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A NON-DETERMINISTIC PARALLEL SORTING ALGORITHM

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ABSTRACT. A *miniswap* S_i, $1 \le i < n$, compares two adjacent keys π_i, π_{i+1} in the sequence $\langle \pi_1, \ldots, \pi_n \rangle$, and transposes them if they are out of order. A *full sweep* is any composition of all $n-1$ possible miniswaps. We prove that the composition of any $n-1$ full sweeps is a sorting function.

Let n be a fixed (through most of our discussion) positive integer. Following de Bruijn [1], we model networks for sorting as discussed, e.g., by Knuth [2, Section 5.3.4] as compositions of *swaps*, where for $1 \le i < j \le n$ the effect of the swap S_{ij} on an arrangement $\pi = \langle \pi_1, \ldots, \pi_n \rangle$ of n distinct keys is given by

$$(\mathrm{S}_{ij}\pi)_k = \pi_k \qquad \text{if } k \ne i \text{ and } k \ne j,$$
$$(\mathrm{S}_{ij}\pi)_{\max(i,j)} = \max(\pi_i, \pi_j), \qquad (\mathrm{S}_{ij}\pi)_{\min(i,j)} = \min(\pi_i, \pi_j).$$

De Bruijn considers an arrangement π as simply a permutation on the set $\{1, \ldots, n\}$, which of course does no real harm, and is certainly the most natural choice of n sortable things. On the other hand, we feel that in principle it would slightly cloud the exposition if we were needlessly to conflate the data type of indices with that of keys. As a compromise, we write constants and variables for keys in bold type; that is, we say that the keys are some n distinct elements $\mathbf{1}, \ldots, \boldsymbol{n}$ from a totally ordered set with $\mathbf{1} < \mathbf{2} < \cdots < \boldsymbol{n}$, and an arrangement is a bijection $\pi : \{1, \ldots, n\} \to \{\mathbf{1}, \ldots, \boldsymbol{n}\}$. Let $\mathcal{S}_n$ be the set of all arrangements.

We will be concerned here entirely with *miniswaps* $\mathrm{S}_{i(i+1)}$ (also discussed by de Bruijn) which compare adjacent keys; we denote a miniswap more concisely as S_i $(1 \le i < n)$.

We are interested in compositions C of miniswaps which *sort*, that is, such that for all $\pi \in \mathcal{S}_n$, $C\pi = \pi^0$, where π^0 denotes the sorted arrangement $\langle \mathbf{1}, \ldots, \boldsymbol{n} \rangle$.

We denote (to begin with—other notations will follow) composition in diagrammatic order of functions from $\mathcal{S}_n$ to $\mathcal{S}_n$ by infixed semicolon:

$$[C; D]\pi = D(C\pi),$$

using for clarity brackets to group compositions and parentheses to group applications.

Key words and phrases. sorting networks, adjacent transposition sorting, comparators, swaps, miniswaps, sweeps.

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De Bruijn introduces a partial order on arrangements, which we denote here by $\preccurlyeq$, defined by

$$\sigma \preccurlyeq \pi \iff \sigma = E\pi \quad \text{for some composition } E \text{ of swaps,}$$

and observes that the sorted arrangement π^0 is least in the partial order. He proves [1, Theorem 4.2] that miniswaps (and consequently compositions of miniswaps) are monontone with respect to $\preccurlyeq$:

$$\text{if } \sigma \preccurlyeq \pi \text{ then } \mathrm{S}_i\sigma \preccurlyeq \mathrm{S}_i\pi.$$

It is essential to the truth of this result that the partial order on arrangements be defined in terms of swaps not restricted to being mini.

De Bruijn then proves, generalizing a discovery of Knuth's, a result [1, Theorem 6.2] which we repeat here, because the strong form we shall need below is given by de Bruijn only as an aside.

Proposition 1 (de Bruijn). *If C is a composition of miniswaps, and if D arises from C by inserting extra swaps, then $D\pi \preccurlyeq C\pi$ for all arrangements π.*

Proof. Let C be written as $C_0; C_1; \ldots; C_m$, each C_k a composition of miniswaps, in such a way that $D = C_0; \mathrm{S}_{i_1j_1}; C_1; \ldots; \mathrm{S}_{i_mj_m}C_m$. We may observe for each segment $\mathrm{S}_{i_kj_k}C_k$, and any arrangements ρ and σ:

$$\begin{aligned} \text{if } \rho \preccurlyeq \sigma, \text{ then } [\mathrm{S}_{i_kj_k}; C_k]\rho &= C_k(\mathrm{S}_{i_kj_k}\rho) \\ &\preccurlyeq C_k\rho \quad \text{by definition of } \preccurlyeq \text{ and monotonicity of } C_k \\ &\preccurlyeq C_k\sigma \quad \text{by monotonicity of } C_k. \end{aligned}$$

So we have, for any arrangement π, successively

$$\begin{aligned} C_0\pi &= C_0\pi, \\ [\mathrm{S}_{i_1j_1}; C_1](C_0\pi) &\preccurlyeq C_1(C_0\pi), \\ &\vdots \\ [\mathrm{S}_{i_mj_m}; C_m](\;\cdots\;) &\preccurlyeq C_m(\;\cdots\;), \end{aligned}$$

that is, $D\pi \preccurlyeq C\pi$. □

Let a *sweep*, W (more explicitly when necessary, an n-sweep) be any composition of zero or more miniswaps S_i, S_{i+1}, ..., S_{j-1} $(1 \le i \le j \le n)$ used once each in any order. We say that W *extends from i to j*, and write $W : i \longrightarrow j$. If $W : 1 \longrightarrow n$ we call it a *full sweep*. The composition of no miniswaps is of course the identity function I on $\mathcal{S}_n$; we call I the empty sweep, other sweeps non-empty, and note that $I : i \longrightarrow i$ for every i.

Lemma 2. *If $V : i \longrightarrow j$ and $W : k \longrightarrow l$ are sweeps with $j < k$, then V and W commute: $V; W = W; V$.*

Proof. Self-evident. □

Call sweeps satisfying the hypothesis of Lemma 2 *disjoint.* In particular, if $k = j+1$ in Lemma 2, denote the common value of $V;W$ and $W;V$ by $V^\frown W$.

We introduce $^\frown$ as a first step towards an algebraic notation for sweeps and their compositions that retains some of the visual appeal of the diagrams used by Knuth and others to exhibit sorting networks—n parallel wires with some pairs (adjacent pairs only as long as we stick to miniswaps) connected by "comparators". Continuing with this plan, for sweeps $V : i \to j$ and $W : j \to k$, we denote $V;W$ by $V \searrow W$ and $W;V$ by $V \swarrow W$.

Lemma 3. *If $U : i \to j$, $V : j \to k$, and $W : k \to l$ are sweeps of which V at least is non-empty, then all four associative laws for $\searrow$ and $\swarrow$ hold:*

(i, ii)
$$[U \searrow V] \searrow W = U \searrow [V \searrow W], \qquad [U \swarrow V] \swarrow W = U \swarrow [V \swarrow W],$$
(iii, iv)
$$[U \searrow V] \swarrow W = U \searrow [V \swarrow W], \qquad [U \swarrow V] \searrow W = U \swarrow [V \searrow W].$$

Proof. Parts (i) and (ii) are by the associativity of functional composition. For (iii),

$$\begin{aligned}[U \searrow V] \swarrow W &= W;U;V \\ &\overset{\text{L2}}{=} U;W;V \qquad \text{because } U \text{ and } W \text{ are disjoint} \\ &= U \searrow [V \swarrow W].\end{aligned}$$

The proof of (iv) is symmetrical to that of (iii). ☐

As a consequence of Lemma 3, any sweep extending from i to j can be written unambiguously without brackets in the form $\mathrm{S}_i \mathbin{\searrow\!\!\!\!\!\swarrow} \cdots \mathbin{\searrow\!\!\!\!\!\swarrow} \mathrm{S}_{j-1}$, where "$\searrow\!\!\!\!\!\swarrow$" stands for "$\searrow$ or $\swarrow$". It is not difficult to see that this expression for a sweep is unique, starting from the observation that definitely $\mathrm{S}_i \searrow \mathrm{S}_{i+1} \neq \mathrm{S}_i \swarrow \mathrm{S}_{i+1}$ for every $i < n-1$.

For $i \leq j$, define the *left-to-right* sweep $Z_i^j : i \to j$ by

$$Z_i^j \overset{\text{def}}{=} \mathrm{S}_i \searrow \mathrm{S}_{i+1} \searrow \cdots \searrow \mathrm{S}_{j-1}.$$

For $n > 1$ and $1 \leq i \leq j \leq n$, let an arrangement σ have $\boldsymbol{n}$ at position i and suppose that some full sweep W takes $\boldsymbol{n}$ from i to j; that is, let $\sigma_i = \boldsymbol{n}$ and $(W\sigma)_j = \boldsymbol{n}$. Then we may write

$$W = P \mathbin{\searrow\!\!\!\!\!\swarrow} Z_i^j \swarrow R.$$

Since $\boldsymbol{n}$ wins every comparison, Z_i^j will be the longest left-to-right sweep starting at position i to be found in W. Consequently we must have $i < j$ (that is, Z_i^j non-empty) unless $i = j = n$; in either case $j > 1$, so that $R : j \to n$ is non-full. Denote by $\overleftarrow{R}$ the "left shift" of R by one position, that is

$$\overleftarrow{R} \overset{\text{def}}{=} \mathrm{S}_{j-1} \mathbin{\searrow\!\!\!\!\!\swarrow} \cdots \mathbin{\searrow\!\!\!\!\!\swarrow} \mathrm{S}_{n-2} : j-1 \to n-1$$

where the succession of $\searrow$'s and $\swarrow$'s is the same as for R. Then we have the following rather specialized but straightforward lemma, saying that left-to-right sweeps can, in a sense, be pulled to the front of certain other sweeps:

Lemma 4. *For* $n > 1$, *let* σ *be an arrangement and* W *a full sweep such that* $\sigma_i = \boldsymbol{n}$, $(W\sigma)_j = \boldsymbol{n}$, *and* $W = P \mathbin{\swarrow\!\!\!\!\!\!\searrow} Z_i^j \swarrow R$. *Then*

$$[[Z_i^j \swarrow R]; Z_j^n]\sigma = [Z_i^n; \overleftarrow{R}]\sigma.$$

Proof. If $j = n$, then $R = I = \overleftarrow{R}$, and the asserted equation holds simply because $Z_i^j; Z_j^n = Z_i^n$. Otherwise, as noted above, $i < j < n$; let $R\langle\sigma_1, \dots, \sigma_n\rangle = \langle\sigma_1, \dots, \sigma_{j-1}, \rho_j, \dots, \rho_n\rangle$. Then

$$\begin{aligned}
[[Z_i^j \swarrow R]; Z_j^n]\sigma &= [Z_i^j; Z_j^n]\langle\sigma_1, \dots, \sigma_{i-1}, \boldsymbol{n}, \sigma_{i+1}, \dots, \sigma_{j-1}, \rho_j, \dots, \rho_n\rangle \\
&= Z_i^n\langle\sigma_1, \dots, \sigma_{i-1}, \boldsymbol{n}, \sigma_{i+1}, \dots, \sigma_{j-1}, \rho_j, \dots, \rho_n\rangle \\
&= \langle\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{j-1}, \rho_j, \dots, \rho_n, \boldsymbol{n}\rangle \\
&= \overleftarrow{R}\langle\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{j-1}, \sigma_j, \dots, \sigma_n, \boldsymbol{n}\rangle \\
&= \overleftarrow{R}(Z_i^n\sigma) \\
&= [Z_i^n; \overleftarrow{R}]\sigma. \qquad \square
\end{aligned}$$

If we compose arbitrarily chosen full sweeps, it is clear that $n - 1$ of them will be sufficient, and may be necessary, to make sure that $\boldsymbol{n}$ arrives at position n. This suggests our theorem:

Theorem 5. *For* $n \geq 1$, *any composition of* $n - 1$ *full* n*-sweeps sorts.*

Proof. We proceed by induction on n; the case $n = 1$ is immediate. For $n \geq 2$, let $W_1, \dots, W_{n-1}$ be any full sweeps, and let σ be any arrangement. Follow the rightward movement of $\boldsymbol{n}$ under the composition $W_1; \cdots; W_{n-1}$, and let its successive positions be $i_1, \dots, i_{n-1}, i_n = n$; that is, define $i_1 < \cdots < \cdots = \cdots = i_n$ (there may be from 0 to $n - 1$ strict inequalities and from $n - 1$ to 0 equalities, but all the inequalities will come first) by the equations $\sigma_{i_1} = \boldsymbol{n}$, $(W_1\sigma)_{i_2} = \boldsymbol{n}, \dots,$ $([W_1; \cdots; W_{n-1}]\sigma)_{i_n} = \boldsymbol{n}$. We may write

$$\begin{matrix}
W_1; & & [P_1 \mathbin{\swarrow\!\!\!\!\!\!\searrow} Z_{i_1}^{i_2} \swarrow R_1]; \\
W_2; & & [P_2 \mathbin{\swarrow\!\!\!\!\!\!\searrow} Z_{i_2}^{i_3} \swarrow R_2]; \\
 & = & \\
\vdots & & \vdots \\
W_{n-1} & & [P_{n-1} \mathbin{\swarrow\!\!\!\!\!\!\searrow} Z_{i_{n-1}}^n].
\end{matrix}$$

Note that $P_2, \dots, P_{n-1}$ are non-empty, but not necessarily P_1.

Obviously our plan is, by $n - 2$ applications of Lemma 4, to collect all the Z's into one solid $Z_{i_1}^n$ at the top which can be skimmed off, leaving behind an instance of sorting only $n - 1$ keys. Two complementary difficulties stand in our way: the operations shown above as $\swarrow\!\!\!\!\!\!\searrow$, of unknown directionality, may prevent Lemma 4 from applying; and in order to appeal to induction we must get down from $(n-1)^2$ to $(n-2)^2$ miniswaps; that is, we need to discard $(n - 1) + (n - 2)$ miniswaps, or $(i_1 - 1) + (n - 2)$ in addition to the $n - i_1$ miniswaps occurring in the Z's.

It is not difficult to see which miniswaps we will be as well or better off without: the rightmost miniswap from each of $P_2, \dots, P_{n-1}$ (each may or must be otiose on account of having $\boldsymbol{n}$ as its right-hand input) and all $i_1 - 1$ miniswaps of P_1. So, for $k = 2, \dots, n-1$, let $\overline{P}_k : 1 \to i_k - 1$ be the sweep such that $P_k = \overline{P}_k \times \mathrm{S}_{i_k - 1}$. Then by Proposition 1, we will have

$$
\begin{array}{c}
[[P_1 \times Z_{i_1}^{i_2} \swarrow R_1]; \\
[P_2 \times Z_{i_2}^{i_3} \swarrow R_2]; \\
\vdots \\
[P_{n-1} \times Z_{i_{n-1}}^{n}]]\sigma
\end{array}
\quad \preccurlyeq \quad
\begin{array}{c}
[[Z_{i_1}^{i_2} \swarrow R_1]; \\
[\overline{P}_2 \frown Z_{i_2}^{i_3} \swarrow R_2]; \\
\vdots \\
[\overline{P}_{n-1} \frown Z_{i_{n-1}}^{n}]]\sigma.
\end{array}
$$

In particular, to show that the left-hand side is π^0, it will be enough to show that the right-hand side is.

Clearly the movement of $\boldsymbol{n}$ is unaffected by these deletions of miniswaps. Hence we may make our $n-2$ applications of Lemma 4 by the calculation

$$
\begin{array}{r}
[[Z_{i_1}^{i_2} \swarrow R_1]; \\
[\overline{P}_2 \frown Z_{i_2}^{i_3} \swarrow R_2]; \\
\vdots \\
[\overline{P}_{n-2} \frown Z_{i_{n-2}}^{i_{n-1}} \swarrow R_{n-2}]; \\
[\overline{P}_{n-1} \frown Z_{i_{n-1}}^{n}]]\sigma
\end{array}
=
\begin{array}{r}
[[Z_{i_1}^{i_2} \swarrow R_1]; \\
[\overline{P}_2 \frown Z_{i_2}^{i_3} \swarrow R_2]; \\
\vdots \\
[\overline{P}_{n-2} \frown Z_{i_{n-2}}^{n}]; \\
[\overline{P}_{n-1} \swarrow \overleftarrow{R}_{n-2}]]\sigma
\end{array}
= \cdots
$$

$$
\cdots =
\begin{array}{r}
[[Z_{i_1}^{i_2} \swarrow R_1]; \\
[\overline{P}_2 \frown Z_{i_2}^{n}]; \\
\vdots \\
[\overline{P}_{n-2} \swarrow \overleftarrow{R}_{n-3}]; \\
[\overline{P}_{n-1} \swarrow \overleftarrow{R}_{n-2}]]\sigma
\end{array}
=
\begin{array}{r}
[Z_{i_1}^{n}; \\
[\overline{P}_2 \swarrow \overleftarrow{R}_1]; \\
\vdots \\
[\overline{P}_{n-2} \swarrow \overleftarrow{R}_{n-3}]; \\
[\overline{P}_{n-1} \swarrow \overleftarrow{R}_{n-2}]]\sigma.
\end{array}
$$

In more detail, we may show the replacement effected in any one step, say for $n - 1 > k \geq 1$, by

$$
\begin{array}{r}
[\cdots \\
[\overline{P}_k \frown Z_{i_k}^{i_{k+1}} \swarrow R_k]; \\
[\overline{P}_{k+1} \frown Z_{i_{k+1}}^{n}]; \\
\cdots]\sigma
\end{array}
=
\begin{array}{r}
[\cdots \\
\overline{P}_k; \\
[Z_{i_k}^{i_{k+1}} \swarrow R_k]; \\
Z_{i_{k+1}}^{n}; \\
\overline{P}_{k+1}; \\
\cdots]\sigma
\end{array}
\overset{\text{L4}}{=}
\begin{array}{r}
[\cdots \\
\overline{P}_k; \\
Z_{i_k}^{n}; \\
\overleftarrow{R}_k; \\
\overline{P}_{k+1}; \\
\cdots]\sigma
\end{array}
=
\begin{array}{r}
[\cdots \\
[\overline{P}_k \frown Z_{i_k}^{n}]; \\
[\overline{P}_{k+1} \swarrow \overleftarrow{R}_k]; \\
\cdots]\sigma.
\end{array}
$$

(When $k = 1$, omit "$\overline{P}_k \frown$" from the first and last expressions, "$\overline{P}_k$;" from the middle two.)

But $Z^n_{i_1}\sigma = \langle \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n, \boldsymbol{n} \rangle$, and the $n-2$ sweeps $\overline{P}_2 \swarrow \overleftarrow{R}_1, \dots, \overline{P}_{n-1} \swarrow \overleftarrow{R}_{n-2} : 1 \to n-1$ may be regarded as so many full $(n-1)$-sweeps whose composition, which in effect acts on $\langle \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n \rangle$, will by induction hypothesis sort it. That is to say,

$$\begin{aligned}
&[[Z^{i_2}_{i_1} \swarrow R_1]; [\overline{P}_2 \frown Z^{i_3}_{i_2} \swarrow R_2]; \cdots ; [\overline{P}_{n-1} \frown Z^n_{i_{n-1}}]]\sigma \\
&\qquad = [[\overline{P}_2 \swarrow \overleftarrow{R}_1]; \cdots ; [\overline{P}_{n-1} \swarrow \overleftarrow{R}_{n-2}]] \langle \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n, \boldsymbol{n} \rangle \\
&\qquad = \langle \mathbf{1}, \dots, \boldsymbol{n} - \mathbf{1}, \boldsymbol{n} \rangle \qquad \text{by induction} \\
&\qquad = \pi^0. \qquad\qquad\qquad\qquad \square
\end{aligned}$$

We could have insisted that P_1 be empty: de Bruijn and Knuth credit R. W. Floyd with the discovery that if a composition of miniswaps sorts the reversed arrangement $\langle \boldsymbol{n}, \dots, \mathbf{1} \rangle$, then it sorts. It seemed to us that P_1 caused too little trouble to justify appeal to another substantial theorem.

The odd-even transposition sort described by Knuth [2, Exercise 5.3.4.37] can be derived from the above algorithm by a particular choice of full sweeps. If we take $W_{\mathbf{zigzag}} \overset{\text{def}}{=} \mathrm{S}_1 \searrow \mathrm{S}_2 \swarrow \mathrm{S}_3 \searrow \mathrm{S}_4 \swarrow \searrow \cdots$, $W_{\mathbf{zagzig}} \overset{\text{def}}{=} \mathrm{S}_1 \swarrow \mathrm{S}_2 \searrow \swarrow \cdots$, then $n-1$ instances of $W_{\mathbf{zigzag}}$ and $W_{\mathbf{zagzig}}$ in alternation (starting with either) provides a way to sort whose redundant comparisons are very evident. We may decompose these sweeps as $W_{\mathbf{zigzag}} = C_{\mathrm{odd}}; C_{\mathrm{even}}$, $W_{\mathbf{zagzig}} = C_{\mathrm{even}}; C_{\mathrm{odd}}$, where

$$\begin{aligned}
C_{\mathrm{odd}} &\overset{\text{def}}{=} \mathrm{S}_1 \frown \mathrm{S}_3 \frown \cdots, \\
C_{\mathrm{even}} &\overset{\text{def}}{=} \mathrm{S}_2 \frown \mathrm{S}_4 \frown \cdots,
\end{aligned}$$

and then the whole sort becomes, say, $C_{\mathrm{odd}}; C_{\mathrm{even}}; C_{\mathrm{even}}; C_{\mathrm{odd}}; C_{\mathrm{odd}}; \cdots$. Applying as often as possible the identity $\mathrm{S}_i; \mathrm{S}_i = \mathrm{S}_i$ (immediate repetition of a swap accomplishes nothing) we can boil this down to $C_{\mathrm{odd}}; C_{\mathrm{even}}; C_{\mathrm{odd}}; C_{\mathrm{even}}; \cdots$, which may be regarded as $\lfloor n/2 \rfloor$ iterations of $W_{\mathbf{zigzag}}$ followed, if n is odd, by an additional C_{odd}; this is the odd-even transposition sort.

Since there are sorting networks (using swaps which are not mini) that sort in time $O(\log^2 n)$ if executed with n-fold parallelism, it is a little hard to imagine practical situations that would make our algorithm in its non-deterministic generality, which plainly takes $\Omega(n)$ time, desirable to use, unless as a by-product of some computation that for its own reasons used $n-1$ rounds of next-neighbor communications between n processors connected in line. We mention one context in which the algorithm would, however, be natural and easy to program: Sabot's parallation model of parallel computation [3] provides, for any (not necessarily associative) binary operation $\oplus$ on a set A, a non-deterministic reduction $\oplus/$ applicable to positive-length vectors of elements of A such that $\oplus/\langle a_1, \dots, a_n \rangle$ denotes $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ computed with some unspecified parenthesization. If we take $\oplus$ to be "swapping concatenation" between non-empty sequences of keys:

$$\langle k_1, \dots, k_m \rangle \oplus \langle k'_1 \dots, k'_{m'} \rangle = \langle k_1, \dots, k_{m-1}, \min(k_m, k'_1), \max(k_m, k'_1), k'_2, \dots, k'_{m'} \rangle,$$

then $\oplus/\big\langle \langle \pi_1 \rangle, \dots, \langle \pi_n \rangle \big\rangle$ computes $W\pi$ for an unspecified full sweep W.

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