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Two models for combinatory logic

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1. Introduction

We consider in this paper two models of combinatoric logic in which the domain is the same: $P(N)$ the power set of N = the set of non negative integers. The first model is the graph model introduced by Scott in [6]; the second is a generalization that we call the hypergraph model. It is related to the graph model roughly as hyperarithmetical reducibility is related to Turing reducibility. We develop some properties of the graph model, most of them taken from [6], and some fundamental features of the hypergraph model. Our aim is to find intrinsic connections between the two models. The main result we have asserts that the hypergraph model can be simulated to some extent in the graph model.

Capital letter $A,B,\ldots,X,Y,Z$, will denote elements of $P(N)$, and small letters x,y,z will denote elements of N. Subsets of $P(N)$ will be denoted with greek letters α,β. We shall use standard notation for the boolean operations on subsets of N or of $P(N)$.

We shall encode finite sequences of integers via the pairing function $J(x,y) = \frac{1}{2}((x+y)^2 + 3x + y)$ (see [2], p. 43). We put $\langle x\rangle = x$ and $\langle x_1,\ldots,x_{n+1}\rangle = J(x_1, \langle x_2,\ldots,x_{n+1}\rangle)$. Note that every x represents some n-tuple for every $n \geq 1$.

As in Rogers [4] the notation D_x denotes the finite set with index x. Note that whenever $y \in D_x$ then $y < x$.

We shall find it convenient to formulate some results in terms of the Cantor topology on $P(N)$. If C and D are finite sets such that $C \cap D = \emptyset$ then the interval $\langle C;D \rangle$ is the set $\{X : C \subseteq X \subseteq \bar{D}\}$. A subset α of $P(N)$ is open in case that for every $X \in \alpha$ there is an interval $\langle C;D \rangle$ such that $X \in \langle C;D \rangle$ and $\langle C;D \rangle \subseteq \alpha$. A subset α of $P(N)$ is dense in case that $\alpha \cap \langle C;D \rangle$ is non empty for every interval $\langle C;D \rangle$. Finally α is nowhere dense in case that for every interval $\langle C;D \rangle$ there is an interval $\langle C_1;D_1 \rangle$ such that $\langle C_1;D_1 \rangle \subseteq \langle C;D \rangle$ and $\langle C_1;D_1 \rangle \subseteq \bar{\alpha}$.

We recall that if α is open and dense then $\bar{\alpha}$ is nowhere dense. If α is the union of a denumerable collection of nowhere dense sets then α is of the first category. Otherwise it of the second category.

We shall need the notion of continuous operator on $P(N)$ but actually the continuity of the operator is defined in terms of another topology. If $F(X_1,\ldots,X_n)$ is an operator defined on elements of $P(N)$ with value also in $P(N)$ we shall say it is continuous if the following condition is satisfied:

$$x \in F(X_1,\ldots,X_n) \text{ iff } \exists y_1 \exists y_2 \ldots \exists y_n \; (x \in F(D_{y_1},\ldots,D_{y_n}))$$

A model for combinatory logic consists of a non empty domain D of individuals and a binary function $f(x,y)$ on D such that there are elements S and K in D and the following identities hold:

$$\text{(S)} \qquad f(f(f(S,x),y),z) = f(f(x,z),f(y,z))$$

$$\text{(K)} \qquad f(f(K,x),y) = x$$

To avoid trivial cases it is convenient to require that the elements S and K are different. In the models we consider in this paper there are many elements with the properties of S and K so we do not require the strong extensionality property that such elements are unique. A weak extensionality property is satisfied by the graph model (see [6]) but apparently no similar result holds for the hypergraph model.

The fundamental theorem of combinatory logic can be formulated as follows: Let D be a model of combinatory logic with application function $f(x,y)$ and let t be a term built using variables, constants from D and the application function. Then if y is any variable there is a term h which is built using the same variables and constants in t, the constants S and K and the application function, but does not contain the variable y such that the following identity holds:

$$f(h,y) = t$$

There are different ways to construct such term h. For a general discussion see [1] Chapter 6A. Note also that different choices of the elements S and K will produce different terms.

The most important notion in a model of combinatory logic is the representation of functions in the model. To make precise this concept we introduce extension of the application function as follows. We put

$f_1(x,y) = f(x,y)$ and $f_{n+1}(x,y_1,\ldots,y_{n+1}) = f_n(f(x,y_1),y_2,\ldots,y_{n+1})$.

Then a function $g(y_1,\ldots,y_n)$ on D is representable in the model in case there is some element $x \in D$ such that $f(x,y_1,\ldots,y_n) = g(y_1,\ldots,y_n)$. From the fundamental theorem it follows that every function defined by a term t built using variables, constants and the application function is representable.

2. The Graph Model

The domain D of this model is the set P(N). Functions on this set will be called operators so the application function of this model is an operator of two variables. This operator - that we shall call the graph application operator - is denoted in the form (ZY) and it is defined as follows:

$$(ZY) = \{x : \exists y\ (\langle y,x\rangle \in Z \wedge D_y \subseteq Y)\}$$

We shall follow the usual conventions by omitting parentheses with the understanding that the association is to the left. Note that with this notation the fact that the operator $F(Y_1,\ldots,y_n)$ is representable in the model means that for some set Z

$$F(Y_1,\ldots,Y_n) = (ZY_1 \ldots Y_n)$$

Note also that

$$(ZY_1 \ldots Y_n) =$$
$$\{x : \exists y_1 \ldots \exists y_n (\langle y_1,\ldots,y_n,x\rangle \in Z \wedge D_{y_1} \subseteq Y_1 \wedge \ldots \wedge D_{y_n} \subseteq Y_n)\} \quad .$$

We must show first that this application operator is actually a model of combinatory logic. This follows immediately from the following theorem.

Theorem 1. Let $F(Y_1,\ldots,Y_n)$ be a continuous operator on P(N). Then there is a set A such that

$$F(Y_1,\ldots Y_n) = (AY_1 \ldots Y_n) \quad .$$

Take $A = \{\langle y_1,\ldots,y_n,x\rangle : x \in F(D_{y_1},\ldots,D_{y_n})\}$ and the theorem follows from the preceding remark.

Now since the operators $F_1(X,Y,Z) = (XZ(YZ))$ and $F_2(X,Y) = X$ are clearly continuous it follows from the theorem the existence of elements S and K satisfying the axioms of combinatory logic.

On the other hand the graph application operator itself is continuous, and it is clear that continuous operators are closed under composition.

Corollary. An operator on P(N) is representable in the graph model if and only if it is continuous.

This result does not give any information about the set A that represents a particular continuous operator. We shall call such a set a graph of the operator. In case an operator has a graph that is a recursively enumerable set we shall say it is a RE operator or more precisely that it is a RE operator in the graph model.

Now the graph application operator (ZY) is RE in the graph model. For a graph for this operator is the following RE set:

$$\{\langle y_1, y_2, x\rangle : y_1 = 2^{\langle y_2, x\rangle}\} \quad .$$

In general if t is a term built out of variables $Y_1,\ldots,Y_n$, RE operators and RE sets then the operator $F(Y_1,\ldots,Y_n) = t$ is RE in the graph model. For in this case the condition defining the graph in Theorem 1 is actually an RE predicate so the graph is an RE set.

It follows from this that we may assume there are elements S and K satisfying the axioms of combinatory logic that are RE sets. And in general all the combinators in the sense of combinatory logic are presented by RE sets.

Another important RE operator is related with the weak extensional properties of this model. In case Z_1 and Z_2 are sets such that for every set Y, $(Z_1Y) = (Z_2Y)$ we shall write $Z_1 \approx Z_2$. Now define a RE set L as follows:

$$L = \{\langle y,z,x\rangle : \exists v\ (y = 2^{\langle v,x\rangle} \wedge D_v \subseteq D_z)\}$$

It follows that for arbitrary Z_1 and Z_2, $(LZ_1) \approx Z_1$ and furthermore $Z_1 \approx Z_2$ if and only if $L(Z_1) = L(Z_2)$.

Another important operator in this model is the minimal fixed point operator. Given any Z the operator (ZY) as a function of Y is monotone so it has a minimal fixed point. Actually such minimal fixed point as a function of Z is also continuous so it is represented in the model. Analysis of the usual proof shows that it is a RE operator in the graph model.

Scott has proved a stronger result, namely that the graph of the minimal fixed point operator can be defined explicitly using elements S and K and the graph application function. We reproduce his argument here.

As usual I denotes a RE set such that (IX) = X for all X. Let F(Z) be the following RE operator.

$$F(Z) = (S(KZ)(SII)(S(KZ)(SII))) \quad .$$

By straightforward computation it follows that if F(Z) = Y then Y = (ZY) so F(Z) computes a fixed point of Z. We shall show that F(Z) is actually a minimal fixed point of Z. Let (ZY) = Y for some Y and to get a contradiction assume it is not the case that $F(Z) \subseteq Y$. Hence there is $x \in F(Z)$ but $x \notin Y$, so there is y such $\langle y,x\rangle \in (S(KZ)(SII))$ and $D_y \subseteq (S(KZ)(SII))$. Choose one such x such that the corresponding y is minimal. Then $x \in (S(KZ)(SII)D_y)$ so $x \in Z(D_y D_y)$. Since we are assuming $x \notin Y = (ZY)$ it follows that it is not the case that $(D_y D_y) \subseteq Y$ so there is a pair $\langle v,w\rangle \in D_y$, $D_v \subseteq D_y$ and $w \notin Y$. Since $v < y$ and $D_y \subseteq (S(KZ)(SII))$ this contradicts the minimality of y.

We can give a general rule for the existence of RE operators that will be useful later. For that purpose let us define a G-form as an expression in set variables $Y_1,\ldots,Y_n$ and number variables $x_1,\ldots,x_k$ built out of atomic formulas: i) $f(x_1,\ldots,x_k) \in t$ where f is a recursive function and t is a term containing set variables, RE operators and RE sets; and ii) recursively enumerable predicates in the number variables. The admissible operations in the form are disjunction, conjunction, bounded universal quantification and existential quantification.

Theorem 2. Let $R(Y_1,\ldots,Y_n, x_1,\ldots,x_k)$ be a G-form. Then there is a RE set A such that $(AY_1 \ldots Y_n) = \{\langle x_1,\ldots,x_k\rangle : R(Y_1,\ldots,Y_n, x_1,\ldots,x_k)\}$.

A proof by induction on the construction of the form can be given but a more direct argument is possible. Replace in the form every occurrence of Y_j by D_{y_j}. This gives a recursively enumerable predicate $R_1(y_1,\ldots,y_n, x_1,\ldots,x_k)$ and then the set A can be defined as follows:

$$A = \{\langle y_1,\ldots,y_k, x_1,\ldots,x_n\rangle : R_1(y_1,\ldots,y_n, x_1,\ldots,x_k)\} .$$

3. The Hypergraph Model

The form of the definition of graph application operator suggests a generalization using function quantification. It is not clear that such extension will produce a model of combinatory logic but we shall show this is actually the case. In general the operators represented in the new model are not continuous and this of course may be considered an important disadvantage. At any rate we shall show the operators are not completely discontinuous since they can be simulated in the graph model on a large subdomain of P(N).

We shall need here some extra notation. If f(x) is a total numerical function we put $\bar{f}(x) = \langle x, f(0), f(1), \ldots, f(x-1) \rangle$ so $\bar{f}(0) = 0$. If $x = \langle y,z \rangle$ then $l(x) = y$ and $ll(x) = l(z)$ if $y > 1$, $= z$ otherwise. If $x = \langle n, x_1, \ldots, x_n \rangle$ and $y = \langle m, y_1, \ldots, y_m \rangle$ then $x * y = \langle n+m, x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$. We shall write $x \cdot y$ in place of $x * \langle 1,y \rangle$.

The hypergraph application operator will be denoted in the form [ZY]. The definition of the operator is as follows:

$$[ZY] = \{x : \forall f \exists v \exists y \; (\langle \bar{f}(v), y, x \rangle \in Z \wedge D_y \subseteq Y)\} .$$

We shall omit brackets with the convention that the association is to the left. In some cases we may use both the parenthesis notation and the bracket notation. The convention is that inside parentheses we must replace parentheses and inside brackets we must replace brackets. For instance the expression (XY[ZYX]) is actually $((XY)[[ZY]X])$.

We shall prove first this is actually a model of combinatory logic. We choose to prove a more general theorem from which the existence of combinators follows.

Lemma 1. There is a RE set A such that for arbitrary Z_1, Z_2 and Y

$$[(AZ_1Z_2)Y] = [Z_1Y] \cap [Z_2Y]$$

Let d(z) be a recursive function such that whenever g(x) and f(x) are functions such that $g(x) = f(x+1)$ then $d(\bar{f}(v+1)) = \bar{g}(v)$. Introduce the following predicates:

$$R_0(u,y) \equiv l(u) > 0 \wedge ll(u) > 1 \wedge y = 0$$

$$R_1(Z,u,y,x) \equiv l(u) > 0 \wedge ll(u) = 0 \wedge \langle y,d(u),x\rangle \in Z$$

$$R_2(Z,u,y,x) \equiv l(u) > 0 \wedge ll(u) = 1 \wedge \langle y,d(u),x\rangle \in Z$$

Take now A as the RE set such that

$$(AZ_1Z_2) = \{\langle u,y,x\rangle \,;\, R_0(u,y) \vee R_1(Z_1,u,y,x) \vee R_2(Z_2,u,y,x)\} .$$

This A exists by Theorem 2 and it is easy to check that it satisfies the conditions of the Lemma.

We define an H-form as an expression in set variables $Y_1,\ldots,Y_n$ and number variables $x_1,\ldots,x_k$, $n \geq 1$, $k \geq 1$, built out of the following atomic formulas: i) $x_i \in X_j$; ii) $P(x_1,\ldots,x_k)$ where P is a recursively enumerable predicate. We allow in H-forms the following operations; conjunction and function quantification. The latter is understood as follows: if $R(Y_1,\ldots,Y_n, v, x_1,\ldots,x_k)$ is H-form then $\forall f \exists v\, R(Y_1,\ldots,Y_n, \bar{f}(v), x_1,\ldots,x_k)$ is also H-form.

Theorem 3. Let $R(X_1,\ldots,X_n,Y, x_1,\ldots,x_k)$ be H-form, $n \geq 0$, $k \geq 1$. There exists a RE set A such that

$$[(AX_1 \ldots X_n)Y] = \{\langle x_1,\ldots,x_k\rangle : R(X_1,\ldots,X_n,Y, x_1,\ldots,x_k)\}$$

The cases in which the form is atomic follow easily using Theorem 2. And conjunction can be handled using Theorem 2 with Lemma 1. So we need only consider the case in which function quantification is used.

Consider the form $\forall f \exists v R(X_1,\ldots,X_n,Y, \bar{f}(v), x_1,\ldots,x_k)$ and assume A is a RE set such that

$$[(AX_1 \ldots X_n)Y] = \{\langle v,x_1,\ldots,x_k\rangle : R(X_1,\ldots,X_n,Y,v,x_1,\ldots,x_k)\} .$$

Then the form is actually equivalent to the following

$$\forall f \exists v \forall g \exists w \exists y (\langle \bar{g}(w),y,\bar{f}(v),x_1,\ldots,x_k\rangle \in (AX_1 \ldots X_n) \wedge D_y \subset Y) .$$

Now by standard permutation and contraction of quantifiers the expression is equivalent to

$$\forall f \exists v \exists y (\langle d_1(\bar{f}(v)),y, d_2(\bar{f}(v)), x_1,\ldots,x_k\rangle \in (AX_1 \ldots X_n) \wedge D_y \subseteq Y)$$

where $d_1(u)$ and $d_2(u)$ are recursive functions. Hence to satisfy the theorem we take a RE set A_1 such that:

$$(A_1X \ldots X_n) = \{\langle u,y,x\rangle : \langle d_1(u),y,d_2(u),x_1,\ldots,x_k\rangle \in (AX_1 \ldots X_n)\}$$

Next we extend the notion of H-form by allowing other constructions. In each case we show that the new construction is equivalent to a form in the original sense. So Theorem 3 can be applied to the enlarged forms.

a) We can use number quantifiers in H-forms. In fact it is well known that such quantifiers can be expressed using the universal function quantifier.

b) We can use Π^1_1-predicates in H-forms since such predicates can be defined by applying function quantifiers to recursively enumerable predicates.

c) We can use expression of the form $f(x_1,\ldots,x_k) \varepsilon t$ where f is any hyperarithmetical function and t is a term in which variables are used, the constants are Π^1_1-sets and the operations are the graph application operator or the hypergraph application operator. Any such expression can be expanded using the definitions to a form containing quantifiers and conjunction.

d) Disjunction. Let $R(X_1,\ldots,X_n, x_1,\ldots,x_k)$ be

$$R_1(X_1,\ldots,X_n, x_1,\ldots,x_k) \vee R_2(X_1,\ldots,X_n, x_1,\ldots,x_k)$$

where R_1 and R_2 are forms. First note there is a RE set B such that $[(BZ_1Z_2)Y] = [Z_1Y] \cup [Z_2Y]$. To show there is such B we use Theorem 2 to get RE set B_1 such that

$$(B_1Z_1Z_2) = \{\langle u,y,x,z\rangle : (\langle u,y,x\rangle \varepsilon Z_1 \wedge z = 0) \vee (\langle u,y,x\rangle \varepsilon Z_2 \vee z = 1)\}.$$

Now note that $[Z_1Y] \cup [Z_2Y] = \{x : \exists z(\langle x,z\rangle \varepsilon [(B_1Z_1Z_2)Y])$ so B exists by Theorem 3 applied to expressions covered in a) and c). Also by Theorem 3 applied to H-forms R_1 and R_2 with a dummy variable Y there are RE sets A_1 and A_2 such that:

$$[(A_1X_1 \ldots X_n)\emptyset] = \{\langle x_1,\ldots,x_k\rangle : R_1(X_1,\ldots,X_n,x_1,\ldots,x_k)\}$$

$$[(A_2X_1 \ldots X_n)\emptyset] = \{\langle x_1,\ldots,x_k\rangle : R_2(X_1,\ldots,X_n,x_1,\ldots,x_k)\}$$

Hence the form R is equivalent to

$$\langle x_1,\ldots,x_k\rangle \varepsilon [(B(A_1X_1 \ldots X_n)(A_2X_1 \ldots X_n))\emptyset]$$

and it is H-form by c) .

Theorem 4. For each $n \geq 1$ there is a RE set B_n such that $(ZX_1 \ldots X_n) = [(B_nZ)X_1 \ldots X_n]$.

For $n = 1$ take B_1 such that $(B_1Z) = \{<0,y,x> ; <y,x> \in Z\}$. For n+1 we use Theorem 3 to get a RE set A such that

$$[(AZX_1 \ldots X_n)X_{n+1}] = [ZX_1 \ldots X_{n+1}]$$

hence take B_{n+1} such that $(B_{n+1}Z) = B_n(AZ)$.

Theorem 5. Let $R(X_1,\ldots,X_n,x_1,\ldots,x_k)$ be H-form. There is a RE set A such that $[AX_1 \ldots X_n] = \{<x_1,\ldots,x_k> : R(X_1,\ldots,X_n,x_1,\ldots,x_k)\}$.

If $n = 1$ the result is given by Theorem 3. Otherwise apply Theorem 3 and then Theorem 4 to eliminate the expression with graph application.

From Theorem 5 it follows that the hypergraph application operator defines a model of combinatory logic, and that every combinator is actually represented by some RE set.

We complete this section showing that the minimal fixed point for the hypergraph model is representable in the model. Such minimal fixed point always exists since the operators representable in the model are monotone. But since they are not continuous we cannot depend in any construction of the fixed point from below. Actually we must use the construction in which the fixed point is obtained as the intersection of all sets containing its own image under the operator.

Theorem 6. There is a RE set A such that for arbitrary Z, $[Z[AZ]] = [AZ]$ and whenever $[ZY] = Y$ then $[AZ] \subseteq Y$.

Define the operator F(Z) by the following condition:

$$x \in F(Z) \text{ iff } \forall Y([ZY] \subseteq Y \rightarrow x \in Y)$$

We need only to show that F(Z) is represented in the hypergraph model by a RE set A. Using the definitions and exporting quantifiers we get

$$x \in F(Z) \text{ iff } \forall Y \exists z \forall f \exists v \exists y((\langle \bar{f}(v), y, z\rangle \in Z \wedge D_y \subseteq Y \wedge z \notin Y) \vee x \in Y)$$

In the expression in the right it is possible to replace the quantifier $\forall Y$ by a universal function quantifier. After this by standard permutation and contraction of quantifier we get H-form R(Z,x) such that

$$x \in F(Z) \text{ iff } R(Z,x)$$

hence RE set A exists by Theorem 5.

4. Relation Between the Models

Every operator representable in the graph model can be represented in the hypergraph model. But the converse is not true. For instance if A is the set of all triples $\langle\langle 1,v\rangle, 2^v, x\rangle$ then $[AN] = N$ but $[AX] = \emptyset$ for any X different from N. So we may ask to what extent it is possible to simulate operators of the hypergraph model by operators of the graph model.

We make precise this idea by introducing the following definition. $\delta(Z,X) = \{Y : [ZY] = (XY)\}$. Now given a set Z we can find a set Z_1 such that $\delta(Z,Z_1)$ contains all finite sets. We may take for instance $Z_1 = \{\langle y,x\rangle : x \in [ZD_y]$ and then there is a RE set A such that $[AZ] = Z_1$. The set Z_1 is to some extent unique. For if Z_2 is another set such that $\delta(Z,Z_2)$ contains all finite sets then $Z_1 \approx Z_2$ in the graph model, hence $\delta(Z,Z_1) = \delta(Z,Z_2)$.

We note also that in case $\delta(Z,Z_1)$ contains all finite sets then the operator [ZY] with variable Y is continuous if and only if $\delta(Z,Z_1) = P(N)$.

In the preceding construction the set Z_1 simulates the set Z as far as the latter defines a continuous operator. We shall show now how to construct for every Z another set Z_1 such that $\delta(Z,Z_1)$ is of second category. The construction is essentially a forcing argument of the type introduced in [7].

We define by transfinite induction a set $T_p(Z)$ where p denotes ordinals. The defining rules are as follows:

T1) If $\exists w(\langle u,w,x\rangle \in Z \wedge D_w \subseteq D_y)$ then $\langle u,y,x\rangle \in T_p(Z)$ for all ordinals p.

T2) If for every number j, and for every z such that $D_z \cap D_y = \emptyset$ there is $q < p$ such that $Ew(\langle u \cdot j,w,x\rangle \in T_q(Z) \wedge D_w \cap D_z = \emptyset)$ then $\langle u,y,x\rangle \in T_p(Z)$.

It follows immediately from these rules that whenever $\langle u,y,x\rangle \in T_p(Z)$ and $D_y \subseteq D_z$ then $\langle u,z,x\rangle \in T_p(Z)$.

We shall say that an interval $\langle D_y;D_v\rangle$ forces the pair $\langle u,x\rangle$ in case one of the two following conditions is satisfied.

F1) There is some ordinal p such that $\langle u,y,x\rangle \in T_p(Z)$ and for every ordinal $q < p$ and for every z such that $D_z \cap D_v = \emptyset$ $\langle u,z,x\rangle \notin T_q(Z)$.

F2) There is a number j such that for every ordinal p and for every z such that $D_z \cap D_v = \emptyset$, $\langle u \cdot j,z,x\rangle \notin T_p(Z)$ and $\langle u,z,x\rangle \notin T_p(Z)$.

Theorem 7. For every pair u,x and every interval $\langle D_y;D_v\rangle$ there is a subinterval that forces $\langle u,x\rangle$.

In case there is y_1 such that $D_{y_1} \cap D_v = \emptyset$ and for some p, $\langle u, y_1,x\rangle \in T_p(Z)$ we take y_1 such that p is minimal and then $\langle D_y \cup D_{y_1};D_v\rangle$ forces $\langle u,x\rangle$.

Otherwise there is no p such that $\langle u,y,x\rangle \in T_p(Z)$. Then by rule T2) there is j and v_1 such that $D_y \cap D_{v_1} = \emptyset$ and for all ordinals p and all z such that $D_z \cap D_{v_1} = \emptyset$ $\langle u \cdot j,z,x\rangle \notin T_p(Z)$. Then $\langle D_y;D_v \cup D_{v_1}\rangle$ forces $\langle u,x\rangle$.

We shall say that a set X forces a pair $\langle u,x\rangle$ in case X belongs to some interval that forces $\langle u,x\rangle$. The collection of all X that forces a pair $\langle u,x\rangle$ is open, and by Theorem 7 it is dense.

We shall say that a set X is generic in case it forces all pair $\langle u,x\rangle$. It follows that the collection of all non generic set is of first category.

Theorem 8. Let X be a generic set. If $\langle u,y,x\rangle \in T_p(Z)$ where $D_y \subseteq X$ then either there is w such that $D_w \subseteq D_y$ and $\langle u,w,x\rangle \in Z$ or for every number j there is $q < p$ and z such that $D_z \subseteq X$ and $\langle u \cdot j,z,x\rangle \in T_q(Z)$.

Assume there is no w such that $D_w \subseteq D_y$ and $\langle u,w,x\rangle \in Z$. Then $p > 0$ and for any given j there is an interval $\langle D_z;D_v\rangle$ that contains X and forces $\langle u \cdot j,x\rangle$. By T2 there is w such that $D_w \cap D_v = \emptyset$ and $\langle u \cdot j,w,x\rangle \in T_{q'}(Z)$ for $q' < p$. Hence F2 does not apply and $\langle u \cdot j,z,x\rangle \in T_q(Z)$ for $q \leq q' < p$.

Theorem 9. Let X be a generic set such that there is no p and y such that $\langle u,y,x\rangle \in T_p(Z)$ and $D_y \subseteq X$. Then there is j such that there is no p and y such that $\langle u \cdot j,y,x\rangle \in T_p(Z)$ and $D_y \subseteq X$.

Take any interval that contains X and forces $\langle u,x\rangle$. Since F1 is impossible F2 holds and this implies the theorem.

Theorem 10. Let X be a generic set. The following conditions are equivalent:

$$\text{i)} \quad \exists p \exists y\ (\langle 0,y,x\rangle \in T_p(Z) \wedge D_y \subseteq X)$$

$$\text{ii)} \quad \forall f \exists v \exists y (\langle \bar{f}(v),y,x\rangle \in Z \wedge D_y \subseteq X) .$$

Assume i) holds. Then by Theorem 8 given any function f(v) a value v must exist such that $\langle \bar{f}(v),w,x\rangle \in T_0(Z)$ and $D_w \subseteq X$, so

ii) follows. Assume now i) is false. Then by Theorem 9 there is a function f(v) such that $\forall v \forall w(D_w \subseteq X \to \langle \bar{f}(v), w, x\rangle \notin T_0(Z))$ so ii) is false.

Now define $Z_1 = \{\langle u,x\rangle : \exists p(\langle 0,y,x\rangle \in T_p(Z))\}$.

Corollary. $\delta(Z,Z_1)$ contains all generic set so its complement is of first category.

A more detailed analysis of the construction of T(Z) shows that this is actually the minimal fixed point of an operator which is RE in the hypergraph model. Hence there is a RE set A such that $[AZ] = Z_1$. In case Z is RE set then Z_1 is Π^1_1-set.

5. Reduction

Both models are useful to define general forms of reduction that under special restrictions become well known reducibilities of recursive function theory. If α is any subset of P(N) we define the relation $X \leq_g^{\alpha} Y$ to hold between sets X and Y exactly in the case there is some set $Z \in \alpha$, such that $(ZY) = X$. The case in which α is the set of RE sets it is well known in the literature as enumeration reducibility and is denoted $X \leq_e Y$ in [4]. Similarly we define $X \leq_{hg}^{\alpha} Y$ to hold in case ther is $Z \in \alpha$ such that $[ZY] = X$. The case in which α is the set of RE sets has been called hyperenumeration reducibility in [5] and denoted $X \leq_{he} Y$.

Under proper restrictions on α these relations become partial orders and induce partitions whose elements are called degrees. We are interested in some evaluation of the number of degrees containing total functions. We shall show that a classical result of enumeration reducibility can be generalized to the hypergraph model. This is also a generalization of the main result of. [5]

We shall say that a set X is single-valued in case that whenever $\langle x,y\rangle \in X$ and $\langle x,z\rangle \in X$ then $y = z$. In case X is single-valued and for every x there is some y such that $\langle x,y\rangle \in X$ we shall say it is total.

Theorem 11. Let Z be a given set. The collection of all sets Y such that (ZY) is not single-valued or there is a cofinite extension Y_1 of Y such that (ZY_1) is single-valued is open and dense.

Note that every cofinite set belongs to the collection, so it is clearly dense. To show it is open first consider the case in which (ZY) is not single-valued. Then there is a finite subset Y_1 of Y such that (ZY_1) is not single-valued, hence the interval $\langle Y_1;\emptyset\rangle$ is contained in the collection. In case there is a cofinite extension Y_1 of Y such that (ZY_1) is single-valued the interval $\langle\emptyset;\bar{Y}_1\rangle$ is contained in the collection.

Theorem 12. Let α be a denumerable subset of P(N) closed under enumeration reducibility. The collection of all Y such that there is some X, X is total, $X \leq_g^\alpha Y$ and $X \notin \alpha$ is of first category.

Note that in case (ZY) = X where $Z \in \alpha$ and Y is cofinite then $X \leq_e Z$ so $X \in \alpha$. For each $Z \in \alpha$ call β_Z the collection of Theorem 11. So in case there is X such that for some $Z \in \alpha$, (ZY) = X, X is total but $X \notin \alpha$, then $Y \in \bar{\beta}_Z$ and since each $\bar{\beta}_Z$ is nowhere dense, the theorem follows.

Theorem 13. Let α be a denumerable subset of P(N), closed under hyperenumeration reducibility. The collection of all Y such that there is some X, X is total, $X \leq_{hg}^\alpha Y$ and $X \notin \alpha$ is of first category.

For each Z let Z_g be a set such that $\overline{\delta(Z,Z_g)}$ is of first category and $Z_g \leq_{he} Z$ (see remark at the end of section 4). Call α_g the collection of all Y such that there is some $Z \in \alpha$ and $Y \leq_e Z_g$. Hence $\alpha_g \subseteq \alpha$. Call β the collection of all Y such that there is some X,X is total, $X \leq_{hg}^\alpha Y$ and $X \notin \alpha$. And call

β_g the collection of all Y such that there is X, X is total $X \leq_g^{\alpha_g} Y$ and $X \notin \alpha_g$. By Theorem 12 β_g is of first category hence $\beta \cap \beta_g$ is also of first category.

Now $\beta = (\beta - \beta_g) \cup (\beta \cap \beta_g)$ and for any $Y \in (\beta - \beta_g)$ there is $Z \in \alpha$ such that $Y \in \overline{\delta(Z, Z_g)}$ hence $(\beta - \beta_g)$ is also of first category. It follows that β is of first category.

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