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Domains for Logic Programming

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A Theory of Lexical Semantics

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Abstract

The linguistic theory of Richard Montague (variously referred to as Montague Grammar or Montague Semantics) provides a comprehensive formalized account of natural language semantics. It appears to be particularly applicable to the problem of natural language understanding by computer systems. However the theory does not deal with meaning at the lexical level. With few exceptions, lexical items are treated simply as unanalyzed basic expressions. As a result, comparison of distinct lexical meanings or of semantic expressions containing these lexical meanings falls outside the theory. In this paper, I attempt to provide a compatible theory of lexical semantics which may serve as an extension of Montague Semantics.

1 INTRODUCTION

Over the fifteen year period from 1955 to 1970, Richard Montague advanced a precise and elegant theory of natural language semantics [10]. In my opinion, this theory and the subsequent developments inspired by it offer the most promising approach to the realization of computer understanding of natural language. However analysis of meaning at the lexical level is outside the theory. Therefore relations between expressions in the object language, such as entailment or contradiction, cannot be determined directly from the theory. Examples given below will make this clear. They are prefaced by a brief description of Montague Semantics.

According to Montague's theory, a sentence in the object language is analyzed by first producing its structural description. A structural description is an expression

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consisting of lexical items (i.e., basic expressions) and structural operations which successively combine subexpressions to generate the sentence. Then, beginning with the meanings of the lexical items, semantic operations are invoked in one-to-one correspondence with the structural operations. The semantic operations combine meanings into successively larger structures, finally resulting in the meaning of the sentence.

In keeping with the view of semantics as parallel to syntax, the syntactically primitive lexical items are considered semantically primitive as well. Thus in PTQ (a fragment of English formulated by Montague to demonstrate his theory) man translates to *man′* in the Intensional Logic.[1] Presumably the meaning of *man′* stands in certain relations to other meanings, but this is outside the scope of the theory.[2]

The complex meanings constructed by Montague's theory are expressions (or terms) in the algebra that models the object language. Therefore these meanings can be further processed according to the laws of the model (e.g., first-order logic).

For example, from the meanings of (1) "Mary dates an actor" and (2) "Every actor is a male" Montague's theory can deduce the meaning of (3) "Mary dates a male". Without (2) however, the deduction of (3) is not possible. Any English speaker could deduce (3) directly from (1), since a part of the meaning of actor is male. But in Montague's system, the translations *actor′* and *male′* are unanalyzed.

To make the point more strongly, an English speaker would also deduce "Mary accompanies an actor" from (1) since accompany is part of the meaning of date. To permit this deduction in Montague's system one might add the sentences "Every person who dates a person accompanies a person", "Mary is a person" and "Every actor is a person". A more common method is to add a meaning postulate or fact to the model such as "$\forall x \forall y[date'_*(y)(x) \rightarrow accompany'_*(y)(x)]$."[3] This is the beginning of what can be called a knowledge base.

Thus to emulate the deduction capability of an English speaker, the meanings of lexical items are specified implicitly by a number of postulates or facts residing in a knowledge base in the model.

Although not always explicit, this is the conventional approach taken in systems

[1]The Intensional Logic (IL) is a tensed modal language to which PTQ is translated. The interpretation of IL provides an indirect interpretation of PTQ. Because it is similar in syntax and semantics to familiar first-order and modal logics, IL provides a perspicuous means for interpreting PTQ. Therefore, the translation of a PTQ expression to IL is used to describe the meaning of that expression.

[2]Strictly speaking, nonlogical constants are treated in this manner. Logical constants, such as every, he and be receive special treatment.

[3]If δ is a relation in IL that translates a transitive or intransitive verb, then δ_* is the corresponding (extensional) relation on individuals (see [4,10]).

that deal with the semantics of lexical items. The postulates may take the form of logical expressions, graphs or other relational data structures. The large number of postulates required and their ad hoc nature limit the value of this conventional approach.

An alternative, to be developed in this paper, is to treat the meanings of lexical items as decomposable. In this approach, each lexical item is given a semantic value whose constituents convey its essential meaning. It is similar to Katz' sequences of semantic markers [7] but differs in these important ways. First, the entities that play the role of semantic markers are fixed and well defined for a given realization, being derived from empirical data. Second, they are not burdened with any ideal properties such as universality. Third, they are clustered into "orthogonal dimensions of meaning" which can be processed independently of one another.

This leads to a representation of meaning that has the structure of a multidimensional space with an orthogonal basis. This structure would appear to be easier to deal with computationally than the complex relational network which is characteristic of the conventional approach.

An embedded system that provides meanings for lexical items in the manner indicated will be called a *lexicon*. The role and the importance of the lexicon in the process of deduction should be evident. Deduction involves comparing meanings to determine equivalence (synonymy), inclusion (entailment), and exclusion (contradiction and anomaly). The way in which the meanings of lexical items are represented (encoded) by the lexicon can significantly facilitate or impede the process.

The lexicon does not eliminate the need for a knowledge base. Rather it makes the knowledge base an independent embedded system with a different role. To define each lexical item in its domain, the lexicon employs those distinguishing properties that are sufficient to differentiate between nonsynonymous lexical items.[4] For example, porpoise is sufficiently defined as a totally aquatic, toothed, small (200-600 pound) mammal. That porpoises are playful, nonagressive and have been known to rescue drowning humans is considered encyclopedic information, not appropriate for a lexicon.[5] Encyclopedic data resides in the knowledge base. This data is used in extended deduction,[6] not necessary for direct linguistic competence.

[4]This includes not only purely semantic data, but also syntactic category and feature data. The latter are especially important for differentiating homographs, distinct lexical items that are structurally identical. For example, "set" is a lexical structure that represents a number of distinct lexical items that are differentiated by syntactic category and feature data.

[5]This general characterization of a lexicon should not be interpreted as taking a position with regard to the "minimal description principle" [5], which holds that a lexicon should be restricted to information necessary and sufficient to distinguish between lexical items contained therein. The theory to be developed is completely neutral on this issue.

[6]For purposes of this discussion, a distinction is made between direct deduction and extended

The objective of this paper is to develop a theory of lexical semantics that is consistent with the principle of compositionality and that leads to a representation for meaning that facilitates recognition of direct entailment.

Section 2 develops the underlying concept of a semantic domain and its basis. This provides a framework for definition of a representation of meaning presented in Section 3. A lexicon is defined in terms of this representation. Section 4 illustrates how these concepts can be applied to construct a lexicon from empirical linguistic data. Section 5 considers how lexical meaning relates to the higher level meaning constructs of Montague Semantics.

Throughout these discussions, little or no consideration is given to practical algorithms. Neither is any assessment made of computational complexity. These issues will be dealt with in subsequent papers.

2 SEMANTIC DOMAIN

This section defines underlying concepts. The first is the semantic domain, a collection of related meanings. Since meaning is susceptible in general to unlimited refinement, the number of possible meanings in some semantic domains is infinite. A finite approximation to a semantic domain, called a reduced semantic domain, is constructed by partitioning the semantic domain into equivalence classes, using a special partition called a basis. Distinguished elements of the reduced domain, called elementary subsets, are then defined. They play an important role in the development of a representation for meaning.

2.1 A semantic domain is a collection of subsets interpreted as references or extensions. Consider the set H of all humans living at this moment. Certain subsets of H provide extensions or meanings of English words and phrases: e.g., boy, Canadian, blue collar worker.

Or, consider the set $H \times H$ of all pairs of humans. Subsets of $H \times H$ serve as meanings of English words and phrases (understood as binary relations) such as father, sister, friend, manager.

The subsets of a given set S thus provide extensions for all the properties of members of S. The members of S are not restricted to concrete existent entities as in the examples just given. Members of S may be nonexistent (e.g., fictitious or imaginary)

deduction. Direct deduction is the immediate unmediated (usually unconscious) generation of information that characterizes real time linguistic performance. Extended deduction is generation of information mediated by (usually conscious) thought. No claim is made that the two kinds of deduction are disjoint. The terms are used only to distinguish major regions of a continuum.

or they may be abstractions. S may be infinite as well as finite.

Set inclusion in a semantic domain is viewed as meaning inclusion or *entailment*. That is, $x \subseteq y$ is interpreted as x entails y in the sense that membership in the subset x implies membership in the subset y. For example, the meaning of father entails the meaning of parent because the extension of father in $H \times H$ is contained in the extension of parent.

This use of entailment is a generalization of its conventional use as a relation between sentences. The precise nature of this generalization will be described in Section 5.

The notion of a semantic domain is formalized in the following definition.

DEFINITION. Let S be a set and Su be the power set of S. A *semantic domain* is defined to be the algebra of subsets, $\mathbf{Su} := (Su, \cup, \cap, 0, 1)$ where $\cup, \cap, 0, 1$ are the operations set union, set intersection, null set and unit set, respectively.[7] Set inclusion, the partial order on Su, is denoted $\subseteq$.

As with any algebra, a suitable subset of a semantic domain can be regarded as a semantic domain and semantic domains can be combined to form a semantic domain.

2.2 A partition of S is a set of nonempty pairwise disjoint elements of Su whose union is equal to S. A partition will be written $P = \{p^j | j \in J\}$ where J is a set indexing P.

Let $P_1 = \{p_1^j | j \in J_1\}$ and $P_2 = \{p_2^j | j \in J_2\}$ be partitions of S. The *product* of P_1 and P_2 is defined $P_1 \otimes P_2 := \{p_1^j \cap p_2^k | j \in J_1, k \in J_2\}$. In general, $P_1 \otimes \cdots \otimes P_{n-1} \otimes P_n$ is defined to be $(P_1 \otimes \cdots \otimes P_{n-1}) \otimes P_n$. It is obvious that the product is associative and commutative. The product will also be written $\otimes\{P_i | 1 \le i \le n\}$.

2.2.1 EXAMPLE. Let $S = \mathbf{N}_0$, the non-negative integers. Let $P_1 = \{\{i | i = 0, mod4\}$, $\{i | i = 1, mod4\}$, $\{i | i = 2, mod4\}$, $\{i | i = 3, mod4\}\}$ and $P_2 = \{\{i | \textit{is-prime}(i)\}, \{i | \neg \textit{is-prime}(i)\}\}$. Then $P_1 \otimes P_2 = \{\{i | i = 0, mod4 \wedge \textit{is-prime}(i)\}, \{i | i = 0, mod4 \wedge \neg \textit{is-prime}(i)\}, \ldots, \{i | i = 3, mod4 \wedge \neg \textit{is-prime}(i)\}\}$. Note that $p_1^1 \cap p_2^1 = \emptyset$; that is, the conjunction $i = 0, mod4 \wedge \textit{is-prime}(i)$ is logically impossible. Thus, while P_1 and P_2 are partitions of S, $P_1 \otimes P_2$ is not.

2.2.2 EXAMPLE. Let S be the set of possible shoes. Let $P_1 = \{men's, women's\}$, $P_2 = \{large, medium, small\}$, and $P_3 = \{white, black, red, green, blue\}$ be partitions of S. Then $P_1 \otimes P_2 \otimes P_3 = \{$*men's-large-white-shoes*, *men's-large-black-shoes*, $\ldots$, *women's-small-blue-shoes*$\}$. While *men's-large-green-shoes* may be unusual, they are logically possible, and so are all other combinations. Therefore, $P_1 \otimes P_2 \otimes P_3$ is a

[7]The notation "$X := Y$" means that X is defined to be equal to Y; "$X :\Leftrightarrow Y$" means that X is defined to be logically equivalent to Y; etc.

partition of S.

It should be pointed out that *white* and other properties used in this example are not words, but symbols naming subsets of S.

2.3 In the first example a block of $P_1 \otimes P_2$ was found to be empty because it is logically impossible for an element of S to occupy it. Therefore $P_1 \otimes P_2$ is not a partition of S. On the other hand, in the second example every block of $P_1 \otimes P_2 \otimes P_3$ is occupied, and so $P_1 \otimes P_2 \otimes P_3$ partitions S. If P_1, P_2 are partitions of S and $P_1 \otimes P_2$ is also, then P_1 and P_2 are said to be independent. This concept is generalized as follows.

DEFINITION. Let $\{P_i | i \in I\}$ be a set of partitions of S. Then $\{P_i | i \in I\}$ is an *independent* set iff (if and only if) for any finite $I' \subseteq I$, $\otimes\{P_i | i \in I'\}$ is also a partition of S.

Obviously, $\otimes\{P_i | i \in I'\}$ is a partition of S iff whenever $j_i \in J_i$, $\bigcap_{i \in I'} p_i^{j_i} \neq \emptyset$. Thus independence ensures that residence in a block of one partition cannot restrict residence in any block of any other partition.

An independent set of partitions of S will be called a *basis* of S.

2.4 DEFINITION. Let $B = \{P_i | i \in I_B\}$ be a basis of S. An *elementary subset* of S defined by B is a subset x of S that can be written in the form $x = \bigcap_{i \in I_B} \bigcup_{j \in J_i^x} p_i^j$ where $J_i^x \subseteq J_i$ for all $i \in I_B$ and $J_i^x = J_i$ for all but finitely many $i \in I_B$. This form will be called the *standard form* for elementary subset x relative to basis B. The conjunct $\bigcup_{j \in J_i^x} p_i^j$ will be called the *ith component* of x relative to basis B.

Obviously, given a nonempty elementary subset x, the standard form for x relative to a basis B is unique. It will be convenient to define I_B^x: $i \in I_B^x :\Leftrightarrow J_i^x \neq J_i$. Then $x = \bigcap_{i \in I_B^x} \bigcup_{j \in J_i^x} p_i^j$. Also $x = 0$ iff $J_i^x = \emptyset$ for some $i \in I_B^x$.

LEMMA. Let x and y be elementary subsets of S defined by basis B with standard forms $\bigcap_{i \in I_B} \bigcup_{j \in J_i^x} p_i^j$ and $\bigcap_{i \in I_B} \bigcup_{j \in J_i^y} p_i^j$, respectively. Then $x \cap y$ is an elementary subset with standard form $\bigcap_{i \in I_B} \bigcup_{j \in (J_i^x \cap J_i^y)} p_i^j$.
proof: $x \cap y = [\bigcap_{i \in I_B} \bigcup_{j \in J_i^x} p_i^j] \cap [\bigcap_{i \in I_B} \bigcup_{j \in J_i^y} p_i^j] = \bigcap_{i \in I_B} [(\bigcup_{j \in J_i^x} p_i^j) \cap (\bigcup_{j \in J_i^y} p_i^j)] = \bigcap_{i \in I_B} \bigcup_{j_1 \in J_i^x, j_2 \in J_i^y} (p_i^{j_1} \cap p_i^{j_2}) = \bigcap_{i \in I_B} \bigcup_{j \in (J_i^x \cap J_i^y)} p_i^j$, since $p_i^{j_1} \cap p_i^{j_2} = p_i^{j_1}$ if $j_1 = j_2$ and 0 otherwise. Therefore $x \cap y$ is an elementary subset with the ith component indexed by $J_i^{x \cap y} = J_i^x \cap J_i^y$.

2.5 DEFINITION. The *algebra of elementary subsets*, $\mathbf{Es_B} := (Es_B, \cap, 0, 1)$ where $\cap, 0, 1$ are as defined for **Su**.

The binary operation of $\mathbf{Es_B}$ acts componentwise on standard forms to yield standard forms. Set inclusion is a partial order on Es_B. Moreover, $x \subseteq y$ iff $J_i^x \subseteq J_i^y$ for all $i \in I_B$. Equivalently, $x \subseteq y$ iff $I_B^y \subseteq I_B^x \wedge J_i^x \subseteq J_i^y$ for all $i \in I_B^y$.

2.6 Let Su_B be the closure of Es_B under set union. Su_B is also closed under finite intersection since Es_B is.

DEFINITION. The *subset algebra of S defined by* basis B, $\mathbf{Su_B} := (Su_B, \cup, \cap, 0, 1)$ where all operations are as defined for **Su**.

2.7 Observe that **Su** and $\mathbf{Su_B}$ can be viewed as lattices with order $\leq$ defined as set inclusion. Similarly, $\mathbf{Es_B}$ can be viewed as a meet semilattice, also with set inclusion as the order. It will be convenient to take this view in much of the discussion to follow. In particular, it will be said that *y covers x*, written $x \prec y$, iff $x < z \leq y$ implies $z = y$; and x is an *atom* iff $0 \prec x$. An *interval* $[x, y]$ is a sublattice containing x and y in which every element z satisfies $x \leq z \leq y$. A *chain* is a sublattice for which $\leq$ is a total order. A chain between elements y and $x \leq y$ is an interval $[x, y]$ of a chain.

No restrictions have been placed on the cardinalities of Su, Su_B, and Es_B. However, at this point a finiteness assumption will be adopted.

finite chain assumption (F): Any chain between any two elements of Su_B is finite.

Since $\mathbf{Su_B}$ is a sublattice of **Su**, it is distributive and bounded. The finite chain assumption implies that $\mathbf{Su_B}$ is finite. $\mathbf{Es_B}$ must also be finite. Of course, no restriction is placed on **Su**.

It is possible that a weaker assumption would be adequate for purposes of the following discussions.[8] However, adoption of F will avoid complications. Moreover, F seems quite reasonable for a theory of natural language.

2.8 Let B be a basis of S, $\mathbf{Su_B}$ be the subset algebra defined by B and A be the set of atoms of $\mathbf{Su_B}$. Define g_B: $S \to A$ to be the map from an element of S to the atom containing it. g_B can be extended to a map $Su \to Su_B$ by defining $g_B(x) := \bigcup_{u \in x} g_B(u)$. Then g_B induces a partition on semantic domain **Su**. The blocks of this partition correspond to the elements of Su_B. $\mathbf{Su_B}$ will be called the *reduced semantic domain* defined by basis B.

$\mathbf{Su_B}$ can be visualized as a space of dimension equal to the cardinality of I_B. Each $P_i \in B$ is regarded as a "dimension of meaning". The $p_i^j \in P_i$ are mutually antonymous "primitive" meanings. The p_i^j play a role analogous to orthogonal functions in that if x is an elementary subset with standard form $\bigcap_{i \in I_B} \bigcup_{j \in J_i^x} p_i^j$ and $p_q^r \in P_q$, $P_q \in B$,

[8] Another possible assumption is: (F′) Any chain between any two elements of $Su_B - \{1\}$ is finite. (See 2.10 for a definition of $\dot{Su_B}$.) F′ would allow an infinite partition of 1. This would permit an unbounded number of first level domains.

then $p^r_q \cap x \neq 0$ iff $r \in J^x_q$. An elementary subset is analogous to a convex subspace of $\mathbf{Su_B}$; it represents a conjunction of "primitive" meanings from various dimensions of meaning.

A reduced semantic domain is thought of as a finite realization of a semantic domain. The basis B determines the precision to which meanings in the semantic domain **Su** can be expressed or differentiated in $\mathbf{Su_B}$. The following definitions relativize relations between meanings to the reduced semantic domain. Let $x, y \in Su$. *x entails y relative to B* $:\Leftrightarrow g_B(x) \subseteq g_B(y)$. *$x$ is synonymous with y relative to B* $:\Leftrightarrow g_B(x) = g_B(y)$. *$x$ is anomalous* or *contradictory relative to B* $:\Leftrightarrow g_B(x) = 0$.

On this view, synonymy and entailment are not absolute, but relative to a particular realization. In a coarse realization (e.g., that of a child) large sets of meanings (i.e., elements of Su) may be synonymous. As the realization is refined (e.g., by learning), previously synonymous meanings are differentiated. In the limit (as $\mathbf{Su_B}$ approaches **Su**) synonymy is equivalent to identity.

2.9 Let $P = \{p^j | j \in J\}$ be a partition of $Y \subseteq S$ and let $X \subseteq Y$. Define the restriction of P to X: $P\!\upharpoonright_X := \{p^j \cap X | j \in J\}$. Note that $P\!\upharpoonright_X$ is a partition of X iff $p^j \cap X \neq 0$ for all $j \in J$. Let $B = \{P_i | i \in I_B\}$ be a basis of Y. Define the restriction of B to X: $B\!\upharpoonright_X := \{P_i\!\upharpoonright_X | i \in I_B\}$. $B\!\upharpoonright_X$ is a basis of X iff $P_i\!\upharpoonright_X$ is a partition of X for all $i \in I_B$ and the set $\{P_i\!\upharpoonright_X | i \in I_B\}$ is independent.

2.10 Let $\mathbf{Es_B}$ be the algebra of elementary subsets defined by basis B and a_1, a_2 be atoms of $\mathbf{Es_B}$. Let $B' \neq B$ be a basis of $X \subseteq S$ such that $B'\!\upharpoonright_{a_1}$ is a basis of a_1 but $B'\!\upharpoonright_{a_2}$ is not a basis of a_2. For example, it may be that X contains meanings (i.e., subsets of S) relating to animate entities while $S - X$ contains meanings relating to inanimate entities. If B' is defined by meanings relevant to animate entities, it will not in general be a basis of $S - X$.

Let $B'\!\upharpoonright_{a_1} = B_1$. B_1 determines an algebra of elementary subsets, $\mathbf{Es_{B_1}} = (Es_{B_1}, \cap, 0, 1)$ where 1 has the value a_1. It is natural to define a combined algebra $\mathbf{Es}_\mathcal{B} := (Es_B \cup Es_{B_1}, \cap, 0, 1)$ where $\mathbf{Es_B}$ is embedded in the interval $[0, 1]$ and $\mathbf{Es_{B_1}}$ is embedded in the interval $[0, a_1]$ such that the covering relation is preserved for all nonzero elements.

$\mathbf{Su}_\mathcal{B}$ is defined to be the subset algebra with universe equal to the closure of $Es_B \cup Es_{B_1}$ under set union.

EXAMPLE. Let $B = \{P_1, P_2\}$, $P_1 = \{NT, T\}$, $P_2 = \{NP, P\}$. Suppose that $B' = \{Q_1, Q_2\}$, where $Q_1 = \{SL, PH\}$ and $Q_2 = \{NTT, TT\}$, and that $B'\!\upharpoonright_{a_3}$ and $B'\!\upharpoonright_{a_4}$ are bases of a_3 and a_4, respectively. Suppose further that $B'\!\upharpoonright_{a_1}$ and $B'\!\upharpoonright_{a_2}$ are not bases. The resulting partitions of S are diagrammed in Figure 1. The semilattice $\mathbf{Es}_\mathcal{B}$ is shown in Figure 2.

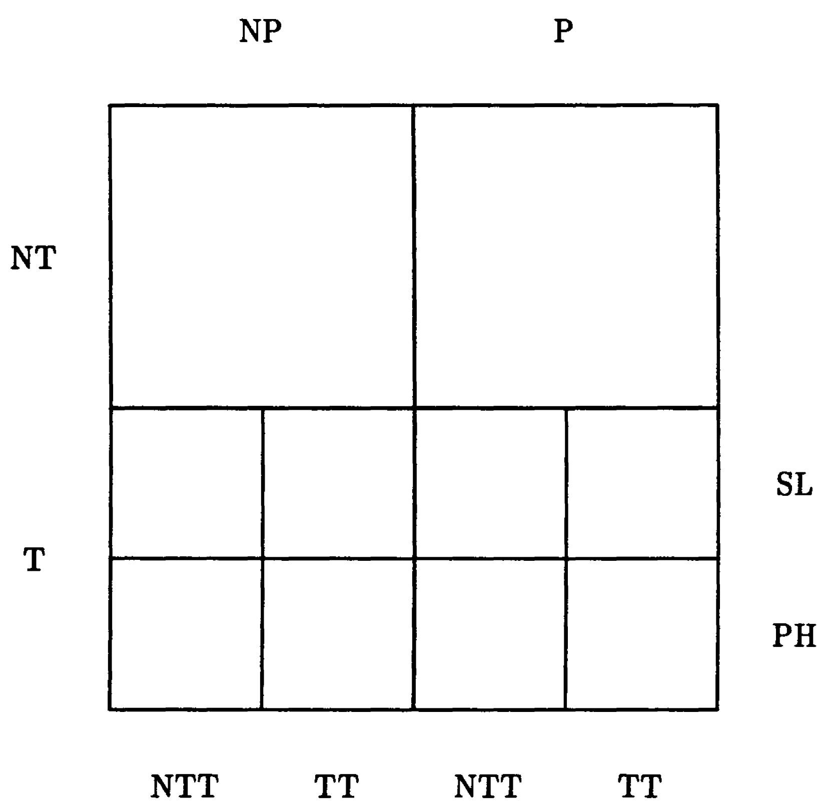

Figure 1: Partitions of S.

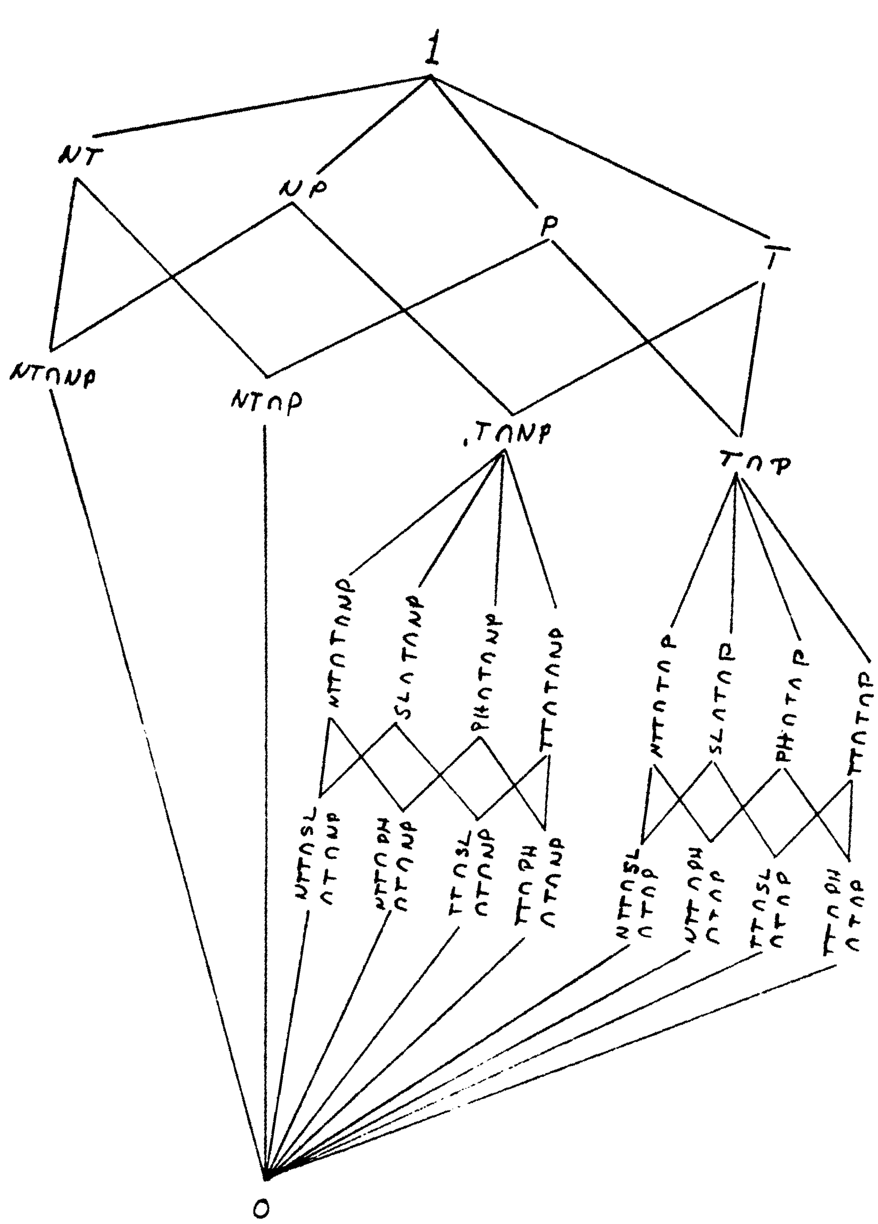

Figure 2: The semilattice $\mathbf{Es}_B$.

This situation is generalized as follows. Let T be a tree indexing defined in the usual way: (i) $T \subset \mathbf{N}^*$ where $\mathbf{N}$ denotes the positive integers; (ii) $\alpha, \beta \in \mathbf{N}^*$ and $\alpha.\beta \in T$ implies $\alpha \in T$; (iii) $\alpha \in \mathbf{N}^*$, $b \in \mathbf{N}$ and $\alpha.b \in T$ implies $\forall c \in \mathbf{N}$: $c < b \Rightarrow \alpha.c \in T$.

Let $\mathcal{B} = \{B_\alpha | \alpha \in T\}$ be a system of bases such that $B = B_\epsilon$ is a basis of S (ϵ denotes the empty string) and $B_{\alpha.b}$ is a basis of $a_{\alpha.b}$, an atom of $\mathbf{Es}_{\mathbf{B}_\alpha}$. $\mathcal{B}$ will be called an *extended basis* of S. The algebra of elementary subsets is defined: $\mathbf{Es}_\mathcal{B} := (Es_\mathcal{B}, \cap, 0, 1)$ with $Es_\mathcal{B} = \bigcup_{\alpha \in T} Es_{B_\alpha}$ where $\mathbf{Es}_{\mathbf{B}_\alpha}$ is embedded in $[0, a_\alpha]$ such that the covering relation is preserved for all nonzero elements. $a_\epsilon = a$ is taken to be 1; thus $\mathbf{Es}_\mathbf{B}$ is embedded in $[0, 1]$. The operation $\cap$ is defined as follows. Let $\alpha, \beta \in T$, $x = \bigcap_{i \in I_{B_\alpha}} \bigcup_{j \in J^x_{\alpha,i}} p^j_{\alpha,i}$ and $y = \bigcap_{i \in I_{B_\beta}} \bigcup_{j \in j^y_{\beta,i}} p^j_{\beta,i}$. Then

$$x \cap y := \begin{cases} \bigcap_{i \in I_{B_\alpha}} \bigcup_{j \in J^x_{\alpha,i} \cap J^y_{\alpha,i}} p^j_{\alpha,i} & \text{if } \alpha = \beta \\ x & \text{if } \alpha = \beta.b.\gamma \text{ and } y \cap a_{\beta.b} = a_{\beta.b} \\ y & \text{if } \beta = \alpha.b.\gamma \text{ and } x \cap a_{\alpha.b} = a_{\alpha.b} \\ 0 & \text{otherwise} \end{cases}$$

Thus $\cap$ is identical to set intersection.

2.11 Each interval $[0, a_{\alpha.b}] \subseteq \mathbf{Es}_\mathcal{B}$, where $a_{\alpha.b}$ is an atom of $\mathbf{Es}_{\mathbf{B}_\alpha}$ and $B_\alpha \in \mathcal{B}$, will be called an *elementary domain*. An element $x \in Es_\mathcal{B}$ will be said to *belong to* elementary domain $[0, a_\alpha]$ iff $x \in [0, a_\alpha)$ and for any elementary domain $[0, a_\beta]$, $x \in [0, a_\beta)$ implies $a_\alpha \leq a_\beta$. That is, x belongs to $[0, a_\alpha]$ iff $[0, a_\alpha]$ is the smallest elementary domain properly containing x.

2.12 Let $Su_\mathcal{B}$ be the closure of $Es_\mathcal{B}$ under set union. Then $\mathbf{Su}_\mathcal{B}$ is defined to be the subset algebra $(Su_\mathcal{B}, \cup, \cap, 0, 1)$ where $\cup, \cap, 0, 1$ are defined as for $\mathbf{Su}$. The set A of atoms of $\mathbf{Su}_\mathcal{B}$ consists of atoms defined by bases in $\mathcal{B}$ and not further decomposed. That is, an atom $a_{\alpha.b}$ defined by basis $B_\alpha \in \mathcal{B}$ is an atom of $\mathbf{Su}_\mathcal{B}$ just in case $\alpha.b$ is maximal in T ($\forall c \in \mathbf{N} : \alpha.b.c \notin T$). The map $g_\mathcal{B}$ is defined similarly to g_B, and allows definition of entailment, synonymy and contradiction or anomaly relative to the extended basis $\mathcal{B}$.

3 NORMAL FORM

In this section a unique representation, or normal form, for elements of the reduced semantic domain, $\mathbf{Su}_\mathcal{B}$, is defined. Then an algebra of normal forms is developed. The definition of a lexicon is based upon this algebra.

3.1 The elementary subsets of $Su_\mathcal{B}$ have several important properties. First, the standard form provides a unique representation. Second, the set of elementary subsets is closed under set intersection. The intersection of standard forms is the componentwise intersection and is again a standard form. Third, every element of $Su_{\mathcal{B}}$ is a union of elementary subsets.

It is natural to ask if the standard form could be generalized to a representation or code for all meanings. In simplest form, does an arbitrary element y of $Su_{\mathcal{B}}$ have a unique representation $sup\{x_k \in Es_{\mathcal{B}} | k \in K^y\}$? Unfortunately, no. An arbitrary element of $Su_{\mathcal{B}}$ may have a number of representations as a union of elementary subsets.

EXAMPLE. Let $\mathcal{B} = B = \{P_1, P_2\}$, $P_1 = \{p_1^1, p_1^2\}$ and $P_2 = \{p_2^1, p_2^2\}$. The following all represent the same subset (each expression in brackets is a standard form):
(i) $[p_1^1 \cap (p_2^1 \cup p_2^2)] \cup [p_1^2 \cap p_2^1]$;
(ii) $[p_1^1 \cap p_2^2] \cup [(p_1^1 \cup p_1^2) \cap p_2^1]$;
(iii) $[p_1^1 \cap (p_2^1 \cup p_2^2)] \cup [(p_1^1 \cup p_1^2) \cap p_2^1]$;
(iv) $[p_1^1 \cap p_2^1] \cup [p_1^1 \cap p_2^2] \cup [p_1^2 \cap p_2^1]$.

But one of the alternatives might be selected as a normal form. Possibilities include the following. (a) The union of maximal elementary subsets. It will be shown that this form is unique for any subset of $Su_{\mathcal{B}}$. In the example it is (iii). (b) The union of maximal disjoint elementary subsets. This form is not unique. Both (i) and (ii) in the example satisfy this description. (c) The union of minimal (atomic) elementary subsets. This form is unique. In the example it is (iv). Of these possibilities, the union of maximal elementary subsets offers uniqueness and structural simplicity and will therefore be adopted.

A normal form for elements of $Su_{\mathcal{B}}$ will permit testing for entailment, synonymy, and anomaly or contradiction to be performed by simply checking whether certain meanings are identical or not.

3.2 It follows from the definition of $\mathbf{Su}_{\mathcal{B}}$ that there exists an embedding ϕ_1 of $\mathbf{Es}_{\mathcal{B}}$ as a meet semilattice into $\mathbf{Su}_{\mathcal{B}}$. Therefore, element $x \in Es_{\mathcal{B}}$ can be identified with its image $\phi_1(x) \in Su_{\mathcal{B}}$. x will be said to be a *maximal* elementary subset of $y \in Su_{\mathcal{B}}$ iff $x \leq y$ and for any elementary subset z, $x \leq z \leq y$ implies $z = x$. The properties of maximal elementary subsets will be developed in a lattice (the ideal lattice) in which the elementary subsets are distinguished elements.

3.3 Let $X \subseteq Es_{\mathcal{B}}$. The *order ideal* generated by X, $I(X) := \{y \in Es_{\mathcal{B}} - \{0\} | y \leq x$ for some $x \in X\}$. If $X = \{x\}$ then $I(X)$ is *principal* and is written $I(x)$.

Since unions and intersections of order ideals are again order ideals, the set of all order ideals ordered by set inclusion is a lattice. This lattice, $\mathcal{I}$, is called the *ideal lattice* of $\mathbf{Es}_{\mathcal{B}}$. It contains the zero element $\emptyset$ and unit element $Es_{\mathcal{B}} - \{0\}$.

Obviously, ϕ_2: $\mathbf{Es}_{\mathcal{B}} \to \mathcal{I}$ defined $\phi_2(x) := I(x)$ is an embedding of $\mathbf{Es}_{\mathcal{B}}$ as a meet semilattice into $\mathcal{I}$. $x \in Es_{\mathcal{B}}$ will be identified with its image $\phi_2(x) \in \mathcal{I}$.

It follows from its definition (and assumption F) that $\mathcal{I}$ is a finite distributive lattice.

3.4 The following results from lattice theory [1,6] will be used. Let $\mathbf{L}$ be a lattice. An element $x \in \mathbf{L}$ is (join) *irreducible* iff $\forall y, z \in \mathbf{L}$: $x = y \cup z$ implies either $x = y$ or $x = z$. An expression $x = x_1 \cup \cdots \cup x_m$, where $x_1, \ldots, x_m$ are irreducible, is a *decomposition* of x. If no x_k can be eliminated, the decomposition is *irredundant.*

If $\mathbf{L}$ is a finite distributive lattice then every element has a *unique irredundant decomposition.* Moreover, if x is irreducible and $x \leq x_1 \cup \cdots \cup x_m$, where $x_1, \ldots, x_m$ are arbitrary elements, then $x \leq x_k$ for some k, $1 \leq k \leq m$.

3.5 All the results of 3.4 hold for $\mathcal{I}$ and $\mathbf{Su}_{\mathcal{B}}$ since both are finite distributive lattices.

The irreducible elements of $\mathcal{I}$ are precisely the principal ideals, i.e., the images of elementary subsets. Consider nonzero ideal $I(X) \in \mathcal{I}$ where $X \subseteq Es_{\mathcal{B}}$. Then $z \prec I(X)$ iff $z = I(X) - \{x\}$ for $x \in X$. Therefore $I(X)$ is irreducible iff $X = \{x\}$, i.e., iff $I(X)$ is principal.

Define the set of atoms of $\mathbf{Su}_{\mathcal{B}}$, $A := \{a \in Su_{\mathcal{B}} | 0 \prec a\}$. That these are the only nonzero irreducible elements of $\mathbf{Su}_{\mathcal{B}}$ can be seen as follows. Every element of $Su_{\mathcal{B}}$ is either an elementary subset or a join of elementary subsets. The irreducible elements must therefore be elementary subsets. Let x be a nonzero elementary subset belonging to $[0, a_\alpha]$. Then $x = \bigcap_{i \in I^x_{B_\alpha}} \bigcup_{j \in J^x_{\alpha,i}} p^j_{\alpha,i} = \bigcup_{(j_i) \in \prod_{i \in I^x_{B_\alpha}} J^x_{\alpha,i}} \bigcap_{i \in I^x_{B_\alpha}} p^{j_i}_{\alpha,i}$. Obviously, x is irreducible only if $|\prod_{i \in I^x_{B_\alpha}} J^x_{\alpha,i}| = 1$ only if $\forall i \in I^x_{B_\alpha} [|J^x_{\alpha,i}| = 1]$ only if x is an atom of $\mathbf{Es}_{\mathbf{B}_\alpha}$. Finally, x is irreducible iff x is an atom of $\mathbf{Es}_{\mathbf{B}_\alpha}$ and x is not further decomposed; i.e., iff x is an atom of $\mathbf{Su}_{\mathcal{B}}$.

Therefore every nonzero element of $Su_{\mathcal{B}}$ has a unique decomposition into atoms. For $x \in Su_{\mathcal{B}}$, define $\mathcal{A}(x) := \{a \in A | a \leq x\}$; for $X \subseteq Su_{\mathcal{B}}$, $\mathcal{A}(X) := \bigcup_{x \in X} \mathcal{A}(x)$.

Because $\mathbf{Es}_{\mathcal{B}}$ is embedded in $\mathbf{Su}_{\mathcal{B}}$ as a meet semilattice, and atoms are elementary subsets, it follows that every element of $Es_{\mathcal{B}}$ is also uniquely determined by the atoms it dominates. Since no confusion can result, the function $\mathcal{A}$ will be applied to elements of $Es_{\mathcal{B}}$ as well as elements of $Su_{\mathcal{B}}$. It follows that for $x, y \in Es_{\mathcal{B}}$, $x = y$ iff $\mathcal{A}(x) = \mathcal{A}(y)$ and $x \leq y$ iff $\mathcal{A}(x) \subseteq \mathcal{A}(y)$.

The atoms of $\mathcal{I}$ are elementary subsets and so the definition $\mathcal{A}(x) := \{a \in \mathcal{I} | 0 \prec a \leq x\}$ is also useful. Because not all irreducible elements of $\mathcal{I}$ are atoms, an arbitrary element is not uniquely determined by the atoms it dominates. However $x \leq y$ only if $\mathcal{A}(x) \subseteq \mathcal{A}(y)$ holds for arbitrary $x, y \in \mathcal{I}$.

When $\mathcal{A}$ is used in the following its domain will be clear from the context.

3.6 DEFINITION. σ: $\mathcal{I} \to \mathcal{I}$ is defined $\sigma(x) := \overline{x} := sup\{y \in \mathcal{I} | \mathcal{A}(y) = \mathcal{A}(x)\}$.

It is seen that σ is a closure operation on $\mathcal{I}$. Some properties of this closure operator follow.

LEMMA. (i) All irreducible elements of $\mathcal{I}$ are closed.
(ii) Let $\overline{x}$, $\overline{y} \in \mathcal{I}$. Then $\overline{x} \leq \overline{y}$ iff $\mathcal{A}(\overline{x}) \subseteq \mathcal{A}(\overline{y})$.
(iii) Let $x \in \mathcal{I}$. Viewed as an order ideal of $\mathbf{Es}_\mathcal{B}$, $\overline{x} = \{z \in Es_\mathcal{B} - \{0\} | \ \mathcal{A}(I(z)) \subseteq \mathcal{A}(x)\}$. Or, identifying $\mathbf{Es}_\mathcal{B}$ with its embedding in $\mathcal{I}$, $\overline{x} = \{z \in Es_\mathcal{B} - \{0\} | \ \mathcal{A}(z) \subseteq \mathcal{A}(x)\}$.
(iv) The meet of the closures of two elements of $\mathcal{I}$ is closed and equal to the closure of their meet; i.e., $\overline{x} \cap \overline{y} = \overline{x \cap y}$.
(v) The closure operation induces a congruence on $\mathcal{I}$: $x \approx y :\Leftrightarrow \overline{x} = \overline{y}$. The factor lattice $\mathcal{I}/\approx$ is isomorphic to $\mathbf{2}^A$ (equivalently, $\mathbf{Su}_\mathcal{B}$).
proof: (i) Let $x \in \mathcal{I}$ be irreducible. Let $\overline{x} = x_1 \cup \cdots \cup x_m$ be the decomposition of $\overline{x}$. Since x is irreducible, $x \leq \overline{x}$ implies $x \leq x_k \leq \overline{x}$ for some $k, 1 \leq k \leq m$. Thus $\mathcal{A}(x) \subseteq \mathcal{A}(x_k) \subseteq \mathcal{A}(\overline{x}) = \mathcal{A}(x)$, i.e., $\mathcal{A}(x_k) = \mathcal{A}(x)$. Since x_k and x are irreducible, this implies $x_k = x$. For all $l, 1 \leq l \leq m$, $\mathcal{A}(x_l) \subseteq \mathcal{A}(\overline{x}) = \mathcal{A}(x)$. Since the x_l are irredundant, $x_l = x$. Thus $\overline{x} = x$.
(ii) That $\overline{x} \leq \overline{y}$ implies $\mathcal{A}(\overline{x}) \subseteq \mathcal{A}(\overline{y})$ is obvious. Suppose $\mathcal{A}(\overline{x}) \subseteq \mathcal{A}(\overline{y})$. Then $\mathcal{A}(\overline{x} \cup sup\mathcal{A}(\overline{y})) = \mathcal{A}(\overline{y})$. Therefore $\overline{x} \cup sup\mathcal{A}(\overline{y}) \leq \overline{y}$ by definition of $\overline{y}$. Hence $\overline{x} \leq \overline{y}$.
(iii) Let $\overline{x} = x_1 \cup \cdots \cup x_m$ be the decomposition of $\overline{x}$. Let $z \in Es_\mathcal{B} - \{0\}$ and $z' = I(z)$. Then $z \in \overline{x}$ iff $z' \leq x_k \leq \overline{x}$ for some $k, 1 \leq k \leq m$. By (i) and (ii), $z' \leq x_k$ iff $\mathcal{A}(z') \subseteq \mathcal{A}(x_k)$. Since $\mathcal{A}(x_k) \subseteq \mathcal{A}(\overline{x}) = \mathcal{A}(x)$, $z \in \overline{x}$ iff $\mathcal{A}(I(z)) \subseteq \mathcal{A}(x)$.
(iv) $\overline{x} \cap \overline{y} = \{z \in Es_\mathcal{B} - \{0\} | \mathcal{A}(z) \subseteq \mathcal{A}(x)\} \cap \{z \in Es_\mathcal{B} - \{0\} | \mathcal{A}(z) \subseteq \mathcal{A}(y)\}$
$= \{z \in Es_\mathcal{B} - \{0\} | \mathcal{A}(z) \subseteq \mathcal{A}(x) \wedge \mathcal{A}(z) \subseteq \mathcal{A}(y)\}$
$= \{z \in Es_\mathcal{B} - \{0\} | \mathcal{A}(z) \subseteq \mathcal{A}(x \cap y)\}$
$= \overline{x \cap y}$.
(v) Define $\psi : \mathcal{I} \to \mathbf{2}^A$: $\psi(x) := \mathcal{A}(x)$. Obviously ψ is a lattice homomorphism. Therefore $\approx$ (the kernel of ψ) is a congruence on $\mathcal{I}$ and $\mathcal{I}/\approx \ \cong \mathbf{2}^A$. Since a finite distributive lattice is determined by its irreducible elements, $\mathbf{2}^A \cong \mathbf{Su}_\mathcal{B}$.

3.7 DEFINITION. Let $\overline{x} \in \mathcal{I}$ and $\overline{x} = x_1 \cup \cdots \cup x_m$ be the (unique) decomposition of $\overline{x}$ into irreducible elements. Then the *normal form* of $\overline{x}$ is defined: $\mathcal{N}(\overline{x}) = \{x_1, \ldots, x_m\}$.

The elements of $\mathcal{N}(\overline{x})$ are the maximal elementary subsets of $\overline{x}$. That is, if $y \in Es_\mathcal{B}$ such that $y \leq \overline{x}$ then it follows from 3.4 that $y \leq x_k$ for some $x_k \in \mathcal{N}(\overline{x})$.

Since $\mathcal{I}/\approx \ \cong \mathbf{2}^A$, it can be concluded that every subset of A has a unique representation as the union of maximal elementary subsets.

3.8 An obvious but useful result of 3.6 and 3.7 is the following.

LEMMA. Let $\overline{x}, \overline{y} \in \mathcal{I}$ with normal forms $\mathcal{N}(\overline{x}) = \{x_1, \ldots, x_m\}$ and $\mathcal{N}(\overline{y}) = \{y_1, \ldots, y_n\}$. Then $\overline{x} \cap \overline{y}$ has the normal form $\mathcal{N}(\overline{x} \cap \overline{y}) = \{z_1, \ldots, z_r\}$ satisfying (i) $\forall q \ \exists k, l$

such that $z_q = x_k \cap y_l$ and (ii) $\forall k, l\ \exists q$ such that $x_k \cap y_l \leq z_q$, where $1 \leq q \leq r$, $1 \leq k \leq m$, and $1 \leq l \leq n$.
proof: $\overline{x} \cap \overline{y} = \bigcup_{1 \leq k \leq m, 1 \leq l \leq n}(x_k \cap y_l)$. Since the decomposition of $\overline{x} \cap \overline{y}$ is unique, $\bigcup_{1 \leq k \leq m, 1 \leq l \leq n}(x_k \cap y_l)$ is identical to $\bigcup_{1 \leq q \leq r} z_q$ up to redundancy. The lemma follows.

EXAMPLE. Let $B = \{P_i | i = 1,2\}$, $P_i = \{p_i^j | j = 1,2,3\}$, $x = [p_1^2 \cap (p_2^2 \cup p_2^3)] \cup [(p_1^2 \cup p_1^3) \cap p_2^2] \cup [p_1^3 \cap (p_2^1 \cup p_2^2)]$, $y = [p_2^2] \cup [p_1^3 \cap (p_2^2 \cup p_2^3)]$. Then $x \cap y = [p_1^2 \cap p_2^2] \cup [(p_1^2 \cup p_1^3) \cap p_2^2] \cup [p_1^3 \cap p_2^2]$. Eliminating redundant elementary subsets, $x \cap y = (p_1^2 \cup p_1^3) \cap p_2^2$.

This result is useful because it shows that the normal form of the meet of two closed elements is the set of pairwise meets of the elements of their normal forms, with redundant elements removed. Removal of a redundant element involves recognition that the redundant element entails some other element in the normal form. According to 2.5, this requires only a componentwise comparison. The following definition incorporates this result.

DEFINITION. Let $\overline{x}, \overline{y} \in \mathcal{I}$ with normal forms $\mathcal{N}(\overline{x}) = \{x_1, \ldots, x_m\}$ and $\mathcal{N}(\overline{y}) = \{y_1, \ldots, y_n\}$. Then $\mathcal{N}(\overline{x}) \Delta \mathcal{N}(\overline{y}) := \{x_k \cap y_l | 1 \leq k, k' \leq m, 1 \leq l, l' \leq n$ and $x_k \cap y_l \leq x_{k'} \cap y_{l'}$ implies $k = k'$ and $l = l'\}$.

Thus $\mathcal{N}(\overline{x}) \Delta \mathcal{N}(\overline{y}) = \mathcal{N}(\overline{x} \cap \overline{y})$. Let C be the set of normal forms of closed elements of $\mathcal{I}$. Then (C, Δ) is a meet semilattice.

3.9 Let $x \in \mathcal{I}$. The *pseudocomplement* of x is that element $x^* \in \mathcal{I}$ such that $\forall y \in \mathcal{I} : y \cap x = 0$ iff $y \leq x^*$. Thus, if it exists, $x^* := sup\{y \in \mathcal{I} | x \cap y = 0\}$. Every element of $\mathcal{I}$ has a pseudocomplement (i.e., $\mathcal{I}$ is a pseudocomplemented lattice) because $\mathcal{I}$ is a finite distributive lattice [6].

Because of the structure of $\mathcal{I}$, the pseudocomplement relative to an interval is also of interest.

DEFINITION. Let $\mathcal{B}$ be a system of bases with domain T. Let $\alpha = \beta.b \in T$, a_α be an atom relative to basis B_β and $x \in [0, a_\alpha]$. Then the *pseudocomplement of x in* $[0, a_\alpha]$ is defined $x_\alpha^* := sup\{y \in [0, a_\alpha] | x \cap y = 0\}$.

It follows from the definition that $x_\beta^* = (a_\alpha)_\beta^* \cup x_\alpha^*$. More generally, if $\alpha = b_1.b_2. \cdots .b_m$ then $x^* = \bigcup_{k=1}^m (a_{b_1.\cdots.b_k})_{b_1.\cdots.b_{k-1}}^* \cup x_\alpha^*$. (Note that b_0 is interpreted as the empty string, ϵ.)

3.10 Observe that $\mathcal{A}(x)$ and $\mathcal{A}(x^*)$ partition A. In fact, $z \cap x = 0$ iff $\mathcal{A}(z) \subseteq A - \mathcal{A}(x)$ iff $z \leq \overline{z} \leq \overline{sup(A - \mathcal{A}(x))}$ by 3.6(ii). Therefore $x^* = \overline{sup(A - \mathcal{A}(x))}$.

The following useful identities therefore hold in $\mathcal{I}$. (i) $x^* = \overline{x}^*$; (ii) $x^{**} = \overline{x}$; (iii) $\overline{x \cup x^*} = 1$; (iv) $(x \cup y)^* = x^* \cap y^*$; (v) $(x \cap y)^* = \overline{x^* \cup y^*}$.

It follows that the set of pseudocomplements of $\mathcal{I}$ is identical to the set of closed elements of $\mathcal{I}$.

3.11 Irreducible elements of $\mathcal{I}$ have the simplest pseudocomplements. Consider first the easy case of a single basis, i.e., $\mathcal{B} = B$. Let x be the principal ideal generated by $z = \bigcap_{i \in I_B} \bigcup_{j \in J_i^z} p_i^j \in Es_B$. Let $z_i = \bigcup_{j \in J_i - J_i^z} p_i^j \in Es_B$. Then by 2.4, $z \cap z_i = 0$ for all $i \in I_B$. Moreover, if $y \in Es_B$ such that $z \cap y = 0$ then $y \leq z_i$ for some $i \in I_B$. Since $\mathbf{Es_B}$ is embedded in $\mathcal{I}$ as a meet semilattice, $x \cap I(z_i) = 0$ for all $i \in I_B$ also. By distributivity of $\mathcal{I}$, $x \cap [\bigcup_{i \in I_B} I(z_i)] = 0$. The $I(z_i)$ for $i \in I_B^z$ are nonzero, principal, irredundant and no other irredundant principal ideals can be joined. Therefore, $\bigcup_{i \in I_B^z} I(z_i)$ is the irredundant decomposition of the pseudocomplement of x.

The general case is similar. Let x be the principal ideal generated by $\bigcap_{i \in I_{B_\alpha}} \bigcup_{j \in J_{\alpha,i}^z} p_{\alpha,i}^j \in Es_{\mathcal{B}}$, where $\alpha = b_1.b_2.\cdots.b_m$. Then by 3.9, $x^* = \bigcup_{k=1}^m (a_{b_1.\cdots.b_k})^*_{b_1.\cdots.b_{k-1}} \cup x_\alpha^* = \bigcup_{k=1}^m [\bigcup_{i \in I_{B_{b_1.\cdots.b_{k-1}}}} I(\bigcup_{j \in J_{b_1.\cdots.b_{k-1},i} - J^{a_{b_1.\cdots.b_k}}_{b_1.\cdots.b_{k-1},i}} p^j_{b_1.\cdots.b_{k-1},i})] \cup [\bigcup_{i \in I_{B_\alpha}^z} I(\bigcup_{j \in J_{\alpha,i} - J_{\alpha,i}^z} p_{\alpha,i}^j)]$.

The ideals are all principal and irredundant. Therefore the expression is the decomposition of x^* for x an irreducible element.

3.12 Now consider an arbitrary $x \in \mathcal{I}$. Let $\overline{x} = x_1 \cup \cdots \cup x_m$ be the decomposition of its closure. $x^* = \overline{x}^* = x_1^* \cap \cdots \cap x_m^*$ by 3.10
$= (x_{11} \cup \cdots \cup x_{1n_1}) \cap \cdots \cap (x_{m1} \cup \cdots \cup x_{mn_m})$ by 3.11
$= (x_{11} \cap \cdots \cap x_{m1}) \cup \cdots \cup (x_{1n_1} \cap \cdots \cap x_{mn_m})$ by distributivity
$= z_1 \cup \cdots \cup z_q$
where $z_r = (x_{1r(1)} \cap \cdots \cap x_{mr(m)})$ is irreducible for $1 \leq r \leq q$, and $q = \prod_{k=1}^m n_k$. In general, $z_1 \cup \cdots \cup z_q$ will not be irredundant (see example below). Therefore, the irredundant decomposition of x^* is the join with redundant z_k removed.

EXAMPLE. Let $\mathcal{B} = \{B, B_1, B_2\}$, $B_\alpha = \{P_{\alpha,1}, P_{\alpha,2}\}$ for $\alpha \in \{\epsilon, 1, 2\}$, $P_{\alpha,i} = \{p_{\alpha,i}^1, p_{\alpha,i}^2\}$ for $i \in \{1,2\}$ and $x = p_1^1 \cap p_2^1 \cap p_{1,1}^1 \cup p_1^1 \cap p_2^2 \cap p_{2,1}^1$ (see Figure 3).
Then $x^* = [p_1^1 \cap p_2^1 \cap p_{1,1}^2 \cup p_1^2 \cup p_2^2] \cap [p_1^1 \cap p_2^2 \cap p_{2,1}^2 \cup p_1^2 \cup p_2^1]$
$= [p_1^1 \cap p_2^1 \cap p_{1,1}^2] \cup [p_1^2] \cup [p_1^2 \cap p_2^1] \cup [p_1^1 \cap p_2^2 \cap p_{2,1}^2] \cup [p_1^2 \cap p_2^2]$.

DEFINITION. Let $\overline{x} \in \mathcal{I}$. The *complement* of $\mathcal{N}(\overline{x})$ is defined:
(i) if $\overline{x}$ is principal, generated by $\bigcap_{i \in I_{B_\alpha}} \bigcup_{j \in J_{\alpha,i}^z} p_{\alpha,i}^j \in Es_{\mathcal{B}}$, where $\alpha = b_1.b_2.\cdots.b_m$, so that $\mathcal{N}(\overline{x}) = \{\overline{x}\}$ then
$\sim \mathcal{N}(\overline{x}) := \bigcup_{k=1}^m [\bigcup_{i \in I_{B_{b_1.\cdots.b_{k-1}}}} I(\bigcup_{j \in J_{b_1.\cdots.b_{k-1},i} - J^{a_{b_1.\cdots.b_k}}_{b_1.\cdots.b_{k-1},i}} p^j_{b_1.\cdots.b_{k-1},i})] \cup$
$[\bigcup_{i \in I_{B_\alpha}^z} I(\bigcup_{j \in J_{\alpha,i} - J_{\alpha,i}^z} p_{\alpha,i}^j)]$;
(ii) if $\overline{x}$ is not principal and $\mathcal{N}(\overline{x}) = \{x_1, \ldots, x_m\}$ then
$\sim \mathcal{N}(\overline{x}) := \sim \mathcal{N}(x_1) \Delta \cdots \Delta \sim \mathcal{N}(x_m)$.

Thus $\sim \mathcal{N}(\overline{x}) = \mathcal{N}(\overline{x}^*)$. Hence C is closed under $\sim$.

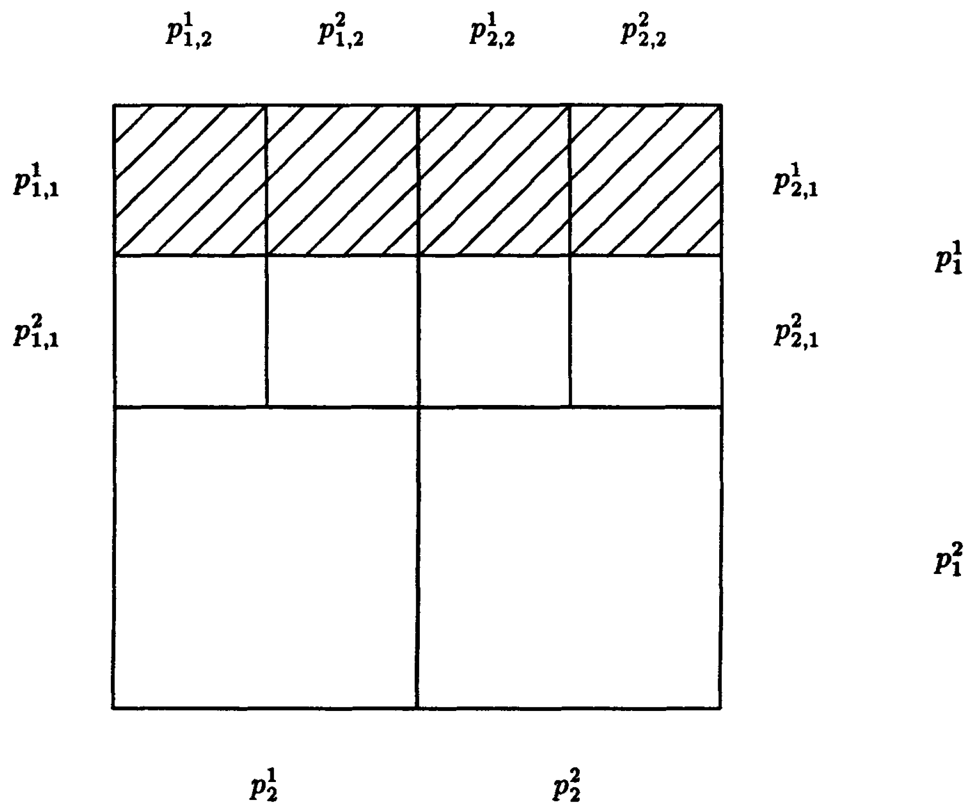

Figure 3: Partitions of S. x is the shaded area.

3.13 DEFINITION. Let $\overline{x}, \overline{y} \in \mathcal{I}$. Then $\mathcal{N}(\overline{x}) \veebar \mathcal{N}(\overline{y}) := \sim (\sim \mathcal{N}(\overline{x}) \Delta \sim \mathcal{N}(\overline{y}))$.

Thus $\mathcal{N}(\overline{x}) \veebar \mathcal{N}(\overline{y}) = \sim (\mathcal{N}(\overline{x}^*) \Delta \mathcal{N}(\overline{y}^*)) = \sim \mathcal{N}(\overline{x}^* \cap \overline{y}^*) = \mathcal{N}((\overline{x}^* \cap \overline{y}^*)^*)$. Hence C is closed under $\veebar$.

THEOREM. $\mathbf{C} := (C, \veebar, \Delta, \sim, 0, 1)$ is a Boolean algebra, the *algebra of normal forms.*
proof: $\mathcal{I}$ is a pseudocomplemented lattice. Let $K := \{x^* | x \in \mathcal{I}\}$. Then $\mathbf{K} := (K, \underline{\sqcup}, \cap, *, 0, 1)$, where $x \underline{\sqcup} y = (x^* \cap y^*)^*$, is a Boolean algebra (see [6], Theorem I.6.4.). Obviously, $\mathbf{C} \cong \mathbf{K}$.

3.14 Having defined a normal form for elements of the reduced semantic domain and an algebra of normal forms, the discussion can turn to the definition of a lexicon.

DEFINITION. A *vocabulary*, V, is a set of lexical items. A *lexicon for V relative to basis $\mathcal{B}$* is a map $v_{\mathcal{B}}$: $V \rightarrow \mathbf{C}$.

The definition of the map $v_{\mathcal{B}}$ is addressed in the next section.

Let *and*, *or* and *not* denote operations on elements of V, intended as logical conjunction, disjunction and negation, respectively. Then $v_{\mathcal{B}}$ can be extended to expressions over V generated by these operations by defining:
$v_{\mathcal{B}}(e_1 \, and \, e_2) := v_{\mathcal{B}}(e_1) \underline{\Delta} v_{\mathcal{B}}(e_2)$
$v_{\mathcal{B}}(e_1 \, or \, e_2) := v_{\mathcal{B}}(e_1) \veebar v_{\mathcal{B}}(e_2)$
$v_{\mathcal{B}}(not \, e_1) :=\sim v_{\mathcal{B}}(e_1)$
where e_1 and e_2 are expressions over V. The images under this extension are the meanings that intuition dictates.

This suggests that an inverse lexicon, $v_{\mathcal{B}}^{-1}$, mapping meanings in C to sets of expressions over V might be defined. The inverse function is significant for translation and language generation. However, it presents some problems since $v_{\mathcal{B}}^{-1}$ is in general only a partial function. The inverse lexicon will be addressed a subsequent paper.

3.15 Finally, entailment, synonymy and contradiction or anomaly relative to the lexicon can be defined for lexical items. (Cf 2.8.) Let e_1, e_2 be Boolean expressions over V. e_1 ***entails*** $e_2 :\Leftrightarrow v_{\mathcal{B}}(e_1 \, and \, e_2) = v_{\mathcal{B}}(e_1)$. e_1 and e_2 are ***synonymous*** $:\Leftrightarrow v_{\mathcal{B}}(e_1) = v_{\mathcal{B}}(e_2)$. e_1 ***contradicts*** e_2 or e_1 is ***anomalous in conjuction with*** $e_2 :\Leftrightarrow v_{\mathcal{B}}(e_1 \, and \, e_2) = 0$.

Observe that consistency with common usage requires the definition: e_1 contradicts e_2 iff $v_{\mathcal{B}}(e_1 \, and \, e_2) = 0$. Therefore anomaly must be a kind of contradiction. The distinction between anomalous contradiction and nonanomalous contradiction is addressed in Section 4.

4 LEXICON CONSTRUCTION

Construction of a lexicon has two parts: (i) computation of an extended basis $\mathcal{B}$, and (ii) computation of the map $v_{\mathcal{B}}$: $V \to \mathbf{C}$.

4.1 The computation of an extended basis assumes a set of distinguishing semantic properties, D, adequate to distinguish between meanings to the desired degree of precision. These properties are provided by empirical linguistic analysis. The analysis may relate to a single language, to a group of languages or to all languages. The analysis may be fixed once for all or it may evolve. The lexicon construction is indifferent to these matters. The construction produces an unambiguous representation for meaning based on the input provided.

Computation of the map $v_{\mathcal{B}}$ assumes a definition or classification of the lexical items of V in terms of the distinguishing properties. Since $v_{\mathcal{B}}$ is a map from lexical items to normal forms of meaning, this computation must determine the normal forms.

A suitable linguistic analysis is the "componential analysis of meaning" described by Eugene Nida [9]. According to Nida, componential analysis consists of the following four linguistic procedures.
(i) Naming. A referent is designated for the lexical item. The referent may be an object, an event, or a condition, including the effect on an audience.
(ii) Paraphrasing. The lexical item is explicated in terms of already known meanings.
(iii) Defining. Using the results of naming and paraphrasing, those properties are extracted that relate a meaning to and differentiate it from other meanings. These properties, which Nida calls "diagnostic components of meaning", are the elements of D.
(iv) Classifying. The lexical items are placed in classes determined by the diagnostic components.

The results of the third and fourth procedures constitute the input to the computation of the extended basis $\mathcal{B}$ and the map $v_{\mathcal{B}}$, respectively.

It seems that these four procedures also describe the process by which a child acquires semantic knowledge. In the child the process is incremental. The linguist on the other hand carries out the procedures on large classes of related meanings, i.e., semantic domains. When considering machine acquisition of semantic knowledge, both possibilities should be kept in mind.

To prevent misunderstanding about the set D, it should be emphasized that the elements of D are semantic constructs. They are denoted by English words and phrases. Nonetheless, they are not to be identified with those words and phrases. The words and phrases are simply convenient mnemonics for codes. Of course, it may happen that a word w is in both D and V, and that the meaning of $w \in V$ is $w \in D$ (more precisely, the code denoted by $w \in D$). This should cause no confusion if the nature of D is properly understood.

The computation of basis $\mathcal{B}$ requires specification of D sufficient to form a poset as follows. The elements of the poset are the terms in elements of D and the operator $\cap$. For $d_1, d_2 \in D$, $d_1 \cap d_2$ denotes the intersection of the extensions of d_1 and d_2; $d_1 = d_2$ is the assertion that the extensions of d_1 and d_2 are identical. One can also view $\cap$ as logical conjunction and $=$ as logical equivalence of properties. The partial order $\leq$ is defined $x \leq y :\Leftrightarrow x \cap y = x$. The zero element 0 denotes the empty set and may be viewed as logical impossibility. The specification of D by the empirical data must be sufficient to identify equal terms: e.g., $x \cap y = x$ or $x \cap y = 0$.

The set A of *atoms* of the poset is defined to contain those terms a such that $0 \prec a$. For any term x, the set of atoms *dominated* by x is defined $\mathcal{A}(x) := \{a \in A | a \leq x\}$. The *rank* of x is defined $r(x) := |\mathcal{A}(x)|$.

As a practical matter, the specification of D should be given in a form that is simple and compatable with the algorithm used to compute $\mathcal{B}$. The design of an optimal input data representation and an efficient algorithm will not be addressed here. Rather an arbitrary presentation of the data (convenient for manual processing) will be used. The computation of $\mathcal{B}$ will only be illustrated. Two examples will be used.

D will be specified by a tree H in which a path represents logical conjunction of the nodes on that path. Each path from the root to a leaf represents an atom. Logically equivalent terms are represented by the structurally simplest equivalent term.

4.2 The first example is taken from Nida [9], where it is used as an illustration of componential analysis. The vocabulary V is a set of names for rigid fasteners; for convenience, however, numerical codes will be used in place of the longer names. The names and their numerical codes are listed in Table 1. The set D of distinguishing properties is given in Table 2 along with abbreviations. The tree H for this example is shown in Figure 4. The atoms are in one-to-one correspondence with the leaves of H. An atom is given by the conjunction of labels of the nodes on a path from the root to a leaf. For example, the leftmost leaf is associated with atom $T \cap TT \cap SL \cap P \cap RD \cap SM$. This tree asserts that each atom is a logically possible conjunction of distinguishing properties and that the atoms span the universe S of meanings relating to rigid fasteners. There are a total of 40 atoms.

The first step in the construction of the basis identifies all partitions of the unit element, RF. An element x is partitioned by the set $\{x_1, \ldots, x_m\} \subseteq D$ if $r(x \cap x_1) + \cdots + r(x \cap x_m) = r(x)$, $r(x \cap x_k) \neq 0$ and $r(x \cap x_k \cap x_l) = 0$ for $1 \leq k, l \leq m$ and $k \neq l$.[9] Therefore, partitions of RF are: $\{T, NT\}$, $\{P, NP\}$, $\{RD, SQ\}$, $\{SM, LG\}$,

[9] More precisely, x is partitioned by the restriction of $\{x_1, \ldots, x_m\}$ to x (see 2.9). No confusion

1	common nail
2	finishing nail
3	slot head wood screw, partial threads
4	slot head wood screw, full threads
5	phillips head wood screw, full threads
6	phillips head wood screw, partial threads
7	machine bolt, square head, full threads
8	machine bolt, square head, partial threads
9	carriage bolt, full threads
10	carriage bolt, partial threads
11	rivet

Table 1: Rigid fastners and their abbreviations.

T	threaded
NT	not threaded
P	pointed
NP	not pointed
TT	threads to top
NTT	threads not to top
RD	round head
SQ	square head
SM	small head
LG	large head
SL	slot drive
PH	phillips drive

Table 2: Distinguishing properties for rigid fasteners and their abbreviations.

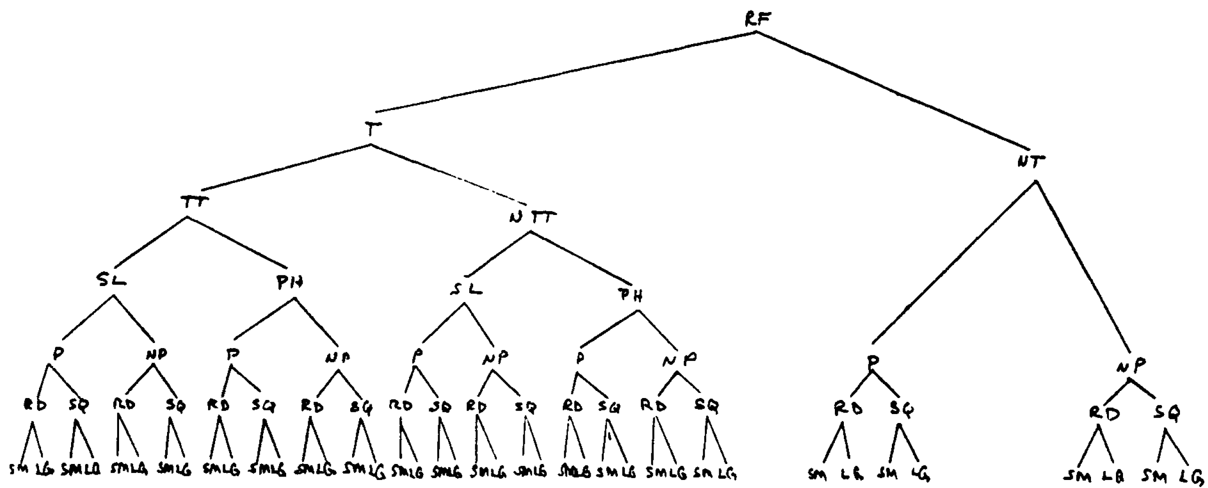

Figure 4: Relations between distinguishing properties for rigid fastners

		P		NP	
		SM	LG	SM	LG
T	RD	4	4	4	4
	SQ	4	4	4	4
NT	RD	1	1	1	1
	SQ	1	1	1	1

Figure 5: First-level basis for RF with ranks of atoms shown.

$\{TT, NTT, NT\}$, $\{SL, PH, NT\}$.

When these partitions are examined further for independence, it is found that $\{T, NT\}$ $\otimes\{P, NP\} \otimes \{RD, SQ\} \otimes \{SM, LG\}$ is a partition of RF and hence these four partitions are independent. Therefore, they comprise the first-level basis. This is diagrammed in Figure 5. This figure represents a four-dimensional cube drawn in two dimensions.

Next, each atom of the first-level basis is considered, in turn, as the unit element. Partitions of some these atoms are identified. For example, the atom $T \cap P \cap RD \cap SM$ is partitioned by $\{TT, NTT\}$ and $\{SL, PH\}$. Since $\{TT, NTT\} \otimes \{SL, PH\}$ is a partition of $T \cap P \cap RD \cap SM$, these two partitions are independent and therefore

will result, however, from this more convenient though less precise language.

	SL	PH
TT	1	1
NTT	1	1

Figure 6: Second-level basis for atoms of rank 4 (each atom has rank 1).

form a basis for $T \cap P \cap RD \cap SM$, shown in Figure 6. A similar result is obtained for the remaining first level atoms of rank 4. This completes the system of bases and gives a definition of $\mathcal{B}$.

Finally, the definitions of members of V in terms of the distinguishing properties are given in tabular form in Table 3. These definitions are sufficient to immediately define $v_{\mathcal{B}}$. The lexicon representation for 4 (slot head wood screw with full threads) is $v_{\mathcal{B}}(4) = [[T \cap P \cap RD \cap SM] \cap [TT \cap SL]] \cup [[T \cap P \cap RD \cap LG] \cap [TT \cap SL]]$.

Now suppose it were desired to define a new vocabulary element, "wood screw", meaning any of the types 3, 4, 5 or 6. Then wood screw would be represented in the lexicon as $[T \cap P \cap RD]$. It follows immediately from the lexicon representations that 4 entails wood screw since $v_{\mathcal{B}}(4) \subset v_{\mathcal{B}}(\text{wood screw})$.

4.3 In the example above, the partitions $\{\{P, NP\}, \{RD, SQ\}, \{SM, LG\}, \{SL, PH, NT\}\}$ could as well have been chosen as the first-level basis. In general, given a particular universe, a basis is not uniquely determined. Various criteria might be used to select a basis. For example, certain distinguishing properties might be recognized as related in a "meaningful" way and preferred as elements in the same partition. This criterion was used implicitly in selecting $\{T, NT\}$ rather that $\{SL, PH, NT\}$.

A quantitative criterion is the extent to which a particular basis subdivides the universe (in a sense, the "information content" of the basis). A basis that achieves a

	1	2	3	4	5	6	7	8	9	10	11
T	No	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No
P	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No	No	No
TT	—	—	No	Yes	Yes	No	Yes	No	Yes	No	—
RD/SQ	RD	RD	RD	RD	RD	RD	SQ	SQ	RD	RD	RD
SM/LG	LG	SM	E	E	E	E	E	E	E	E	E
SL/PH	—	—	SL	SL	PH	PH	E	E	E	E	—

Table 3: Definitions of rigid fasteners. Note: "—" means the property is not applicable; "E" means both values of the property are applicable.

greater subdivision would be preferred. A precise definition of this notion is developed below.

Consider a universe of N entities. With no basis (i.e., subdivision) at all, to find an entity satisfying a particular description, it might be necessary to examine $N-1$ entities. The extent to which a particular basis improves upon this worst case will be taken as a figure of merit for that basis. Specifically, if the maximum number of steps required to find the entity with the basis is n, then $n/(N-1)$ will be taken as the figure of merit for the basis. The smaller the figure of merit, the better the basis. For example, suppose the universe is subdivided into two (equal for simplicity) blocks determined by the presence or absence of some property. To find the entity requires that one block be checked for appropriateness and then the $\frac{N}{2}-1$ entities of the appropriate block be examined. Thus $1+\frac{N}{2}-1$ steps are required, resulting in a figure of merit $\frac{N}{2(N-1)}$. Similarly, a partition of four equal blocks results in a figure of merit $\frac{1}{N-1}(3+\frac{N}{4}-1)=\frac{N+8}{4(N-1)}$; two independent partitions, each of two equal blocks, yields a figure of merit $\frac{1}{N-1}(2+\frac{N}{4}-1)=\frac{N+4}{4(N-1)}$.

Obviously, one of the worst bases is the unit basis consisting only of the unit partition: a single block of N entities. The figure of merit for the unit basis is 1. Just as bad is the zero basis consisting only of the zero partition: N blocks of one entity each. For the zero basis also the figure of merit is 1. One of the best is the basis of $\lceil log_2 N \rceil$ binary partitions, with a figure of merit equal to $\frac{\lceil log_2 N \rceil}{N-1}$.

In general the figure of merit is defined: $M := \frac{1}{N-1}[\sum_{i\in I_B}(|J_i|-1)+\sum_{atom\, a\in B}\frac{r(a)}{N}(r(a)-1)]$ where $\frac{r(a)}{N}$ is taken to be the relative frequency with which atom a is accessed, assuming all entities in the universe are accessed with equal frequency. It appears that $\frac{log_2 N}{N-1}\leq M\leq 1$.

Returning to the example, bases for RF are found to be
$B = \{\{T, NT\}, \{P, NP\}, \{RD, SQ\}, \{SM, LG\}\}$
$B' = \{\{TT, NTT, NT\}, \{P, NP\}, \{RD, SQ\}, \{SM, LG\}\}$
$B'' = \{\{SL, PH, NT\}, \{P, NP\}, \{RD, SQ\}, \{SM, LG\}\}$
Their figures of merit are $M(B) = 0.164$, $M(B') = 0.149$, and $M(B'') = 0.149$ while $M_{min} = (log_2 N)/(N-1) = 0.136$. Thus, by the second criterion B' or B'' would be selected as the first-level basis. The first criterion would lead to selection of B' over B''.

4.4 A second somewhat larger example, also taken from Nida [9], deals with English verbs of motion. The distinguishing properties are listed in Table 4. The index of a property in this list will be used as an abbreviation for that property. For example, "continuous contact with the surface by one then another limb or set of limbs" will be abbreviated "E3a".

As in the first example, additional information about the distinguishing properties is presented in the form of a tree. See Figure 7. Because of the large size, some subtrees are indicated by a triangle containing a label. The details of the subtree are shown in the tree whose root carries that label. Identical subtrees are only detailed once. The rank of a node is given in a small circle adjacent to the node.

Partitions of the unit element are easily found to be $\{A1a, A1b, A1c, A2, A3\}$, $\{B1, B2, B3\}$ and $\{G1, G2, G3\}$. The bases that can be formed from these are
$B = \{\{A1a, A1b, A1c, A2, A3\}\}$ and
$B' = \{\{B1, B2, B3\}, \{G1, G2, G3\}\}$.
The figures of merit for these bases are 0.862 and 0.135, respectively. Therefore, B' is chosen as the first-level basis.

Continuing in this manner, the extended basis shown in Figure 8 is computed.

The vocabulary and definitions of vocabulary elements in terms of the distinguishing properties are shown in Table 5. Minor deviations from Nida's data are indicated. These data immediately determine v_B. For example,
$v_B(\text{climb}) = [B1 \cap G3] \cap [A1a] \cap [C2 \cap D1 \cap E3a] \cap [F1]$ and
$v_B(\text{fall}) = [B3 \cap G2] \cap [A1c] \cap [D2 \cap E1]$.

4.5 Figure 7 is based on certain assumptions about the relations between the distinguishing properties. (Nida does not give any relations between distinguishing properties.) These particular assumptions may not be as good as some others. The effect of the assumptions on the distinguishing properties will affect the "quality" of the lexicon. However, the approach to lexicon construction described here is independent of the definition of any particular set of distinguishing properties. Neither claims nor assumptions are made regarding the universality, the quality or even the validity of

A. Environment
   1. Surfaces
      a. Supporting
      b. Nonsupporting
      c. Between surfaces on different levels
   2. Air
   3. Water
B. Source of energy
   1. Animate being
   2. Animate being and gravity
   3. Gravity
C. Use of limbs for propulsion
   1. All four limbs
   2. All limbs normally in contact with supporting surface (with optional addition of forelimbs for bipeds in climbing)
   3. Forelimbs
D. Points of contact with the surface
   1. Extremities of the limbs
   2. Any parts
   3. Continuous series of points
E. Nature of contact with the surface
   1. No contact during movement
   2. Intermittent contact
   3. Continuous contact
      a. By one and then another limb or set of limbs
      b. By the same or contiguous portion
F. Order of repeated contact between limbs and surface
   1. Alternating
   2. Variable but rhythmic
   3. 1-1-1-1 or 2-2-2-2 or continuous series of short jumps
   4. 1-1-2-2-1-1-2-2
G. Directional orientation
   1. Indeterminant
   2. Down
   3. Up

Table 4: Distinguishing properties for verbs of motion.

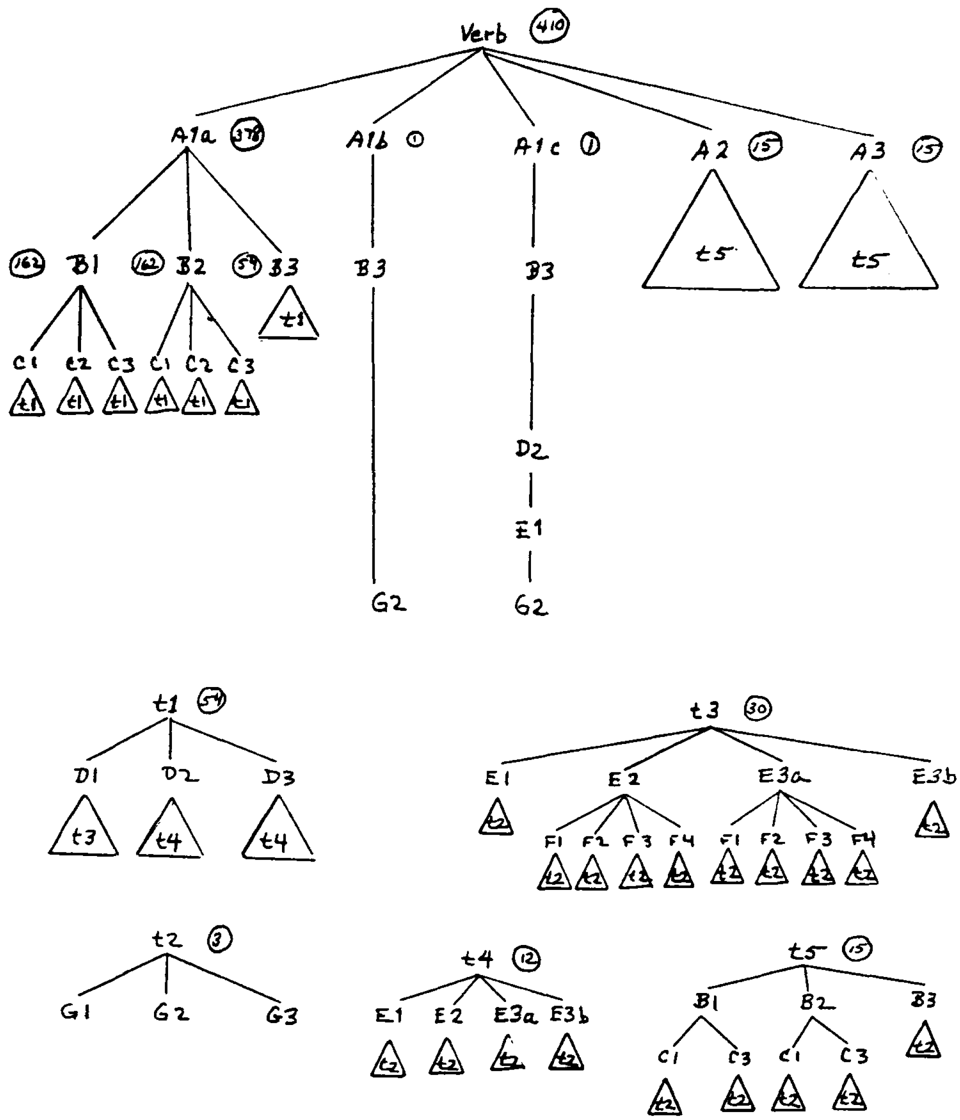

Figure 7: The tree presentation of the distinguishing properties for verbs of motion.

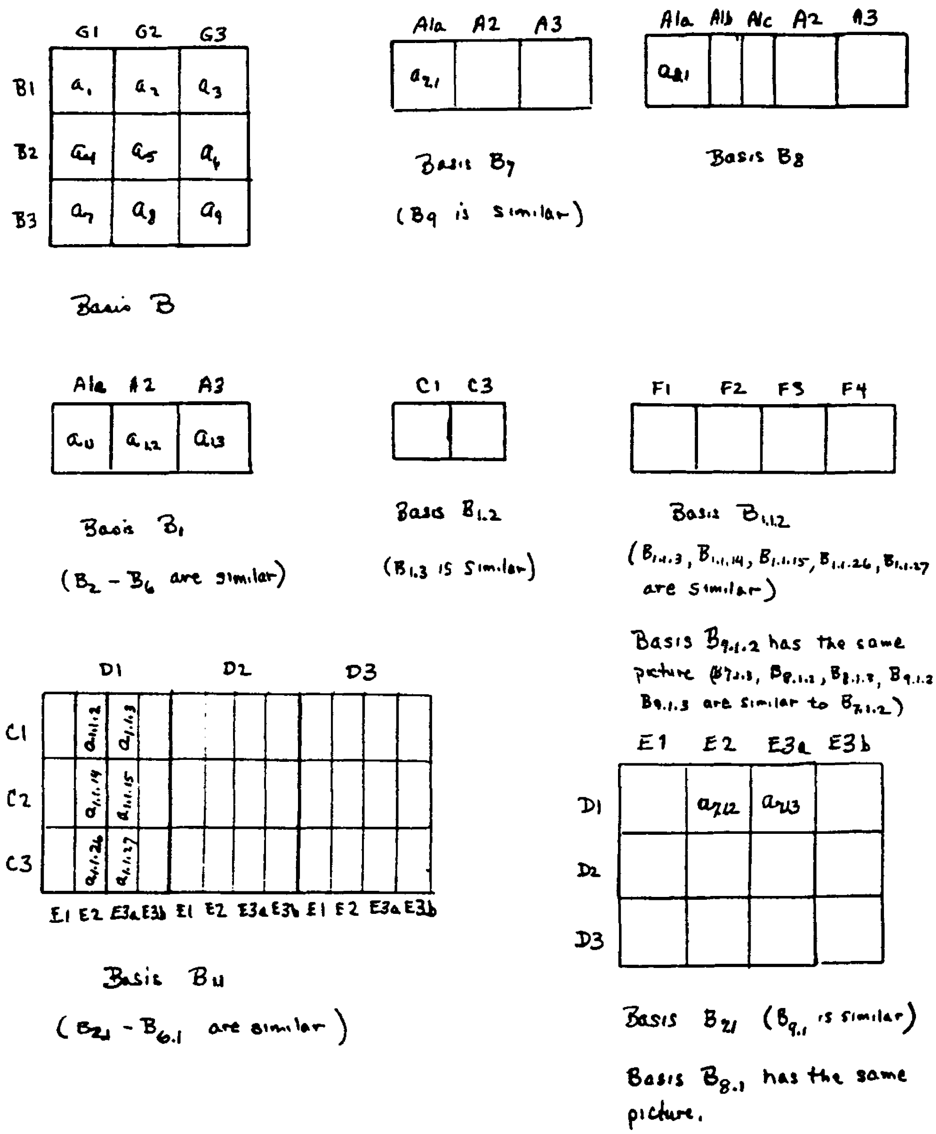

Figure 8: The extended basis for verbs of motion.

	climb	crawl	dance	fly	fall	hop	jump	roll	run	sink	skip	slide	swim	walk
A1a	x	x	x			x	x	x	x		x	x		x
A1b										x				
A1c					x									
A2				x										
A3													x	
B1	x	x	x	x		x	x		x		x		x	x
B2								x				x		
B3					x					x				
C1		x						*					x	
C2	x		*			x	x	*	x		x			x
C3				x				*						
D1	x	x	x			x	x		x		x			x
D2					x							x		
D3								x						
E1					x		x							
E2			x			x			x		x			
E3a	*	x	x											x
E3b								x				x		
F1	x	x							x					x
F2			x											
F3						x								
F4											x			
G1		x	x	x		x	x	x	x		x	x	x	x
G2					x					x				
G3	x													

Table 5: Definition of verbs of motion. ("x" indicates Nida's data; "*" indicates deviations from Nida.)

the distinguishing properties. It might be expected that invalid data will occasionally be used in the construction of a lexicon. The resulting errors in the lexicon will require eventual correction. The means by which this can be accomplished will be considered in a later paper.

It should be appreciated that definition of a "good" set of distinguishing properties for a given semantic domain is a significant task. Much of Nida's book is devoted to describing this task. The difficulty can be illustrated by attempting to add "bounce" to the lexicon of the second example.

Having a person on a trampoline in mind, one might define bounce as $B2 \cap G3 \cap A1a \cap C2 \cap D1 \cap E1$. However, if one thinks of a ball bouncing on the floor, the definition might be $B3 \cap G3 \cap A1a \cap D2 \cap E1$. But the inclusion of $G3$ fails to permit use of the word to describe a ball bouncing off a wall or ceiling!

A solution might be to define bounce$_1$, bounce$_2$ and bounce$_3$ to represent these different senses. But it would be better to admit that the set of distinguishing properties is too limited to accomodate this new lexical item and should be revised. At a minimum, it appears that heading B should be revised:

B. Source of energy
   1. animate source
   2. combination of animate and inanimate sources
   3. inanimate source

B′. Form of energy
   1. potential energy
      a. gravity
      b. elastic
      c. chemical
      d. electrical
   2. kinetic energy
   3. exchange of potential and kinetic energy

In terms of these revised distinguishing properties, the essence of bounce might be rendered as $A1a \cap B'3 \cap D2 \cap (E1 \cup E2)$. Further consideration might reveal this set to be inadequate as well.

4.6 The input data in the examples are very simple. In particular, the vocabulary elements are almost all defined simply as conjunctions of distinguishing properties. Any such conjunction is already a normal form.

In the general case, vocabulary elements may be defined by complex Boolean expressions in the elements of D. Computation of the map v_B then requires that the normal form of each such expression be computed.

This can be accomplished for $w \in V$ as follows. Let x be the Boolean expression in elements of D defining w. If any symbol in x does not denote a block of $\mathcal{B}$ (i.e., a block of some partition in $\mathcal{B}$), that symbol will be equivalent to some expression in the blocks of $\mathcal{B}$. Substitute for all such terms in x to get an equivalent expression x' containing only blocks of $\mathcal{B}$. Let y be the disjunction of conjunctions of blocks of $\mathcal{B}$ equivalent to x'. Then y is a disjunction of elementary subsets, say $y = y_1 \cup \cdots \cup y_n$. It follows that $\mathcal{N}(y_l) = \{y_l\}$ for $1 \leq l \leq n$, and by 3.13 that $\mathcal{N}(y) = \sim (\sim \mathcal{N}(y_1) \Delta \cdots \Delta \sim \mathcal{N}(y_n))$.

4.7 At this point it is possible to address the distinction between anomalous contradiction and nonanomalous contradiction. Consider atoms $a_{\alpha.b}, a_{\beta.c}$ defined by bases B_α, B_β respectively. Bases $B_{\alpha.b}$ and $B_{\beta.c}$ will be said to be *similar* iff (i) $\alpha = \beta$ and there exists a basis B' constructed from elements of D such that $B' \upharpoonright_{a_{\alpha.b}} = B_{\alpha.b}$ and $B' \upharpoonright_{a_{\alpha.c}} = B_{\alpha.c}$; or (ii) B_α is similar to B_β and there exists a basis B' constructed from elements of D such that $B' \upharpoonright_{a_{\alpha.b}} = B_{\alpha.b}$ and $B' \upharpoonright_{a_{\beta.c}} = B_{\beta.c}$. Intuitively, bases $B_{\alpha.b}$ and $B_{\beta.c}$ are similar if they are defined by the same distinguishing properties. Examples of similar bases are found in Figure 8.

If $x, y \in Es_{\mathcal{B}}$ and $x \cap y = 0$ then meanings x and y are contradictory. Suppose that x and y belong to elementary domains $[0, a_\alpha]$ and $[0, a_\beta]$ respectively, where $\alpha = b_1. \cdots .b_k$ and $\beta = c_1. \cdots .c_l$, $k \leq l$. If $B_{b_1.\cdots.b_k}$ is similar to $B_{c_1.\cdots.c_k}$, then x in conjunction with y is not anomalous; otherwise, x in conjunction with y is anomalous.

Now let $x, y \in Su_{\mathcal{B}}$ with normal forms $\mathcal{N}(x) = \{x_1, \ldots, x_m\}$ and $\mathcal{N}(y) = \{y_1, \ldots, y_n\}$. Then x and y are contradictory iff $x \cap y = 0$ and x is anomalous in conjunction with y iff elementary subset x_k is anomalous in conjunction with elementary subset y_l for all k, l such that $1 \leq k \leq m$, $1 \leq l \leq n$.

Thus crawl *and* skip, relative to the basis of the second example defining verbs of motion, is contradictory but not anomalous because the elementary domain to which $v_{\mathcal{B}}$(crawl) belongs has a basis similar to that of the elementary domain to which $v_{\mathcal{B}}$(skip) belongs. But sink *and* skip is both contradictory and anomalous since the elementary domains do not have similar bases. A more intuitively obvious anomaly is green idea, in the sense of green entity *and* idea. Relative to an extended basis defining the meanings of both words, green idea is contradictory and anomalous. This follows from the circumstance that $v_{\mathcal{B}}$(green entity) would belong to some elementary domain dominated by an atom representing concrete physical entities while $v_{\mathcal{B}}$(idea) would belong to some elementary domain dominated by an atom representing abstract entities.

Anomaly, like the related concepts entailment, synonymy and contradiction, is relative to a basis. Unlike these related concepts, anomaly has a further dependence on the set of distinguishing properties chosen to define meaning. While entailment, synonymy and contradiction can be defined in purely mathematical terms, anomaly cannot.

4.8 Finally consider briefly the length of the code for meanings in the semantic domain constructed for verbs of motion. In the extended basis, a path requiring the maximum length code is 1.1.2, containing subcodes for atoms $a_1, a_{1.1}$, and $a_{1.1.2}$ and for subsets of $a_{1.1.2}$. Bases B, B_1 and $B_{1.1}$ define 9, 3 and 36 atoms respectively and basis $B_{1.1.2}$ contains a single partition of 4 blocks. Therefore any subset of $a_{1.1.2}$ can be coded by $\lceil log_2 9\rceil + \lceil log_2 3\rceil + \lceil log_2 36\rceil + 4 = 4 + 2 + 6 + 4 = 16$ bits.

Since the extended basis defines 410 atoms, a lower bound for code length is $\lceil log_2 410\rceil$ $= 9$ bits.

For example, $v_B(\text{climb}) = [B1 \cap G3] \cap [A1a] \cap [C2 \cap D1 \cap E3a] \cap [F1] = a_3 \cap a_{3.1} \cap a_{3.1.15} \cap F1$ which could be coded 0011.01.001111.0001 or in decimal 3.1.15.1.

5 HIGHER LEVEL MEANING

The theory of lexical semantics developed in previous sections deals with entailment, synonymy, contradiction and anomaly relations on lexical meanings. It remains to be considered how these relations on lexical meanings can be used to determine similar relations on complex meanings. Complex meanings are constructed from simpler meanings (the simplest being lexical meanings) by the semantic functors of Montague's theory. Determination of entailment, synonymy, contradiction and anomaly on complex meanings constitute the direct deduction that was claimed in Section 1 to be part of linguistic competence.

The following discussion is couched in terms of the PTQ fragment as presented by Dowty [4].

5.1 It is necessary to define a partial ordering $\leq_1$ on complex meanings that will agree with an intuitive notion of entailment. Each translation into the Intentional Logic is interpreted as a set of some kind (set of individuals, set of propositions, set of properties, function or relation). It is natural therefore to define $\leq_1$ to hold between IL expressions of the same type just when set inclusion holds between their denotations. In all other instances $\leq_1$ is undefined.

Obviously the definition of $\leq_1$ must be consistent with the definition of $\leq$, the partial order on lexical meanings. It is therefore a requirement on the empirical linguistic data, from which both definitions must ultimately derive, to ensure that this is the case. It is sufficient that the linguistic data satisfy the following. Let x, y be expressions in PTQ with translations x', y' respectively.

(i) If x, y are basic common nouns (B_{CN}) or basic intransitive verbs (B_{IV})[10] then it

[10] B_a is the set of basic expressions of syntactic category a. P_a is the set of phrases of syntactic

is necessary that $v_B(x) \leq v_B(y)$ iff $x' \leq_1 y'$.

(ii) If x, y are lexical items that translate to functors of type $\langle a, b\rangle$, then it is necessary that $v_B(x) \leq v_B(y)$ iff $x' \leq_1 y' :\Leftrightarrow \forall z \in ME_a[x'(z) \leq_1 y'(z)]$.[11]

Since it will be assumed that these requirements are satisfied, $\leq_1$ will be written $\leq$ in subsequent discussion.

5.2 The following definition relates entailment between complex expressions to entailment between their constituents.

DEFINITION. Let $x \in ME_{\langle a,b\rangle}$. Then x is *isotone* $:\Leftrightarrow \forall y_1, y_2 \in ME_a[y_1 \leq y_2 \Rightarrow x(y_1) \leq x(y_2)]$ and x is *antitone* $:\Leftrightarrow \forall y_1, y_2 \in ME_a[y_1 \leq y_2 \Rightarrow x(y_2) \leq x(y_1)]$.

Now suppose that $x_1, x_2 \in ME_{\langle a,b\rangle}$ such that $x_1 \leq x_2$ and x_1 is isotone. Suppose further that $y_1, y_2 \in ME_a$ such that $y_1 \leq y_2$. Then it follows from the definition that $x_1(y_1) \leq x_1(y_2) \leq x_2(y_2)$. Thus from x_1 entails x_2 and y_1 entails y_2 the isotone property of x_1 permits $x_1(y_1)$ entails $x_2(y_2)$ to be inferred.

This result can be illustrated with expressions of PTQ.

5.3 First consider the determiners of PTQ, viz., every, a (or an) and the, in the light of the above definition. Their translations into IL are ($\Rightarrow_T$ is the translates-to relation):[12]
every $\Rightarrow_T \lambda P\lambda Q\forall x[P\{x\} \rightarrow Q\{x\}] \equiv every'$
a $\Rightarrow_T \lambda P\lambda Q\exists x[P\{x\} \wedge Q\{x\}] \equiv a'$
the $\Rightarrow_T \lambda P\lambda Q\exists x[\forall y[P\{y\} \leftrightarrow x = y] \wedge Q\{x\}] \equiv the'$
Therefore every man $\Rightarrow_T \lambda Q\forall x[man'(x) \rightarrow Q\{x\}]$. Similarly every human $\Rightarrow_T \lambda Q\forall x[human'(x) \rightarrow Q\{x\}]$. Each of these expressions in IL denotes a set of properties of individuals. Obviously the first contains the second since any property possessed by every human is certainly possessed by every man. Therefore, $[\![\text{man}]\!] \leq [\![\text{human}]\!]$[13] as lexical meanings while $[\![\text{every human}]\!] \leq [\![\text{every man}]\!]$. Since the same argument is valid for all $x, y \in P_{CN}$ such that $[\![x]\!] \leq [\![y]\!]$, it follows that the functor *every'* is antitone.

Similarly, a man $\Rightarrow_T \lambda Q\exists x[man'(x) \wedge Q\{x\}]$ and a human $\Rightarrow_T \lambda Q\exists x[human'(x) \wedge Q\{x\}]$. Now a property possessed by some man is certainly possessed by some human. Therefore $[\![\text{a man}]\!] \leq [\![\text{a human}]\!]$. Since this argument is valid generally, it is concluded

category a. Of course, $B_a \subseteq P_a$.

[11] A functor of type $\langle a, b\rangle$ is one whose argument is of type a and whose result is of type b. The set of expressions in IL of type a is denoted ME_a (meaningful expressions of type a). Therefore, a functor of type $\langle a, b\rangle$ is a member of $ME_{\langle a,b\rangle}$.

[12] See Dowty [4] for details of translation of PTQ into IL.

[13] $[\![x]\!]$ will denote the meaning of x whether in the sense of lexical meaning or in the sense of the interpretation of x' as an expression of IL. Further, $[\![x]\!] \leq [\![y]\!] :\Leftrightarrow [\![x]\!] \subseteq [\![y]\!]$.

that a' is isotone.

Finally, the man $\Rightarrow_T$ $\lambda Q \exists x[\forall y[man'(y) \leftrightarrow x = y] \wedge Q\{x\}]$ and the human $\Rightarrow_T$ $\lambda Q \exists x[\forall y[human'(y) \leftrightarrow x = y] \wedge Q\{x\}]$. In a model containing a single man but many humans, $[\![$the man$]\!] \neq \emptyset$ while $[\![$the human$]\!] = \emptyset$. On the other hand, in a model containing a single human but no man, $[\![$the man$]\!] = \emptyset$ while $[\![$the human$]\!] \neq \emptyset$. Therefore, *the'* can be neither isotone nor antitone in general. However, if $[\![$the man$]\!] \neq \emptyset \neq [\![$the human$]\!]$, then $[\![$the man$]\!] = [\![$the human$]\!]$. This argument holds generally for all common nouns. Therefore, when it occurs in nonvacuous phrases, *the'* is both isotone and antitone; when vacuous phrases are admitted, it is neither.

Quantitative determiners (which are not part of PTQ) can also be defined as logical constants (e.g., see [8]). Using reasoning similar to the above they are classified as follows. For natural numbers n and m: at least n, more than n, infinitely many, less than one-nth and no more than one-nth translate to isotone functors; no, at most n, less than n, (only) finitely many, at least one-nth and more than one-nth translate to antitone functors; the n, the n or more and the n or less translate to functors having the same character as *the'*; (exactly) n, all but n, between n and m and (exactly) one-nth translate to functors that are neither isotone nor antitone.

Possessive determiners such as John's occuring in the sentence "John's car is red" can also be classified. Note that "John's car is red" is equivalent to "The car of John is red" [8]. Therefore it follows that possessive determiners behave like the and translate to functors that are both isotone and antitone when the phrases involved are nonvacuous and neither isotone nor antitone otherwise.

5.4 The term phrases of PTQ are formed from determiners combined with common nouns, from term phrases conjoined by "or", or are basic terms such as John, he_0 and ninety. Each term phrase is interpreted as a set of properties of individuals, where a property of individuals is a function from indices (or "possible worlds") into sets of individuals. At a fixed index such a set of properties is called a *sublimation*. Three kinds of sublimations are distinguished [4]: (i) a sublimation that interprets an IL expression of the form $\lambda Q[Q\{j\}]$ is called an *individual* sublimation; (ii) one that interprets an IL expression of the form $\lambda Q \exists x[man'(x) \wedge Q\{x\}]$ is called an *existential* sublimation; and (iii) one that interprets an IL expression of the form $\lambda Q \forall x[man'(x) \rightarrow Q\{x\}]$ is called a *universal* sublimation. These three kinds interpret all the term phrases of PTQ.

Suppose that Q_1 and Q_2 are properties of individuals such that $Q_1 \subseteq Q_2$ (i.e., $Q_1(i) \subseteq Q_2(i)$ for each index i). Then if $(\lambda Q[Q\{j\}])(Q_1)$ is interpreted as true (i.e., $[\![j]\!] \in [\![Q_1]\!]$), it follows that $(\lambda Q[Q\{j\}])(Q_2)$ must also be interpreted as true (i.e., $[\![j]\!] \in [\![Q_2]\!]$ also). Thus IL expressions which are interpreted by individual sublimations are isotone functors.

Similar arguments show that IL expressions which are interpreted by existential and universal sublimations are isotone functors. It is concluded that all term phrases of PTQ translate to isotone functors.

5.5 This is enough to illustrate how 5.2 mediates determination of entailment between complex meanings. Consider first the example of Section 1. Assume that the lexicon contains the following data: ⟦date⟧ $\leq$ ⟦accompany⟧, ⟦actor⟧ $\leq$ ⟦male⟧, *date′* is isotone, and a' is isotone. Applying the above results to these data it is immediately deduced that ⟦date an actor⟧ $\leq$ ⟦accompany a male⟧. In a similar fashion, from the datum *every′* is antitone, it follows that ⟦date every male⟧ $\leq$ ⟦accompany every actor⟧.

Since *Mary′* is an isotone functor, ⟦Mary dates an actor⟧ $\leq$ ⟦Mary accompanies a male⟧ and ⟦Mary dates every male⟧ $\leq$ ⟦Mary accompanies every actor⟧. Thus the entailments recognized by English speakers are obtained from the theory.

5.6 It is well known that the sentence "Every woman dates a man" permits two distinct readings, depending on the order of quantification. The first, called the *de dicto* reading, can be unambiguously rendered "Every woman dates some man or other but not necessarily the same man". The second, known as the *de re* reading, can also be rendered "Every woman dates a man, the very same one for every woman". The sentence on the de re reading entails the sentence on the de dicto reading. This is true generally; the converse is not.

From 5.2-5.4 it follows that for a fixed reading (either de dicto or de re), it can be derived that "Every woman dates a man" entails "Every debutante accompanies a male". Therefore, "Every woman dates a man" (de re) entails "Every debutante accompanies a male" (de re) which entails "Every debutante accompanies a male" (de dicto). In fact, introduction of the de re - de dicto entailment at any point in this derivation is valid. This is a general condition and therefore it can be concluded that the entailment following from 5.2-5.4 and the de re - de dicto entailment act independently.

5.7 It is reasonable to generalize definition 5.2.

DEFINITION. Let $x \in ME_{\langle a_0,\langle a_1,\ldots,\langle a_{n-1},b\rangle\cdots\rangle}$. Then
x is *isotone in argument* k $:\Leftrightarrow \forall y_k, y'_k \in ME_{a_k}[y_k \leq y'_k \Rightarrow [\forall y_1 \in ME_{a_1} \cdots \forall y_{n-1} \in ME_{a_{n-1}}\ [x(y_0)\cdots(y_{k-1})(y_k)(y_{k+1})\cdots(y_{n-1}) \leq x(y_0)\cdots(y_{k-1})(y'_k)(y_{k+1})\cdots(y_{n-1})]]]$
and
x is *antitone in argument* k $:\Leftrightarrow \forall y_k, y'_k \in ME_{a_k}[y_k \leq y'_k \Rightarrow [\forall y_1 \in ME_{a_1} \cdots \forall y_{n-1} \in ME_{a_{n-1}}\ [x(y_0)\cdots(y_{k-1})(y'_k)(y_{k+1})\cdots(y_{n-1}) \leq x(y_0)\cdots(y_{k-1})(y_k)(y_{k+1})\cdots(y_{n-1})]]]$.

The implication of this definition is a generalization of that given in 5.2. Let $x_1, x_2 \in ME_{\langle a_0,\langle a_1,\ldots,\langle a_{n-1},b\rangle\cdots\rangle}$, $y_k, y'_k \in ME_{a_k}$ and $y_1 \in ME_{a_1}, \ldots, y_{k-1} \in ME_{a_{k-1}}, y_{k+1} \in$

$ME_{a_{k+1}}, \dots, y_{n-1} \in ME_{a_{n-1}}$. If $x_1 \leq x_2$, x_1 is isotone (respectively antitone) in argument k and $y_k \leq y'_k$ (respectively $y'_k \leq y_k$) then $x_1(y_0) \cdots (y_{k-1})(y_k)(y_{k+1}) \cdots (y_{n-1}) \leq x_2(y_0) \cdots (y_{k-1})(y'_k)\ (y_{k+1}) \cdots (y_{n-1})$.

5.8 A further generalization to partial isotone (antitone) property is possible.

DEFINITION. x is *partially isotone in argument* k on $M \subseteq ME^2_{a_k} :\Leftrightarrow \forall (y_k, y'_k) \in M[y_k \leq y'_k \Rightarrow [\forall y_1 \in ME_{a_1} \cdots \forall y_{n-1} \in ME_{a_{n-1}}\ [x(y_0) \cdots (y_{k-1})(y_k)(y_{k+1}) \cdots (y_{n-1}) \leq x(y_0) \cdots (y_{k-1})(y'_k)\ (y_{k+1}) \cdots (y_{n-1})]]]$ and
x is *partially antitone in argument* k on $M \subseteq ME^2_{a_k} :\Leftrightarrow \forall (y_k, y'_k) \in M[y_k \leq y'_k \Rightarrow [\forall y_1 \in ME_{a_1} \cdots \forall y_{n-1} \in ME_{a_{n-1}}\ [x(y_0) \cdots (y_{k-1})(y'_k)(y_{k+1}) \cdots (y_{n-1}) \leq x(y_0) \cdots (y_{k-1})\ (y_k)\ (y_{k+1}) \cdots (y_{n-1})]]]$.

Again let $x_1, x_2 \in ME_{\langle a_0, \langle a_1, \dots, \langle a_{n-1}, b \rangle \cdots \rangle}$, $y_k, y'_k \in ME_{a_k}$ and $y_1 \in ME_{a_1}, \dots, y_{k-1} \in ME_{a_{k-1}}, y_{k+1} \in ME_{a_{k+1}}, \dots, y_{n-1} \in ME_{a_{n-1}}$. If $x_1 \leq x_2$, x_1 is partially isotone (respectively partially antitone) in argument k on M and $(y_k, y'_k) \in M$ such that $y_k \leq y'_k$ (respectively $y'_k \leq y_k$) then $x_1(y_0) \cdots (y_{k-1})(y_k)(y_{k+1}) \cdots (y_{n-1}) \leq x_2(y_0) \cdots (y_{k-1})(y'_k)\ (y_{k+1}) \cdots (y_{n-1})$.

It is an interesting question whether partially isotone (antitone) functors occur in natural languages.

A more complete treatment of entailment (and hence synonymy, contradiction and anomaly) at the level of complex meaning is reserved for a subsequent paper.

6 CONCLUSION

The theory of lexical semantics described in this paper represents lexical meanings as subspaces of a multidimensional semantic space. The space is coordinatized, with each coordinate regarded as an independent "dimension of meaning." Each subspace has a unique representation, called its normal form. A lexicon is defined as a map from a vocabulary to a Boolean algebra of normal forms.

A lexicon can be constructed using data from empirical linguistic analysis. No assumption of universal or ideal semantic categories is made. Data from Nida's componential analysis [9] are used to illustrate the construction. It is an interesting question whether this construction could also support incremental or evolutionary acquisition of semantic knowledge.

This theory of lexical semantics complements Montague semantics. It appears that the theory also complements lexical extensions of Montague semantics such as described by Dowty [2,3]. Entailment at the level of lexical meaning can be determined directly from the normal forms. At higher levels of meaning, entailment can be determined using entailment at lower levels and knowledge of isotone/antitone properties of functors that combine lower level meanings. This is demonstrated in connection with the PTQ fragment [4].

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