

Syracuse University

SURFACE

Physics - Dissertations

College of Arts and Sciences

5-2012

Physics on Noncommutative Spacetimes

Pramod Padmanabhan
Syracuse University

Follow this and additional works at: https://surface.syr.edu/phy_etd



Part of the [Physics Commons](#)

Recommended Citation

Padmanabhan, Pramod, "Physics on Noncommutative Spacetimes" (2012). *Physics - Dissertations*. 122.
https://surface.syr.edu/phy_etd/122

This Dissertation is brought to you for free and open access by the College of Arts and Sciences at SURFACE. It has been accepted for inclusion in Physics - Dissertations by an authorized administrator of SURFACE. For more information, please contact surface@syr.edu.

Abstract

The structure of spacetime at the Planck scale remains a mystery to this date with a lot of insightful attempts to unravel this puzzle. One such attempt is the proposition of a ‘pointless’ structure for spacetime at this scale. This is done by studying the geometry of the spacetime through a noncommutative algebra of functions defined on it. We call such spacetimes ‘noncommutative spacetimes’. This dissertation probes physics on several such spacetimes. These include compact noncommutative spaces called fuzzy spaces and noncompact spacetimes. The compact examples we look at are the fuzzy sphere and the fuzzy Higg’s manifold. The noncompact spacetimes we study are the Groenewold-Moyal plane and the $\mathcal{B}_{\chi\bar{n}}$ plane.

A broad range of physical effects are studied on these exotic spacetimes. We study spin systems on the fuzzy sphere. The construction of Dirac and chirality operators for an arbitrary spin j is studied on both S_F^2 and S^2 in detail. We compute the spectrums of the spin 1 and spin $\frac{3}{2}$ Dirac operators on S_F^2 . These systems have novel thermodynamical properties which have no higher dimensional analogs, making them interesting models.

The fuzzy Higg’s manifold is found to exhibit topology change, an important property for any theory attempting to quantize gravity. We study how this change occurs in the classical setting and how quantizing this manifold smoothens the classical conical singularity. We also show the construction of the star product on this manifold using coherent states on the noncommutative algebra describing this noncommutative space.

On the Moyal plane we develop the LSZ formulation of scalar quantum field theory. We compute scattering amplitudes and remark on renormalization of this theory. We show that the LSZ formalism is equivalent to the interaction representation formalism for computing scattering amplitudes on the Moyal plane. This result is true for on-shell Green’s functions and fails to hold for off-shell Green’s functions.

With the present technology available, there is a scarcity of experiments which directly involve the Planck scale. However there are interesting low and medium energy experiments which put bounds on the validity of established principles which are thought to be violated at the Planck scale. One such principle is the Pauli principle which is expected to be violated on noncommutative spacetimes. We introduce a noncommutative spacetime called the $\mathcal{B}_{\chi\bar{n}}$ plane to show how transitions, not obeying the Pauli principle, occur in atomic systems. On confronting with the data from experiments, we place bounds on the noncommutative parameter.

PHYSICS ON NONCOMMUTATIVE SPACETIMES

By

Pramod Padmanabhan
B.Sc.(Hons.), Chennai Mathematical Institute, 2007

DISSERTATION

Submitted in partial fulfillment of the requirements for the
degree of Doctor of Philosophy in Physics
in the Graduate School of Syracuse University

May 2012

Copyright 2012 Pramod Padmanabhan

All rights reserved

Contents

Acknowledgments	ix
Introduction	x
1 Construction of Fuzzy Spaces and Star Products	1
1.1 Obtaining S_F^2	1
1.1.1 Quantizing the Classical Poisson Structure on S^2	1
1.1.2 Using Harmonic Oscillators	2
1.1.3 From the Equation of S^2 as an embedding in \mathbb{R}^3	4
1.2 Star Products on Fuzzy Spaces	5
1.2.1 Properties of Coherent States(CS)	7
1.2.2 Properties of Star Products	7
1.2.3 The *-Products for S_F^2	8
1.2.4 The CS *-Product on S_F^2	8
1.2.5 Higgs Algebra and its Representation	11
1.2.6 The Higgs Algebra Coherent States	12
1.2.7 Normalization	13
1.2.8 Resolution of Identity	14
1.2.9 The Star Product on the Higg's Manifold	14
2 Spin Systems on S_F^2	16
2.1 The Algebra of S_F^2	16
2.2 The GW Algebra	17
2.3 The Spin System from the Fuzzy GW Algebra	18
2.4 The Dirac and Chirality Operators on S^2	18
2.4.1 ${}^\infty D^j$ and ${}^\infty \gamma^j$ for the Spin $\frac{1}{2}$ Case	18
2.4.2 ${}^\infty D^j$ and ${}^\infty \gamma^j$ on S^2 for the Spin 1 Case	19
2.5 More Dirac Operators on S^2	20
2.5.1 Case 1: $a, b, c \in \mathbb{R}$	21
2.5.2 Case 2: $a, b, c \in \mathbb{C}$	25
2.5.3 Remarks for the Dirac Operators for Higher Spins	26
2.6 Construction of the Fuzzy Spin $\frac{1}{2}$ Dirac and Chirality Operators on S_F^2	26
2.7 Ambiguities in the Fuzzy Spin $\frac{1}{2}$ Dirac and Chirality Operators	27
2.8 The Fuzzy Spin 1 Dirac Operator	28
2.9 Construction of Higher Spin Dirac Operators on S_F^2	30
2.10 Rules for finding the Fuzzy Dirac Operator on S_F^2 for any Spin j	33
2.10.1 Half-Integral Spins	33
2.10.2 Integral Spins	34

2.11	Unitary Inequivalence of the Fuzzy Spin j Dirac Operators	34
2.11.1	Spin 1	34
2.11.2	Spin $\frac{3}{2}$	36
3	Spectrum of Higher Spin Dirac Operators on S_F^2	39
3.1	Analytic Results of the Spectrums of the Spin $\frac{1}{2}$ and Spin 1 Dirac Operators . .	39
3.1.1	Number of Positive Eigenvalues and Zero Modes for the Spin 1 Dirac Operator	41
3.1.2	Remarks on the other two Dirac Operators for the Spin 1 Case	42
3.2	Numerical Results	42
3.2.1	Spectrum of the Traceless Spin 1 Dirac Operator	42
3.2.2	Fit 1	43
3.2.3	Fit 2	46
3.2.4	Scaling Property of the Spectrum of the Spin 1 Dirac Operator	48
3.3	Spectrum of the Spin $\frac{3}{2}$ Dirac Operator	49
4	Thermodynamics of the Spin Systems on S_F^2	50
4.1	The Partition Function and the Mean Energy	51
4.2	Reasons for the Strange Behavior	53
4.3	Zero Modes of the Spin 1 Dirac Operator	54
4.4	Specific Heats of the Two Systems	55
4.5	Entropies of the Two Systems	56
4.6	Deviations from the Ideal Gas Law	56
5	Topology Change through Fuzzy Physics	59
5.1	The Higgs Manifold	59
6	Lehmann Symanzik Zimmermann Formulation of Quantum Field Theory on the Groenewold-Moyal Plane	61
6.1	Twisted Relativistic Quantum Fields on the Moyal plane \mathcal{A}_θ	62
6.2	The Untwisted and Twisted LSZ Reduction Formula	64
6.2.1	The $\theta_{\mu\nu} = 0$ case	65
6.2.2	The $\theta_{\mu\nu} \neq 0$ case	65
6.3	Non-perturbative Computations of the Scattering Amplitudes	66
6.3.1	Method 1	66
6.3.2	Method 2	67
7	Non-Pauli Effects on the $\mathcal{B}_{\chi\vec{n}}$ Plane	73
7.1	The Spacetime $\mathcal{B}_{\chi\vec{n}}$	74
7.2	The Electronic States of Be	77
7.2.1	<i>The Two-Electron Ground State</i>	78
7.2.2	<i>The Two-Electron Excited State</i>	78
7.3	The Non-Pauli Rate	79
7.4	Experiments and Bounds on χ	82
7.4.1	Bounds from The Borexino Experiment	82
7.4.2	Bounds from The Kamiokande Detector	82
7.4.3	Bounds from The NEMO Experiment	82
7.4.4	Bounds from Experiments at Maryland	83

7.4.5	Bounds from The VIP Experiment	83
7.5	Time Quantization	83
Conclusions		85
A	Derivation of the Star Product on the Fuzzy Higg's Manifold	88
B	Proof of the Existence of Traceless Dirac Operators for an Integer Spin j	92
C	Review of Hopf Algebras	95
C.0.1	The Group Algebra $\mathbb{C}G$ of a Group G	96
D	Braid Group and Yang-Baxter Equations	99
D.0.2	Identical Particles and Statistics	99
D.0.3	The Braid Group and Yang-Baxter Equations	101
Bibliography		104

List of Tables

2.1	Traces of the 3 Spin 1 Dirac Operators	36
7.1	Bounds on the noncommutativity parameter χ	83

List of Figures

3.1	Plot of the energy eigenvalues along with the best fit curves for $N = 21$, $N = 35$ and $N = 45$	43
3.2	Fit for the parameter α	45
3.3	Fit for the parameter β	45
3.4	Fit for the parameter η	46
3.5	Plot predicting the energy eigenvalues on ${}^F D_3$ for $N = 60$	47
3.6	E^2 Vs j for the spectrum of the spin 1 Dirac operator.	47
3.7	Plot showing the scaling property of the spectrum of the spin 1 Dirac operator.	48
3.8	Plot showing the scaling property of the spectrum of the spin $\frac{3}{2}$ Dirac operator.	49
4.1	The mean energies of the spin 1 and spin $\frac{1}{2}$ systems.	52
4.2	The crossing of the mean energy curves of the spin 1 and spin $\frac{1}{2}$ systems.	52
4.3	Energy as a function of the degeneracy g for the spin $\frac{1}{2}$ and spin 1 systems.	53
4.4	Ratio of specific heats $\frac{C_v^{\frac{1}{2}}}{C_v^1}$	56
4.5	Deviation from ideal gas law for the spin $\frac{1}{2}$ system.	58
4.6	Deviation from the ideal gas law for the spin 1 system.	58
5.1	Surface plots depicting the change in topology.	60

Acknowledgments

I would like to thank my teacher, friend and advisor Prof. A. P. Balachandran for his constant encouragement, support and patience during these years. I thank him for the freedom he gave me in developing his ideas.

I am greatly indebted to my friends and collaborators in India. I especially thank Prof. T. R. Govindarajan and Prof. Sanatan Digal for letting me visit The Institute of Mathematical Sciences, Chennai several times during the last few years.

Thanks are also due to Prof. Sachin Vaidya and Prof. Kumar Gupta for all their guidance. I would like to express my gratitude to Prof. Amilcar R. de Queiroz for all the discussions we had in Brasil and India.

I also wish to thank my high school teacher, Ravishankar and Prof R. Ananthan of the Physics Society, Chennai. Their enthusiasm in the working of nature has been a great source of inspiration. It is also a pleasure to thank my friends Aditya, Anand, Bhanukiran, Koushik, Krishna and Mohan for all the numerous hours we spent understanding physics and math during the last ten years.

Finally I thank my parents and my sister for helping me pursue my dreams with all the love and support.

Introduction

The question of the structure of spacetime has haunted humans for a long time and it would not be wrong to add that this is still unresolved. Early ideas on the geometry of space emerged from the time of the Greeks especially Euclid whose work in *Elements* [1] gave geometry a closed and complete form. His geometry was based on laws confined to what we now understand as flat space, perhaps the only thing he could perceive at that time. Moreover time had not featured in his studies, but ideas of which were nevertheless being developed in parallel through observations of periodicity in the cosmos. The ideas of spacetime showed up first in physical theories. These can be traced back to Galileo who still considered them as distinct entities. The unified concept of spacetime arose much later and much progress was made during the early part of the twentieth century. Besides unifying time with space, the nineteenth century saw rapid developments in non-Euclidean geometry and Riemannian geometry which will play a crucial role in geometrical theories of gravitation.

The other natural problem humans were intrigued by is the one on the structure of matter. History records that this question was also studied by the Greeks. Progress on the modern understanding of this question took place in the last century with the development of quantum field theory. It should also be mentioned that group theory played an essential role in constructing many of the known quantum field theories which explain matter. These symmetry principles are the ones also used to classify elementary particles we see in particle detectors today. It was thought that the principles of quantum field theory (qft) would be enough to explain the workings of nature from the most fundamental level. However a known force namely that of gravitation could not be confined to this general framework which helped describe the other known forces of nature. This can be looked at as the problem of length(or mass in Planck units) scales and our failure to understand how the large merges with the small. This is the problem of quantum gravity.

This problem occurs at the Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616252(81) \times 10^{-35} \text{ m} \quad (1)$$

where \hbar is the Planck's constant, G is the universal gravitational constant and c is the speed of light. An important issue is the structure of spacetime at such length scales. A speculation is that spacetime becomes noncommutative at these scales. We will explain what it means for spacetime coordinates not to commute but before that we look at how they arise in physics.

The earliest ideas on noncommutative geometry were indicated in a letter from Heisenberg to Peierls [2] in 1930. It included the idea of an effective ultraviolet (UV) cut-off in quantum

field theories. The initial papers on this subject was by Snyder [3] and Yang [4] in the late forties.

Interest in these theories diminished due to the successes of more conservative approaches to the problem like the renormalization program which appeared at around the same time. Later, von Neumann inspired by the foundations of quantum mechanics has initiated studies on describing quantum space in algebraic terms, and named his studies as “pointless geometry” since the notion of a point is lost due to the Heisenberg uncertainty relation. In the 1980’s Connes, Drinfel’d, Rieffel, Woronowicz and other mathematicians influenced by the earlier work of von Nuemann started to develop an operator algebraic theory, generalizing the notion of differential structure to the noncommutative framework, which is now commonly referred to as “noncommutative geometry”.

Noncommutative geometries arise as certain limits in string theory [5]. They occur in the low energy descriptions of a class of brane configurations that have a non-zero B_{NS} -field turned on. Noncommutative manifolds also make their appearance in discussions of brane configurations in external Ramond-Ramond(RR) fields [6].

Noncommutative field theory is also known to appear in condensed matter theory. The well known example of this occurrence is in the theory of electrons in a magnetic field projected to the lowest Landau level as this can naturally be thought of as a noncommutative field theory. Thus these ideas are relevant to the quantum Hall effect [7, 8, 9, 10].

An argument for why spacetime at Planck scales become noncommutative was given by Doplicher, Fredenhagen and Roberts [11]. This argument combines Heisenberg’s uncertainty principle and Einstein’s theory of gravity. It goes as follows. In order to probe physics at the Planck scale we need a probe whose Compton wavelength satisfies

$$\frac{\hbar}{Mc} \leq l_P \quad (2)$$

where l_P is the Planck length given in Eq.(1) and M is mass of the probe. This implies that

$$M \geq \frac{\hbar}{l_P c} \quad (3)$$

which upon substitution of values in SI units turns out to be the Planck mass, $M \approx m_P = 2.17645 \times 10^{-8}$ kg. The Schwarzschild radius for this mass is around 3.21×10^{-35} m which is of the order of l_P . Thus such high masses in small volumes will cause black hole horizons to form. This suggests a fundamental length limiting spatial localization indicating *space-space noncommutativity*. Similar arguments can be made about *time-space noncommutativity*. Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta t \Delta |\vec{x}| \geq l_p^2, \quad l_P = \text{a fundamental length scale.} \quad (4)$$

The essential idea of what it means for spacetime or rather geometry to be noncommutative can be explained in simple terms as follows. Let \mathcal{M} be a manifold and \mathcal{A} be the commutative C^* -algebra of continuous functions on \mathcal{M} . A result due to Gel’fand and Naimark states that it is possible to recover the topology of \mathcal{M} from \mathcal{A} alone. In light of this theorem a noncommutative manifold can be obtained if the commutative C^* -algebra is deformed to a noncommutative C^* -algebra using a parameter which goes to the commutative algebra in the appropriate limit.

The Groenewold-Moyal(GM) plane is the simplest example of such a noncommutative algebra and it also models spacetime uncertainties given in the arguments of Doplicher, Fredenhagen

and Roberts. It is given by

$$[\hat{x}_\mu, \hat{x}_\nu]_\star = i\theta_{\mu\nu}, \quad \mu, \nu = 0, 1, \dots, d \quad (5)$$

where \hat{x}_μ are coordinate functions acting on the algebra of smooth functions on \mathbb{R}^{d+1} denoted by $\mathcal{A}(\mathbb{R}^{d+1})$ and d is the number of spacetime dimensions.

In this thesis we will study physics on several such noncommutative spacetimes and mention the generic nature of such effects. Our studies broadly classifies these noncommutative spacetimes into two kinds: compact noncommutative spacetimes and non-compact noncommutative spacetimes.

The compact noncommutative spaces are called fuzzy spaces. Physics on such spacetimes can be looked at as matrix models. The reason is the noncommutative algebra of functions over them has a sequence of finite-dimensional irreducible representations with increasing dimensions, tending to the commutative algebra of functions in the infinite limit. The examples of such spacetimes we will encounter in this dissertation are the fuzzy 2-sphere (S_F^2) [12] given by

$$[X_i, X_j] = \frac{2}{n} i\epsilon_{ijk} X_k \quad (6)$$

where n is a parameter and the Higg's algebra given by

$$[X_+, X_-] = C_1 Z + C_2 Z^3, \quad [X_\pm, Z] = \pm X_\pm \quad (7)$$

where C_1 and C_2 are two parameters and $X_\pm = X_1 \pm iX_2$. It can be seen that S_F^2 can be got as a special case of the Higg's algebra. In this sense the Higg's algebra is a non-linear deformation of S_F^2 . Note that we have not included time coordinates in these models as the spaces are compact and we avoid unphysical spacetimes. They can be thought of as time-slices. The special feature of fuzzy spaces is that they maintain the underlying symmetries of the commutative theory. This is in contrast to familiar lattice discretization methods where these symmetries are not retained. Lattice discretization methods are effective in studying theories with strong coupling which remains to be studied in the fuzzy cases. The topology and differential geometry in lattice theories are not very rich resulting in serious limitations to the study of finite energy topological excitations on these spaces like solitons, monopoles and instantons. These are overcome in fuzzy spaces given their ability to reflect topology better than the lattice counterparts.

An obvious limitation to fuzzy spaces is that they have to even dimensional as these spaces are obtained by quantizing phase spaces. For doing physics on such spaces we need to extend the concept of Laplacians and Dirac operators to these spaces. This turns not always to be possible limiting the number of such spaces we can work on. In this thesis our focus will be on studying Dirac operators on S_F^2 and the new features they give rise to. In particular we see that we can avoid the fermion doubling problem and show the existence of chirality operators on these spaces.

Fuzzy spaces also exhibit topology change as will be shown using the Higg's algebra. Field theories on such spaces are yet to be explored. However as a preliminary to such explorations, star products are studied on these spaces using the general theory of star products on fuzzy spaces using coherent states.

The other set of noncommutative spacetimes are the noncompact ones. These are not limited to just time-slices and so represent the noncommutative version of the entire commutative spacetime. The GM plane in Eq.(5) is a well studied example of such a spacetime. Physical models on these spacetimes are non-local and provide examples of Lorentz non-invariant theories. This immediately affects the connection between causality and statistics hinting at

possible violations of the Pauli-exclusion principle on such spacetimes. Though this is believed to be a generic feature of such spacetimes we formulate a theory exhibiting this effect on a new noncommutative spacetime called the $\mathcal{B}_{\chi\vec{n}}$ algebra. Its noncommutative algebra is defined by the relations

$$[x_0, x_i] = i\chi\epsilon_{ijk}x_jn_k, \quad (8)$$

$$[x_i, x_j] = 0, \quad i, j = 1, 2, 3 \quad (9)$$

where χ is the noncommutative parameter and n_i are the components of an arbitrary vector in space.

Several approaches to qfts on the GM plane have been made in the past few years [13]. The particular method used in this thesis is that of twisted Poincaré invariance of the GM plane. Due to this the usual statistics operator has to be a twisted to a new one to be compatible with the new symmetry. However in this case the symmetry is not that of a group but a Hopf algebra. A recurring theme in this thesis is that of a Hopf algebra being the symmetry of a noncommutative spacetime.

This dissertation is divided into two parts. The first part deals with compact noncommutative spaces namely fuzzy spaces. The following is a short preview of these topics along with its organization.

Construction of fuzzy spaces and star products: The different ways of obtaining the fuzzy 2-sphere are shown along with a construction which helps obtain other fuzzy spaces like the Higg's algebra which cannot be got from the standard methods. These will be explained in the first section. In the following section we define the noncommutative algebra of functions on these spaces and show the construction of their star products. The associative star product of functions on fuzzy spaces is an important tool in understanding the differential geometric structure and the continuum limits of noncommutative spaces. The star product is defined with the help of the symbol map which maps the algebra of operators on a Hilbert space to the algebra of functions on a manifold \mathcal{M} . This product is no longer the point-wise product of functions of this algebra but instead the *-product induced from the symbol map. We review the properties of these star products and show the construction of the star product on S_F^2 which are now well known. We then construct the star product for the Higg's algebra using coherent states [14].

Spin systems on S_F^2 : This chapter develops the formulation of spin $\frac{1}{2}$ Dirac operators and higher spin Dirac-like operators on S_F^2 [15]. This construction is carried out with a generalization of the Ginsparg-Wilson(GW) algebras on S_F^2 . These algebras were originally introduced in fuzzy spaces to solve the fermion-doubling problem. These operators can be thought of providing analogs of higher spin equations on S_F^2 . This is because a spin j Dirac operator acts on $Mat(2L+1) \otimes \mathbb{C}^{2j+1}$ where $Mat(2L+1)$ is the noncommutative algebra of square matrices of size $2L+1$ describing S_F^2 for a cutoff L . This implies that the operator acts on a complex vector bundle, the sections of which are the wave functions of a spin j particle. It is shown that these operators can be constructed from projectors to these spaces which helps realize a theorem due to Serre and Swan, that all such sections are obtained from projective modules.

There exist in the literature several other constructions of the spin $\frac{1}{2}$ Dirac operator on S_F^2 [16, 17]. However it is not clear how these constructions can be extended to higher spin Dirac-like operators. On the other hand the construction using GW algebras provide a systematic method to construct higher spin Dirac-like operators. In particular for a given spin j we have $2j+1$ Dirac-like systems. These Dirac-like operators come paired with anticommuting chirality operators, thus establishing a whole class of spin systems on S_F^2 . It is shown that in each of these $2j+1$ spin systems, there are $2j+1$ chirality operators and there are an innumerable number of

Dirac-like operators corresponding to each of these chirality operators. It is a remarkable fact that these systems have no higher dimensional Minkowskian analogs.

Computation of the spectrum of spin 1 Dirac operator on S_F^2 : The analytic computation of the spectrum of these operators turns out to be a difficult task as will be explained. They are however amenable to numerical studies. The numerical results for the spectrum of the spin 1 Dirac operator show interesting behavior [18]. It is worth mentioning that the spectrum for different values of the cut-off N is similar to a parabola or a quadratic behavior. It is also shown that the Dirac operators corresponding to the different chirality operators are unitarily inequivalent. This is seen by computing the traces of these operators which are found to be different.

Having studied the pattern of the spectrum in the spin 1 case we also take a look at spectrum of the spin $\frac{3}{2}$ Dirac operator. Various similarities between the two spectrums emerge as a result of numerical studies. A striking pattern, though not very surprising, is that the highest eigenvalues of these operators go as the cut-off L .

An obvious limitations to these studies is availability of computational resources. The sizes of the matrices involved in the computation of the spectrum of a spin j operator is $(2j+1) \times (2L+1)^2$ by $(2j+1) \times (2L+1)^2$ which get very large for large values of the cutoff L . Preliminary studies of the first three spin values show stark similarities hinting at an universal behavior of the spectrums of higher spin Dirac-like operators. However at the moment we leave them as interesting conjectures.

Physics of the spin systems: The thermodynamical properties of the spin systems are studied in detail [19]. We restrict ourselves to the spin $\frac{1}{2}$ and the spin 1 systems. As these systems are chiral systems, we assume that these particles obey fermionic statistics. This is not the only possible choice of statistics on a two dimensional space such as S_F^2 as there are anyonic statistics possible on these lower dimensional spaces. It should also be noted that ideas of antiparticles exist for non-relativistic theories as well [20]. These ideas will just be mentioned in this section as the results obtained are quite general and independent of the choice of statistics.

Several thermodynamic quantities such as the mean energy, the equation of state, the entropy and the specific heats are computed for both the systems. We find a remarkable result where the mean energy of the spin $\frac{1}{2}$ system is more than that of the spin 1 system. This is despite the fact that the spin 1 system contains more number of degrees of freedom than the spin $\frac{1}{2}$ system. We understand this result by computing the number of zero modes of the spin 1 system. This is one of the few analytically possible computations in these systems. We also find a deviation from the ideal gas law of two dimensional spaces. Though we do not know of any physical system exhibiting such strange behavior we provide possible physical applications of these systems.

Topology change through fuzzy spaces: The Higg's algebra(HA) provides an example of topology change via fuzzy spaces [14]. The HA is given in Eq.(7). As it stands we can think of this algebra as a deformation of the $SU(2)$ algebra. There exist other deformations of S_F^2 called the q -deformed spheres which are defined by the $SU_q(2)$ algebra. We can think of the HA as lying in between the Lie and q -deformed algebras. This algebra originally arose as a symmetry algebra in the study of the Kepler problem in curved spaces, particularly on a sphere [21]. Quantum mechanical Hamiltonian of a particle in a Kepler potential on the surface of a sphere has a dynamical symmetry given by the above algebra and can be used to solve the problem exactly.

The HA can be thought of as coming from quantizing the Poisson structure on the following

embedding in \mathbb{R}^3

$$x^2 + y^2 + (z^2 - \mu)^2 = 1 \quad (10)$$

where μ is a free parameter that can be varied. We call this the Higgs manifold, \mathcal{M}_H . Thus the HA provides the noncommutative algebra on \mathcal{M}_H . Its representation theory is studied in detail to show that there exists a sequence of representations similar to the case of S_F^2 , thus giving hope to do field theories on such deformed manifolds or non-linear algebras. Earlier studies on topology change in fuzzy Riemann surfaces can be found in [22]. As fuzzy algebras can be thought of as matrix algebras, it is essential to look at how matrices can encode topological information [22]. We briefly review these works here.

The second part of the thesis is on the physics on non-compact noncommutative spacetimes. In particular we study qfts on the GM plane (Eq.(5)) and formulate a model which violates the Pauli-exclusion principle on the $\mathcal{B}_{\chi\vec{n}}$ plane (Eq.(8)). The respective noncommutative algebras on these spacetimes are denoted by \mathcal{A}_θ and $\mathcal{A}_{\chi\vec{n}}$. As for fuzzy spaces, the usual point-wise products of functions on the corresponding commutative manifolds are deformed to star products which can be written as follows:

$$f \star g(x) := m_\theta (f \otimes g) (x) = m_0 (\mathcal{F}_\theta f \otimes g) (x) \quad (11)$$

and

$$f \star g(x) := m_{\chi\vec{n}} (f \otimes g) (x) = m_0 (F_{\chi\vec{n}} f \otimes g) (x) \quad (12)$$

where

$$m_0 (f \otimes g) (x) = f(x) \cdot g(x) \quad (13)$$

is the usual point-wise product of functions on the corresponding commutative spacetimes. The elements \mathcal{F}_θ and $F_{\chi\vec{n}}$ are called the twist elements and they are given by

$$\mathcal{F}_\theta = \exp \left(\frac{i}{2} \theta^{\mu\nu} \partial_\mu \otimes \partial_\nu \right) \quad (14)$$

and

$$F_{\chi\vec{n}} = \exp \left(\frac{i}{2} \chi \left(\partial_t \otimes \vec{n} \cdot \vec{L} - \vec{n} \cdot \vec{L} \otimes \partial_t \right) \right) \quad (15)$$

where ∂_μ are generators of spacetime translations and $\vec{n} \cdot \vec{L}$ are generators of rotation in \mathbb{R}^3 . As the generators in each of these twists commute with each other they are also known as Abelian twists. Due to this they give rise to associative products. The twist in Eq.(14) is famous in the mathematics literature and is named after its founder as the Drinfel'd twist [23]. These products are non-local in spacetime and thus describe non-local models. They are not Lorentz invariant, though the GM plane is invariant under spacetime translations. Thus physics on such spacetimes are expected to violate locality and causality principles and help model Lorentz non-invariant theories.

There are various approaches to physics on such spacetimes [13]. We follow a particular method of quantization which exploits the twisted Lorentz invariance of these spacetimes [24]. This leads to twisted statistics as explained earlier. We call the qfts constructed using twisted statistics as twisted qfts. The following is a summary of studies carried out on these two noncommutative spacetimes.

Lehmann-Symanzik-Zimmermann(LSZ) formulation of qfts on the GM plane: In the commutative theory, the LSZ reduction formula provides a non-perturbative approach for deriving

the formula for S -matrix elements in scattering theory [25, 26, 27]. A noncommutative generalization of this result for twisted scalar field theories on the GM plane was first given in [28]. The results of this work are applied to the computations of scattering amplitudes in a twisted scalar field theory model [29]. Three methods are employed to achieve this. The simplest method is a non-perturbative one relating the noncommutative and the commutative answers by an overall noncommutative phase depending on the external momenta. The second method is another non-perturbative method arriving at the same conclusion. This result is seen to be in agreement with previous computations using interaction-representation perturbation theory, thus establishing a complete consistency between the two methods on the GM plane. This result is non-trivial in the following way. In the LSZ formulation the *twist* element (to be explained in this chapter) involves the full four-momentum operator P_μ of the interacting theory, whereas in the interaction representation theory the *twist* element involves just the free four-momentum operator P_μ^0 which is far easier to handle using mode expansions of free fields.

The third method involves an all-order perturbative computation of the S -matrix elements to arrive at the same answer thus establishing the equivalence between the two formalisms. The advantage of this computation is that it helps to conclude that the Feynman diagrams involved in the twisted field theory are the same as in the commutative theory. The LSZ formula can be summarized by stating that the coefficient of the poles of the Fourier transform of time-ordered correlation functions give the S -matrix elements. The equivalence between the interaction representation perturbation theory and the LSZ formalism thus holds only for on-shell Green's functions. However this equivalence fails to hold for off-shell Green's functions. This is seen with the perturbative computation of the off-shell Green's functions.

For any field theory to be physically relevant it is important for them to be unitary and renormalizable. Remarks are made on these aspects of twisted field theories using the noncommutative LSZ formalism.

Non-Pauli effects on the $\mathcal{B}_{\chi\vec{n}}$ plane: Physical theories on noncommutative spacetimes are nonlocal and violate causality, two essential ingredients for proving spin-statistics theorems [30]. Thus these theories help testing the validity of the Pauli-exclusion principle. An earlier striking calculation [31] challenging this principle, demonstrates that the core of a fermion on a noncommutative spacetime is not infinitely repulsive. We study Pauli-forbidden transitions in a quantum mechanical model of an atom with two electrons [32, 33]. The system is the tensor product of two hydrogen-like atoms, to keep things simple. We propose a noncommutative spacetime called the $\mathcal{B}_{\chi\vec{n}}$ plane given in Eq.(8) to study this effect. This model depends on an arbitrary vector \vec{n} in space which is an additional parameter in the problem other than the noncommutative parameter χ . This vector is made dynamical as it is thought to be influenced by the movement of the earth which is a natural thing to do in this setting. The final amplitudes are however found after averaging over \vec{n} . We compute branching ratios to remove dependence on effects independent of noncommutativity. These are the best possible quantities to confront with experiments.

There are numerous experiments testing the validity of this principle [34, 35, 36, 37, 38, 39]. The comparison of results from our model with the numbers from these experiments help put bounds on the noncommutative parameter χ . The best bound obtained is $\chi \gtrsim 10^{24}\text{TeV}$ suggesting an energy beyond Planck scale. These come from forbidden processes occurring in neutrino experiments. This also suggests a further check on its validity.

We also provide a brief survey of other theories modeling Pauli-forbidden processes without the explicit use of noncommutative spacetimes. Extensive work on this subject was carried out by Greenberg and coworkers [40].

Finally we look at other aspects of this new noncommutative spacetime $\mathcal{B}_{\chi\vec{n}}$. We show that time is quantized on these spacetimes and remark on its possible consequences.

Chapter 1

Construction of Fuzzy Spaces and Star Products

The principal way of constructing fuzzy spaces is from the fundamental observation of Kostant [41], Kirillov [42] and Berezin [43] that coadjoint orbits of Lie groups are symplectic manifolds possessing a Poisson structure which can then be quantized using the usual techniques of quantization. This can be thought of phase space quantization. This also means that the classical manifold has to be even dimensional. The task then is to obtain representations for this noncommutative algebra to obtain the quantum version in the most useful form to do physics. We then say that we have made the classical manifold fuzzy. The reason it is fuzzy is because we can no longer localize points on this manifold. The process of quantizing involves a parameter which when taken to 0 helps us retrieve the classical manifold.

Simple and semi-simple Lie groups have coadjoint and adjoint orbits which are compact. In this case the fuzzy manifold is a finite-dimensional matrix algebra on which the Lie group acts in simple ways. They retain the symmetries of the classical manifold which is one of the positive features of fuzzy manifolds. We illustrate these ideas through S_F^2 . We do this in three different ways.

1.1 Obtaining S_F^2

1.1.1 Quantizing the Classical Poisson Structure on S^2

The orbit of $SU(2)$ through the Pauli matrix σ_3 or any of its multiples $\lambda\sigma_3$ ($\lambda \neq 0$) is the set

$$\lambda g \sigma_3 g^{-1} : g \in SU(2).$$

The symplectic form is $\sqrt{j(j+1)}d(\cos\theta) \wedge d\phi$ where θ and ϕ are the coordinates on S^2 . Quantization of the Poisson structure associated with S^2 gives its fuzzy version namely S_F^2 . This is seen as follows. Consider the Poisson structure on S^2 given by

$$\{f, g\} = \Omega^{\mu\nu} \partial_\mu f \partial_\nu g \tag{1.1}$$

where f and g are two functions on S^2 with a radius r . They are functions of the coordinates θ and ϕ . $\Omega^{\mu\nu}$ is the symplectic form with the following non-vanishing components

$$\Omega^{\theta\phi} = -\frac{1}{\sqrt{j(j+1)}\sin\theta}, \quad \Omega^{\phi\theta} = \frac{1}{\sqrt{j(j+1)}\sin\theta}. \quad (1.2)$$

If we now consider the Poisson brackets of

$$\begin{aligned} x_1 &= r \sin\theta \cos\phi \\ x_2 &= r \sin\theta \sin\phi \\ x_3 &= r \cos\theta \end{aligned} \quad (1.3)$$

we get

$$\{x_i, x_j\} = -\frac{r}{\sqrt{j(j+1)}}\epsilon_{ijk}x_k, \quad i, j = 1, 2, 3. \quad (1.4)$$

As $r \rightarrow 0$ we retrieve the commutative sphere, S^2 . Quantization of this manifold gives spin j representations of $SU(2)$. This can be seen if we define

$$x_i = -\frac{r}{\sqrt{j(j+1)}}L_i \quad (1.5)$$

which gives

$$\{L_i, L_j\} = \epsilon_{ijk}L_k. \quad (1.6)$$

Upon quantization which is replacing the Poisson bracket with $i \times [,]$ we get the familiar angular momentum algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad (1.7)$$

and

$$L_i^2 \mathbb{I} = j(j+1)\mathbb{I} \quad (1.8)$$

which is the spin j representation of $SU(2)$.

1.1.2 Using Harmonic Oscillators

There is a well known descent chain from \mathbb{C}^2 to the 3-sphere S^3 and then to S^2 which we quantize to obtain the fuzzy 2-sphere, S_F^2 . Consider \mathbb{C}^2 with the origin removed, $\mathbb{C}^2 - 0$. We have the fibration

$$\mathbb{R} \rightarrow \mathbb{C}^2 - 0 \rightarrow S^3 = \langle \zeta = \frac{z}{|z|} \rangle \quad (1.9)$$

with z labeling the coordinates on \mathbb{C}^2 . Now S^3 is a $U(1)$ -bundle over S^2 . This is also known as Hopf fibration [44]. If $\zeta \in S^3$, then $\vec{x}(\zeta) = \zeta^\dagger \vec{\sigma} \zeta$, where $\sigma_i, i = 1, 2, 3$ are the Pauli matrices, is invariant under the the $U(1)$ action $\zeta \rightarrow \zeta e^{i\theta}$ and is a real normalized three-vector:

$$\vec{x}(\zeta)^* = \vec{x}(\zeta), \quad \vec{x}(\zeta) \cdot \vec{x}(\zeta) = 1. \quad (1.10)$$

Thus $\vec{x}(\zeta) \in S^2$ and we have the Hopf fibration

$$U(1) \rightarrow S^3 \rightarrow S^2, \quad \zeta \rightarrow \vec{x}(\zeta). \quad (1.11)$$

Fuzzy S^3 is obtained by replacing $\frac{z_i}{|z|}$ by $a_i \frac{1}{\sqrt{\hat{N}}}$ where $\hat{N} = \sum_j a_j^\dagger a_j$ is the number operator:

$$\frac{z_i}{|z|} \rightarrow a_i \frac{1}{\sqrt{\hat{N}}}, \quad \frac{z_i^*}{|z|} \rightarrow \frac{1}{\sqrt{\hat{N}}} a_i^\dagger, \quad \hat{N} = \sum_j a_j^\dagger a_j, \quad \hat{N} \neq 0. \quad (1.12)$$

The quantum condition $\hat{N} \neq 0$ means that the vacuum is omitted from the Hilbert space, so that it is the orthogonal complement of the vacuum in Fock space. This omission is like the deletion of 0 from \mathbb{C}^2 . The problem with this is that the oscillators will create it from any $\hat{N} = n$ state and hence they do not have the infinite-dimensional Fock space to act on and thus do not get finite-dimensional models for S_F^3 .

This problem however vanishes for S_F^2 as here $x_i(\zeta)$ becomes the operator x_i given by

$$x_i(\zeta) \rightarrow x_i = \frac{1}{\sqrt{\hat{N}}} a^\dagger \sigma_i a \frac{1}{\sqrt{\hat{N}}} = \frac{1}{\hat{N}} a^\dagger \sigma_i a, \quad \hat{N} \neq 0. \quad (1.13)$$

Now

$$[x_i, \hat{N}] = 0 \quad (1.14)$$

which allows us to restrict x_i to a subspace \mathcal{H}_n of the Fock space where $\hat{N} = n \neq 0$. This space is $(n+1)$ -dimensional and is spanned by the orthonormal vectors

$$\frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1!} \sqrt{n_2!}} |0\rangle \equiv |n_1, n_2\rangle, \quad n_1 + n_2 = n. \quad (1.15)$$

The x_i act irreducibly on this space and generate the full matrix algebra $Mat(n+1)$.

The generators of $SU(2)$ can be constructed using the Schwinger construction given by:

$$L_i = a^\dagger \frac{\sigma_i}{2} a, \quad [L_i, L_j] = i\epsilon_{ijk} L_k. \quad (1.16)$$

When these operators act on the space spanned by the vectors $|n_1, n_2\rangle$ we see that they have angular momentum $\frac{n}{2}$:

$$L_i L_i |_{\mathcal{H}_n} = \frac{n}{2} \left(\frac{n}{2} + 1 \right) \mathbb{I}|_{\mathcal{H}_n}. \quad (1.17)$$

As

$$x_i |_{\mathcal{H}_n} = \frac{2}{n} L_i |_{\mathcal{H}_n}, \quad (1.18)$$

we get the following algebra for x_i :

$$[x_i, x_j] |_{\mathcal{H}_n} = \frac{2}{n} i\epsilon_{ijk} x_k |_{\mathcal{H}_n}, \quad (1.19)$$

which is the algebra of S_F^2 . We have

$$\left(\sum_i x_i^2 \right) |_{\mathcal{H}_n} = \left(1 + \frac{2}{n} \right) \mathbb{I}|_{\mathcal{H}_n}. \quad (1.20)$$

which gives the radius of S_F^2 and this becomes 1 as $n \rightarrow \infty$. Also in this limit the algebra of x_i becomes commutative.

1.1.3 From the Equation of S^2 as an embedding in \mathbb{R}^3

S^2 of radius r can be considered as an embedding in (submanifold of) \mathbb{R}^3 described by the following equation

$$x_1^2 + x_2^2 + x_3^2 = r^2. \quad (1.21)$$

The Poisson bracket on such a surface is given by

$$\{f, g\} = \frac{\partial(C, f, g)}{\partial(x_1, x_2, x_3)} \quad (1.22)$$

where f and g are two functions on S^2 and C is given by

$$C = x_1^2 + x_2^2 + x_3^2. \quad (1.23)$$

This can be thought of as the Jacobian of the coordinate transformation between x_1, x_2, x_3 to C, f and g . Explicitly it is given by

$$\frac{\partial(C, f, g)}{\partial(x_1, x_2, x_3)} = \det \begin{pmatrix} \frac{\partial C}{\partial x_1} & \frac{\partial C}{\partial x_2} & \frac{\partial C}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{pmatrix}. \quad (1.24)$$

Applying this formula to S^2 we get

$$\{x_i, x_j\} = 2\epsilon_{ijk}x_k \quad (1.25)$$

which upon quantization gives S_F^2 . The advantage of this method lies in the fact that this can be used to get the noncommutative Poisson algebra for any surface in \mathbb{R}^3 . However this is only at the classical level as we are not guaranteed a representation theory of these algebras in the most general case. This implies that we will not have the fuzzy versions of all these surfaces. (However see [22].) This necessarily means that we can obtain fuzzy versions of manifolds which are not necessarily coadjoint orbits of semi-simple Lie groups. An example of this kind is given by the Higg's manifold which we define as

$$x_1^2 + x_2^2 + (x_3 - \mu)^2 = 1 \quad (1.26)$$

where μ is a parameter. This equation describes deformed spheres in \mathbb{R}^3 . The transformations leaving this surface invariant do not have a group structure, instead they have the structure of a non-linear algebra. For variations of the parameter μ we will see that this surface exhibits topology change (See Chapter 5 of this thesis). The noncommutative Poisson algebra on this manifold is given by

$$\{x_1, x_2\} = 4x_3^3 - 4\mu x_3 \quad (1.27)$$

$$\{x_3, x_1\} = 2x_2 \quad (1.28)$$

$$\{x_2, x_3\} = 2x_1. \quad (1.29)$$

Upon quantization this gives the Higg's algebra [21] which is

$$[X_+, X_-] = C_1 Z + C_2 Z^3, \quad [Z, X_\pm] = \pm X_\pm \quad (1.30)$$

where $X_\pm = X_1 \pm iX_2$, $Z = X_3$ and C_1 and C_2 are two parameters. Note that this form of the algebra is a slightly modified version of the classical Poisson algebra where only one parameter

was used. The representations of this algebra are similar to those of $SU(2)$ [45] as we shall soon see when we study star products on the Higg's manifold.

It should be noted that the first method and this method are not completely independent. We can arrive at the Poisson bracket in Eq.(1.1) from the Poisson bracket in Eq.(1.22) if we make the identification

$$\Omega^{ij} = \epsilon_{ijk} \frac{\partial C}{\partial x_k}. \quad (1.31)$$

This method gives us a systematic way of obtaining symplectic forms restricted to embeddings in \mathbb{R}^3 .

The other advantage of this method is that this process is not restricted to just even dimensional manifolds. This method can be extended to higher dimensional manifolds except that now we will not be working with Poisson algebras but generalizations of the same to structures called the Nambu brackets [46]. For example this method can be a possible way to obtain S_F^3 through the following process. Consider the 3-sphere as an embedding in \mathbb{R}^4 which gives

$$C = x_1^2 + x_2^2 + x_3^2 + x_4^2. \quad (1.32)$$

Using this we can write the Nambu brackets restricted to this submanifold as

$$\{f, g, h\} = \frac{\partial (C, f, g, h)}{\partial (x_1, x_2, x_3, x_4)}. \quad (1.33)$$

Applying this to the 3-sphere we get

$$\{x_i, x_j, x_k\} = 2\epsilon_{ijkl}x_l, \quad i, j, k, l = 1, 2, 3, 4. \quad (1.34)$$

However these structures are at the classical level. A satisfactory representation theory consisting of a sequence of finite dimensional representations are not known for this algebra. This difficulty must be overcome to obtain a S_F^3 where we can do reasonable physics.

This process of obtaining classical noncommutative algebras, using higher dimensional versions of the Nambu brackets, can clearly be extended to arbitrary embeddings in higher dimensional spaces. We are however restricted by suitable representation theory for these algebras to complete the process of fuzzification.

1.2 Star Products on Fuzzy Spaces

The algebra of smooth functions on a manifold \mathcal{M} under point-wise multiplication is commutative. In deformation quantization [47], this point-wise product is deformed to a noncommutative (but still associative) product called the \star -product.

The existence of such products were studied by Weyl, Wigner, Groenewold and Moyal [48, 49, 50]. They noted that if there is a linear injection ψ of an algebra \mathcal{A} into smooth functions $C^\infty(\mathcal{M})$ on a manifold \mathcal{M} , then the product in \mathcal{A} can be transported to the image $\psi(\mathcal{A})$ of \mathcal{A} in $C^\infty(\mathcal{M})$ using the map. This gives the \star -product.

We take \mathcal{A} to be the algebra of bounded operators on a Hilbert space closed under the Hermitian conjugation of $*$. This is an example of a $*$ -algebra. In general, \mathcal{A} can be a generic $*$ -algebra, that is an algebra closed under an anti-linear involution:

$$a, b \in \mathcal{A}, \lambda \in \mathbb{C} \Rightarrow a^*, b^* \in \mathcal{A}, (ab)^* = b^*a^*, (\lambda a)^* = \lambda^*a^*. \quad (1.35)$$

A two-sided ideal \mathcal{A}_0 of \mathcal{A} is a subalgebra of \mathcal{A} with the property

$$a_0 \in \mathcal{A}_0 \Rightarrow \alpha a_0 \text{ and } a_0 \alpha \in \mathcal{A}_0, \forall \alpha \in \mathcal{A}. \quad (1.36)$$

That is $\mathcal{A}\mathcal{A}_0, \mathcal{A}_0\mathcal{A} \subseteq \mathcal{A}_0$. A two-sided $*$ -ideal \mathcal{A}_0 by definition is itself closed under $*$ as well.

An element of the quotient $\mathcal{A}/\mathcal{A}_0$ is the equivalence class

$$\{\alpha + \mathcal{A}_0 \subset \mathcal{A}\} = \{[\alpha + a_0] | a_0 \in \mathcal{A}_0\}. \quad (1.37)$$

If \mathcal{A}_0 is a two-sided ideal, $\mathcal{A}/\mathcal{A}_0$ is itself an algebra with the sum and the product

$$(\alpha + \mathcal{A}_0) + (\beta + \mathcal{A}_0) = \alpha + \beta + \mathcal{A}_0, \quad (1.38)$$

$$(\alpha + \mathcal{A}_0)(\beta + \mathcal{A}_0) = \alpha\beta + \mathcal{A}_0. \quad (1.39)$$

If \mathcal{A}_0 is a two-sided $*$ -ideal, then $\mathcal{A}/\mathcal{A}_0$ is a $*$ -algebra with the $*$ -operation

$$(\alpha + \mathcal{A}_0)^* = \alpha^* + \mathcal{A}_0. \quad (1.40)$$

We note that the product and $*$ are independent of the choice of the representatives α, β from the equivalence classes $\alpha + \mathcal{A}_0$ and $\beta + \mathcal{A}_0$ because \mathcal{A}_0 is a two-sided ideal. So they make sense for $\mathcal{A}/\mathcal{A}_0$.

Let $C^\infty(\mathcal{M})$ denote the complex-valued smooth functions on a manifold \mathcal{M} . Complex conjugation $\bar{}$ is defined on these functions. It sends a function f to its complex conjugate \bar{f} .

We consider the linear maps

$$\psi : \mathcal{A} \rightarrow C^\infty(\mathcal{M}), \quad (1.41)$$

$$\psi \left(\sum \lambda_i a_i \right) = \sum \lambda_i \psi(a_i), \quad a_i \in \mathcal{A}, \lambda_i \in \mathbb{C}. \quad (1.42)$$

The kernel of such a map is given by the set

$$Ker \psi = \langle \alpha_0 \in \mathcal{A} | \psi(\alpha_0) = 0 \rangle. \quad (1.43)$$

ψ descends to a linear map, called Ψ , from $\mathcal{A}/Ker \psi = \{\alpha + Ker \psi : \alpha \in \mathcal{A}\}$ to $C^\infty(\mathcal{M})$:

$$\Psi(\alpha + Ker \psi) = \psi(\alpha). \quad (1.44)$$

$\psi(\alpha)$ does not depend on the choice of the representative α from $\alpha + Ker \psi$ due to the definition of $Ker \psi$. Clearly Ψ is an injective map from $\mathcal{A}/Ker \psi$ to $C^\infty(\mathcal{M})$. If $Ker \psi$ is a two-sided ideal then this map is from an algebra to $C^\infty(\mathcal{M})$. Using this fact we define a product, also denoted by $*$, on $\Psi(\mathcal{A}/Ker \psi) = \psi(\mathcal{A}) \subseteq C^\infty(\mathcal{M})$:

$$\Psi(\alpha + Ker \psi) * \Psi(\beta + Ker \psi) = \Psi((\alpha + Ker \psi)(\beta + Ker \psi)). \quad (1.45)$$

or

$$\psi(\alpha) * \psi(\beta) = \psi(\alpha\beta). \quad (1.46)$$

With this product, $\psi(\mathcal{A})$ is an algebra ($\psi(\mathcal{A}, *)$ isomorphic to $\mathcal{A}/Ker \psi$).

We assume that $\mathcal{A}/Ker \psi$ is a $*$ -algebra and that Ψ preserves the stars on $\mathcal{A}/Ker \psi$ and $C^\infty(\mathcal{M})$, the $*$ on the latter being complex conjugation denoted by $\bar{}$:

$$\Psi((\alpha + Ker \psi)^*) = \overline{\Psi(\alpha + Ker \psi)}, \quad (1.47)$$

$$\psi(\alpha^*) = \overline{\psi(\alpha)}. \quad (1.48)$$

Such ψ and Ψ are said to be $*$ -morphisms from \mathcal{A} and $\mathcal{A}/Ker \psi$ to $(\psi(\mathcal{A}), *)$. The two algebras $\mathcal{A}/Ker \psi$ and $(\psi(\mathcal{A}), *)$ are said to be $*$ -isomorphic.

1.2.1 Properties of Coherent States(CS)

The Baker-Campbell-Hausdorff (BCH) formula is given by

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]} \quad (1.49)$$

for two operators \hat{A}, \hat{B} if

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0. \quad (1.50)$$

For one oscillator with annihilation-creation operators a, a^\dagger , the coherent state

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}|0\rangle, \quad z \in \mathbb{C} \quad (1.51)$$

has the properties

$$a|z\rangle = z|z\rangle; \quad \langle z'|z\rangle = e^{\frac{1}{2}(z-\bar{z}')^2}. \quad (1.52)$$

The CS are overcomplete, with the resolution of identity

$$\mathbb{I} = \int \frac{d^2z}{\pi} |z\rangle\langle z|, \quad d^2z = dx_1 dx_2, \quad \text{where } z = \frac{x_1 + ix_2}{\sqrt{2}}. \quad (1.53)$$

A central property of the CS is the following: an operator \hat{A} is determined just by its diagonal matrix elements

$$A(z, \bar{z}) = \langle z|\hat{A}|z\rangle, \quad (1.54)$$

that is by its ‘‘symbol’’ A , a function on \mathbb{C} with values $A(z, \bar{z}) = \langle z|\hat{A}|z\rangle$ [51]. \hat{A} is certainly determined by the collection of all its matrix elements $\langle \bar{\eta}|\hat{A}|\zeta\rangle$ or equally by

$$e^{\frac{1}{2}(|\eta|^2 + |\zeta|^2)} \langle \bar{\eta}|\hat{A}|\zeta\rangle = \langle 0|e^{\eta a^\dagger} \hat{A} e^{\zeta a}|0\rangle. \quad (1.55)$$

The right hand side (for appropriate \mathcal{A} is seen to be a holomorphic function of η and ζ , or equally well of

$$u = \frac{\eta + \zeta}{2}, \quad v = \frac{\eta - \zeta}{2i}. \quad (1.56)$$

Holomorphic functions are globally determined by their values for real arguments. Hence the function \tilde{A} defined by

$$\tilde{A} = \langle 0|e^{\eta a^\dagger} \hat{A} e^{\zeta a}|0\rangle \quad (1.57)$$

is globally determined by its values for u, v real or $\eta = \bar{\zeta}$. Thus $\langle \zeta|\hat{A}|\zeta\rangle$ determines \hat{A} as claimed. Explicit formulas for \hat{A} in terms of $\langle \zeta|\tilde{A}|\zeta\rangle$ [52].

1.2.2 Properties of Star Products

The trace of operators has the fundamental property

$$Tr \hat{A}\hat{B} = Tr \hat{B}\hat{A} \quad (1.58)$$

which leads to the general cyclic identities

$$Tr \hat{A}_1 \cdots \hat{A}_n = Tr \hat{A}_n \hat{A}_1 \cdots \hat{A}_{n-1}. \quad (1.59)$$

We can show that

$$Tr \hat{A}\hat{B} = \int \frac{d^2z}{\pi} A * B(z, \bar{z}). \quad (1.60)$$

From this follows

$$\int \frac{d^2 z}{\pi} (A_1 * A_2 * \cdots * A_n)(z, \bar{z}) = \int \frac{d^2 z}{\pi} (A_n * A_1 * \cdots * A_{n-1})(z, \bar{z}). \quad (1.61)$$

These identities can be shown for specific choices of the $*$ -product.

We provide specific examples of the star product in the following sections. We start with the known example of the star product on S_F^2 , and then consider the star product on the fuzzy Higg's manifold, \mathcal{M}_H .

1.2.3 The $*$ -Products for S_F^2

Star products for Kähler manifolds are well known by now. The approach shown here is due to Prešnajder [17, 53].

We will now use the notation for S_F^2 as obtained by using harmonic oscillators in Sec.(1.1.2). Let P_n be the orthogonal projection operator to the subspace with $N = n$. The fuzzy sphere algebra is then the algebra with elements $P_n \gamma(a_i^\dagger a_j) P_n$ where γ is any polynomial in $a_i^\dagger a_j$. As any such polynomial commutes with N , if γ and δ are two such polynomials,

$$P_n \gamma(a_i^\dagger a_j) P_n P_n \delta(a_i^\dagger a_j) P_n = P_n \gamma(a_i^\dagger a_j) \delta(a_i^\dagger a_j) P_n. \quad (1.62)$$

This algebra, more precisely, is the orthogonal direct sum $Mat(n+1) \oplus 0$ where $Mat(n+1)$ acts on the $\hat{N} = n$ subspace and is the fuzzy sphere. But the extra 0 is entirely harmless. There are two kinds of $*$ -products namely the coherent state product and the Weyl $*$ -product. We present here just the CS $*$ -product.

1.2.4 The CS $*$ -Product on S_F^2

There are now two oscillators a_1, a_2 , so the CS are labelled by two complex variables, being

$$|Z_1, Z_2\rangle = e^{Z a^\dagger - \bar{Z} a} |0\rangle, \quad Z = (Z_1, Z_2). \quad (1.63)$$

We use capital Z 's for unnormalized Z 's and z 's for normalized ones: $z = \frac{Z}{|Z|}$, $|Z|^2 = \sum |Z_i|^2$.

The normalized CS $|z\rangle_n$ for S_F^2 , as one can guess, are obtained by projection from $|Z\rangle$,

$$|z\rangle_n = \frac{P_n |Z\rangle}{|\langle P_n | Z \rangle|} = \frac{\left(\sum_i z_i a_i^\dagger \right)^n}{\sqrt{n!}} |0\rangle, \quad (1.64)$$

where we have used

$$P_n |Z\rangle = \frac{\left(Z_i a_i^\dagger \right)^n}{n!} |0\rangle. \quad (1.65)$$

These are called Perelomov states [51].

For an operator $P_n \hat{A} P_n$, the CS symbol has the value

$$\langle Z | P_n \hat{A} P_n | Z \rangle = \frac{|z|^{2n}}{n!} \langle z | \hat{A} | z \rangle_n \quad (1.66)$$

at Z . We know that the diagonal CS expectation values $\langle z | P_n \hat{A} P_n | z \rangle_n$ determines $P_n \hat{A} P_n$ uniquely and there is a $*$ -product for S_F^2 . Call this product the $*_C$ -product to denote that it is a CS $*$ -product.

We can find it explicitly as follows [17, 54, 55]. For $n = 1$ (spin $\frac{n}{2} = \frac{1}{2}$), a basis for 2×2 matrices is

$$\{\sigma_A : \sigma_0 = \mathbf{1}, \sigma_i (i = 1, 2, 3) = \text{Pauli Matrices}, \text{Tr} \sigma_A \sigma_B = 2\delta_{AB}\}. \quad (1.67)$$

Let

$$|i\rangle = a_i^\dagger |0\rangle, \quad i = 1, 2 \quad (1.68)$$

be a pair of orthonormal vectors for $n = 1$. A general operator is

$$\hat{F} = f_A \hat{\sigma}_A, \quad \hat{\sigma}_A = a^\dagger \sigma_A a|_{n=1}, \quad f_A \in \mathbb{C} \quad (1.69)$$

and $\hat{\sigma}_A |i\rangle = |j\rangle (\sigma_A)_{ji}$. In above, by $a^\dagger \sigma_A a|_{n=1}$, we mean the restriction of $a^\dagger \sigma_A a$ to the subspace with $n = 1$.

Call the CS symbol of $\hat{\sigma}_A$ for $n = 1$ as $\tilde{\chi}_A$:

$$\tilde{\chi}_A(z) = \langle z | \hat{\sigma}_A | z \rangle, \quad \tilde{\chi}_0(z) = 1, \quad \tilde{\chi}_i = \bar{z} \sigma_i z, \quad i = 1, 2, 3. \quad (1.70)$$

The $*$ -product for $n = 1$ now follows:

$$\tilde{\chi}_A * \tilde{\chi}_B(z) = \langle z | \hat{\sigma}_A \hat{\sigma}_B | z \rangle. \quad (1.71)$$

Write

$$\begin{aligned} \sigma_A \sigma_B &= \delta_{AB} + E_{ABi} \chi_i(z) \\ &:= \tilde{\chi}_A(z) \tilde{\chi}_B(z) + \kappa_{AB}(z). \end{aligned} \quad (1.72)$$

Let us use the notation

$$n_i = \tilde{\chi}_i(z), \quad n_0 = 1. \quad (1.73)$$

\vec{n} is the coordinate on S^2 : $\vec{n} \cdot \vec{n} = 1$. Then

$$n_A *_C n_B(z) = n_A n_B + K_{AB}(n), \quad \kappa_{AB}(z) := K_{AB}(n). \quad (1.74)$$

This K_{AB} has a particular significance for complex analysis. Since $\tilde{\chi}_0(z) = 1$, $\tilde{\chi}_0(z) * \tilde{\chi}_A = \tilde{\chi}_0 \tilde{\chi}_A$ and

$$K_{0A} = 0. \quad (1.75)$$

The components $K_{ij}(n)$ of K can be calculated. Let $\theta(\alpha)$ be the spin 1 angular momentum matrices:

$$\theta(\alpha)_{ij} = -i\epsilon_{\alpha ij}. \quad (1.76)$$

Then

$$K_{ij}(\vec{n}) = \frac{[\vec{\theta} \cdot \vec{n} (\vec{\theta} \cdot \vec{n} - 1)]_{ij}}{2} \quad (1.77)$$

where $\vec{\theta} \cdot \vec{n} := \theta(\alpha) n_\alpha$. The eigenvalues of $\vec{\theta} \cdot \vec{n}$ are $\pm 1, 0$ and $K_{ij}(\vec{n})$ is the projection operator to the eigenspace $\vec{\theta} \cdot \vec{n} = -1$,

$$K(\vec{n})^2 = K(\vec{n}). \quad (1.78)$$

This is related to the complex structure of S^2 in the projective module picture, an overview of which will be given in the following chapter.

The vector space for angular momentum $\frac{n}{2}$ is the n -fold symmetric tensor product of the spin- $\frac{1}{2}$ vector spaces. The general linear operator on this space can be written as

$$\hat{F} = f_{A_1 A_2 \dots A_n} \hat{\sigma}_{A_1} \otimes \hat{\sigma}_{A_2} \otimes \dots \otimes \hat{\sigma}_{A_n} \quad (1.79)$$

where f is totally symmetric in its indices. Its symbol is thus

$$F(\vec{n}) = f_{A_1 A_2 \dots A_n} n_{A_1} n_{A_2} \dots n_{A_n}, \quad n_0 := 1. \quad (1.80)$$

The symbol of another operator

$$\hat{G} = g_{B_1 B_2 \dots B_n} n_{B_1} n_{B_2} \dots n_{B_n}. \quad (1.81)$$

Since

$$\hat{F}\hat{G} = f_{A_1 A_2 \dots A_n} g_{B_1 B_2 \dots B_n} \sigma_{A_1} \sigma_{B_1} \otimes \sigma_{A_2} \sigma_{B_2} \otimes \dots \otimes \sigma_{A_n} \sigma_{B_n}, \quad (1.82)$$

we have that

$$F * G(\vec{n}) = f_{A_1 A_2 \dots A_n} g_{B_1 B_2 \dots B_n} \prod_i (n_{A_i} n_{B_i} + K_{A_i B_i}) \quad (1.83)$$

or

$$\begin{aligned} F * G(\vec{n}) &= FG(\vec{n}) + \sum_{m=1}^n \frac{n!}{m!(n-m)!} f_{A_1 A_2 \dots A_m A_{m+1} \dots A_n} \times \\ &n_{A_{m+1}} n_{A_{m+2}} \dots n_{A_n} \times K_{A_1 B_1}(\vec{n}) K_{A_2 B_2}(\vec{n}) \dots K_{A_m B_m}(\vec{n}) \times \\ &g_{B_1 B_2 \dots B_m B_{m+1} \dots B_n} n_{B_{m+1}} n_{B_{m+2}} \dots n_{B_n}. \end{aligned} \quad (1.84)$$

Now as f and g are symmetric in indices, there is the expression

$$\partial_{A_1} \partial_{A_2} \dots \partial_{A_m} F(\vec{n}) = \frac{n!}{(n-m)!} f_{A_1 A_2 \dots A_m A_{m+1} \dots A_n} n_{A_{m+1}} n_{A_{m+2}} \dots n_{A_n}, \quad (1.85)$$

where $\partial_{A_i} \equiv \frac{\partial}{\partial n_{A_i}}$ for F and a similar expression for G . Hence

$$\begin{aligned} F *_C G(\vec{n}) &= \sum_{m=0}^n \frac{(n-m)!}{n!m!} (\partial_{A_1} \partial_{A_2} \dots \partial_{A_m} F)(\vec{n}) \\ &\times K_{A_1 B_1}(\vec{n}) K_{A_2 B_2}(\vec{n}) \dots K_{A_m B_m}(\vec{n}) (\partial_{B_1} \partial_{B_2} \dots \partial_{B_m} G)(\vec{n}) \end{aligned} \quad (1.86)$$

is the CS $*$ -product on S_F^2 . Here the $m=0$ term is to be understood as $FG(\vec{n})$, the pointwise product of F and G evaluated at \vec{n} .

For large n , this product is an expansion in powers of $\frac{1}{n}$, the leading term giving the commutative product. Thus the algebra S_F^2 is in some sense a deformation of the commutative algebra of functions $C^\infty(S^2)$. But as the maximum angular momentum in F and G is n , we get only the spherical harmonics Y_{lm} , $l \in \{0, 1, \dots, n\}$ in their expansion. For this reason, F and G span a finite-dimensional subspace of $C^\infty(S^2)$ and S_F^2 is not properly a deformation of the commutative algebra $C^\infty(S^2)$.

In the next section we will study how the $*$ -product is constructed for a non-linear algebra like the fuzzy Higg's manifold.

1.2.5 Higgs Algebra and its Representation

We have shown in the previous section as to how the cubic Poisson bracket can be induced by a surface that is quartic in z . When we quantize this non-linear bracket we get the HA. The interest in studying these nonlinear algebras, apart from the physical applications [21, 56], is that we can construct unitary finite or infinite dimensional representations. These and other interesting aspects were studied for many of these nonlinear structures, collectively called the polynomial algebras, by various authors [57, 58, 45, 59, 60, 61]. In what follows we will explicitly state the representations of importance to us.

Let X_+, X_-, Z be the generators of a three dimensional polynomial algebra. This algebra is defined by the following commutation relations:

$$[X_+, X_-] = C_1 Z + C_2 Z^3 \equiv f(Z), \quad [Z, X_\pm] = \pm X_\pm. \quad (1.87)$$

In the above C_1 and C_2 are arbitrary constants. It is straight forward to check that the Jacobi identity is preserved. When $C_2 = 0$ and $C_1 = 2$ or $C_1 = -2$, we have the $\mathfrak{su}(2)$ or $\mathfrak{su}(1, 1)$ algebra respectively. We will be interested in the cubic algebra that is treated as a deformation of the $\mathfrak{su}(2)$ algebra. Hence we will consider finite dimensional representations only.

The finite dimensional irreducible representations of the HA are characterized, like in $SU(2)$, by an integer or half integer j of dimension $2j + 1$.

$$\begin{aligned} Z |j, m\rangle &= m |j, m\rangle, \\ X_+ |j, m\rangle &= \sqrt{g(j) - g(m)} |j, m + 1\rangle. \end{aligned} \quad (1.88)$$

The structure function $g(Z)$ is chosen such that $f(Z) = g(Z) - g(Z-1)$. Note that $g(Z)$ is defined only upto the addition of a constant. Later we will see this freedom plays an important role in arriving at the one parameter family of surfaces namely Eq. (10). For arbitrary polynomials $f(Z)$, one can solve and find solutions for $g(Z)$ [45]. The fact that we can write $f(Z)$ as difference of structure functions $g(Z)$ enables one to find the Casimir \mathcal{C} of the algebra in an almost trivial way. The Casimir \mathcal{C} is:

$$\mathcal{C} = \frac{1}{2} [\{X_+, X_-\} + g(Z) + g(Z-1)], \quad (1.89)$$

$$\mathcal{C} |j, m\rangle = g(j) |j, m\rangle \quad (1.90)$$

where the curly brackets denote anti-commutator. It is easy to verify $[\mathcal{C}, X_\pm] = [\mathcal{C}, Z] = 0$. So far we have not specified what the explicit form of $g(Z)$ is and without further ado we state for our case of HA:

$$g(Z) = C_0 + \frac{C_1}{2} Z(Z+1) + \frac{C_2}{4} Z^2(Z+1)^2. \quad (1.91)$$

Here C_0 is a constant. Now the Casimir as a function of Z alone assumes a form of a single or double well potential depending on the values of the parameters. The physical meaning of this behavior can be understood from the work of Rocek [58]. The condition for finite dimensional representations is also discussed in [58]. In our case we note that $g(Z) = g(-Z-1)$, which is also the condition for the case of the $SU(2)$ algebra. This makes the function g periodic and hence we can be sure that we have finite dimensional representations for the choice of parameters we will make for our Higgs algebra.

Applying Eq. (1.91) to Eq. (1.89) and then comparing it with Eq. (10), we get $C_0 = \mu^2$, $C_1 = -2(2\mu + 1)$, and $C_2 = 4$. Let us note that though there is a singularity in the continuum limit, in the discrete case we have a valid representation theory as we vary the parameters.

This looks like a novel resolution of singularity. Similar behavior was noted in [22] where, the topology changes from a sphere to a torus with a degenerate surface at a transition point in the parameter space.

Now we will construct the CS for this nonlinear algebra to get a better understanding of the semiclassical behavior.

1.2.6 The Higgs Algebra Coherent States

The field CS [62, 63] and their generalizations [64, 65, 51] been extensively studied from various aspects, motivated mainly by applications to quantum optics. But, we are interested in them as providing appropriate semiclassical descriptions of the nonlinear algebra. As is well known there are two types of CS. (1) those that are ‘‘annihilation operator’’ eigenstates also known as Barut-Girardello CS [64] (2) states obtained through the action of the displacement operator also known as Perelomov states [51]. The first is useful when considering non compact groups like $SU(1, 1)$ and the second for compact ones.

We consider the finite dimensional representation of the Higgs algebra as we want to view it as a deformation of the fuzzy sphere algebra. Hence, we resort to the construction via the displacement operator. One should keep in mind that since our algebra is nonlinear one cannot attach any group theoretical interpretation to such states. The actual procedure should be viewed as an algebraic construction and has been carried out in [66, 67].

Since, the algebra under study is not a Lie algebra, a straightforward application of the Perelomov prescription is also not possible, wherein essential use of the Baker-Campbell-Hausdorff (BCH) formula is made. To get around it we find a new operator X_- such that $[X_+, X_-] = 2Z$. Let $X_- = X_- G(\mathcal{C}, Z)$; substituting this in the commutator relation we get

$$X_+ X_- G(\mathcal{C}, Z) - X_- X_+ G(\mathcal{C}, Z + 1) = 2Z. \quad (1.92)$$

Choose the *ansatz* for G of the form

$$G(\mathcal{C}, Z) = \frac{-Z(Z + 1) + \lambda}{\mathcal{C} - g(Z - 1)}, \quad (1.93)$$

where λ is an arbitrary constant. Now that we have the ‘ladder’ operators that obey the $\mathfrak{su}(2)$ algebra, we can use the Perelomov prescription. The CS are given by

$$|\zeta\rangle = e^{\zeta X_+ - \zeta^* X_-} |j, -j\rangle. \quad (1.94)$$

Disentangling the above exponential, using the BCH formula for $\mathfrak{su}(2)$ and $X_- |j, -j\rangle = 0$ we find the expression for the CS acquires the form

$$|\zeta\rangle = N^{-1}(|\zeta|^2) e^{\zeta X_+} |j, -j\rangle. \quad (1.95)$$

where $N^{-1}(|\zeta|^2)$ is the normalization constant that is yet to be determined and $\zeta \in \mathbb{C}$. Notice that the ladder operators that form the ‘Lie algebra’ are not mutually adjoint. The above state is to be viewed as ‘non-linear $\mathfrak{su}(2)$ coherent state’ and are very similar in spirit to the CS of nonlinear oscillators [68] and extensively used in quantum optics.

Now we will study whether the above definition of CS is suitable. The requirements for $|\zeta\rangle$ to be CS have been enunciated by Klauder [69]: (1) $|\zeta\rangle$ should be normalizable, (2) $|\zeta\rangle$ should be continuous in ζ , (3) $|\zeta\rangle$ should satisfy resolution of identity. We will consider normalization and resolution of identity in the following.

1.2.7 Normalization

To find the normalization constant $N^{-1}(|\zeta|^2)$ we compute the scalar product of HACS and set it equal to 1. We get

$$\begin{aligned}
N^2(|\zeta|^2) &= \langle j, -j | e^{\bar{\zeta}X_-} e^{\zeta X_+} | j, -j \rangle, \\
&= 1 + \sum_{n=1}^{2j} \frac{|\zeta|^{2n}}{(n!)^2} \prod_{\ell=0}^{n-1} K_{j, -j+\ell} \prod_{p=0}^{n-1} H_{j, -j+n-p}, \\
&= 1 + \sum_{n=1}^{2j} \frac{|\zeta|^{2n}}{(n!)^2} \prod_{\ell=0}^{n-1} (K_{j, -j+\ell})^2, \\
&= 1 + \sum_{n=1}^{2j} |\zeta|^{2n} \binom{2j}{n} \\
&\quad \prod_{\ell=0}^{n-1} \left(\frac{C_1}{2} + \frac{C_2}{4} [2j(j-\ell) - \ell(\ell+1)] \right). \tag{1.96}
\end{aligned}$$

In the above expression $K_{j,m} \equiv H_{j,m+1} = \sqrt{g(j) - g(m)}$. Observe that the expression under the product is quadratic in ℓ and can be factorized.

$$\begin{aligned}
N^2(|\zeta|^2) &= 1 + \sum_{n=1}^{2j} |\zeta|^{2n} \binom{2j}{n} \prod_{\ell=0}^{n-1} (\ell - A_+)(\ell - A_-), \\
&= 1 + \sum_{n=1}^{2j} |\zeta|^{2n} D_n, \tag{1.97}
\end{aligned}$$

where

$$A_{\pm} = - \left[\left(j + \frac{1}{2} \right) \pm \sqrt{\left(j + \frac{1}{2} \right)^2 + (2j^2 + \frac{2C_1}{C_2})} \right]. \tag{1.98}$$

Taking the ratio of D_{n+1}/D_n we get

$$\frac{D_{n+1}}{D_n} = \frac{(n - A_+)(n - A_-)(2j - n)}{(n + 1)}. \tag{1.99}$$

It can be seen that this is the condition for the generalized hypergeometric series¹ for

$${}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2).$$

Hence, the final expression for the normalization constant of the HA is

$$N^2(|\zeta|^2) = {}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2). \tag{1.100}$$

¹The generalized hypergeometric series is defined by ${}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{n! (b_1)_n (b_2)_n \dots (b_s)_n} z^n$ where $(a)_n = a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ $n = 1, 2, \dots$, is the shifted factorial.

1.2.8 Resolution of Identity

The resolution of identity is one very important criterion that any CS must satisfy:

$$\frac{1}{\pi} \int |\zeta\rangle d\mu(\zeta, \bar{\zeta}) \langle \zeta| = \mathbb{I}. \quad (1.101)$$

The integration is over the complex plane. Introducing the HACS in the above equation and writing the resulting equation in angular coordinates, $\zeta = r e^{i\theta}$ ($0 \leq \theta < 2\pi$), brings us to

$$\sum_{n=0}^{2j} \int dr \frac{\rho(r^2)}{N^2(r^2)} \frac{r^{2n+1}}{(n!)^2} X_+^n |j, -j\rangle \langle -j, j| X_-^n = \mathbb{I}. \quad (1.102)$$

We know that the angular momentum states are complete and hence for the above equality to hold the integral should be equal to one. Defining $\tilde{\rho}(r^2) \equiv \rho(r^2)/N^2(r^2)$ and simplifying the product as shown in the previous subsection we have

$$\int_0^\infty dr r^{2n+1} \tilde{\rho}(r^2) = \Gamma(n+1) \times \frac{\Gamma(A'_+ - n + 1) \Gamma(A'_- - n + 1) \Gamma(2j - n + 1)}{\Gamma(2j + 1) \Gamma(A'_+ + 1) \Gamma(A'_- + 1)}, \quad (1.103)$$

Where $A'_\pm = -A_\pm$. Making a change of variable, $r^2 = x$ and replacing the discrete variable n by the complex one ($s - 1$) we notice that the weight function $\tilde{\rho}(x)$ and the r.h.s. of the above equation become a Mellin transform related pair [70]. The unknown function $\tilde{\rho}(x)$ can be read off from tables of Mellin transforms [71]. For the sake of completeness we reproduce the relevant formula below

$$\int_0^\infty dx x^{s-1} G_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q, 0 \end{matrix} \right. \right) = \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)}, \quad (1.104)$$

where the r.h.s. is the s -dependent part of the weight function. $G_{m,n}^{p,q}$ is called the Meijer-G function and more details can be found in [72]. Casting Eq. (1.103) in the above standard form we find that

$$\rho(|\zeta|^2) = \frac{{}_3F_0(A'_+, A'_-, -2j; 0; -|\zeta|^2)}{\Gamma(2j+1)\Gamma(A'_++1)\Gamma(A'_-+1)} G_{0,0}^{1,3} \left[-|\zeta|^2 \left| \begin{matrix} -2j-1, -(A'_++1), -(A'_-+1) \\ 0 \end{matrix} \right. \right]. \quad (1.105)$$

At $\mu = 1$, this function is well behaved, leaving no trace of the conical singularity encountered in the continuum. This can be seen as a consequence of quantizing the Higgs manifold using the HACS.

1.2.9 The Star Product on the Higg's Manifold

The technique for obtaining star product for the HA, follows Grosse and Presnajder [73] and we refer to it for details regarding the use of CS in this construction. Suffice it to mention here that CS ensures that the product obtained is associative.

The algebra of functions on the Higgs manifold, \mathcal{M}_H , is commutative under point-wise multiplication. When we quantize this manifold, this point-wise product is deformed to an associative star product which is noncommutative.

We consider the algebra of operators, \mathcal{A} , generators of which satisfy the HA. These operators act on some Hilbert space. We then use the symbol to map these operators to the functions on the Higgs manifold. The symbol map is defined as follows

$$\phi : \mathcal{A} \rightarrow \mathcal{M}_H . \quad (1.106)$$

We use the HACS to define the symbol map in this case:

$$\begin{aligned} \phi(\hat{\alpha}) &\equiv \langle \zeta | e^{\alpha_- X_-} e^{\alpha_0 Z} e^{\alpha_+ X_+} | \zeta \rangle = N^{-2} \langle j, -j | e^{\alpha_0 Z} e^{-\alpha_0 Z} e^{(\alpha_- + \zeta^*) X_-} e^{\alpha_0 Z} e^{(\alpha_+ + \zeta) X_+} | j, -j \rangle , \\ &= N^{-2} e^{-j\alpha_0} \langle j, -j | e^{(\alpha_- + \zeta^*) e^{\alpha_0} X_-} e^{(\alpha_+ + \zeta) X_+} | j, -j \rangle , \quad (1.107) \\ &= e^{-j\alpha_0} \frac{{}_3F_0(-A_+, -A_-, -2j; 0; -(\alpha_+ + \zeta)(\alpha_- + \zeta^*) e^{\alpha_0})}{{}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2)} , \end{aligned}$$

where A_+ and A_- are as defined in Sec.(1.2.7). In the above derivation we have made use of the identity

$$e^{\alpha Z} X_+ e^{-\alpha Z} = e^{\alpha} X_+ . \quad (1.108)$$

We will use this identity in simplifying the symbol of the product of two general operators labeled by $\hat{\alpha}$ and $\hat{\beta}$.

We now compute the following to find the star product in terms of deformations of the point-wise product:

$$\phi(\hat{\alpha}\hat{\beta}) = N^{-2} \langle \zeta | e^{\alpha_- X_-} e^{\alpha_0 Z} e^{\alpha_+ X_+} e^{\beta_- X_-} e^{\beta_0 Z} e^{\beta_+ X_+} | \zeta \rangle . \quad (1.109)$$

We give the final result of this matrix element²;

$$\begin{aligned} \phi(\hat{\alpha}\hat{\beta}) &= N^{-2} \phi(\hat{\alpha})\phi(\hat{\beta}) + N^{-2} e^{-j(\alpha_0 + \beta_0)} \left[\phi(\hat{\alpha})e^{j\alpha_0} + \chi(\hat{\alpha})e^{j\alpha_0} \right. \\ &\quad \left. + e^{j(\alpha_0 + \beta_0)} \{ \phi(\hat{\alpha})\phi(\hat{\beta}) + \chi(\hat{\alpha})\chi(\hat{\beta}) + \phi(\hat{\alpha})\chi(\hat{\beta}) + \chi(\hat{\alpha})\phi(\hat{\beta}) \} + L \right] . \quad (1.110) \end{aligned}$$

In this expression,

$$\chi(\hat{\alpha}) = e^{-j\alpha_0} \sum_{i=1}^{2j} \frac{(\alpha_- + \zeta^*)^i (-\zeta)^i e^{i\alpha_0}}{(i!)^2} \prod_{l=0}^{i-1} K_{j, -j+l}^2 \sum_{k=0}^{i-1} \frac{(\alpha_+ + \zeta)^k}{(-\zeta)^k} \binom{i}{k} , \quad (1.111)$$

$$\chi(\hat{\beta}) = e^{-j\beta_0} \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i (-\zeta^*)^i e^{i\beta_0}}{(i!)^2} \prod_{l=0}^{i-1} K_{j, -j+l}^2 \sum_{k=0}^{i-1} \frac{(\beta_- + \zeta^*)^k}{(-\zeta^*)^k} \binom{i}{k} , \quad (1.112)$$

and

$$\begin{aligned} L &= \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i (\alpha_- + \zeta^*)^i e^{i(\alpha_0 + \beta_0)}}{(i!)^2} \prod_{l=0}^{i-1} K_{j, -j+l}^2 \left[{}_3F_0(-A_+, -A_-, i-2j; 0; -\alpha_+(\alpha_- + \zeta^*) e^{\alpha_0}) \right. \\ &\quad \left. + \sum_{m=1}^{i-1} \frac{\beta_-^m}{(\alpha_- + \zeta^*)^m e^{m\alpha_0}} \binom{i}{m} {}_3F_0(-A_+, -A_-, i-2j-m; 0; -\alpha_+(\alpha_- + \zeta^*) e^{\alpha_0}) \right] + 1 . \quad (1.113) \end{aligned}$$

The computations involve some non-trivial simplifications to bring it to Eq. (1.110).

We see that the first term in Eq. (1.110) is the point-wise product of the the two symbols and the term in the bracket gives the deformations.

As the star product was computed using the HACS we can be sure that they are well behaved at the conical singularity seen in the continuum.

²The derivation is shown in Appendix A

Chapter 2

Spin Systems on S_F^2

The Dirac and chirality operators are central for fundamental physics and also in noncommutative geometry, where it is used to formulate metrical, differential geometric and bundle-theoretic ideas following Connes' approach [74].

The theory of these operators on the fuzzy sphere S_F^2 can be formulated using the Ginsparg-Wilson(GW) algebra, or the approach of [75, 76] The GW algebra was originally encountered in the context of lattice gauge theories [77] where it was formulated in order to avoid the fermion doubling problem. The fact that this algebra appears naturally in the fuzzy case is interesting. In particular we shall see that it provides a way to formulate the Dirac and chirality operators for any non-zero spin. The latter in turn leads to a Dirac-like equation for any spin on S^2 and \mathbb{R}^2 with its associated chirality operator.

We shall hereafter refer to these Dirac-like equations and their chiralities just as Dirac and chirality operators.

These Dirac and chirality operators remind one of the Duffin-Kemmer, Rarita-Schwinger and Bargmann-Wigner equations. The relation between these well-known equations and those found here remain to be explored.

2.1 The Algebra of S_F^2

The algebra for the fuzzy sphere is characterized by a cut-off angular momentum L and is the full matrix algebra $Mat(2L+1) \equiv M_{2L+1}$ of $(2L+1) \times (2L+1)$ matrices. They can be generated by the $(2L+1)$ -dimensional irreducible representation (IRR) of $SU(2)$ with the standard angular momentum basis. The latter is represented by the angular momenta L_i^L acting on the left on $Mat(2L+1)$: If $\alpha \in Mat(2L+1)$,

$$L_i^L \alpha = L_i \alpha \tag{2.1}$$

$$[L_i^L, L_j^L] = i\epsilon_{ijk} L_k^L \tag{2.2}$$

$$(L_i^L)^2 = L(L+1)\mathbf{1} \otimes \mathbf{1} \tag{2.3}$$

where L_i are the standard angular momentum matrices for angular momentum L .

Since L_i^L acts on $Mat(2L+1)$ which is a vector space of dimension $(2L+1)^2$, it can be thought of as a square matrix of size $(2L+1)^2$. It is given by

$$L_i^L = L_i \otimes \mathbf{1} \quad (2.4)$$

where L_i is i -th component of the spin L representation of $SU(2)$.

We can also define right angular momenta L_i^R :

$$L_i^R \alpha = \alpha L_i, \alpha \in M_{2L+1} \quad (2.5)$$

$$[L_i^R, L_j^R] = -i\epsilon_{ijk} L_k^R \quad (2.6)$$

$$(L_i^R)^2 = L(L+1)\mathbf{1} \otimes \mathbf{1}. \quad (2.7)$$

Just as L_i^L is given by Eq.(2.4), we can write the matrix realization of L_i^R on $Mat(2L+1)$ as

$$L_i^R = \mathbf{1} \otimes L_i^T \quad (2.8)$$

where L_i^T is the transpose of the i -th component of the spin L representation of $SU(2)$.

We also have

$$[L_i^L, L_j^R] = 0. \quad (2.9)$$

The operator $\mathcal{L}_i = L_i^L - L_i^R$ is the fuzzy version of orbital angular momentum. They satisfy the $SU(2)$ angular momentum algebra

$$[\mathcal{L}_i, \mathcal{L}_j] = i\epsilon_{ijk} \mathcal{L}_k \quad (2.10)$$

In the continuum, S^2 can be described by the unit vector $\hat{x} \in S^2$, where $\hat{x} \cdot \hat{x} = 1$. Its analogue on S_F^2 is $\frac{L_i^L}{L}$ or $\frac{L_i^R}{L}$ such that

$$\lim_{L \rightarrow \infty} \frac{L_i^{L,R}}{L} = \hat{x}_i. \quad (2.11)$$

This shows that $L_i^{L,R}$ do not have continuum limits. But $\mathcal{L}_i = L_i^L - L_i^R$ does and becomes the orbital angular momentum as $L \rightarrow \infty$:

$$\lim_{L \rightarrow \infty} L_i^L - L_i^R = -i(\vec{r} \wedge \vec{\nabla})_i. \quad (2.12)$$

2.2 The GW Algebra

In algebraic terms, the GW algebra \mathcal{A} is the unital $*$ -algebra over \mathbb{C} , generated by two $*$ -invariant involutions Γ, Γ' .

$$\mathcal{A} = \{\Gamma, \Gamma' : \Gamma^2 = \Gamma'^2 = 1, \Gamma^* = \Gamma, \Gamma'^* = \Gamma'\} \quad (2.13)$$

In any $*$ -representation on a Hilbert space, $*$ becomes the adjoint \dagger .

Each representation of Eq.(2.13) is a particular realization of the GW algebra. Representations of interest in fuzzy physics are generally reducible.

2.3 The Spin System from the Fuzzy GW Algebra

Consider the following two elements constructed out of Γ, Γ' :

$$\Gamma_1 = \frac{1}{2}(\Gamma + \Gamma'), \quad (2.14)$$

$$\Gamma_2 = \frac{1}{2}(\Gamma - \Gamma'). \quad (2.15)$$

It follows from Eq.(2.13) that $\{\Gamma_1, \Gamma_2\} = 0$. This suggests that for suitable choices of Γ, Γ' , one of these operators may serve as the Dirac operator and the other as the chirality operator provided they have the right continuum limits after suitable scaling.

2.4 The Dirac and Chirality Operators on S^2

We can construct a set of anti-commuting operators and call them the Dirac and chirality operators after checking that they have the right properties. Consider

$${}^\infty D^j = (\Sigma_i - {}^\infty \gamma^j \Sigma_i {}^\infty \gamma^j)(\mathcal{L}_i + \Sigma_i) \quad (2.16)$$

where ${}^\infty \gamma^j$ satisfies $({}^\infty \gamma^j)^2 = 1$ and $({}^\infty \gamma^j)^\dagger = ({}^\infty \gamma^j)$. $\vec{\Sigma}$ is the spin j representation of $SU(2)$. It is easy to check that this form of ${}^\infty D^j$ in Eq.(2.16) implies that

$$\{{}^\infty D^j, {}^\infty \gamma^j\} = 0 \quad (2.17)$$

as ${}^\infty \gamma^j$ commutes with the total angular momentum $J_i = \mathcal{L}_i + \Sigma_i$. This follows from the following operator identity:

$$\{A, BC\} = \{A, B\}C - B\{A, C\} \quad (2.18)$$

Thus ${}^\infty D^j$ and ${}^\infty \gamma^j$ are Dirac and chirality operators.

2.4.1 ${}^\infty D^j$ and ${}^\infty \gamma^j$ for the Spin $\frac{1}{2}$ Case

Let us now explicitly construct ${}^\infty D^j$ and ${}^\infty \gamma^j$ for the spin $\frac{1}{2}$ case.

In the fuzzy case $\vec{\sigma} \cdot \vec{L}^L = L$ on the $L + \frac{1}{2}$ space and $\vec{\sigma} \cdot \vec{L}^L = -(L + 1)$ on the $L - \frac{1}{2}$ space. Thus taking their continuum limits gives us $\vec{\sigma} \cdot \hat{x} = \pm 1$ on these two spaces. An alternative way to find the eigenvalues of $\vec{\sigma} \cdot \hat{x}$ without taking continuum limits of the fuzzy case is by noting that we can choose the direction of \hat{x} to be along the third direction, which implies the eigenvalues of $\vec{\sigma} \cdot \hat{x}$ are just the eigenvalues of σ_3 namely ± 1 . This will be used extensively when we generalize to higher spins.

Using $\vec{\sigma} \cdot \hat{x}$, we can construct the projectors onto the two spaces with $\vec{\sigma} \cdot \hat{x} = \pm 1$:

$$P_1 = \frac{1 + \vec{\sigma} \cdot \hat{x}}{2} \quad (2.19)$$

and

$$P_{-1} = \frac{1 - \vec{\sigma} \cdot \hat{x}}{2} \quad (2.20)$$

Now for any projector P , $1 - 2P$ is an idempotent:

$$(1 - 2P)^2 = 1. \quad (2.21)$$

Thus from Eq.(2.19) and Eq.(2.20), we can read off the two chirality operators as $\pm\vec{\sigma} \cdot \hat{x}$.

The Dirac operators corresponding to these two chirality operators are the same due to the form of the Dirac operator given by Eq.(2.16).

We can compute ${}^\infty D^{\frac{1}{2}}$ using the algebra of the Pauli matrices. That gives us

$$\sigma_i - (\vec{\sigma} \cdot \hat{x})\sigma_i(\vec{\sigma} \cdot \hat{x}) = \sigma_i - x_i(\vec{\sigma} \cdot \hat{x}) \quad (2.22)$$

and thus from Eq.(2.16),

$${}^\infty D^{\frac{1}{2}} = \vec{\sigma} \cdot \vec{\mathcal{L}} + \frac{1}{2} \quad (2.23)$$

which is the well-known continuum Dirac operator for spin $\frac{1}{2}$ on S^2 [78]

2.4.2 ${}^\infty D^j$ and ${}^\infty \gamma^j$ on S^2 for the Spin 1 Case

Just as for the spin $\frac{1}{2}$ case, we can find the chirality operators in the continuum for the spin 1 case by noting that the eigenvalues of $\vec{\Sigma} \cdot \hat{x}$ are ± 1 and 0. We then write the projectors to the spaces where $\vec{\Sigma} \cdot \hat{x}$ takes these three values and by writing these projectors as $\frac{1+\gamma}{2}$ we can read off the three chirality operators. They are

$${}^\infty \gamma_1^1 = 1 - 2(\vec{\Sigma} \cdot \hat{x})^2, \quad (2.24)$$

$${}^\infty \gamma_2^1 = (\vec{\Sigma} \cdot \hat{x})^2 + (\vec{\Sigma} \cdot \hat{x}) - 1, \quad (2.25)$$

$${}^\infty \gamma_3^1 = (\vec{\Sigma} \cdot \hat{x})^2 - (\vec{\Sigma} \cdot \hat{x}) - 1. \quad (2.26)$$

The Dirac operator corresponding to Eq.(2.24) is found to be

$${}^\infty D_1^1 = \vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2. \quad (2.27)$$

The Dirac operator in Eq.(2.27) is found using the algebra of the spin 1 matrices [79] which is used to simplify

$$[\Sigma_i - (1 - 2(\vec{\Sigma} \cdot \hat{x})^2)\Sigma_i(1 - 2(\vec{\Sigma} \cdot \hat{x})^2)](\mathcal{L}_i + \Sigma_i). \quad (2.28)$$

We simplify the term in the square bracket after writing it in the form

$$[2\Sigma_i(\vec{\Sigma} \cdot \hat{x})^2 + (\vec{\Sigma} \cdot \hat{x})^2\Sigma_i - 4(\vec{\Sigma} \cdot \hat{x})^2\Sigma_i(\vec{\Sigma} \cdot \hat{x})^2]. \quad (2.29)$$

The first two terms in the above expression can be simplified using

$$\Sigma_i \Sigma_k \Sigma_j = \frac{i}{3}\varepsilon_{ikj} + \frac{1}{2}(\delta_{ik}\Sigma_j + \delta_{kj}\Sigma_i) + i\varepsilon_{ijm}Q_{km} \quad (2.30)$$

where Q_{km} is a symmetric tensor. This identity gives the sum of the first two terms as

$$A + B = 2\Sigma_i + 2(\vec{\Sigma} \cdot \hat{x})x_i \quad (2.31)$$

where A and B are the first two terms in Eq.(2.29). The identity in Eq.(2.30) can also be used to simplify the third term in Eq.(2.29) and we get

$$C = 2\Sigma_i(\vec{\Sigma} \cdot \hat{x})^2 + 2x_i(\vec{\Sigma} \cdot \hat{x}) - 4i\varepsilon_{ikm}Q_{jm}(\vec{\Sigma} \cdot \hat{x})^2x_kx_j. \quad (2.32)$$

Using Eq.(2.30), we can simplify this further to

$$C = 3x_i(\vec{\Sigma} \cdot \hat{x}) + \Sigma_i + 2i\varepsilon_{ijm}Q_{km}x_kx_j - 4i\varepsilon_{ikm}Q_{jm}(\vec{\Sigma} \cdot \hat{x})^2x_kx_j. \quad (2.33)$$

To evaluate this, we need to simplify the last term in the expression. That can be done using the following identities:

$$\Sigma_l \Sigma_n = \frac{2}{3} \delta_{ln} + \frac{i}{2} \varepsilon_{lno} \Sigma_o + Q_{ln} \quad (2.34)$$

and

$$\begin{aligned} Q_{jm} Q_{ln} &= \frac{1}{6} (\delta_{jl} \delta_{mn} + \delta_{jn} \delta_{lm} - \frac{2}{3} \delta_{jm} \delta_{ln}) \\ &\quad - \frac{1}{4} (\delta_{jl} Q_{mn} + \delta_{jn} Q_{lm} + \delta_{mn} Q_{jl} + \delta_{ml} Q_{jn} - \frac{4}{3} \delta_{jm} Q_{ln} - \frac{4}{3} \delta_{ln} Q_{jm}) \\ &\quad + \frac{i}{8} (\delta_{jl} \varepsilon_{mnp} \Sigma_p + \delta_{jn} \varepsilon_{mlp} \Sigma_p + \delta_{ml} \varepsilon_{jnp} \Sigma_p + \delta_{mn} \varepsilon_{jlp} \Sigma_p). \end{aligned} \quad (2.35)$$

On using these two identities, the last term in Eq.(2.33) becomes

$$2i \varepsilon_{ikm} Q_{lm} x_k x_l - x_i (\vec{\Sigma} \cdot \hat{x}) + \Sigma_i \quad (2.36)$$

This can then be substituted in Eq.(2.33) to get

$$C = 4x_i (\vec{\Sigma} \cdot \hat{x}). \quad (2.37)$$

With this, we obtain the following simple expression for $A + B - C$:

$$A + B - C = \Sigma_i - x_i (\vec{\Sigma} \cdot \hat{x}) \quad (2.38)$$

Multiplying this with $(\vec{\mathcal{L}}_i + \vec{\Sigma}_i)$ gives the Dirac operator in Eq.(2.27).

Next we write down the Dirac operators corresponding to the other chirality operators.

The Dirac operators corresponding to Eq.(2.25) and Eq.(2.26) are found to be

$${}^\infty D_2^1 = (\vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2) + 2(\vec{\Sigma} \cdot \hat{x}) + \{\vec{\Sigma} \cdot \vec{\mathcal{L}}, \vec{\Sigma} \cdot \hat{x}\} \quad (2.39)$$

and

$${}^\infty D_3^1 = (\vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2) - 2(\vec{\Sigma} \cdot \hat{x}) - \{\vec{\Sigma} \cdot \vec{\mathcal{L}}, \vec{\Sigma} \cdot \hat{x}\}. \quad (2.40)$$

These are found using the algebra of spin 1 matrices [79] as before.

These are the continuum limits which guide us in finding the fuzzy spin 1 Dirac operators.

2.5 More Dirac Operators on S^2

The construction used for the continuum Dirac operators in the previous section are not the only possible first order differential operators anticommuting with the chirality operators. We now show that there exist more such operators which can qualify as the Dirac operators. In what follows we show how to find out these operators in the spin 1 case. Using these ideas as a guiding principle, we generalize this construction to an arbitrary spin j .

We start with a chirality operator ${}^\infty \gamma^1$. This can be any one of the three chirality operators for the spin 1 case given in Eq.(2.24)-Eq.(2.26). We do not specify which one of these three operators we are dealing with now.

The Dirac operator corresponding to this chirality operator is given by Eq.(2.16) which we write down here again:

$${}^\infty D^1 = (\Sigma_i - {}^\infty \gamma^1 \Sigma_i {}^\infty \gamma^1) (\mathcal{L}_i + \Sigma_i). \quad (2.41)$$

Now consider a polynomial in $\vec{\Sigma} \cdot \hat{x}$ given by

$$P = a + b \left(\vec{\Sigma} \cdot \hat{x} \right) + c \left(\vec{\Sigma} \cdot \hat{x} \right)^2 \quad (2.42)$$

where a, b, c are constant numbers. They can even be functions of the cutoff L , but we do not consider that case here. We will consider these numbers to either belong to \mathbb{R} or \mathbb{C} . We discuss the two cases separately. Note that this polynomial is the most general polynomial in $\vec{\Sigma} \cdot \hat{x}$ for the spin 1 case. This follows from the relation $(\vec{\Sigma} \cdot \hat{x})^3 = \vec{\Sigma} \cdot \hat{x}$. This relation can be seen explicitly by choosing \hat{x} to be along the third direction making $\vec{\Sigma} \cdot \hat{x} = \Sigma_3$. the matrix for Σ_3 is the diagonal one with 1, 0, -1 as the diagonal entries. The relation then trivially follows. However in order to generalize to higher spins, we look for a more systematic way to arrive at this relation. This can indeed be found by looking at the chirality operators given by Eq.(2.24)-Eq.(2.26). The square of each of these operators is 1. That immediately gives a constraint relation in $\vec{\Sigma} \cdot \hat{x}$. However each of the chirality operators are quadratic in $\vec{\Sigma} \cdot \hat{x}$ and not all of them give non-trivial constraints between the powers of $\vec{\Sigma} \cdot \hat{x}$. It can be checked by trial and error that $(\infty\gamma_1^1)^2 = 1$ gives the required constraint and so does $(\infty\gamma_2^1)^2 = (\infty\gamma_3^1)^2$. Others give trivial constraints between the powers of $\vec{\Sigma} \cdot \hat{x}$. To check these relations we just need to choose \hat{x} along the third direction.

This systematic procedure of obtaining non-trivial constraints between the powers of $\vec{\alpha} \cdot \hat{x}$ (where $\vec{\alpha}$ is now the matrices of an higher spin j), is essential for writing the most general polynomials for higher spins. We will write down the general polynomials for an arbitrary spin j when we consider this construction for higher spins.

Having obtained the most general polynomials in $\vec{\Sigma} \cdot \hat{x}$ for the spin 1 case, we can write down the other Dirac operators. To do this we consider the cases where the coefficients in the polynomials in $\vec{\Sigma} \cdot \hat{x}$ belong to \mathbb{R} or \mathbb{C} separately.

2.5.1 Case 1: $a, b, c \in \mathbb{R}$

We construct three sets of operators which anticommute with $\infty\gamma^1$.

Consider the first set

$$\infty D_P^1 = P \infty D^1 + \infty D^1 P. \quad (2.43)$$

This operator is hermitian as $\vec{\Sigma} \cdot \hat{x}$ is hermitian and $\{\infty D_P^1, \infty\gamma^1\} = 0$ as $[\infty\gamma^1, P] = 0$.

The second set consists of operators of the form

$$\infty D_P^1 = P \infty D^1 P. \quad (2.44)$$

It is easy to check that $\{\infty D_P^1, \infty\gamma^1\} = 0$.

The third set of operators are

$$\infty D_P^1 = P_1 \infty D^1 P_2 + P_2 \infty D^1 P_1 \quad (2.45)$$

where $P_i = a_i + b_i \left(\vec{\Sigma} \cdot \hat{x} \right) + c_i \left(\vec{\Sigma} \cdot \hat{x} \right)^2$ $i = 1, 2$ are two different polynomials differing in their coefficients.

Let us simplify these operators further by writing polynomials in $\vec{\Sigma} \cdot \hat{x}$ as polynomials in $\infty\gamma_i^1$ for $i = 1, 2, 3$ which are the chirality operators in the spin 1 case on S^2 . We have from Eq.(2.24)-Eq.(2.26) that

$$\left(\vec{\Sigma} \cdot \hat{x} \right)^2 = \frac{1 - \infty\gamma_1^1}{2} \quad (2.46)$$

and

$$\vec{\Sigma} \cdot \hat{x} = \frac{\infty\gamma_2^1 - \infty\gamma_3^1}{2}. \quad (2.47)$$

We can now write

$$P = a + \frac{b}{2} (\infty\gamma_2^1 - \infty\gamma_3^1) + \frac{c}{2} (1 - \infty\gamma_1^1). \quad (2.48)$$

With this consider

$$\begin{aligned} \infty D_P^1 &= P \infty D^1 P \\ &= \left[\left(a + \frac{c}{2} \right) + \frac{b}{2} \infty\gamma_2^1 - \frac{b}{2} \infty\gamma_3^1 - \frac{c}{2} \infty\gamma_1^1 \right] \infty D^1 \\ &\times \left[\left(a + \frac{c}{2} \right) + \frac{b}{2} \infty\gamma_2^1 - \frac{b}{2} \infty\gamma_3^1 - \frac{c}{2} \infty\gamma_1^1 \right]. \end{aligned} \quad (2.49)$$

Now choose ∞D^1 to be ∞D_1^1 that is the Dirac operator corresponding to the chirality operator, $\infty\gamma_1^1$. Using the property that these two operators anticommute with each other and the fact that $\infty\gamma_1^1 \infty\gamma_2^1 = \infty\gamma_3^1$ we find that

$$\infty D_P^1 = (a^2 + ac) \infty D_1^1 + \frac{ab}{2} [\{\infty\gamma_2^1, \infty D_1^1\} - \{\infty\gamma_3^1, \infty D_1^1\}]. \quad (2.50)$$

In arriving at this formula we have also used the identity that

$$\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1 = 0. \quad (2.51)$$

The proof of this identity is as follows:

$$\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1 = \infty\gamma_1^1 [\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1] \infty\gamma_1^1. \quad (2.52)$$

But the right hand side can be written as

$$\begin{aligned} \infty\gamma_1^1 [\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1] \infty\gamma_1^1 &= \infty\gamma_2^1 \infty\gamma_1^1 \infty D_1^1 \infty\gamma_1^1 \infty\gamma_2^1 \\ &+ \infty\gamma_3^1 \infty\gamma_1^1 \infty D_1^1 \infty\gamma_1^1 \infty\gamma_3^1 \\ &= -[\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1]. \end{aligned} \quad (2.53)$$

Thus

$$\infty\gamma_2^1 \infty D_1^1 \infty\gamma_2^1 + \infty\gamma_3^1 \infty D_1^1 \infty\gamma_3^1 = 0. \quad (2.54)$$

Let us simplify ∞D_P^1 by substituting Eq.(2.27) for ∞D_1^1 and Eq.(2.25) and Eq.(2.26) for $\infty\gamma_2^1$ and $\infty\gamma_3^1$ respectively. We then get

$$\{\infty\gamma_2^1, \infty D_1^1\} = \{(\vec{\Sigma} \cdot \hat{x})^2, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} + \{\vec{\Sigma} \cdot \hat{x}, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} + 4(\vec{\Sigma} \cdot \hat{x})^2 + 2\vec{\Sigma} \cdot \hat{x} - 2\vec{\Sigma} \cdot \vec{\mathcal{L}} - 4 \quad (2.55)$$

and

$$\{\infty\gamma_3^1, \infty D_1^1\} = \{(\vec{\Sigma} \cdot \hat{x})^2, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} - \{\vec{\Sigma} \cdot \hat{x}, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} + 4(\vec{\Sigma} \cdot \hat{x})^2 - 2\vec{\Sigma} \cdot \hat{x} - 2\vec{\Sigma} \cdot \vec{\mathcal{L}} - 4. \quad (2.56)$$

Thus

$$\{\infty\gamma_2^1, \infty D_1^1\} - \{\infty\gamma_3^1, \infty D_1^1\} = 2\{\vec{\Sigma} \cdot \hat{x}, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} + 4\vec{\Sigma} \cdot \hat{x}. \quad (2.57)$$

Using these we get

$$\infty D_P^1 = (a^2 + ac) \infty D_1^1 + ab\{\vec{\Sigma} \cdot \hat{x}, \vec{\Sigma} \cdot \vec{\mathcal{L}}\} + 2ab\vec{\Sigma} \cdot \hat{x}. \quad (2.58)$$

In a similar way

$$\begin{aligned} {}^\infty D_P^1 &= P {}^\infty D_1^1 + {}^\infty D_1^1 P \\ &= \left[\left(a + \frac{c}{2} \right) + \frac{b}{2} {}^\infty \gamma_2^1 - \frac{b}{2} {}^\infty \gamma_3^1 - \frac{c}{2} {}^\infty \gamma_1^1 \right] {}^\infty D_1^1 \\ &\quad + {}^\infty D_1^1 \left[\left(a + \frac{c}{2} \right) + \frac{b}{2} {}^\infty \gamma_2^1 - \frac{b}{2} {}^\infty \gamma_3^1 - \frac{c}{2} {}^\infty \gamma_1^1 \right]. \end{aligned} \quad (2.59)$$

This can be simplified to

$${}^\infty D_P^1 = (2a + c) {}^\infty D_1^1 + \frac{b}{2} [\{ {}^\infty \gamma_2^1, {}^\infty D_1^1 \} - \{ {}^\infty \gamma_3^1, {}^\infty D_1^1 \}]. \quad (2.60)$$

By a similar manipulation as before we obtain

$${}^\infty D_P^1 = (2a + c) {}^\infty D_1^1 + b \{ \vec{\Sigma} \cdot \hat{x}, \vec{\Sigma} \cdot \vec{\mathcal{L}} \} + 2b \vec{\Sigma} \cdot \hat{x}. \quad (2.61)$$

Transformations between the different sets of Dirac Operators

The polynomials in $\vec{\Sigma} \cdot \hat{x}$ for $a, b, c \in \mathbb{R}$ form a closed algebra. The multiplication of any two polynomials gives another polynomial in this algebra. Thus to transform from one Dirac operator to another we need to multiply the Dirac operators by some polynomial in $\vec{\Sigma} \cdot \hat{x}$ and its hermitian conjugate. Thus let

$$P' = a' + b' \vec{\Sigma} \cdot \hat{x} + c' (\vec{\Sigma} \cdot \hat{x})^2 \quad (2.62)$$

then

$$(P')^\dagger {}^\infty D_1^1 P' = (P_0)^\dagger P^\dagger {}^\infty D_1^1 P(P_0) \quad (2.63)$$

where

$$P_0 = a_0 + a_1 \vec{\Sigma} \cdot \hat{x} + a_2 (\vec{\Sigma} \cdot \hat{x})^2 \quad (2.64)$$

and

$$P = a + b \vec{\Sigma} \cdot \hat{x} + c (\vec{\Sigma} \cdot \hat{x})^2. \quad (2.65)$$

As the coefficients are real the operators are hermitian.

By computing $P_0 P = P'$ and comparing coefficients we have

$$a_0 a = a' \quad (2.66)$$

$$a_1 a + (a_0 + a_2) b + a_1 c = b' \quad (2.67)$$

$$a_2 a + a_1 b + (a_0 + a_2) c = c' \quad (2.68)$$

These transformations can be thought of as merely arising due to a change in the coefficients of the polynomials. To find out the group connected to these transformations let us represent the polynomial with its coefficients as a 3 by 1 column matrix. The transformations in Eq.(2.66-2.68) can be written as

$$\begin{pmatrix} a_0 & 0 & 0 \\ a_1 & a_0 + a_2 & a_1 \\ a_2 & a_1 & a_0 + a_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}. \quad (2.69)$$

It is easy to see that the set of these 3 by 3 matrices form a group. It is generated by

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.70)$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.71)$$

and

$$J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (2.72)$$

The generators commute with each other and satisfy the following algebra

$$(J_1)^n = \begin{cases} J_2 & n \text{ is even} \\ J_1 & n \text{ is odd.} \end{cases} \quad (2.73)$$

$$(J_2)^n = J_2 \quad \forall n \in \mathbb{Z}^+. \quad (2.74)$$

$J_1 = \vec{\Sigma} \cdot \hat{x}$ and $J_2 = (\vec{\Sigma} \cdot \hat{x})^2$ realize this algebra. So we have if

$$e^{[kI+l\vec{\Sigma}\cdot\hat{x}+m(\vec{\Sigma}\cdot\hat{x})^2]} = P_0 \quad (2.75)$$

then

$$k = \ln a_0 \quad (2.76)$$

$$l = \ln \sqrt{\frac{a + \frac{a_2}{a_0} + \frac{a_1}{a_0}}{1 + \frac{a_2}{a_0} - \frac{a_1}{a_0}}} \quad (2.77)$$

$$m = \ln \sqrt{\left(1 + \frac{a_2}{a_0} + \frac{a_1}{a_0}\right) \left(a + \frac{a_2}{a_0} - \frac{a_1}{a_0}\right)}. \quad (2.78)$$

Solving Eq.(2.66-2.68) for a_0, a_1, a_2 in terms of a, b, c and a', b', c' we get

$$a_0 = \frac{a'}{a} \quad (2.79)$$

$$a_1 = \frac{1}{[b^2 - (a+c)^2]} [c'b - b'c + a'b - b'a] \quad (2.80)$$

$$a_2 = \frac{1}{[b^2 - (a+c)^2]} \left[b'b - c'c + a'c - c'a + \frac{a'}{a}(c^2 - b^2) \right]. \quad (2.81)$$

The singularities in these expressions imply that we cannot transform to operators where $a = 0$ or $b = \pm(a+c)$.

We note that then it is not possible to transform from ${}^\infty D_1^1$ given by Eq.(2.27) to ${}^\infty D_2^1$ given by Eq.(2.39) or ${}^\infty D_3^1$ given by Eq.(2.40). This is because

$${}^\infty D_P^1 = (a^2 + ac) {}^\infty D_1^1 + \frac{ab}{2} [\{{}^\infty \gamma_2^1, {}^\infty D_1^1\} - \{{}^\infty \gamma_3^1, {}^\infty D_1^1\}] \quad (2.82)$$

which requires us to go from an operator with $a = 1$, $b = 0$, $c = 0$ to an operator where $a^2 + ac = 1$ and $ab = 1$ which implies that $a^2 + ac = ab$ or $b = a + c$ which is a singular point in these transformations. Similarly to go to ${}^\infty D_3^1$ we will have $b = -(a + c)$ which is again a singular point as we have seen.

The transformations we just studied help us to transform between the set of Dirac operators given by the set in Eq.(2.44). It should be noted that it is not possible to go from an operator in the set given by Eq.(2.44) to an operator given in Eq.(2.43). It is also not possible to transform from an operator in Eq.(2.44) to one given in Eq.(2.45). Thus it is clear that there is no transformation between the Dirac operator in Eq.(2.27) to the Dirac operators given in Eq.(2.39) and Eq.(2.40).

It is easy to see that we can transform between two Dirac operators, one of which is in the set of Dirac operators given by Eq.(2.43) and the other is in the set given by Eq.(2.45). The transformations between two Dirac operators in the set given by Eq.(2.43) are more restricted as we can only go from a given operator in this set to another operator characterized by a polynomial which is the inverse of the given polynomial.

These transformations help define an equivalence relation among all these Dirac operators. They then split this space of Dirac operators into equivalence classes. It is also clear that since these transformations are not unitary, these operators are unitarily inequivalent to each other.

2.5.2 Case 2: $a, b, c \in \mathbb{C}$

In this case the three sets of Dirac operators given by Eq.(2.43), Eq.(2.44) and Eq.(2.45) become

$${}^\infty D_P^1 = P^\dagger {}^\infty D^1 + {}^\infty D^1 P, \quad (2.83)$$

$${}^\infty D_P^1 = P^\dagger {}^\infty D^1 P \quad (2.84)$$

and

$${}^\infty D_P^1 = P_1 {}^\infty D^1 P_2 + P_2^\dagger {}^\infty D^1 P_1^\dagger \quad (2.85)$$

where $P^\dagger = a^* + b^* \vec{\Sigma} \cdot \hat{x} + c^* (\vec{\Sigma} \cdot \hat{x})^2$, a^* , b^* and c^* denote complex conjugates.

We can simplify Eq.(2.84) as we did in the real case to obtain

$$\begin{aligned} {}^\infty D_P^1 &= |a|^2 {}^\infty D_1^1 + a^* b {}^\infty D_1^1 (\vec{\Sigma} \cdot \hat{x}) + b^* a (\vec{\Sigma} \cdot \hat{x}) {}^\infty D_1^1 \\ &+ c^* a (\vec{\Sigma} \cdot \hat{x})^2 {}^\infty D_1^1 + a^* c {}^\infty D_1^1 (\vec{\Sigma} \cdot \hat{x})^2. \end{aligned} \quad (2.86)$$

Similarly for Eq.(2.83) we obtain

$$\begin{aligned} {}^\infty D_P^1 &= \operatorname{Re} \left(a + \frac{c}{2} \right) {}^\infty D_1^1 + \frac{b^*}{2} {}^\infty \gamma_2^1 {}^\infty D_1^1 + \frac{b}{2} {}^\infty D_1^1 {}^\infty \gamma_2^1 - \frac{b^*}{2} {}^\infty \gamma_3^1 {}^\infty D_1^1 \\ &- \frac{b}{2} {}^\infty D_1^1 {}^\infty \gamma_3^1 + \operatorname{Im} c {}^\infty \gamma_1^1 {}^\infty D_1^1. \end{aligned} \quad (2.87)$$

The group of transformations between these operators become unitary groups. Since the group is generated by three abelian generators, the group is $U(1) \times U(1) \times U(1)$. With the realization of this algebra with the operators $\vec{\Sigma} \cdot \hat{x}$ and $(\vec{\Sigma} \cdot \hat{x})^2$, these transformations depend on \hat{x} , thus making this group local. The Dirac operators in this case are all unitarily equivalent resulting in the same spectrum for these operators.

It should be noted here that the Dirac operators given in Eq.(2.39) and Eq.(2.40) are not of the form given in Eq.(2.86) and Eq.(2.87). This proves unitary inequivalence between these

operators and the Dirac operator given by Eq.(2.27). Note that this proof is in the continuum case. We will see later that this inequivalence can also be proved in the fuzzy case by a simple evaluation of the traces of the corresponding fuzzy operators.

These results can be extended to higher spins as we shall now briefly see.

2.5.3 Remarks for the Dirac Operators for Higher Spins

To extend the previous results in the spin 1 case to higher spins we need to first study the construction of the most general polynomials in $\vec{\alpha} \cdot \hat{x}$. We just write down the results in what follows.

The constraint relations among the powers of $\vec{\alpha} \cdot \hat{x}$ can be obtained from the identities satisfied by chirality operators for higher spins such as $(\infty\gamma_i^j)^2 = (\infty\gamma_k^j)^2$ where j is the spin and i and k take values between $1, \dots$ and $2j + 1$. Note that these are just indices to denote the $2j + 1$ chirality operators associated to each spin j . We give an example of such a constraint for the spin $\frac{3}{2}$ case.

$$16(\vec{\chi} \cdot \hat{x})^4 - 40(\vec{\chi} \cdot \hat{x})^2 + 9 = 0 \quad (2.88)$$

where $\vec{\chi}$ now denotes the matrices of the spin $\frac{3}{2}$ representation of $SU(2)$. This constraint implies that the most general polynomial in $\vec{\alpha} \cdot \hat{x}$ for the spin $\frac{3}{2}$ case is third order in $\vec{\alpha} \cdot \hat{x}$.

This result can be easily generalized to an arbitrary spin j . The answer is that the most general polynomials in $\vec{\alpha} \cdot \hat{x}$ is of order $2j$ in $\vec{\alpha} \cdot \hat{x}$. This can be seen due to the fact that the chirality operators for a spin j are of order $2j$ in $\vec{\alpha} \cdot \hat{x}$.

The analysis of finding the transformation group between the different Dirac operators can be carried out in these cases as well. For the case where the coefficients of the polynomials belong to \mathbb{C} , the group is seen to be $[U(1)]^{2j+1}$ where the power denotes the number of times we have to take the direct product between the $U(1)$ group.

It is important to note here that this analysis can be carried out even for the spin $\frac{1}{2}$ case. In this case the polynomials are first order in $\vec{\sigma} \cdot \hat{x}$. But for the spin $\frac{1}{2}$ system the chirality operator is given by $\vec{\sigma} \cdot \hat{x}$ and hence such an analysis only yields multiples of the Dirac operator anticommuting with $\vec{\sigma} \cdot \hat{x}$.

2.6 Construction of the Fuzzy Spin $\frac{1}{2}$ Dirac and Chirality Operators on S_F^2

The construction is based on the GW algebra of [80, 81]. First we note that if P is a projector, then,

$$P^2 = P \quad (2.89)$$

and $\gamma = 2P - 1$ is an idempotent:

$$\gamma^2 = 1. \quad (2.90)$$

We now construct Γ, Γ' from suitable projectors.

Consider $Mat(2L + 1) \otimes \mathbb{C}^2$. The spin $\frac{1}{2}$ IRR of $SU(2)$ acts on \mathbb{C}^2 . It has the standard Lie algebra basis $\frac{\sigma_i}{2}$, σ_i being the Pauli matrices. The projector coupling the left angular momentum and this spin $\frac{1}{2}$ to its maximum value $L + \frac{1}{2}$ is

$$P_{L+\frac{1}{2}}^L = \frac{\vec{\sigma} \cdot \vec{L}^L + L + 1}{2L + 1}. \quad (2.91)$$

Hence the corresponding idempotent is

$$\Gamma_{L+\frac{1}{2}}^L = \frac{\vec{\sigma} \cdot \vec{L}^L + \frac{1}{2}}{L + \frac{1}{2}}. \quad (2.92)$$

The projector $P_{L+\frac{1}{2}}^R$ coupling the right angular momentum and spin $\frac{1}{2}$ to $L + \frac{1}{2}$ is obtained by changing \vec{L}^L to $-\vec{L}^R$ in the above expression:

$$P_{L+\frac{1}{2}}^R = \frac{-\vec{\sigma} \cdot \vec{L}^R + L + 1}{2L + 1}. \quad (2.93)$$

The minus sign is because of the minus sign in Eq.(2.6).

The corresponding idempotent is

$$\Gamma_{L+\frac{1}{2}}^R = \frac{-\vec{\sigma} \cdot \vec{L}^R + \frac{1}{2}}{L + \frac{1}{2}}. \quad (2.94)$$

Identifying $\Gamma_{L+\frac{1}{2}}^{L,R}$ with Γ, Γ' , we get

$$\Gamma_1 = \frac{1}{2} \left[\frac{\vec{\sigma} \cdot \vec{L} + 1}{L + \frac{1}{2}} \right] \quad (2.95)$$

and

$$\Gamma_2 = \frac{1}{2} \left[\frac{\vec{\sigma} \cdot (\vec{L}^L + \vec{L}^R)}{L + \frac{1}{2}} \right] \quad (2.96)$$

Now as $L \rightarrow \infty$,

$$2L\Gamma_1 \rightarrow \vec{\sigma} \cdot \vec{L} + 1 \quad (2.97)$$

and

$$\Gamma_2 \rightarrow \vec{\sigma} \cdot \hat{x}. \quad (2.98)$$

These are the correct Dirac and chirality operators on S^2 and so we can regard $2L\Gamma_1$ as the fuzzy Dirac operator (upto a finite scaling) and Γ_2 as its chirality operator.

2.7 Ambiguities in the Fuzzy Spin $\frac{1}{2}$ Dirac and Chirality Operators

Having looked at the construction of the spin $\frac{1}{2}$ Dirac operator as given in [82], we now consider other possibilities for constructing the same Dirac operator. This observation turns out to be crucial in finding the Dirac operator for higher spins.

The projectors $P_{L+\frac{1}{2}}^{L,R}$ are not the only projectors with rotational invariance. We can also consider the two projectors to the $L - \frac{1}{2}$ space, obtained by coupling the left and right angular momenta $L_i^{L,R}$ and spin $\frac{1}{2}$. These are,

$$P_{L-\frac{1}{2}}^L = - \left(\frac{\vec{\sigma} \cdot \vec{L}^L - L}{2L + 1} \right), \quad (2.99)$$

and

$$P_{L-\frac{1}{2}}^R = - \left(\frac{-\vec{\sigma} \cdot \vec{L}^R - L}{2L+1} \right). \quad (2.100)$$

This gives us two new generators, $\Gamma_{L-\frac{1}{2}}^{L,R}$, to the GW algebra. Thus there are a total of four rotationally invariant idempotents which we list in the following table

$$P_{L+\frac{1}{2}}^{L,R} : \quad \Gamma_{L+\frac{1}{2}}^L \quad \Gamma_{L+\frac{1}{2}}^R \quad (2.101)$$

$$P_{L-\frac{1}{2}}^{L,R} : \quad \Gamma_{L-\frac{1}{2}}^L \quad \Gamma_{L-\frac{1}{2}}^R \quad (2.102)$$

The negatives of these idempotents are also idempotents, but that is a trivial ambiguity.

Now a GW algebra is generated by any pair from this table. However if we adopt the two left or the two right as Γ and Γ' , then Γ_1 and Γ_2 have no suitable continuum limit. We can see this from choosing as our generators either $\Gamma_{L\pm\frac{1}{2}}^L$ or $\Gamma_{L\pm\frac{1}{2}}^R$. We observe that $\Gamma_{L+\frac{1}{2}}^L = -\Gamma_{L-\frac{1}{2}}^L$ and $\Gamma_{L+\frac{1}{2}}^R = -\Gamma_{L-\frac{1}{2}}^R$, which as remarked above is a trivial ambiguity. So clearly we cannot construct suitable GW algebras from such pairs of idempotents.

But if we now use the two operators $\Gamma_{L+\frac{1}{2}}^L$ and $\Gamma_{L-\frac{1}{2}}^R$ and consider the combination $(L + \frac{1}{2})(\Gamma_{L+\frac{1}{2}}^L - \Gamma_{L-\frac{1}{2}}^R)$, we get the Dirac operator given in Eq.(2.97). As we saw earlier in section 2 [82], this Dirac operator is found by adding $\Gamma_{L+\frac{1}{2}}^L$ and $\Gamma_{L+\frac{1}{2}}^R$ and scaling as $L \rightarrow \infty$. The corresponding chirality operator is got from $\frac{\Gamma_{L+\frac{1}{2}}^L + \Gamma_{L-\frac{1}{2}}^R}{2}$ as this goes to the correct limit as $L \rightarrow \infty$ which is $\sigma \cdot \hat{x}$. The other possibility of combining $\Gamma_{L-\frac{1}{2}}^L$ and $\Gamma_{L+\frac{1}{2}}^R$ also exists and it is easy to see that $-(L + \frac{1}{2})(\Gamma_{L-\frac{1}{2}}^L - \Gamma_{L+\frac{1}{2}}^R)$ also goes to the Dirac operator given by Eq.(2.97) while $\frac{\Gamma_{L-\frac{1}{2}}^L + \Gamma_{L+\frac{1}{2}}^R}{2}$ goes to the corresponding chirality operator. This exhausts all the possible combinations.

We again note here that we can only construct our desired Dirac and chirality operators by choosing one Γ from the second column and one from the third column of Eq.(2.101) and Eq.(2.102) as we will not get a differential operator in the continuum if we choose them from the same column.

The fact that there exist all these possibilities for combining various generators of the GW algebra for obtaining the fuzzy Dirac and chirality operators imply that we should take care while writing the corresponding versions of higher spin Dirac and chirality operators as not all of them may go to correct continuum limits. In the case of spin $\frac{1}{2}$, all the possibilities go to the correct continuum limit, but as we shall soon see, this fails in the case of higher spins. This calls for a rule to construct the fuzzy versions of these operators, which we shall formulate after studying the spin 1 case in detail. We shall also see later that this becomes essential for finding the Dirac operators in the continuum for higher spins.

2.8 The Fuzzy Spin 1 Dirac Operator

Consider $Mat(2L+1) \otimes \mathbb{C}^3$, where $Mat(2L+1)$ is the carrier space of spin $L \otimes L$ representation of $SU(2)$ acting on left and right and \mathbb{C}^3 is the carrier space of the spin 1 representation of $SU(2)$. When a spin L couples with spin 1, we have three possible spaces labeled by the values of the total angular momentum $L+1, L$ and $L-1$. So we have six projectors and as in Eq.(2.101)

and Eq.(2.102) we can construct the corresponding generators of the GW algebra. Thus we have a table similar to the one in Eq.(2.101) and Eq.(2.102):

$$P_{L+1}^{L,R} : \quad \Gamma_{L+1}^L \quad \Gamma_{L+1}^R \quad (2.103)$$

$$P_L^{L,R} : \quad \Gamma_L^L \quad \Gamma_L^R \quad (2.104)$$

$$P_{L-1}^{L,R} : \quad \Gamma_{L-1}^L \quad \Gamma_{L-1}^R \quad (2.105)$$

The three projectors corresponding to the left angular momentum coupling to spin 1 are

$$P_{L+1}^L = \frac{(\vec{\Sigma} \cdot \vec{L}^L + L + 1)(\vec{\Sigma} \cdot \vec{L}^L + 1)}{(L + 1)(2L + 1)} \quad (2.106)$$

$$P_L^L = -\frac{(\vec{\Sigma} \cdot \vec{L}^L - L)(\vec{\Sigma} \cdot \vec{L}^L + L + 1)}{L(L + 1)} \quad (2.107)$$

$$P_{L-1}^L = \frac{(\vec{\Sigma} \cdot \vec{L}^L - L)(\vec{\Sigma} \cdot \vec{L}^L + 1)}{(2L + 1)L} \quad (2.108)$$

while the corresponding right projectors are obtained from above by substituting \vec{L}^L by $-\vec{L}^R$.

Writing each projector as $\frac{1+\Gamma}{2}$ and \vec{L} as \vec{L}^L or $-\vec{L}^R$, we can find the generators of the GW algebra for each of the projectors above. Let us write down the relevant generators whose combinations give the fuzzy Dirac and chirality operators having the right continuum limits which we found in the previous section.

$$\Gamma_{L+1}^L = \frac{2(\vec{\Sigma} \cdot \vec{L}^L + L + 1)(\vec{\Sigma} \cdot \vec{L}^L + 1) - (L + 1)(2L + 1)}{(L + 1)(2L + 1)} \quad (2.109)$$

$$\Gamma_{L+1}^R = \frac{2(-\vec{\Sigma} \cdot \vec{L}^R + L + 1)(-\vec{\Sigma} \cdot \vec{L}^R + 1) - (L + 1)(2L + 1)}{(L + 1)(2L + 1)} \quad (2.110)$$

$$\Gamma_{L-1}^L = \frac{2(\vec{\Sigma} \cdot \vec{L}^L - L)(\vec{\Sigma} \cdot \vec{L}^L + 1) - L(2L + 1)}{L(2L + 1)} \quad (2.111)$$

$$\Gamma_{L-1}^R = \frac{2(\vec{\Sigma} \cdot \vec{L}^R + L)(\vec{\Sigma} \cdot \vec{L}^R - 1) - L(2L + 1)}{L(2L + 1)} \quad (2.112)$$

We can immediately see that $\frac{\Gamma_{L-1}^L \pm \Gamma_{L+1}^R}{2}$, are chirality and Dirac operators (the latter upto an overall constant) for the fuzzy sphere by checking their continuum limits. Thus as $L \rightarrow \infty$,

$$\frac{\Gamma_{L-1}^L + \Gamma_{L+1}^R}{2} \rightarrow (\vec{\Sigma} \cdot \hat{x})^2 - \vec{\Sigma} \cdot \hat{x} - 1, \quad (2.113)$$

which is a chirality operator for the spin 1 case in the continuum which we encountered in the previous section. Also

$$\lim_{L \rightarrow \infty} L \left(\frac{\Gamma_{L-1}^L - \Gamma_{L+1}^R}{2} \right) = -(\vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2) \quad (2.114)$$

is the corresponding Dirac operator as $L \left(\frac{\Gamma_{L-1}^L - \Gamma_{L+1}^R}{2} \right)$ anti-commutes with $\frac{\Gamma_{L-1}^L + \Gamma_{L+1}^R}{2}$. The Dirac operator got from the fuzzy case in Eq.(2.114) is unitarily equivalent to the one got in

Eq.(2.39). Eq.(2.114) can be seen by substituting the expressions for Γ_{L-1}^L and Γ_{L+1}^R from Eq.(2.111) and Eq.(2.110) respectively and grouping terms similar in the order of \vec{L}^L and \vec{L}^R .

Here we note the order L term in the expression

$$L \frac{((\vec{\Sigma} \cdot \vec{L}^L)^2 - (\vec{\Sigma} \cdot \vec{L}^R)^2)}{(L+1)(2L+1)} \quad (2.115)$$

got by grouping the second order terms. As $L \rightarrow \infty$ this term goes to $-\frac{(\vec{\Sigma} \cdot \hat{x})}{2}$. This can be understood easily by noting that $[L_i^L, L_j^L] = i\varepsilon_{ijk}L_k^L$, produces first order terms in L and these commutators arise when we expand $L_i^L L_j^L$ as a sum of a commutator and an anticommutator.

Similarly we find the chirality and Dirac operators $\frac{\Gamma_{L+1}^L + \Gamma_{L-1}^R}{2}$ for the fuzzy sphere(the latter upto a constant) and their continuum limits.

$$\frac{\Gamma_{L+1}^L + \Gamma_{L-1}^R}{2} \rightarrow (\vec{\Sigma} \cdot \hat{x})^2 + \vec{\Sigma} \cdot \hat{x} - 1 \quad (2.116)$$

and

$$L \left(\frac{\Gamma_{L+1}^L - \Gamma_{L-1}^R}{2} \right) \rightarrow (\vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2) + 2(\vec{\Sigma} \cdot \hat{x}) + \{\vec{\Sigma} \cdot \vec{\mathcal{L}}, \vec{\Sigma} \cdot \hat{x}\} \quad (2.117)$$

as $L \rightarrow \infty$.

We can also see that γ_1 in Eq.(2.24) is got by taking the continuum limit of

$$\frac{\Gamma_L^L + \Gamma_L^R}{2} \quad (2.118)$$

where

$$\Gamma_L^L = \frac{-2(\vec{\Sigma} \cdot \vec{L}^L - L)(\vec{\Sigma} \cdot \vec{L}^L + L + 1) - L(L+1)}{L(L+1)} \quad (2.119)$$

$$\Gamma_L^R = \frac{2(\vec{\Sigma} \cdot \vec{L}^R + L)(-\vec{\Sigma} \cdot \vec{L}^R + L + 1) - L(L+1)}{L(L+1)} \quad (2.120)$$

This implies $L \left(\frac{\Gamma_L^L - \Gamma_L^R}{2} \right)$ goes to the corresponding Dirac operator. Thus $\frac{\Gamma_L^L + \Gamma_L^R}{2}$ and constant times $\frac{\Gamma_L^L - \Gamma_L^R}{2}$ can also serve as chirality and Dirac operators.

The continuum limit of the combination $\Gamma_L^R + \Gamma_{L+1}^L$ goes to $\vec{\Sigma} \cdot \hat{x} - (\vec{\Sigma} \cdot \hat{x})^2$ which is not part of the chiralities we obtained in the continuum in section 2.4.2. They are not unitarily to equivalent to any of those obtained in section 2.4.2 either. Other combinations like $\Gamma_L^R + \Gamma_{L-1}^L$ go to a chirality we do not have in the continuum as formulated in section 2.4.2. The combinations anticommuting with these namely $L(\Gamma_L^R - \Gamma_{L+1}^L)$ and $L(\Gamma_L^R + \Gamma_{L-1}^L)$ do not have proper continuum limits, in fact they diverge. Hence we discard these combinations.

2.9 Construction of Higher Spin Dirac Operators on S_F^2

The projectors to spaces, got by coupling L to higher spins contain more factors increasing the order in $\vec{L}^{L,R}$ and making the expressions look complicated. We observe the kind of terms

that can emerge from simplifying these expressions and formulate rules to take their continuum limits.

We first carefully look at the spin $\frac{3}{2}$ case and use this to generalize to terms emerging from higher spins. We have eight projectors in this case which are $P_{L+\frac{3}{2}}^{L,R}$, $P_{L+\frac{1}{2}}^{L,R}$, $P_{L-\frac{1}{2}}^{L,R}$, $P_{L-\frac{3}{2}}^{L,R}$. We can construct the generators of the GW algebra from each of these projectors and thus construct a table similar to that shown in Eq.(2.103)-Eq.(2.105). From this table, let us take the relevant Γ operators whose combination gives us the fuzzy Dirac operator. Consider

$$\Gamma_{L+\frac{3}{2}}^L = \frac{(2\vec{\chi} \cdot \vec{L}^L - L + 3)(2\vec{\chi} \cdot \vec{L}^L + L + 4)(2\vec{\chi} \cdot \vec{L}^L + 3L + 3) - 6(L + 1)(2L + 3)(2L + 1)}{6(L + 1)(2L + 1)(2L + 3)} \quad (2.121)$$

$$\Gamma_{L-\frac{3}{2}}^R = \frac{(-2\vec{\chi} \cdot \vec{L}^R - L + 3)(-2\vec{\chi} \cdot \vec{L}^R + L + 4)(2\vec{\chi} \cdot \vec{L}^R + 3L) - 6L(2L - 1)(2L + 1)}{6L(2L + 1)(2L - 1)} \quad (2.122)$$

where $\vec{\chi}$ denotes the matrices of the spin $\frac{3}{2}$ representation of $SU(2)$. Now as $L \rightarrow \infty$,

$$\Gamma_{L+\frac{3}{2}}^L + \Gamma_{L-\frac{3}{2}}^R \rightarrow \frac{8(\vec{\chi} \cdot \hat{x})^3 - 2\vec{\chi} \cdot \hat{x} + 12(\vec{\chi} \cdot \hat{x})^2 - 27}{24}. \quad (2.123)$$

This expression can be obtained directly on S^2 by considering the projector to the space where $\vec{\chi} \cdot \hat{x}$ takes the value of $\frac{3}{2}$.

The Dirac operator corresponding to this can be got from taking the continuum limit of $L \left(\Gamma_{L+\frac{3}{2}}^L - \Gamma_{L-\frac{3}{2}}^R \right)$. We will look at the possible terms we will be coming across in the process of taking the limits of the Dirac operators. In the case of spin $\frac{3}{2}$, we see the following term:

$$L \left(\frac{(\vec{\chi} \cdot \vec{L}^L)^3 - (\vec{\chi} \cdot \vec{L}^R)^3}{L^3} \right) \quad (2.124)$$

There is also a constant factor multiplying this. However this is not important for us right now as we are formulating rules for taking continuum limits of such terms.

Let us see how to take this continuum limit. For this consider

$$\frac{(\vec{\chi} \cdot \vec{L}^L)^3}{L^2} = \frac{1}{L^2} (\vec{\chi} \cdot \vec{\mathcal{L}} + \vec{\chi} \cdot \vec{L}^R)^3 \quad (2.125)$$

$$= \frac{1}{L^2} \left[(\vec{\chi} \cdot \vec{\mathcal{L}})^3 + (\vec{\chi} \cdot \vec{L}^R)^2 (\vec{\chi} \cdot \vec{\mathcal{L}}) + \{ \vec{\chi} \cdot \vec{\mathcal{L}}, \vec{\chi} \cdot \vec{L}^R \} \vec{\chi} \cdot \vec{\mathcal{L}} \right. \\ \left. + (\vec{\chi} \cdot \vec{\mathcal{L}})^2 \vec{\chi} \cdot \vec{L}^R + (\vec{\chi} \cdot \vec{L}^R)^3 + \{ \vec{\chi} \cdot \vec{\mathcal{L}}, \vec{\chi} \cdot \vec{L}^R \} \vec{\chi} \cdot \vec{L}^R \right]. \quad (2.126)$$

Here we have written $\vec{L}^L = \vec{\mathcal{L}} + \vec{L}^R$ where $\vec{\mathcal{L}}$ is the first order differential operator in the continuum. In the previous equation we note that the $(\vec{\chi} \cdot \vec{L}^R)^3$ term cancels the $-(\vec{\chi} \cdot \vec{L}^R)^3$ in equation Eq.(2.124). When $L \rightarrow \infty$, the order 1 terms in \vec{L}^R go away. The $(\vec{\chi} \cdot \vec{\mathcal{L}})^3$ also goes away in the continuum as we take the limit. So we are left with the following terms that have a non-zero limit

$$\frac{(\vec{\chi} \cdot \vec{L}^L)^3}{L^2} = \frac{1}{L^2} [\{ \vec{\chi} \cdot \vec{\mathcal{L}}, (\vec{\chi} \cdot \vec{L}^R)^2 \} + (\vec{\chi} \cdot \vec{L}^R) (\vec{\chi} \cdot \vec{\mathcal{L}}) (\vec{\chi} \cdot \vec{L}^R)]. \quad (2.127)$$

This is the following self-adjoint operator in the continuum:

$$\{ \vec{\chi} \cdot \vec{\mathcal{L}}, (\vec{\chi} \cdot \hat{x})^2 \} + (\vec{\chi} \cdot \hat{x}) (\vec{\chi} \cdot \vec{\mathcal{L}}) (\vec{\chi} \cdot \hat{x}). \quad (2.128)$$

The other terms we find in the expression for the fuzzy Dirac operator for the spin $\frac{3}{2}$ case involve powers of \vec{L}^L and \vec{L}^R less than 3 and their continuum limits were already found while we evaluated the corresponding continuum limits in the spin 1 and the spin $\frac{1}{2}$ case.

At this point we make a crucial observation that the limits we are taking are all independent of the algebra of the spin matrices $\vec{\chi}$. This is the reason why we need not bother about the order 1 and 2 terms in the spin $\frac{3}{2}$ case, though the spin matrices $\vec{\chi}$ are different from those in the spin 1 case.

We are interested in finding the limits of expressions of the form Eq.(2.124), which are similar in the case of all spins, but with higher powers of \vec{L}^L and \vec{L}^R .

Consider first

$$\begin{aligned} \frac{(\vec{\alpha} \cdot \vec{L}^L)^4 - (\vec{\alpha} \cdot \vec{L}^R)^4}{L^3} &= \frac{1}{L^3} \left((\vec{\alpha} \cdot (\vec{\mathcal{L}} + \vec{L}^R))^4 - (\vec{\alpha} \cdot \vec{L}^R)^4 \right) \\ &= \frac{1}{L^3} \left([(\vec{\alpha} \cdot \vec{\mathcal{L}})^2 + (\vec{\alpha} \cdot \vec{L}^R)^2 + \{\vec{\alpha} \cdot \vec{L}^R, \vec{\alpha} \cdot \vec{\mathcal{L}}\}] \right. \\ &\quad \times \left. [(\vec{\alpha} \cdot \vec{\mathcal{L}})^2 + (\vec{\alpha} \cdot \vec{L}^R)^2 + \{\vec{\alpha} \cdot \vec{L}^R, \vec{\alpha} \cdot \vec{\mathcal{L}}\}] - (\vec{\alpha} \cdot \vec{L}^R)^4 \right) \end{aligned} \quad (2.129)$$

where $\vec{\alpha}$ now denotes the matrices of spin $j(j > \frac{3}{2})$ representation of $SU(2)$. In the above expression, only the order 3 terms in L^R have a non-zero continuum limit. The $(\vec{\alpha} \cdot \vec{L}^R)^4$ term cancels just as it did in expression Eq.(2.124). The terms with non-zero limit are

$$\frac{1}{L^3} [\{\vec{\alpha} \cdot \vec{L}^R, \vec{\alpha} \cdot \vec{\mathcal{L}}\}(\vec{\alpha} \cdot \vec{L}^R)^2 + (\vec{\alpha} \cdot \vec{L}^R)^2 \{\vec{\alpha} \cdot \vec{L}^R, \vec{\alpha} \cdot \vec{\mathcal{L}}\}] \quad (2.130)$$

As $L \rightarrow \infty$ this term goes to the following non zero, self-adjoint expression

$$\{\vec{\alpha} \cdot \vec{\mathcal{L}}, (\vec{\alpha} \cdot \hat{x})^3\} + \{\vec{\alpha} \cdot \vec{\mathcal{L}}, \vec{\alpha} \cdot \hat{x}(\vec{\alpha} \cdot \vec{\mathcal{L}})\vec{\alpha} \cdot \hat{x}\} \quad (2.131)$$

Looking at this pattern and using the fact that we are just applying the binomial expansion in this computation, we can write a general rule for computing the continuum limit for order n terms. For this we consider

$$\frac{1}{L^{n-1}} [(\vec{\alpha} \cdot \vec{L}^L)^n - (\vec{\alpha} \cdot \vec{L}^R)^n] \quad (2.132)$$

Again we write $\vec{L}^L = \vec{\mathcal{L}} + \vec{L}^R$ and expand $(\vec{\alpha} \cdot \vec{L}^L)^n$ using the binomial expansion. As in previous cases the $(\vec{\alpha} \cdot \vec{L}^R)^n$ term gets canceled and we need to pick only the order $n-1$ terms in \vec{L}^R as these are the only terms having a non-zero continuum limit. Since the continuum operator has to be self-adjoint and the terms occurring in the expansion are all those occurring in a binomial expansion, it is easy to see that the terms having a non-zero limit can be given as the following sum:

$$\frac{1}{L^{n-1}} \left(\sum_{k=0}^{n-1} (\vec{\alpha} \cdot \vec{L}^R)^{n-1-k} (\vec{\alpha} \cdot \vec{\mathcal{L}}) (\vec{\alpha} \cdot \vec{L}^R)^k \right) \quad (2.133)$$

It is clear from this expression that we only have terms of order $n-1$ in \vec{L}^R here and we immediately see the continuum limit of this expression as

$$\sum_{k=0}^{n-1} (\vec{\alpha} \cdot \hat{x})^{n-1-k} (\vec{\alpha} \cdot \vec{\mathcal{L}}) (\vec{\alpha} \cdot \hat{x})^k. \quad (2.134)$$

Thus when considering the expression for the Dirac operator for any spin j , the highest order term in $\vec{\alpha} \cdot \vec{L}^L$ has a power $n = 2j$ and other terms decrease from $2j$ to 1. We have just seen how

to take the continuum limit of each of these terms with our general rules. We also encounter polynomials in L in these expressions whose limits are easy to take. Apart from grouping terms of similar order as in Eq.(2.124), we will also encounter $\vec{\alpha} \cdot \vec{L}^{L,R}$ of various orders which cannot be grouped as in Eq.(2.124). The process of taking limits for such terms is straightforward as they result in powers of $\vec{\alpha} \cdot \hat{x}$.

2.10 Rules for finding the Fuzzy Dirac Operator on S_F^2 for any Spin j

2.10.1 Half-Integral Spins

In this case, we have an even number of projectors and hence an even number of chiralities in the continuum. We can easily find all the chiralities in the continuum as they are just got from constructing projectors to various spaces labeled by the eigenvalues of $\vec{\alpha} \cdot \hat{x}$.

Next we list the projectors in the fuzzy case and construct the corresponding GW systems for each of them. So we have tables similar to the ones in Eq.(2.103)-Eq.(2.105). Then we consider the construction of the correct combination of the generators of the various GW systems, which go to the chiralities found in the continuum previously, as we take the continuum limit.

The claim is: The chiralities got from the projectors to the spaces labeled by j and $-j$ in the continuum are got by taking the continuum limits of

$$\frac{\Gamma_{L+j}^L + \Gamma_{L-j}^R}{2} \quad (2.135)$$

and

$$\frac{\Gamma_{L-j}^L + \Gamma_{L+j}^R}{2} \quad (2.136)$$

respectively.

We now prove this claim:

Consider spin j coupling to the orbital part l . Then if we project to the $l + j - k$ space, it is easy to see that

$$\text{Spectrum of } \vec{\alpha} \cdot \vec{L}^L \in l_j + \frac{k}{2}[k - 1 - 2l - 2j] \quad (2.137)$$

where $k = 0, 1, \dots, 2j$. We use this spectrum to construct the projectors to the above spaces.

It then follows from definition that

$$\frac{\Gamma_{l+j}^L + \Gamma_{l-j}^R}{2} = P_{l+j}^L + P_{l-j}^R - 1 \quad (2.138)$$

where $P^{L,R}$ denotes the left or right projector to the corresponding space, indicated in the suffix. Taking the continuum limit, we get

$$\lim_{l \rightarrow \infty} P_{l+j}^L + P_{l-j}^R - 1 = \prod_{k=1}^{2j} \frac{(\vec{\alpha} \cdot \hat{x} - j + k)}{k} + \prod_{k=0}^{2j-1} \frac{(-\vec{\alpha} \cdot \hat{x} - j + k)}{(-2j + k)} - 1. \quad (2.139)$$

Pulling out the minus signs in the second expression we get

$$\lim_{l \rightarrow \infty} P_{l+j}^L + P_{l-j}^R - 1 = \frac{\prod_{k=1}^{2j} (\vec{\alpha} \cdot \hat{x} - j + k)}{(2j)!} + (-1)^{4j} \frac{\prod_{k=0}^{2j-1} (\vec{\alpha} \cdot \hat{x} + j - k)}{(2j)!} - 1. \quad (2.140)$$

Since $4j$ is even for both integral and half-integral j , observing that $\prod_{k=1}^{2j} \vec{\alpha} \cdot \hat{x} - (j - k) = \prod_{k=0}^{2j-1} \vec{\alpha} \cdot \hat{x} + (j - k)$, we get

$$\lim_{l \rightarrow \infty} \frac{\Gamma_{l+j}^L + \Gamma_{l-j}^R}{2} = 2 \frac{\prod_{k=1}^{2j} (\vec{\alpha} \cdot \hat{x} - j + k)}{(2j)!} - 1. \quad (2.141)$$

This is exactly the expression for the chirality operator got in the continuum from the projector to the space where $\vec{\alpha} \cdot \hat{x} = j$.

Now since, $L \left(\frac{\Gamma_{L+j}^L - \Gamma_{L-j}^R}{2} \right)$ and $\frac{\Gamma_{L+j}^L + \Gamma_{L-j}^R}{2}$ anticommute in the fuzzy case, they will continue to do so as we take the continuum limit. So we can be sure that

$$L \left(\frac{\Gamma_{L+j}^L - \Gamma_{L-j}^R}{2} \right) \quad (2.142)$$

gives us the fuzzy Dirac operator corresponding to this chirality.

We can follow the same procedure to get the remaining fuzzy Dirac and chirality operators, exhausting all possibilities.

2.10.2 Integral Spins

In this case, we have an odd number of projectors and hence an odd number of chiralities in the continuum. We then proceed as we did for the case of half-integral spins and we note that all the arguments go through, except when it comes to the Dirac operator corresponding to the chirality obtained from the projector to the space where $\vec{\alpha} \cdot \hat{x} = 0$. In this case, we construct the fuzzy analogues from the generators of the GW system obtained from the left and right projectors to the $L + 0$ space alone. We cannot mix the generators of the GW system got from this projector with the generators obtained from the projectors to other spaces as we get diverging continuum limits.

2.11 Unitary Inequivalence of the Fuzzy Spin j Dirac Operators

We show that for a given spin j the Dirac operators corresponding to the different chirality operators are unitarily inequivalent. We do this by computing the trace of the fuzzy versions of these Dirac operators. By continuity this inequivalence extends to the continuum limit as well. In what follows we illustrate the inequivalence for the spin 1 and spin $\frac{3}{2}$ case alone.

2.11.1 Spin 1

For the spin 1 case the combination which leads to the desired Dirac and chirality operators are

$${}^F D_1^1 = L \left(\frac{\Gamma_{L+1}^L - \Gamma_{L-1}^R}{2} \right), \quad (2.143)$$

$${}^F D_2^1 = L \left(\frac{\Gamma_{L-1}^L - \Gamma_{L+1}^R}{2} \right) \quad (2.144)$$

and

$${}^F D_3^1 = L \left(\frac{\Gamma_L^L - \Gamma_L^R}{2} \right). \quad (2.145)$$

with

$${}^F \gamma_1^1 = \left(\frac{\Gamma_{L+1}^L + \Gamma_{L-1}^R}{2} \right), \quad (2.146)$$

$${}^F \gamma_2^1 = \left(\frac{\Gamma_{L-1}^L + \Gamma_{L+1}^R}{2} \right) \quad (2.147)$$

and

$${}^F \gamma_3^1 = \left(\frac{\Gamma_L^L + \Gamma_L^R}{2} \right) \quad (2.148)$$

as their corresponding chirality operators.

The trace of the fuzzy Dirac operators in Eq.(2.143)-Eq.(2.145) and Eq.(2.155)-Eq.(2.158) can be computed analytically by using the formula

$$\text{tr}(A \otimes B) = \text{tr}(A) \cdot \text{tr}(B) \quad (2.149)$$

where A and B are square matrices. Since the Dirac operators we construct act on $Mat(2L + 1) \otimes \mathbb{C}^{2j+1}$, they are of the form of tensor products and hence we can apply this formula to analytically compute their traces.

The trace is a rotationally invariant object leading to

$$\text{tr}((L_1^L)^2) = \text{tr}((L_2^L)^2) = \text{tr}((L_3^L)^2) \quad (2.150)$$

and

$$\text{tr}(\Sigma_1^2) = \text{tr}(\Sigma_2^2) = \text{tr}(\Sigma_3^2) = 2. \quad (2.151)$$

The above equations hold due to the fact that the three generators of any representation of the $SU(2)$ algebra have the same trace because of rotational invariance.

The trace of $(L_i^R)^2$ is the same as the trace of $(L_i^L)^2$ for $i = 1, 2, 3$. However we have the following general result

$$\text{tr}(L_2^R)^{2k+1} = -\text{tr}(L_2^L)^{2k+1} \quad (2.152)$$

for $k \in \mathbb{Z}$. This is seen due to the fact that $\text{tr}(L_2)^{2k+1} = -\text{tr}(L_2^T)^{2k+1}$ ¹ where T denotes transpose.

The trace of $(L_i^L)^2$ varies according to whether L is integer or half-integer. When L is an integer

$$\text{tr}((L_i^L)^2) = \frac{1}{3}L(L+1)(2L+1)^2 \quad (2.153)$$

and when L is an half-integer

$$\text{tr}((L_i^L)^2) = \frac{1}{3}L(L+1)(L+2)(2L+1). \quad (2.154)$$

It is simple to see that $\vec{\Sigma} \cdot \vec{L}^L$ and $\vec{\Sigma} \cdot \vec{L}^R$ are traceless. This is because these are generators of the $SU(2)$ Lie algebra.

Using these identities we write down the traces of our 3 spin 1 Dirac operators in Table(2.1).

¹Due to this result we have $\text{tr}(\vec{\Sigma} \cdot \vec{L}^L)^n = (-1)^n \text{tr}(\vec{\Sigma} \cdot \vec{L}^R)^n$ for $n > 1$ and Σ_i are the spin matrices of an arbitrary spin j

Dirac Operator	$L \in \mathbb{Z}$	$L \in \frac{\mathbb{Z}}{2}$
${}^F D_1^1$	$4L(2L + 1)$	$2L(5L + 1)$
${}^F D_2^1$	$-4L(2L + 1)$	$-2L(5L + 1)$
${}^F D_3^1$	0	0

Table 2.1: Traces of the 3 Spin 1 Dirac Operators

The trace of the Dirac operator is the sum of its eigenvalues. The availability of these exact trace formulas are helpful in verifying the spectrum of these operators found numerically in the next chapter.

The operators ${}^F D_1^1$ and ${}^F D_2^1$ have non-zero trace implying the existence of unpaired eigenstates or zero modes.

To check the unitary equivalence of the 3 Dirac operators, it is a necessary, though not sufficient condition that the traces of the 3 operators be the same. Since the trace formulas show the traces are not the same, they provide an analytic proof for the unitary inequivalence of the 3 Dirac operators confirming numerical results presented in the next chapter.

2.11.2 Spin $\frac{3}{2}$

There are four chirality operators in this case obtained from the projectors to the four subspaces got by combining the angular momentums L and $\frac{3}{2}$. The relevant combinations of the fuzzy GW algebra generators in this case are the following: The four Dirac operators are

$${}^F D_1^{\frac{3}{2}} = L \left(\frac{\Gamma_{L+\frac{3}{2}}^L - \Gamma_{L-\frac{3}{2}}^R}{2} \right), \quad (2.155)$$

$${}^F D_2^{\frac{3}{2}} = L \left(\frac{\Gamma_{L+\frac{1}{2}}^L - \Gamma_{L-\frac{1}{2}}^R}{2} \right), \quad (2.156)$$

$${}^F D_3^{\frac{3}{2}} = L \left(\frac{\Gamma_{L-\frac{3}{2}}^L - \Gamma_{L+\frac{3}{2}}^R}{2} \right), \quad (2.157)$$

$${}^F D_4^{\frac{3}{2}} = L \left(\frac{\Gamma_{L-\frac{1}{2}}^L - \Gamma_{L+\frac{1}{2}}^R}{2} \right) \quad (2.158)$$

and the corresponding chirality operators are

$${}^F \gamma_1^{\frac{3}{2}} = \left(\frac{\Gamma_{L+\frac{3}{2}}^L + \Gamma_{L-\frac{3}{2}}^R}{2} \right), \quad (2.159)$$

$${}^F \gamma_2^{\frac{3}{2}} = \left(\frac{\Gamma_{L+\frac{1}{2}}^L + \Gamma_{L-\frac{1}{2}}^R}{2} \right), \quad (2.160)$$

$${}^F\gamma_3^{\frac{3}{2}} = \left(\frac{\Gamma_{L-\frac{3}{2}}^L + \Gamma_{L+\frac{3}{2}}^R}{2} \right), \quad (2.161)$$

$${}^F\gamma_4^{\frac{3}{2}} = \left(\frac{\Gamma_{L-\frac{1}{2}}^L + \Gamma_{L+\frac{1}{2}}^R}{2} \right). \quad (2.162)$$

The remaining Dirac operators are got from the following GW algebra generators

$$\Gamma_{L+\frac{3}{2}}^L = \frac{(2\vec{\chi} \cdot \vec{L}^L - L + 3)(2\vec{\chi} \cdot \vec{L}^L + L + 4)(2\vec{\chi} \cdot \vec{L}^L + 3L + 3) - 6(L + 1)(2L + 3)(2L + 1)}{6(L + 1)(2L + 1)(2L + 3)} \quad (2.163)$$

$$\Gamma_{L-\frac{3}{2}}^R = \frac{(-2\vec{\chi} \cdot \vec{L}^R - L + 3)(-2\vec{\chi} \cdot \vec{L}^R + L + 4)(2\vec{\chi} \cdot \vec{L}^R + 3L) - 6L(2L - 1)(2L + 1)}{6L(2L + 1)(2L - 1)} \quad (2.164)$$

where $\vec{\chi}$ are the matrices of the spin $\frac{3}{2}$ representation of $SU(2)$ algebra. They are got from the following generators of the GW algebra:

$$\Gamma_{L-\frac{3}{2}}^L = \frac{-(2\vec{\chi} \cdot \vec{L}^L - 3L)(2\vec{\chi} \cdot \vec{L}^L - L + 3)(2\vec{\chi} \cdot \vec{L}^L + L + 4) - 6L(2L - 1)(2L + 1)}{6L(2L + 1)(2L - 1)} \quad (2.165)$$

$$\Gamma_{L+\frac{3}{2}}^R = \frac{(-2\vec{\chi} \cdot \vec{L}^R - L + 3)(-2\vec{\chi} \cdot \vec{L}^R + L + 4)(-2\vec{\chi} \cdot \vec{L}^R + 3L + 3) - 6(L + 1)(2L + 3)(2L + 1)}{6(L + 1)(2L + 1)(2L + 3)} \quad (2.166)$$

$$\Gamma_{L+\frac{1}{2}}^L = \frac{-(2\vec{\chi} \cdot \vec{L}^L - 3L)(2\vec{\chi} \cdot \vec{L}^L + L + 4)(2\vec{\chi} \cdot \vec{L}^L + 3L + 3) - 2L(2L + 3)(2L + 1)}{2L(2L + 1)(2L + 3)} \quad (2.167)$$

$$\Gamma_{L-\frac{1}{2}}^R = \frac{-(2\vec{\chi} \cdot \vec{L}^R + 3L)(-2\vec{\chi} \cdot \vec{L}^R - L + 3)(-2\vec{\chi} \cdot \vec{L}^R + 3L + 3) - 2(L + 1)(2L - 1)(2L + 1)}{2(L + 1)(2L + 1)(2L - 1)} \quad (2.168)$$

$$\Gamma_{L-\frac{1}{2}}^L = \frac{(2\vec{\chi} \cdot \vec{L}^L - 3L)(2\vec{\chi} \cdot \vec{L}^L - L + 3)(2\vec{\chi} \cdot \vec{L}^L + 3L + 3) - 2(L + 1)(2L - 1)(2L + 1)}{2(L + 1)(2L + 1)(2L - 1)} \quad (2.169)$$

$$\Gamma_{L+\frac{1}{2}}^R = \frac{(2\vec{\chi} \cdot \vec{L}^R + 3L)(-2\vec{\chi} \cdot \vec{L}^R + L + 4)(-2\vec{\chi} \cdot \vec{L}^R + 3L + 3) - 2L(2L + 3)(2L + 1)}{2L(2L + 1)(2L + 3)} \quad (2.170)$$

To compute the trace of the Dirac operator given by Eq.(2.155) for the spin $\frac{3}{2}$ case we need to use the formula [79, 83, 84]

$$\text{tr}(L_i L_j L_k) = i \frac{L(L + 1)(2L + 1)}{6} \epsilon_{ijk} \quad (2.171)$$

where L_i is the i -th component of the angular momentum generator of the $SU(2)$ Lie algebra.

The expression for the trace of the Dirac operator in Eq.(2.155) is given by

$$\text{tr}({}^F D_1^{\frac{3}{2}}) = 2L(2L + 1) \left[\frac{2L^2 + 7L + 6}{2L + 3} - \frac{2L^2 - 3L + 1}{2L - 1} \right]. \quad (2.172)$$

The traces of the remaining three spin $\frac{3}{2}$ Dirac operators are

$$\text{tr}({}^F D_1^{\frac{3}{2}}) = -\text{tr}({}^F D_2^{\frac{3}{2}}) = 2L(2L+1) \left[\frac{2L^2 + 7L + 6}{2L+3} - \frac{2L^2 - 3L + 1}{2L-1} \right] \quad (2.173)$$

and

$$\text{tr}({}^F D_3^{\frac{3}{2}}) = -\text{tr}({}^F D_4^{\frac{3}{2}}) = 2L(2L+1) \quad (2.174)$$

establishing the unitary inequivalence of the four spin $\frac{3}{2}$ Dirac operators.

Chapter 3

Spectrum of Higher Spin Dirac Operators on S_F^2

The computation of the spectrum of the spin 1 Dirac operator is a hard problem to do analytically due to the presence of $\vec{\Sigma} \cdot \hat{x}$ terms in the expressions of the operators in the continuum. This problem persists for all higher spin Dirac operators on S_F^2 for arbitrary spin j . This problem is particularly not present for the case of the spin $\frac{1}{2}$ Dirac operator whose spectrum has been computed in several different ways [16, 82].

We study general properties of the spectrum of the higher spin Dirac operators. These are the computations we can perform analytically to study about the spectrum of these operators. We will numerically study the spectrums of the spin 1 and the spin $\frac{3}{2}$ case in what follows. Wherever possible we compare the spectrums to the spin $\frac{1}{2}$ case.

3.1 Analytic Results of the Spectrums of the Spin $\frac{1}{2}$ and Spin 1 Dirac Operators

The spectrum of the spin $\frac{1}{2}$ Dirac operator can be found analytically [82]. In the GW approach to constructing the Dirac operator, the spin $\frac{1}{2}$ system has the same spectrum both in the continuum and the fuzzy level. To illustrate the method of finding the spectrum, we consider the spin $\frac{1}{2}$ Dirac operator in the continuum:

$${}^\infty D^{\frac{1}{2}} = \vec{\sigma} \cdot \vec{\mathcal{L}} + 1. \quad (3.1)$$

In the above equation $\vec{\mathcal{L}}$ is the orbital angular momentum got by taking the continuum limit of $\vec{L}^L - \vec{L}^R$. $\vec{\sigma}$ are the spin $\frac{1}{2}$ Pauli matrices. The total angular momentum \vec{J} given by

$$\vec{J} = \frac{\vec{\sigma}}{2} + \vec{\mathcal{L}}$$

commutes with the Dirac operator. We can use its eigenvalues to label the eigenstates of the Dirac operator. For given cut-off angular momentum L , the spectrum of the orbital angular

momentum is given by

$$\vec{\mathcal{L}} \in \{0, 1, \dots, 2L\}. \quad (3.2)$$

Given this we can find the spectrum of the total angular momentum \vec{J} to be

$$\vec{J} \in \{\frac{1}{2}, \frac{3}{2}, \dots, 2L - \frac{1}{2}, 2L + \frac{1}{2}\}. \quad (3.3)$$

Each value of the total angular momentum \vec{J} can be got from two different orbital angular momentum except the top mode whose \vec{J} value is $2L + \frac{1}{2}$. From this we can count the total number of eigenvalues for a given cut-off L with the help of the following sum:

$$\sum_{j=\frac{1}{2}}^{j=2L-\frac{1}{2}} 2(2j) + 2L + \frac{1}{2} = 2(2L + 1)^2. \quad (3.4)$$

The spectrum of the Dirac operator in Eq.(3.1) can be got by noting that this operator can be written as

$${}^\infty D^{\frac{1}{2}} = \vec{J}^2 - \vec{\mathcal{L}}^2 + \frac{1}{4}. \quad (3.5)$$

As $[\vec{J}^2, \vec{\mathcal{L}}^2] = 0$, we can write the spectrum of ${}^\infty D^{\frac{1}{2}}$ as

$$\text{Spectrum of } {}^\infty D^{\frac{1}{2}} = j(j+1) - l(l+1) + \frac{1}{4}. \quad (3.6)$$

As mentioned before each j comes from two different l values except the top mode. Thus we have for the spectrum of ${}^\infty D^{\frac{1}{2}}$:

$${}^\infty D^{\frac{1}{2}} \left(\begin{array}{l} = j + \frac{1}{2}; \quad \text{if } l = j - \frac{1}{2} \\ = -j - \frac{1}{2}; \quad \text{if } l = j + \frac{1}{2}. \end{array} \right) \quad (3.7)$$

The spectrum has the chiral nature as expected. Note that there are no zero modes for the spin $\frac{1}{2}$ Dirac operator. The computation of the spectrum in the spin $\frac{1}{2}$ case is easy due to the form of ${}^\infty D^{\frac{1}{2}}$ as given by Eq.(3.1). This however is not true for the Dirac operator of the spin 1 case given by Eq.(2.27). This is due to the presence of the term $\vec{\Sigma} \cdot \hat{x}$ which does not commute with $\vec{\Sigma} \cdot \vec{\mathcal{L}}$ making the analytic computation difficult. This is the reason why we take to numerical methods to achieve this. Nevertheless we can still get some vital information about the spectrum of the spin 1 Dirac operator by analytic methods.

The total angular momentum \vec{J} given by

$$\vec{J} = \vec{\Sigma} + \vec{\mathcal{L}}$$

commutes with the Dirac operator in Eq.(2.27) just as the corresponding total angular momentum does in the spin $\frac{1}{2}$ case. The spectrum of the orbital angular momentum $\vec{\mathcal{L}}$ is the same as in the spin $\frac{1}{2}$ case given by Eq.(3.2). The spectrum of \vec{J} is now given by

$$\vec{J} \in \{0, 1, 2, \dots, 2L - 1, 2L, 2L + 1\}. \quad (3.8)$$

In this case each value of \vec{J} comes from three different orbital angular momenta namely $j - 1$, j and $j + 1$ except three j values. $j = 0$ comes from only one state. $j = 2L$ comes from 2

states and $j = 2L + 1$ comes from only one state. These are easy to check as they involve the simple angular momentum addition rules. With this information we can count the number of eigenvalues for each cut-off L with the following sum:

$$1 + \sum_{j=1}^{j=2L-1} 3(2j+1) + 2(4L+1) + 2(2L+1) + 1 = 3(2L+1)^2. \quad (3.9)$$

This is exactly the number of eigenvalues we expect from each cut-off L for the spin 1 case as this is the size of the matrix for the Dirac operator for each L . This method gives us the degeneracy for each \vec{J} value. The energy eigenvalues are found as a function of the total angular momentum.

3.1.1 Number of Positive Eigenvalues and Zero Modes for the Spin 1 Dirac Operator

Out of the three Dirac operators in the spin 1 case we will consider the traceless Dirac operator (See Table 2.1). The trace equation gives us an easy and elegant way to count the number of different non-zero positive and negative eigenvalues as well as the number of zero modes for each cut-off angular momentum L .

The zero modes can be counted as follows: $j = 0$ comes from just one orbital angular momentum state and so it can not result in a positive or negative eigenvalue of ${}^F D_3^1$ and hence it must only be 0 due to the traceless nature of the Dirac operator. This contributes 1 zero mode for each L . Similar argument holds for $j = 2L + 1$ which contributes $2(2L + 1) + 1$ zero modes for each L . For values of j between 1 and $2L - 1$ there is a contribution of $2j + 1$ zero modes for each of the j values. The value $j = 2L$ comes from two orbital angular momenta. The chiral nature of the spectrum prevent the eigenvalues corresponding to these two sets from being 0. However this is not an obvious fact and it can happen that the eigenvalues corresponding to one of the orbital angular momentum is 0, which will make the other set also 0. This observation is important to see that zero modes are possible even in half-integral spin systems.¹ Summing all this we find that there are exactly $(2L + 1)^2 + 2$ zero modes for each L .

In a similar way we can find the number of positive eigenvalues. When we do this we find there is a contribution of $2j + 1$ eigenvalues for values of j between 1 and $2L$. Summing these we get $4L^2 + 4L$. As the Dirac operator is traceless, the same argument holds for the negative eigenvalues giving a total of $4L^2 + 4L$ eigenvalues for each L . It is easy to see that the sum of the positive, negative eigenvalues and zero eigenvalues give $3(2L + 1)^2$ as the total number of eigenvalues as expected for each cut-off L .

These arguments can again be easily extended to the spectrum of higher spin Dirac operators on S_F^2 . However for the case of half-integer spins, the number of zero modes cannot be counted this way as there are an even number of irreducible subspaces when a half-integral angular momentum combines with an integer angular momentum. Nevertheless their existence cannot be ruled out by this argument.

Finally we count the number of different positive eigenvalues we expect to find for the spin 1 Dirac operator for each cut-off L . Since there are $2L + 2$ values the total angular momentum j can take, out of which 2 of them can only contribute to the zero modes for each L , we can

¹This also means that we cannot use this method to obtain the exact number of zero modes for other integer spins. It only gives a lower bound on the number of zero modes for integer spins more than 1.

conclude that there are $2L$ different positive eigenvalues for each L . The degeneracies of each of them can easily be read off as $2j + 1$ according to the corresponding value j takes.

3.1.2 Remarks on the other two Dirac Operators for the Spin 1 Case

So far the arguments in this section were for the traceless Dirac operator in Eq.(2.27). These arguments do not hold for the Dirac operators in Eq.(2.39) and Eq.(2.40) as they have positive and negative traces respectively. These are given in table 2.1.

Consider the Dirac operator with the positive trace whose continuum value is given by Eq.(2.39). In this case too we have for the spectrum of the total angular momentum

$$\text{Spec } \vec{J} \in \{0, 1, \dots, 2L - 1, 2L, 2L + 1\} \quad (3.10)$$

as before. However in this case we cannot say that the states corresponding to $j = 2L + 1$ and $j = 0$ correspond to zero modes. This is because of the non-zero trace. They now have some positive energy say E_0 and E_{2L+1} . We then have the following equation

$$E_0 + (4L + 3)E_{2L+1} = 4L(2L + 1) \quad (3.11)$$

for integral values of L and

$$E_0 + (4L + 3)E_{2L+1} = 2L(5L + 1) \quad (3.12)$$

for half-integral values of L . The $4L + 3$ states with energy E_{2L+1} correspond to unpaired eigenstates. The eigenstates corresponding to these states are labeled by different values of J_3 . They represent an edge effect in the problem.

Since the Dirac operators corresponding to spin $\frac{3}{2}$ have non-zero trace given by Eq.(2.173) and Eq.(2.174), they have unpaired eigenstates as well. They correspond to an edge effect as well.

Having studied the general nature of the spectrum of the spin j Dirac operator, we compute the eigenvalues numerically in the next section. In particular we will find a relation between the eigenvalue and its multiplicity for a given cut-off L . This is equivalent to finding the eigenvalues as a function of total angular momentum j for each cut-off L .

3.2 Numerical Results

3.2.1 Spectrum of the Traceless Spin 1 Dirac Operator

We compute the eigenvalues of the the three Dirac operators in Eq.(2.143)-Eq.(2.145) numerically. The size of each of these operators is $3N^2$ where $N = 2L + 1$ and L is the cutoff. It is clear from the dimensions of these matrices ($\sim 9N^4$) that we cannot go to arbitrarily large values of N . Even for $L = 22$, the size of the matrix becomes 6075×6075 which is difficult to handle numerically within the resources available to us. For large values of N number of computational steps increase which will lead to growth of systematic error. However, the patterns emerging from the spectrum we computed so far strongly suggest what the behavior would be at higher values of N . This circumvents computational problems and helps us predict the behavior as we go close to the continuum. This is particularly important given the problems in handling very large matrices.

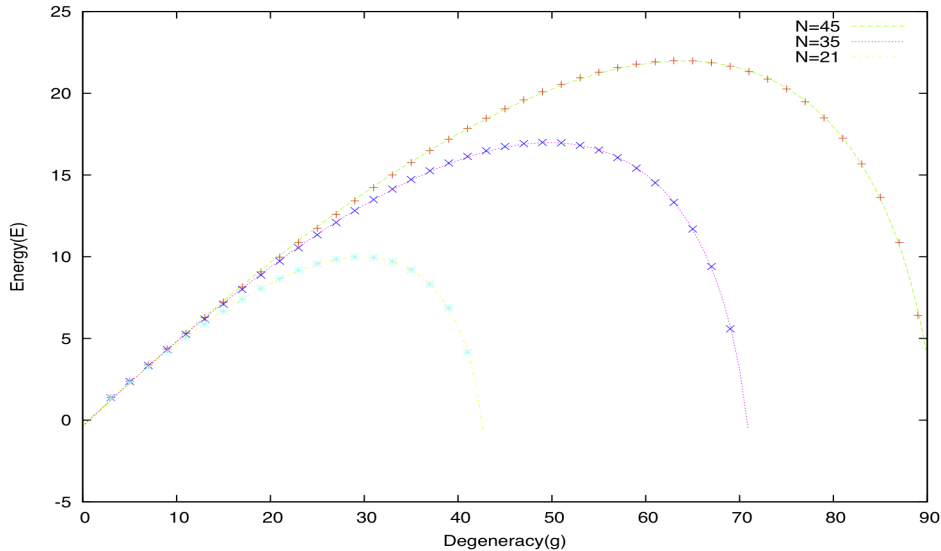


Figure 3.1: Plot of the energy eigenvalues along with the best fit curves for $N = 21$, $N = 35$ and $N = 45$.

The nature of the spectrum was discussed in the previous section and we confirm those results numerically. The spectrum of ${}^F D_3^1$ is similar to that of fermions with equal number of positive and negative energy eigenvalues. This is a reflection of the existence of the chirality operator given by Eq.(2.148), which anticommutes with ${}^F D_3^1$. Apart from the non-zero eigenvalues there also exist a number of zero modes. We find exactly $(2L + 1)^2 + 2$ zero modes for each cut-off L as we explained in the previous section. The number of positive eigenvalues is also as expected.

We find two fits for this case. One is for the positive eigenvalues and the other is for the square of the positive eigenvalues.

3.2.2 Fit 1

We work with only the positive eigenvalues of the spectrum. As the operator is traceless we have the same pattern for the negative eigenvalues and so we do not use them to fit curves. Then we find the degeneracies of each of the positive eigenvalues. Note from the discussion in the earlier section, that there can only be odd degeneracies for our system as the total angular momentum j takes integral values. The plot for the energy vs the degeneracies is shown in figure 3.1. It shows the data points for three different values of N (namely $N = 21$, $N = 35$ and $N = 45$) along with the best fit curves.

By inspection we found the curve has a mirror symmetry about some principal axis.

Next we try to find a universal curve that will fit the data(eigenvalues) of different cutoffs, just by changing the value of the cutoff. To this end we analyzed the data in a rotated frame in which the data was found to have reflection symmetry around the rotated y-axis. Given that the data for small (E, g) is independent of the cutoff(this can be seen in figure 3.1 where for

small values of g the three sets of data points lie almost on top of each other on a straight line), we found a unique rotation angle to rotate all the results for different cut-off L . After observing the reflection symmetry of (E', g') we tried to fit the data with a polynomial with only even powers. To our surprise we found an excellent fit with just a parabola for all different cutoff values. Higher powers in the function did not make any further improvement in the fitting. The parameters of the parabola run with the cut off. We also find excellent fit for these parameters as a function of the cutoff L .

We now elaborate this method. The plot of (E, g) is rotated to a new set of variables (E', g') . This set of points is then fitted with the curve

$$E' = \alpha(g' + \eta)^2 + \beta. \quad (3.13)$$

Here α , β and η are expected to vary with the cut-off L . The relation between (E', g') and (E, g) is given by

$$\begin{pmatrix} E' \\ g' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} E \\ g \end{pmatrix}. \quad (3.14)$$

The angle $\theta = 2.26159$ radians. This angle is a constant for different values of L . This can be seen as a consequence of the data points lying on top of each other for small values of g as seen in figure 3.1. Note that E' and g' are not energies and degeneracies respectively. We just need to use the transformation in Eq.(3.14) to get the relation between the energies and the degeneracies.

Our next task is to find α , β and η as functions of $N = 2L + 1$. They are found by fitting the quadratic form (Eq.(3.13)) to the rotated curves for different values of N . We find them to follow simple relations. These are shown in the figures 3.2-3.4. The exact functions we found were:

$$\alpha = \frac{0.863569}{N^{0.930775}} - 0.00141635, \quad (3.15)$$

$$\beta = -1.45123N + 0.333497 \quad (3.16)$$

and

$$\eta = -1.16288N - 0.555529. \quad (3.17)$$

The numbers may look uninteresting but if we could fit these functions after we find these parameters for more values of N we could converge onto some special numbers. We did not attempt this in this dissertation. The relations are simple enough to imply something deeper in the spectrum. More exact numbers could help in the quest for an analytic solution of this problem.

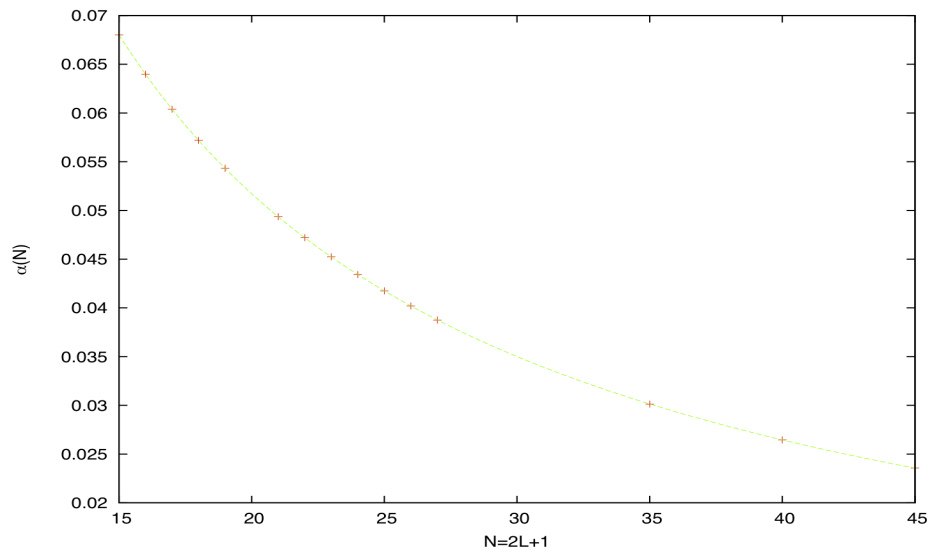
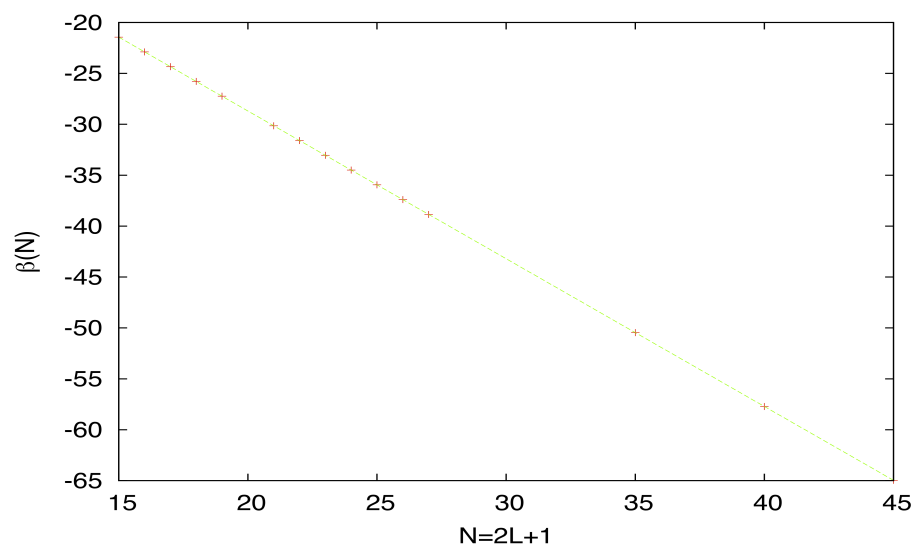
We mention here that if we insist that the $E'(g')$ curve pass through the origin, then only two out of the three parameters will be independent. Considering β and η to be independent parameters we get reasonably good fits. The fitted values of β and η were found to be again linear in N . The importance of a two parameter fit however is that all different N spectra can be scaled to a universal curve.

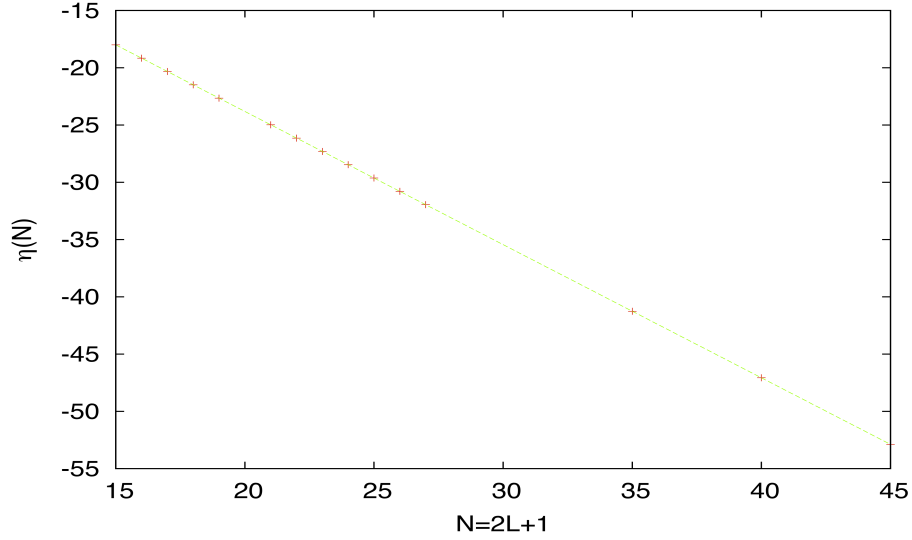
We can now write down the exact relation between E and g based on our numerical fits:

$$E = \frac{\sqrt{b(g)^2 - 4ac(g)} - b(g)}{2a} \quad (3.18)$$

where

$$a = \alpha \cos^2 \theta, \quad (3.19)$$

Figure 3.2: Fit for the parameter α .Figure 3.3: Fit for the parameter β .

Figure 3.4: Fit for the parameter η .

$$b(g) = 2\alpha \cos \theta (\sin \theta g + \eta) + \sin \theta \quad (3.20)$$

and

$$c(g) = \alpha (\sin \theta g + \eta)^2 + \beta - \cos \theta g. \quad (3.21)$$

Having found this relation between E and g , we can now find the eigenvalues for arbitrarily large values of N . If we diagonalize ${}^F D_3^1$ for such large values of N it would take a lot of memory on the computer and is subject to a lot of numerical error. But we can get around this with our relation between E and g . Figure 3.5 shows the eigenvalues for $N = 60$. Note that though the curve looks continuous, we have seen in the previous section that degeneracies are allowed to take only odd integral values. The maximum degeneracy for a given N is $2N - 1$. Starting from 3 we can allow g to vary till $2N - 1$ through odd integers and find the corresponding eigenvalues using Eq.(3.18). In an equivalent manner we can find the energy eigenvalues as a function of the total angular momentum j by simply substituting $g = 2j + 1$ in Eq.(3.18).

3.2.3 Fit 2

The square of the spectrum of the spin 1 Dirac operator was plotted with the total angular momentum j . This curve has a fit which is similar to the square of the spectrum of the spin $\frac{1}{2}$ Dirac operator due to Watamura [16]. This is shown in the figure 3.6. The curve which fits this data is given by

$$E^2 = b \left(j + \frac{1}{2} \right) \left[1 + \frac{\{1 - (j + \frac{1}{2})^2\}}{(a-1)(a+1)} \right] \quad (3.22)$$

where $b = 1$ and $a \sim N$.

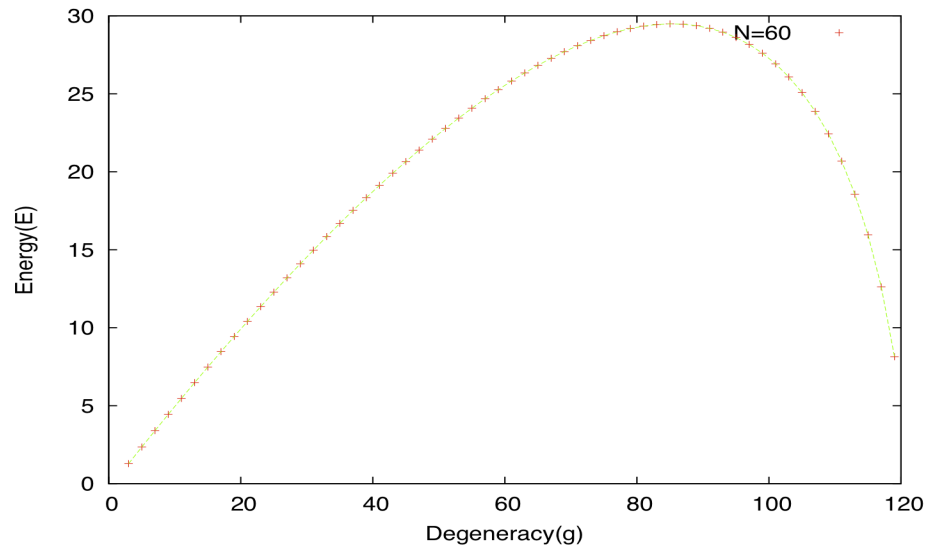


Figure 3.5: Plot predicting the energy eigenvalues on ${}^F D_3$ for $N = 60$.

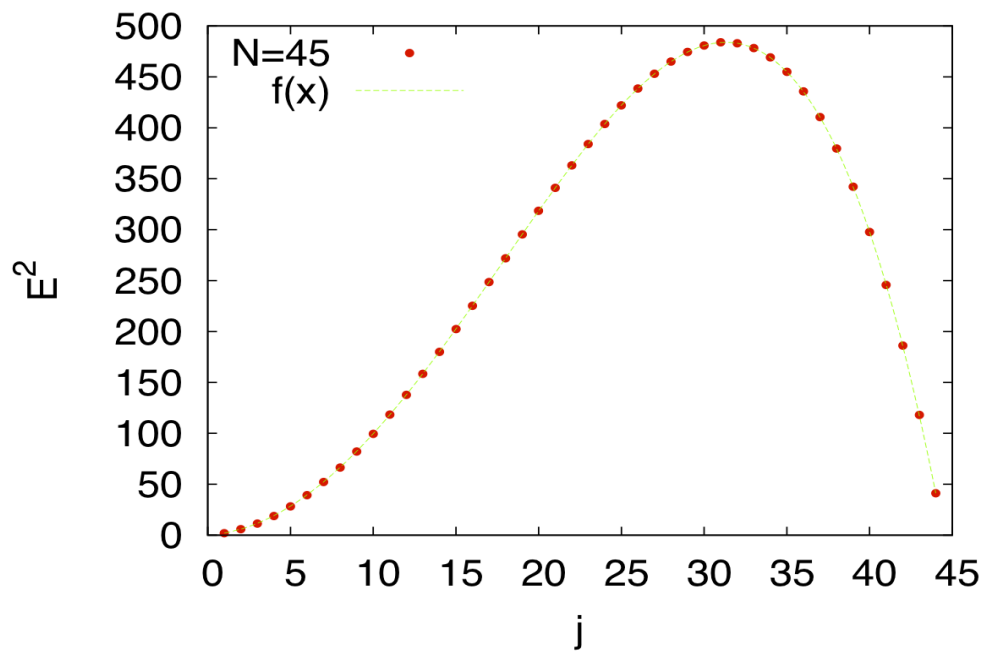


Figure 3.6: E^2 Vs j for the spectrum of the spin 1 Dirac operator.

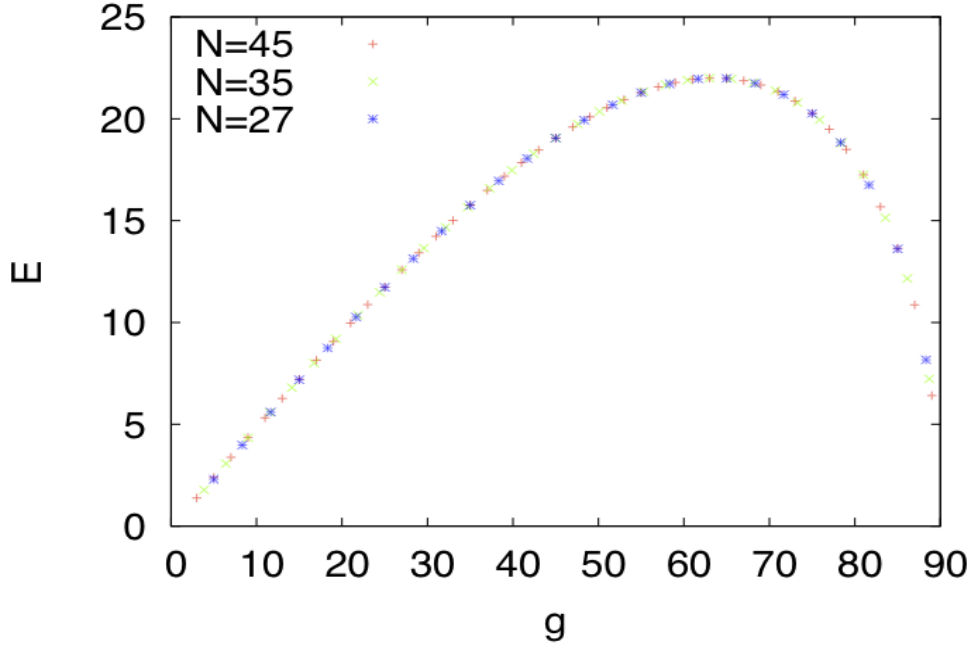


Figure 3.7: Plot showing the scaling property of the spectrum of the spin 1 Dirac operator.

A quantum particle on the continuum sphere, S^2 has energy eigenvalues given by $l(l+1)$. These are the eigenvalues of the Laplacian on the sphere which is a second order differential operator. The eigenvalues of the square of the continuum limit of the spin $\frac{1}{2}$ Dirac operator on S_F^2 [16, 15, 82, 17] also gives a spectrum similar to that of the standard Laplacian on S^2 apart from a additional constant. This additional constant can be interpreted as the scalar curvature according to the Lichnerowicz formula for the square of a general Dirac operator. In the Minkowskian case this is analogous to the square of the Dirac operator giving the Laplacian on that space. This leads to each component of the Dirac spinor satisfying the Klein-Gordon equation. We can view the Laplacian of the standard sphere as an analog of the Klein-Gordon equation on the sphere as this gives the $SU(2)$ covariant dispersion relation on S^2 . Note that we can add additional constants to this Laplacian as they are rotationally invariant. The plot shows that this is true for the traceless spin 1 Dirac operators on S_F^2 as well, inspite of their continuum limits, given by Eq.(2.27)-Eq.(2.40), containing $\vec{\Sigma} \cdot \hat{x}$ terms which makes the square of these operators look complicated. (Note that we do still get $l(l+1)$, but with additional terms containing $\vec{\Sigma} \cdot \hat{x}$ which makes the analytical computation of the spectrum difficult.) We can expect similar behavior for higher spins as well but we do not have a proof for this statement.

3.2.4 Scaling Property of the Spectrum of the Spin 1 Dirac Operator

The spectrum of the spin 1 Dirac operator exhibits a scaling property similar to the one shown by the spectrum of the spin $\frac{3}{2}$ case. This is seen by plotting the spectrum for $N = 27, 35$ and $N = 45$. This is shown in the figure 3.7.

The scaling property of the spectrums of the higher spin Dirac operators seem to be an

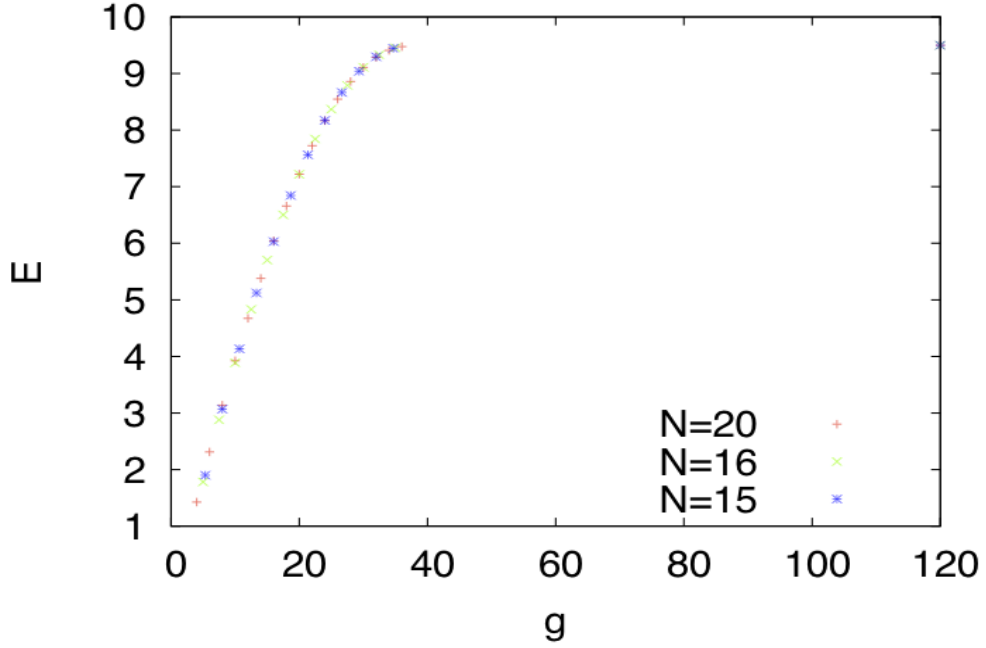


Figure 3.8: Plot showing the scaling property of the spectrum of the spin $\frac{3}{2}$ Dirac operator.

universal feature for these systems. At this point we have no analytic proof of this statement and leave it as a conjecture.

3.3 Spectrum of the Spin $\frac{3}{2}$ Dirac Operator

The spectrum of the spin $\frac{3}{2}$ Dirac operator given in Eq.(2.155) was found numerically. The highest eigenvalue was to be L with a degeneracy of $6N$. The size of the matrices in the computation of the spectrum of the spin $\frac{3}{2}$ Dirac operators are $4(2L+1)^2$. Naturally we could not go to large enough $N = 2L+1$ values due to limitations of resources. Due to this constraint we did not find a curve that fits this spectrum. Instead we studied the scaling property of the spectrum by looking at the spectrums for $N = 15, 16$ and $N = 20$. We find a remarkable scaling property shown in figure 3.8.

The scaling process is as follows. We plot the E vs g for an arbitrary value of N . Call this value $N_1 = 2L_1 + 1$. The curve for another value of $N = N_2 = 2L_2 + 1$ is got by plotting $\frac{N_2}{N_1} E_{N_1}$ vs $\frac{L_2}{L_1} g_{N_1}$ where E_{N_1} and g_{N_1} are the energy eigenvalues and degeneracies for $N = N_1$. The idea of scaling is crucial as it helps predict the spectrum for larger values of N , a problem which is numerically difficult to handle. Though we do not find the exact relation between E and g for the spin $\frac{3}{2}$, the scaling compensates for this.

Chapter 4

Thermodynamics of the Spin Systems on S_F^2

We probe the thermodynamics of spin 1 and spin $\frac{1}{2}$ particles on the fuzzy sphere. We find several counterintuitive results which we will present in this chapter. We work with the spectrum of the Dirac operators for the spin 1 and spin $\frac{1}{2}$ case. This is natural to do as the Dirac operator is fundamental to physics and is useful in formulating metrical, differential geometric and bundle-theoretic ideas. Moreover in Connes' approach to noncommutative geometry [74], the Dirac operator gains fundamental significance as part of the spectral triple in formulating the spectral action principle [85].

Using this spectrum we first compute the partition function for a system of spin 1 particles on S_F^2 . For doing this we need to assume the particles obey a particular statistics. As we are dealing with a chiral system we assume that the particles obey the Fermi-Dirac statistics. However it should be noted that the conventional proofs of the spin-statistics theorem hold in relativistic quantum field theories (qft's) in three or more dimensions. They use the axioms of local relativistic qft's. For comprehensive proofs see [30, 117]. Field theory on the fuzzy sphere is not a relativistic one as the symmetry group of the underlying theory is $SU(2)$. This being the case there is no well defined spin-statistics relation on the fuzzy sphere. However there are spin statistics relations which do not require relativity and which are topological [20, 86, 87]. General theory for quantum statistics in 2 spatial dimensions have also been discussed [88]. The non-triviality in two spatial dimensions arises due to topology of the configuration space of indistinguishable particles living on such a space. The fundamental group for such a configuration space $(\frac{(\mathbb{R}^2)^N - \Delta}{S_N})$, where $\Delta = (\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) | \vec{r}_i \in \mathbb{R}^2 \text{ and } \vec{r}_i = \vec{r}_j \text{ for some } i \neq j$ and S_N is the symmetric group of N particles.) is the braid group B_N . For the case of S^2 instead of \mathbb{R}^2 , the fundamental group is still the braid group with an additional constraint [89, 81]. These considerations allow for the possibility of the assumption of anyonic statistics [88, 91, 92, 93] in our case, but we do not consider these possibilities in this dissertation and only briefly remark about them.

4.1 The Partition Function and the Mean Energy

As explained, we assume the spin 1 particles to obey fermionic statistics. The grand-canonical partition function is given by

$$\ln Z = \sum_{i=1} g_i \ln \left(1 + e^{-\beta(E_i - \mu)} \right) \quad (4.1)$$

where g_i is the degeneracy of the i th level, E_i is the energy of the i th level, μ is the chemical potential and $\beta = \frac{1}{k_B T}$. In the commutative case when there is no cut-off, the product over i extends till infinity, but here we are restricted by the cut-off angular momentum L .

For the spin 1 case we numerically computed the spectrum of the Dirac operator given by Eq.(2.145) in [18]. We do not know how to find an analytic expression for the spectrum and so we compute the partition function numerically. (See however [18] for an analytic expression for the spectrum of the spin 1 Dirac operator derived as a result of the numerical computations.) The analogous situation for the spin $\frac{1}{2}$ case is far better as we know its complete spectrum analytically for arbitrarily large cut-off L .

From the grand-canonical partition function in Eq.(4.1) we can use the standard formula to compute the mean energy which is

$$\langle E_1 \rangle = -\frac{\partial \ln Z}{\partial \beta} = \sum_{i=1}^{(2L+1)^2-1} \frac{g_i E_i}{e^{\beta(E_i - \mu)} + 1}. \quad (4.2)$$

In what follows we take the Boltzmann constant $k_B = 1$ and the chemical potential $\mu = 0$. The above expression for the mean energy is used for both the spin 1 and the spin $\frac{1}{2}$ cases. For the spin $\frac{1}{2}$ case it becomes

$$\langle E_{\frac{1}{2}} \rangle = \sum_{j=1}^{2L-\frac{1}{2}} \frac{1}{2} \frac{(2j+1)^2}{e^{\beta(j+\frac{1}{2})} + 1}. \quad (4.3)$$

Note that in the above Eqns.(4.2, 4.3) the sums are restricted by the cut-off L . The mean energies for both the cases were computed for various temperatures from 0.1 to 50. We show only these plots here though we did go to higher values of temperature and found nothing new. The plot for the mean energies of both the spin 1 and spin $\frac{1}{2}$ systems is shown in figure 4.1. The value of cut-off is $L = \frac{59}{2}$.

In figure 4.1, the green curve shows the mean energy for the spin $\frac{1}{2}$ system as a function of temperature and the red one shows the corresponding curve for the spin 1 system. The curves become flat for higher values of temperature. This is due to the presence of the cut-off angular momentum in our sum. If we go to higher values of temperature this flattening occurs towards the higher temperatures considered. The plot clearly shows that the mean energy of the spin $\frac{1}{2}$ system is much higher than the spin 1 system. This is inspite of the spin 1 system having more number of degrees of freedom than the spin $\frac{1}{2}$ system. We know of no such analogous behavior in higher dimensions.

Another interesting feature in the behavior of these curves is the crossing of the two curves for low values of temperature. This is not clear in figure 4.1 but is shown explicitly in figure 4.2. This plot shows that the mean energy of the spin $\frac{1}{2}$ system is smaller than the spin 1 system till about $T = 10.77$ after which it stays above the spin 1 curve.

We now try to explain the cause of this unusual behavior by looking closely at the distributions of the eigenvalues of the Dirac operators of the two systems.

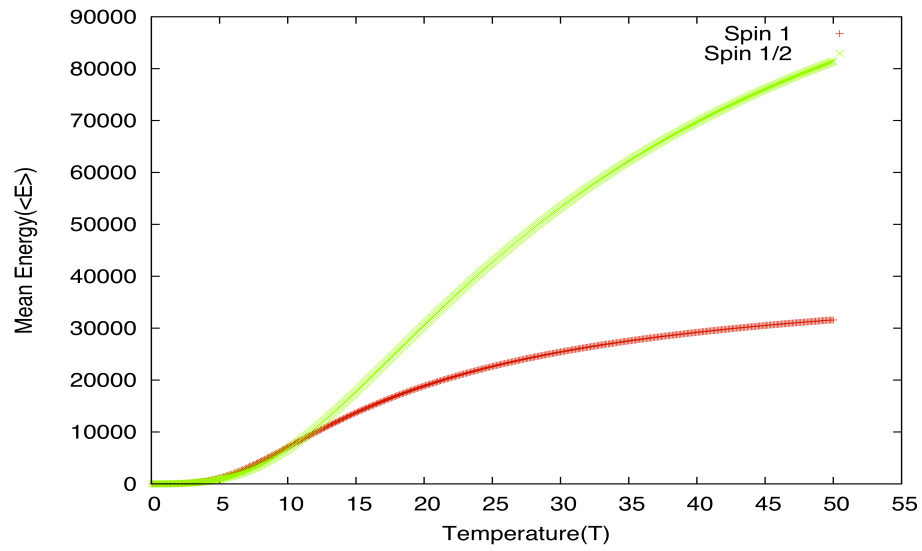


Figure 4.1: The mean energies of the spin 1 and spin $\frac{1}{2}$ systems.

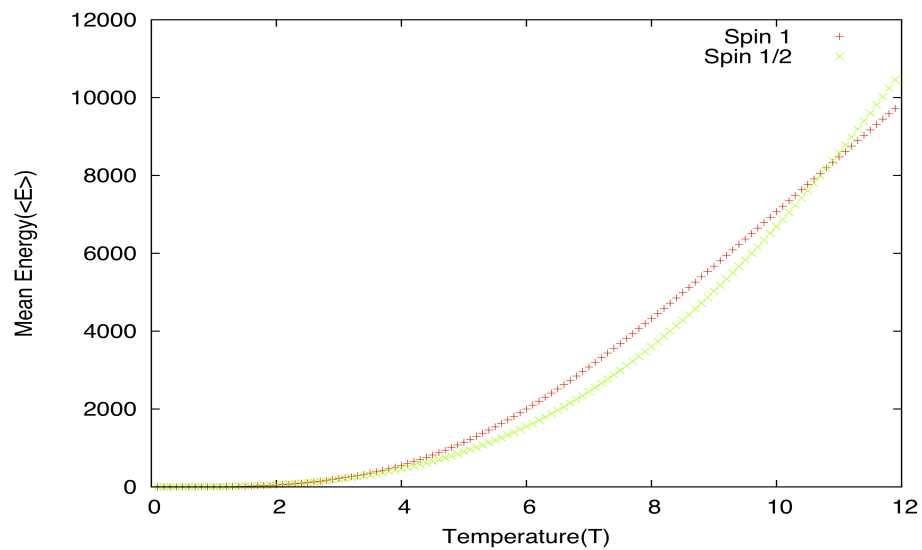


Figure 4.2: The crossing of the mean energy curves of the spin 1 and spin $\frac{1}{2}$ systems.

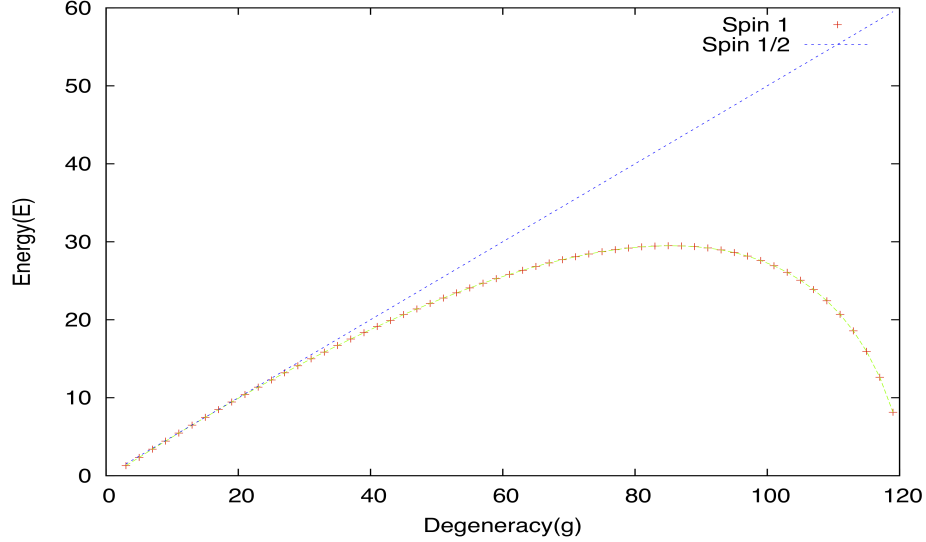


Figure 4.3: Energy as a function of the degeneracy g for the spin $\frac{1}{2}$ and spin 1 systems.

4.2 Reasons for the Strange Behavior

The main reason can be understood once we look at the spectrum of the Dirac operator in the two cases.

Using the expressions for the energy as a function of the degeneracy we can study the differences between the two systems. The relation is given by Eq.(3.7) for the spin $\frac{1}{2}$ case and Eq.(3.18) for the spin 1 case.

This plot of the energy eigenvalues as a function of their degeneracies is shown in figure 4.3.

For a given cut-off L , figure 4.3 clearly indicates that eigenvalues of the spin $\frac{1}{2}$ Dirac operator exceeds that of the spin 1 Dirac operator except for small values of the degeneracy g . The plot in figure 4.3 is shown only for positive values of the energy eigenvalue E . In the spin $\frac{1}{2}$ case the energy eigenvalues linearly increase with the degeneracy g and so the maximum eigenvalue occurs for the j value $2L - \frac{1}{2}$. We have ignored the maximum j value of $2L + \frac{1}{2}$ as they correspond to unpaired eigenstates of the Dirac operator, which will be inconsistent given the chirality of the spin $\frac{1}{2}$ system squares to 1. This is a feature of the operator $\vec{\sigma} \cdot \vec{\mathcal{L}}$ in the spin $\frac{1}{2}$ Dirac operator which is

$${}^{\infty}D_{\frac{1}{2}} = \vec{\sigma} \cdot \vec{\mathcal{L}} + \frac{1}{2}, \quad (4.4)$$

where $\vec{\sigma}$ are the Pauli matrices.

In the spin 1 case the maximum eigenvalue is L and this occurs for some intermediate value of the degeneracy g as can be seen in figure 4.3. The reason why the spin 1 Dirac operator consists of all eigenvalues ranging from $-L$ to $+L$ for a given L can be seen by looking at the operator in the continuum given by :

$${}^{\infty}D_{\frac{1}{3}} = \vec{\Sigma} \cdot \vec{\mathcal{L}} - (\vec{\Sigma} \cdot \hat{x})^2 + 2 \quad (4.5)$$

where $\vec{\Sigma}$ are the matrices of the spin 1 representation of $SU(2)$. The term $\vec{\Sigma} \cdot \hat{x}$ makes the analytic computation of the spectrum in the spin 1 case difficult when compared to the spin $\frac{1}{2}$ case. We believe this term to be also the cause of the varied spectrum of the spin 1 system.

4.3 Zero Modes of the Spin 1 Dirac Operator

The spectrum of the spin 1 Dirac operator consists of a number of zero modes for each cut-off angular momentum L . The number of such zero eigenvalues follows a simple power law as a function of L . This number was found to be $(2L + 1)^2 + 2$. This is an exact result and can be found analytically as explained in [18]. This has also been verified numerically. With this result it follows immediately that the number of positive eigenvalues of the spin 1 system are $(2L + 1)^2 - 1$. The spin $\frac{1}{2}$ Dirac operator has no zero modes as it has non-zero trace. Removing the states corresponding to the top mode gives us $4L^2 + 2L$ states with positive eigenvalues. The zero modes in the spin 1 case drastically reduce the total number of states corresponding to positive eigenvalues to $(2L + 1)^2 - 1$ but this is still more than the corresponding number of states in the spin $\frac{1}{2}$ case.

The counting of the zero modes and the behavior of the spectrum with degeneracy in the two cases justify the counter-intuitive behavior of the mean energies.

We now digress a bit to remark about the plot in figure 4.3. We try to speculate the energy versus degeneracies curves for higher spin Dirac operators. To do this we first find the number of zero modes for higher spin Dirac operators. We will compute this for the integer spin case.

To construct higher spin Dirac operators on S_F^2 , we need to construct operators acting on $Mat(2L+1) \otimes C^{2k+1}$ where k is the desired spin. The spectrum of these operators will in general be hard to compute due to the presence of $\vec{\Sigma} \cdot \hat{x}$ terms just as in the spin 1 case.

The analytic computation of the number of zero modes was given in [18]. We extend those arguments to higher spins in the following. Consider the spectrum of the total angular momentum j for a given cut-off L :

$$\text{Spec } \vec{J} \in \{0, 1, 2, \dots, 2L - k, \dots, 2L, \dots, 2L + k\}. \quad (4.6)$$

For an even-integer spin k , the number of zero-modes can be found by computing the following sum

$$\sum_{j=0}^{2L-k} (2j + 1) + \sum_{j=-\frac{k-2}{2}}^{\frac{k}{2}} [2(2L + 2j) + 1] = (2L + 1)^2 + k^2 + k. \quad (4.7)$$

For an odd-integer spin k , this number is

$$\sum_{j=0}^{2L-k-1} (2j + 1) + \sum_{j=-\frac{k+1}{2}}^{\frac{k-1}{2}} [2(2L + 2j + 1) + 1] = (2L + 1)^2 + k^2 + k. \quad (4.8)$$

These computations hold as there exists a traceless Dirac operator for all integer spin Dirac operators on S_F^2 .¹ This is because we can construct an integer spin Dirac operator from the following combination of generators of GW algebra [15]:

$$D_k = L \left(\frac{\Gamma_L^L - \Gamma_L^R}{2} \right). \quad (4.9)$$

¹This proof is shown in Appendix B

As $tr(\Gamma_L^L) = tr(\Gamma_L^R)$, this operator is traceless.

In the case of the Dirac operators for half-integral spins, there exists no such combinations of generators of GW algebras which have 0 trace. This makes the number of states with positive energy eigenvalues for spin k and spin $k - \frac{1}{2}$, for integer k , comparable.

We can then go on to compute their mean energies and compare them. We suspect $\langle E_{k-\frac{1}{2}} \rangle > \langle E_k \rangle$ to hold but we have no analytic proof for this. We could however compute the spectrums for the two Dirac operators numerically and carry out this comparison, but we do not do this here.

The reason why this is interesting is the following. It seems from the plot in figure 4.3 that the behavior of $\frac{E}{g}$ for small values of g is similar for higher spins as well. We leave this as a conjecture as we have no analytic proof for this but do have strong reasons to suspect so.

It is also very likely that the plots of E versus g for higher spins will fall below the $E = \frac{g}{2}$ curve. This is expected due to the fact that higher spin Dirac operators contain $\vec{\Sigma} \cdot \hat{x}$ terms along with $\vec{\Sigma} \cdot \vec{\mathcal{L}}$ [15]. $\vec{\Sigma}$ is the $2k + 1$ dimensional representation of $SU(2)$ for some spin k . The $\vec{\Sigma} \cdot \hat{x}$ term disrupts the linearity between the energy and degeneracy. It is easy to see this as a linear relation between the energy and the degeneracy is only possible for a Dirac operator which has just a $\vec{\Sigma} \cdot \vec{\mathcal{L}}$ term apart from constant terms. This can be seen analytically for any given spin k by looking at the spectrum of $\vec{\Sigma} \cdot \vec{\mathcal{L}}$:

$$\text{Spec } \vec{\Sigma} \cdot \vec{\mathcal{L}} = (k - m)(2j - k + m) - k^2 - m \quad m \in \{0, 1, \dots, 2k\}. \quad (4.10)$$

Only the spin $\frac{1}{2}$ Dirac operator contains just the $\vec{\sigma} \cdot \vec{\mathcal{L}}$ term leading to the linear relation between its energy and their multiplicities.

The $\vec{\Sigma} \cdot \hat{x}$ terms are present in the spin 1 case and it was remarked that these terms cause the energy versus degeneracy curve in figure 4.3. As these terms also occur for higher spin Dirac operators we expect a similar behavior from these systems. The reason why they occur for all higher spin Dirac operators is because of the fact that the fuzzy versions of these higher spin Dirac operators contain terms of the form

$$\frac{(\vec{\Sigma} \cdot \vec{L}^L)^n - (\vec{\Sigma} \cdot \vec{L}^R)^n}{L^{n-1}}.$$

The continuum limit of these terms contain $\vec{\Sigma} \cdot \hat{x}$ terms. This is explained in detail in [15].

The preceding statements prove the non-linearity between the energy and their degeneracies for all Dirac operators other than the spin $\frac{1}{2}$ system. They however do not show that these curves fall below the corresponding curve for the spin $\frac{1}{2}$ system. A complete answer to this question would only come from a numerical analysis of this system and at present we leave this question as a worthy one to explore in the future.

4.4 Specific Heats of the Two Systems

The specific heat is defined as the derivative of the mean energy with respect to temperature. A straightforward computation gives the specific heat as

$$C_v = \frac{1}{T^2} \sum_i \frac{g_i E_i^2 e^{\beta E_i}}{(e^{\beta E_i} + 1)^2}. \quad (4.11)$$

As expected here too we find the specific heat of the spin $\frac{1}{2}$ system to be more than that of the spin 1 system. This is shown in figure 4.4.

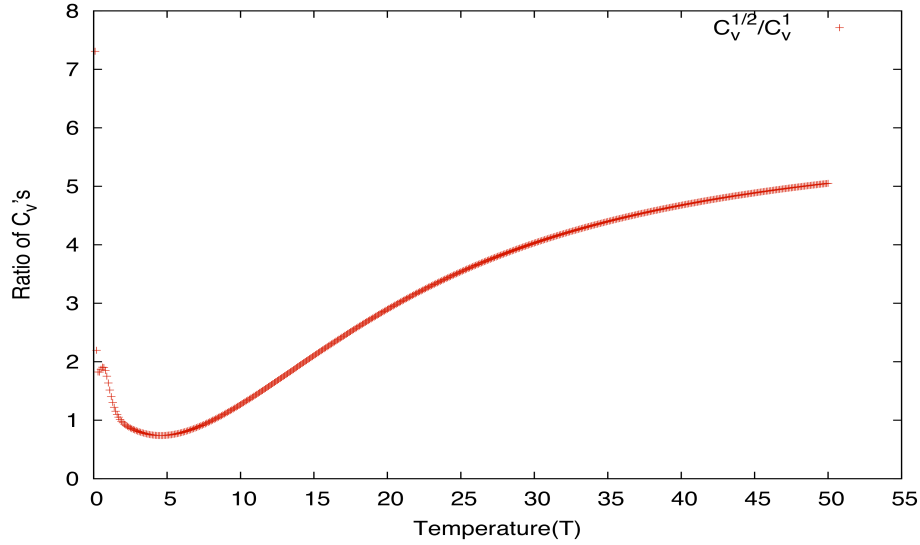


Figure 4.4: Ratio of specific heats $\frac{C_v^{1/2}}{C_v^1}$

There is a region till $T = 10.77$ where the specific heat of the spin 1 system is more than that of the spin $\frac{1}{2}$ system. This can be seen as a result of the crossing of the mean energy curves for the two systems as shown in figure 4.2.

4.5 Entropies of the Two Systems

The entropy is given by the equation

$$S = \sum_i g_i \ln(1 + e^{-\beta E_i}) + \frac{1}{T} \sum \frac{E_i}{1 + e^{-\beta E_i}}. \quad (4.12)$$

This follows from

$$S = \ln Z + \beta \langle E \rangle. \quad (4.13)$$

From these formulas it can be easily seen that the entropy of a spin 1 system is less than that of a spin $\frac{1}{2}$ system.

4.6 Deviations from the Ideal Gas Law

For a system of non-interacting massless particles obeying Fermionic statistics, the mean energy goes as T^4 in 3+1 dimensions. This can be seen as follows:

$$\langle E \rangle = \int d^3 p \frac{p}{e^{\beta p} + 1} \quad (4.14)$$

where p is the energy of the massless particle. This is the dispersion law for a massless particle on a flat space which has the Poincare group has its group of symmetries. We now substitute

$$x = \frac{p}{T}$$

to find

$$\langle E \rangle \propto T^4 \int_0^\infty dx \frac{x^3}{e^x + 1}. \quad (4.15)$$

In a similar manner, in 2+1 dimensions, the mean energy goes as T^3 . This law holds however only for a system living on a flat 2+1 dimensional spacetime.

In the case of the spin $\frac{1}{2}$ system living on S^2 or S_F^2 the mean energy is given by

$$\langle E_{\frac{1}{2}} \rangle = \sum_{j=0}^{2L+\frac{1}{2}} \frac{1}{2} \frac{(2j+1)^2}{e^{\frac{2j+1}{2T}} + 1}. \quad (4.16)$$

We have cut-off the sum with a cut-off L . If we arbitrarily increase the value of the cut-off L we will find the sum replaced by an integral over j and the upper limit in the sum goes to ∞ . In the above equation make the substitution

$$\frac{2j+1}{T} = x. \quad (4.17)$$

This makes the sum

$$\langle E_{\frac{1}{2}} \rangle = \frac{T^2}{2} \sum_{x=\frac{1}{T}}^{\frac{4L+2}{T}} \frac{x^2}{e^{\frac{x}{2}} + 1}. \quad (4.18)$$

As the limits of the sum depend on the temperature T we get no definite relation between the mean energy and temperature. It should be noted that the upper limit is dependent on T due to the cut-off L . We can remove this by allowing L to go to ∞ . In such a case, as already mentioned the sum becomes an integral making the dependence go as T^3 . This still does not remove the T dependence from the lower limit of the integral. This is due to the dispersion relation for the spin $\frac{1}{2}$ particle which goes as $j + \frac{1}{2}$. The additional $\frac{1}{2}$ can be attributed to the curvature of the sphere the system lives on. This results in the deviation from ideal gas law on 2+1 dimensional space.

Similar arguments hold for the spin 1 case also. This can be easily seen from our analytic expressions for the spectrum of the spin 1 Dirac operator as seen in the previous section. We do not write the simple details of this here.

The deviations for the spin $\frac{1}{2}$ and the spin 1 system are shown in the plots in figures 4.5 and 4.6.

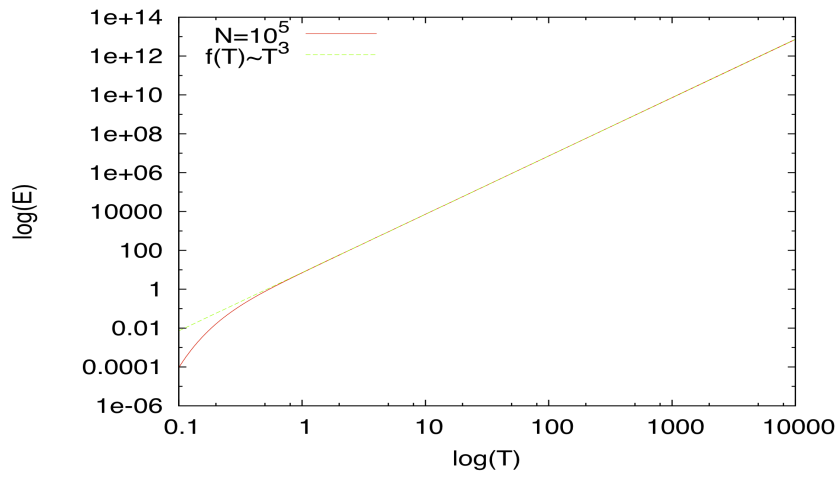


Figure 4.5: Deviation from ideal gas law for the spin $\frac{1}{2}$ system.

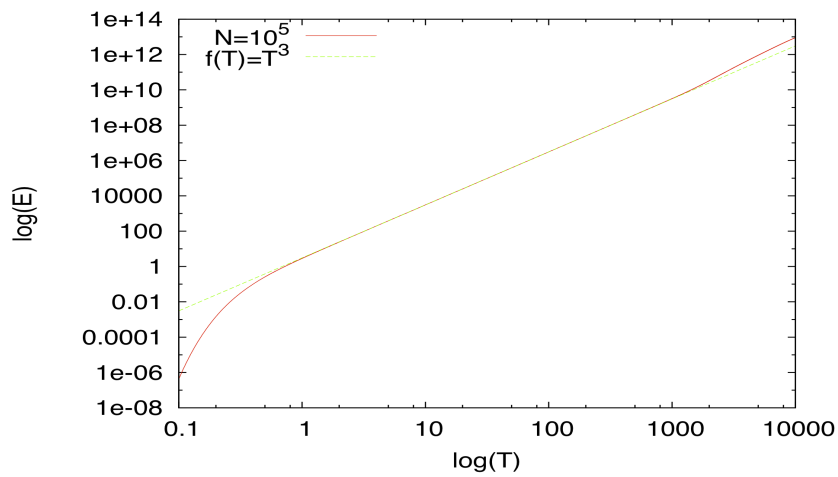


Figure 4.6: Deviation from the ideal gas law for the spin 1 system.

Chapter 5

Topology Change through Fuzzy Physics

For each time, space-time is a manifold of the form $\Sigma \times \mathbb{R}$, where Σ is a space-like surface at time t . In theories which do not involve gravity, at each time t , the space-like surface is homeomorphic to Σ . This means that the topology of Σ does not change. In a theory involving quantum gravity, we should allow for this possibility. Topology change can manifest itself in different forms- creation of baby universes, production of topological defects(cosmic strings and domain walls), changes in genus (production of wormholes and topological geons).

Several works have investigated the effects of topology change in classical and quantum gravity. In the usual canonical approach to gravity only the metric of the spatial manifold Σ appears as a degree of freedom and receives a quantum treatment. The topology of Σ is usually treated as a classical entity. We need to search for theories where topology can be canonically quantized and if possible separated from other degrees of freedom like the metric and other fields. Thus topology and quantum gravity must be intimately related and there is better hope to understand this in the $2 + 1$ setting as 2-dimensional compact(connecting) manifolds are classified. We do not attempt to find such a theory in this chapter but we show how topology change can arise in the classical setting using noncommutative algebras with a simple example. We use the Higg's algebra for this purpose. This quantum version of this algebra and the star product on this fuzzy space was studied in chapter 1.

In this chapter we see how the Higg's algebra exhibits topology change, a crucial thing to study in a diffeomorphism invariant theory of gravity. We will describe the manifold underlying this algebra and see how the topology change occurs.

The Higg's algebra and it's representations were studied in Chapter 1. We also saw there that this classical singularity is smoothened out in the fuzzy version of this manifold.

5.1 The Higgs Manifold

We consider the following embedding in \mathbb{R}^3 ,

$$x^2 + y^2 + (z^2 - \mu)^2 = 1. \tag{5.1}$$

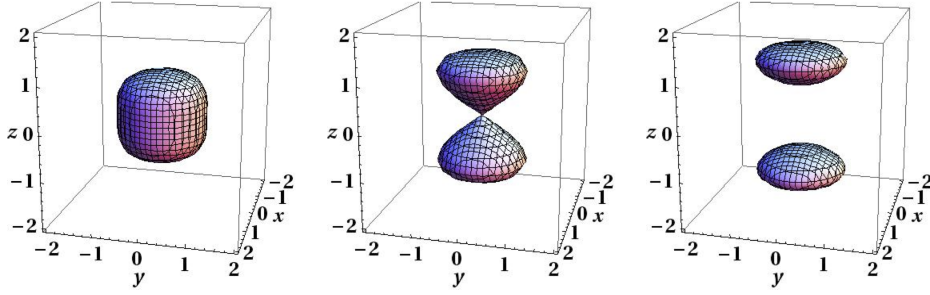


Figure 5.1: Surface plots depicting the change in topology.

This is the surface we call as the Higgs manifold, \mathcal{M}_H . μ is a parameter which can be varied. We now analyze this equation for different values of μ .

For $\mu = 1$, it is easy to see that there is a singular point ($x = y = z = 0$) where the surface degenerates. But in the discrete case, the representations do not display any difficulty at this value. When $\mu < -1$ there are no solutions. For $-1 < \mu < 1$ we have a deformed sphere, but still symmetric under rotations about the z -axis. The surface becomes two disconnected spheres for $\mu > 1$. These are explicitly shown in the figures below for specific values of μ .

The singularity

We use cylindrical coordinates to show the conical singularity arising at $\mu = 1$ and $x = y = z = 0$ as shown in Fig. (1b). The equation of the surface becomes

$$r^2 + (z^2 - 1)^2 = 1. \quad (5.2)$$

This acts as a constraint giving z in terms of r . Substituting this in the line element of Euclidean 3-space, in cylindrical coordinates, $ds^2 = dr^2 + r^2 d\phi^2 + dz^2$, we get the induced metric on this surface.

$$ds^2 = \left[1 + \frac{r^2}{4(1-r^2)[1-\sqrt{1-r^2}]} \right] dr^2 + r^2 d\phi^2. \quad (5.3)$$

As $r \rightarrow 0$, the first term of the metric approaches $\frac{3}{2}$. This implies a scaling of r by $\sqrt{\frac{3}{2}}$. This in turn induces a scaling of ϕ by $\sqrt{\frac{2}{3}}$. Thus the new ϕ coordinate has range from 0 to $2\pi(1 - \sqrt{\frac{2}{3}})$, making the origin a conical singularity. This singularity cannot be removed by a coordinate transformation.

Chapter 6

Lehmann Symanzik Zimmermann Formulation of Quantum Field Theory on the Groenewold-Moyal Plane

Quantum field theories (qft's) on the Moyal plane, \mathcal{A}_θ have been extensively studied in the past [94, 95, 96, 74]. Different approaches have been used to study them. The initial ones starting from [94] were based on the star product approach. There were others using the Seiberg-Witten map [97] of the noncommutative theory to a commutative one. Most of these approaches were plagued by the phenomenon of UV/IR mixing as was first shown in [94]. There were also questions regarding the renormalizability of these field theories. The approaches of [96, 98] restored renormalizability by using a different propagator and interaction for these theories. They also proved that their formulation of scalar field theory is renormalizable to all orders [99, 100]. In another line of development, with the appearance of the possibility of a twisted action of the Lorentz group on the Moyal plane [24], it was quickly realized by Balachandran and coworkers that the statistics of the quantum fields have to be twisted in order to be compatible with the deformed symmetry group of the noncommutative spacetime [101]. As a consequence the twisted perturbative S-matrix was shown to be independent of the noncommutative parameter $\theta_{\mu\nu}$ [102] in the absence of gauge fields. However when there is an interaction among non-abelian gauge and matter fields, the θ -dependence and UV/IR mixing reappear [103].

Qft's on the Moyal plane can be extended to include gauge fields as well [104]. The gauge fields in the approaches of [104] are not twisted unlike the matter fields and so the gauge group remains the same as in the commutative theory. This circumvents a problem faced in alternative formulations of gauge theories on the Moyal plane where the finite-dimensional Lie algebra of the group of the gauge theory gets enlarged into an infinite dimensional algebra. One important consequence of the twisted field approach of [104] to gauge theories is the addition of a central element to the spacetime symmetry algebra of the system. This results in a new deformed Hopf

algebra with a new coproduct. This coproduct does not obey the coassociativity¹ condition. This makes the spacetime also nonassociative [104].

In this chapter, we concentrate on the twisted scalar field theory on the Moyal plane in the absence of gauge fields. We compute the S-matrix elements of this theory using the LSZ reduction formula for the noncommutative case developed in [28]. It was remarked in [28] that these amplitudes can be computed using the perturbation theory of Wightman functions [105] with appropriate modifications. However here we do not use the Wightman function perturbation theory, but instead present two nonperturbative ways of computing the scattering amplitudes. The methods relate the commutative and noncommutative scattering amplitudes. When the in- and out- states are momentum eigenstates, the θ -dependence is in the form of an overall phase multiplying the commutative scattering amplitude. It represents a time delay [106]. The corresponding θ -dependence via the perturbative interaction representation S-matrix elements appears in the form of the same overall phase so that both approaches are mutually consistent.

We emphasize that the emergence of this consistency is nontrivial since the systematic formulation of the interaction representation from the Heisenberg representation for the Moyal plane is not easy as we indicate later.

In what follows we first review the formulation of twisted scalar field theory on \mathcal{A}_θ before computing the scattering amplitudes using the twisted LSZ formula.

6.1 Twisted Relativistic Quantum Fields on the Moyal plane \mathcal{A}_θ

The Gronenwold-Moyal or Moyal plane is the algebra \mathcal{A}_θ of smooth functions on \mathbb{R}^{d+1} with a twisted (star) product. It can be written as [24, 23, 107]

$$f \star g := m_\theta(f \otimes g)(x) = m_0(F_\theta f \otimes g)(x) \quad (6.1)$$

where $m_0(f \otimes g)(x) := f(x) \cdot g(x)$ stands for the usual pointwise multiplication of the commutative algebra \mathcal{A}_0 ,

$$F_\theta = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right), \quad (6.2)$$

is called the *Drinfel'd twist element* and $\theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constant}$. P. Watts [108] and R. Oeckl [109] were the first to observe that the star product in Eq.(6.1) can be cast using an F_θ .

We next briefly explain the notion of twisted Poincaré symmetry for the Moyal plane².

The proper orthochronous Poincaré group \mathcal{P}_+^\uparrow acts on multiparticle states through a coproduct which is a homomorphism from \mathcal{CP}_+^\uparrow to $\mathcal{CP}_+^\uparrow \otimes \mathcal{CP}_+^\uparrow$ where \mathcal{CP}_+^\uparrow is the group algebra of \mathcal{P}_+^\uparrow [110]. The factors in the tensor product here act through unitary representations of the Poincaré group on the single particle Hilbert spaces. On the noncommutative spacetime the coproduct should be compatible with the twisted multiplication map. The work of Aschieri *et al.* [107] and Chaichian *et al.* [24] based on Drinfel'd's original work [23] shows that $h \in \mathcal{P}_+^\uparrow$ acts on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ compatibly with m_θ i.e.,

$$m_\theta(\Delta_\theta(h)f \otimes g) = h \cdot m_\theta(f \otimes g), \quad f, g \in \mathcal{A}_\theta(\mathbb{R}^{d+1}) \quad (6.3)$$

¹Defined in Appendix C

²Hopf algebras and twisted and braid statistics are explained in detail in Appendices C and D

if its coproduct is given by

$$\Delta_\theta(h) = F_\theta^{-1}(h \otimes h)F_\theta, \quad (6.4)$$

where $F_\theta = e^{-\frac{i}{2}\hat{P}_\mu \otimes \theta_{\mu\nu} \hat{P}_\nu}$ and \hat{P}_μ is the generator of translations. It is realized as $-i_\mu$ on functions. Thus $\Delta_\theta(h)$ is a twisted version of the standard coproduct $\Delta_0(h) = h \otimes h$.

Next we define the notion of twisted statistics on the Moyal plane.

The action of the twisted coproduct is not compatible with the standard flip or statistics operator defined by τ_0 . The operator τ_0 flips two elements of $V \otimes V$ where V is a representation space for $\mathbb{C}\mathcal{P}_\pm^\uparrow$:

$$\tau_0(f \otimes g) = g \otimes f \quad (6.5)$$

where $f, g \in \mathcal{A}_0$. Now $\tau_0 F_\theta = F_\theta^{-1} \tau_0$ so that $\tau_0 \Delta_\theta(h) \neq \Delta_\theta(h) \tau_0$. This shows that the usual statistics operator is not compatible with the twisted coproduct. Hence it should be changed in quantum theory. Now the new ‘‘twisted’’ statistics operator [111]

$$\tau_\theta \equiv F_\theta^{-1} \tau_0 F_\theta, \quad \tau_\theta^2 = \mathbf{1} \otimes \mathbf{1} \quad (6.6)$$

does commute with the twisted coproduct,

$$\Delta_\theta(h) = F_\theta^{-1} h \otimes h F_\theta. \quad (6.7)$$

Hence τ_θ is an appropriate twisted flip operator and twisted bosons and fermions are to be defined using the projectors $\frac{1}{2}(\mathbb{I} \pm \tau_\theta)$ respectively.

We now define twisted quantum fields ϕ_θ which we will use throughout the rest of this paper. Here for simplicity, we assume that they are scalar fields. They are ‘‘covariant’’ [112] under the twisted action of the Poincaré group and incorporate the above twisted statistics in their creation and annihilation operators. Their star products have the important self-reproducing property

$$\phi_\theta \star \phi_\theta \star \cdots \phi_\theta(x) = (\phi_0(x) \phi_0(x) \cdots \phi_0(x))_\theta \quad (6.8)$$

where on the right, ϕ_0 's are first multiplied as ordinary fields and then finally twisted as the subsequent θ indicates.

Consider a free untwisted ($\theta_{\mu\nu} = 0$) scalar field, ϕ_0 of mass m . It has the mode expansion

$$\phi_0(x) = \int d\mu(p) (a_0(p) e_p(x) + a_0^\dagger(p) e_{-p}(x)) \quad (6.9)$$

where $e_p(x) = e^{-ip \cdot x}$, $p \cdot x = p_0 x_0 - \vec{p} \cdot \vec{x}$, $d\mu(p) = \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}}$, $p_0 = |\sqrt{\vec{p}^2 + m^2}|$. The creation and annihilation operators satisfy the standard commutation relations, the nonvanishing commutator being

$$a_0(p) a_0^\dagger(q) - a_0^\dagger(q) a_0(p) = (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (6.10)$$

The one-particle states are defined as

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}} a_0^\dagger(p) |0\rangle, \quad (6.11)$$

with $E_{\vec{p}} = p_0$. The scalar product between two such states is given by

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (6.12)$$

The completeness relation for the 1-particle states is given by

$$\mathbb{I}_{1\text{-particle}} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}|. \quad (6.13)$$

The quantum mechanical two-particle bosonic states for $\theta_{\mu\nu} = 0$ can be constructed from ϕ_0 as:

$$\begin{aligned} \langle 0|\phi_0(x_1)\phi_0(x_2)\sqrt{2E_{\vec{q}}}\sqrt{2E_{\vec{p}}}\ a_0^\dagger(q)\ a_0^\dagger(p)|0\rangle &= (1 + \tau_0)(e_{\vec{p}} \otimes e_{\vec{q}})(x_1, x_2) \\ &\equiv \langle x_1, x_2|p, q\rangle_{S_0}, \end{aligned} \quad (6.14)$$

$$|p, q\rangle_{S_0} = \sqrt{2E_{\vec{q}}}\sqrt{2E_{\vec{p}}}\ a_0^\dagger(q)a_0^\dagger(p)|0\rangle_{S_0}, \quad (6.15)$$

where τ_0 is the commutative flip operator. Here the right hand side is symmetric in x_1 and x_2 .

The two-particle states in non-commutative quantum field theory should obey twisted statistics. Using Eq.(6.9) as a guide, we can construct the twisted scalar quantum field $\phi_\theta(x)$ as

$$\phi_\theta(x) = \int d\mu(p)(a_\theta(p)\ e_p(x) + a_\theta^\dagger(p)\ e_{-p}(x)) \quad (6.16)$$

It is possible to write the twisted creation and annihilation operators $a_\theta^\dagger(p), a_\theta(p)$ in terms of the untwisted operators in Eq.(6.9). The transformation connecting the twisted and untwisted creation and annihilation operators is called the ‘‘dressing transformation’’ [113, 114] and is given by

$$a_\theta(p) = a_0(p)\ e^{-\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu}. \quad (6.17)$$

Using the above twisted field, we can construct twisted two-particle states as in Eq.(6.14):

$$\begin{aligned} \langle 0|\phi_\theta(x_1)\phi_\theta(x_2)\sqrt{2E_{\vec{q}}}\sqrt{2E_{\vec{p}}}\ a_\theta^\dagger(q)\ a_\theta^\dagger(p)|0\rangle &= (1 + \tau_\theta)(e_{\vec{p}} \otimes e_{\vec{q}})(x_1, x_2) \\ &\equiv \langle x_1, x_2|p, q\rangle_{S_\theta}, \end{aligned} \quad (6.18)$$

$$|p, q\rangle_{S_\theta} = \sqrt{2E_{\vec{q}}}\sqrt{2E_{\vec{p}}}\ a_\theta^\dagger(q)a_\theta^\dagger(p)|0\rangle_{S_\theta}, \quad (6.19)$$

where τ_θ is the twisted flip operator given in Eq.(6.6). Note that the reversed ordering of p, q as we go from LHS to RHS really matters here [115]. From Eq.(6.18) we can deduce the relations [111, 112]

$$a_\theta^\dagger(p)\ a_\theta^\dagger(q) = e^{ip_\mu\theta^{\mu\nu}q_\nu}\ a_\theta^\dagger(q)\ a_\theta^\dagger(p), \quad (6.20)$$

$$a_\theta(p)\ a_\theta(q) = e^{ip_\mu\theta^{\mu\nu}q_\nu}\ a_\theta(q)\ a_\theta(p) \quad (6.21)$$

Here P_μ is the four-momentum operator:

$$P_\mu = \int \frac{d^3p}{(2\pi)^3}(a_0^\dagger(p)\ a_0(p))p_\mu = \int \frac{d^3p}{(2\pi)^3}(a_\theta^\dagger(p)\ a_\theta(p))p_\mu. \quad (6.22)$$

Note that both the twisted and untwisted 4-momentum operators are the same since $p_\mu\theta^{\mu\nu}P_\nu$ commutes with $a_0^\dagger(p)\ a_0(p)$.

We can write the twisted quantum field in terms of the untwisted one with the help of the dressing transformation as

$$\phi_\theta(x) = \phi_0(x)e^{\frac{i}{2}\overleftarrow{\delta}_\mu\theta^{\mu\nu}P_\nu} \quad (6.23)$$

6.2 The Untwisted and Twisted LSZ Reduction Formula

The LSZ formalism for computing scattering amplitudes is non-perturbative. There are two ways to arrive at the formula for scattering amplitudes [25, 26, 27]. We use the approach given in [27]. After discussing it briefly for $\theta_{\mu\nu} = 0$, we recall [28], where the twisted LSZ reduction formula was derived.

6.2.1 The $\theta_{\mu\nu} = 0$ case

Consider an interacting quantum field theory whose Hamiltonian H can be split as

$$H = H_0 + H_I \quad (6.24)$$

where H_0 is the free Hamiltonian for a massive field and H_I is the interaction part. H_0 is used to define the states in the infinite past and infinite future. The in- and out-states of the theory are eigenstates of the full Hamiltonian H , which evolve like free states in the infinite past and future. On the other hand, free states are eigenstates of the free Hamiltonian H_0 , whose evolutions are governed by H_0 itself. The LSZ formalism works with the in- and out-states. There are creation-annihilation operators $a_0^{\dagger\text{in(out)}}(k)$, $a_0^{\text{in(out)}}(k)$ which create the in- and out- states. Note that these are not the free creation-annihilation operators. They are used in the mode expansion of the in- and out-fields. They help create the in(out) states $|k_1, k_2, \dots, k_N; \text{in(out)}\rangle$.

The interacting vacuum is unique after a phase choice.

The LSZ reduction formula for $\theta_{\mu\nu} = 0$ can be now written as

$$\langle k'_N, \dots, k'_1; \text{out} | k_M, \dots, k_1; \text{in} \rangle = \int \mathcal{I} G_{N+M}^0(x'_1, \dots, x'_N; x_1, \dots, x_M), \quad (6.25)$$

where

$$\mathcal{I} = \prod_{i=1}^N d^4 x'_i \prod_{j=1}^M d^4 x_j e^{-i(k_j \cdot x_j - k'_i \cdot x'_i)} i(\partial_i^2 + m^2) i(\partial_j^2 + m^2) \quad (6.26)$$

and

$$G_{N+M}^0(x'_1, \dots, x'_N; x_1, \dots, x_M) = \langle \Omega | T [\phi_0(x'_1) \cdots \phi_0(x'_N) \phi_0(x_1) \cdots \phi_0(x_M)] | \Omega \rangle \quad (6.27)$$

where $|\Omega\rangle$ is the interacting vacuum and $G_{N+M}^0(x'_1, \dots, x'_N; x_1, \dots, x_M)$ is the Green's function for M in-fields and N out-fields. The proof is standard and can be found in textbooks like [27].

Now we write down the twisted LSZ formula.

6.2.2 The $\theta_{\mu\nu} \neq 0$ case

It was argued in [28] that the relations between the twisted in- and out-creation-annihilation operators and the free creation-annihilation operators are:

$$a_\theta^{\text{in, out}}(k) = a_0^{\text{in, out}}(k) e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} \hat{P}_\nu}, \quad (6.28)$$

$$a_\theta^{\dagger\text{in, out}}(k) = a_0^{\dagger\text{in, out}}(k) e^{\frac{i}{2} k_\mu \theta^{\mu\nu} \hat{P}_\nu}. \quad (6.29)$$

Thus as remarked above, the in- and out-fields can be obtained from the commutative ones from the formula

$$\phi_\theta^{\text{in, out}} = \phi_0^{\text{in, out}} e^{\frac{1}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \hat{P}_\nu}. \quad (6.30)$$

The twisted in- and out-states are created using the twisted in- and out creation-annihilation operators. The twisted LSZ reduction formula is given by [28]

$$\langle \theta | k'_N, \dots, k'_1; \text{out} | k_M, \dots, k_1; \text{in} \rangle_\theta = \int \mathcal{I} G_{N+M}^\theta(x'_1, \dots, x'_N; x_1, \dots, x_M), \quad (6.31)$$

where \mathcal{I} is defined in Eq.(6.26), and

$$G_{N+M}^\theta(x'_1, \dots, x'_N; x_1, \dots, x_M) = T \left[e^{-\frac{i}{2} [\sum_{i < j} \partial_{z_{i,\mu}} \theta^{\mu\nu} \partial_{z_{j,\nu}}]} \right. \\ \left. \times W_{N+M}^0(z_1, \dots, z_N; z_{N+1}, \dots, z_{N+M}) \right] \quad (6.32)$$

with

$$z_i = x'_i, \quad i \leq N; \quad z_{N+i} = x_i, \quad i \leq M. \quad (6.33)$$

In the above $W_{N+M}^0(z_1, \dots, z_N; z_{N+1}, \dots, z_{N+M})$ is the Wightman function for $\theta_{\mu\nu} = 0$ given by

$$W_{N+M}^0(z_1, \dots, z_N; z_{N+1}, \dots, z_{N+M}) = \langle \Omega | \phi_0(z_1) \cdots \phi_0(z_{N+M}) | \Omega \rangle \quad (6.34)$$

where $|\Omega\rangle$ is the exact vacuum of the fully interacting theory, the arguments of the fields are given in Eq.(6.33) and ϕ_0 's are the fully interacting commutative quantum fields.

We will use this formula to evaluate scattering amplitudes in the noncommutative case.

6.3 Non-perturbative Computations of the Scattering Amplitudes

In this section, in order to avoid index cluttering, we use notations such as

$$p_i \wedge p_j \equiv p_{i,\mu} \theta^{\mu\nu} p_{j,\nu}, \quad \partial \wedge P = \partial_\mu \theta^{\mu\nu} P_\nu \quad (6.35)$$

where i, j stand for particle labels, and μ, ν as usual stand for spacetime components.

6.3.1 Method 1

The in- and out- states for the twisted case are

$$|p_M, \dots, p_1; \text{in}\rangle_\theta = \sqrt{(2E_{\vec{p}_1}) \cdots (2E_{\vec{p}_M})} a_\theta^{\dagger \text{in}}(p_1) \cdots a_\theta^{\dagger \text{in}}(p_M) |\Omega\rangle \\ = \sqrt{(2E_{\vec{p}_1}) \cdots (2E_{\vec{p}_M})} a_0^{\dagger \text{in}}(p_1) \cdots a_0^{\dagger \text{in}}(p_M) |\Omega\rangle e^{\frac{i}{2} \sum_{i < j \leq M} p_i \wedge p_j} \quad (6.36)$$

and

$$|p'_1, \dots, p'_N; \text{out}\rangle_\theta = \sqrt{(2E_{\vec{p}'_1}) \cdots (2E_{\vec{p}'_N})} a_\theta^{\dagger \text{out}}(p'_N) \cdots a_\theta^{\dagger \text{out}}(p'_1) |\Omega\rangle \\ = \sqrt{(2E_{\vec{p}'_1}) \cdots (2E_{\vec{p}'_N})} a_0^{\dagger \text{out}}(p'_N) \cdots a_0^{\dagger \text{out}}(p'_1) |\Omega\rangle e^{\frac{i}{2} \sum_{i < j \leq N} p'_i \wedge p'_j} \quad (6.37)$$

respectively.

It can now be immediately seen that the twisted scattering amplitude in terms of the untwisted scattering amplitude can be obtained by using the definition of the LSZ S -matrix:

$$S_\theta(p'_N, \dots, p'_1; p_M, \dots, p_1) = \theta \langle p'_N, \dots, p'_1; \text{out} | p_M, \dots, p_1; \text{in} \rangle_\theta. \quad (6.38)$$

By using the definition of the twisted in- and out- states given by Eq.(6.36) and Eq.(6.37) respectively, we see that

$$\theta \langle p'_N, \dots, p'_1; \text{out} | p_M, \dots, p_1; \text{in} \rangle_\theta = e^{\frac{i}{2} [\sum_{i < j \leq M} p_i \wedge p_j - \sum_{i < j \leq N} p'_i \wedge p'_j]} \times \\ \langle p'_N, \dots, p'_1; \text{out} | p_M, \dots, p_1; \text{in} \rangle_0. \quad (6.39)$$

Thus the twisted scattering amplitude for any process is given by

$$S_\theta(p'_N, \dots, p'_1; p_M, \dots, p_1) = e^{\frac{i}{2}[\sum_{i < j \leq M} p_i \wedge p_j - \sum_{i < j \leq N} p'_i \wedge p'_j]} \times S_0(p'_N, \dots, p'_1; p_M, \dots, p_1). \quad (6.40)$$

This relation between the commutative and the noncommutative scattering amplitudes is the same as the one obtained via the interaction representation formalism [102, 116].

We note that this method is non-perturbative and is completely independent of the interaction term in the scalar field theory considered.

The scattering amplitude on the Moyal plane given by Eq.(6.40) also shows that the twisted S -matrix is unitary in a trivial way, since the commutative S -matrix is unitary.

6.3.2 Method 2

In this second method we will find the same result via the reduction formula. It brings out the difference between scattering amplitudes and off-shell Green's functions.

The computation shown here closely follows the derivation of the reduction formula given in [25].

Here we will consider as an example the time ordered product of four fields representing a process of two particles going into two other particles described by the correlation function

$$G_{2+2}^0(x'_1, x'_2; x_1, x_2) = \langle \Omega | T (\phi_0(x'_1) \phi_0(x'_2) \phi_0(x_1) \phi_0(x_2)) | \Omega \rangle \quad (6.41)$$

which is the appropriate Green's function for the untwisted case. The Green's functions for the twisted case is obtained by replacing the commutative fields by the noncommutative ones and G_{2+2}^0 by G_{2+2}^θ . The procedure involves finding the pole structure in momentum space of the Fourier transform of $G_{2+2}^0(x'_1, x'_2; x_1, x_2)$.

We first consider the commutative case.

$$\theta_{\mu\nu} = 0$$

Let us consider the general off-shell Fourier transforms

$$\int \prod_{i=1}^j d^4 x'_i e^{i p'_i \cdot x'_i} G_{N+M}^0(x'_1, \dots, x'_N; x_1, \dots, x_M) = \tilde{G}_0^{(j)}(p'_1, \dots, p'_j, \dots, x'_N, x_1, \dots, x_M). \quad (6.42)$$

Consider Fourier transforming $G_{2+2}^0(x'_1, x'_2; x_1, x_2)$ in just x'_1 . Assume without loss of generality that x_1^0 is associated with an outgoing particle. Split the x_1^0 -integral into three regions as follows:

$$\left(\int_{T_+}^{\infty} dx_1^0 + \int_{-T_-}^{T_-} dx_1^0 + \int_{T_-}^{T_+} dx_1^0 \right) d^3 x'_1 e^{i p_1^0 x_1^0 - i \vec{p}'_1 \cdot \vec{x}'_1} G_{2+2}^0(x'_1, x'_2; x_1, x_2). \quad (6.43)$$

Here $T_+ \gg \max(x_2^0, x_1^0, x_2^0)$ and $T_- \ll \min(x_2^0, x_1^0, x_2^0)$. Since $T_+ \geq x_1^0 \geq T_-$ is a finite interval, the corresponding integral will not give any pole. A pole comes from single particle insertion in the integral over $x_1^0 \geq T_+$ in G_{2+2}^0 as we now show following [25]. In the integration between the limits T_+ and $+\infty$, $\phi(x'_1)$ stands to the extreme left inside the time-ordering so that

$$G_{2+2}^0(x'_1, x'_2; x_1, x_2) = \int \frac{d^3 q_1}{(2\pi)^3} \frac{1}{2E_{\vec{q}_1}} \langle \Omega | \phi_0(x'_1) | q_1 \rangle \langle q_1 | T (\phi_0(x'_2) \phi_0(x_1) \phi_0(x_2)) | \Omega \rangle + \text{OT} \quad (6.44)$$

where OT stands for the other terms. These other terms include those which arise from the omitted time orderings.

The matrix element of the field $\phi_0(x'_1)$ can be written as

$$\begin{aligned} \langle \Omega | e^{iP \cdot x'_1} \phi_0(0) e^{-iP \cdot x'_1} | E_{\vec{q}_1}, \vec{q}_1 \rangle &= \langle \Omega | \phi_0(0) | E_{\vec{q}_1}, \vec{q}_1 \rangle e^{-iq_1 \cdot x'_1} |_{q_1^0 = E_{\vec{q}_1}} \\ &= \langle \Omega | \phi_0(0) | q_1^0, \vec{q}_1 = 0 \rangle e^{-iq_1 \cdot x'_1} |_{q_1^0 = E_{\vec{q}_1}} \end{aligned} \quad (6.45)$$

where $E_{\vec{q}_1} = \sqrt{\vec{q}_1^2 + m^2}$. In obtaining the above relation we have used the Lorentz invariance of the vacuum and of $\phi_0(0)$ [25]. Thus

$$\langle \Omega | \phi_0(x'_1) | E_{\vec{q}_1}, \vec{q}_1 \rangle = \sqrt{Z} e^{-i(E_{\vec{q}_1} x_1^0 - \vec{q}_1 \cdot \vec{x}'_1)} \quad (6.46)$$

where

$$\langle \Omega | \phi_0(0) | q_1^0, \vec{q}_1 = 0 \rangle = \sqrt{Z} \quad (6.47)$$

and $q_1^0 > 0$. In the above \sqrt{Z} is the field-strength renormalization factor. So the integral between T_+ and $+\infty$ becomes

$$\sqrt{Z} \frac{1}{2E_{\vec{p}'_1}} \int_{T_+}^{\infty} dx_1^0 e^{i(p_1^0 - E_{\vec{p}'_1} + i\epsilon)x_1^0} \langle p'_1 | T(\phi_2' \phi_1 \phi_2) | \Omega \rangle + \text{OT} \quad (6.48)$$

where $\epsilon > 0$ is the adiabatic cut-off and $\phi_0(x_i) = \phi_i$. Performing the x_1^0 integral we get

$$\tilde{G}_0^{(1)}(p'_1, x'_2, x_1, x_2) = \sqrt{Z} \frac{i}{2E_{\vec{p}'_1}} \frac{e^{i(p_1^0 - E_{\vec{p}'_1} + i\epsilon)T_+}}{(p_1^0 - E_{\vec{p}'_1} + i\epsilon)} \langle p'_1 | T(\phi_2' \phi_1 \phi_2) | \Omega \rangle + \text{OT} \quad (6.49)$$

which as $p_1^0 \rightarrow E_{\vec{p}'_1}$, becomes

$$\tilde{G}_0^{(1)}(p'_1, x'_2, x_1, x_2) = \sqrt{Z} \frac{i}{p_1^2 - m^2 - i\epsilon} \langle p'_1 | T(\phi_2' \phi_1 \phi_2) | \Omega \rangle + \text{OT}. \quad (6.50)$$

In the integration over $(-\infty, T_-)$, $\phi_0(x'_1)$ will stand to the extreme right in the time ordered product, so the one-particle state contribution comes from

$$\langle q_1 | \phi_0(x'_1) | \Omega \rangle = \sqrt{Z} e^{i(E_{\vec{q}_1} x_1^0 - \vec{q}_1 \cdot \vec{x}'_1)}. \quad (6.51)$$

The energy denominator is thus $\frac{1}{p_1^0 + E_{\vec{p}'_1} - i\epsilon}$ and has no pole for $p_1^0 > 0$. Thus the answer for the pole is given by Eq.(6.50).

For the two-particle scattering $p_1, p_2 \rightarrow p'_1, p'_2$, we can now proceed similarly. The poles appear in both p_1^0 and p_2^0 when both x_1^0 and x_2^0 integrations are large:

$$x_1^0, x_2^0 \gg T_1 \gg x_1^0, x_2^0. \quad (6.52)$$

So for these poles

$$\begin{aligned} \tilde{G}_0^{(2)}(p'_1, p'_2, x_1, x_2) &= \int_{T_+}^{\infty} dx_1^0 dx_2^0 d^3x'_1 d^3x'_2 e^{ip'_1 \cdot x'_1 + ip'_2 \cdot x'_2} \frac{1}{2!} \left(\frac{1}{(2\pi)^3} \right)^2 \frac{d^3q_1 d^3q_2}{(2E_{\vec{q}_1})(2E_{\vec{q}_2})} \times \\ &\quad \langle \Omega | \phi_0(x'_1) \phi_0(x'_2) | \vec{q}_1 \vec{q}_2 \rangle \langle \vec{q}_1 \vec{q}_2 | T(\phi_1 \phi_2) | \Omega \rangle + \text{OT}. \end{aligned} \quad (6.53)$$

Here T_+ is considered to be very large. We set $\phi_0(x'_1), \phi_0(x'_2)$ to be out fields. As we set $|\vec{q}_2\vec{q}_1\rangle$ to $|\vec{q}_2\vec{q}_1\rangle_{\text{out}}$ for large T_+ , only $\langle\Omega|\phi_0^{\text{out}+}(x'_1)\phi_0^{\text{out}+}(x'_2)|\vec{q}_2\vec{q}_1\rangle_{\text{out}}$, where $\phi_0^{\text{out}+}$ is the annihilation part of the out-field, contributes. Thus there is no time-ordering needed involving these out-fields. So we have

$$\tilde{G}_0^{(2)}(p'_1, p'_2, x_1, x_2) = \int_{T_+}^{\infty} d^4x'_1 d^4x'_2 e^{ip'_1 \cdot x'_1 + ip'_2 \cdot x'_2} \frac{1}{2!} \left(\frac{1}{(2\pi)^3} \right)^2 \left(\frac{d^3q_1}{2E_{\vec{q}_1}} \right) \left(\frac{d^3q_2}{2E_{\vec{q}_2}} \right) \times \langle\Omega|\phi_0^{\text{out}}(x'_1)\phi_0^{\text{out}}(x'_2)|\vec{q}_2\vec{q}_1\rangle_{\text{out}} \langle\vec{q}_2\vec{q}_1|T(\phi_1\phi_2)|\Omega\rangle. \quad (6.54)$$

Now

$$\langle\Omega|\phi_0^{\text{out}}(x'_1)\phi_0^{\text{out}}(x'_2)|\vec{q}_2\vec{q}_1\rangle_{\text{out}} = \langle\Omega|\phi_0^{\text{out}}(x'_1)|\vec{q}_1\rangle\langle\Omega|\phi_0^{\text{out}}(x'_2)|\vec{q}_2\rangle + \vec{q}_2 \leftrightarrow \vec{q}_1. \quad (6.55)$$

Thus Eq.(6.50) generalizes to

$$\tilde{G}_0^{(2)}(p'_1, p'_2, x_1, x_2) = \left[\sqrt{Z} \left(\frac{i}{p_1'^2 - m^2 - i\epsilon} \right) \right] \left[\sqrt{Z} \left(\frac{i}{p_2'^2 - m^2 - i\epsilon} \right) \right] \times \langle\vec{p}'_1\vec{p}'_2|T(\phi_1\phi_2)|\Omega\rangle + \text{OT}. \quad (6.56)$$

With similar calculations for incoming poles, with $x_1^0, x_2^0 \ll T_- \ll x_1^0, x_2^0$,

$$\tilde{G}_0^{(4)}(p'_1, p'_2, p_1, p_2) = \prod_{i=1}^2 \prod_{j=1}^2 \left[\sqrt{Z} \left(\frac{1}{p_i'^2 - m^2 - i\epsilon} \right) \right] \left[\sqrt{Z} \left(\frac{1}{p_j^2 - m^2 - i\epsilon} \right) \right] \times \langle\vec{p}'_1\vec{p}'_2 | p_1 p_2\rangle_{\text{in}} \quad (6.57)$$

as required.

$\theta_{\mu\nu} \neq 0$

We will work along lines similar to the one followed for the commutative case to arrive at the twisted version of Eq.(6.57). However the process we consider in the noncommutative case will not be a 2-particle scattering process as chosen in the commutative case. Instead we consider a process where M particles go into N particles.

We introduce the following notations:

$$\hat{p} \text{ is an on-shell momentum} = (E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}, \vec{p}) \quad (6.58)$$

$$p \text{ is a generic 4-momentum, with } p^0 > 0. \quad (6.59)$$

Completeness

The completeness relations for the twisted in- and out-states are the same as in the commutative case, since the noncommutative phases cancel each other. Hence

$$a_{\theta}^{\dagger, \text{in}, \text{out}}(p_N) \cdots a_{\theta}^{\dagger, \text{in}, \text{out}}(p_1) |\Omega\rangle \langle\Omega| a_{\theta}^{\text{in}, \text{out}}(p_1) \cdots a_{\theta}^{\text{in}, \text{out}}(p_N) = a_0^{\dagger, \text{in}, \text{out}}(p_N) \cdots a_0^{\dagger, \text{in}, \text{out}}(p_1) |\Omega\rangle \langle\Omega| a_0^{\text{in}, \text{out}}(p_1) \cdots a_0^{\text{in}, \text{out}}(p_N). \quad (6.60)$$

From Eq.(6.60) follow both the resolution of identity given below and hence completeness for the twisted in- and out-states.

Resolution of Identity

Consider

$$I' = \sum_N \frac{1}{N!} \left(\int \prod_{i=1}^N \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right) a_\theta^{\dagger \text{in, out}}(p_N) \cdots a_\theta^{\dagger \text{in, out}}(p_1) |\Omega\rangle \langle \Omega| a_\theta^{\text{in, out}}(p_1) \cdots a_\theta^{\text{in, out}}(p_N). \quad (6.61)$$

This is independent of $\theta_{\mu\nu}$ due to Eq.(6.60) and hence is the resolution of identity:

$$I' = I = \sum_N \frac{1}{N!} \left(\int \prod_{i=1}^N \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right) a_0^{\dagger \text{in, out}}(p_N) \cdots a_0^{\dagger \text{in, out}}(p_1) |\Omega\rangle \langle \Omega| a_0^{\text{in, out}}(p_1) \cdots a_0^{\text{in, out}}(p_N). \quad (6.62)$$

For the scattering process of M particles to N particles, the twisted $N + M$ -point Green's function we need to look at is

$$G_{N+M}^\theta(x'_1, \dots, x'_N; x_1, \dots, x_M) = \langle \Omega | T(\phi_\theta(x'_1) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle. \quad (6.63)$$

This is Fourier transformed by integrating with respect to the measure

$$\left(\prod_i d^4 x'_i \right) \left(\prod_j d^4 x_j \right) e^{i(\sum_{i \leq N} p'_i \cdot x'_i - \sum_{j \leq M} p_j \cdot x_j)}.$$

Integration over x_i, x'_i gives $\tilde{G}_\theta^{N+M}(p'_1 \cdots, p'_N, p_1 \cdots, p_M)$ and the residue at the poles in all the momenta multiplied together gives the scattering amplitude. This is just the noncommutative version of the LSZ reduction formula. We show that we obtain the same answer as Method I for the S-matrix elements in this way.

Pole in just p'_1

Fourier transform just in x'_1 to obtain

$$\tilde{G}_\theta^{(1)}(p'_1, \dots, x'_N, x_1, \dots, x_M) = \int d^4 x'_1 e^{i(p'_1 x'_1 - \vec{p}'_1 \cdot \vec{x}'_1)} \times \langle \Omega | T(\phi_\theta(x'_1) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle. \quad (6.64)$$

With $T_+ \gg x_N^0 \cdots x_2^0, x_M^0, \dots, x_1^0$, we isolate the term with pole in $\tilde{G}_\theta^{(1)}$:

$$\begin{aligned} \tilde{G}_\theta^{(1)}(p'_1, \dots, x'_N, x_1 \cdots x_M) &= \sqrt{Z} \int_{T_+}^\infty dx_1^0 d^3 x'_1 e^{i(p'_1 x'_1 - \vec{p}'_1 \cdot \vec{x}'_1)} \times \\ &\langle \Omega | \phi_\theta^{\text{out}}(x'_1) T(\phi_\theta(x'_2) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle + \text{OT} \\ &= \sqrt{Z} \int_{T_+}^\infty dx_1^0 d^3 x'_1 \frac{1}{(2\pi)^3} \frac{d^3 q_1}{2E_{\vec{q}_1}} e^{i(p'_1 x'_1 - \vec{p}'_1 \cdot \vec{x}'_1)} \times \\ &\langle \Omega | \phi_\theta^{\text{out}}(x'_1) | \hat{q}_1 \rangle \langle \hat{q}_1 | T(\phi_\theta(x'_2) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle + \text{OT} \end{aligned} \quad (6.65)$$

where

$$\langle \Omega | \phi_\theta^{\text{out}}(x'_1) | \hat{q}_1 \rangle = \langle \Omega | \phi_0^{\text{out}}(x'_1) | \hat{q}_1 \rangle \quad (6.66)$$

as the twist gives just 1 in this case. This can be seen by writing ϕ_θ^{out} as $e^{\frac{1}{2} \partial_\mu \theta^{\mu\nu} P_\nu} \phi_0^{\text{out}}$ and acting with P_ν on $\langle \Omega |$.

Repeating the same procedure as in that of the commutative case, we can extract the pole $\frac{1}{p_1^2 - m^2 - i\epsilon}$ and its coefficient.

Extracting poles at p'_1, p'_2

In this case we are led to

$$\begin{aligned} \tilde{G}_\theta^{(2)}(p'_1, p'_2, x'_3, \dots, x'_N, x_1, \dots, x_M) &= \int_{T_+}^\infty d^4 x'_1 d^4 x'_2 e^{ip'_1 \cdot x'_1 + ip'_2 \cdot x'_2} (\sqrt{Z})^2 \frac{d^3 \hat{q}_1 d^3 \hat{q}_2}{2!(2E_{\vec{q}_1})(2E_{\vec{q}_2})} \times \\ &\langle \Omega | \phi_\theta^{\text{out}}(x'_1) \phi_\theta^{\text{out}}(x'_2) | \hat{q}_1, \hat{q}_2 \rangle \langle \hat{q}_1, \hat{q}_2 | T(\phi_\theta(x'_3) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle + \text{OT}. \end{aligned} \quad (6.67)$$

Note that there is no twist in $|\hat{q}_1, \hat{q}_2\rangle$ and $\langle \hat{q}_2, \hat{q}_1|$ (See Eq.(6.62)).

We now compute the matrix element of the two out-fields.

$$\begin{aligned} \langle \Omega | \phi_\theta^{\text{out}}(x'_1) \phi_\theta^{\text{out}}(x'_2) | \hat{q}_1, \hat{q}_2 \rangle &= \int \left(\frac{1}{(2\pi)^3} \right)^2 \frac{d^3 p''_1}{\sqrt{2E_{\vec{p}''_1}}} \frac{d^3 p''_2}{\sqrt{2E_{\vec{p}''_2}}} e^{-ip''_1 \cdot x'_1 - ip''_2 \cdot x'_2} \times \\ &\sqrt{2E_{\vec{q}_1}} \sqrt{2E_{\vec{q}_2}} \langle \Omega | \left(a_0^{\text{out}}(p''_1) e^{-\frac{i}{2} \hat{p}''_1 \wedge P} \right) \left(a_0^{\text{out}}(p''_2) e^{-\frac{i}{2} \hat{p}''_2 \wedge P} \right) a_0^{\dagger \text{out}}(\hat{q}_2) a_0^{\dagger \text{out}}(\hat{q}_1) | \Omega \rangle \\ &= \int \left(\frac{1}{(2\pi)^3} \right)^2 \frac{d^3 p''_1}{\sqrt{2E_{\vec{p}''_1}}} \frac{d^3 p''_2}{\sqrt{2E_{\vec{p}''_2}}} e^{-ip''_1 \cdot x'_1 - ip''_2 \cdot x'_2} e^{-\frac{i}{2} \hat{p}''_1 \wedge (-\hat{p}''_2 + \hat{q}_1 + \hat{q}_2)} e^{-\frac{i}{2} \hat{p}''_2 \wedge (\hat{q}_1 + \hat{q}_2)} \times \\ &\sqrt{2E_{\vec{q}_1}} \sqrt{2E_{\vec{q}_2}} \langle \Omega | a_0^{\text{out}}(p''_1) a_0^{\text{out}}(p''_2) a_0^{\dagger \text{out}}(q_2) a_0^{\dagger \text{out}}(q_1) | \Omega \rangle. \end{aligned} \quad (6.68)$$

The matrix element becomes

$$\begin{aligned} \langle \Omega | a_0^{\text{out}}(p''_1) a_0^{\text{out}}(p''_2) a_0^{\dagger \text{out}}(q_2) a_0^{\dagger \text{out}}(q_1) | \Omega \rangle &= (2\pi)^3 (2\pi)^3 \left[\delta^3(\vec{p}''_1 - \vec{q}_1) \delta^3(\vec{p}''_2 - \vec{q}_2) \right. \\ &\quad \left. + \delta^3(\vec{p}''_1 - \vec{q}_2) \delta^3(\vec{p}''_2 - \vec{q}_1) \right] \end{aligned} \quad (6.69)$$

which means that the whole matrix element is 0 unless

$$\hat{p}''_1 + \hat{p}''_2 = \hat{q}_1 + \hat{q}_2. \quad (6.70)$$

So the noncommutative phase can be simplified according to

$$e^{-\frac{i}{2} \hat{p}''_1 \wedge (-\hat{p}''_2 + \hat{p}''_1 + \hat{p}''_2)} e^{-\frac{i}{2} \hat{p}''_2 \wedge (\hat{p}''_1 + \hat{p}''_2)} = e^{-\frac{i}{2} \hat{p}''_2 \wedge \hat{p}''_1}. \quad (6.71)$$

Integrations over \vec{x}'_1, \vec{x}'_2 give δ -functions setting

$$\vec{p}''_1 = \vec{p}'_1, \quad \vec{p}''_2 = \vec{p}'_2 \quad (6.72)$$

and hence

$$\hat{p}''_1 = \hat{p}'_1, \quad \hat{p}''_2 = \hat{p}'_2. \quad (6.73)$$

Thus the noncommutative phase becomes $e^{-\frac{i}{2} \hat{p}'_2 \wedge \hat{p}'_1}$.

Since

$$\text{out} \langle \hat{q}_1, \hat{q}_2 | \rightarrow \text{out} \langle \hat{p}'_1, \hat{p}'_2 | \quad (6.74)$$

and due to the identity

$$\text{out} \langle \Omega | a_0^{\text{out}}(q_1) a_0^{\text{out}}(q_2) = \text{out} \langle \Omega | a_0^{\text{out}}(q_2) a_0^{\text{out}}(q_1) \quad (6.75)$$

we end up with

$$\tilde{G}_\theta^{(2)}(p'_1, p'_2, \dots, x'_N, x_1, \dots, x_M) = \frac{\sqrt{Z}}{p_1'^2 - m^2 - i\epsilon} \frac{\sqrt{Z}}{p_2'^2 - m^2 - i\epsilon} e^{-\frac{i}{2} p'_2 \wedge \hat{p}'_1} \times \\ \text{out} \langle \hat{p}'_1 \hat{p}'_2 | T(\phi_\theta(x'_3) \cdots \phi_\theta(x'_N) \phi_\theta(x_1) \cdots \phi_\theta(x_M)) | \Omega \rangle + \text{OT}. \quad (6.76)$$

The phase can be absorbed to get the twisted out-state

$$\langle \Omega | a_\theta^{\text{out}}(\hat{p}'_2) a_\theta^{\text{out}}(\hat{p}'_1). \quad (6.77)$$

Thus the two-particle residue gives the answer appropriate for the one obtained in Eq.(6.40).

This can be easily generalized to N outgoing particles. For this purpose it, is enough to prove that the phases associated with the outgoing fields give the appropriate phases. This phase comes from manipulating

$$\langle \Omega | a_\theta^{\text{out}}(\hat{p}'_1) a_\theta^{\text{out}}(\hat{p}'_2) \cdots a_\theta^{\text{out}}(\hat{p}'_N) | \hat{q}_1 \cdots \hat{q}_N \rangle \quad (6.78)$$

and

$$\langle \hat{q}_1 \cdots \hat{q}_N | a_\theta^\dagger(\hat{p}'_N) \cdots a_\theta^\dagger(\hat{p}'_1) | \Omega \rangle. \quad (6.79)$$

They have phases related by a complex conjugation. They can be calculated by moving the twists of $a_\theta(\hat{p}')$ to the left in Eq.(6.78) and to the right in Eq.(6.79). This will give the appropriate phase as seen in Eq.(6.40).

We can proceed in a similar manner for incoming particles as well where the conjugates of Eq.(6.78) and Eq.(6.79) appear. Putting all this together, the final answer is easily seen to be the same as the one obtained in Eq.(6.40).

Chapter 7

Non-Pauli Effects on the $\mathcal{B}_{\chi\vec{n}}$ Plane

The spin-statistics theorem in three or more dimensions has been proved in many ways in local relativistic qft's. It assumes its comprehensive form in the work of Doplicher and Roberts [117, 118]. It states that identical tensorial particles are bosons and identical spinorial particles are fermions. The proofs of this theorem require the axioms of local relativistic qft's. Deep extensions of the theorem to qft's on gravitational backgrounds exist [119, 120], but they too require spacetime commutativity and a form of locality.

It is reasonable to expect that the spin-statistics connection and its emergent physics can get modified in models where spacetime commutativity and locality do not hold. A suggestion was made along these lines for qft's on the Moyal plane [111, 121]. A subsequent paper by Banerjee et al [31] developed this idea and showed in a striking calculation that the Pauli repulsion between fermions, infinite for zero separation on commutative spacetimes, softens to a finite value on the Moyal plane. Applications of this effect to statistical mechanics, superconductivity, and Chandrasekhar limit either exist or are in progress [122].

In this chapter we explicitly consider Pauli-forbidden transitions which are not considered in earlier works.

With precision experiments at increasingly shorter length and time scales, it is now timely to question principles of local qft's such as Lorentz invariance, *CPT* theorem and the spin-statistics connection. As regards the last, there exist excellent experiments on Pauli-forbidden transitions, but there is a scarcity of good models to confront data, those of Greenberg and coworkers being among the exceptions. These are reported or reviewed in [123, 124, 40] where also much existing information is surveyed. A desirable model will have a small parameter χ , $\chi = 0$ giving back the standard treatment. Here we develop such an approach adapted to treat Pauli-violating atomic and nuclear transitions.

Our model is based on a spacetime $\mathcal{B}_{\chi\vec{n}}$ different from the Moyal plane \mathcal{A}_θ . The latter also seems to predict the exotic effects we look for, but the calculations get complicated. Just as in the case of \mathcal{A}_θ , $\mathcal{B}_{\chi\vec{n}}$ too can be described in terms of a Drinfel'd twist element $F_{\chi\vec{n}}$. So the Poincaré group algebra $\mathbb{C}\mathcal{P}$ can act on $\mathcal{B}_{\chi\vec{n}}$ as a Hopf algebra if its coproduct is deformed. Compatibility with this action requires that we deform the standard symmetrization or flip

operator τ_0 to

$$\tau_{\chi\vec{n}} = F_{\chi\vec{n}}^{-1} \tau_0 F_{\chi\vec{n}}. \quad (7.1)$$

That changes the symmetrization and anti-symmetrization of wave functions and leads to novel physics. The details we need about the modified flip $\tau_{\chi\vec{n}}$ and the deformed Hopf algebra of $\mathbb{C}\mathcal{P}$ are in section 7.1.

A typical Pauli-forbidden transition can occur in neutral beryllium with two electrons in the ground state and the remaining two electrons in the excited state: the transition of the excited electrons to the ground state is Pauli-forbidden on the commutative spacetime \mathcal{B}_0 . But it occurs on $\mathcal{B}_{\chi\vec{n}}$ and we calculate its rate. It involves new physics, relying on the fact that the direction of the unit vector \vec{n} effectively changes with earth's rotation and movements. These are very swift events for noncommutative corrections induced by χ , so that the sudden approximation is appropriate to treat χ -dependent atomic or nuclear phenomena. (We do not consider TeV scale gravity [125].)

When \vec{n} changes to \vec{m} by earth's fast motions, twisted fermions with $\tau_{\chi\vec{n}} = -1$ in the sudden approximation become admixtures of both twisted fermions and twisted bosons ($\tau_{\chi\vec{m}} = \mp 1$) leading to the above process.

Section 7.2 describes the two-electron energy eigenstates on $\mathcal{B}_{\chi\vec{n}}$.

In section 7.3, we calculate what becomes of these state vectors when \vec{n} rapidly changes to \vec{m} . We explicitly find the twisted Bose components induced in certain twisted Fermi levels of $\mathcal{B}_{\chi\vec{n}}$. This enables us to calculate the rate R of transition of the excited electrons to the fully occupied ground level for a sufficiently generic perturbation. R depends on \vec{n}, \vec{m} , but since \vec{m} and \vec{n} keep changing, we average them to get an average rate $\langle R \rangle$.

Comparison with experiments are best done by developing a formula for a branching ratio B where the effects not specific to noncommutativity may largely cancel. So we divide $\langle R \rangle$ by a typical rate for an allowed atomic or nuclear transition and find a B . It is $O((\chi\Delta E)^2)$ where ΔE is a suitable energy difference.

The expression for B and the available atomic and nuclear experiments give bounds on χ . The use of B away from its original context is justified as remarked above, B being a ratio. In any case, our bounds are rough. They are reported in section 7.4. The best ones come from neutrino signals of forbidden processes [34, 35, 39] and give $\chi \gtrsim 10^{24} \text{TeV}$. This does seem an excessively stringent bound suggesting further checks on its validity. As it stands, it suggests an energy scale beyond Planck scale.

The focus of section 7.5 shifts away from Pauli principle and probes other features of $\mathcal{B}_{\chi\vec{n}}$. We show that time translation gets quantized on $\mathcal{B}_{\chi\vec{n}}$ in units of χ . Elsewhere this effect has been discussed in detail [126, 127, 128] and it has been proved that energy is conserved only mod $\frac{2\pi}{\chi}$ in scattering processes. A formal scattering theory has also been developed. Thus $\mathcal{B}_{\chi\vec{n}}$ predicts much new physics. Its potential applications to higher dimensional models is also pointed out in section 7.5.

7.1 The Spacetime $\mathcal{B}_{\chi\vec{n}}$

The elements of $\mathcal{B}_{\chi\vec{n}}$ are functions on the Minkowski space M^4 . If x_μ are coordinate functions transforming under the Poincaré group \mathcal{P} in the standard manner, the algebra $\mathcal{B}_{\chi\vec{n}}$ is characterized by the relations

$$[x_0, x_i] = i\chi\epsilon_{kij}n_k x_j, \quad (7.2)$$

$$[x_i, x_j] = 0, \quad i, j = 1, 2, 3 \quad (7.3)$$

where x_0 is the time function and \vec{n} is a fixed three-dimensional unit vector.

A product map $m_{\chi\vec{n}}$ of two functions f, g , which leads to Eq.(7.2) is given by

$$m_{\chi\vec{n}}(f \otimes g) = f e^{\frac{1}{2}\chi(\overleftarrow{\partial}_t \vec{n} \cdot \vec{L} - \vec{n} \cdot \overleftarrow{L} \partial_t)} g \quad (7.4)$$

where $\vec{L} = -i\chi\vec{x} \wedge \vec{\nabla}$ is orbital angular momentum and generates rotations. The product in Eq.(7.4), is associative since $[\partial_t, \vec{n} \cdot \vec{L}] = 0$. Equation (7.4) defines $\mathcal{B}_{\chi\vec{n}}$.

We can write Eq.(7.4) in terms of the twist element

$$F_{\chi\vec{n}} = e^{\frac{1}{2}\chi(\partial_t \otimes \vec{n} \cdot \vec{L} - \vec{n} \cdot \vec{L} \otimes \partial_t)} \quad (7.5)$$

as follows:

$$m_{\chi\vec{n}} = m_0 \cdot F_{\chi\vec{n}}, \quad (7.6)$$

$$m_{\chi\vec{n}}(f \otimes g) = m_0[F_{\chi\vec{n}}f \otimes g] \quad (7.7)$$

where m_0 is point-wise multiplication :

$$m_0(f \otimes g)(p) = f(p)g(p), \quad p = \text{a point of } M^4. \quad (7.8)$$

The algebra $\mathcal{B}_{\chi\vec{n}}$ is well-suited for deforming dynamics with spherical symmetry as in atomic physics with its central potentials. For the same reason, it is well-adapted to deform quantum fields on black hole backgrounds. The Moyal plane is awkward to deal with in either case (See however [122]).

In a generic representation carrying the action of $\mathbb{C}\mathcal{P}$, \vec{L} becomes the rotation generator \vec{J} and $i\partial_t$ the translation generator P_0 . If $G_{\chi\vec{n}}$ is the generic form of $F_{\chi\vec{n}}$, then

$$G_{\chi\vec{n}} = e^{-\frac{i}{2}\chi(P_0 \otimes \vec{n} \cdot \vec{J} - \vec{n} \cdot \vec{J} \otimes P_0)}. \quad (7.9)$$

Drinfel'd's original work [23] and subsequent developments by Aschieri et al. [107] and Chaichian et al. [24] show that $\mathbb{C}\mathcal{P}$ acts as a Hopf algebra HP if its coproduct is modified by the Drinfel'd twist $G_{\chi\vec{n}}$ to $\Delta_{\chi\vec{n}}$:

$$\Delta_{\chi\vec{n}}(g) := G_{\chi\vec{n}}^{-1}(g \otimes g)G_{\chi\vec{n}}, \quad g \in \mathcal{P}. \quad (7.10)$$

For $\chi = 0$, when noncommutativity is absent, symmetrization and anti-symmetrization is achieved using the projectors $\frac{1 \pm \tau_0}{2}$. τ_0 here is the flip operator: if \mathcal{H} is a Hilbert space carrying a representation of \mathcal{P} or one of its subgroups, and $\alpha, \beta \in \mathcal{H}$, $\tau_0(\alpha \otimes \beta) = \beta \otimes \alpha$. This flip commutes with $\Delta_0(g)$ and is Poincaré invariant for $\chi = 0$.

But for $\chi \neq 0$,

$$\tau_0 G_{\chi\vec{n}} = G_{\chi\vec{n}}^{-1} \tau_0 \quad (7.11)$$

and τ_0 fails to commute with $\Delta_{\chi\vec{n}}(g)$: the projectors $\frac{1 \pm \tau_0}{2}$ are not Poincaré invariant for $\chi \neq 0$. Hence we must deform τ_0 suitably. Such a deformed flip operator is the twisted flip operator

$$\tau_{\chi\vec{n}} = G_{\chi\vec{n}}^{-1} \tau_0 G_{\chi\vec{n}} = G_{\chi\vec{n}}^{-2} \tau_0, \quad \tau_{\chi\vec{n}}^2 = 1. \quad (7.12)$$

Thus if \mathcal{H} is a representation space for $\mathbb{C}\mathcal{P}$ or one of its generic subgroups, and $\alpha \otimes \beta \in \mathcal{H} \otimes \mathcal{H}$, the twisted bosons and fermions are images of $\mathcal{H} \otimes \mathcal{H}$ under the projectors $\frac{1 \pm \tau_{\chi\vec{n}}}{2}$:

$$\text{Twisted Bosons: } \mathcal{H} \otimes_{S_{\chi\vec{n}}} \mathcal{H} := \frac{1 + \tau_{\chi\vec{n}}}{2} \mathcal{H} \otimes \mathcal{H} \quad (7.13)$$

$$\text{Twisted Fermions: } \mathcal{H} \otimes_{A_{\chi\vec{n}}} \mathcal{H} := \frac{1 - \tau_{\chi\vec{n}}}{2} \mathcal{H} \otimes \mathcal{H}. \quad (7.14)$$

We note that \mathcal{H} can be the Hilbert space of an electron with spin in the central potential of a nucleus. The single particle symmetry group G we then focus on is $SU(2) \times \mathbb{R}$ where $SU(2)$ is the (two-fold cover of the) rotation group acting also on spin and rotating around the nuclear center, and \mathbb{R} is the time translation group. The generator P_0 of \mathbb{R} is the single-particle Hamiltonian:

$$P_0 \equiv H = \frac{\vec{p}^2}{2\mu} - \frac{Ze^2}{r}, \quad (7.15)$$

$$Z = \text{Nuclear charge,}$$

$$\vec{r} = \text{relative coordinates,}$$

$$\mu = \text{reduced mass.}$$

For this paper, the Hopf algebra of interest is the group algebra $\mathbb{C}(SU(2) \times \mathbb{R})$ where \mathbb{R} is time translation, along with the coproduct $\Delta_{\chi\vec{n}}$. We are interested in its concrete realization, denoted here as $H_\chi(SU(2) \times \mathbb{R})$, on multi-electron states. We now describe a convenient basis for this Hilbert space and evaluate the coproducts $\Delta_{\chi\vec{n}}(H)$ and $\Delta_{\chi\vec{n}}(J_i)$ of the Hamiltonian and angular momentum in this basis.

The single particle basis we choose consists of eigenstates of H and is

$$|N, l\rangle \otimes |\alpha\rangle_{\vec{n}} \equiv |N, l, \alpha\rangle_{\vec{n}}, \quad \alpha = \pm 1 \quad (7.16)$$

where N and l are the principal quantum number and orbital angular momentum and $|\alpha\rangle_{\vec{n}}$ denotes the eigenstates of $\vec{\sigma} \cdot \vec{n}$ (σ_i being Pauli matrices) with eigenvalues α :

$$H|N, l, \alpha\rangle_{\vec{n}} = E_N|N, l, \alpha\rangle_{\vec{n}}, \quad (7.17)$$

$$E_N = -\frac{Z \times 13.6}{N^2} \text{eV} = \text{energy for principal quantum number } N$$

$$\vec{\sigma} \cdot \vec{n}|N, l, \alpha\rangle_{\vec{n}} = \alpha|N, l, \alpha\rangle_{\vec{n}}. \quad (7.18)$$

The state vector $|N, l, \alpha\rangle_{\vec{n}}$ is $|N, l\rangle \otimes |\alpha\rangle_{\vec{n}}$ where the spin vector $|\alpha\rangle_{\vec{n}}$ can be constructed as follows. Let $g(\vec{n}) \in SU(2)$ (in its defining representation) such that [44, 110]

$$g(\vec{n})\sigma_3g(\vec{n})^\dagger = \vec{\sigma} \cdot \vec{n} \quad (7.19)$$

and let

$$\sigma_3|\alpha\rangle_{\hat{k}} = \alpha|\alpha\rangle_{\hat{k}}, \quad \hat{k} = (0, 0, 1) \quad (7.20)$$

so that

$$|+\rangle_{\hat{k}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle_{\hat{k}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (7.21)$$

Then

$$g(\vec{n})|\alpha\rangle_{\hat{k}} = |\alpha\rangle_{\vec{n}}. \quad (7.22)$$

Note that $g(\vec{n})$ is not unique as both $g(\vec{n})$ and $g(\vec{n})e^{i\sigma_3\theta}$ rotate σ_3 to $\vec{\sigma} \cdot \vec{n}$. This ambiguity will disappear when we compute rates. We also do not need an explicit choice of $g(\vec{n})$ to calculate rates.

Next we calculate $\Delta_{\chi\vec{n}}(H)$ and $\Delta_{\chi\vec{n}}(\vec{J})$.

As for $\Delta_{\chi\vec{n}}(H)$ and $\Delta_{\chi\vec{n}}(\vec{n} \cdot \vec{J})$, they are not affected by χ since H and $\vec{n} \cdot \vec{J}$ commute and $G_{\chi\vec{n}}$ contains only these operators. (Hereafter $G_{\chi\vec{n}}$ denotes Eq.(7.9) on the electronic states with spin included.) Thus

$$\Delta_{\chi\vec{n}}(H) = H \otimes 1 + 1 \otimes H, \quad (7.23)$$

$$\Delta_{\chi\vec{n}}(\vec{n} \cdot \vec{J}) = \vec{n} \cdot \vec{J} \otimes 1 + 1 \otimes \vec{n} \cdot \vec{J}. \quad (7.24)$$

The coproduct for the remaining components of \vec{J} can be evaluated as follows. Let \vec{n}^a , ($a = 1, 2$), \vec{n} be an orthonormal positively oriented coordinate system so that $\vec{n}^1 \wedge \vec{n}^2 = \vec{n}$, and let

$$\vec{n}^{\pm} \cdot \vec{J} = (\vec{n}^1 \pm i\vec{n}^2) \cdot \vec{J}.$$

Then

$$[\vec{n} \cdot \vec{J}, \vec{n}^{(\pm)} \cdot \vec{J}] = \pm \vec{n}^{(\pm)} \cdot \vec{J}, \quad (7.25)$$

$$[\vec{n}^{(+)} \cdot \vec{J}, \vec{n}^{(-)} \cdot \vec{J}] = 2 \vec{n} \cdot \vec{J}. \quad (7.26)$$

From, this it follows that

$$\Delta_{\chi\vec{n}}(\vec{n}^{(\pm)} \cdot \vec{J}) = \vec{n}^{(\pm)} \cdot \vec{J} \otimes e^{\mp \frac{i}{2}\chi P_0} + e^{\pm \frac{i}{2}\chi P_0} \otimes \vec{n}^{(\pm)} \cdot \vec{J}. \quad (7.27)$$

7.2 The Electronic States of Be

The nucleus of Be has $Z = 4$. We put two of the four electrons of neutral Be in the $N = 1$ level. The remaining two are put in the $N = 2, l = 0$ level. The choice $l = 0$ for all these levels is deliberate as it greatly simplifies the calculations.

The equations Eq.(7.23), Eq.(7.24) show that energy and $\vec{n} \cdot \vec{J}$ are additive in the twist antisymmetrized levels $\frac{1-\tau_{\chi\vec{n}}}{2} (|N, l, \alpha\rangle_{\vec{n}} \otimes |N', l', \alpha'\rangle_{\vec{n}})$. We have

$$\Delta_{\chi\vec{n}}(H) \frac{1-\tau_{\chi\vec{n}}}{2} |N, 0, \alpha\rangle_{\vec{n}} \otimes |N', 0, \alpha'\rangle_{\vec{n}} = (E_N + E_{N'}) \frac{1-\tau_{\chi\vec{n}}}{2} |N, 0, \alpha\rangle_{\vec{n}} \otimes |N', 0, \alpha'\rangle_{\vec{n}}, \quad (7.28)$$

$$\Delta_{\chi\vec{n}}(\vec{n} \cdot \vec{J}) \frac{1-\tau_{\chi\vec{n}}}{2} |N, 0, \alpha\rangle_{\vec{n}} \otimes |N', 0, \alpha'\rangle_{\vec{n}} = \frac{1}{2}(\alpha + \alpha') \frac{1-\tau_{\chi\vec{n}}}{2} |N, 0, \alpha\rangle_{\vec{n}} \otimes |N', 0, \alpha'\rangle_{\vec{n}}. \quad (7.29)$$

As for $\Delta_{\chi\vec{n}}(\vec{n}^{(\pm)} \cdot \vec{J})$, we find,

$$\Delta_{\chi\vec{n}}(\vec{n}^{(+)} \cdot \vec{J}) \left\{ \begin{array}{l} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}}, \\ |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}} \end{array} \right\} = \left\{ \begin{array}{l} e^{\frac{i}{2}\chi E_N} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}}, \\ e^{-\frac{i}{2}\chi E_{N'}} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}} \end{array} \right\}; \quad (7.30)$$

$$\Delta_{\chi\vec{n}}(\vec{n}^{(+)} \cdot \vec{J}) \left\{ \begin{array}{l} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}}, \\ |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}} \end{array} \right\} = \left\{ \begin{array}{l} 0, \\ e^{-\frac{i}{2}\chi E_{N'}} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}} \\ + e^{\frac{i}{2}\chi E_N} |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}} \end{array} \right\}; \quad (7.31)$$

$$\Delta_{\chi\vec{n}}(\vec{n}^{(-)} \cdot \vec{J}) \left\{ \begin{array}{l} |N, 0, +1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}}, \\ |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, +1\rangle_{\vec{n}} \end{array} \right\} = \left\{ \begin{array}{l} e^{\frac{i}{2}\chi E_{N'}} |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}}, \\ e^{-\frac{i}{2}\chi E_N} |N, 0, -1\rangle_{\vec{n}} \otimes |N', 0, -1\rangle_{\vec{n}} \end{array} \right\}; \quad (7.32)$$

$$\Delta_{\chi\bar{n}}(\bar{n}^{(-)} \cdot \vec{J}) \left\{ \begin{array}{l} |N, 0, +1\rangle_{\bar{n}} \otimes |N', 0, +1\rangle_{\bar{n}}, \\ |N, 0, -1\rangle_{\bar{n}} \otimes |N', 0, -1\rangle_{\bar{n}} \end{array} \right\} = \left\{ \begin{array}{l} e^{\frac{i}{2}\chi E_{N'}} |N, 0, -1\rangle_{\bar{n}} \otimes |N', 0, +1\rangle_{\bar{n}} \\ + e^{-\frac{i}{2}\chi E_N} |N, 0, +1\rangle_{\bar{n}} \otimes |N', 0, -1\rangle_{\bar{n}}, \\ 0 \end{array} \right\}. \quad (7.33)$$

7.2.1 *The Two-Electron Ground State*

For $\chi = 0$, it is unique, being the (untwisted) spin-singlet state,

$$\begin{aligned} \frac{1-\tau_0}{\sqrt{2}} |1, 0, +1\rangle_{\bar{n}} \otimes |1, 0, -1\rangle_{\bar{n}} &= \frac{1}{\sqrt{2}} [|1, 0, +1\rangle_{\bar{n}} \otimes |1, 0, -1\rangle_{\bar{n}} \\ &- |1, 0, -1\rangle_{\bar{n}} \otimes |1, 0, +1\rangle_{\bar{n}}] \end{aligned} \quad (7.34)$$

with energy $2E_{10}$.

As χ is changed away from 0, thus vector is deformed to

$$\begin{aligned} |1\ 1\rangle_{\chi\bar{n}} &= \frac{1-\tau_{\chi\bar{n}}}{\sqrt{2}} |1, 0, +1\rangle_{\bar{n}} |1, 0, -1\rangle_{\bar{n}} \\ &= \frac{1}{\sqrt{2}} [|1, 0, +1\rangle_{\bar{n}} |1, 0, -1\rangle_{\bar{n}} - e^{i\chi E_1} |1, 0, -1\rangle_{\bar{n}} |1, 0, +1\rangle_{\bar{n}}]. \end{aligned}$$

Its energy still remains $2E_{10}$ in view of Eq.(7.28).

No new linearly independent state appears by continuity: if they had appeared, then as $\chi \rightarrow 0$, the ground state would not be unique. We can verify this assertion by calculating $\frac{1-\tau_{\chi\bar{n}}}{\sqrt{2}} |1, 0, \alpha\rangle_{\bar{n}} \otimes |1, 0, \alpha'\rangle_{\bar{n}}$ for any choice of α, α' and verifying that it is either proportional to Eq.(7.35) or zero.

The values of $\Delta_{\chi\bar{n}}(\bar{n} \cdot \vec{J})$ and $\Delta_{\chi\bar{n}}(\bar{n}^{(\pm)} \cdot \vec{J})$ on $|1\ 1\rangle_{\chi\bar{n}}$ are also zero from Eq.(7.29), Eq.(7.30) and Eq.(7.32). So it is a twisted spin-singlet with zero (twisted) value for total angular momentum.

7.2.2 *The Two-Electron Excited State*

The actual Pauli-forbidden transition we will calculate will use the excited state

$$\frac{1-\tau_{\chi\bar{n}}}{\sqrt{2}} [|2, 0, +1\rangle_{\bar{n}} |3, 0, +1\rangle_{\bar{n}}] \quad (7.35)$$

which is part of a (twisted!) spin triplet with orbital angular momentum 0 and energy $E_2 + E_3$.

For completeness, we here list all the spin triplet and singlet components of the states with energy $E_2 + E_3$.

The triplet vectors

$$\Delta_{\chi\bar{n}}(\bar{n} \cdot \vec{J}) = 1 : \frac{1}{\sqrt{2}} \left[|2, 0, +1\rangle_{\bar{n}} |3, 0, +1\rangle_{\bar{n}} - e^{\frac{i}{2}\chi(E_3-E_2)} |3, 0, +1\rangle_{\bar{n}} |2, 0, +1\rangle_{\bar{n}} \right]. \quad (7.36)$$

$$\Delta_{\chi\bar{n}}(\bar{n} \cdot \vec{J}) = 0 : \frac{1}{2} \left[\begin{array}{l} e^{i\chi \frac{E_3}{2}} |2, 0, -1\rangle_{\bar{n}} |3, 0, +1\rangle_{\bar{n}} - e^{-\frac{i}{2}\chi E_2} |3, 0, +1\rangle_{\bar{n}} |2, 0, -1\rangle_{\bar{n}} \\ + e^{-\frac{i}{2}\chi E_2} |2, 0, +1\rangle_{\bar{n}} |3, 0, -1\rangle_{\bar{n}} - e^{-\frac{i}{2}\chi E_3} |3, 0, -1\rangle_{\bar{n}} |2, 0, +1\rangle_{\bar{n}} \end{array} \right]. \quad (7.37)$$

$$\Delta_{\chi\vec{n}}(\vec{n} \cdot \vec{J}) = -1 : \frac{1}{\sqrt{2}} \left[|2, 0, -1\rangle_{\vec{n}} |3, 0, -1\rangle_{\vec{n}} - e^{-\frac{i}{2}\chi(E_3-E_2)} |3, 0, -1\rangle_{\vec{n}} |2, 0, -1\rangle_{\vec{n}} \right]. \quad (7.38)$$

The singlet vector

$$\begin{aligned} \Delta_{\chi\vec{n}}(\vec{n} \cdot \vec{J}) &= \Delta_{\chi\vec{n}}(\vec{n}^{(\pm)} \cdot \vec{J}) = 0 \\ &: \frac{1}{2} \left[e^{i\chi\frac{E_2}{2}} |2, 0, -1\rangle_{\vec{n}} |3, 0, +1\rangle_{\vec{n}} - e^{-\frac{i}{2}\chi E_3} |2, 0, +1\rangle_{\vec{n}} |3, 0, -1\rangle_{\vec{n}} \right. \\ &+ \left. e^{-\frac{i}{2}\chi E_3} |2, 0, +1\rangle_{\vec{n}} |3, 0, -1\rangle_{\vec{n}} - e^{\frac{i}{2}\chi E_2} |3, 0, -1\rangle_{\vec{n}} |2, 0, +1\rangle_{\vec{n}} \right]. \end{aligned} \quad (7.39)$$

7.3 The Non-Pauli Rate

This section contains the formula Eq.(7.54) for confrontation with experiments. The rest of this section is a derivation of this formula.

Spin Overlaps

The basic transition we focus on is from the triplet excited state Eq.(7.36) for twist $G_{\chi\vec{n}}$ to the ground state levels for twist $G_{\chi\vec{m}}$. That involves the calculation of the overlap ${}_{\vec{m}}\langle\alpha'|\alpha\rangle_{\vec{n}}$ which follows from Eq.(7.22):

$${}_{\vec{m}}\langle\alpha'|\alpha\rangle_{\vec{n}} = \left(g(\vec{m})^\dagger g(\vec{n}) \right)_{\alpha'\alpha}. \quad (7.40)$$

This expression depends on the choice of $g(\vec{n})$, $g(\vec{m})$. But in rates, we get its squared modulus. That depends only on $\vec{m} \cdot \vec{n}$:

$$|{}_{\vec{m}}\langle\alpha'|\alpha\rangle_{\vec{n}}|^2 = \frac{1}{2} \left[1 + (-1)^{\frac{(\alpha'-\alpha)}{2}} \vec{m} \cdot \vec{n} \right]. \quad (7.41)$$

Here is a simple proof of Eq.(7.41). The R.H.S is

$$g(\vec{m})_{\alpha'\rho}^\dagger g(\vec{n})_{\rho\alpha} g(\vec{n})_{\alpha\lambda}^\dagger g(\vec{m})_{\lambda\alpha'}$$

for fixed α, α' and summed ρ, λ . Consider $\alpha = \alpha' = 1$:

$$\begin{aligned} |{}_{\vec{m}}\langle+1|+1\rangle_{\vec{n}}|^2 &= Tr \left(g(\vec{n}) \frac{1+\tau_3}{2} g(\vec{n})^\dagger \right) \left(g(\vec{m}) \frac{1+\tau_3}{2} g(\vec{m})^\dagger \right) \\ &= \frac{1}{4} Tr [1 + \vec{n} \cdot \vec{\tau}] [1 + \vec{m} \cdot \vec{\tau}] \\ &= \frac{1}{2} [1 + \vec{m} \cdot \vec{n}] \end{aligned}$$

In a similar way we can establish Eq.(7.41) for any α, α' .

We can now see the root of the non-Pauli transition. consider the *twist symmetrized* ground state for twist along \vec{m} :

$$\frac{1+\tau_{\chi\vec{m}}}{\sqrt{2}} |1, 0, \alpha\rangle_{\vec{m}} |1, 0, \beta\rangle_{\vec{m}} = \frac{1}{\sqrt{2}} \left[|1, 0, \alpha\rangle_{\vec{m}} |1, 0, \beta\rangle_{\vec{m}} + e^{i\chi E_1 \frac{\alpha-\beta}{2}} |1, 0, \beta\rangle_{\vec{m}} |1, 0, \alpha\rangle_{\vec{m}} \right]. \quad (7.42)$$

It is part of the spin triplet which with the twist antisymmetrized singlet gives the four two-electron ground states.

The normalized radial wave function for principal quantum number N can be denoted by $|N\rangle$. It is independent of the twist direction. The tensor product $|N\rangle \otimes |M\rangle$ can then be written as $|N, M\rangle$.

Now a generic perturbation, call it V_0 , will have a non-zero radial matrix element $\langle 1\ 1|V_0|2\ 3\rangle$ where V_0 is regarded as spin-independent for illustration. Then the Pauli-forbidden amplitudes are roughly proportional to this factor multiplied by spin overlaps $\vec{m}\langle\alpha\ \beta|\frac{(1+\tau_{\chi\vec{m}})}{\sqrt{2}}|+1\ +1\rangle_{\vec{n}}$: the spin-statistics connection does not permit $\frac{(1+\tau_{\chi\vec{m}})}{\sqrt{2}}|\alpha\ \beta\rangle_{\vec{m}}$. But we will see that these overlaps are not zero. So there are Pauli-forbidden transitions.

For $\chi = 0$, let V_0 be a generic spin-independent perturbation of the two-electron Hamiltonian. We do not show its dependence on electron coordinates, but we can assume it to be symmetric in them as it preserves statistics:

$$[V_0, \tau_0] = 0. \quad (7.43)$$

For $\chi \neq 0$, we have to modify V_0 to $V_{\chi\vec{n}}$:

$$V_{\chi\vec{n}} = \frac{1}{2} [V_0 + \tau_{\chi\vec{n}}V_0\tau_{\chi\vec{n}}] \quad (7.44)$$

so that it preserves the twisted statistics. As V_0 is an external perturbation which causes transitions between levels, it can be time-dependent. The perturbation has additional time dependence as \vec{n} changes with time.

The perturbed two-electron Hamiltonian is

$$H' = \Delta_{\chi\vec{n}}(H) + V_{\chi\vec{n}}. \quad (7.45)$$

Let $\vec{\rho}(t)$ be a time-dependent unit vector which at $t = t_i$ is \vec{n} and at time $t = t_f$ is \vec{m} . To leading order in $V_{\chi\vec{n}}$, the transition matrix element from an initial state $|I\rangle$ of energy E_I at time t_i to an orthogonal final state $|F\rangle$ of energy E_F at time t_f is

$$-ie^{-i(t_f-t_i)E_f}\langle F|\int_{t_i}^{t_f} d\tau e^{i\tau H}V_{\chi\vec{\rho}(\tau)}e^{-i\tau H}|I\rangle.$$

For us

$$|I\rangle = \frac{1-\tau_{\chi\vec{n}}}{\sqrt{2}}|2, 0, +1\rangle_{\vec{n}} |3, 0, +1\rangle_{\vec{n}}, \quad (7.46)$$

with $E_I = E_2 + E_3$.

For $|F\rangle$, we choose a *Pauli-forbidden* ground state

$$\frac{1+\tau_{\chi\vec{m}}}{\sqrt{2}}|1, 0, \alpha\rangle_{\vec{m}} |1, 0, \alpha'\rangle_{\vec{m}}$$

(This vector is not normalized if $\alpha = \alpha'$. We will fix that problem later.)

From Eq.(7.44), we can see that $V_{\chi\vec{n}} = V_0 + O(\chi)$. The explicit calculations below show that the amplitude is $O(\chi)$ if $V_{\chi\vec{n}}$ is approximated by V_0 . So we approximate $V_{\chi\vec{n}}$ by V_0 in Eq.(7.44) neglecting terms of $O(\chi)$.

As V_0 is symmetric in electron coordinates, for the radial matrix element, $\langle 1\ 1|V_0|2, 3\rangle = \langle 1\ 1|V_0|3, 2\rangle$.

We now use this identity to simplify the probability for transition P_χ to any Pauli-forbidden ground state. That is obtained from modulus squared of the amplitude by summing over $|F\rangle$ after normalizing them. But the projector to the Pauli-forbidden ground states is

$$Q = |1, 1\rangle\langle 1, 1|_{\mathbb{I}_{\text{spin}}} - |1, 1\rangle_{\chi\vec{m}} \chi\vec{m} \langle 1, 1| \quad (7.47)$$

where \mathbb{I}_{spin} is the unit operator on spin space.

Thus the probability of interest is

$$P_\chi = \langle I | \left(\int_{t_i}^{t_f} d\tau e^{i\tau 2E_1} V_0(\tau) e^{-i\tau(E_2+E_3)} \right)^* Q \left(\int_{t_i}^{t_f} e^{i\tau 2E_1} V_0(\tau) e^{-i\tau(E_2+E_3)} \right) | I \rangle. \quad (7.48)$$

This simplifies to the following on using the symmetry of V_0 :

$$P_\chi = |\langle 1, 1 | \int_{t_i}^{t_f} d\tau e^{i\tau 2E_1} V_0(\tau) e^{-i\tau(E_2+E_3)} | 2, 3 \rangle|^2 \times P_{\text{SPIN}}^\chi \quad (7.49)$$

where

$$P_{\text{SPIN}}^\chi = \frac{1}{2} \left| \left(1 - e^{i\frac{\chi}{2}(E_3-E_2)} \right) \right|^2 \left[1 - \frac{1}{2} \left| \left(\vec{m} \langle + - | - e^{-i\chi E_1} \vec{m} \langle - + | \right) | + \rangle \vec{n} \right|^2 \right]. \quad (7.50)$$

As claimed, P_χ is $O(\chi^2)$.

P_{SPIN}^χ can be evaluated using Eq.(7.41). The result is

$$P_{\text{SPIN}}^\chi = 2 \sin^2\left(\frac{\chi}{4}\Delta E\right) \left[1 - \frac{1}{4}(1 - (\vec{m} \cdot \vec{n})^2)(1 - \cos(\chi E_1)) \right] \quad (7.51)$$

where $\Delta E = E_3 - E_2$.

Here since \vec{n} and \vec{m} vary, it is best to average over them using the rotationally invariant measure. We first average over \vec{m} by integrating over its polar and azimuthal angles θ_m, ϕ_m using the standard measure

$$\frac{d\omega_m}{4\pi}, \quad d\omega_m = d\cos\theta_m d\phi_m.$$

Then

$$\int \frac{d\omega_m}{4\pi} \mathbb{I} = 1, \quad \int \frac{d\omega_m}{4\pi} m_i = 0, \quad \int \frac{d\omega_m}{4\pi} m_i m_j = \frac{1}{3} \delta_{ij} \quad (7.52)$$

giving for the average $\langle P_\chi \rangle$ of P_χ ,

$$\langle P_\chi \rangle = \left\{ |\langle 1, 1 | \int_{t_i}^{t_f} e^{i\tau 2E_1} V_0(\tau) e^{-i\tau(E_2+E_3)} | 2, 3 \rangle|^2 \right\} \times \left\{ \frac{1}{3} (5 + \cos(\chi E_1)) \sin^2\left(\frac{\chi}{4}\Delta E\right) \right\}. \quad (7.53)$$

There is no need to average over \vec{n} as this is \vec{n} -independent.

The magnitude of the prefactor in braces is that of a typical probability for a Pauli-allowed process. Thus the branching ratio of a Pauli-forbidden to a Pauli-allowed process is

$$B_\chi = \frac{1}{3} (5 + \cos(\chi E_1)) \sin^2\left(\frac{\chi}{4}\Delta E\right), \quad \Delta E = E_3 - E_2. \quad (7.54)$$

It is independent of t_i, t_f . It is this expression we use to confront experiments as it is a ratio and may not be sensitive to the details of its derivation.

7.4 Experiments and Bounds on χ

The experiments searching for Pauli-forbidden transitions can be broadly classified into atomic and nuclear experiments. Here we discuss each experiment separately.

Some of the above experiments give only lifetimes for the forbidden processes. To obtain the branching ratios in such cases we multiply the given rate with the typical lifetimes for such processes. In the case of an atomic process, we use the number 10^{-16} seconds and for a nuclear process we use 10^{-23} seconds for typical lifetimes.

7.4.1 Bounds from The Borexino Experiment

The Borexino collaboration has used its counting test facility to obtain limits on the violation of the Pauli exclusion principle (PEP) using nuclear transitions in ^{12}C and ^{16}O nuclei. The method is to search for γ , n , p and/or α emitted in a non-Paulian transition of $1P$ shell nucleons to the filled $1S_{1/2}$ shell in nuclei. Various stringent bounds were obtained as a result.

We use the following result from the Borexino experiment [34]:

$$\tau(^{12}\text{C} \rightarrow ^{12}\tilde{\text{C}} + \gamma) \geq 2.1 \times 10^{27} \text{ years.} \quad (7.55)$$

In the above process, $^{12}\tilde{\text{C}}$ denotes an anomalous carbon nucleus with an extra nucleon in the filled K shell of ^{12}C . This corresponds to a branching ratio of the order of 10^{-58} . We take ΔE for this process to be of the order of 1MeV to get a bound on χ .

7.4.2 Bounds from The Kamiokande Detector

In this experiment searches were made for forbidden transitions in ^{16}O nuclei and they obtain a bound on the ratio of forbidden transitions to normal transitions. The bound for this transition is $< 2.3 \times 10^{-57}$ [35]. Again for this process ΔE is assumed to be of the order of 1MeV.

7.4.3 Bounds from The NEMO Experiment

Similar to nucleon transitions, experiments searching for Pauli-forbidden atomic transitions have also been performed. The NEMO collaboration [37] searches for anomalous $^{12}\hat{\text{C}}$ atoms which are those with 3 K -shell electrons. The method used is the γ ray activation analysis in a sample of boron where the impurity carbon has been removed radiochemically. The bound on the existence of such atoms is given by the ratio of abundances of $^{12}\hat{\text{C}}$ to ^{12}C : it is $< 2.5 \times 10^{-12}$. It corresponds to a limit on the lifetime with respect to violation of the Pauli principle by electrons in a carbon atom of $\tau \geq 2 \times 10^{21}$ years. We take ΔE for this process to be 272 eV to calculate a bound on χ .

The NEMO-2 collaboration has also performed nucleon transition experiments [37] and the limit obtained is

$$\tau(^{12}\text{C} \rightarrow ^{12}\tilde{\text{C}} + \gamma) \geq 4.2 \times 10^{24} \text{ years.} \quad (7.56)$$

This corresponds to a branching ratio of the order $< 10^{-55}$ if we assume ΔE for this process to be of the order of 1MeV.

Experiment	Type	Bound on χ (Length scales)	Bound on χ (Energy scales)
Borexino	Nuclear	$\lesssim 10^{-43}$ m	$\gtrsim 10^{24}$ TeV
Kamiokande	Nuclear	10^{-42} m	10^{23} TeV
NEMO	Atomic	10^{-12} m	10^5 eV
NEMO-2	Nuclear	10^{-41} m	10^{22} TeV
Maryland	Atomic	10^{-20} m	10 TeV
VIP	Atomic	10^{-21} m	100 TeV

Table 7.1: Bounds on the noncommutativity parameter χ

7.4.4 Bounds from Experiments at Maryland

Atomic transition experiments have been conducted by Ramberg and Snow in Maryland using copper (Cu) atoms. The idea here is to introduce new electrons into a copper strip and to look for the K X-rays that would be emitted if one of these electrons were to be captured by a Cu atom and cascade down to the $1S$ state despite the fact that the $1S$ level was already filled with two electrons. The probability for this to occur was found to be less than 1.76×10^{-26} [38]. This corresponds to a lifetime of $\tau > 8.36 \times 10^3$ years. We assume ΔE for this process to be of the order of 1.5KeV.

7.4.5 Bounds from The VIP Experiment

An improved version of the experiment at Maryland has been performed by the VIP collaboration [39]. They improved the limit obtained by Ramberg and Snow at Maryland by a factor of about 40. The limit on the probability of PEP violating interactions between external electrons and copper is found to be less than 4.5×10^{-28} . Here again we take ΔE to be of the order of 1.5KeV.

The bounds are summarized in Table (7.1).

7.5 Time Quantization

The algebra $B_{\chi\vec{n}}$ leads to time-quantization in units of χ and therefore [126, 127] energy nonconservation: it is conserved only mod $\frac{2\pi}{\chi}$. An effect of this sort was first discovered by Chaichian [128] for a cylindrical noncommutative spacetime. Quantum physics on such spacetime including scattering theory was later developed in [127].

Time quantization comes about as follows. From Eq.(7.2), one sees that x_0 generates rotations around \vec{n} and that $e^{i\frac{2\pi}{\chi}x_0}$, being 2π rotation, acts as identity on x_i . Being a time exponential, it also commutes with momentum operators. Thus it is in the center of the algebra generated by $B_{\chi\vec{n}}$ and by its momentum operators. Hence it is a multiple of the identity in an irreducible representation of the latter:

$$e^{i\frac{2\pi}{\chi}x_0} = e^{i\phi}\mathbb{I}, \quad (7.57)$$

$e^{i\phi}$ being, characteristic of the representation.

A consequence of Eq.(7.57) is that the spectrum $\text{spec } x_0$ of x_0 is quantized:

$$\text{Spec } x_0 = \chi \left(\mathbb{Z} + \frac{\phi}{2\pi} \right). \quad (7.58)$$

As explained in [127, 126], a quantum field ψ is defined only on the spectrum of time operator x_0 . Time translations are from one point of this spectrum to another, so that only the time translations

$$(e^{i\chi P_0})^N, \quad N \in \mathbb{Z}$$

exist on quantum fields.

But then P_0 and $P_0 + \frac{2\pi}{\chi}M$, $M \in \mathbb{Z}$ generate the same time translation. Due to this we can anticipate energy conservation only mod $\frac{2\pi}{\chi}$ in scattering processes. This anticipation is correct. In [126], scattering theory with time quantization has been developed and energy is found to be conserved only mod $\frac{2\pi}{\chi}$.

An interesting application of such time quantization is to extra-dimensional models. Thus for example if spacetime is $M^4 \times S^1$ where M^4 is our four-dimensional spacetime, and the time operator x_0 fails to commute with the $e^{i\phi}$ which generates the algebra of functions on S^1 ,

$$x_0 e^{i\phi} = e^{i\phi} x_0 + \chi e^{i\phi}, \quad (7.59)$$

then scattering theory on M^4 will conserve energy only mod χ . No further interaction is needed for this energy nonconservation to occur.

Such energy nonconservation can be tested by experiments. Unfortunately, we know of no recent experiment to test energy conservation.

Conclusions

In this dissertation we have studied physical systems on 4 different noncommutative spacetimes. Two of these are given by noncommutative matrix algebras on compact spaces called fuzzy spaces. The fuzzy spacetimes have only the space slices made noncommutative. The examples we have looked at are the fuzzy sphere S_F^2 and the fuzzy Higg's manifold which can be thought of as fuzzified versions of a deformed sphere (or correspondingly a nonlinear deformation of the algebra of S_F^2). The remaining two noncommutative spacetimes are the Moyal plane and the $\mathcal{B}_{\chi\bar{n}}$ plane which are given by noncommutative algebras on noncompact spacetimes. There is both space-space noncommutativity and time-space noncommutativity for the Moyal plane whereas the latter is absent in the $\mathcal{B}_{\chi\bar{n}}$ plane.

Spin systems were studied on S_F^2 showing novel low dimension phenomena. The problems explored were the construction of Dirac operators for an arbitrary spin j , the numerical computation of the spectra of these operators and the thermodynamical properties of these systems. The Dirac operators for an arbitrary spin j were constructed using the Ginsparg-Wilson (GW) algebra. Each of these operators come paired with an anticommuting chirality operator making these systems chiral. This construction holds even for the case where the spin j is any integer. The spin 1 Dirac and chirality operators were studied in detail. These operators were constructed both on S_F^2 and its continuum version S^2 . The general rules for obtaining the fuzzy Dirac and chirality operators were given for both integral and half-integral spins. The rules for taking the continuum limits of these operators were also shown.

It was seen that for a given spin j there exist $2j + 1$ pairs of Dirac and chirality operators. These operators were shown to be unitarily inequivalent in the fuzzy case by showing that they have different traces.

It was also found that on S^2 there exist several other Dirac operators for each spin j anticommuting with the same chirality operator. The construction of these operators involved finding polynomials in $\vec{\alpha} \cdot \hat{x}$ where $\vec{\alpha}$ is the spin j matrix representation of $SU(2)$. The general rules of this construction for an arbitrary spin j were shown. These operators were also classified according to whether they were unitarily equivalent to the given Dirac operator or not. The transformations between these operators form a group. The group in the case of the set of unitarily equivalent operators was found to be $[U(1)]^{2j+1}$ where the power denotes the number of times we have to take the direct product between $U(1)$. This analysis also helps prove the unitary inequivalence of the $2j + 1$ Dirac operators on S^2 for a given spin j . It should be noted that these features of lower dimensional field theories on compact spaces have no known higher dimensional analogs.

Among the three Dirac operators constructed on S_F^2 , one was found to be traceless. We numerically found the spectrum of this traceless operator. The computation of the spectrum is a difficult problem to do analytically due to the presence of $\vec{\Sigma} \cdot \hat{x}$ terms in the continuum expressions of the higher spin Dirac operators. However transferring this task to the computer does not make things easier due to the size of the huge matrices ($3 \times (2L + 1)^2$ for a given cutoff L) involved in the computations. The spectrum was found to possess scaling properties which helped us circumvent the computational problems involved in handling large matrices. This feature of the spectrum of the spin 1 Dirac operator helps us predict the spectrum for arbitrarily large values of the cutoff L . We also found striking fits for the spectra for different cutoffs L . The square of the spectrum of the spin 1 Dirac operator was found to have a spectrum very similar to the square of the spectrum of the spin $\frac{1}{2}$ operator. A similar scaling behavior was seen for the spectrum of the spin $\frac{3}{2}$ Dirac operator as well. Extensions of these results to higher spins were also remarked upon.

The other important feature of the spectrums was the order of energy values for increasing spin. It was found that the spectrum of the spin $\frac{1}{2}$ Dirac operator was found to have higher energy eigenvalues than that of the spin 1 Dirac operator which in turn had larger energy eigenvalues than that of the spin $\frac{3}{2}$ Dirac operator. This was understood by counting the zero modes in each case. The spin $\frac{1}{2}$ operator does not have zero modes whereas they exist for cases of spin 1 and spin $\frac{3}{2}$ Dirac operators. A general method for finding the lower bound on the number of zero modes for any spin j was also given.

This hierarchy in the energy eigenvalues of the Dirac operators for different spins was used to show interesting thermodynamical properties exhibited by these systems. The chiral nature of these systems made us choose fermionic statistics for computing the partition functions for these systems. These partition functions were then used to compute the mean energies, specific heats and entropies of these systems. The result of the study was that $\langle E_{\frac{1}{2}} \rangle > \langle E_1 \rangle$. The same holds for the specific heats and the entropies. This led us to conjecture an interesting result that the entropies of spin systems on S_F^2 , and by continuity also on S^2 , has an upper bound given by the entropy of the spin $\frac{1}{2}$ system. This implies that an increase in the number of degrees of freedom does not increase the entropy of these systems.

In the case of the spin 1 system we also found that the equation of state deviated from the ideal gas law. The deviation was attributed to the curvature of the underlying manifold on which the spin system lived.

The other fuzzy space we studied was the fuzzy Higg's manifold. This manifold was found to exhibit topology change where a deformed sphere split into two different deformed spheres after pinching themselves off from a conical singularity. This was studied by parametrizing the Higg's manifold and studying the change in topology by varying this parameter. It was seen that though this was classically true the representations of this noncommutative algebra did not show any peculiarities at the singularity showing the smoothening of the singularity at the quantum level. We then looked at the construction of star products on this nonlinear algebra using the coherent states on this algebra.

We then studied the problem of computing scattering amplitudes on the Moyal plane using the LSZ formalism. This was done non-perturbatively by two different methods. It was found that the noncommutative scattering amplitude was a phase, depending on the external momenta of the incoming particles, times the commutative scattering amplitude. This result was earlier shown to be true via the formalism of interaction representation perturbation theory. This thus proves the equivalence of the two formalisms at the on-shell level on the Moyal plane. This is a non-trivial result as the same does not hold for the off-shell Green's functions. Remarks were also

made on the on-shell renormalization of these theories being very similar to the corresponding commutative theories.

Finally we studied the Pauli-violating transitions on noncommutative spacetimes. To do this we introduced a new noncommutative algebra called the $\mathcal{B}_{\chi\vec{n}}$ plane. We studied the forbidden transitions with the beryllium atom as an example. The model we considered included another parameter \vec{n} apart from the noncommutative parameter χ (which is the analog of θ of the Moyal plane but with different dimensions). This vector \vec{n} was taken to be a dynamical quantity affected by the motions of system in the universe. Due to the really short times scales involved in these processes we averaged over this vector \vec{n} to obtain the final rates. These expressions were then compared with available experiments testing the Pauli principle. The best bound we obtained was $\chi \gtrsim 10^{24}$ TeV from the Borexino experiment. This suggests a new scale beyond the Planck scale.

Preliminary remarks were also made on modification of the potential for systems with twisted statistics. The modified potential introduces higher order corrections to the transition rates. We however did not compute those in this thesis. We studied further properties of the $\mathcal{B}_{\chi\vec{n}}$ plane. We found that time is quantized on this noncommutative space-time. Time was found to be quantized in units of the noncommutative parameter χ . This also leads to energy nonconserving processes on these spacetimes or more precisely energy is conserved only mod $\frac{2\pi}{\chi}$.

Appendix A

Derivation of the Star Product on the Fuzzy Higg's Manifold

We show the derivation of Eq.(1.110) in this appendix.

Consider the symbol of the product of two operators $\hat{\alpha}$ and $\hat{\beta}$ where both are functions of the generators of the Higgs algebra.

$$\begin{aligned}
\phi(\hat{\alpha}\hat{\beta}) &= \langle \zeta | e^{\alpha-X_-} e^{\alpha_0 Z} e^{\alpha+X_+} e^{\beta-X_-} e^{\beta_0 Z} e^{\beta-X_-} | \zeta \rangle \\
&= (1 + |\zeta|^2)^{-2j} \langle j, -j | e^{(\alpha_+ + \zeta^*)X_-} e^{\alpha_0 Z} e^{\alpha+X_+} e^{\beta-X_-} e^{\beta_0 Z} e^{(\beta_+ + \zeta)X_+} | j, -j \rangle \\
&= (1 + |\zeta|^2)^{-2j} \langle j, -j | e^{\alpha_0 Z} e^{-\alpha_0 Z} e^{(\alpha_+ + \zeta^*)X_-} e^{\alpha_0 Z} e^{\alpha+X_+} e^{\beta-X_-} e^{\beta_0 Z} e^{(\beta_+ + \zeta)X_+} e^{-\beta_0 Z} e^{\beta_0 Z} | j, -j \rangle \\
&= (1 + |\zeta|^2)^{-2j} e^{-j(\alpha_0 + \beta_0)} \langle j, -j | e^{(\alpha_+ + \zeta^*)e^{\alpha_0} X_-} e^{\alpha+X_+} e^{\beta-X_-} e^{(\beta_+ + \zeta)e^{\beta_0} X_+} | j, -j \rangle \quad (\text{A.1})
\end{aligned}$$

where we have used

$$e^{\alpha Z} X_+ e^{-\alpha Z} = e^{\alpha} X_+ \quad (\text{A.2})$$

and

$$e^{\alpha Z} X_- e^{-\alpha Z} = e^{-\alpha} X_-. \quad (\text{A.3})$$

Let us simplify this expression further

$$\begin{aligned}
\phi(\hat{\alpha}\hat{\beta}) &= (1 + |\zeta|^2)^{-2j} e^{-j(\alpha_0 + \beta_0)} \langle j, -j | e^{(\alpha_+ + \zeta^*)e^{\alpha_0} X_-} e^{\alpha+X_+} e^{\beta-X_-} \\
&\quad \times \left(1 + \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} X_+^i \right) | j, -j \rangle \\
&= (1 + |\zeta|^2)^{-2j} e^{-j(\alpha_0 + \beta_0)} \left[\phi'(\hat{\alpha}) e^{j\alpha_0} + \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \right. \\
&\quad \left. \times \prod_{l=0}^{i-1} K_{j, -j+l} \langle j, -j | e^{(\alpha_+ + \zeta^*)e^{\alpha_0} X_-} e^{\alpha+X_+} e^{\beta-X_-} | j, -j + i \rangle \right] \quad (\text{A.4})
\end{aligned}$$

where

$$\phi'(\hat{\alpha}) e^{j\alpha_0} = \langle j, -j | e^{(\alpha_+ + \zeta^*)e^{\alpha_0} X_-} e^{\alpha+X_+} | j, -j \rangle. \quad (\text{A.5})$$

The second term in the square bracket becomes

$$\sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \beta_-^i \prod_{l=0}^{i-1} K_{j,-j+l}^2 \langle j, -j | e^{(\alpha_- + \zeta^*)X_-} e^{\alpha_+ X_+} | j, -j \rangle \quad (\text{A.6})$$

which can be written as

$$\phi''(\hat{\beta}) e^{j\beta_0} \phi'(\hat{\alpha}) e^{j\alpha_0}. \quad (\text{A.7})$$

Thus we have

$$\phi(\hat{\alpha}\hat{\beta}) = \frac{(1 + |\zeta|^2)^{-2j}}{\langle \zeta | \zeta \rangle} \left[\phi'(\hat{\alpha}) e^{-j\beta_0} + \phi'(\hat{\alpha}) \phi''(\hat{\beta}) \right]. \quad (\text{A.8})$$

Let us simplify $\phi'(\hat{\alpha})$

$$\phi'(\hat{\alpha}) = e^{-j\alpha_0} \left[1 + \sum_{i=1}^{2j} \frac{\alpha_+^i (\alpha_- + \zeta^*)^i}{(i!)^2} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2 \right]. \quad (\text{A.9})$$

This can be written as

$$\phi'(\hat{\alpha}) = e^{-j\alpha_0} \left[1 + \sum_{i=1}^{2j} \frac{(\alpha_+ + \zeta - \zeta)(\alpha_- + \zeta^*)^i}{(i!)^2} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2 \right]. \quad (\text{A.10})$$

The reason for doing this will be clear in a few steps. We have

$$[(\alpha_+ + \zeta - \zeta)(\alpha_- + \zeta^*)]^i = (\alpha_- + \zeta^*)^i \left[(\alpha_+ + \zeta)^i + \sum_{k=0}^{i-1} (-1)^{i-k} \binom{i}{k} (\alpha_+ + \zeta)^k \zeta^{i-k} \right]. \quad (\text{A.11})$$

Substituting this back into the expression for $\phi'(\hat{\alpha})$ we get

$$\begin{aligned} \phi'(\hat{\alpha}) = e^{-j\alpha_0} & \left[1 + \sum_{i=1}^{2j} \frac{(\alpha_- + \zeta^*)^i (\alpha_+ + \zeta)^i}{(i!)^2} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2 \right. \\ & \left. + \sum_{i=1}^{2j} \sum_{k=0}^{i-1} \frac{(\alpha_- + \zeta^*)^i (-1)^{i-k} \binom{i}{k} (\alpha_+ + \zeta)^k \zeta^{i-k}}{(i!)^2} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2 \right]. \end{aligned} \quad (\text{A.12})$$

Write this as

$$\phi'(\hat{\alpha}) = \phi(\hat{\alpha}) + \chi(\hat{\alpha}) \quad (\text{A.13})$$

where

$$\phi(\hat{\alpha}) = e^{-j\alpha_0} \left[1 + \sum_{i=1}^{2j} \frac{(\alpha_+ + \zeta)^i (\alpha_- + \zeta^*)^i}{(i!)^2} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2 \right] \quad (\text{A.14})$$

and

$$\chi(\hat{\alpha}) = e^{-j\alpha_0} \sum_{i=1}^{2j} \sum_{k=0}^{i-1} (-1)^{i-k} \binom{i}{k} (\alpha_- + \zeta^*)^i (\alpha_+ + \zeta)^k \zeta^{i-k} e^{i\alpha_0} \prod_{l=0}^{i-1} K_{j,-j+l}^2. \quad (\text{A.15})$$

In a similar way

$$\phi''(\hat{\beta}) = \phi(\hat{\beta}) + \chi(\hat{\beta}). \quad (\text{A.16})$$

Apart from these terms we have an additional term which is got by evaluating the expression

$$\begin{aligned}
& \langle j, -j | e^{(\alpha_- + \zeta^*)e^{\alpha_0} X_-} e^{\alpha_+ X_+} \sum_{m=0}^{i-1} \frac{\beta_-^m}{m!} X_-^m | j, -j + i \rangle = \\
& \quad \langle j, -j | e^{(\alpha_- + \zeta^*)e^{\alpha_0} X_-} e^{\alpha_+ X_+} | j, -j + i \rangle \\
& \quad + \sum_{m=1}^{i-1} \frac{\beta_-^m}{m!} \langle j, -j | e^{(\alpha_- + \zeta^*)e^{\alpha_0} X_-} e^{\alpha_+ X_+} X_-^m | j, -j + i \rangle
\end{aligned} \tag{A.17}$$

Let us evaluate the additional terms. This will give a simplified version of Eq.(1.113). Consider

$$\begin{aligned}
G &= \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \prod_{l=0}^{i-1} K_{j, -j+l} \langle j, -j | e^{(\alpha_- + \zeta^*)e^{\alpha_0} X_-} e^{\alpha_+ X_+} | j, -j + i \rangle \\
&= \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{(i!)^2} (\alpha_- + \zeta^*)^i e^{i(\alpha_0 + \beta_0)} \prod_{l=0}^{i-1} K_{j, -j+l}^2 \\
&+ \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \prod_{l=0}^{i-1} K_{j, -j+l} \times \sum_{m=1}^{2j-i} \frac{\alpha_+^m (\alpha_- + \zeta^*)^{i+m}}{m! (i+m)!} e^{(i+m)\alpha_0} \\
&\times \prod_{p=0}^{m-1} K_{j, -j+i+p} \prod_{q=0}^{i+m-1} K_{j, -j+q} \\
&= {}_3F_0 \left(-A_+, -A_-, 2j; 0; (\beta_+ + \zeta)(\alpha_- + \zeta^*)e^{\alpha_0 + \beta_0} \right) - 1 \\
&+ \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} (\alpha_- + \zeta^*)^i e^{i\alpha_0} e^{i\beta_0} \prod_{l=0}^{i-1} K_{j, -j+l} \\
&\times \sum_{m=1}^{2j-i} \frac{\alpha_+^m (\alpha_- + \zeta^*)^m}{m! (i+m)!} e^{m\alpha_0} \prod_{p=0}^{m-1} K_{j, -j+i+p} \prod_{q=0}^{i+m-1} K_{j, -j+q} \\
&= {}_3F_0 \left(-A_+, -A_-, 2j; 0; (\beta_+ + \zeta)(\alpha_- + \zeta^*)e^{\alpha_0 + \beta_0} \right) - 1 \\
&+ \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} (\alpha_- + \zeta^*)^i e^{i\alpha_0} e^{i\beta_0} \prod_{l=0}^{i-1} K_{j, -j+l}^2 \\
&\times \sum_{m=1}^{2j-i} \frac{\alpha_+^m (\alpha_- + \zeta^*)^m}{m! (i+m)!} e^{m\alpha_0} \prod_{p=0}^{m-1} K_{j, -j+i+p}^2
\end{aligned} \tag{A.18}$$

Observe that

$$\sum_{m=1}^{2j-i} \frac{1}{(i+m)!} = \frac{1}{i!} \sum_{m=1}^{2j-i} \frac{1}{\prod_{q=1}^m (i+q)}. \tag{A.19}$$

Substituting this we have have

$$\begin{aligned}
G &= {}_3F_0\left(-A_+, -A_-, -2j; 0; (\beta_+ + \zeta)(\alpha_- + \zeta^*)e^{\alpha_0 + \beta_0}\right) - 1 \\
&+ \left[{}_3F_0\left(-A_+, -A_-, -2j; 0; (\beta_+ + \zeta)(\alpha_- + \zeta^*)e^{\alpha_0 + \beta_0}\right) - 1 \right] \left[{}_4F_1(C, D, E, F; i; \alpha_+(\alpha_- + \zeta^*)e^{\alpha_0}) - 1 \right] \\
&= \left[{}_3F_0\left(-A_+, -A_-, -2j; 0; (\beta_+ + \zeta)(\alpha_- + \zeta^*)e^{\alpha_0 + \beta_0}\right) - 1 \right] \\
&\times {}_4F_1(C, D, E, F; i; \alpha_+(\alpha_- + \zeta^*)e^{\alpha_0})
\end{aligned} \tag{A.20}$$

where C, D, E, F are the roots of $K_{j, -j+i+p}^2$. This is because $K_{j, -j+i+p}^2$ is a fourth order polynomial in p .

The other deformation term is

$$\begin{aligned}
&\sum_{m=1}^{i-1} \frac{\beta_-^m}{m!} \langle j, -j | e^{(\alpha_- + \zeta^*)e^{\alpha_0} X_-} e^{\alpha_+ X_+} X_-^m | j, -j + i \rangle = \\
&\sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \prod_{l=0}^{i-1} K_{j, -j+l} \sum_{m=1}^{i-1} \frac{\beta_-^m}{m!} \prod_{k=0}^{m-1} H_{j, -j+i-k} \\
&\times \frac{(\alpha_- + \zeta^*)^{i-m}}{(i-m)!} \prod_{q=0}^{i-m-1} H_{j, -j+i-m-q} e^{(i-m)\alpha_0} + \sum_{i=1}^{2j} \frac{(\beta_+ + \zeta)^i}{i!} e^{i\beta_0} \\
&\times \prod_{k=0}^{i-1} K_{j, -j+l} \sum_{m=1}^{i-1} \frac{\beta_-^m}{m!} \prod_{k=0}^{m-1} H_{j, -j+i-k} \sum_{p=1}^{2j-i+m} \frac{\alpha_+^p}{p!} \prod_{q=0}^{p-1} K_{j, -j+i-m+q} \\
&\times \frac{(\alpha_- \zeta^*)^{i-m+p}}{(i-m+p)!} e^{(i-m+p)\alpha_0} \prod_{s=0}^{i-m+p-1} H_{j, -j+i-m+p-s}
\end{aligned} \tag{A.21}$$

We do not simplify further. Easy ways to simplify this expression is by making approximations like assuming the value of C_2 is small compared to the value of C_1 . This expression is of $O(C_2^4)$.

The appearance of hypergeometric series in the expressions for star products on noncommutative spacetimes(for the case of the Moyal plane, the exponential is also a hypergeometric series of the form ${}_rF_s$ for specific values of r and s) suggests that the star products of more general deformations of S_F^2 will also be of a similar form but with different hypergeometric series with higher values of r and s .

Appendix B

Proof of the Existence of Traceless Dirac Operators for an Integer Spin j

In this appendix we show the existence of traceless Dirac operators in the integer spin case. We write down the exact operator for an arbitrary integer spin j .

Consider the addition of two angular momenta, one of which is the spin j angular momenta, $\vec{\Sigma}$ and the other is the left orbital angular momenta, \vec{L}^L . The angular momentum addition rules tell us that there are $2j + 1$ values possible for the total angular momentum $\vec{J} = \vec{\Sigma} + \vec{L}^L$. These values are

$$\begin{aligned}
 & L + j + 0 \\
 & L + j - 1 \\
 & L + j - 2 \\
 & \vdots \\
 & L + j - (j - 1) \\
 & L \\
 & L + j - (j + 1) \\
 & L + j - (j + 2) \\
 & \vdots \\
 & L + j - (2j - 1) \\
 & L - j
 \end{aligned} \tag{B.1}$$

The value of $\vec{\Sigma} \cdot \vec{L}^L$ on a space where $\vec{J} = L + j - k$ is given by

$$\vec{\Sigma} \cdot \vec{L}^L = \frac{k(k-1)}{2} - kj + L(j-k) ; k \in \{0, 1, \dots, 2j\}. \tag{B.2}$$

For $k = j$ we have

$$\vec{\Sigma} \cdot \vec{L}^L = -\frac{j(j+1)}{2}. \tag{B.3}$$

The projector to the space where $\vec{J} = L$ is given by

$$P_L^L = \frac{\prod_{k=0}^{j-1} \left[\vec{\Sigma} \cdot \vec{L}^L - \frac{k(k-1)}{2} + kj - L(j-k) \right] \prod_{k=j+1}^{2j} \left[\vec{\Sigma} \cdot \vec{L}^L - \frac{k(k-1)}{2} + kj - L(j-k) \right]}{\prod_{k=0}^{j-1} \left[-\frac{j(j+1)}{2} - \frac{k(k-1)}{2} + kj - L(j-k) \right] \prod_{k=j+1}^{2j} \left[-\frac{j(j+1)}{2} - \frac{k(k-1)}{2} + kj - L(j-k) \right]} \quad (\text{B.4})$$

The corresponding projector which acts on the right, that is, P_L^R is got by replacing $\vec{\Sigma} \cdot \vec{L}^L$ by $-\vec{\Sigma} \cdot \vec{L}^R$.

We can get the generators of the GW algebra from these left and right projectors as follows

$$\Gamma_L^L = 2P_L^L - 1 \quad (\text{B.5})$$

and

$$\Gamma_L^R = 2P_L^R - 1. \quad (\text{B.6})$$

It was proved in chapter 2 that the Dirac operator corresponding to this algebra is got from the following combination of the generators of this GW algebra :

$${}^F D_L^j = \frac{L}{2} (\Gamma_L^L - \Gamma_L^R) = L (P_L^L - P_L^R). \quad (\text{B.7})$$

Let us simplify this expression.

Define

$$f(k) := -\frac{k(k-1)}{2} + kj - L(j-k) \quad (\text{B.8})$$

then the numerator of P_L^L is given by

$$\text{Nr of } P_L^L = \sum_{m=0}^{2j} \left(\prod_{i=1}^m \sum_{k_i=0}^{2j} f(k_i) \right) (\vec{\Sigma} \cdot \vec{L}^L)^{2j-m} \quad (\text{B.9})$$

where the restricted sum $\tilde{\sum}$ is defined for $k_i \neq j \forall i \in \{1, \dots, m\}$ and $k_i \neq k_n$ if $i = n \forall i, n \in \{1, \dots, m\}$.

The expression for the right projector is given by

$$P_L^R = \frac{\prod_{k=0}^{j-1} \left[\vec{\Sigma} \cdot \vec{L}^R + \frac{k(k-1)}{2} - kj + L(j-k) \right] \prod_{k=j+1}^{2j} \left[\vec{\Sigma} \cdot \vec{L}^R + \frac{k(k-1)}{2} - kj + L(j-k) \right]}{\prod_{k=0}^{j-1} \left[-\frac{j(j+1)}{2} + \frac{k(k-1)}{2} - kj + L(j-k) \right] \prod_{k=j+1}^{2j} \left[-\frac{j(j+1)}{2} + \frac{k(k-1)}{2} - kj + L(j-k) \right]} \quad (\text{B.10})$$

The numerator of P_L^R is given by

$$\text{Nr of } P_L^R = \sum_{m=0}^{2j} \left(\prod_{i=1}^m (-1)^m \sum_{k_i=0}^{2j} f(k_i) \right) (\vec{\Sigma} \cdot \vec{L}^R)^{2j-m} \quad (\text{B.11})$$

where the definition of the restricted sum is the same as before.

To study the trace of the Dirac operator it is sufficient to look at the numerator of the two projectors as the denominators are just constants and are the same for both the projectors.

Thus the numerator of the Dirac operator obtained from these projectors

$$\text{Nr. of } {}^F D_L^j = L \sum_{m=0}^{2j} \left[\left(\prod_{i=1}^m \sum_{k_i=0}^{2j} f(k_i) \right) (\vec{\Sigma} \cdot \vec{L}^L)^{2j-m} - \left(\prod_{i=1}^m (-1)^m \sum_{k_i=0}^{2j} f(k_i) \right) (\vec{\Sigma} \cdot \vec{L}^R)^{2j-m} \right]. \quad (\text{B.12})$$

Splitting the odd and even powers of $\vec{\Sigma} \cdot \vec{L}^L$ and $\vec{\Sigma} \cdot \vec{L}^R$ we have

$$\begin{aligned} \text{Nr. of } {}^F D_L^j &= L \left[\sum_{r=0}^j \left(\prod_{i=1}^{2r} \sum_{k_i=0}^{2j} f(k_i) \right) \left\{ (\vec{\Sigma} \cdot \vec{L}^L)^{2j-2r} - (\vec{\Sigma} \cdot \vec{L}^R)^{2j-2r} \right\} \right. \\ &\quad \left. + \sum_{r=1}^j \left(\prod_{i=1}^{2r-1} \sum_{k_i=0}^{2j} f(k_i) \right) \left\{ (\vec{\Sigma} \cdot \vec{L}^L)^{2j-2r+1} + (\vec{\Sigma} \cdot \vec{L}^R)^{2j-2r+1} \right\} \right]. \end{aligned} \quad (\text{B.13})$$

Using the fact that $\text{tr}(\vec{\Sigma} \cdot \vec{L}^L)^n = (-1)^n \text{tr}(\vec{\Sigma} \cdot \vec{L}^R)^n$ for $n > 1$, it follows that the trace of the numerator of ${}^F D_L^j$ is 0.

For the sake of completeness we write down the full expression of ${}^F D_L^j$:

$$\begin{aligned} {}^F D_L^j &= \frac{L}{\sum_{m=0}^{2j} \left(\prod_{i=1}^m \sum_{k_i=0}^{2j} f(k_i) \right) \left(-\frac{j(j+1)}{2} \right)^{2j-m}} \\ &\quad \times \left[\sum_{r=0}^j g_r^{\text{even}}(L) \left\{ (\vec{\Sigma} \cdot \vec{L}^L)^{2j-2r} - (\vec{\Sigma} \cdot \vec{L}^R)^{2j-2r} \right\} \right. \\ &\quad \left. + \sum_{r=1}^j g_r^{\text{odd}}(L) \left\{ (\vec{\Sigma} \cdot \vec{L}^L)^{2j-2r+1} + (\vec{\Sigma} \cdot \vec{L}^R)^{2j-2r+1} \right\} \right] \end{aligned} \quad (\text{B.14})$$

where

$$g_r^{\text{even}}(L) = \left\{ \begin{array}{l} 1, \quad r = 0 \\ \prod_{i=1}^{2r} \left(\sum_{k_i=0}^{2j} f(k_i) \right), \quad r \in \{1, \dots, j\} \end{array} \right\}, \quad (\text{B.15})$$

$$g_r^{\text{odd}}(L) = \prod_{i=1}^{2r-1} \sum_{k_i=0}^{2j} f(k_i), \quad r \in \{1, \dots, j\} \quad (\text{B.16})$$

and

$$f(k_i) = -\frac{k_i(k_i - 1)}{2} + k_i j - L(j - k_i). \quad (\text{B.17})$$

The restricted sum operates as defined before.

Using the rules for taking continuum limits as explained in chapter 2, we can use the above expression to find the Dirac operator on S^2 for any integer spin j .

Appendix C

Review of Hopf Algebras

This appendix gives an overview of Hopf algebras in general. The ideas are illustrated through the group algebra which is an example of a Hopf algebra.

A $*$ -Hopf algebra H is an associative, $*$ -algebra with unit element e such that:

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) \text{ for } \alpha, \beta, \gamma \in H \quad (\text{C.1})$$

$$* : \alpha \in H \rightarrow \alpha^* \in H, \quad (\text{C.2})$$

$$(\alpha\beta)^* = \beta^* \alpha^*, \alpha, \beta \in H, \quad (\text{C.3})$$

$$(\lambda\alpha)^* = \bar{\lambda}\alpha^*, \lambda \in \mathbb{C}, \alpha \in H. \quad (\text{C.4})$$

$$e^* = e, \alpha e = e\alpha = \alpha, \forall \alpha \in H. \quad (\text{C.5})$$

It is equipped with a coproduct or comultiplication Δ , a counit ϵ and an antipode S with the following properties:

1) The coproduct

$$\Delta : H \rightarrow H \otimes H \quad (\text{C.6})$$

is a $*$ -homomorphism.

2) The counit

$$\epsilon : H \rightarrow \mathbb{C} \quad (\text{C.7})$$

is a $*$ -homomorphism subject to the conditions

$$(id \otimes \epsilon) \Delta = id = (\epsilon \otimes id) \Delta. \quad (\text{C.8})$$

3) The antipode

$$S : H \rightarrow H \quad (\text{C.9})$$

is a $*$ -antihomomorphism,

$$S(\alpha\beta) = S(\beta)S(\alpha) \quad (\text{C.10})$$

subject to the following conditions. Define

$$m_r(\xi \otimes \eta \otimes \rho) = \eta\xi \otimes \rho, \quad (\text{C.11})$$

$$m'_r(\xi \otimes \eta \otimes \rho) = \rho \otimes \eta\xi. \quad (\text{C.12})$$

Then

$$m_r[(S \otimes id \otimes id)(id \otimes \Delta)\Delta(\alpha)] = e \otimes \alpha, \quad (\text{C.13})$$

$$m'_r[(id \otimes S \otimes id)(id \otimes \Delta)\Delta(\alpha)] = \alpha \otimes e, \quad \alpha \in H. \quad (\text{C.14})$$

4) It is convenient to write multiplication in terms of the multiplication map m :

$$m(\alpha \otimes \beta) = \alpha\beta. \quad (\text{C.15})$$

Then we can also define $m_{13} \otimes m_{24}$:

$$(m_{13} \otimes m_{24})(\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \alpha_4) = \alpha_1\alpha_3 \otimes \alpha_2\alpha_4. \quad (\text{C.16})$$

This operation extends by linearity to all of $H \otimes H \otimes H \otimes H$. Then we require that

$$\Delta m(\alpha \otimes \beta) = (m_{13} \otimes m_{24})(\Delta \otimes \Delta)(\alpha \otimes \beta). \quad (\text{C.17})$$

This is a compatibility condition between m and Δ .

The statement that Δ is a $*$ -homomorphism is complete only if a $*$ -operation is defined on $H \otimes H$. The following two choices are possible:

$$(\alpha \otimes \beta)^* = \alpha^* \otimes \beta^*, \quad (\text{C.18})$$

$$(\alpha \otimes \beta)^* = \beta^* \otimes \alpha^*. \quad (\text{C.19})$$

The coassociativity of the coproduct is given by the requirement that

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta. \quad (\text{C.20})$$

The lack of coassociativity leads to quasi-Hopf structures.

A *bi-algebra* \mathcal{B} has less structure than a Hopf algebra. It is an algebra as it has the multiplication map and it is a coalgebra as it has the coproduct which is a homomorphism from \mathcal{B} to $\mathcal{B} \otimes \mathcal{B}$.

m and Δ fulfill the compatibility condition Eq.(C.17). The bialgebra does not have the antipode.

There are several known examples of Hopf algebras with applications in physics. One such known example is $U_q(SL(2, \mathbb{C}))$. The Poincaré algebra with the twisted coproduct is another such example. It is a non-trivial example as the twisted coproduct of the Poincaré group element involves momenta which cannot be expressed in terms of group elements.

C.0.1 The Group Algebra $\mathbb{C}G$ of a Group G

The group algebra of a group G consists of the linear combinations

$$\int_G d\mu(g)\alpha(g)g, \quad d\mu(g) = \text{Haar measure on } G \quad (\text{C.21})$$

of elements g of G , α being any smooth \mathbb{C} -valued smooth function on G . The algebra product is induced from the group product:

$$\int_G d\mu(g)\alpha(g)g \int_G d\mu(g')\beta(g')g' := \int_G d\mu(g) \int_G d\mu(g')\alpha(g)\beta(g')(gg'). \quad (\text{C.22})$$

We will henceforth omit the symbol G under the integrals.

The right hand side of the above equation is

$$\int d\mu(s) (\alpha *_C \beta)(s) s \quad (\text{C.23})$$

where $*_C$ is the convolution product:

$$(\alpha *_C \beta)(s) = \int d\mu(g) \alpha(g) \beta(g^{-1}s). \quad (\text{C.24})$$

The convolution algebra consists of smooth functions α on G with $*_C$ as their product. Under the map

$$\int d\mu(g) \alpha(g) g \rightarrow \alpha, \quad (\text{C.25})$$

the product of two elements of the group algebra goes over to $\alpha *_C \beta$ so that the group algebra and the convolution algebra are isomorphic. We call either as G^* or $\mathbb{C}G$.

The group algebra $\mathbb{C}G$ has several important properties which are significant for physics. The group algebra can be made into a Hopf algebra by defining the coproduct Δ , the counit ϵ and the antipode S as follows:

$$\Delta(g) = g \otimes g, \quad (\text{C.26})$$

$$\epsilon(g) = 1 \in \mathbb{C}, \quad (\text{C.27})$$

$$S(g) = g^{-1}. \quad (\text{C.28})$$

Here ϵ is the trivial one-dimensional representation of G and S maps g to its inverse. Δ and S satisfy all the consistency conditions required of Hopf algebras. These can be easily verified.

Let us see how the above notions extend to the group algebra $\mathbb{C}G$.

The counit extends to $\mathbb{C}G$ as follows:

$$\epsilon : \int d\mu(g) \alpha(g) g \rightarrow \int d\mu(g) \alpha(g). \quad (\text{C.29})$$

This is a one-dimensional representation.

In quantum theory we need ϵ to define state vectors such as the vacuum and operators invariant by G . Under $\mathbb{C}G$, they transform by the representation ϵ .

The antipode map S extends by linearity to an antihomomorphism from $\mathbb{C}G$ to $\mathbb{C}G$ as follows:

$$S : \hat{\alpha} = \int d\mu(g) \alpha(g) g \rightarrow S(\hat{\alpha}) = \int d\mu(g) \alpha(g) S(g), \quad (\text{C.30})$$

$$S(\hat{\alpha}\hat{\beta}) = S(\hat{\beta})S(\hat{\alpha}). \quad (\text{C.31})$$

The compatibility condition between the multiplication map m and Δ is fulfilled for $\mathbb{C}G$ by linearity as follows:

$$m \left(\int d\mu(g) \alpha(g) g \otimes \int d\mu(h) \beta(h) h \right) = \int d\mu(g) d\mu(h) \alpha(g) \beta(h) gh. \quad (\text{C.32})$$

Then

$$\Delta \circ m(a \otimes b) = (m_{13} \otimes m_{24})(\Delta \otimes \Delta)(a \otimes b). \quad (\text{C.33})$$

Here

$$(m_{13} \otimes m_{24})(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = a_1 a_3 \otimes a_2 a_4. \quad (\text{C.34})$$

Let us check this for $g, h \in G$,

$$\text{Left Hand Side} = \Delta(gh) = (gh) \otimes (gh).$$

$$\text{Right Hand Side} = m_{13} \otimes m_{24}(g \otimes g \otimes h \otimes h) = (gh) \otimes (gh) = \text{Left Hand Side}.$$

The coproduct also extends linearly to $\mathbb{C}G$ in a trivial way as seen in the compatibility conditions above.

In quantum theory, we are generally interested in unitary representations U of G , which means that

$$U(g^{-1}) = U(G^*),$$

* denoting the adjoint operation.

We want to express this equation in terms of a *-operation from $\mathbb{C}G$ to $\mathbb{C}G$. It is to be an anti-linear anti-homomorphism just as the adjoint operation: $(\lambda\hat{\alpha})^* = \bar{\lambda}\hat{\alpha}^*$, $(\hat{\alpha}\hat{\beta})^* = \hat{\beta}^*\hat{\alpha}^*$, $\lambda \in \mathbb{C}$, $\hat{\alpha}, \hat{\beta}, \hat{\alpha}^*, \hat{\beta}^* \in \mathbb{C}G$.

Such a * exists for $\mathbb{C}G$:

$$\left(\int d\mu(g)\alpha(g)g \right)^* = \int d\mu(g)\bar{\alpha}(g)g^{-1}. \quad (\text{C.35})$$

Note that S and * are different operations since for a given complex α ,

$$S \left(\int d\mu(g)\alpha(g)g \right) = \int d\mu(g)\alpha(g)g^{-1} \neq \int d\mu(g)\bar{\alpha}(g)g^{-1}. \quad (\text{C.36})$$

The unitarity condition on the representation U now translates to the condition

$$U(\hat{\alpha}^*) = U(\hat{\alpha})^*. \quad (\text{C.37})$$

U is thus a *-representation of $\mathbb{C}G$.

Appendix D

Braid Group and Yang-Baxter Equations

In this appendix we give a brief review of the braid group and the Yang-Baxter equations.

D.0.2 Identical Particles and Statistics

Consider a particle in quantum theory with its associated space V of state vectors. Let a symmetry group G act on V by a representation ρ .

The associated two-particle vector space is based on $V \otimes V$. For the conventional choice of the coproduct Δ ,

$$\Delta(g) = g \otimes g, g \in G \subset \mathbb{C}G,$$

the symmetry group G acts on $\xi \otimes \eta \in V \otimes V$ according to

$$\xi \otimes \eta \rightarrow (\rho \otimes \rho)\Delta(g)\xi \otimes \eta = \rho(g)\xi \otimes \rho(g)\eta.$$

Let τ be the flip operator:

$$\tau(\xi \otimes \eta) = \eta \otimes \xi. \tag{D.1}$$

It is to be a linear operator so that

$$\tau \sum_i \xi_i \otimes \eta_i = \sum_i \eta_i \otimes \xi_i. \tag{D.2}$$

Note that τ is not an element of $(\rho \otimes \rho)\Delta(\mathbb{C}G)$:

$$\tau \notin \{(\rho \otimes \rho)\Delta(g^*) | g^* \in \mathbb{C}G\} \equiv (\rho \otimes \rho)\Delta(\mathbb{C}G). \tag{D.3}$$

Now τ commutes with $(\rho \otimes \rho)\Delta(g)$:

$$(\rho \otimes \rho)\Delta(g)\xi \otimes \eta = \rho(g)\eta \otimes \rho(g)\xi = \tau(\rho \otimes \rho)\Delta(g)\xi \otimes \eta. \tag{D.4}$$

Hence also

$$\tau(\rho \otimes \rho)\Delta(\mathbb{C}G) = (\rho \otimes \rho)\Delta(\mathbb{C}G)\tau. \tag{D.5}$$

Further

$$\tau^2 = \mathbf{1} \otimes \mathbf{1} \quad (\text{D.6})$$

so that τ generates the permutation group S_2 .

Now since τ commutes with the action of $\mathbb{C}G$ on $V \otimes V$, the eigenspaces of τ for eigenvalues ± 1 are invariant by the action of $\mathbb{C}G$. These eigenspaces are obtained by symmetrization and anti-symmetrization:

$$V \otimes_S V = \frac{1+\tau}{2} V \otimes V = \left\{ \frac{1+\tau}{2} \xi \otimes \eta = \frac{1}{2} (\xi \otimes \eta + \eta \otimes \xi), \xi, \eta \in V \right\} \quad (\text{D.7})$$

$$V \otimes_A V = \frac{1-\tau}{2} V \otimes V = \left\{ \frac{1-\tau}{2} \xi \otimes \eta = \frac{1}{2} (\xi \otimes \eta - \eta \otimes \xi) \right\}. \quad (\text{D.8})$$

$V \otimes_S V$ describes bosons and $V \otimes_A V$ describes fermions.

We can easily extend this argument to N -particle states. Thus on $V \otimes V \otimes \cdots \otimes V$, we can define transposition operators

$$\tau_{i,i+1} = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1) \text{ factors}} \otimes \tau \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(N-i-1) \text{ factors}}. \quad (\text{D.9})$$

They generate the full permutation group and commute with the action of $\mathbb{C}G$. Each subspace of $V \otimes V \otimes \cdots \otimes V$ transforming by an irreducible representation of S_N is invariant under the action of $\mathbb{C}G$. They define particles with definite statistics. Bosons and fermions are obtained by the representations where $\tau_{i,i+1} \rightarrow \pm \mathbf{1}$ respectively. The corresponding vector spaces are the symmetrized and anti-symmetrized tensor products $V \otimes_{S,A} V \otimes_{S,A} \cdots \otimes_{S,A} V \equiv V^{\otimes_S}, V^{\otimes_A}$.

In quantum theory, there is the further assumption that all observables commute with $\tau_{i,i+1}$. Hence the above symmetrized and anti-symmetrized subspaces are invariant under the full observable algebra.

We can proceed as follows to generalize these considerations to any coproduct. The coproduct can be written as a series:

$$\Delta(\eta) = \sum_{\alpha} \eta_{\alpha}^{(1)} \otimes \eta_{\alpha}^{(2)} \equiv \eta_{\alpha}^{(1)} \otimes \eta_{\alpha}^{(2)} \equiv \eta^{(1)} \otimes \eta^{(2)}. \quad (\text{D.10})$$

Such a notation is called the Sweedler notation. Then to every coproduct Δ , there is another coproduct Δ^{op} ('op' stands for opposite) given by:

$$\Delta^{\text{op}}(\eta) = \eta_{\alpha}^{(2)} \otimes \eta_{\alpha}^{(1)} \equiv \eta^{(2)} \otimes \eta^{(1)}. \quad (\text{D.11})$$

Suppose that Δ and Δ^{op} are equivalent in the following sense: There exists an R -matrix $R \in H \otimes H$ such that

- i) R is invertible,
- ii) $\Delta^{\text{op}}(\eta)R = R\Delta(\eta)$,

and fulfills also certain relations called the Yang-Baxter relations then the Hopf algebra is said to be quasi-triangular.

For the simple case of $\Delta(g) = g \otimes g$, $R = 1 \otimes 1$. If i) is true, and ρ is the representation of H , $(\rho \otimes \rho)\Delta(\alpha)$ commutes with $\tau(\rho \otimes \rho)R$ as we now show. Therefore at least for two identical particles, we can use τR in place of τ to define statistics. We have

$$(\rho \otimes \rho)\Delta(\eta)\tau[(\rho \otimes \rho)R]v \otimes w = (\rho \otimes \rho)\Delta(\eta)\tau \left[\rho(r_{\alpha}^{(1)})v \right] \otimes \left[\rho(r_{\alpha}^{(2)})w \right] = \rho \left(\eta_{\beta}^{(1)} r_{\alpha}^{(2)} \right) w \otimes \rho \left(\eta_{\beta}^{(2)} r_{\alpha}^{(1)} \right) v \quad (\text{D.12})$$

while

$$\begin{aligned}
\tau[(\rho \otimes \rho)](\rho \otimes \rho)\Delta(\eta)v \otimes w &= \tau(\rho \otimes \rho)\Delta^{\text{op}}(\eta)\rho\left(r_\alpha^{(1)}\right)v \otimes \rho\left(r_\alpha^{(2)}\right)w & (\text{D.13}) \\
&= \tau\rho\left(\eta_\beta^{(2)}r_\alpha^{(1)}\right)v \otimes \rho\left(\eta_\beta^{(1)}r_\alpha^{(2)}\right)w \\
&= \rho\left(\eta_\beta^{(1)}r_\alpha^{(2)}\right)w \otimes \rho\left(\eta_\beta^{(2)}r_\alpha^{(1)}\right)v
\end{aligned}$$

which is the same as the previous equation thus proving the result. Here we have used $R = r_\alpha^{(1)} \otimes r_\alpha^{(2)}$.

We can next decompose $V \otimes V$ into irreducible subspaces of τR . The observables are required to commute with τR . These irreducible subspaces describe particles of definite statistics.

We can generalize τR to N -particle sectors. Thus let

$$R_{i,i+1} = \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{(i-1) \text{ factors}} \otimes R \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{(N-i-1) \text{ factors}}. \quad (\text{D.14})$$

Then

$$\mathcal{R}_{i,i+1} = \tau_{i,i+1}(\rho \otimes \rho \otimes \cdots \otimes \rho)R_{i,i+1} \quad (\text{D.15})$$

generalizes $\tau_{i,i+1}$ of $\mathbb{C}G$ on $V \otimes V \otimes \cdots \otimes V$ (N -factors). the group it generates replaces S_N . We will see that it is the braid group \mathcal{B}_N .

As we really have different $\tau_{i,i+1}$ for different N , we sometimes call $\tau_{i,i+1}(\rho \otimes \rho \otimes \cdots \otimes \rho)R_{i,i+1}$ as $\rho \otimes \rho \otimes \cdots \otimes \rho[\tau_{i,i+1}R_{i,i+1}]$:

$$\tau_{i,i+1}(\rho \otimes \rho \otimes \cdots \otimes \rho)R_{i,i+1} \equiv \rho \otimes \rho \otimes \cdots \otimes \rho[\tau_{i,i+1}R_{i,i+1}]. \quad (\text{D.16})$$

Also, $\mathcal{R}_{i,i+1}$ depends on N , but for simplicity we do not show this dependence.

The square of τR (or $\tau_{i,i+1}R_{i,i+1}$) is not necessarily identity. There could be representations where its eigenvalues are phases leading to anyons.

D.0.3 The Braid Group and Yang-Baxter Equations

As we saw in the previous section the braid group \mathcal{B}_N generalizes the statistics group S_N when the symmetry algebra is generalized to a quasi-triangular Hopf algebra. For the Hopf algebra to be quasi-triangular, it is necessary that R fulfills an additional relation called the Yang-Baxter relation. it can be derived by requiring that $\mathcal{R}_{i,i+1}$ generate \mathcal{B}_N .

Consider

$$u \otimes v \otimes w \in V \otimes V \otimes V.$$

We can shuffle the left had side into its anti-cyclic form in two ways:

First apply \mathcal{R}_{12} , then \mathcal{R}_{23} and finally \mathcal{R}_{12} again. On applying \mathcal{R}_{12} , we get

$$v_\alpha \otimes u_\alpha \otimes w,$$

repeated indices being summed. Next under \mathcal{R}_{23} , this becomes

$$v_\alpha \otimes w_\beta \otimes u_{\alpha,\beta}.$$

Finally on applying \mathcal{R}_{12} once more, we find

$$w_{\beta,\alpha} \otimes v_{\alpha,\beta} \otimes u_{\alpha,\beta}.$$

The operator $\mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}$ also shuffles the left-hand side to the anti-cyclic form:

$$\begin{aligned}\mathcal{R}_{23} &: u \otimes v \otimes w \rightarrow u \otimes w'_\alpha \otimes v'_\alpha \\ \mathcal{R}_{12}\mathcal{R}_{23} &: u \otimes v \otimes w \rightarrow w'_{\alpha,\beta} \otimes u'_\beta \otimes v'_\alpha \\ \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23} &: u \otimes v \otimes w \rightarrow w'_{\alpha,\beta} \otimes v'_{\alpha,\beta} \otimes u'_{\alpha,\beta}.\end{aligned}$$

On requiring that the final result in either case is the same, we get

$$\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}. \quad (\text{D.17})$$

In the N -particle sectors, this generalizes to

$$\mathcal{R}_{i,i+1}\mathcal{R}_{i+1,i+2}\mathcal{R}_{i,i+1} = \mathcal{R}_{i+1,i+2}\mathcal{R}_{i,i+1}\mathcal{R}_{i+1,i+2}. \quad (\text{D.18})$$

The group generated by $\mathcal{R}_{i,i+1}$ with these relations is known as the braid group \mathcal{B}_N . If the additional relation

$$\mathcal{R}_{i,i+1}^2 = 1 \quad (\text{D.19})$$

is imposed, it becomes the permutation group S_N .

The Yang-Baxter equation is a relation in terms of $R_{i,i+1}$. This can be got as follows.

Since $\tau_{i,i+1}$ generate S_N , they too fulfill

$$\tau_{i,i+1}\tau_{i+1,i+2}\tau_{i,i+1} = \tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2} \quad (\text{D.20})$$

in addition to being idempotent.

Further

$$\tau_{ij} = \tau_{ji} = \tau_{ij}^{-1}, \quad i \neq j, \quad (\text{D.21})$$

$$\tau_{ij}\rho(R_{i\beta}) = \rho(R_{j\beta})\tau_{ij}, \quad \text{if } \beta \notin \{i, j\}, \quad (\text{D.22})$$

$$\tau_{ij}\rho(R_{\alpha i}) = \rho(R_{\alpha j})\tau_{ij}, \quad \text{if } \alpha \notin \{i, j\}, \quad (\text{D.23})$$

and

$$\tau_{ij}\rho(R_{ij}) = \rho(R_{ji})\tau_{ij}. \quad (\text{D.24})$$

These identities can be easily proved by acting on a generic vector in $V \otimes V \otimes \dots \otimes V$. They express the fact that τ_{ij} changes either index of $\rho(R_{\alpha\beta})$ which is $i(j)$ to $j(i)$.

Now consider the braid relation

$$\tau_{i,i+1}\rho(R_{i,i+1})\tau_{i+1,i+2}\rho(R_{i+1,i+2})\tau_{i,i+1}\rho(R_{i,i+1}) = \tau_{i+1,i+2}\rho(R_{i+1,i+2})\tau_{i,i+1}\rho(R_{i,i+1})\tau_{i+1,i+2}\rho(R_{i+1,i+2}). \quad (\text{D.25})$$

The τ 's can be moved to the extreme left using the relations written above to get

$$\text{L.H.S} = \tau_{i,i+1}\tau_{i+1,i+2}\tau_{i,i+1}\rho(R_{i+1,i+2}R_{i,i+2}R_{i+1,i+2}). \quad (\text{D.26})$$

Similarly for the right hand side we get

$$\text{R.H.S} = \tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2}\rho(R_{i,i+1}R_{i,i+2}R_{i+1,i+2}). \quad (\text{D.27})$$

Therefore

$$\rho(R_{i+1,i+2}R_{i,i+2}R_{i,i+1}) = \rho(R_{i,i+1}R_{i,i+2}R_{i+1,i+2}). \quad (\text{D.28})$$

This is correct in any representation ρ . So it is natural to require that it holds for R_{ij} itself, regarding it as an element of $H \otimes H \otimes \cdots \otimes H$.

The relations

$$R_{i+1,i+2}R_{i,i+2}R_{i,i+1} = R_{i,i+1}R_{i,i+2}R_{i+1,i+2} \quad (\text{D.29})$$

are known as the (quantum) Yang-Baxter equation. For $i = 1$, they read

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}. \quad (\text{D.30})$$

There is another way to derive this relation. Δ^{op} is $R\Delta R^{-1}$ and both Δ and Δ^{op} satisfy the identity coming from coassociativity. That imposes a condition on R . It is fulfilled when R satisfies the Yang-Baxter relation. If the relation is not fulfilled then R does not satisfy the Yang-Baxter relation and thus gives rise to quasi-Hopf algebras. Such algebras occur for noncommutative gauge theories [104].

There exists a vast literature on Hopf algebras some of which are purely from the math point of view and some which give comprehensive introductions to physicists. These are [129, 130, 131, 132, 133, 134, 110, 135, 136, 137, 138, 139, 140, 141, 142, 143].

Bibliography

- [1] *Euclid's Elements All thirteen books in one volume, Based on Heath's translation*, Green Lion Press; *The Elements: Books I-XIII-Complete and Unabridged*, (2006) Translated by Sir Thomas Heath, Barnes and Noble.
- [2] R. Jackiw, Nucl. Phys. Proc. Suppl. **108**, 30 (2002) and hep-th/0110057; *Letter of Heisenberg to Peierls* (1930), W. Pauli, Scientific Correspondence, Vol. II, p.15, Ed. K. von Meyenn, Springer-Verlag, (1985); *Letter of Pauli to Oppenheimer* (1946), W. Pauli, Scientific Correspondence, Vol. III, p.380, Ed. K. von Meyenn, Springer-Verlag, (1993).
- [3] H. S. Snyder, *Quantized Space-Time*, Phys. Rev. 71, 3841 (1947).
- [4] C. N. Yang, *On Quantized Space-Time*, Phys. Rev. 72, 874874 (1947).
- [5] N. Seiberg, E. Witten, *String Theory and Noncommutative Geometry*, JHEP 9909:032, (1999) and hep-th/9908142v3.
- [6] R.C. Myers, *Dielectric-Branes*, JHEP 9912 (1999) 022 and hep-th/9910053v2.
- [7] R. Prange, S. Girvin, *The Quantum Hall Effect*, Springer, New York, (1987).
- [8] D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, G. Montambaux, *Poisson versus GOE Statistics in Integrable and Non-Integrable Quantum Hamiltonian*, Europhys. Lett. 22 537, (1993) and cond-mat/9301005v1.
- [9] J. Bellissard, A. van Elst, H. Schulz-Baldes, *The Non-Commutative Geometry of the Quantum Hall Effect*, J. Math. Phys. 35, 5373 (1994), and cond-mat/9411052v1.
- [10] A. P. Balachandran, K. S. Gupta, S. Kurkcuoglu, *Interacting Quantum Topologies and the Quantum Hall Effect*, Int. J. Mod. Phys. A 23:1327-1336, (2008) and hep-th/0707.1219v2.
- [11] S. Doplicher, K. Fredenhagen, J. E. Roberts, *Spacetime Quantization Induced by Classical Gravity*, Phys. Lett. B **331**, 33-44 (1994).
- [12] J. Madore, *The Fuzzy Sphere*, Class. Quantum Grav. 9 69. (1992).
- [13] M. R. Douglas, N. A. Nekrasov, *Noncommutative Field Theory*, Rev. Mod. Phys. 73 (2001); A. P. Balachandran, P. Padmanabhan, *The Groenewold - Moyal Plane and its Quantum Physics*, hep-th/0908.3888v1; A. P. Balachandran, A. Joseph, P. Padmanabhan, *Causality and Statistics on the Groenewold-Moyal plane*, hep-th/0905.0876v1; R. J. Szabo, *Quantum Field Theory on Non-commutative Spaces*, Phys. Rept. **378**, 207 (2003) and hep-th/0109162.

- [14] T. R. Govindarajan, P. Padmanabhan, T. Shreecharan, *Beyond Fuzzy Spheres*, J. Phys. A 43: 205203, (2010) and hep-th/0906.1660v2.
- [15] A. P. Balachandran, P. Padmanabhan, *Spin j Dirac Operators on the Fuzzy 2-Sphere*, JHEP 0909:120, (2009) and hep-th/0907.2977v2.
- [16] U. Carow-Watamura and S. Watamura, *Chirality and Dirac Operator on Noncommutative Sphere*, Commun. Math. Phys. 183 (1997) 365 and hep-th/9605003; *Noncommutative Geometry and Gauge Theory on Fuzzy Sphere*, Commun. Math. Phys. 212 (2000) 395 and hep-th/9801195.
- [17] P. Prešnajder, *The Origin of Chiral Anomaly and the Noncommutative Geometry*, J. Math. Phys. 41, 2789 (2000) and hep-th/9912050.
- [18] S. Digoal, P. Padmanabhan, *Spectrum of Spin 1 Dirac Operators on the Fuzzy 2-Sphere* hep-th/1004.3252v3.
- [19] S. Digoal, P. Padmanabhan, *Unusual Thermodynamics on the Fuzzy 2-Sphere*, JHEP 1010:091, (2010) and hep-th/1006.4792v2.
- [20] A.P. Balachandran, A. Daughton, Z.C. Gu, G. Marmo, R.D. Sorkin, A.M. Srivastava, *Spin Statistics Theorems without Relativity or Field Theory*, Int. J. Mod. Phys. A 8: 2993-3044, (1993).
- [21] P. W. Higgs, *Dynamical Symmetries in a Spherical Geometry. I*, J. Phys. A **12** (1979) 309.
- [22] J. Arnlind, M. Bordemann, L. Hofer, J. Hoppe, and H. Shimada, *Fuzzy Riemann Surfaces*, JHEP 0906:047, (2009) and hep-th/0602290.
- [23] V. G. Drinfel'd, *Almost Cocommutative Hopf Algebras*, Leningrad Math. J. 1 (1990), 321-332, *Quasi-Hopf Algebras*, Leningrad Math. J. 1, 1419-1457 (1990).
- [24] M. Chaichian *et al.*, *On a Lorentz Invariant Interpretation of Noncommutative Space-Time and its Implications on Noncommutative QFT*, Phys. Lett. B **604**, 98 (2004), and hep-th/0408069; M. Chaichian, P. Presnajder, A. Tureanu, *New Concept of Relativistic Invariance in NC space-time: Twisted Poincaré Symmetry and its Implications*, Phys. Rev. Lett. 94, 151602 (2005) and hep-th/0409096.
- [25] M. E. Peskin, D. V. Schroeder, *An Introduction to Quantum Field Theory*, Boulder, CO, Westview, (1995).
- [26] S. Weinberg, *The Quantum Theory of Fields*, Vol 1. Chap. 10, Cambridge Univ. Press, (2000).
- [27] M. Srednicki, *Quantum Field Theory*, Cambridge Univ. Press, (2007).
- [28] A. P. Balachandran, T. R. Govindarajan, S. Vaidya, *Spontaneous Symmetry Breaking in Twisted Noncommutative Quantum Theories*, Phys. Rev. D 79: 105020, (2009) and hep-th/0901.1712v4.
- [29] A. P. Balachandran, P. Padmanabhan, A. R. de Queiroz, *The LSZ S-Matrix Elements on the Moyal Plane*, To appear in Phys. Rev. D, (2011) hep-th/1104.1629v2.

- [30] R.F. Streater, A.S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin, (1964).
- [31] B. Chakraborty, S. Gangopadhyay, A. G. Hazra, F. G. Scholtz, *Twisted Galilean Symmetry and the Pauli principle at Low Energies*, J. Phys. A **39**, 9557 (2006) and hep-th/0601121.
- [32] A. P. Balachandran, A. Joseph, P. Padmanabhan, *Non-Pauli Transitions From Spacetime Noncommutativity*, Phys. Rev. Lett. 105, 051601 (2010) and hep-th/1003.2250v3.
- [33] A. P. Balachandran, P. Padmanabhan, *Non-Pauli Effects from Noncommutative Spacetimes*, JHEP 1012:001, (2010) and hep-th/1006.1185v2.
- [34] H. O. Back *et al.* [Borexino Collaboration], *New Experimental Limits on Violations of the Pauli Exclusion Principle obtained with the Borexino Counting Test Facility*, Eur. Phys. J. C **37**, 421 (2004).
- [35] Y. Suzuki *et al.* [Kamiokande Collaboration], *Study of invisible nucleon decay, $N \rightarrow \nu \bar{\nu} \bar{\nu}$, and a forbidden nuclear transition in the Kamiokande detector*, Phys. Lett. B **311**, 357 (1993).
- [36] A. S. Barabash *et al.*, *Search for Anomalous Carbon Atoms Evidence of Violation of the Pauli Principle during the period of Nucleosynthesis*, JETP Lett. **68** (1998) 112
- [37] R. Arnold *et al.*, *Testing the Pauli Exclusion Principle with the NEMO-2 Detector*, Eur. Phys. J. A **6**, 361 (1999).
- [38] E. Ramberg, G. A. Snow, *A New Experimental Limit on Small Violation of the Pauli Principle*, Phys. Lett. B **238**, 438 (1990).
- [39] S. Bartalucci *et al.*, *New Experimental Limit on the Pauli Exclusion Principle Violation by Electrons*, Phys. Lett. B **641**, 18 (2006).
- [40] O. W. Greenberg, *Particles with Small Violations of Fermi or Bose Statistics*, Phys. Rev. D **43**, 4111-4120 (1991); *Theories of Violation of Statistics*, hep-th/0007054v2.
- [41] B. Kostant, *Lecture Notes in Mathematics*, Vol.170, p. 87, Springer-Verlag (1970).
- [42] A.A. Kirillov, *Encyclopedia of Mathematical Sciences*, Vol. 4, p. 230.
- [43] F. A. Berezin, *General Concept of Quantization*, Commun. Math. Phys. **40**, 153 (1975).
- [44] A. P. Balachandran, G. Marmo, B. S. Skagerstam, A. Stern, *Classical Topology and Quantum States*, World Scientific, (1991).
- [45] C. Delbecq, C. Quesne, *Representation Theory and q -Boson Realizations of Witten's $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ deformations*, Phys. Lett. **B 300** (1993) 227.
- [46] Y. Nambu, *Generalized Hamiltonian Dynamics*, Phys. Rev. D **7**, 24052412 (1973).
- [47] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization 1, Deformations of Symplectic Structures*, Annals Phys. **111**, 61 (1978), *Quantum Mechanics as a Deformation of Classical Mechanics*, Lett. Math. Phys. **1**, 521 (1977).

- [48] H. Weyl, *Gruppentheorie und Quantenmechanik* or *The Theory of Groups and Quantum Mechanics*, New York, Dover Publications, (1950); H. Weyl, *Quantum Mechanics and Group Theory*, Z. Phys. 46, 1 (1927).
- [49] H. J. Groenewold, *On the principles of Elementary Quantum Mechanics*, Physica 12, 405 (1946).
- [50] J. E. Moyal, *Quantum Mechanics as a Statistical Theory*, Proc. Cambridge Phil. Soc. 45, 99 (1949).
- [51] A. M. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag, New-York, (1986).
- [52] G. Alexanian, A. Pinzul, A. Stern, *Generalized Coherent State Approach to Star Products and Applications to Fuzzy Sphere*, Nucl. Phys. B 600, 531 (2001) and hep-th/0010187.
- [53] A. B. Hammou, M. Lagraa, M. M. Sheikh-Jabbari, *Coherent State Induced Star-Product on R_λ^3 and the Fuzzy Sphere*, Phys. Rev. D 66, 025025 (2002) and hep-th/0110291v3.
- [54] A. P. Balachandran, B. P. Dolan, J. Lee, X. Martin, D. O'Connor, *Fuzzy Complex Projective Spaces and their Star-products*, J. Geom. Phys. 43 (2002) 184-204 and hep-th/0107099v2.
- [55] A. P. Balachandran, S. Kurkcuglu, E. Rojas, *The Star Product on the Fuzzy Supersphere*, JHEP 0207 (2002) 056 and hep-th/0204170v2.
- [56] H. I. Leemon, *Dynamical Symmetries in a Spherical Geometry. II*, J. Phys. A **12** (1979) 489.
- [57] T. L. Curtright, C. K. Zachos, *Deforming Maps for Quantum Algebras*, Phys. Lett. B **243** (1990) 237.
- [58] M. Roček, *Representation Theory of the Nonlinear $SU(2)$ Algebra*, Phys. Lett. B **255**, (1991) 554.
- [59] R. Floreanini, L. Lapointe, L. Vinet, *The Polynomial $SU(2)$ Symmetry Algebra of the Two-Body Calogero Model*, Phys. Lett. B **389** (1996) 327.
- [60] B. Abdesselam, J. Beckers, A. Chakrabarti, N. Debergh, *On Nonlinear Angular Momentum Theories, their Representations and associated Hopf Structures*, J. Phys. A **29** (1996) 3075.
- [61] V. Sunilkumar, *Aspects of Polynomial Algebras and their Physical Applications*, Thesis in University of Hyderabad, math-ph/0203047.
- [62] R. J. Glauber, *Photon Correlations*, Phys. Rev. Lett. **10** (1963) 84.
- [63] E. C. G. Sudarshan, *Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams*, Phys. Rev. Lett. **10** (1963) 277.
- [64] A. O. Barut, L. Girardello, *New Coherent States Associated with Non-Compact Groups*, Commun. Math. Phys. **21** (1971) 41.

- [65] W. M. Zhang, D. H. Feng, R. Gilmore, *Coherent States: Theory and Applications*, Rev. Mod. Phys. **62** (1990) 867.
- [66] V. Sunilkumar, B. A. Bambah, R. Jagannathan, P. K. Panigrahi, V. Srinivasan, *Coherent States of Nonlinear Algebras: Applications to Quantum Optics*, J. Opt. **B 2** (2000) 126.
- [67] M. Sadiq, A. Inomata, *Coherent States for Polynomial $\mathfrak{su}(2)$ Algebra*, J. Phys. **A 40** (2007) 11105.
- [68] R. L. M. Filho, W. Vogel, *Nonlinear Coherent States*, Phys. Rev. **A 54** (1996) 4560; V. I. Manko, G. Marmo, E. C. G. Sudarshan, F. Zaccaria, *f-Oscillators and Nonlinear Coherent States*, Phys. Scripta **55** (1997) 528.
- [69] J. R. Klauder, *Continuous-Representation Theory. II. Generalized Relation between Quantum and Classical dynamics*, J. Math. Phys. **4** (1963) 1058.
- [70] J-M. Sixdeniers, K. A. Penson, A. I. Solomon, *Mittag-Leffler Coherent States*, J. Phys. **A 32** (1999) 7543.
- [71] F. Oberhettinger, *Tables of Mellin transforms*, Springer-Verlag, New-York, (1974)
- [72] A. M. Mathai, R. K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, LNM **348**, Springer-Verlag, Heidelberg and New-York, (1973).
- [73] H. Grosse, P. Prešnajder, *The Construction of Noncommutative Manifolds using Coherent States*. Lett. Math. Phys. **28** (1993) 239-250.
- [74] A. Connes, *Noncommutative Geometry*, Academic Press, London, 1994
- [75] H. Grosse, C. Klimck, P. Prenajder, *Topologically Nontrivial Field Configurations in Noncommutative Geometry*, Commun. Math. Phys. 507 (1996) and hep-th/9510083.
- [76] B. P. Dolan, I. Huet, S. Murray, D. O. Connor, *Noncommutative Vector Bundles over Fuzzy CP^N and their Covariant Derivatives*, JHEP 0707:007 (2007) and hep-th/0611209, (2006).
- [77] P. H. Ginsparg and K. G. Wilson, *A Remnant of Chiral Symmetry on the Lattice*, Phys. Rev. D25, 2649 (1982).
- [78] C. Jayawardena, *Schwinger Model on S^2* , Helvetica Physica Acta, Vol 61(1988) 636-711.
- [79] D. A. Varshalovich, A. N. Moskalev, V. K. Khersonskii, *Quantum Theory of Angular Momentum*, World Scientific (1987).
- [80] A. P. Balachandran, G. Immirzi, *The Fuzzy Ginsparg-Wilson Algebra: A Solution of the Fermion Doubling Problem*, Phys. Rev. D68 (2003) 065023 and hep-th/0301242v2.
- [81] A. P. Balachandran, T. R. Govindarajan, B. Ydri, *The Fermion Doubling Problem and Noncommutative Geometry*, Mod. Phys. Lett. A15 (2000) 1279 and hep-th/9911087v2; *Fermion doubling problem and noncommutative geometry II*, hep-th/0006216v1.

- [82] A. P. Balachandran, S. Kurkcuoglu, S. Vaidya, *Lectures on Fuzzy and Fuzzy SUSY Physics*, World Scientific (2007).
- [83] P. R. Subramanian, V. Devanathan, *Trace Techniques for Angular Momentum Operators*, J. Phys. A, Math. Gen. **7** 1995-2007 (1974).
- [84] J. Thakur, *General Expression for the Trace of the Product of an Arbitrary Number of Components of Angular Momentum*, Ind. J. Pure and Appl. Phys. **13** 792-793 (1975).
- [85] A. H. Chamseddine, A. Connes, *The Spectral Action Principle*, Commun. Math. Phys. **186**:731-750, (1997) and arXiv:hep-th/9606001v1.
- [86] R.D. Tscheuschner, *Towards a Topological Spin - Statistics Relation in Quantum Field Theory*, Int. J. Theor. Phys. **28**: 1269-1310, (1989).
- [87] D. Finkelstein, J. Rubinstein, *Connection between Spin, Statistics, and Kinks*, J. Math. Phys. **9**: 1762-1779, (1968).
- [88] Y.S. Wu, *General Theory for Quantum Statistics in Two-Dimensions*, Phys. Rev. Lett. **52**: 2103-2106, (1984).
- [89] J.S. Birman, *Braids, Links, and Mapping Class Groups*, Princeton University Press, (1974).
- [90] A.P. Balachandran, T. Einarsson, T.R. Govindarajan, R. Ramachandran, *Statistics and Spin on Two-Dimensional Surfaces*, Mod. Phys. Lett. A **6**: 2801-2810, (1991).
- [91] A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistics*, Lecture Notes in Physics, (1992).
- [92] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific, (1990).
- [93] A. Khare, *Fractional Statistics and Quantum Theory*, World Scientific, (2005).
- [94] S. Minwalla, M. V. Raamsdonk, N. Seiberg, *Noncommutative Perturbative Dynamics*, JHEP 0002:020, (2000) and hep-th/9912072v2.
- [95] E. Akofor, A. P. Balachandran, A. Joseph, *Quantum Fields on the Groenewold-Moyal Plane*, Int. J. Mod. Phys. A **23**: 1637-1677, (2008) and hep-th/0803.4351v2.
- [96] V. Rivasseau, *Non-commutative Renormalization*, hep-th/0705.0705v1.
- [97] N. Seiberg, E. Witten, *String Theory and Noncommutative Geometry*, JHEP 9909:032, (1999) and hep-th/9908142v3.
- [98] H. Grosse, R. Wulkenhaar, *Progress in Solving a Noncommutative Quantum Field Theory in Four Dimensions*, hep-th/0909.1389v1.
- [99] H. Grosse, R. Wulkenhaar, *Renormalisation of ϕ^4 -Theory on Noncommutative R^4 in the Matrix Base*, Comm. Math. Phys. **256**: 305-374, (2005) and hep-th/0401128v2.
- [100] R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret, *Renormalization of Non-Commutative ϕ_4^4 Field Theory in x Space*, Comm. Math. Phys. **267**: 515-542, (2006) and hep-th/0512271v2.

- [101] A. P. Balachandran, B. A. Qureshi, A. Pinzul, S. Vaidya, *QED on the Groenewold Moyal Plane*, hep-th/0608138v2.
- [102] A. P. Balachandran, A. Pinzul, B. Qureshi, *UV-IR Mixing in Non-Commutative Plane*, Phys. Lett. B 634: 434-436, (2006) and hep-th/0508151v2.
- [103] A. P. Balachandran, A. Pinzul, B. A. Qureshi, S. Vaidya, *Twisted Gauge and Gravity Theories on the Groenewold-Moyal Plane*, Phys. Rev. D 76: 105025, (2007) and hep-th/0708.0069v2.
- [104] A. P. Balachandran, B. A. Qureshi, *Poincare' Quasi-Hopf Symmetry and Non-Associative Spacetime Algebra from Twisted Gauge Theories*, Phys. Rev. D 81: 065006, (2010) and hep-th/0903.0478v3.
- [105] A. Ostendorf, *Feynman Rules for Wightman functions*, Ann. Inst. Henri Poincaré (A) Physique Théorique, 40 No. 3 (1984); O. Steinmann, *Perturbation Theory of Wightman Functions*, Comm. Math. Phys. Volume 152, No. 3 (1993), 627-645.
- [106] D. Buchholz, S. J. Summers, *Scattering in Relativistic Quantum Field Theory: Fundamental Concepts and Tools*, math-ph/0509047v1.
- [107] P. Aschieri *et al.*, *A Gravity Theory on Noncommutative Spaces*, Class. Quant. Grav. **22**, 3511 (2005) and hep-th/0504183.
- [108] P. Watts, *Noncommutative String Theory, the R-Matrix, and Hopf Algebras*, Phys. Lett. B **474**, 295 (2000) and hep-th/9911026.
- [109] R. Oeckl, *Untwisting Noncommutative R^d and the Equivalence of Quantum Field Theories*, Nucl. Phys. B **581**, 559 (2000) and hep-th/0003018.
- [110] A. P. Balachandran, S. G. Jo, G. Marmo, *Group Theory and Hopf Algebras: Lectures for Physicists*, World Scientific, (2010).
- [111] A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi, S. Vaidya, *Statistics and UV-IR mixing with Twisted Poincare Invariance*, Phys. Rev. D 75, 045009 (2007) and hep-th/0608179.
- [112] A. P. Balachandran, A. Ibort, G. Marmo, M. Martone, *Covariant Quantum Fields on Noncommutative Spacetimes*, JHEP(in press), and hep-th/1009.5136v1.
- [113] H. Grosse, *On the Construction of Möller Operators for the Nonlinear Schrödinger Equation*, Phys. Lett. B 86, 267 (1979).
- [114] A. B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. 120, 253 (1979); L. D. Faddeev, Sov. Sci. Rev. 0 1 (1980) 107.
- [115] A. P. Balachandran, A. Pinzul, B. A. Qureshi, *Twisted Poincare Invariant Quantum Field Theories* Phys. Rev. D 77, 025021, (2008) and hep-th/0708.1779.
- [116] M. Leineker, A. R. Queiroz, A. E. Santana, C. de A. Siqueira, *Thermofield Dynamics for Twisted Poincare-Invariant Field Theories: Wick Theorem and S-matrix*, hep-th/1008.1242v2.

- [117] S. Doplicher, J. E. Roberts, *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Comm. Math. Phys. 131: No. 1 51-107 (1990).
- [118] S. Doplicher, R. Haag, J. E. Roberts, *Local observables and particle statistics. II*, Comm. Math. Phys. 35: No. 1, 49-85 (1974); *Local Observables And Particle Statistics. I.*, Comm. Math. Phys. 23: No. 3, 199-230 (1971).
- [119] R. Brunetti, K. Fredenhagen, R. Verch, *The Generally Covariant Locality Principle – A New Paradigm for Local Quantum Physics*, Comm. Math. Phys. 237 (2003) 31-68 and math-ph/0112041v1.
- [120] R. Verch, *A Spin-Statistics Theorem for Quantum Fields on Curved Spacetime Manifolds in a Generally Covariant Framework*, Comm. Math. Phys. 223: 261-288, (2001) and math-ph/0102035v2.
- [121] A. P. Balachandran, G. Mangano, A. Pinzul, S. Vaidya, *Spin and Statistics on the Groenewold-Moyal Plane: Pauli-Forbidden Levels and Transitions*, Int. J. Mod. Phys. A 21 (2006) 3111-3126 and hep-th/0508002v2.
- [122] P. Basu, R. Srivastava, S. Vaidya, *Thermal Correlation Functions of Twisted Quantum Fields*, hep-th/1003.4069; P. Basu, B. Chakraborty, S. Vaidya, *Fate of the Superconducting Ground State on the Moyal Plane*, hep-th/0911.4581; N. Acharyya, S. Vaidya *Accelerated Observer in Noncommutative Spacetime*, hep-th/1005.4666v2.
- [123] O. W. Greenberg, R. N. Mohapatra, *Local Quantum Field Theory of Possible Violation of the Pauli Principle*, Phys. Rev. Lett. 59, 2507-2510 (1987); *Difficulties with a Local Quantum Field Theory of Possible Violation of the Pauli Principle*, Phys. Rev. Lett. 62, 712-714 (1989).
- [124] O. W. Greenberg, R. N. Mohapatra, *Phenomenology of Small Violations of Fermi and Bose Statistics*, Phys. Rev. D 39, 2032-2038 (1989).
- [125] L. Randall, R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. 83, 3370 (1999) and hep-ph/9905221.
- [126] A. P. Balachandran, T. R. Govindarajan, A. G. Martins, Paulo Teotonio-Sobrinho, *Time-Space Noncommutativity: Quantized Evolutions*, JHEP11 (2004) 068; A. P. Balachandran, T. R. Govindarajan, A. G. Martins, Paulo Teotonio-Sobrinho, C. Molina, *Unitary Quantum Physics with Time-Space Noncommutativity*, Journal of Physics: Conference Series 24 (2005) 179 202.
- [127] A. P. Balachandran, A. G. Martins, Paulo Teotonio-Sobrinho, *Discrete Time Evolution and Energy Nonconservation in Noncommutative Physics*, JHEP 05 (2007) 066.
- [128] M. Chaichian, A. Demichev, P. Presnajder, *Field Theory on Noncommutative Space-Time and the Deformed Virasoro Algebra*, hep-th/0003270v2.
- [129] S. Majid, *Foundations of Quantum Group Theory*, Cambridge Univ. Press, (1995). - 640 p.
- [130] J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhauser, (2001).

-
- [131] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge Univ. Press, (1992).
- [132] M. E. Sweedler, *Hopf Algebras*, WA Benjamin New York, (1969).
- [133] A. U. Klimyk, K. Schmudgen, *Quantum Groups and their Representations*, Springer, (1997).
- [134] C. Gomez, M. Ruiz-Altaba, G. Sierra, *Quantum Groups in Two-Dimensional Physics*, Cambridge Univ. Press, (1996).
- [135] D. Kreimer, *Knots and Feynman Diagrams*, Cambridge Lecture Notes in Physics, (2000).
- [136] H. D. Doebner, J. D. Hennig (eds.), *Quantum Groups*, Springer, (1990).
- [137] V. Chari, A. N. Pressley, *A Guide to Quantum Groups*, Cambridge Univ. Press, (1995).
- [138] C. Kassel, *Quantum Groups*, Springer-Verlag, (1995).
- [139] Yu. I. Manin, *Topics in Noncommutative Geometry*, Princeton Univ. Press, (1991).
- [140] Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, CRM, (1988).
- [141] L. C. Biedenharn, M. A. Lohe, *Quantum Group Symmetry and q-Tensor Algebras*, World Scientific, (1995).
- [142] G. Landi, *An Introduction to Non-commutative Spaces and their Geometries*, Springer-Verlag, (1997).
- [143] J. Madore, *An Introduction to Non-commutative Differential Geometry and its Physical Applications*, Cambridge Univ. Press, (1995).

VITA

NAME OF AUTHOR: Pramod Padmanabhan

PLACE OF BIRTH: Chennai, Tamil Nadu, India

DATE OF BIRTH: August 17, 1985

DEGREES AWARDED:

B.Sc.(Hons.), Physics, Chennai Mathematical Institute, Chennai, India (2007)

PROFESSIONAL EXPERIENCE:

Syracuse University, Department of Physics, Graduate Student Instructor/Assistant & Graduate Student Researcher (2007-2011)

HONORS AND AWARDS:

Gold Medalist, Chennai Mathematical Institute, Chennai, India (May 2007)