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Khovanov Homology and Conway Mutation

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Abstract

In this article, we present an easy example of mutant links with different Khovanov homology. The existence of such an example is important because it shows that Khovanov homology cannot be defined with a skein rule similar to the skein relation for the Jones polynomial.

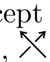
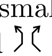
1 Introduction

In [Kh] M. Khovanov assigned to the diagram D of an oriented link L a bigraded chain complex $\mathcal{C}^{*,*}(D)$, with a differential d that maps the chain group $\mathcal{C}^{i,j}(D)$ into $\mathcal{C}^{i+1,j}(D)$. He proved that the homotopy equivalence class of graded chain complex $\mathcal{C}^{*,*}(D)$ only depends on the oriented link L. In particular, the homology groups $\mathcal{H}^{i,j}(D)$ (considered up to isomorphism) and the graded Poincaré polynomial

$$Kh(L)(t,q) := \sum_{i,j} t^i q^j \dim_{\mathbb{Q}}(\mathcal{H}^{i,j}(D) \otimes \mathbb{Q}) \in \mathbb{Z}[t, t^{-1}, q, q^{-1}]$$

are link invariants. The aim of this paper is to give an example of oriented mutant links which are separated by the polynomial Kh and to prove that consequently the invariant Kh does not satisfy a skein relation similar to the skein relation for the Jones polynomial.

2 Skein equivalence

In this section we briefly recall the definition of skein equivalence given in [Ka]. A triple (L_+, L_-, L_0) of oriented links is called a *skein triple*, if the oriented links L_+, L_- and L_0 possess diagrams which are mutually identical except in a small neighborhood, where they are respectively consistent with  ,  and  .

Definition 1 *The* skein equivalence *is the minimal (with respect to set-theoretical inclusion) equivalence relation "$\sim$" on the set of oriented links such that*

1. $L \sim L'$ *when* L *and* L' *are isotopic,*
2. $L_0 \sim L_0'$ *and* $L_- \sim L_-'$ *imply* $L_+ \sim L_+'$,
3. $L_0 \sim L_0'$ *and* $L_+ \sim L_+'$ *imply* $L_- \sim L_-'$,

for any two skein triples (L_+, L_-, L_0) *and* (L'_+, L'_-, L'_0).

It is easy to see that such a minimal relation as postulated in the definition actually exists. The definition is motivated by the following: Assume we are given an invariant f of oriented links, such as the Jones polynomial, which takes values in an arbitrary ring R and satisfies a relation

$$\alpha f(L_+) + \beta f(L_-) + \gamma f(L_0) = 0,$$

where $\alpha, \beta \in R^*$ and $\gamma \in R$. Then $f(L_+)$ is determined by $f(L_0)$ and $f(L_-)$, and $f(L_-)$ is determined by $f(L_0)$ and $f(L_+)$. The minimality of "$\sim$" implies:

Theorem 1 *Let* L *and* L' *be skein equivalent. Then* $f(L) = f(L')$.

3 Conway mutation

The mutation of links was originally defined in [Co]. We will use the definition given in [Mu]. In Figure 1, the rectangular boxes represent an oriented 2-tangle T. Let h_1, h_2 and h_3 be the half-turns about the indicated axes.

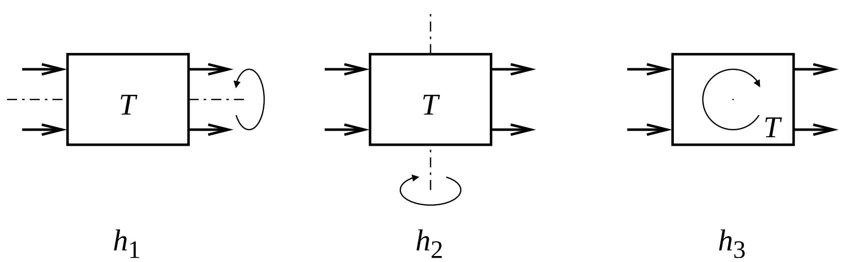


Figure 1: The half-turns h_1, h_2 and h_3

Define three involutions ρ_1, ρ_2 and ρ_3 on the set of oriented 2-tangles by $\rho_1 T := h_1(T)$, $\rho_2 T := -h_2(T)$ and $\rho_3 T := -h_3(T)$ (where $-h_2(T)$ and $-h_3(T)$ are the oriented 2-tangles obtained from $h_2(T)$ and $h_3(T)$ by re-

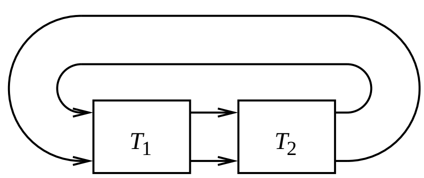


Figure 2: The closure of the composition of T_1 and T_2

versing the orientations of all strings). For two oriented 2-tangles T_1 and T_2, denote by T_1T_2 the composition of T_1 and T_2 and by $(T_1T_2)^\wedge$ the closure of T_1T_2 (see Figure 2).

Definition 2 *Two oriented links* L *and* L' *are called* Conway mutants *if there are two oriented 2-tangles* T_1 *and* T_2 *such that for an involution* ρ_i ($i = 1, 2, 3$) *the links* L *and* L' *are respectively isotopic to* $(T_1 T_2)^\wedge$ *and* $(T_1 \rho_i T_2)^\wedge$.

Theorem 2 *Let* L *and* L' *be Conway mutants. Then* L *and* L' *are skein equivalent.*

Proof. The proof goes by induction on the number n of crossings of T_2. For $n \leq 1$, T_2 and $\rho_i T_2$ are isotopic, whence $L \sim L'$. For $n > 1$, modify a crossing of T_2 to obtain a skein triple of tangles (T_+, T_-, T_0) (with either $T_+ = T_2$ or $T_- = T_2$, depending on whether the crossing is positive or negative). Denote by (L_+, L_-, L_0) and (L'_+, L'_-, L'_0) the skein triples corresponding to (T_+, T_-, T_0) and $(\rho_i T_+, \rho_i T_-, \rho_i T_0)$ respectively (i.e. $L_+ = (T_1 T_+)^\wedge$, $L_- = (T_1 T_-)^\wedge$ and so on). By induction, $L_0 \sim L'_0$. Therefore, by the definition of skein equivalence, $L_+ \sim L'_+$ if and only if $L_- \sim L'_-$. In other words, switching a crossing of T_2 does not affect the truth or falsity of the assertion. Since T_2 can be untied by switching crossings, we are back in the case $n \leq 1$. $\square$

4 Mutant links with different Khovanov homology

Let $V(L)(q) := Kh(L)(-1, q)$ denote the graded Euler characteristic of $\mathcal{C}(D)$ and $W(L)(t) := Kh(L)(1, q)$ the ordinary (ungraded) Poincaré polynomial. As is shown in [Kh], V is just an unnormalized version of the Jones polynomial. By the results of sections 2 and 3, the Jones polynomial is invariant under Conway mutation. On the other hand, the following theorem gives an example of mutant links which are separated by W.

Theorem 3 *Let* K_i ($i = 1, 2$) *be a* $(2, n_i)$ *torus link,* $n_i > 2$. *Then the oriented links*

$$L := \circlearrowleft \sqcup (K_1 \sharp K_2) \quad \textit{and} \quad L' := K_1 \sqcup K_2$$

are Conway mutants with different W *polynomial. Here,* $\circlearrowleft$ *denotes the trivial knot and* $K_1 \sharp K_2$ *is the connected sum of the oriented links* K_1 *and* K_2. *Note that the connected sum is well-defined even if* K_i *has two components, because in this case the link* K_i *is symmetric in its components.*

Proof. From Figure 3 it is apparent that L and L' are Conway mutants. The Khovanov complex of the trivial knot is

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{A} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

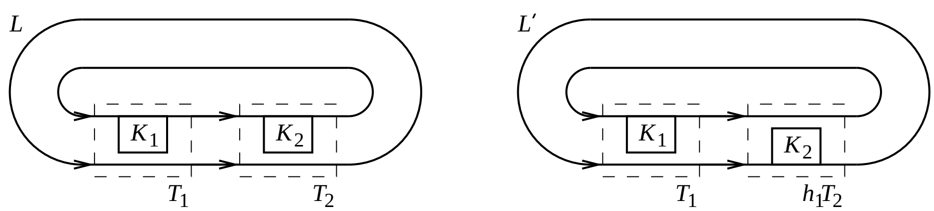


Figure 3: L and L' are Conway mutants

and rank($\mathcal{A}$) = 2, whence $W(\bigcirc) = 2$. By [Kh, Proposition 33], Kh is multiplicative under disjoint union, and so $W(L) = 2W(K_1 \sharp K_2)$. On the other hand, by [Kh, Proposition 35],

$$W(K_i) = 2 + t^{-2} + t^{-3} + \ldots + t^{-(n_i-1)} + t^{-n_i}$$

if n_i is odd and

$$W(K_i) = 2 + t^{-2} + t^{-3} + \ldots + t^{-(n_i-1)} + 2t^{-n_i}$$

if n_i is even. Since $n_i > 2$, $W(K_i)$ is not divisible by 2. But then $W(L') = W(K_1)W(K_2)$ is not divisible by 2 and hence $W(L') \neq W(L)$. □

Theorems 1, 2 and 3 immediatly imply:

Corollary 1 *The W polynomial does not satisfy a relation of the kind mentioned in section 2.*

Remark. Theorem 3 remains true if we also allow torus links K_i with $n_i < -2$ (this may be seen using [Kh, Corollary 11], which relates the Khovanov homology of a link to the Khovanov homology of its mirror image). The condition $|n_i| > 2$ is necessary. In fact, if one of the $|n_i|$ is ≤ 1, then the corresponding torus link K_i is trivial and hence L and L' are isotopic. If $n_2 = 2$, then L and L' look as is shown in Figure 4. Note that both $L - L_0$ and $L' - L'_0$ are isotopic to the link $\bigcirc \sqcup K_1$. Using [Kh, Corollary 10], one can show that both $\mathcal{H}^{i,j}(L)$ and $\mathcal{H}^{i,j}(L')$ are isomorphic to $\mathcal{H}^{i+2,j+5}(\bigcirc \sqcup K_1) \oplus \mathcal{H}^{i,j+1}(\bigcirc \sqcup K_1)$. The cases $n_2 = -2$ and $n_1 = \pm 2$ are similar.

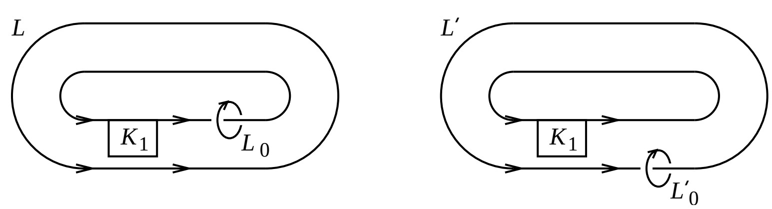


Figure 4: L and L' for the case $n_2 = 2$

5 Computer Calculations with KHOHO

Tables 1 and 2 show the Khovanov homology of L and L' for the case $n_1 = n_2 = 3$. The tables where generated using A. Shumakovitch's program KhoHo [Sh]. The entry in the i-th column and the j-th row looks like $\frac{a[b]}{c}$, where a is the rank of the homology group $\mathcal{H}^{i,j}$, b the number of factors $\mathbb{Z}/2\mathbb{Z}$ in the decomposition of $\mathcal{H}^{i,j}$ into p-subgroups, and c the rank of the chain group $\mathcal{C}^{i,j}$. The numbers above the horizontal arrows denote the ranks of the chain differentials.

In the examples, only 2-torsion occurs. It has been conjectured by A. Shumakovitch that this is actually true for arbitrary links. The reader may verify that not only the dimensions but also the torsion parts of the $\mathcal{H}^{i,j}$ are different for L and L'.

The dimensions of the $\mathcal{C}^{i,j}$ agree because there is a natural one-to-one correspondence between the Kauffman states of L and L' (which re-proves the fact that the Jones polynomial is invariant under Conway mutation).

We do not know the answer to the following question:

Question: Does there exist a pair of mutant oriented knots with distinct Khovanov homology?

According to the database of A. Shumakovitch, no such pair of knots with 13 or less crossings exists. In particular, the Kinoshita-Terasaka knot and the Conway knot (the knots depicted in Figure 5) are mutant knots with the same Khovanov homology (see Table 3).

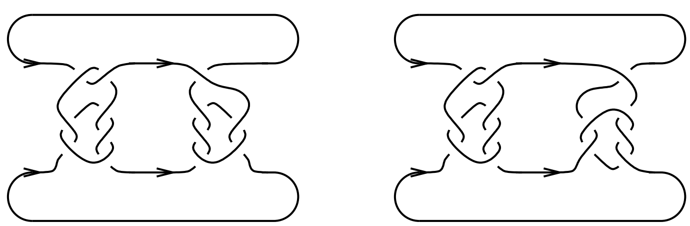


Figure 5: The Kinoshita-Terasaka knot and the Conway knot

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	-6	-5	-4	-3	-2	-1	0
-2							$\frac{\mathbf{1}}{\mathbf{1}}$
-4				$\frac{0}{2} \xrightarrow{2}$	$\frac{0}{6} \xrightarrow{4}$	$\frac{0}{6} \xrightarrow{2}$	$\frac{\mathbf{2}}{\mathbf{4}}$
-6	$\frac{0}{1} \xrightarrow{1}$	$\frac{0}{6} \xrightarrow{5}$	$\frac{0}{15} \xrightarrow{10}$	$\frac{0}{28} \xrightarrow{18}$	$\frac{\mathbf{2}}{\mathbf{33}} \xrightarrow{13}$	$\frac{0}{18} \xrightarrow{5}$	$\frac{\mathbf{1}}{\mathbf{6}}$
-8	$\frac{0}{6} \xrightarrow{6}$	$\frac{0}{30} \xrightarrow{24}$	$\frac{0}{60} \xrightarrow{36}$	$\frac{0}{74} \xrightarrow{38}$	$\frac{\mathbf{2[2]}}{\mathbf{54}} \xrightarrow{14}$	$\frac{0}{18} \xrightarrow{4}$	$\frac{0}{4}$
-10	$\frac{0}{15} \xrightarrow{15}$	$\frac{0}{60} \xrightarrow{45}$	$\frac{\mathbf{1}}{\mathbf{90}} \xrightarrow{44}$	$\frac{\mathbf{2}}{\mathbf{74}} \xrightarrow{28}$	$\frac{\mathbf{0[2]}}{\mathbf{33}} \xrightarrow{5}$	$\frac{0}{6} \xrightarrow{1}$	$\frac{0}{1}$
-12	$\frac{0}{20} \xrightarrow{20}$	$\frac{\mathbf{1}}{\mathbf{60}} \xrightarrow{39}$	$\frac{\mathbf{1[1]}}{\mathbf{60}} \xrightarrow{20}$	$\frac{\mathbf{2}}{\mathbf{28}} \xrightarrow{6}$	$\frac{0}{6}$		
-14	$\frac{0}{15} \xrightarrow{15}$	$\frac{\mathbf{2[1]}}{\mathbf{30}} \xrightarrow{13}$	$\frac{\mathbf{0[1]}}{\mathbf{15}} \xrightarrow{2}$	$\frac{0}{2}$			
-16	$\frac{\mathbf{1}}{\mathbf{6}} \xrightarrow{5}$	$\frac{\mathbf{1[1]}}{\mathbf{6}}$					
-18	$\frac{\mathbf{1}}{\mathbf{1}}$						

Table 1: Ranks of $\mathcal{H}^{i,j}$ and $\mathcal{C}^{i,j}$ and ranks of the chain differentials for the disjoint union of the unknot and the granny-knot

	-6	-5	-4	-3	-2	-1	0
-2							**1/1**
-4				0/2 $\xrightarrow{2}$	0/6 $\xrightarrow{4}$	0/6 $\xrightarrow{2}$	**2/4**
-6	0/1 $\xrightarrow{1}$	0/6 $\xrightarrow{5}$	0/15 $\xrightarrow{10}$	0/28 $\xrightarrow{18}$	**2/33** $\xrightarrow{13}$	0/18 $\xrightarrow{5}$	**1/6**
-8	0/6 $\xrightarrow{6}$	0/30 $\xrightarrow{24}$	0/60 $\xrightarrow{36}$	0/74 $\xrightarrow{38}$	**2[2]/54** $\xrightarrow{14}$	0/18 $\xrightarrow{4}$	0/4
-10	0/15 $\xrightarrow{15}$	0/60 $\xrightarrow{45}$	**1/90** $\xrightarrow{44}$	**2/74** $\xrightarrow{28}$	**0[2]/33** $\xrightarrow{5}$	0/6 $\xrightarrow{1}$	0/1
-12	0/20 $\xrightarrow{20}$	0/60 $\xrightarrow{40}$	**0[2]/60** $\xrightarrow{20}$	**2/28** $\xrightarrow{6}$	0/6		
-14	0/15 $\xrightarrow{15}$	**2[1]/30** $\xrightarrow{13}$	**0[1]/15** $\xrightarrow{2}$	0/2			
-16	0/6 $\xrightarrow{6}$	**0[2]/6**					
-18	**1/1**						

Table 2: Ranks of $\mathcal{H}^{i,j}$ and $\mathcal{C}^{i,j}$ and ranks of the chain differentials for the disjoint union of two trefoil knots

	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
9								$\frac{0}{1}\xrightarrow{1}$	$\frac{0}{6}\xrightarrow{5}$	$\frac{0}{15}\xrightarrow{10}$	$\frac{0}{20}\xrightarrow{10}$	$\frac{0}{16}\xrightarrow{6}$	$\frac{\mathbf{1}}{\mathbf{8}}\xrightarrow{1}$	$\frac{0}{1}$
7						$\frac{0}{1}\xrightarrow{1}$	$\frac{0}{13}\xrightarrow{12}$	$\frac{0}{67}\xrightarrow{55}$	$\frac{0}{180}\xrightarrow{125}$	$\frac{0}{279}\xrightarrow{154}$	$\frac{0}{261}\xrightarrow{107}$	$\frac{\mathbf{1}}{\mathbf{149}}\xrightarrow{41}$	$\frac{\mathbf{0}[\mathbf{1}]}{\mathbf{45}}\xrightarrow{4}$	$\frac{0}{4}$
5					$\frac{0}{5}\xrightarrow{5}$	$\frac{0}{60}\xrightarrow{55}$	$\frac{0}{315}\xrightarrow{260}$	$\frac{0}{918}\xrightarrow{658}$	$\frac{0}{1609}\xrightarrow{951}$	$\frac{0}{1743}\xrightarrow{792}$	$\frac{\mathbf{1}}{\mathbf{1164}}\xrightarrow{371}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{461}}\xrightarrow{89}$	$\frac{0}{95}\xrightarrow{6}$	$\frac{0}{6}$
3				$\frac{0}{10}\xrightarrow{10}$	$\frac{0}{136}\xrightarrow{126}$	$\frac{0}{789}\xrightarrow{663}$	$\frac{0}{2558}\xrightarrow{1895}$	$\frac{0}{5054}\xrightarrow{3159}$	$\frac{\mathbf{1}}{\mathbf{6264}}\xrightarrow{3104}$	$\frac{\mathbf{2}}{\mathbf{4873}}\xrightarrow{1767}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{2333}}\xrightarrow{565}$	$\frac{0}{656}\xrightarrow{91}$	$\frac{0}{95}\xrightarrow{4}$	$\frac{0}{4}$
1			$\frac{0}{10}\xrightarrow{10}$	$\frac{0}{168}\xrightarrow{158}$	$\frac{0}{1141}\xrightarrow{983}$	$\frac{0}{4224}\xrightarrow{3241}$	$\frac{0}{9479}\xrightarrow{6238}$	$\frac{\mathbf{2}}{\mathbf{13406}}\xrightarrow{7166}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{12038}}\xrightarrow{4871}$	$\frac{\mathbf{1}[\mathbf{2}]}{\mathbf{6788}}\xrightarrow{1916}$	$\frac{0}{2333}\xrightarrow{417}$	$\frac{0}{461}\xrightarrow{44}$	$\frac{0}{45}\xrightarrow{1}$	$\frac{0}{1}$
-1		$\frac{0}{6}\xrightarrow{6}$	$\frac{0}{119}\xrightarrow{113}$	$\frac{0}{972}\xrightarrow{859}$	$\frac{0}{4234}\xrightarrow{3375}$	$\frac{0}{10982}\xrightarrow{7607}$	$\frac{\mathbf{1}}{\mathbf{17827}}\xrightarrow{10219}$	$\frac{\mathbf{3}[\mathbf{1}]}{\mathbf{18408}}\xrightarrow{8186}$	$\frac{\mathbf{2}[\mathbf{1}]}{\mathbf{12038}}\xrightarrow{3850}$	$\frac{0}{4873}\xrightarrow{1023}$	$\frac{0}{1164}\xrightarrow{141}$	$\frac{0}{149}\xrightarrow{8}$	$\frac{0}{8}$	
-3	$\frac{0}{1}\xrightarrow{1}$	$\frac{0}{43}\xrightarrow{42}$	$\frac{0}{474}\xrightarrow{432}$	$\frac{0}{2560}\xrightarrow{2128}$	$\frac{0}{7916}\xrightarrow{5788}$	$\frac{\mathbf{2}}{\mathbf{14976}}\xrightarrow{9186}$	$\frac{\mathbf{2}[\mathbf{1}]}{\mathbf{17827}}\xrightarrow{8639}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{13406}}\xrightarrow{4766}$	$\frac{0}{6264}\xrightarrow{1498}$	$\frac{0}{1743}\xrightarrow{245}$	$\frac{0}{261}\xrightarrow{16}$	$\frac{0}{16}$		
-5	$\frac{0}{5}\xrightarrow{5}$	$\frac{0}{118}\xrightarrow{113}$	$\frac{0}{897}\xrightarrow{784}$	$\frac{0}{3492}\xrightarrow{2708}$	$\frac{\mathbf{1}}{\mathbf{7916}}\xrightarrow{5207}$	$\frac{\mathbf{1}[\mathbf{2}]}{\mathbf{10982}}\xrightarrow{5774}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{9479}}\xrightarrow{3704}$	$\frac{0}{5054}\xrightarrow{1350}$	$\frac{0}{1609}\xrightarrow{259}$	$\frac{0}{279}\xrightarrow{20}$	$\frac{0}{20}$			
-7	$\frac{0}{10}\xrightarrow{10}$	$\frac{0}{162}\xrightarrow{152}$	$\frac{0}{897}\xrightarrow{745}$	$\frac{\mathbf{1}}{\mathbf{2560}}\xrightarrow{1814}$	$\frac{\mathbf{2}[\mathbf{1}]}{\mathbf{4234}}\xrightarrow{2418}$	$\frac{\mathbf{1}}{\mathbf{4224}}\xrightarrow{1805}$	$\frac{0}{2558}\xrightarrow{753}$	$\frac{0}{918}\xrightarrow{165}$	$\frac{0}{180}\xrightarrow{15}$	$\frac{0}{15}$				
-9	$\frac{0}{10}\xrightarrow{10}$	$\frac{0}{118}\xrightarrow{108}$	$\frac{\mathbf{1}}{\mathbf{474}}\xrightarrow{365}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{972}}\xrightarrow{606}$	$\frac{0}{1141}\xrightarrow{535}$	$\frac{0}{789}\xrightarrow{254}$	$\frac{0}{315}\xrightarrow{61}$	$\frac{0}{67}\xrightarrow{6}$	$\frac{0}{6}$					
-11	$\frac{0}{5}\xrightarrow{5}$	$\frac{0}{43}\xrightarrow{38}$	$\frac{\mathbf{1}[\mathbf{1}]}{\mathbf{119}}\xrightarrow{80}$	$\frac{0}{168}\xrightarrow{88}$	$\frac{0}{136}\xrightarrow{48}$	$\frac{0}{60}\xrightarrow{12}$	$\frac{0}{13}\xrightarrow{1}$	$\frac{0}{1}$						
-13	$\frac{0}{1}\xrightarrow{1}$	$\frac{\mathbf{1}}{\mathbf{6}}\xrightarrow{4}$	$\frac{0}{10}\xrightarrow{6}$	$\frac{0}{10}\xrightarrow{4}$	$\frac{0}{5}\xrightarrow{1}$	$\frac{0}{1}$								

Table 3: Ranks of $\mathcal{H}^{i,j}$ and $\mathcal{C}^{i,j}$ and ranks of the chain differentials for either the Kinoshita-Terasaka knot or the Conway knot (both have the same Khovanov homology)