W$, then*

$$\nabla \left(\kappa w^2 + |\nabla w|^2 \right) = \frac{w^2}{u^2} \nabla \left(\kappa u^2 + |\nabla u|^2 \right)$$

and

$$\kappa w^2 + |\nabla w|^2 = \bar{\mu}(u) v^2 + |\nabla v|_F^2$$

is a constant on M when $k > 1$ or $S \neq \emptyset$.

Proof. The first equality follows from the calculation

$$\begin{aligned} \nabla \left(\kappa w^2 + |\nabla w|^2 \right) &= w^2 \nabla \kappa + 2\kappa w \nabla w + 2wQ(\nabla w) \\ &= w^2 \nabla \kappa + 2w\kappa g(\nabla w, \nabla u) \frac{\nabla u}{|\nabla u|^2} + 2w \frac{g(\nabla w, \nabla u)}{|\nabla u|^2} Q(\nabla u) \\ &= \frac{w^2}{u^2} \left(u^2 \nabla \kappa + 2\kappa u \nabla u + 2uQ_B(\nabla u) \right) \\ &= \frac{w^2}{u^2} \nabla \left(\kappa u^2 + |\nabla u|^2 \right). \end{aligned}$$

For the second note that if $w = \pi_1^*(u) \cdot \pi_2^*(v)$ for $v \in W(F; -\bar{\mu}(u)g_F)$ then

$$\kappa w^2 + |\nabla w|^2 = \bar{\mu}(u) v^2 + |\nabla v|_F^2$$

defines a function on F . In the case when $\bar{\mu}(u)$ is constant on M , i.e., $k > 1$ or $S \neq \emptyset$, we have

$$\nabla |\nabla v|_F^2 = 2\nabla_{\nabla v} \nabla v = -2v\bar{\mu}(u) \nabla v$$

which shows that $\kappa w^2 + |\nabla w|^2$ is constant on F and thus on M . \square

We saw in Proposition 5.1 that it is also necessary to compute $\bar{\mu}(z)$ even though z is not an element in W .

Proposition 6.2. *If $\dim W(M; q) = k + 1 \geq 2$ and $\dim W(B; q_B) \geq 2$, then*

$$\nabla \bar{\mu}(z, z) = \frac{z^2}{u^2} \nabla \bar{\mu}(u) + 2\frac{z}{u} (\kappa - \kappa_B) (u \nabla z - z \nabla u)$$

and

$$\nabla \bar{\mu}(u, z) = \frac{z}{u} \nabla \bar{\mu}(u) + (\kappa - \kappa_B) (u \nabla z - z \nabla u)$$

where κ_B is the κ defined on B using $W(B; q_B)$.

Proof. Let $z \in W(B; q_B)$ and $K = u \nabla z - z \nabla u$ be the corresponding Killing field. Then we have

$$\begin{aligned} \nabla \bar{\mu}(u, z) &= (\nabla \kappa) uz + \kappa (\nabla u) z + \kappa u \nabla z + \nabla_{\nabla z} \nabla u + \nabla_{\nabla u} \nabla z \\ &= (\nabla \kappa) uz + 2\kappa (\nabla u) z + \kappa K + 2\frac{z}{u} \nabla_{\nabla u} \nabla u + \frac{1}{u} \nabla_K \nabla u \\ &= \frac{z}{u} \nabla (\bar{\mu}(u)) + \kappa K + Q_B(K) \\ &= \frac{z}{u} \nabla (\bar{\mu}(u)) + \kappa K - \kappa_B K \\ &= \frac{z}{u} \nabla (\bar{\mu}(u)) + (\kappa - \kappa_B) (u \nabla z - z \nabla u). \end{aligned}$$

Similarly we have

$$\nabla \left(\kappa z^2 + |\nabla z|^2 \right) = \frac{z^2}{u^2} \nabla \left(\kappa u^2 + |\nabla u|^2 \right) + 2 \frac{z}{u} (\kappa - \kappa_B) (u \nabla z - z \nabla u)$$

which shows the desired identities. \square

Corollary 6.3. *Assume that M is simply connected such that $\dim W(M; q) = k + 1 \geq 2$ and $S = \emptyset$. If $\bar{\mu}(u)$ is constant and $\dim W(B; q_B) \geq 2$, then $\kappa \neq \kappa_B$.*

Proof. In case $\bar{\mu}(u)$ is constant and $\kappa = \kappa_B$ the above Proposition 6.2 implies that $\bar{\mu} = \bar{\mu}_B$ defines a quadratic form on $W(B; q_B)$. If $\dim W(B; q_B) \geq 2$ it is then possible to find $z \in W(B; q_B) - \text{span}\{u\}$ such that $\bar{\mu}(u, z) = 0$. Proposition 5.1 then implies that $w = z + uv \in W$ when $v \in W(F; -\bar{\mu}(u) g_F)$. But that contradicts Theorem 5.4. \square

6.2. Invariant Groups. We start by checking how such isometries interact with elements of $W(M; q)$.

Proposition 6.4. *If $h \in \text{Iso}(M)$ and $h^*q = q$, then $h^* : W(M; q) \rightarrow W(M; q)$ preserves the characteristic constant/function $\bar{\mu}$.*

Proof. Let $w \in W(M; q)$. Since h is an isometry we have

$$\begin{aligned} \text{Hess}(w \circ h) &= h^*(\text{Hess}w) \\ &= (w \circ h)(h^*q) \\ &= (w \circ h)q. \end{aligned}$$

This shows that $h^* : W(M; q) \rightarrow W(M; q)$. Next we note that

$$\begin{aligned} \bar{\mu}(w \circ h) &= (\kappa \circ w)(w \circ h)^2 + |\nabla(w \circ h)|^2 \\ &= (\kappa \circ w)(w \circ h)^2 + |Dh^{-1}((\nabla w) \circ h)|^2 \\ &= (\kappa \circ w)(w \circ h)^2 + |(\nabla w) \circ h|^2 \\ &= \bar{\mu}(w) \circ h \end{aligned}$$

which shows that $\bar{\mu}$ is preserved by h . \square

Let $\Gamma \subset \text{Iso}(M, g)$ be a subgroup that preserves the quadratic form q . Define

$$W(M; q, \Gamma) = \{w \in W(M; q) : w \circ h = w, \text{ for all } h \in \Gamma\} \subset W(M; q)$$

as the fixed point set of the action of Γ on $W(M; q)$. In this case Γ will preserve the foliations \mathcal{F} and $\widehat{\mathcal{F}}$ defined by the subspace $W(M; q, \Gamma)$ since

$$Dh^{-1}((\nabla w)|_h) = \nabla(w \circ h) = \nabla w.$$

Thus Γ preserves the distributions and fixes u_p . In particular, it induces an action on B that leaves u as well as q_B invariant.

The next result follows almost directly from Theorems 5.3 and 5.4.

Proposition 6.5. *Let (M, g) be complete and simply connected and assume that $\dim W(M; q, \Gamma) = k + 1 \geq 2$. If $k > 1$ or $S \neq \emptyset$, then $\bar{\mu}(u)$ is constant on M , and*

$$W(M; q, \Gamma) = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u) g_F)\}.$$

Furthermore, Γ acts trivially on F .

Moreover, if Γ acts properly and only has principal isotropy, then

$$(M/\Gamma, g) = (B/\Gamma) \times_u F$$

and

$$\pi^*(W(M/\Gamma; q)) = W(M; q, \Gamma)$$

where $\pi : M \rightarrow M/\Gamma$ is the quotient map.

Proof. When $S \neq \emptyset$ we can immediately apply Theorem 5.3 with $W = W(M; q, \Gamma)$ and so the only item left to check is that Γ acts trivially on F . We note that Γ must leave all elements in $W(F; -\bar{\mu}(u)g_F)$ invariant. However, a close inspection of all cases shows that the only situation where a nontrivial subgroup of $\text{Iso}(F, g_F)$ fixes all elements in $W(F; -\bar{\mu}(u)g_F)$ is when $F = \mathbb{R}$ and $\bar{\mu}(u) > 0$ so that all solutions are periodic. However, that case has been eliminated by our assumptions.

When $S = \emptyset$ we are in the case of Theorem 5.4 and the same proof we just gave works since

$$W(M; q, \Gamma) = \{\pi_1^*(u) \cdot \pi_2^*(v) : v \in W(F; -\bar{\mu}(u)g_F)\}.$$

The last statement when there is only principal isotropy follows easily from previous ones. \square

From this we can extract information about the case where M isn't simply connected. In that case we obtain a covering map $\pi : \widetilde{M} \rightarrow M$ and can think of $\Gamma = \pi_1(M)$ as acting by isometries on the universal covering \widetilde{M} . Moreover this action will clearly preserve the pull back of any quadratic form on M .

Corollary 6.6. *Assume that (M, g) is complete and that q is a quadratic form on M . If $\dim W(M; q) = k + 1 \geq 2$, then*

$$(M, g) = (B/\Gamma) \times_u F$$

where the universal covering has a warped product splitting $\widetilde{M} = B \times_u F$ coming from $W(M; q, \Gamma)$ with $\Gamma = \pi_1(M)$, provided $F \neq \mathbb{R}$. Moreover,

$$\pi^*(W(M; q)) = W(\widetilde{M}; q, \Gamma).$$

Remark 6.7. We now discuss the special case when $F = \mathbb{R}$. Note that this case is nontrivial even when the characteristic function is constant. The case where it isn't constant does happen and is related to the problem of finding coexisting solutions to Hill's equation. Specifically, it is possible to choose $\tau(t)$ as in Example 5.5 to be periodic with period 2π and such that the solutions space $W(\mathbb{R}; -\tau dt^2)$ consists of 2π periodic functions (see [MW, Chapter 7].) This gives us non-simply connected examples of the form $M = B \times_u \mathbb{S}^1$ with $\dim W(M; q) = 2$.

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