Syracuse University
SURFACE

Electrical Engineering and Computer Science - Technical Reports
College of Engineering and Computer Science

6-1990

Taxonomic Reasoning and Lexical Semantics

William C. Purdy
Syracuse University, wcpurdy@ecs.syr.edu

Follow this and additional works at: https://surface.syr.edu/eecs_techreports

Part of the Computer Sciences Commons

Recommended Citation

Purdy, William C., "Taxonomic Reasoning and Lexical Semantics" (1990). *Electrical Engineering and Computer Science - Technical Reports*. 89.
https://surface.syr.edu/eecs_techreports/89

This Report is brought to you for free and open access by the College of Engineering and Computer Science at SURFACE. It has been accepted for inclusion in Electrical Engineering and Computer Science - Technical Reports by an authorized administrator of SURFACE. For more information, please contact surface@syr.edu.

SU-CIS-90-28

Taxonomic Reasoning and Lexical Semantics

William C. Purdy

June 1990

School of Computer and Information Science
Syracuse University
Suite 4-116
Center for Science and Technology
Syracuse, NY 13244-4100

(315) 443-2368

Taxonomic Reasoning and Lexical Semantics

William C. Purdy*

June 1990

*School of Computer and Information Science, Syracuse University

Abstract

Taxonomic reasoning is used in many applications, including many-sorted logic, knowledge bases, document retrieval, and natural language processing. These various applications have been dealt with independently. Because they have so much in common, a general approach to taxonomic reasoning would seem to be justified.

This paper presents a theory of lexical semantics as an example of such a general approach. The theory defines a representation and an algebra for that representation. The operations of the algebra are inherently parallel, making them well matched to the capabilities of modern computer systems.

1. Introduction

This paper presents a theory of lexical semantics as an example of a general approach to taxonomic reasoning. Since lexical semantics subsumes many other varieties of taxonomic systems, these results are readily adapted to other applications.

Taxonomic systems occur in diverse guises in computational systems. Some examples are the following.

- Sorts in automated reasoning systems using many-sorted logic
- Types used to denote classes of entities in knowledge bases
- Indexes or key words in document retrieval systems
- Properties in property inheritance systems
- Lexical components or markers in natural language processing systems

Because the applications are quite distinct, these systems have been dealt with independently. However viewed abstractly, they are quite similar; therefore it would seem useful also to consider them as varieties of the same abstract structure.

In the broadest sense, a *taxonomic system* is any subdivision of a set of entities that conveys information about those entities. The subdivisions are related by inclusion, exclusion, and overlap. To minimize the quantity of data, an entity is listed as a member only of the smallest subdivisions which contain it. Similarly, only immediate inclusion is given explicitly. Therefore, although not explicit, membership in a given subdivision may entail membership or nonmembership in other subdivisions. The process of inferring what is entailed by the taxonomic system is called *taxonomic reasoning.*

Construction of a taxonomic system for a given population consists of two parts.

1. (a) Identification of relevant or diagnostic *features* and specification of their Boolean relationships (inclusion, exclusion, overlap)

   (b) Boolean description of each member of the population in terms of the features

2. (a) Construction, using the features, of a structure that will facilitate taxonomic reasoning

   (b) Definition of a mapping from population members into the structure

The first part is application specific. For example, in the case of lexical semantics, it requires an empirical linguistic analysis. Methods are assumed to exist for carrying out this part. The paper will not deal with it further. The second part is general to all the examples enumerated above. In the following sections, a theory is developed for this part of the construction.

The theory defines a representation in which lexical expressions are modeled as subsets or subspaces of a multi-dimensional semantic space. A unique representation or normal form is defined which may be viewed as a code for the subspaces of the semantic space. A Boolean algebra of normal forms is developed, in which lexical entailment is Boolean inclusion. A property of the representation makes the algebraic operations inherently parallel.

The presentation in the body of the paper is informal, making use of examples to illustrate the theory and to indicate the range of applicability. Formal definitions and proofs in support of the presentation are given in the Appendix.

2. A Model of Lexical Semantics

The theory is introduced by an example from a traditional domain: English words for kinship. The kinship vocabulary and its definition, shown in Figure 1, are taken from Nida [6]. The elements of the vocabulary are listed at the top of the table. Lexical features that distinguish between the elements of the vocabulary appear along the left side of the table. The body of the table indicates those features that characterize each vocabulary element. This example is restricted to consanguineal kinship (c-kinship). However, partial consanguineal relations will be added in Section 5 to further illustrate the theory.

Restricting consideration to nominal domains does not indicate a limitation of the theory. Rather the domains are chosen to make the general approach to taxonomic systems easily understandable. Other domains, such as verbs and determiners, can be dealt with similarly [7].

C-kinship can be modeled by a relational structure. For example, the denotation of **father** is defined by the expression[1] $\mathsf{father}(x, y) \leftrightarrow \mathsf{male}(x) \wedge \mathsf{prec}(x, y) \wedge \mathsf{L0}(x, y)$ where $\mathsf{prec}(x, y)$ asserts that x is of the generation preceding that of y and $\mathsf{L0}(x, y)$ asserts a direct lineal relation between x and y. If male is modified so that $\mathsf{male}(x, y)$ is taken to assert that x is male, and application is defined to distribute over Boolean operations, the above can be written more compactly $\mathsf{father}(x, y) \leftrightarrow (\mathsf{male} \wedge \mathsf{prec} \wedge \mathsf{L0})(x, y)$. If all expressions are so treated, the variable symbols are no longer needed. That is, $\mathsf{father} \leftrightarrow \mathsf{male} \wedge \mathsf{prec} \wedge \mathsf{L0}$ conveys the same information.[2]

A relation R_1 is said to be contained by or included in a relation R_2 if for all pairs (x, y), $\mathsf{R}_1(x, y) \rightarrow \mathsf{R}_2(x, y)$, or in variable-free form, $\mathsf{R}_1 \rightarrow \mathsf{R}_2$. To illustrate this, c-kinship can be extended to include the lexical items **self, parent, child, sibling**

[1] Denotations are written in sans serif type.

[2] The modification of male is called *homogenization* by Quine. In terms of Quine's functors, male has been replaced by *inv Pad* male. Further discussion of homogenization and its role in elimination of variables can be found in [8], pp. 283-288.

and **immediate family**, defined as follows:

self $\leftrightarrow$ same $\wedge$ L0

parent $\leftrightarrow$ prec $\wedge$ L0

child $\leftrightarrow$ succ $\wedge$ L0

sibling $\leftrightarrow$ same $\wedge$ L1

immediate family $\leftrightarrow$ L0 $\vee$(same $\wedge$ L1)

From the definitions of these new lexical items it can be inferred for example that sister $\rightarrow$ sibling, i.e., sister is included in sibling. Similarly, it can be inferred that sibling $\rightarrow$ immediate family, i.e., sibling is included in immediate family.

This suggests a way to model entailment between lexical items. Using componential analysis [6] or semantic field analysis [4] one can identify lexical features that distinguish between members of a set of related lexical items (a "semantic domain" or "semantic field"). C-kinship is an example. The derived relational structure can then model the semantic domain, providing denotations for the lexical features and the lexical items.

The Boolean model cannot express some assertions that can be expressed in first-order logic. For example, using first-order logic one can assert that the parent relation is the converse of the child relation:

$$\forall x \forall y [\text{parent}(x,y) \leftrightarrow \text{child}(y,x)]$$

Or, it can be asserted that the uncle relation entails a brother relation:

$$\forall x \forall y [\text{uncle}(x,y) \rightarrow \exists z [\text{brother}(x,z)]]$$

But the Boolean model has the advantage of simplicity: entailment is simply set inclusion.

Specifically, let H be a set of individuals. The power set $2^{H \times H}$ represents the set of all binary relations on H. Let $S \subseteq H \times H$ be a subset of consanguineal pairs such that $\{\mathsf{prec},\mathsf{same},\mathsf{succ}\}$ partitions S. That is,

1. $\mathsf{prec} \cup \mathsf{same} \cup \mathsf{succ} = S$
2. $\mathsf{prec} \cap (\mathsf{same} \cup \mathsf{succ}) = \emptyset$, $\mathsf{same} \cap \mathsf{succ} = \emptyset$
3. $\mathsf{prec} \neq \emptyset$, $\mathsf{same} \neq \emptyset$, $\mathsf{succ} \neq \emptyset$

Let $\{\mathsf{L0},\mathsf{L1},\mathsf{L2}\}$ and $\{\mathsf{male},\mathsf{female}\}$ also partition S.

S can be diagrammed as in Figure 2a or, to suggest a multidimensional space, as in Figure 2b. In this multidimensional space, subspaces or subsets are denotations of c-kinship relations. For example, the subspace $\mathsf{parent} = \mathsf{prec} \cap \mathsf{L0}$ is the denotation of **parent**. When the denotation of a lexical item includes several cells (e.g., $\mathsf{cousin}{=}\mathsf{L2}$), this is indicated by labeling each of the cells with that lexical item. Some examples of subspaces are given in Figure 3.

Thus a subspace can be viewed as the extension or meaning of the associated lexical item. Moreover, relations between subspaces can be viewed as relations between meanings. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be any c-kinship lexical items, and R_1 and R_2 their respective denotations (subspaces). Then $\mathbf{R}_1$ entails $\mathbf{R}_2$ if and only if $\mathsf{R}_1 \subseteq \mathsf{R}_2$. That is, subspace inclusion can be viewed as entailment or meaning inclusion. Similarly, subspace exclusion (disjointness) can be viewed as contradiction. The intersection of two subspaces can be viewed as the meaning common to the corresponding lexical items. In the multidimensional space, inclusion, exclusion, intersection and the like can be determined quite directly. The examples of Figure 4 illustrate this.

The partitions that subdivide the multidimensional space in the preceding example have an important property that was not made explicit. Residence in any given block of the partition $\{\mathsf{prec},\mathsf{same},\mathsf{succ}\}$ does not restrict residence in any block of the partition $\{\mathsf{L0},\mathsf{L1},\mathsf{L2}\}$. A similar assertion holds for any subset of the three partitions. This property is called "independence."

More precisely, let $B = \{P_i | i \in I\}$ be a set of partitions of a set S, where $P_i = \{p_i^j | j \in J_i\}$. Then B will be said to be *independent* if and only if for any selection of $j_i \in J_i$, for each $i \in I$, $\bigcap_{i \in I} p_i^{j_i}$ is nonempty.[3]

Independence means that the set of partitions contains no redundancy. Each partition contributes information in every case. If one visualizes the atomic cells of the multidimensional space, independence implies that some individuals occupy every cell. Put another way, no cell represents a logically impossible condition. This is not to be confused with "lexical gaps," which are breaks in a pattern of related lexical items [4, 5]. It may be that a particular cell is the denotation of no lexical item; but it is the denotation of some expression or paraphrase. Thus independence does not imply no lexical gaps; rather it implies no "logical gaps."

An independent set of partitions of a set S will be called a *basis of* S. The partitions of a basis of S define dimensions of S. Their blocks correspond to the coordinate values. Thus each partition can be viewed as a dimension of meaning. The blocks can be viewed as mutually antonymous "primitive" meanings.

Geometrically each block can be thought of as a hyperplane orthogonal to a coordinate axis. These hyperplanes are the simplest subspaces. Next in order of simplicity are those subspaces that can be expressed as the intersection of such hyperplanes, one or the union of several from each dimension.

In the c-kinship space defined previously, prec corresponds to a plane orthogonal to the "generation" axis. The intersection of prec, L0 (a plane orthogonal to the "lineality" axis) and male $\cup$ female (union of planes orthogonal to the "gender" axis) is the subspace previously identified as the extension of **parent**. Such subspaces will be called "elementary subsets." They are analogous to convex subspaces because they can have no "inside corners." But they are not exactly convex subspaces because

[3]To simplify the present discussion it is assumed that all partitions as well as all sets of partitions are finite. This assumption is not necessary and is not made in the Appendix.

they need not be connected. Equivalently, a subspace x is an elementary subset if and only if for some reordering of the blocks of each partition, x becomes a rectangular polyhedron. Thus parent and cousin are elementary subsets. So is prec∪succ, although not connected. But immediate family is not an elementary subset. It has inside corners and so cannot be formed by intersecting sets of planes orthogonal to the coordinate axes.

More precisely, if $B = \{P_i | i \in I\}$ is a basis of S where $P_i = \{p_i^j | j \in J_i\}$, then an *elementary subset of S relative to the basis B* is a subspace x that can be represented $x = \bigcap_{i \in I} \bigcup_{j \in J_i^x} p_i^j$ where $J_i^x \subseteq J_i$. This representation is called the *standard form for x*. The conjunct $\bigcup_{j \in J_i^x} p_i^j$ is called the *i-th component of x*.

Thus the i-th component of an elementary subset is formed by taking the union of some of the planes orthogonal to the i-th coordinate. The elementary subset is the intersection of its components.

An equivalent representation is $x = \bigcap_{i \in I^x} \bigcup_{j \in J_i^x} p_i^j$ where $i \in I^x$ if and only if $J_i^x \neq J_i$. For example, the expression L0 represents the same elementary subset that (prec ∪ same ∪ succ) ∩ L0 ∩ (male ∪ female) does. This is called the *abbreviated standard form for x*.

It is shown in the Appendix that the standard form for elementary subset x is unique. It follows that the abbreviated standard form is also unique.

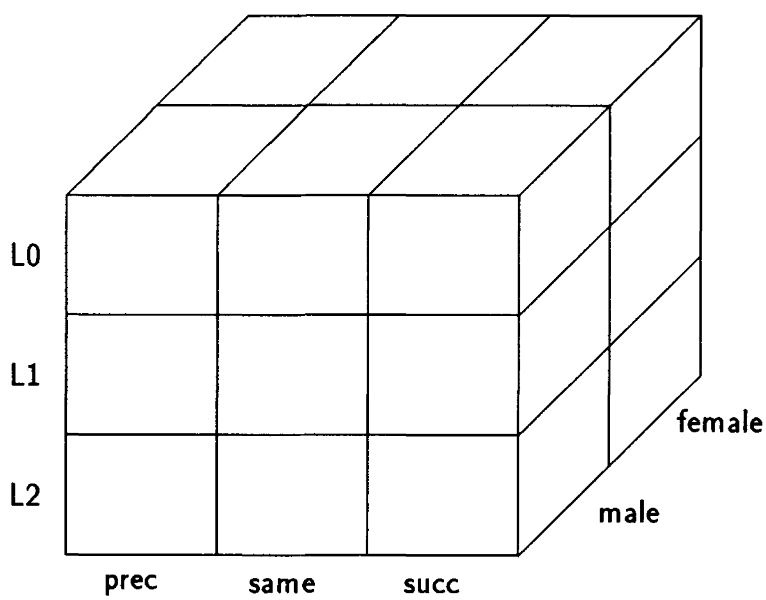
The smallest nonempty elementary subsets are the intersections of hyperplanes where exactly one hyperplane is orthogonal to each coordinate axis. These elementary subsets are called *atoms*. For example, father = prec ∩ L0 ∩ male is an atom.

	father	mother	uncle	aunt	brother	sister	son	daughter	nephew	niece	cousin
male	X		X		X		X		X		X
female		X		X		X		X		X	X
prec.gen.	X	X	X	X							X
same gen.					X	X					X
succ.gen.							X	X	X	X	X
dir.lin.	X	X					X	X			
once rem.			X	X	X	X			X	X	
twice rem.											X
consang.	X	X	X	X	X	X	X	X	X	X	X
affinal			X	X					X	X	

Figure 1: Definition of Kinship Relations (from Nida)

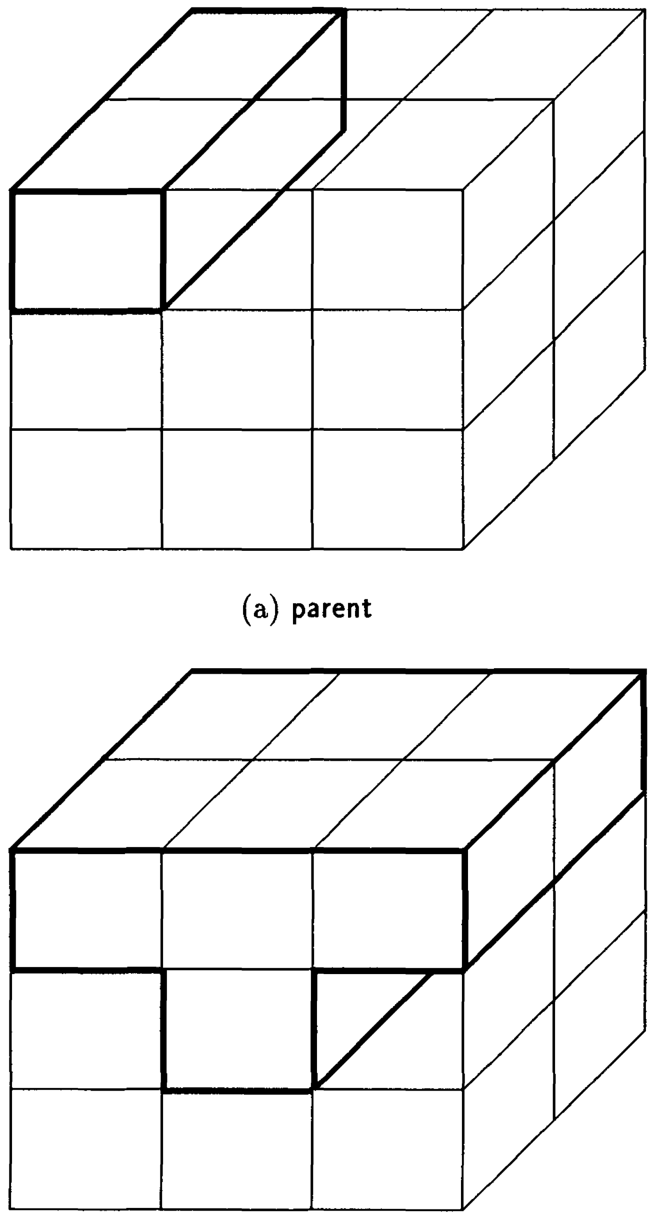
	male			female		
L0	father	self	son	mother	self	daughter
L1	uncle	brother	nephew	aunt	sister	niece
L2	cousin	cousin	cousin	cousin	cousin	cousin
	prec	same	succ	prec	same	succ

(a) Planar Representation



(b) Spatial Representation

Figure 2: C-Kinship as a Multidimensional Space



(a) parent

(b) immediate family

Figure 3: Subspaces of the C-Kinship Semantic Space

Example 1. **Father** entails **parent**

(1) **father** $\mapsto \mathsf{prec} \cap \mathsf{L0} \cap \mathsf{male}$

(2) **parent** $\mapsto \mathsf{prec} \cap \mathsf{L0}$

(3) $\mathsf{prec} \cap \mathsf{L0} \cap \mathsf{male} \subseteq \mathsf{prec} \cap \mathsf{L0}$

Example 2. **Child** entails **immediate family**

(1) **child** $\mapsto \mathsf{succ} \cap \mathsf{L0}$

(2) **immediate family** $\mapsto \mathsf{L0} \cup \mathsf{same} \cap \mathsf{L1}$

(3) $\mathsf{succ} \cap \mathsf{L0} \subseteq \mathsf{L0} \subseteq \mathsf{L0} \cup \mathsf{same} \cap \mathsf{L1}$

Example 3. **Uncle** entails $\neg$ **immediate family**

(1) **uncle** $\mapsto \mathsf{prec} \cap \mathsf{L1} \cap \mathsf{male}$

(2) **immediate family** $\mapsto \mathsf{L0} \cup \mathsf{same} \cap \mathsf{L1}$

(3) $(\mathsf{prec} \cap \mathsf{L1} \cap \mathsf{male}) \cap (\mathsf{L0} \cup \mathsf{same} \cap \mathsf{L1})$
$= (\mathsf{prec} \cap \mathsf{L0} \cap \mathsf{L1} \cap \mathsf{male}) \cup (\mathsf{prec} \cap \mathsf{same} \cap \mathsf{L1} \cap \mathsf{male})$
$= 0$

Figure 4: Entailment as inclusion

3. A Normal Form

An arbitrary subspace is a union of elementary subsets. Clearly, any subspace is a union of atoms. But in general, there are many distinct sets of elementary subsets each having as its union the same subspace. For example, $\{\mathsf{prec}\cap\mathsf{L0}, \mathsf{same}\cap\mathsf{L0}, \mathsf{succ}\cap\mathsf{L0}, \mathsf{same}\cap\mathsf{L1}\}$, $\{(\mathsf{prec}\cup\mathsf{succ})\cap\mathsf{L0}, \mathsf{same}\cap(\mathsf{L0}\cup\mathsf{L1})\}$, $\{\mathsf{L0}, \mathsf{same}\cap\mathsf{L1}\}$, and $\{\mathsf{L0}, \mathsf{same}\cap(\mathsf{L0}\cup\mathsf{L1})\}$ are each a set of elementary subsets whose union is **immediate family**. The last set is special however in that each of its members is maximal.

If x is an arbitrary subspace and y is an elementary subset contained in x, then y is *maximal in* x if no other elementary subset z in x properly contains y. That is, if for every elementary subset $z \subseteq x$, $y \subseteq z \subseteq x$ implies $z = y$, then y is maximal in x.[4]

It is shown in the Appendix that if x is an arbitrary subspace the set of elementary subsets that are maximal in x is unique. Thus any subspace is the union of a unique set of maximal elementary subsets, each of which has a unique standard form. The set of maximal elementary subsets of a subspace therefore constitutes a unique representation or *normal form* for that subspace. Consequently each extension or meaning has a normal form.

Continuing the c-kinship example, **immediate family** has the normal form $\{\mathsf{L0}, \mathsf{same}\cap(\mathsf{L0}\cup\mathsf{L1})\}$. Notice that no elementary subset in **immediate family** properly contains either of the elementary subsets in the normal form. Moreover, every elementary subset in **immediate family** is contained in one of the elementary subsets in the normal form.

The normal form of a subspace x will be denoted $\mathcal{N}(x)$.

Having defined a normal form for subspaces of the multidimensional space of lexical meaning, the next task is to identify useful operations under which the set of normal forms is closed. This will be done by first defining intersection and complement for

[4]It may be helpful for readers familiar with switching theory to think of "maximum elementary subset" as a generalization of "prime implicant."

elementary subsets. Then these operations are generalized to arbitrary subspaces. Finally a union operation is defined. The presentation will continue to be informal. However, the results obtained as well as all subsequent results leading to a Boolean algebra of normal forms are proved in the Appendix.

In the simple case of elementary subsets, geometric intuition may be invoked. Let x and y be elementary subsets with standard forms $\bigcap_{i\in I}\bigcup_{j\in J_i^x} p_i^j$ and $\bigcap_{i\in I}\bigcup_{j\in J_i^y} p_i^j$ respectively. One is easily convinced by geometric considerations that $x \cap y$ is also an elementary subset and moreover that its standard form is $\bigcap_{i\in I}\bigcup_{j\in J_i^x\cap J_i^y} p_i^j$. (See Figure 5 for an example.) That is, intersection of elementary subsets is computed componentwise. For the simple case where x and y are elementary subsets, define $\mathcal{N}(x)\underline{\triangle}\mathcal{N}(y) = \{x\}\underline{\triangle}\{y\} = \{x \cap y\}$.

Now consider the elementary subset $z_i = \bigcup_{j\in(J_i - J_i^x)} p_i^j$. This is the union of hyperplanes, orthogonal to the i-th coordinate axis, that do not intersect the elementary subset x. It is obvious from geometric considerations that $x \cap z_i = 0$ (the *null subspace*). This also follows from the previous result, since for each $i \in I$: $J_i^x \cap (J_i - J_i^x) = \emptyset$. The distributive law holds for the multidimensional space, and therefore $x \cap (\bigcup_{i\in I} z_i) = 0$ as well. Further, $x \cup (\bigcup_{i\in I} z_i) = 1$ (the *unit subspace*, i.e., the denotation of the entire semantic domain under consideration). Thus, $\bigcup_{i\in I} z_i$ is the complement of subspace x. (See Figure 6 for an example.) The complement will be written $-x$. Of course, $-x$ is not in general an elementary subset. But notice that the z_i for $i \in I^x$ are maximal in $-x$ and are irredundant. Therefore, $\{z_i | i \in I^x\} = \mathcal{N}(-x)$. For the special case where x is an elementary subset, define $\sim \mathcal{N}(x) = \{\bigcup_{j\in(J_i - J_i^x)} p_i^j | i \in I^x\}$. Then if x is an elementary subset, $\mathcal{N}(-x) =\sim \mathcal{N}(x)$.

At this point, an intersection operation, $\underline{\triangle}$, and a complement operation, $\sim$, have been defined for elementary subsets.

Next consider arbitrary subspaces x and y with $\mathcal{N}(x) = \{x_1, x_2, \ldots, x_m\}$ and $\mathcal{N}(y) = \{y_1, y_2, \ldots, y_l\}$. Since by definition $x = x_1 \cup x_2 \cup \cdots \cup x_m$ and $y = y_1 \cup y_2 \cup \cdots \cup y_l$, it follows by distributivity that $x \cap y = \bigcup_{1 \leq r \leq m, 1 \leq q \leq l} x_r \cap y_q$. Each of the $x_r \cap y_q$ is an elementary subset. Moreover, the set $\{x_r \cap y_q | 1 \leq r \leq m, 1 \leq q \leq l\}$ contains all the maximal elementary subsets in $x \cap y$. It does not, however, contain *only* the maximal elementary subsets. (For an example, see Figure 7.) Therefore, letting **irr** be the operation that removes subsumed elements, $\mathcal{N}(x \cap y) = \mathbf{irr}\{x_r \cap y_q | 1 \leq r \leq m, 1 \leq q \leq l\}$. Define $\mathcal{N}(x) \underline{\wedge} \mathcal{N}(y) = \mathbf{irr}\{x_r \cap y_q | 1 \leq r \leq m, 1 \leq q \leq l\}$. Then the set of normal forms is closed under $\underline{\wedge}$ and $\mathcal{N}(x \cap y) = \mathcal{N}(x) \underline{\wedge} \mathcal{N}(y)$.

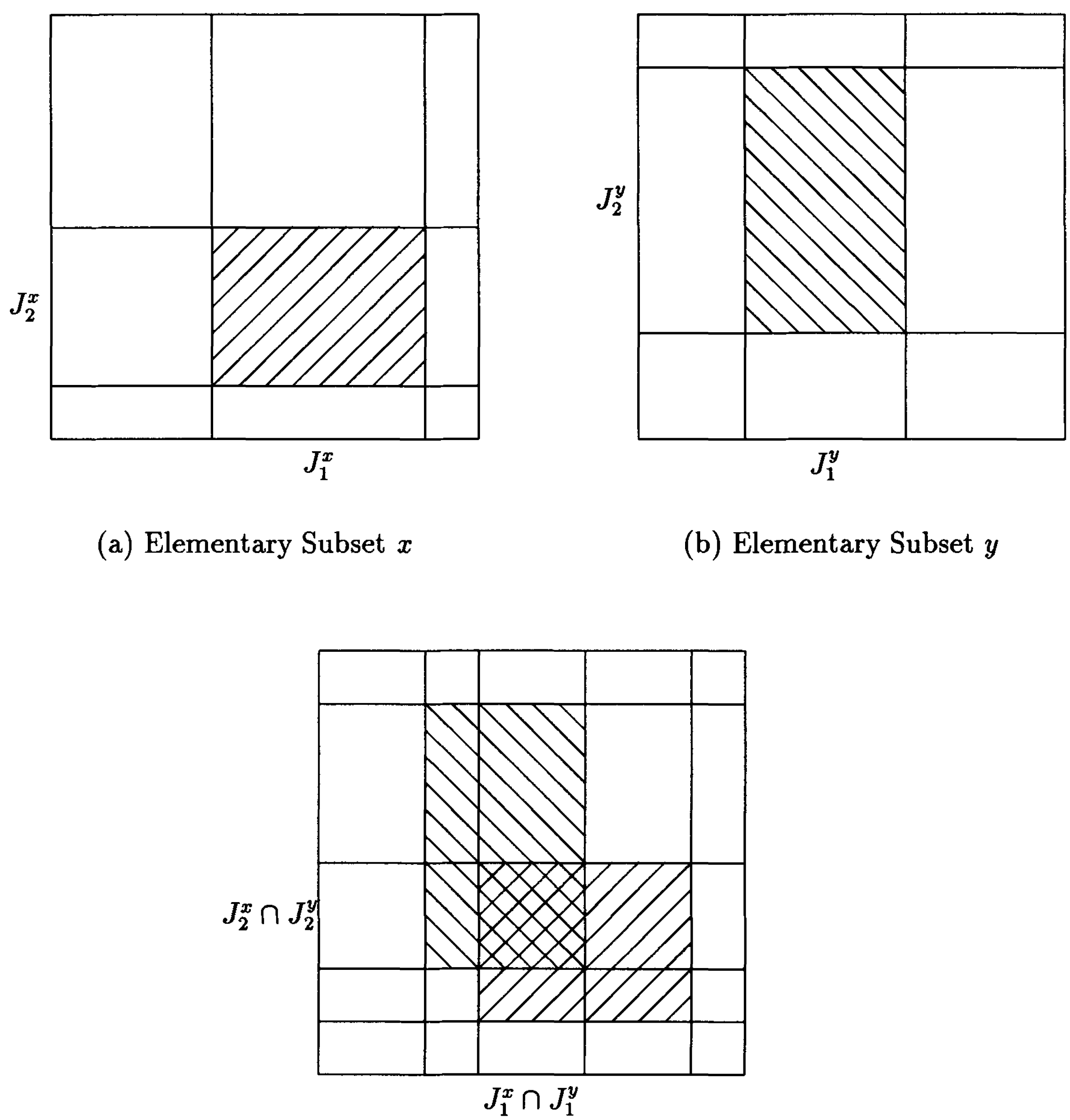
By De Morgan's law, $-x = -x_1 \cap -x_2 \cap \cdots \cap -x_m$, where each $-x_r$ is the complement of an elementary subset, viz., x_r. Applying the result for intersection of normal forms, $\mathcal{N}(-x) = \mathcal{N}(-x_1) \underline{\wedge} \cdots \underline{\wedge} \mathcal{N}(-x_m)$ or $\sim \mathcal{N}(x) = \sim \mathcal{N}(x_1) \underline{\wedge} \cdots \underline{\wedge} \sim \mathcal{N}(x_m)$. Thus $\sim$ is defined for arbitrary subspaces as well as elementary subsets.

Thus the set of normal forms is closed under a complement operation $\sim$ and an intersection operation $\underline{\wedge}$. Next a union operation for normal forms is defined in terms of these operations. Since $x \cup y = -(-x \cap -y)$ by De Morgan's law, $\mathcal{N}(x \cup y) = \sim (\sim \mathcal{N}(x) \underline{\wedge} \sim \mathcal{N}(y))$. Therefore a union operation for normal forms is defined $\mathcal{N}(x) \underline{\vee} \mathcal{N}(y) = \sim (\sim \mathcal{N}(x) \underline{\wedge} \sim \mathcal{N}(y))$.

These results may be summarized as follows. Given a multidimensional space of lexical meaning defined by some basis, the set of normal forms along with operations $\underline{\wedge}$, $\underline{\vee}$ and $\sim$ form a Boolean algebra.

Inclusion between normal forms can be defined: $\mathcal{N}(x) \leq \mathcal{N}(y)$ if and only if $\mathcal{N}(x) \underline{\wedge} \mathcal{N}(y) = \mathcal{N}(x)$. Thus $\mathcal{N}(x) \leq \mathcal{N}(y)$ is equivalent to $x \subseteq y$.

Two examples based on c-kinship will illustrate these operations. (See Figure 7.) Each demonstrates computation of a union of subspaces. In both cases the resulting subspace is immediate family.

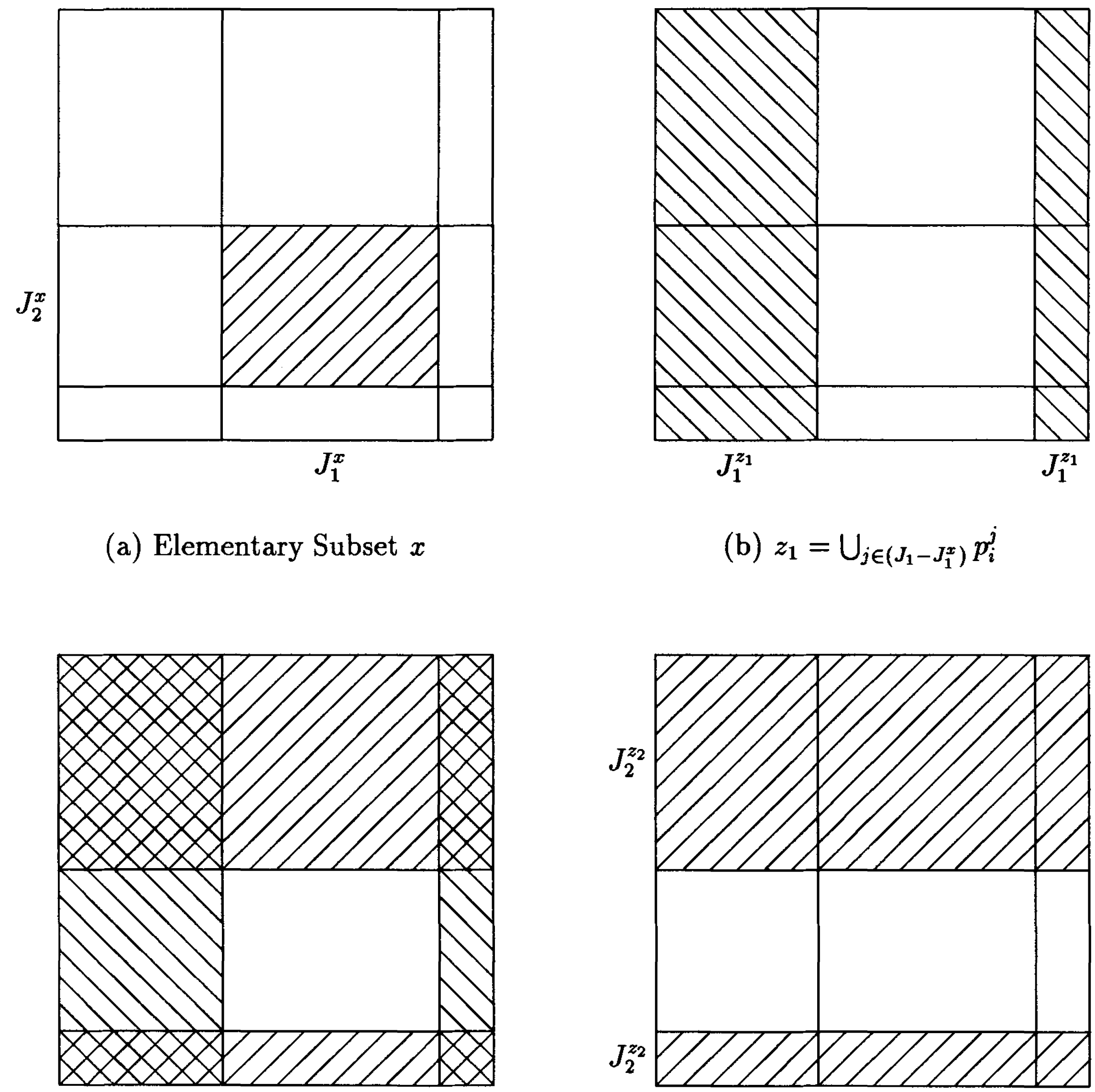


(a) Elementary Subset x

(b) Elementary Subset y

(c) Intersection of x and y

Figure 5: Example of Intersection of Elementary Subsets



(a) Elementary Subset x

(b) $z_1 = \bigcup_{j \in (J_1 - J_1^x)} p_i^j$

(d) $-x = z_1 \cup z_2$

(c) $z_2 = \bigcup_{j \in (J_2 - J_2^x)} p_i^j$

Figure 6: Example of Complement of Elementary Subset

Example 1.

$$
\begin{aligned}
&\text{Let } \mathcal{N}(x) = \{\mathsf{L0}\} \text{ and } \mathcal{N}(y) = \{\mathsf{same} \cap \mathsf{L1}\} \\
&\text{Then } \sim \mathcal{N}(x \cup y) = \{\mathsf{L1} \cup \mathsf{L2}\} \triangle \{\mathsf{prec} \cup \mathsf{succ}, \mathsf{L0} \cup \mathsf{L2}\} \\
&\qquad\qquad\qquad\quad = \mathbf{irr}\{\mathsf{L2}, (\mathsf{prec} \cup \mathsf{succ}) \cap (\mathsf{L1} \cup \mathsf{L2})\} \\
&\text{Hence } \mathcal{N}(x \cup y) \quad = \{\mathsf{L0} \cup \mathsf{L1}\} \triangle \{\mathsf{L0}, \mathsf{same}\} \\
&\qquad\qquad\qquad\quad = \mathbf{irr}\{\mathsf{L0}, \mathsf{same} \cap (\mathsf{L0} \cup \mathsf{L1})\} \\
&\qquad\qquad\qquad\quad = \{\mathsf{L0}, \mathsf{same} \cap (\mathsf{L0} \cup \mathsf{L1})\}
\end{aligned}
$$

The result is the set of maximal elementary subsets of the subspace **immediate family**.

Example 2.

$$
\begin{aligned}
&\text{Let } \mathcal{N}(x) = \{(\mathsf{prec} \cup \mathsf{same}) \cap \mathsf{L0}, \mathsf{same} \cap (\mathsf{L0} \cup \mathsf{L1})\} \text{ and } \mathcal{N}(y) = \{\mathsf{succ} \cap \mathsf{L0}\} \\
&\text{Then } \sim \mathcal{N}(x \cup y) = \{\mathsf{succ}, \mathsf{L1} \cup \mathsf{L2}\} \triangle \{\mathsf{prec} \cup \mathsf{succ}, \mathsf{L2}\} \triangle \{\mathsf{prec} \cup \mathsf{same}, \mathsf{L1} \cup \mathsf{L2}\} \\
&\qquad\qquad\qquad\quad = \mathbf{irr}\{\mathsf{L2}, (\mathsf{prec} \cup \mathsf{same}) \cap \mathsf{L2}, (\mathsf{prec} \cup \mathsf{succ}) \cap (\mathsf{L1} \cup \mathsf{L2}), \\
&\qquad\qquad\qquad\qquad\quad \mathsf{prec} \cap (\mathsf{L1} \cup \mathsf{L2}), \mathsf{succ} \cap \mathsf{L2}, \mathsf{succ} \cap (\mathsf{L1} \cup \mathsf{L2})\} \\
&\qquad\qquad\qquad\quad = \{\mathsf{L2}, (\mathsf{prec} \cup \mathsf{succ}) \cap (\mathsf{L1} \cup \mathsf{L2})\} \\
&\text{Hence } \mathcal{N}(x \cup y) \quad = \{\mathsf{L0} \cup \mathsf{L1}\} \triangle \{\mathsf{same}, \mathsf{L0}\} \\
&\qquad\qquad\qquad\quad = \mathbf{irr}\{\mathsf{L0}, \mathsf{same} \cap (\mathsf{L0} \cup \mathsf{L1})\} \\
&\qquad\qquad\qquad\quad = \{\mathsf{L0}, \mathsf{same} \cap (\mathsf{L0} \cup \mathsf{L1})\}
\end{aligned}
$$

Again the result is the normal form of subspace **immediate family**.

Figure 7: Boolean Operations on Normal Forms

4. The Lexicon

Given a set of lexical items, such as the words denoting c-kinship, distinguishing lexical features can be determined by linguistic analysis. These lexical features can then be organized into sets whose denotations partition the universe modeling the lexical items. It is possible to select a subset of these partitions that has the property of independence. Such a set is a basis. It structures the universe to yield a multidimensional space. Subspaces of the multidimensional space are uniquely represented by normal forms, for which a Boolean algebra can be defined. The multidimensional space so formed will be called a *semantic space.* A general algorithm for basis construction is presented in the Appendix.

The structure of a semantic space can be encoded using the index sets $\{J_i | i \in I\}$. For example, the standard form (or abbreviated standard form) for an elementary subset x can be encoded as a sequence of binary strings, the i-th string representing J_i^x. The normal form for an arbitrary subspace y can be encoded as the sequence of codes for its maximal elementary subsets in lexical order.

Linguistic analysis provides definitions of the lexical items in terms of (specifically, as Boolean functions of) the lexical features. These definitions can be used to define a mapping from lexical items to normal forms (or codes for the normal forms) of the semantic space. This mapping will be called a *lexicon* for the vocabulary of lexical items.

Let the mapping be denoted v. Then the following definitions can be made. Relative to the basis that defines the semantic space, lexical items x and y are synonymous if and only if $v(x) = v(y)$; x and y are contradictory if and only if $v(x) \triangle v(y) = 0$; x entails y if and only if $v(x) \leq v(y)$, that is, if and only if $v(x) \triangle v(y) = v(x)$ or equivalently, $v(x) \triangle \sim v(y) = 0$.

v can be extended to Boolean expressions over lexical items (of the same type) by defining $v(x\ \mathbf{or}\ y) = v(x)\underline{\vee}v(y)$, $v(x\ \mathbf{and}\ y) = v(x)\underline{\Delta}v(y)$, and $v(\mathbf{not}\ x) =\sim v(x)$.

Definition of a lexicon for c-kinship is given in Figure 8.

It is to be noted that the basis selected for the semantic space will determine the precision of the meanings associated with the lexical items. Therefore, meaning equivalence and meaning inclusion are understood relative the basis. Equivalence or inclusion relative to a given basis may not hold relative to a refinement of that basis. Thus a notion of learning or development is inherent in this theory.

While this approach to lexical semantics seems to have a desirable simplicity, its expressiveness is limited relative to that of first-order logic. For example, logic permits assertions such as $\mathsf{parent}(x, y) \leftrightarrow \mathsf{child}(y, x)$ and $\mathsf{uncle}(x, y) \rightarrow \exists z[\mathsf{brother}(x, z)]$. A semantic space cannot explicitly represent such knowledge. However, as the next definition of c-kinship demonstrates, it is sometimes possible to implicitly represent such knowledge.

Consider a set $S \subseteq H \times H$ comprising three generations of blood kin. For $i = 1, 2, 3$, define:

$\mathsf{L}i = \{(x, y) \in S |$the join of x and y in the family tree is a distance i from $x\}$

$\mathsf{R}i = \{(x, y) \in S |$the join of x and y in the family tree is a distance i from $y\}$

It will be assumed that S is partitioned by $P_1 = \{\mathsf{L0}, \mathsf{L1}, \mathsf{L2}\}$, $P_2 = \{\mathsf{R0}, \mathsf{R1}, \mathsf{R3}\}$ and $P_3 = \{\mathsf{male}, \mathsf{female}\}$. As a consequence, $B = \{P_1, P_2, P_3\}$ is a basis of S. The semantic space is shown in Figure 9.

This basis defines a space that is better than the first one in several ways. First, the meanings are grouped more simply: cousin occupies just two atoms; immediate family is now an elementary subset, viz., $(\mathsf{L0} \cup \mathsf{L1}) \cap (\mathsf{R0} \cup \mathsf{R1})$. Second, $\mathsf{L}i \cap \mathsf{R}j$ is converse to $\mathsf{L}j \cap \mathsf{R}i$. For example, $\mathsf{L1} \cap \mathsf{R2}$ is the extension of **uncle or aunt**. The converse c-kinship relation is **nephew or niece** which has the extension $\mathsf{L2} \cap \mathsf{R1}$. Thus knowledge about converse c-kinship relations is implicit in this semantic space. Third, $\mathsf{L}i \cap \mathsf{R}j$ where $i \neq 0 \neq j$ implies the existence of a sibling relation.

The basis defining this space and the underlying linguistic analysis seem to more fully represent the meanings of c-kinship relations. It is likely that a similar circumstance will obtain in most semantic domains. Therefore, the selection of lexical features underlying construction of a lexicon would appear to require experience and good judgment.

$B = \{P_1, P_2, P_3\}$
$P_1 = \{\mathsf{prec}, \mathsf{same}, \mathsf{succ}\}$
$P_2 = \{\mathsf{L0}, \mathsf{L1}, \mathsf{L2}\}$
$P_3 = \{\mathsf{male}, \mathsf{female}\}$

v: **father** $\mapsto \mathsf{prec} \cap \mathsf{L0} \cap \mathsf{male}$
mother $\mapsto \mathsf{prec} \cap \mathsf{L0} \cap \mathsf{female}$
uncle $\mapsto \mathsf{prec} \cap \mathsf{L1} \cap \mathsf{male}$
aunt $\mapsto \mathsf{prec} \cap \mathsf{L1} \cap \mathsf{female}$
brother $\mapsto \mathsf{same} \cap \mathsf{L1} \cap \mathsf{male}$
sister $\mapsto \mathsf{same} \cap \mathsf{L1} \cap \mathsf{female}$
son $\mapsto \mathsf{succ} \cap \mathsf{L0} \cap \mathsf{male}$
daughter $\mapsto \mathsf{succ} \cap \mathsf{L0} \cap \mathsf{female}$
nephew $\mapsto \mathsf{succ} \cap \mathsf{L1} \cap \mathsf{male}$
niece $\mapsto \mathsf{succ} \cap \mathsf{L1} \cap \mathsf{female}$
cousin $\mapsto \mathsf{L2}$
self $\mapsto \mathsf{same} \cap \mathsf{L0}$
parent $\mapsto \mathsf{prec} \cap \mathsf{L0}$
child $\mapsto \mathsf{succ} \cap \mathsf{L0}$
sibling $\mapsto \mathsf{same} \cap \mathsf{L1}$
immediate family $\mapsto \mathsf{L0} \cup \mathsf{same} \cap \mathsf{L1}$

Figure 8: Lexicon for C-Kinship

	male			female		
L0	self	father	gfather	self	mother	gmother
L1	son	brother	uncle	daughter	sister	aunt
L2	grandson	nephew	cousin	gdaughter	niece	cousin
	R0	R1	R2	R0	R1	R2

Figure 9: A Second Basis for C-Kinship

5. Extended Bases

Each of the bases considered thus far consists of a single set of partitions. In the general case, a basis consists of several sets of partitions. The former are called *simple*, the latter *extended* bases. In this section, the way in which extended bases arise and their structure will be shown with the help of an extension of consanguineal kinship to include partial consanguineal relations.

The bases considered thus far cannot represent half blood relationships, for example, **half-brother**. The extension of c-kinship to include new lexical items denoting half blood relationships will be referred to as extended consanguineal kinship or ec-kinship. New features must be defined sufficient to differentiate between half and full blood relationships. This will be accomplished by specifying not only the length of the path from individual x to a nearest common ancestor of individuals x and y, but also the kinds of ancestors on that path. For example,

$\mathsf{LMP} = \{(x,y) \in S|$the path from x to a nearest common ancestor of x and y contains x, the mother of x and the maternal grandfather of x; and there is no other path of length 2$\}$

$\mathsf{LMB} = \{(x,y) \in S|$the paths from x to nearest common ancestors of x and y contains x, the mother of x and both maternal grandparents of $x\}$

$\mathsf{LP} = \{(x,y) \in S|$the path from x to a nearest common ancestor of x and y contains x and the father of x; and there is no other path of length 1$\}$

$\mathsf{L} = \{(x,y) \in S|$the join of x and y is $x\}$

These features form a partition $P_1 = \{\mathsf{L,LM,LP,LB,LMM,LMP,LMB,LPM,LPP,LPB}\}$. A partition $P_2 = \{\mathsf{R,RM,RP,RB,RMM,RMP,RMB,RPM,RPP,RPB}\}$ is defined analogously for the right member y. A third partition is $P_3 = \{\mathsf{male,female}\}$. The subdivision of S produced by these partitions is shown in Figure 10.

It is apparent from the figure that these partitions are not independent since many male | female

	R	RM	RP	RB	RMM	RMP	RMB	RPM	RPP	RPB
L	self	∅	father	∅	∅	matern gfather	∅	∅	patern gfather	∅
LM	son	half brother	∅	∅	half uncle	∅	∅	half uncle	∅	∅
LP	son	∅	half brother	∅	∅	half uncle	∅	∅	half uncle	∅
LB	∅	∅	∅	brother	∅	∅	uncle	∅	∅	uncle
LMM	gson	half nephew	∅	∅	half cousin	∅	∅	half cousin	∅	∅
LMP	gson	∅	half nephew	∅	∅	half cousin	∅	∅	half cousin	∅
LMB	∅	∅	∅	nephew	∅	∅	cousin	∅	∅	cousin
LPM	gson	half nephew	∅	∅	half cousin	∅	∅	half cousin	∅	∅
LPP	gson	∅	half nephew	∅	∅	half cousin	∅	∅	half cousin	∅
LPB	∅	∅	∅	nephew	∅	∅	cousin	∅	∅	cousin

Figure 10: Partitions of the Ec-Kinship Universe

of the cells are empty. Therefore they do not form a basis of S. P_1 and P_3 are independent, but neither P_1 and P_2 nor P_1, P_2 and P_3 are. The result is that distinct standard forms do not represent distinct elementary subsets. For example, $\mathsf{father} = \mathsf{L} \cap \mathsf{RP} \cap \mathsf{male} = (\mathsf{L} \cup \mathsf{LM}) \cap \mathsf{RP} \cap \mathsf{male} = (\mathsf{L} \cup \mathsf{LM}) \cap (\mathsf{RP} \cup \mathsf{RB}) \cap \mathsf{male} = \cdots$.

To remedy this, a basis is formed from P_1 and P_3. $B = \{P_1, P_3\}$ will be called the *first level basis.* Next each subdivision defined by B is examined. These subdivisions are called the *atoms defined by* B. Consider the atom $a_7 = \mathsf{LB} \cap \mathsf{male}$. Blocks of P_2 that have nonempty intersection with this atom are RB, RMB and RPB. Moreover, {RB,RMB,RPB} partitions a_7. $B_7 = \{\{\mathsf{RB},\mathsf{RMB},\mathsf{RPB}\}\}$ will be called a *second level basis.* Each of the atoms defined by B may have a basis. In the present example the second level bases are denoted $B_1, B_2, \ldots, B_{20}$. There are only two levels. The subdivision produced by this system of bases is shown in Figure 11.

The collection $\{B, B_1, B_2, \ldots, B_{20}\}$ will be referred to as an *extended basis* of S. An extended basis can be indexed by a tree domain. That is, the bases may be viewed as labels on the nodes of a tree whose root has the first level basis as its label.

Such an embedding of semantic spaces is typical. A simple example is the following. The domain of physical entities might be partitioned by $P_1 = \{\mathsf{animal},\mathsf{vegetable},\mathsf{mineral}\}$ and again by $P_2 = \{\mathsf{count},\mathsf{mass}\}$. Assuming that every combination is possible, $\{P_1, P_2\}$ is a first level basis of the domain, defining six atoms: $\mathsf{animal} \cap \mathsf{count}$, $\mathsf{animal} \cap \mathsf{mass}$, ..., $\mathsf{mineral} \cap \mathsf{mass}$. Each atom is itself a domain and can be partitioned by attributes appropriate to it. Hence each atom has a (in general distinct) basis. This subdivision can continue through a number of levels. Even the simple taxonomy of Schubert's Steamroller [10] contains at least two levels.

While the extended basis shown in Figure 11 does indeed yield a multidimensional space, it does not structure the subspaces neatly. For example,
$\mathsf{half\text{-}cousin} = [(\mathsf{LMM} \cap (\mathsf{male} \cup \mathsf{female})) \cap (\mathsf{RMM} \cup \mathsf{RPM})] \cup [(\mathsf{LMP} \cap (\mathsf{male} \cup \mathsf{female})) \cap$

<table>
<tr><th></th><th colspan="24">male</th><th colspan="24">female</th></tr>
<tr><td>L</td><td colspan="6">R
self</td><td colspan="6">RP
father</td><td colspan="6">RMP
matern gfather</td><td colspan="6">RPP
patern gfather</td><td colspan="6">R
self</td><td colspan="6">RM
mother</td><td colspan="6">RMM
matern gmother</td><td colspan="6">RPM
patern gmother</td></tr>
<tr><td>LM</td><td colspan="6">R
son</td><td colspan="6">RM
half brother</td><td colspan="6">RMM
half uncle</td><td colspan="6">RPM
half uncle</td><td colspan="6">R
daughter</td><td colspan="6">RM
half sister</td><td colspan="6">RMM
half aunt</td><td colspan="6">RPM
half aunt</td></tr>
<tr><td>LP</td><td colspan="6">R
son</td><td colspan="6">RP
half brother</td><td colspan="6">RMP
half uncle</td><td colspan="6">RPP
half uncle</td><td colspan="6">R
daughter</td><td colspan="6">RP
half sister</td><td colspan="6">RMP
half aunt</td><td colspan="6">RPP
half aunt</td></tr>
<tr><td>LB</td><td colspan="8">RB
brother</td><td colspan="8">RMB
uncle</td><td colspan="8">RPB
uncle</td><td colspan="8">RB
sister</td><td colspan="8">RMB
aunt</td><td colspan="8">RPB
aunt</td></tr>
<tr><td>LMM</td><td colspan="6">R
gson</td><td colspan="6">RM
half nephew</td><td colspan="6">RMM
half cousin</td><td colspan="6">RPM
half cousin</td><td colspan="6">R
gdau'ter</td><td colspan="6">RM
half niece</td><td colspan="6">RMM
half cousin</td><td colspan="6">RPM
half cousin</td></tr>
<tr><td>LMP</td><td colspan="6">R
gson</td><td colspan="6">RP
half nephew</td><td colspan="6">RMP
half cousin</td><td colspan="6">RPP
half cousin</td><td colspan="6">R
gdau'ter</td><td colspan="6">RP
half niece</td><td colspan="6">RMP
half cousin</td><td colspan="6">RPP
half cousin</td></tr>
<tr><td>LMB</td><td colspan="8">RB
nephew</td><td colspan="8">RMB
cousin</td><td colspan="8">RPB
cousin</td><td colspan="8">RB
niece</td><td colspan="8">RMB
cousin</td><td colspan="8">RPB
cousin</td></tr>
<tr><td>LPM</td><td colspan="6">R
gson</td><td colspan="6">RM
half nephew</td><td colspan="6">RMM
half cousin</td><td colspan="6">RPM
half cousin</td><td colspan="6">R
gdau'ter</td><td colspan="6">RM
half niece</td><td colspan="6">RMM
half cousin</td><td colspan="6">RPM
half cousin</td></tr>
<tr><td>LPP</td><td colspan="6">R
gson</td><td colspan="6">RP
half nephew</td><td colspan="6">RMP
half cousin</td><td colspan="6">RPP
half cousin</td><td colspan="6">R
gdau'ter</td><td colspan="6">RP
half niece</td><td colspan="6">RMP
half cousin</td><td colspan="6">RPP
half cousin</td></tr>
<tr><td>LPB</td><td colspan="8">RB
nephew</td><td colspan="8">RMB
cousin</td><td colspan="8">RPB
cousin</td><td colspan="8">RB
niece</td><td colspan="8">RMB
cousin</td><td colspan="8">RPB
cousin</td></tr>
</table>

Figure 11: Ec-Kinship as a Multidimensional Space

$(\mathsf{RMP} \cup \mathsf{RPP})] \cup [(\mathsf{LPM} \cap (\mathsf{male} \cup \mathsf{female})) \cap (\mathsf{RMM} \cup \mathsf{RPM})] \cup [(\mathsf{LPP} \cap (\mathsf{male} \cup \mathsf{female})) \cap (\mathsf{RMP} \cup \mathsf{RPP})]$.

A similar deficiency was found in the first basis for c-kinship:
$\{\{\mathsf{prec,same,succ}\}, \{\mathsf{L0,L1,L2}\}, \{\mathsf{male,female}\}\}$.
The alternative basis
$\{\{\mathsf{L0,L1,L2}\}, \{\mathsf{R0,R1,R2}\}, \{\mathsf{male,female}\}\}$
yielded a neater structure. This latter basis can be taken as a first level basis and refined by second level bases to distinguish between half and full blood relationships. The resulting extended basis for ec-kinship is:

$B = \{\{\mathsf{L0,L1,L2}\}, \{\mathsf{R0,R1,R2}\}, \{\mathsf{male,female}\}\}$

$B_3 = \{\{\mathsf{RMP,RPP}\}\}$

$B_5 = \{\{\mathsf{LM,LP,LB}\}\}$

$B_6 = \{\{\mathsf{LM,LP,LB}\}, \{\mathsf{RMX,RPX}\}\}$

$B_8 = \{\{\mathsf{LMM,LMP,LMB,LPM,LPP,LPB}\}\}$

$B_9 = \{\{\mathsf{LMM,LMP,LMB,LPM,LPP,LPB}\}, \{\mathsf{RMX,RPX}\}\}$

where $\mathsf{RMX} = \mathsf{RMM} \cup \mathsf{RMP} \cup \mathsf{RMB}$ and similarly for RPX. The modified multidimensional space is shown in Figure 12.

Relative to this basis,
$\mathsf{half\text{-}cousin} = (\mathsf{L2} \cap \mathsf{R2} \cap (\mathsf{male} \cup \mathsf{female})) \cap (\mathsf{LMM} \cup \mathsf{LMP} \cup \mathsf{LPM} \cup \mathsf{LPP})$.

It should be pointed out that all the results stated earlier for a simple basis hold as well for an extended basis. Each subspace has a normal form. The Boolean operations (suitably extended to observe the embedded structure of the multidimensional space) and the set of normal forms yield a Boolean algebra. See the Appendix for definitions and proofs.

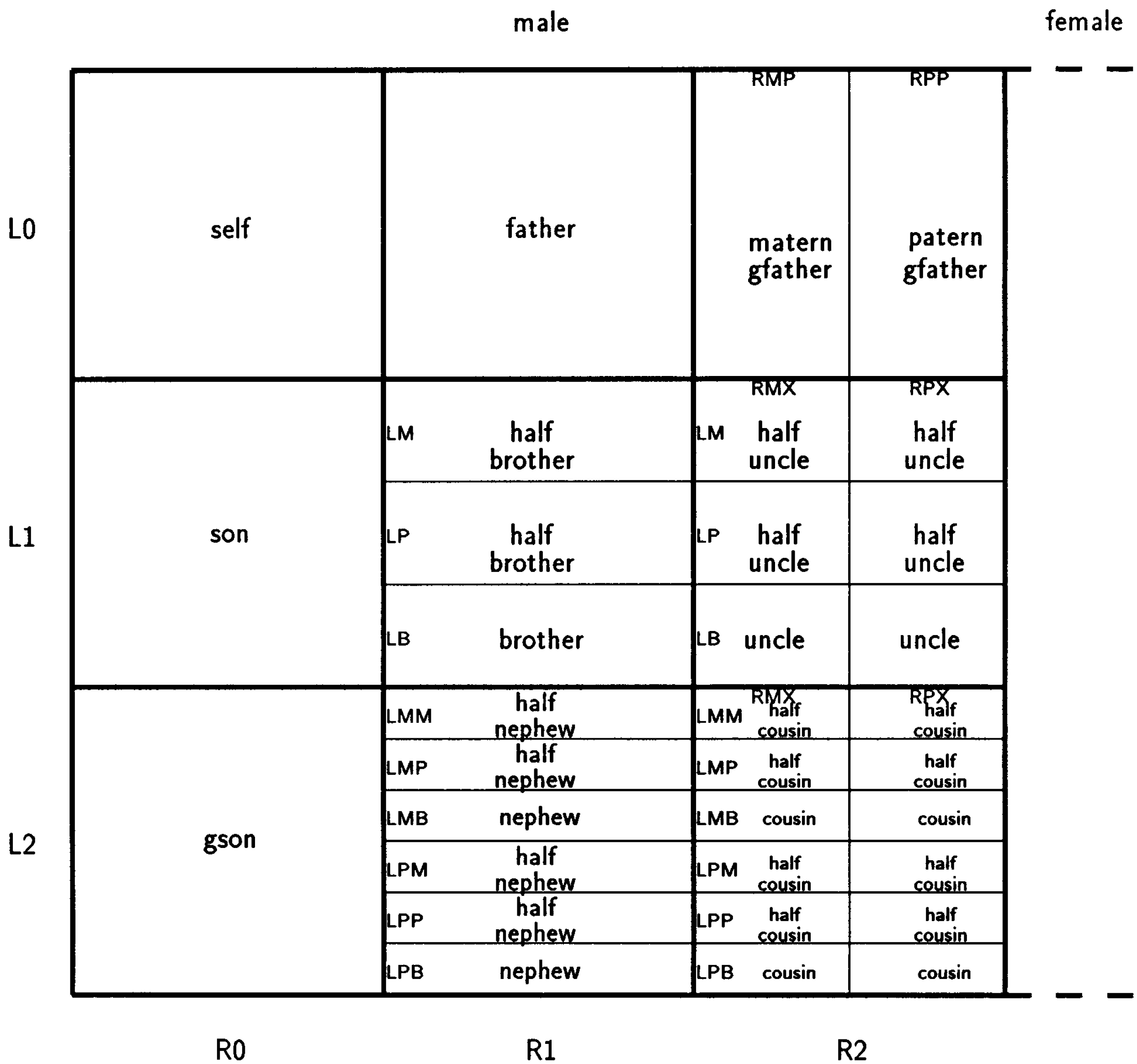


Figure 12: Another Basis for Ec-Kinship

6. Discussion

The theory developed in the preceding sections can be adapted readily to applications other than lexical semantics. In general, this simply involves restriction of the theory. For example, sorts are interpreted as a subsets of the model universe. They therefore correspond to the nominal domains used to illustrate lexical semantics. In the simplest systems sorts are constrained to form a hierarchy [11]. In more complex systems sorts form partially ordered sets or complete lattices [2]. The subsort relation corresponds to meaning inclusion. The most general common subsort or meet of two sorts corresponds to the conjunction of lexical items. Construction of a sort semantic space and a sort lexicon proceeds exactly as in the case of lexical semantics.

The advantage enjoyed by many-sorted logic has been demonstrated with problems such as "Schubert's Steamroller" [10] which, although challenging to single-sorted theorem provers, are nonetheless relatively small. With small problems, the sort computations are not burdensome no matter how performed. Typically it is adequate to make the taxonomy part of the problem statement. Consequently the usefulness of the approach presented here is not apparent. However with much larger real-world problems this solution is no longer feasible. Occurring as part of the "inner loop," sort computations will constitute a significant computational burden.

What is needed is an encapsulated subsystem or "black box" dedicated to reasoning about the particular taxonomy. Subsystems of this kind have been suggested by Stickel under the heading of *theory resolution* [9]. The design of such subsystems is precisely the concern of this paper.

An important consideration is the complexity of the structures and operations involved. It follows immediately from Appendix A3 that construction of a basis for a semantic domain is NP-hard since the Boolean Satisfiability Problem (SAT) reduces to the problem of basis construction. In this regard, semantic spaces as representations for taxonomies fare no worse than logic or semantic nets. By the same reasoning, computation of the normal form of an *arbitrary* expression is also NP-hard. Nonetheless, some useful computations are of polynomial complexity. In particular, determination of subsumption (subsort, meaning inclusion) involves checking that each component of the code for the subsumed taxon is a subset of the corresponding component of the code for the subsuming taxon. Therefore this computation is of order n, where n is the dimension of the semantic space. Similarly, computation of the code for the meet of two taxa involves componentwise intersection of their codes, also of order n.

The conclusion with regard to basis construction can be ameliorated by two further observations. First, exponentially complex computations are infeasible only if the size of the input is large. Indeed an exponential computation may be more efficient than a polynomial computation on small inputs. This is the case with many human capabilities. Perhaps because many applications of taxonomic reasoning are related to human activities, the individual bases and associated semantic subspaces that arise in these applications tend to be small. Second, the construction of a basis is performed only once.

The situation is similar to compiling a program written in a higher level language. Compilation is in general more complex than interpretive execution of the program. But execution of the compiled image is much less complex than the interpretive execution. Therefore a somewhat higher one-time cost is accepted in exchange for a significantly lower recurring cost.

Another important consideration is parallelization of the operations. This consideration differentiates semantic spaces from most other representations. The independence of the dimensions of a semantic space makes the operations inherently parallel. As a result, this approach to taxonomic reasoning is well matched to the resources of advanced computer systems. The decomposition of the operations resulting from the independence property also closely relates semantic spaces to connectionist theory and design.

Appendix

This appendix formalizes the definitions given in the body of the paper, and gives proofs for the claims made there. The first section deals with the semantic space as a model for lexical domains. The second section defines the normal form of subspaces and develops a Boolean algebra of normal forms. The third section shows how extended bases can be constructed.

A1. Semantic Space

Let S be a nonempty set and Su be the power set of S. Su is viewed as the set of properties of the members of S.

DEFINITION 1 *Let* $\{p^j | j \in J\}$ *be a subset of* Su *and* P *be a function from* 2^J *into* Su *such that for any* $J' \subseteq J$, $P(J') := \bigcup\{p^j | j \in J'\}$.[5] *If* $j \in J$, $P(\{j\})$ *is written* $P(j)$. P *is called a* partition of S *if it satisfies:*
(i) $P(J) = S$
(ii) $P(J') = \emptyset$ *iff* $J' = \emptyset$
(iii) $\forall J', J'' \in 2^J : P(J') \cap P(J'') = P(J' \cap J'')$

This definition is equivalent to the one used in the body of the paper. It is introduced here because it results in more succinct expressions. However, where convenient the usual notation $P = \{p^j | j \in J\}$ will also be used.

DEFINITION 2 *Let* $B = \{P_i | i \in I_B\}$ *be a set of partitions of* S. *A subset* x *of* S *is called an* elementary subset of S defined by B *if it can be written* $x = \bigcap_{i \in I_B^x} P_i(J_i^x)$ *where* $I_B^x \subseteq I_B$ *is finite.*

Notice that $\bigcap_{i \in I_B^x} P_i(J_i^x) = \bigcap_{i \in I_B} P_i(J_i^x)$ where $\forall i \notin I_B^x : J_i^x := J_i$. The latter form is

[5] The notation "$X := Y$" means that X is defined to be equal to Y; "$X :\Leftrightarrow Y$" means that X is defined to be logically equivalent to Y; etc.

called the *standard form for x relative to B*. The former form is called the *abbreviated standard form for x relative to B*. The conjunct $P_i(J_i^x)$ is called the ith *component* of x. The set of all elementary subsets defined by B is denoted Es_B.

LEMMA 3 *Let $B = \{P_i | i \in I_B\}$ be a set of partitions of S and $x = \bigcap_{i \in I_B} P_i(J_i^x)$, $y = \bigcap_{i \in I_B} P_i(J_i^y)$ be elementary subsets of S defined by B. Then $x \cap y$ is an elementary subset and $x \cap y = \bigcap_{i \in I_B} P_i(J_i^x \cap J_i^y)$.*

proof: $x \cap y = (\bigcap_{i \in I_B} P_i(J_i^x)) \cap (\bigcap_{i \in I_B} P_i(J_i^y)) = \bigcap_{i \in I_B}(P_i(J_i^x) \cap P_i(J_i^y)) = \bigcap_{i \in I_B} P_i(J_i^x \cap J_i^y)$ by Definition 1. □

Thus intersection of elementary subsets is computed componentwise. Since Es_B is closed under set intersection, it forms a meet semilattice, ordered by set inclusion, denoted $\mathbf{Es_B}$. It has the zero element $\emptyset$ and the unit element S, denoted 0 and 1 respectively.

Let Su_B be the closure of Es_B under finite set union. Then Su_B forms a lattice, denoted $\mathbf{Su_B}$. Since it is a sublattice of the subset lattice formed by Su, it is distributive. $\mathbf{Es_B}$ is embedded as a meet semilattice in $\mathbf{Su_B}$.

DEFINITION 4 *Let $B = \{P_i | i \in I_B\}$ be a set of partitions of S. B is called a* basis of S *if $\forall x = \bigcap_{i \in I_B^x} P_i(J_i^x) \in Es_B : x = 0$ iff $\exists i \in I_B^x : J_i^x = \emptyset$.*

LEMMA 5 *Let B be a basis of S and $x = \bigcap_{i \in I_B} P_i(J_i^x)$ be a nonzero elementary subset. Let $q \in I_B$ and $r \in J_q$. Then $P_q(r) \cap x \neq 0$ iff $r \in J_q^x$.*

proof: Since B is a basis, $x \neq 0$ iff $\forall i \in I_B : J_i^x \neq \emptyset$. Then $P_q(r) \cap x \neq 0$ iff $\{r\} \cap J_q^x \neq \emptyset$, ie., iff $r \in J_q^x$. □

THEOREM 6 *Let B be a basis of S and $x = \bigcap_{i \in I_B} P_i(J_i^x)$ be a nonzero elementary subset. Then the standard form for x relative to B is unique. It follows that the abbreviated standard form for x is unique as well.*

proof: Suppose that $\bigcap_{i \in I_B} P(J_i^1)$ and $\bigcap_{i \in I_B} P(J_i^2)$ are standard forms for x. Let $q \in I_B$ and $r \in (J_q^1 \oplus J_q^2)$. By Lemma 5, $r \in J_q^1$ iff $P_q(r) \cap x \neq 0$ iff $r \in J_q^2$. Therefore $J_q^1 \oplus J_q^2 = \emptyset$ and the two standard forms are identical. □

LEMMA 7 *Let B be a basis of S and $x = \bigcap_{i \in I_B} P_i(J_i^x)$, $y = \bigcap_{i \in I_B} P_i(J_i^y)$ be nonzero elementary subsets of S defined by B. Then $x \subseteq y$ iff $\forall i \in I_B : J_i^x \subseteq J_i^y$. Equivalently, $x \subseteq y$ iff $I_B^y \subseteq I_B^x \wedge \forall i \in I_B^y : J_i^x \subseteq J_i^y$.*

proof: $x \subseteq y$ iff $x \cap y = x$. $x \cap y = \bigcap_{i \in I_B} P_i(J_i^x \cap J_i^y)$ by Lemma 3. Since the standard form is unique (Theorem 6), $\forall i \in I_B : J_i^x \cap J_i^y = J_i^x$. I.e., $J_i^x \subseteq J_i^y$. □

EXAMPLE. Let $S = \mathbf{N}$, the non-negative integers. Let $P_1 = \{\{i | i = 0, mod4\}, \{i | i = 1, mod4\}, \{i | i = 2, mod4\}, \{i | i = 3, mod4\}\}$ and $P_2 = \{\{i | \textit{is-prime}(i)\}, \{i | \neg \textit{is-prime}(i)\}\}$. Then P_1 and P_2 are partitions of S. But note that $p_1^1 \cap p_2^1 = 0$ since the conjunction $i = 0, mod4 \wedge \textit{is-prime}(i)$ is logically impossible. Thus, while P_1 and P_2 are partitions of S, $\{P_1, P_2\}$ is not a basis of S.

EXAMPLE. Let $S = \mathbf{N}_+$, the positive integers, and let π_i denote the ith prime. Let $B = \{P_i | i \in I_B\}$, where $I_B = \mathbf{N}_+$. Let $P_i = \{p_i^j | j \in J_i\}$ where $J_i = \mathbf{N}$. Let $p_i^j = \{n \in S | divides(\pi_i^j, n) \wedge \neg divides(\pi_i^{j+1}, n)\}$ for $j \neq 0$, and $p_i^0 = \{n \in S | \neg divides(\pi_i, n)\}$. Then B is a basis of S.

If x and y are elements of $\mathbf{Es_B}$, y *covers* x, written $x \prec y$, iff $\forall z \in \mathbf{Es_B} : x < z \leq y$ implies $z = y$. x is an *atom* iff $0 \prec x$.

It is not necessary that atoms exist in $\mathbf{Es_B}$. In the second example above, $\mathbf{Es_B}$ has no atoms.

Let P be a partition of $Y \subseteq S$ and let $X \subseteq Y$. Define the restriction of P to X: $P\upharpoonright_X (J^1) := P(J^1) \cap X$. Note that $P\upharpoonright_X$ may fail to be a partition of X because it does not satisfy the conditions of Definition 1. Let $B = \{P_i | i \in I_B\}$ be a basis of Y. Define the restriction of B to X: $B\upharpoonright_X := \{P_i \upharpoonright_X | i \in I_B\}$. $B\upharpoonright_X$ may fail to be a basis of X because some $P_i\upharpoonright_X$ is not a partition of X or because the condition of Definition 4 is not satisfied.

Let Es_B be the set of elementary subsets defined by basis B and a_1, a_2 be atoms of $\mathbf{Es_B}$. Let B' be a basis of $X \subseteq S$ such that $B' \cap B = \emptyset$. Suppose that $B'\upharpoonright_{a_1}$ is a basis of a_1 but $B'\upharpoonright_{a_2}$ is not a basis of a_2. It may be that B' defines properties that are relevant to members of a_1 but inconsistent with members of a_2. For example, properties peculiar to animate entities would be inconsistent if applied to inanimate entities.

Let $B'\upharpoonright_{a_1} = B_1$. B_1 determines a semilattice of elementary subsets, $\mathbf{Es_{B_1}}$, with unit element a_1. B and B' together determine a combined semilattice $\mathbf{Es}_{\mathcal{B}}$, where $\mathbf{Es_B}$ is embedded in the interval $[0, 1]$ and $\mathbf{Es_{B_1}}$ is embedded in the interval $[0, a_1]$ such that the covering relation is preserved for all nonzero elements.

EXAMPLE. Let $B = \{P_1, P_2\}$, $P_1 = \{NT, T\}$, $P_2 = \{NP, P\}$. Suppose that $B' = \{Q_1, Q_2\}$, where $Q_1 = \{SL, PH\}$ and $Q_2 = \{NTT, TT\}$, and that $B'\upharpoonright_{a_3}$ and $B'\upharpoonright_{a_4}$ are bases of a_3 and a_4, respectively. Suppose further that $B'\upharpoonright_{a_1}$ and $B'\upharpoonright_{a_2}$ are not bases. The resulting partitions of S form three bases: one *first level* basis and two *second level* bases. They can be diagrammed as shown in Figure 13.[6]

This situation is generalized as follows. Let T be a tree indexing defined in the usual way: (i) $T \subset \mathbf{N}_+^*$, where $\mathbf{N}_+$ denotes the positive integers and $*$ denotes the Kleene closure; (ii) $\alpha, \beta \in \mathbf{N}_+^*$ and $\alpha.\beta \in T$ implies $\alpha \in T$; (iii) $\alpha \in \mathbf{N}_+^*$, $b \in \mathbf{N}_+$ and $\alpha.b \in T$

[6]This example is part of an example in [6] dealing with a taxonomy of rigid fasteners. The distinguishing properties are: not threaded (NT), threaded (T), not pointed (NP), pointed (P), slot drive (SL), Phillips drive (PH), not threaded to top (NTT) and threaded to top (TT).

implies $\forall c \in \mathbf{N}_+$: $c < b \Rightarrow \alpha.c \in T$.

Let $\mathcal{B} = \{B_\alpha | \alpha \in T\}$ be a system of bases such that $B = B_\epsilon$ is a basis of S (ϵ denotes the empty string) and $B_{\alpha.b}$ is a basis of $a_{\alpha.b}$, an atom of $\mathbf{Es}_{\mathbf{B}_\alpha}$. $\mathcal{B}$ is called an *extended basis* of S.

Define $Es_{\mathcal{B}} := \bigcup_{\alpha \in T} Es_{B_\alpha}$. Set intersection is given as follows. Let $\alpha, \beta \in T$, $x = \bigcap_{i \in I_{B_\alpha}} P_{\alpha,i}(J^x_{\alpha,i})$ and $y = \bigcap_{i \in I_{B_\beta}} P_{\beta,i}(J^y_{\beta,i})$. Then

$$x \cap y := \begin{cases} \bigcap_{i \in I_{B_\alpha}} P_{\alpha,i}(J^x_{\alpha,i} \cap J^y_{\alpha,i}) & \text{if } \alpha = \beta \\ x & \text{if } \alpha = \beta.b.\gamma \text{ and } y \cap a_{\beta.b} = a_{\beta.b} \\ y & \text{if } \beta = \alpha.b.\gamma \text{ and } x \cap a_{\alpha.b} = a_{\alpha.b} \\ 0 & \text{otherwise} \end{cases}$$

Thus $Es_{\mathcal{B}}$ forms a meet semilattice, denoted $\mathbf{Es}_{\mathcal{B}}$. As before, $\mathbf{Es}_{\mathbf{B}_\alpha}$ is embedded in $[0, a_\alpha]$ such that the covering relation is preserved for all nonzero elements. $a_\epsilon = a$ is taken to be 1; thus $\mathbf{Es}_{\mathbf{B}}$ is embedded in $[0, 1]$.

Let $Su_{\mathcal{B}}$ be the closure of $Es_{\mathcal{B}}$ under finite set union. Then $\mathbf{Su}_{\mathcal{B}}$ is a distributive lattice. The (possibly empty) set A of atoms of $\mathbf{Su}_{\mathcal{B}}$ consists of atoms defined by bases in $\mathcal{B}$ and not further decomposed. That is, an atom $a_{\alpha.b}$ defined by basis $B_\alpha \in \mathcal{B}$ is an atom of $\mathbf{Su}_{\mathcal{B}}$ just in case α is maximal in T (i.e., $\alpha.1 \notin T$).

$\mathbf{Su}_{\mathbf{B}}$ can be visualized as a space of dimension equal to the cardinality of I_B. The $P_i(j)$ are coordinate values that define hyperplanes in this space. Each $P_i \in B$ is regarded as a "dimension of meaning". The $P_i(j)$ are mutually antonymous "primitive meanings." Elementary subsets are the elementary concepts, defined by these primitive meanings, from which arbitrarily complex (finite) concepts can be constructed.

A2. Normal Form

In this section a unique representation, or normal form, for elements of $\mathbf{Su}_{\mathcal{B}}$ is defined. Then an algebra of normal forms is defined.

An elementary subset x is *maximal in* $y \in Su_{\mathcal{B}}$ iff $x \subseteq y$ and for any elementary subset z, $x \subseteq z \subseteq y$ implies $z = x$. The properties of maximal elementary subsets will be developed in a lattice (the ideal lattice) in which the elementary subsets are distinguished elements.

DEFINITION 8 *Let* $X \subseteq Es_{\mathcal{B}}$*. The* order ideal *generated by* X *is defined* $I(X) := \{y \in Es_{\mathcal{B}} - \{0\} | y \subseteq x \text{ for some } x \in X\}$*. If* $X = \{x\}$ *then* $I(X)$ *is* principal *and is written* $I(x)$*. If* X *is finite then* $I(X)$ *is* finitely generated.

Since unions and intersections of order ideals are again order ideals, the set of all order ideals ordered by set inclusion is a lattice. This lattice is called the *ideal lattice* of $\mathbf{Es}_{\mathcal{B}}$. It contains the zero element $\emptyset$ and unit element $Es_{\mathcal{B}} - \{0\}$. The finitely generated ideals of $\mathbf{Es}_{\mathcal{B}}$ form a sublattice, denoted $\mathbf{H}_{\mathcal{B}}$, of the ideal lattice. Since $\mathbf{H}_{\mathcal{B}}$ is a sublattice of $2^{Es_{\mathcal{B}}-\{0\}}$, it is a distributive lattice. $\mathbf{Es}_{\mathcal{B}}$ is embedded as a meet semilattice in $\mathbf{H}_{\mathcal{B}}$ by the mapping $x \mapsto I(x)$.

The next three paragraphs review relevant facts from lattice theory about finite decomposition [1, 3].

Let $\mathbf{L}$ be a lattice. An element $x \in \mathbf{L}$ is *(join) irreducible* iff $\forall y, z \in \mathbf{L}$: $x = y \cup z$ implies either $x = y$ or $x = z$. An expression $x = x_1 \cup \cdots \cup x_k$, where $x_1, \ldots, x_k$ are irreducible, is a *(finite) decomposition* of x. If no x_k can be eliminated, the decomposition is *irredundant.* If x has a decomposition, it has an irredundant decomposition, formed by deleting superfluous elements.

Now let $\mathbf{L}$ be a distributive lattice. *If* $x \in \mathbf{L}$ *is irreducible and* $x \leq x_1 \cup \cdots \cup x_k$*,* *where* $x_1, \ldots, x_k$ *are arbitrary elements, then* $x \leq x_q$ *for some* q, $1 \leq q \leq k$. Since $\mathbf{L}$ is distributive, $x = x \cap (x_1 \cup \cdots \cup x_k) = x \cap x_1 \cup \cdots \cup x \cap x_k$. Since x is irreducible, $\exists q : x = x \cap x_q$. Thus $x \leq x_q$.

If $x \in \mathbf{L}$ *has an irredundant decomposition, it is unique.* Suppose x has two distinct irredundant decompositions $x = x_1 \cup \cdots \cup x_k = y_1 \cup \cdots \cup y_l$. Let $x_q \notin \{y_1, \ldots, y_l\}$. Then $x_q \leq y_1 \cup \cdots \cup y_l$ implying $\exists r : x_q \leq y_r$. Similarly, $y_r \leq x_1 \cup \cdots \cup x_k$ which implies $\exists t : y_r \leq x_t$. Thus $x_q \leq y_r \leq x_t$ yielding a contradiction since $t = q$ implies that $x_q = y_r$ and $t \neq q$ implies that x_q is redundant.

Since $\mathbf{H}_{\mathcal{B}}$ and $\mathbf{Su}_{\mathcal{B}}$ are distributive lattices, all the above results apply.

The irreducible elements of $\mathbf{H}_{\mathcal{B}}$ are precisely the principal ideals, i.e., the images of elementary subsets. To see this, consider nonzero ideal $I(X) \in \mathbf{H}_{\mathcal{B}}$ where $X \subseteq Es_{\mathcal{B}} - \{0\}$. Then $z \prec I(X)$ iff $z = I(X) - \{x\}$ for $x \in X$. Therefore $I(X)$ is irreducible iff $I(X) = I(x)$ for $x \in Es_{\mathcal{B}} - \{0\}$, i.e., iff $I(X)$ is principal.

Every element x of $\mathbf{H}_{\mathcal{B}}$ is a finitely generated ideal. Let $x = I(\{x_1, \ldots, x_k\})$. Then $x = I(x_1) \cup \cdots \cup I(x_k)$ is a decomposition of x. By the above results, x has a unique irredundant decomposition. In the sequel it will be assumed that the generators given for an element of $\mathbf{H}_{\mathcal{B}}$ are irredundant and therefore unique.

DEFINITION 9 *Let* $x \in \mathbf{H}_{\mathcal{B}}$. *The* pseudocomplement *of* x *is that element* $x^* \in \mathbf{H}_{\mathcal{B}}$ *such that* $\forall y \in \mathbf{H}_{\mathcal{B}} : y \cap x = 0$ *iff* $y \subseteq x^*$. *Thus, if it exists,* $x^* := sup\{y \in \mathbf{H}_{\mathcal{B}} | x \cap y = 0\}$.

Because of the structure of $\mathbf{H}_{\mathcal{B}}$, the pseudocomplement relative to an interval is useful.

DEFINITION 10 *Let* $\mathcal{B}$ *be a system of bases with domain* T. *Let* $\alpha = \beta.b \in T$, a_α *be an atom defined by basis* B_β *and* $x \in [0, a_\alpha]$. *Then the* pseudocomplement of x in $[0, a_\alpha]$ *is defined* $x^*_\alpha := sup\{y \in [0, a_\alpha] | x \cap y = 0\}$.

LEMMA 11 *Let* $\alpha = b_1.b_2.\cdots.b_m$. *Then* $x^* = \bigcup_{k=1}^{m} (a_{b_1.\cdots.b_k})^*_{b_1.\cdots.b_{k-1}} \cup x^*_\alpha$. *(Note that* b_0 *is interpreted as the empty string,* ϵ.*)*

proof: Let $\alpha = \beta.b$. Then it follows from $sup\{y \in [0, a_\beta] | x \cap y = 0\} = sup\{y \in [0, a_\alpha] | x \cap y = 0\} \cup sup\{y \in [0, a_\beta] | a_\alpha \cap y = 0\}$ that $x^*_\beta = (a_\alpha)^*_\beta \cup x^*_\alpha$. The lemma follows by induction. □

It will now be shown that $\mathbf{H}_\mathcal{B}$ is pseudocomplemented.

LEMMA 12 *Every irreducible element of* $\mathbf{H}_\mathcal{B}$ *has a pseudocomplement.*

proof: First consider the pseudocomplement in an interval with a single basis B. Let $I(x)$ be the principal ideal generated by $x = \bigcap_{i \in I_B^x} P(J_i^x) \in Es_B$. Define $z_i := P(J_i - J_i^x) \in Es_B$. Then by Lemma 3, $x \cap z_i = 0$ for all $i \in I_B^x$. Moreover, if $y \in Es_B$ such that $x \cap y = 0$ then $\exists i \in I_B^x : y \subseteq z_i$. Since $\mathbf{Es_B}$ is embedded in $\mathbf{H_B}$ as a meet semilattice, $I(x) \cap I(z_i) = 0$ for all $i \in I_B^x$ also. By distributivity of $\mathbf{H_B}$, $I(x) \cap [\bigcup_{i \in I_B^x} I(z_i)] = 0$.

Let $I(\{y_1, \ldots, y_l\}) \in \mathbf{H_B}$ be an arbitrary nonzero element such that $x \cap y = 0$. By distributivity, $I(x) \cap I(y_r) = 0$ for all $1 \leq r \leq l$, and hence $x \cap y_r = 0$ in $\mathbf{Es_B}$. Then $\exists i \in I_B^x : y_r \subseteq z_i$. Therefore $\forall r : I(y_r) \subseteq \bigcup_{i \in I_B^x} I(z_i)$, and so $I(\{y_1, \ldots, y_l\}) \subseteq \bigcup_{i \in I_B^x} I(z_i)$. Consequently $\bigcup_{i \in I_B^x} I(z_i)$ is the pseudocomplement of $I(x)$.

The general case is similar. Let $I(x)$ be the principal ideal generated by $\bigcap_{i \in I_{B_\alpha}} P_{\alpha,i}(J^x_{\alpha,i}) \in Es_\mathcal{B}$, where $\alpha = b_1.b_2.\cdots.b_m$. Then by Lemma 11, $I(x)^* = \bigcup_{k=1}^{m} I(a_{b_1.\cdots.b_k})^*_{b_1.\cdots.b_{k-1}} \cup I(x)^*_\alpha = \bigcup_{k=1}^{m} [\bigcup_{i \in I_{B_{b_1.\cdots.b_{k-1}}}} I(P_{b_1.\cdots.b_{k-1},i}(J_{b_1.\cdots.b_{k-1},i} - J^{a_{b_1.\cdots.b_k}}_{b_1.\cdots.b_{k-1},i}))] \cup [\bigcup_{i \in I^x_{B_\alpha}} I(P_{\alpha,i}(J_{\alpha,i} - J^x_{\alpha,i}))]$. □

THEOREM 13 $\mathbf{H}_\mathcal{B}$ *is a pseudocomplemented lattice.*

proof: Consider an arbitrary $x \in \mathbf{H}_{\mathcal{B}}$. Let $x = x_1 \cup \cdots \cup x_k$ be its decomposition.

(i) $x \cap (x_1^* \cap \cdots \cap x_k^*) = (x_1 \cup \cdots \cup x_k) \cap (x_1^* \cap \cdots \cap x_k^*) = (x_1 \cap x_1^* \cap \cdots \cap x_k^*) \cup \cdots \cup (x_k \cap x_1^* \cap \cdots \cap x_k^*) = 0$

(ii) Let $y \in \mathbf{H}_{\mathcal{B}}$ such that $x \cap y = 0$. Then $\forall q : x_q \cap y = 0$ which implies $\forall q : y \subseteq x_q^*$, ie., $y \subseteq (x_1^* \cap \cdots \cap x_k^*)$. Thus $x^* = x_1^* \cap \cdots \cap x_k^*$. □

EXAMPLE. Let $\mathcal{B} = \{B, B_1, B_2\}$, $B_\alpha = \{P_{\alpha,1}, P_{\alpha,2}\}$ for $\alpha \in \{\epsilon, 1, 2\}$, $P_{\alpha,i} = \{p^1_{\alpha,i}, p^2_{\alpha,i}\}$ for $i \in \{1,2\}$ and $x = p_1^1 \cap p_2^1 \cap p_{1,1}^1 \cup p_1^1 \cap p_2^2 \cap p_{2,1}^1$ (see Figure 14).
Then $x^* = [p_1^1 \cap p_2^1 \cap p_{1,1}^2 \cup p_1^2 \cup p_2^2] \cap [p_1^1 \cap p_2^2 \cap p_{2,1}^2 \cup p_1^2 \cup p_2^1]$
$= [p_1^1 \cap p_2^1 \cap p_{1,1}^2] \cup [p_1^2] \cup [p_1^2 \cap p_2^1] \cup [p_1^1 \cap p_2^2 \cap p_{2,1}^2] \cup [p_1^2 \cap p_2^2]$.

LEMMA 14 *Every elementary subset of* $\mathbf{Su}_{\mathcal{B}}$ *has a complement.*

proof: The proof follows that of Lemma 12, with the observation that in $\mathbf{Su}_{\mathcal{B}}$, with x and z_i as defined there, $x \cup \bigcup_{i \in I^x_{\mathcal{B}}} z_i = 1$. □

THEOREM 15 $\mathbf{Su}_{\mathcal{B}}$ *is a Boolean lattice.*

proof: A proof similar to that of Theorem 13, using Lemma 14, shows that every element of $\mathbf{Su}_{\mathcal{B}}$ has a complement. Since $\mathbf{Su}_{\mathcal{B}}$ is distributive, complements are unique. □

DEFINITION 16 $\sigma : \mathbf{H}_{\mathcal{B}} \to \mathbf{H}_{\mathcal{B}}$ *is defined* $\sigma(x) := \overline{x} := x^{**}$.

That σ is a closure operation on $\mathbf{H}_{\mathcal{B}}$ can be seen as follows. By Definition 9, (i) $x \subseteq x^{**}$ and (ii) $x \subseteq y \Rightarrow y^* \subseteq x^*$. From (i), $x^* \subseteq x^{***}$; from (i) and (ii), $x^{***} \subseteq x^*$; hence $x^* = x^{***}$. Thus $x \subseteq \overline{x}$, $x \subseteq y \Rightarrow \overline{x} \subseteq \overline{y}$ and $\overline{\overline{x}} = \overline{x}$.

The quotient lattice formed by the closed elements of $\mathbf{H}_{\mathcal{B}}$ with set inclusion as the order is denoted $\mathbf{H}_{\mathcal{B}}/\sigma$. The meet is $x \wedge y = x \cap y$. The join is $x \vee y = (x^* \cap y^*)^*$.

It will now be shown that $\mathbf{H}_\mathcal{B}/\sigma \cong \mathbf{Su}_\mathcal{B}$.

LEMMA 17 *$\phi : \mathbf{H}_\mathcal{B} \to \mathbf{Su}_\mathcal{B}$ defined $\phi(I(X)) = \bigcup X$ is a homomorphism of $\mathbf{H}_\mathcal{B}$ onto $\mathbf{Su}_\mathcal{B}$. Moreover, $\phi(I(X)) = 0$ iff $I(X) = 0$ and $\phi(I(X)^*) = \phi(I(X))'$.*

proof: (i) If $x \in \mathbf{Su}_\mathcal{B}$ then $x = \bigcup X$, where $X \subseteq \mathbf{Es}_\mathcal{B}$ is finite. But $I(X) \in \mathbf{H}_\mathcal{B}$ and $\phi(I(X)) = x$. Therefore ϕ is onto.
(ii) $\phi(I(X) \cup I(Y)) = \phi(I(X \cup Y)) = \bigcup(X \cup Y) = \bigcup X \cup \bigcup Y = \phi(I(X)) \cup \phi(I(Y))$.
(iii) $\phi(I(X) \cap I(Y)) = \phi(I(Z))$ where $Z = \mathbf{irr}\{x \cap y | x \in X, y \in Y\}$ and $\mathbf{irr}$ reduces a set to its irredundant elements. $\phi(I(Z)) = \bigcup Z = (\bigcup X) \cap (\bigcup Y) = \phi(I(X)) \cap \phi(I(Y))$.
(iv) $I(X) = 0$ implies $X = \emptyset$ implies $\bigcup X = \emptyset$ implies $\phi(I(X)) = 0$. On the other hand, $I(X) \neq 0$ implies $X \neq \emptyset$ implies $\bigcup X \neq \emptyset$ implies $\phi(I(X)) \neq 0$.
(v) To see that $\phi(I(X)^*) = \phi(I(X))'$, let $y \in \mathbf{H}_\mathcal{B}$ such that $\phi(y) = \phi(I(X))'$. Then $\phi(I(X) \cap y) = \phi(I(X)) \cap \phi(y) = 0$. By (iv), $I(X) \cap y = 0$ and therefore $y \subseteq I(X)^*$, implying $\phi(y) \subseteq \phi(I(X)^*)$. Since $\phi(I(X)^*) \cap \phi(I(X)) = 0$ implies $\phi(I(X)^*) \subseteq \phi(y)$, it follows that $\phi(I(X)^*) = \phi(I(X))'$. □

THEOREM 18 *$\mathbf{H}_\mathcal{B}/\sigma \cong \mathbf{Su}_\mathcal{B}$. Moreover, if $I(X) \in \mathbf{H}_\mathcal{B}/\sigma$ then X is exactly the set of elementary subsets maximal in $\bigcup X \in \mathbf{Su}_\mathcal{B}$.*

proof: Let ϕ_σ denote ϕ restricted to $\mathbf{H}_\mathcal{B}/\sigma$. ϕ_σ is an isomorphism if it is 1:1 and onto. ϕ_σ is onto since ϕ is, and for any $I(X) \in \mathbf{H}_\mathcal{B}$, $\phi(I(X)^{**}) = \phi(I(X))'' = \phi(I(X))$. To see that ϕ_σ is 1:1, suppose $\phi(I(X)^{**}) = \phi(I(Y)^{**})$. By Lemma 17, $\phi(I(X)^{**} \cap I(X)^*) = 0$ implies $\phi(I(Y)^{**} \cap I(X)^*) = 0$ implies $I(Y)^{**} \cap I(X)^* = 0$ implies $I(Y)^{**} \subseteq I(X)^{**}$. A symmetrical argument yields $I(X)^{**} \subseteq I(Y)^{**}$. Then $I(X)^{**} = I(Y)^{**}$. Thus $\mathbf{H}_\mathcal{B}/\sigma \cong \mathbf{Su}_\mathcal{B}$.

Now let $x = \bigcup X \in \mathbf{Su}_\mathcal{B}$, and $I(Z) = I(X)^{**} \in \mathbf{H}_\mathcal{B}/\sigma$. Suppose $y \in \mathbf{Es}_\mathcal{B}$ such that $y \subseteq x$. Then $y \cap x' = 0$ and therefore $\phi(I(y) \cap I(X)^*) = 0$. This implies that

$I(y) \cap I(X)^* = 0$ and therefore $I(y) \subseteq I(X)^{**}$. But then $y \in I(X)^{**} = I(Z)$ and hence $\exists z \in Z : y \subseteq z$. Thus the elements of Z are exactly the maximal elementary subsets of $\bigcup Z \in \mathbf{Su}_{\mathcal{B}}$. $\square$

Therefore the set of maximal elementary subsets of any subspace of $\mathbf{Su}_{\mathcal{B}}$ is exactly the unique set of irredundant generators of the corresponding closed order ideal of $\mathbf{H}_{\mathcal{B}}$.

EXAMPLE. Let $B = \{P_i | i = 1, 2\}$, $P_i = \{p_i^j | j = 1, 2, 3\}$, $x = [p_1^2 \cap (p_2^2 \cup p_2^3)] \cup [(p_1^2 \cup p_1^3) \cap p_2^2] \cup [p_1^3 \cap (p_2^1 \cup p_2^2)]$, $y = [p_2^2] \cup [p_1^3 \cap (p_2^2 \cup p_2^3)]$. Then $x \cap y = (p_1^2 \cup p_1^3) \cap p_2^2$ and $x \cup y = [p_1^3] \cup [(p_1^2 \cup p_1^3) \cap (p_2^2 \cup p_2^3)] \cup [p_2^2]$. The elementary subsets forming each union are maximal. Therefore the ideals generated by the elementary subsets in the unions for x and y are in $\mathbf{H_B}/\sigma$. Combining these ideals under the operations $\wedge$ and $\vee$, one can see that the results are the ideals generated by the elementary subsets that are maximal in $x \cap y$ and $x \cup y$, respectively.

DEFINITION 19 *Let* $x = \bigcup X \in \mathbf{Su}_{\mathcal{B}}$. *Let* $\overline{I(X)} = I(x_1) \cup \cdots \cup I(x_k)$ *be the irredundant decomposition of* $\overline{I(X)} \in \mathbf{H}_{\mathcal{B}}$ *into irreducible elements. Then the* normal form *of* x *is defined* $\mathcal{N}(x) := \{x_1, \ldots, x_k\}$.

Operations on normal forms are defined to parallel operations of $\mathbf{H}_{\mathcal{B}}/\sigma$.

DEFINITION 20 *Let* $x, y \in \mathbf{Su}_{\mathcal{B}}$ *with normal forms* $\mathcal{N}(x) = \{x_1, \ldots, x_k\}$ *and* $\mathcal{N}(y) = \{y_1, \ldots, y_l\}$. *Then* $\mathcal{N}(x) \underline{\wedge} \mathcal{N}(y) := \mathbf{irr}\{x_q \cap y_r | 1 \leq q \leq k, 1 \leq r \leq l\}$.

Note that Lemma 7 asserts that the operation **irr** involves only componentwise Boolean operations on elementary subsets.

DEFINITION 21 *Let* $x \in \mathbf{Su}_{\mathcal{B}}$. *The* complement *of* $\mathcal{N}(x)$ *is defined as follows.*
(i) If $x = \bigcap_{i \in I_{B_\alpha}} P_{\alpha,i}(J^x_{\alpha,i}) \in Es_{\mathcal{B}}$, *where* $\alpha = b_1.b_2.\cdots.b_m$, *so that* $\mathcal{N}(x) = \{x\}$ *then*

$\sim \mathcal{N}(x) := \bigcup_{k=1}^{m}[\bigcup_{i\in I_{B_{b_1,\dots,b_{k-1}}}} I(P_{b_1,\dots,b_{k-1},i}(J_{b_1,\dots,b_{k-1},i} - J_{b_1,\dots,b_{k-1},i}^{a_{b_1,\dots,b_k}}))] \cup [\bigcup_{i\in I_{B_\alpha}^x} I(P_{\alpha,i}(J_{\alpha,i}$
$-J_{\alpha,i}^x))]$.
(ii) If $x \notin Es_{\mathcal{B}}$ and $\mathcal{N}(x) = \{x_1, \dots, x_k\}$, then $\sim \mathcal{N}(x) := \sim \mathcal{N}(x_1) \underline{\wedge} \cdots \underline{\wedge} \sim \mathcal{N}(x_k)$.

DEFINITION 22 *Let $x, y \in \mathbf{Su}_{\mathcal{B}}$ with normal forms $\mathcal{N}(x) = \{x_1, \dots, x_k\}$ and $\mathcal{N}(y) = \{y_1, \dots, y_l\}$. Then $\mathcal{N}(x) \underline{\vee} \mathcal{N}(y) := \sim (\sim \mathcal{N}(x) \underline{\wedge} \sim \mathcal{N}(y))$.*

Thus the algebra with universe equal to the set of normal forms of elements of $\mathbf{Su}_{\mathcal{B}}$ and signature $\{\underline{\vee}, \underline{\wedge}, \sim, 0, 1\}$ is a Boolean algebra, the *algebra of normal forms.*

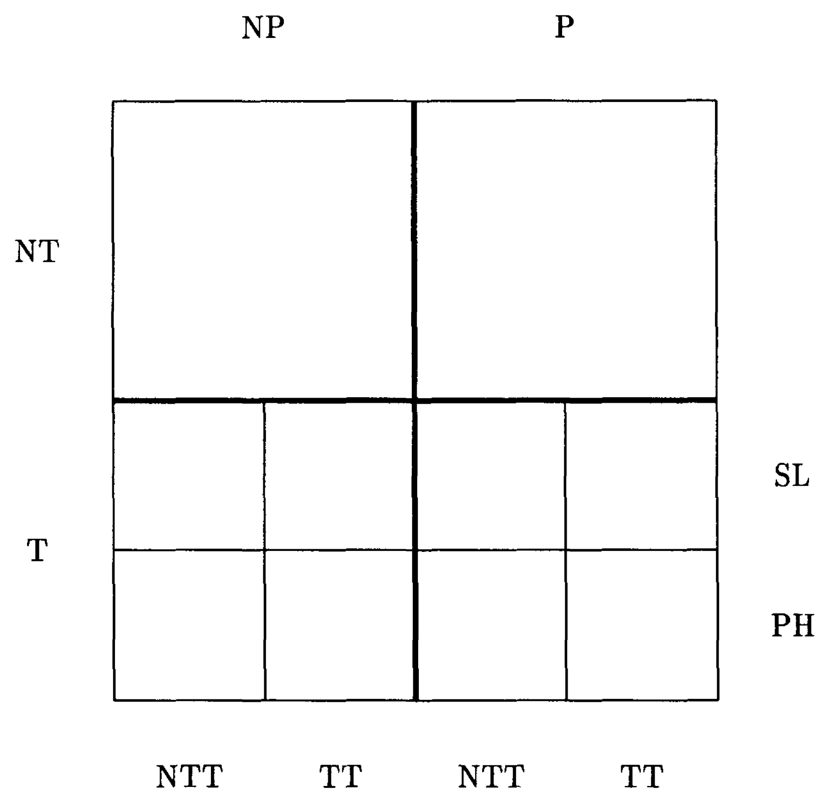


Figure 13: An Example of Embedded Bases.

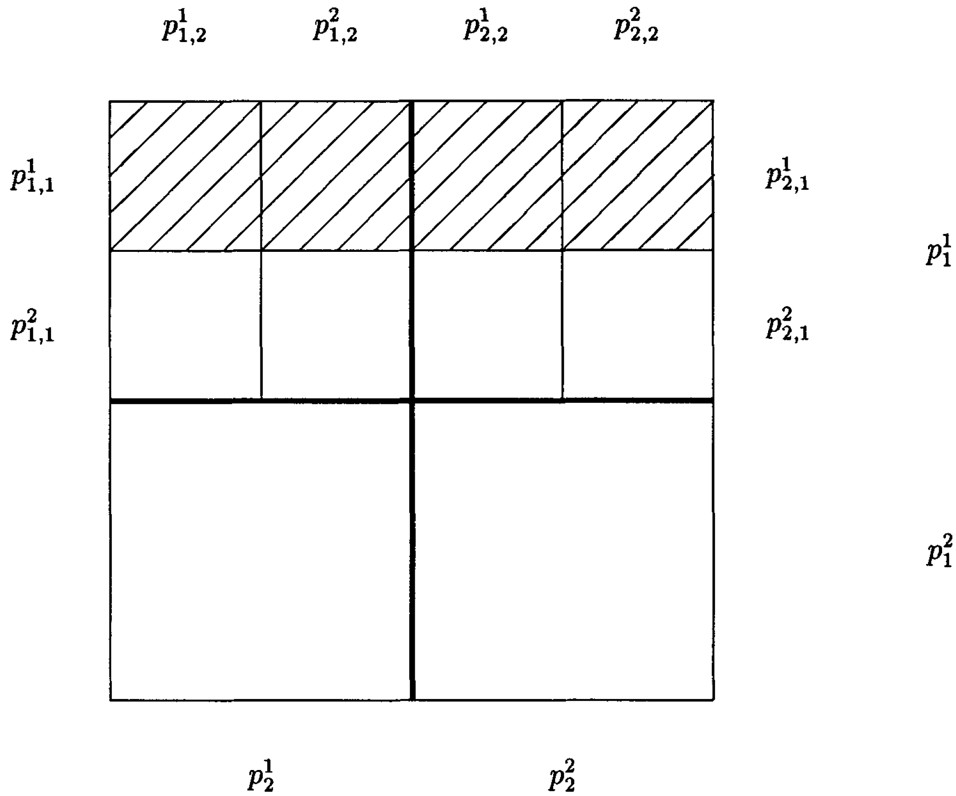


Figure 14: Example Illustrating Pseudocomplement (x is the shaded area).

A3. Basis Construction

The objective of this section is to show how algorithms for basis construction can be defined. It is assumed that an appropriate analysis has yielded a set of diagnostic features, $\Delta = \{d_1, d_2, \ldots, d_n\}$, for the population of interest, and a set of Boolean formulas, Γ, relating these features.

First some notation is defined. Let $\Lambda = \Delta \cup \{\sim d | d \in \Delta\}$, the set of literals. Let F be the conjunctive normal form of $\bigwedge \Gamma$.[7] Then $F = \bigwedge_{1 \leq i \leq m} \bigvee_{1 \leq j \leq l_i} b_{ij}$ where each $b_{ij} \in \Lambda$. Hence $F = \bigwedge_{1 \leq i \leq m} \sim \bigwedge_{1 \leq j \leq l_i} \sim b_{ij} = \bigwedge_{1 \leq i \leq m} \sim \bigwedge B_i$. Each B_i then represents a constraint on the set of features Δ, viz., that $\bigwedge B_i$ is unsatisfiable.

Let Υ be the set of nonempty subsets of Λ. Define $T : \Upsilon \rightarrow \Upsilon$ as follows.

1. For each $A \in \Upsilon : A \subseteq T(A)$
2. For each constraint B and $b \in B : \sim b \in T(B - \{b\})$

Define $T^* : \Upsilon \rightarrow \Upsilon$ recursively as follows.

1. For each $A \in \Upsilon : T(A) \subseteq T^*(A)$
2. For each $B \subseteq T^*(A) : T(B) \subseteq T^*(A)$

The significance of T^* is that $\bigwedge A \subseteq \bigwedge T^*(A)$. Note that if T^* is considered a table, rows such that $T^*(A) = A$ are trivial and could be indicated by their absence. Moreover, rows such that for some $d \in \Delta$, $d, \sim d \in T^*(A)$ are never accessed and so could be deleted.

Now given any ordering of Δ, a *binary* extended basis can be constructed as follows. It is assumed without loss of generality that none of the features are trivial, i.e., that no constraint is a singleton.

[7] For a set $X = \{x_1, \ldots, x_k\}$, $\bigwedge X := x_1 \wedge \cdots \wedge x_k$. $\bigvee X$ is defined similarly.

1. The root basis is $B = \{P_0\}$, where $P_0 = \{\bigwedge T^*(\{d_1\}), \bigwedge T^*(\{\sim d_1\})\}$

2. Let $a_{\alpha.j}$ be an arbitrary atom defined by basis B_α. Let d_k be the first feature not in $a_{\alpha.j}$. Then $B_{\alpha.j} = \{P_{\alpha.j}\}$ where $P_{\alpha.j} = \{\bigwedge T^*(a_{\alpha.j} \cup \{d_k\}), \{\bigwedge T^*(a_{\alpha.j} \cup \{\sim d_k\})$

In general it is desirable to modify this construction. Usually the application provides some information about "meaningful" partitions. This information can be added as a set Ξ of assertions of the form $X = X_1 \dot{\cup} \cdots \dot{\cup} X_k$, meaning that $\{\bigwedge X_1, \ldots, \bigwedge X_k\}$ is a partition of $\bigwedge X$. These assertions mandate use of the associated partitions.

Further modification can be based on the observation that if every atom defined by a basis B_α is partitioned by P, then B_α may be replaced by $B_\alpha \cup \{P\}$. Thus the algorithm can be modified to make each basis of maximum dimension.

References

[1] Aigner, Martin: 1979, *Combinatorial Theory*, Springer-Verlag.

[2] Cohn, A. G.:1985, "On the solution of Schubert's Steamroller in many-sorted logic," *Proceedings of the Ninth International Joint Conference on Artificial Intelligence,* Los Angeles, California, 1169-1174.

[3] Gratzer, George: 1978, *General Lattice Theory*, Birkhauser Verlag.

[4] Lehrer, A.: 1974, *Semantic Fields and Lexical Structure*, North-Holland Publishing Company.

[5] Lyons, John: 1977, *Semantics*, Volume I, Cambridge University Press.

[6] Nida, Eugene A.: 1975, *Componential Analysis of Meaning*, Mouton.

[7] Purdy, William C.: 1990, *A Lexical Extension of Montague Semantics*, Report SU-CIS-90-07 School of Computer and Information Science, Syracuse University.

[8] Quine, W. V.: 1982, *Methods of Logic*, Fourth Edition, Harvard University Press.

[9] Stickel, Mark E.: 1985, "Automated Deduction by Theory Resolution," *Journal of Automated Reasoning* 1, 333-355.

[10] Stickel, Mark E.: 1986, "Schubert's Steamroller Problem: Formulations and Solutions," *Journal of Automated Reasoning* 2, 89-101.

[11] Walther, Christoph: 1987, *A Many-Sorted Calculus Based on Resolution and Paramodulation*, Pitman.