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Almost Split Morphisms, Preprojective Algebras and Multiplication Maps of Maximal Rank

Steven P. Diaz
Syracuse University

Mark Kleiner
Syracuse University

ALMOST SPLIT MORPHISMS, PREPROJECTIVE ALGEBRAS AND MULTIPLICATION MAPS OF MAXIMAL RANK

STEVEN P. DIAZ AND MARK KLEINER

Abstract. With a grading previously introduced by the second-named author, the multiplication maps in the preprojective algebra satisfy a maximal rank property that is similar to the maximal rank property proven by Hochster and Laksov for the multiplication maps in the commutative polynomial ring. The result follows from a more general theorem about the maximal rank property of a minimal almost split morphism, which also yields a quadratic inequality for the dimensions of indecomposable modules involved.

1. Introduction

Let k be a field and let R be the polynomial ring in n commuting variables over k. Let R_i be its i^{th} graded piece consisting of homogeneous polynomials of degree i. A result of Hochster and Laksov [4] says that if $i \geq 2$ and $V \subset R_i$ is a general subspace then the natural multiplication map from $V \otimes R_1$ to R_{i+1} has maximal rank, that is is either injective or surjective, and it is not known what happens if one replaces R_1 by R_d for $d > 1$. One may wonder which other graded rings have a similar property.

In [5] a new grading on the preprojective algebra was introduced. In this paper we show that with this grading, the preprojective algebra of a finite quiver without oriented cycles satisfies a property analogous to the Hochster-Laksov property for polynomial rings, and much of our proof is quite similar to their proof. At one point the proof for preprojective algebras becomes easier than the proof for polynomial rings: some of the more complicated dimension counts needed for polynomial rings are not needed for preprojective algebras. This allows us to obtain a result for preprojective algebras that is stronger than the analogous result for polynomial rings.

The key to making things work is the fact that the multiplication-by-arrow maps into a fixed homogeneous component of the infinite dimensional (in general) preprojective algebra give rise to a minimal right almost split morphism of modules over the finite dimensional path algebra of the quiver [5], which implies the maximal rank property. In fact we show that a minimal right almost split morphism $g : B \to C$ of finite dimensional modules over a k-algebra satisfies a maximal rank property analogous to the Hochster-Laksov property for polynomial rings, and if C is not projective and $B_1, \ldots, B_l$ are the nonisomorphic indecomposable summands of B then $\dim_k C < (\dim_k B_1)^2 + \cdots + (\dim_k B_l)^2$. We do not know what happens if multiplication by arrows is replaced by multiplication by paths of fixed length greater than one.

There is a natural dual to the Hochster-Laksov maximal rank property, and the two properties always occur simultaneously. We give two explanations of this fact, one general homological and the other based on the vector space duality $\mathrm{D} = \mathrm{Hom}_k(\ , k)$. As a consequence, the multiplication-by-arrow maps out of a fixed homogeneous component of the preprojective algebra satisfy the dual Hochster-Laksov maximal rank property.

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The organization of the paper is as follows. In Section 2 we prove a theorem that gives a general situation in which one can obtain a maximal rank property analogous to the Hochster-Laksov property for polynomial rings. This general situation does not include the polynomial ring as a special case. In Section 3 we review some facts about almost split morphisms and preprojective algebras and then show that almost split morhisms in general and the preprojective algebra in particular fit into the general set up of Section 2. In Section 4 we use the material in Sections 2 and 3 to obtain results for the preprojective algebra that look very analogous to the Hochster-Laksov result for polynomial rings. Then we conclude with some examples to illustrate the results.

In this paper for simplicity we work over a fixed algebraically closed field k and dim always means $\dim_k$. For unexplained terminology we refer the reader to [1].

2. The General Theorem

Let $V_1, V_2, ..., V_l, W_1, W_2, ..., W_l, U$ be finite dimensional vector spaces. Let T be a linear transformation from the direct sum of the tensor products $V_1 \otimes W_1$, $V_2 \otimes W_2$, ... ,$V_l \otimes W_l$ to U.

$$T : \bigoplus_{i=1}^{l} (V_i \otimes W_i) \to U \tag{2.1}$$

Definition 2.1. *We say that T satisfies the right omnipresent maximal rank property if and only if for every choice of subspaces $W_i' \subset W_i$ for $i = 1, ..., l$ the restriction of T to the direct sum of the tensor products $V_1 \otimes W_1', V_2 \otimes W_2', ..., V_l \otimes W_l'$ has maximal rank, that is, is either injective or surjective.*

Notice that if T satisfies the right omnipresent maximal rank property then so does its restriction to $\oplus_{i=1}^{l} V_i \otimes W_i'$, and T must itself have maximal rank. Every injective T satisfies the right omnipresent maximal rank property. The interesting case is when T is surjective but not injective.

Denote by $\mathrm{End}\,(V_i)$ the k-algebra of linear operators on V_i. The tensor product $V_i \otimes W_i$ is a left $\mathrm{End}\,(V_i)$-module by means of $\varphi_i \cdot (v_i \otimes w_i) = \varphi_i(v_i) \otimes w_i, \varphi_i \in \mathrm{End}\,(V_i), v_i \in V_i, w_i \in W_i$. Applying this to each term of the direct sum one obtains a bilinear evaluation map

$$e : \prod_{i=1}^{l} \mathrm{End}\,(V_i) \times \bigoplus_{i=1}^{l} (V_i \otimes W_i) \to \bigoplus_{i=1}^{l} (V_i \otimes W_i).$$

Denote $\prod_{i=1}^{l} \mathrm{End}\,(V_i)$ by B. The map e defines a structure of a left B-module on $\bigoplus_{i=1}^{l} (V_i \otimes W_i)$. Notice that B has dimension $\Sigma(\dim V_i)^2$. Let P_i be the projective space of one dimensional subspaces of $V_i \otimes W_i$ and let P be the projective space of one dimensional subspaces of $\bigoplus_{i=1}^{l} (V_i \otimes W_i)$. Notice that P has dimension $\Sigma(\dim V_i \dim W_i) - 1$. We shall study the product $B \times P$ together with its two projection maps π_1 onto B and π_2 onto P.

Since the evaluation map e is bilinear, we may conclude that the inverse image under e of $\mathrm{Ker}\, T$ is a Zariski closed subset of the domain of e. Furthermore using bilinearity again we see that $e^{-1}(\mathrm{Ker}\, T)$ is the affine cone over a Zariski closed subset of $B \times P$. We denote this subset by Y. For each i from 1 to l let X_i be an irreducible quasiprojective subset of P_i and let $C(X_i)$ be its corresponding affine cone in $V_i \otimes W_i$. Let X be the irreducible quasiprojective subset of P corresponding to $C(X_1) \times C(X_2) \times ... \times C(X_l)$. Notice that $\dim X = \sum_{i=1}^{l} \dim C(X_i) - 1$.

Theorem 2.1. *Assume that T satisfies the right omnipresent maximal rank property and that $\sum_{i=1}^{l} \dim C(X_i) \leq \dim U$. Then $\pi_1(\pi_2^{-1}(X) \cap Y)$ is contained in a proper Zariski closed subset of B.*

If T is injective then Y is empty and the result trivially follows. Thus we may assume that T is surjective. To proceed with the proof we shall divide Y into two pieces based on the following easy statement, and then deal with each piece separately. Denote by D the contravariant functor $\mathrm{Hom}_k(\ ,k)$.

Lemma 2.2. *Let V and W be k-vector spaces and let $\alpha : V \otimes W \to \mathrm{Hom}_k(\mathrm{D}\,V, W)$ be the k-linear map given by $\alpha(v \otimes w)(f) = f(v)w, v \in V, w \in W, f \in \mathrm{D}\,V$ (α is an isomorphism if $\dim V < \infty$). For $x \in V \otimes W$ denote by $\mathrm{End}(V)x$ the cyclic $\mathrm{End}(V)$-submodule of $V \otimes W$ generated by x. Then $\mathrm{End}(V)x = V \otimes \mathrm{Im}\,\alpha(x)$.*

Proof. We have $x = \sum_{i=1}^{s} v_i \otimes w_i$. If s is the smallest possible, the sets of vectors $\{v_1, \ldots, v_s\}$ and $\{w_1, \ldots, w_s\}$ are linearly independent [2, Theorem (1.2a), p. 142] so $\mathrm{Im}\,\alpha(x)$ is the span of $\{w_1, \ldots, w_s\}$ and the rest is clear. $\square$

Definition 2.2. *Let $\alpha_i : V_i \otimes W_i \to \mathrm{Hom}_k(\mathrm{D}\,V_i, W_i)$ be the k-linear map described in Lemma 2.2, $i = 1, \ldots, l$. If $x_i \in V_i \otimes W_i$ and $0 \neq c \in k$, then $\mathrm{Im}\,\alpha_i(x_i) = \mathrm{Im}\,\alpha_i(cx_i)$, so if $p \in P$ is represented by $[x_1, \ldots, x_l]$, $x_i \in V_i \otimes W_i$, then for each i the subspace $\mathrm{Im}\,\alpha_i(x_i)$ of W_i is independent of the choice of representative for p. We set $Y_1 = \{(b,p) \in Y : \sum_{i=1}^{l}(\dim V_i)(\mathrm{rank}\,\alpha_i(x_i)) < \dim U\}$ and $Y_2 = Y - Y_1$.*

Lemma 2.3. *$Y = Y_1 \cup Y_2$ where Y_1 is a closed subset of Y, and Y_2 is an open subset of Y.*

Proof. That $Y = Y_1 \cup Y_2$ is obvious. For the other two statements consider the projection onto the second factor $\pi_2 : B \times P \to P$ and note that Y_1 (Y_2) is the intersection of Y with the inverse image of the closed (open) subset of P consisting of points corresponding to tuples of tensors satisfying $\sum_{i=1}^{l}(\dim V_i)(\mathrm{rank}\,\alpha_i(x_i)) < (\geq) \dim U$. $\square$

Lemma 2.4. *Assume that T satisfies the right omnipresent maximal rank property. Suppose that $(b,p) \in Y_1$ and $b = [\varphi_1, \varphi_2, ..., \varphi_l]$. Then for some i, φ_i is not an isomorphism.*

Proof. If $x = [x_1, \ldots, x_l]$ represents p, then $T(bx) = T\left([\varphi_1 \cdot x_1, \ldots, \varphi_l \cdot x_l]\right) = 0$ because $(b,p) \in Y_1$. By Lemma 2.2, $Bx = \bigoplus_{i=1}^{l} \mathrm{End}(V_i)x_i = \bigoplus_{i=1}^{l} V_i \otimes \mathrm{Im}\,\alpha_i(x_i)$, so the restriction of T to Bx is injective because $(b,p) \in Y_1$ and T satisfies the right omnipresent maximal rank property. Since $T(bx) = 0$ then $bx = [\varphi_1 \cdot x_1, \ldots, \varphi_l \cdot x_l] = 0$ whence $\varphi_i \cdot x_i = 0$ for all i. Because p is a point in a projective space, at least one x_i is not equal to 0. For this i, φ_i is not an isomorphism. $\square$

Lemma 2.5. *Assume that T satisfies the right omnipresent maximal rank property and that $\sum_{i=1}^{l} \dim C(X_i) \leq \dim U$. Suppose $\pi_2^{-1}(X) \cap Y_2$ is nonempty. Then $\pi_2^{-1}(X) \cap Y_2$ has Krull dimension at most $\Sigma(\dim V_i)^2 - 1$, one less than the dimension of B.*

Proof. As with Lemma 2.3 we consider the projection map onto the second factor $\pi_2 : B \times P \to P$. Let $X_2 \subset X$ be the set of points p such that $\sum_{i=1}^{l}(\dim V_i)(\mathrm{rank}\,\alpha_i(x_i)) \geq \dim U$ for any $x = [x_1, \ldots, x_l]$ representing p. Since X_2 is open in X and by assumption nonempty, $\dim X_2 = \dim X$. Pick any point p in X_2 and, identifying B with $B \times \{p\}$, consider the composite $T' : B \to U$ of T and the k-linear map $B \to \bigoplus_{i=1}^{l}(V_i \otimes W_i)$ sending b to bx. Clearly $\mathrm{Ker}\,T' = \pi_2^{-1}(p) \cap Y_2$ and $\mathrm{Im}\,T' = T(Bx) = T(\bigoplus_{i=1}^{l}(V_i \otimes \mathrm{Im}\,\alpha_i(x_i)))$ (use Lemma 2.2). From the assumptions on the ranks of the $\alpha_i(x_i)$ and that T satisfies the right omnipresent maximal rank property, we conclude that T' is surjective. We then conclude that $\dim(\pi_2^{-1}(p) \cap Y_2) = \dim B - \dim U$.

Having computed the dimensions of the fibers of $\pi_2^{-1}(X) \cap Y_2$ over X_2 we then see that the dimension of $\pi_2^{-1}(X) \cap Y_2$ equals $\dim X + \dim B - \dim U = \sum_{i=1}^{l} \dim C(X_i) - 1 + \dim B - \dim U \leq \dim B - 1 = \Sigma(\dim V_i)^2 - 1$. $\square$

The proof of the theorem is now easy.

Proof of theorem. By Lemma 2.4 the image of $\pi_2^{-1}(X) \cap Y_1$ in B will be contained in the proper closed subset of B consisting of points where at least one φ_i is not an isomorphism. By Lemma 2.5 $\pi_2^{-1}(X) \cap Y_2$ has dimension less than that of B and so its closure also does. Since π_1 is a projective morphism, the image in B of the closure of $\pi_2^{-1}(X) \cap Y_2$ will be closed and have dimension less than that of B. Since by Lemma 2.3 $Y = Y_1 \cup Y_2$ we are done. □

Corollary 2.6. *Assume that T satisfies the right omnipresent maximal rank property. Make a choice of subspaces $Z_i \subset V_i \otimes W_i$, $i = 1, ..., l$. Then there exists a dense Zariski open subset $A \subset B$ such that if $[\varphi_1, ..., \varphi_l] \in A$ then the restriction of T to the direct sum of the $\varphi_i(Z_i)$ has maximal rank.*

Proof. We first do the case where $\sum_{i=1}^{l} \dim(Z_i) \leq \dim(U)$. In Theorem 2.1 set $Z_i = C(X_i)$. Choose A to be the complement of any proper Zariski closed subset of B containing $\pi_1(\pi_2^{-1}(X) \cap Y)$. For $[\varphi_1, ..., \varphi_l] \in A$, $\bigoplus_{i=1}^{l} \varphi_i(Z_i)$ intersects the kernel of T only in 0. Thus the restriction of T to $\bigoplus_{i=1}^{l} \varphi_i(Z_i)$ is injective. When $\sum_{i=1}^{l} \dim(Z_i) = \dim(U)$ it is also surjective.

For the case where $\sum_{i=1}^{l} \dim(Z_i) > \dim(U)$ choose subspaces $Z_i' \subset Z_i$ such that $\sum_{i=1}^{l} \dim(Z_i') = \dim(U)$. By the previous case we find A such that if $[\varphi_1, ..., \varphi_l] \in A$ then the restriction of T to $\bigoplus_{i=1}^{l} \varphi_i(Z_i')$ is surjective, so the restriction of T to $\bigoplus_{i=1}^{l} \varphi_i(Z_i)$ is also surjective. □

Corollary 2.7. *Assume that T is surjective and satisfies the right omnipresent maximal rank property. Fix integers a_i, $0 \leq a_i \leq (\dim V_i)(\dim W_i)$, $i = 1, ..., l$, such that $\sum a_i = \dim U$. For each i choose a_i linearly independent elements $m(i, j)$, $1 \leq j \leq a_i$, of $V_i \otimes W_i$. Then there exists a dense Zariski open subset $A \subset B$ such that if $[\varphi_1, ..., \varphi_l] \in A$ then the elements $T(\varphi_i(m(i, j)))$ form a basis for U.*

Proof. In Corollary 2.6 set Z_i equal to the span of the $m(i, j)$'s. □

Definition 2.3. *We say that T satisfies the left general maximal rank property if and only if for a general choice of subspaces $V_i' \subset V_i$ for $i = 1, ..., l$ the restriction of T to $\oplus_{i=1}^{l}(V_i' \otimes W_i)$ has maximal rank, that is, is either injective or surjective.*

By a general choice of subspaces we mean the following. Once the dimensions of the V_i''s to be chosen are fixed, the set of all possible choices of V_i''s can be identified with a product of Grassmanians. We mean that there exists a Zariski open dense subset of that product such that if the choice of V_i''s comes from that set, then the restriction of T has maximal rank.

Similar to Definitions 2.1 and 2.3 one can define what it means for the map T of (2.1) to satisfy the left omnipresent or right general maximal rank property. With these definitions, we leave it to the reader to interchange appropriately the words "left" and "right" in the above assertions and obtain true statements. Of course, this comment also applies to the remainder of the section.

Corollary 2.8. *If T satisfies the right omnipresent maximal rank property then T satisfies the left general maximal rank property.*

Proof. Make a choice of subspaces $V_i' \subset V_i$ for $i = 1, ...l$. In Corollary 2.6 set $Z_i = V_i' \otimes W_i$. Notice that $\varphi_i(V_i' \otimes W_i) = \varphi_i(V_i') \otimes W_i$. A general tuple of endomorphisms $[\varphi_1, ..., \varphi_l]$ applied to a specific tuple of subspaces $[V_1', ..., V_l']$ gives a general tuple of subspaces. □

In Section 4 we will give examples to show that the right omnipresent maximal rank property does not imply the left omnipresent maximal rank property and the right general maximal rank property does not imply the left general maximal rank property.

We now indicate how to dualize the above results of this section. Let $V_1, V_2, ..., V_l, W_1, W_2, ..., W_l$, Q be finite dimensional vector spaces and let

$$S : Q \to \bigoplus_{i=1}^{l} (V_i \otimes W_i) \tag{2.2}$$

be a linear transformation.

Definition 2.4. *We say that S satisfies the right omnipresent maximal rank property if and only if for every choice of subspaces $W_i' \subset W_i$ for $i = 1, ..., l$ the composition of S with the linear transformation*

$$\oplus_{i=1}^{l}(1_{V_i} \otimes \tau_i) : \oplus_{i=1}^{l}(V_i \otimes W_i) \to \oplus_{i=1}^{l}(V_i \otimes (W_i/W_i'))$$

has maximal rank, that is, is either injective or surjective, where $\tau_i : W_i \to W_i/W_i'$ is the natural projection. And we say that S satisfies the left general maximal rank property if and only if for a general choice of subspaces $V_i' \subset V_i$ for $i = 1, ..., l$ the composition of S with the linear transformation

$$\oplus_{i=1}^{l}(\sigma_i \otimes 1_{W_i}) : \oplus_{i=1}^{l}(V_i \otimes W_i) \to \oplus_{i=1}^{l}((V_i/V_i') \otimes W_i)$$

has maximal rank, where $\sigma_i : V_i \to V_i/V_i'$ is the natural projection.

The following lemma shows that the question of whether a map of the type (2.1) satisfies the omnipresent or general maximal rank property is equivalent to the same question for a map of the type (2.2).

Lemma 2.9. *Let*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & B' \arrow[d, "i"'] & & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "q"'] & C \arrow[r] & 0 \\
 & & B'' \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

be an exact diagram in an abelian category. Then gi is monic (epi) if and only if qf is monic (epi).

Proof. By the 3×3 lemma the following commutative diagram is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker} gi \arrow[r] \arrow[d, "h"] & B' \arrow[r, "gi"] \arrow[d, "i"] & \operatorname{Im} gi \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r, "g"] \arrow[d, "q"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Coker} h \arrow[r, "r"] \arrow[d] & B'' \arrow[r] \arrow[d] & \operatorname{Coker} j \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence gi is monic if and only if $\operatorname{Ker} gi = 0$, if and only if p is iso, if and only if qf is monic. The rest of the proof is similar. $\square$

Corollary 2.10. (a) *If a linear transformation $T : \bigoplus_{i=1}^{l}(V_i \otimes W_i) \to U$ is surjective, it satisfies the left omnipresent (general) maximal rank property if and only if so does the inclusion $\operatorname{Ker} T \to \bigoplus_{i=1}^{l}(V_i \otimes W_i)$.*

(b) *If a linear transformation $S : Q \to \bigoplus_{i=1}^{l}(V_i \otimes W_i)$ is injective, then it satisfies the left omnipresent (general) maximal rank property if and only if so does the projection $\bigoplus_{i=1}^{l}(V_i \otimes W_i) \to \operatorname{Coker} S$.*

Proof. The statement follows immediately from Lemma 2.9. $\square$

We note that if the map T of Corollary 2.10(a) is injective, it satisfies both the right omnipresent and left general maximal rank property, and if T is neither surjective nor injective then it satisfies neither of the properties. A similar remark applies to the map S of Corollary 2.10(b).

A different way to relate the maps of the types (2.1) and (2.2) is through the vector space duality D.

Proposition 2.11. *A linear transformation $T : \oplus_{i=1}^{l}(V_i \otimes W_i) \to U$ satisfies the left omnipresent (general) maximal rank property if and only if so does its dual $\mathrm{D}\, T : \mathrm{D}\, U \to \oplus_{i=1}^{l}(\mathrm{D}\, V_i \otimes \mathrm{D}\, W_i)$.*

Proof. For $i = 1, \dots, l$ let X_i be a subspace of V_i and $f_i : X_i \to V_i$ the inclusion map, then $\mathrm{D}\, f_i : \mathrm{D}\, V_i \to \mathrm{D}\, X_i$ is an epimorphism with $\operatorname{Ker} \mathrm{D}\, f_i = X_i^{\perp} = \{\phi \in \mathrm{D}\, V_i : \phi(X_i) = 0\}$ so that $\mathrm{D}\, X_i \cong \mathrm{D}\, V_i / X_i^{\perp}$. Therefore $T \circ (\oplus_{i=1}^{l}(f_i \otimes 1_{W_i}))$ is monic (epi) if and only if $(\oplus_{i=1}^{l}(\mathrm{D}\, f_i \otimes 1_{\mathrm{D}\, W_i})) \circ \mathrm{D}\, T$ is epi (monic), if and only if $(\oplus_{i=1}^{l}(\psi_i \otimes 1_{\mathrm{D}\, W_i})) \circ \mathrm{D}\, T$ is epi (monic), where $\psi_i : \mathrm{D}\, V_i \to \mathrm{D}\, V_i / X_i^{\perp}$ is the natural projection. Note that X_i runs through the set of all subspaces of V_i if and only if $X_i^{\perp}$ runs through the set of all subspaces of $\mathrm{D}\, V_i$. Hence T satisfies the left omnipresent maximal rank property if and only if so does $\mathrm{D}\, T$. For a fixed sequence of nonnegative integers $d_i \le n_i = \dim V_i$, the l-tuple $(X_1, \dots, X_i, \dots, X_l)$ runs through a dense open set of the product of Grassmanians $\prod_{i=1}^{l} G(d_i, V_i)$ if and only if $(X_1^{\perp}, \dots, X_i^{\perp}, \dots, X_l^{\perp})$ runs through the corresponding dense open set of the product of Grassmanians $\prod_{i=1}^{l} G(n_i - d_i, \mathrm{D}\, V_i)$ under the isomorphism that is the product of the natural isomorphisms $\mathrm{D} : G(d_i, V_i) \to G(n_i - d_i, \mathrm{D}\, V_i)$, see [3, p. 200]. Therefore T satisfies the left general maximal rank property if and only if so does $\mathrm{D}\, T$. $\square$

We end this section with a lemma showing that the right omnipresent maximal rank property puts a restriction on the relative sizes of the vector spaces involved.

Lemma 2.12. (a) *If $T : \oplus_{i=1}^{l} V_i \otimes W_i \to U$ satisfies the right omnipresent maximal rank property and T is surjective but not injective, then* $\dim U < \sum_{i=1}^{l}(\dim V_i)^2$.

(b) *If $S : Q \to \oplus_{i=1}^{l} V_i \otimes W_i$ satisfies the right omnipresent maximal rank property and S is injective but not surjective, then* $\dim Q < \sum_{i=1}^{l}(\dim V_i)^2$.

Proof. (a) Suppose to the contrary that $\dim U \geq \sum_{i=1}^{l}(\dim V_i)^2$. Let $\{v_{i1}, ..., v_{ik_i}\}$ be a basis for V_i. Express some nonzero element of $\operatorname{Ker} T$ in the form

$$\left(\sum_{j=1}^{k_1} v_{1j} \otimes w_{1j}, \ldots, \sum_{j=1}^{k_i} v_{ij} \otimes w_{ij} \ldots, \sum_{j=1}^{k_l} v_{lj} \otimes w_{lj}\right)$$

and let W_i' equal the span of $\{w_{i1}, ..., w_{ik_i}\}$. Then $\dim \oplus_{i=1}^{l}(V_i \otimes W_i') \leq \dim U$ but the restriction of T to $\oplus_{i=1}^{l}(V_i \otimes W_i')$ is neither surjective nor injective because its kernel is not zero.

(b) Follows from (a) and Proposition 2.11. □

3. Almost Split Morphisms and Preprojective Algebras

We apply the results of Section 2 to representations of algebras which provide a large supply of linear transformations of the form $\oplus_{i=1}^{l}(V_i \otimes W_i) \to U$ or $Q \to \oplus_{i=1}^{l}(V_i \otimes W_i)$. Let Λ be an associative k-algebra, let $\operatorname{mod}\Lambda$ be the category of finite dimensional left Λ-modules, and let $g : B \to C$ and $f : A \to B$ be morphisms in $\operatorname{mod}\Lambda$. Replacing B with an isomorphic module if necessary, we may assume that $B = V_1^{n_1} \oplus \cdots \oplus V_l^{n_l}$ where $V_1, \ldots, V_l$ are nonisomorphic indecomposable Λ-modules, $l, n_1, \ldots, n_l$ are nonegative integers, and V^m stands for the direct sum of m copies of V. For $i = 1, \ldots, l$ denote by W_i the k-space with a basis $e_{i1}, \ldots, e_{in_i}$, and for each $j = 1, \ldots, n_i$ denote by $h_{ij} : V_i \to V_i \otimes ke_{ij}$ the isomorphism of Λ-modules sending each $v \in V_i$ to $v \otimes e_{ij}$. Let

$$h : B \to \oplus_{i=1}^{l}(V_i \otimes W_i) \tag{3.1}$$

be the isomorphism in $\operatorname{mod}\Lambda$ induced by the h_{ij}'s. Denote by $g_{ij} : V_i \to C$ and $f_{ij} : A \to V_i$ the morphisms in $\operatorname{mod}\Lambda$ induced by g and f, respectively, and consider the morphisms $T_i : V_i \otimes W_i \to C$ and $S_i : A \to V_i \otimes W_i$ defined by $T_i(v \otimes e_{ij}) = g_{ij}(v), v \in V_i$, and $S_i(a) = (f_{ij}(a) \otimes e_{ij}), a \in A$, respectively. Let

$$T : \oplus_{i=1}^{l}(V_i \otimes W_i) \to C \quad \text{and} \quad S : A \to \oplus_{i=1}^{l}(V_i \otimes W_i) \tag{3.2}$$

be the morphisms in $\operatorname{mod}\Lambda$ induced by the T_i's and S_i's, respectively. It is straight forward to check that

$$g = Th \quad \text{and} \quad S = hf. \tag{3.3}$$

Proposition 3.1. *Let $g : B \to C$ and $f : A \to B$ be morphisms in* $\operatorname{mod}\Lambda$ *with $B = \oplus_{i=1}^{l} V_i^{n_i}$ where $V_1, \ldots, V_l$ are nonisomorphic indecomposable Λ-modules. Let h be the isomorphism in (3.1), let T and S be the morphisms in (3.2) constructed from g and f, respectively.*

If g is a minimal right almost split morphism in $\operatorname{mod}\Lambda$ *then:*

(a) *T satisfies the right omnipresent maximal rank property.*
(b) *For a general choice of k-subspaces $U_i \subset V_i$, the restriction of g to $\oplus_{i=1}^{l} U_i^{n_i}$ has maximal rank.*
(c) *If g is surjective then* $\dim C < \sum_{i=1}^{l}(\dim V_i)^2$.

If f is a minimal left almost split morphism in $\operatorname{mod}\Lambda$ *then:*

(d) *S satisfies the right omnipresent maximal rank property.*

(e) *For a general choice of k-subspaces $U_i \subset V_i$, denote by $\sigma_i : V_i \to V_i/U_i$ the natural projection. Then the linear transformation* $(\oplus_{i=1}^{l} \sigma_i^{n_i}) \circ f$ *has maximal rank.*

(f) *If f is injective then* $\dim A < \sum_{i=1}^{l} (\dim V_i)^2$.

If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is an almost split sequence in* $\operatorname{mod} \Lambda$ *then:*

(g) $\dim B < 2 \sum_{i=1}^{l} (\dim V_i)^2 - 1$.

Proof. (a) Since g is minimal right almost split, so is T by (3.3). If W_i' is a subspace of W_i, the Λ-module $V_i \otimes W_i'$ is a direct summand of $V_i \otimes W_i$. Hence the restriction of T to $\oplus_{i=1}^{l} (V_i \otimes W_i')$ is an irreducible morphism and thus is either a monomorphism or an epimorphism [1, Ch. V, Theorem 5.3(a) and Lemma 5.1(a)], so (a) holds. According to Corollary 2.8, T satisfies the left general maximal rank property. In view of the structure of the isomorphisms h_{ij} constructed above, we conclude that (b) holds. Part (c) is a direct consequence of (a), formula (3.3), and Lemma 2.12(a).

(d) The proof is similar to that of (a) using the analogous properties of minimal left almost split morphisms.

(e) If f is surjective, the statement is clear. If f is not surjective, it is injective, and so is S in view of formulas (3.3). By (d) and Corollary 2.10(b), the projection $\oplus_{i=1}^{l} (V_i \otimes W_i) \to \operatorname{Coker} S$ satisfies the right omnipresent maximal rank property. By Corollary 2.8, it satisfies the left general maximal rank property, and so does S by Corollary 2.10(b). Then f satisfies the desired property in view of formulas (3.3).

Another way to prove (d) and (e) is to note that both $\mathrm{D} f$ and $\mathrm{D} S$ are minimal right almost split morphisms in $\operatorname{mod} \Lambda^{\mathrm{op}}$, and then use (a), Corollary 2.8, and Proposition 2.11 together with formulas (3.3).

(f) The proof is similar to that of (c), using Lemma 2.12(b).

(g) The formula follows from (c) and (f). □

Remark 3.1. (a) Lemma 2.12 holds when k is an arbitrary field. Hence so do parts (a), (c), (d), (f), and (g) of Proposition 3.1; moreover, they hold if $\operatorname{mod} \Lambda$ is replaced by any full subcategory of an abelian category closed under extensions and direct summands where the objects and morphism sets are finite dimensional k-vector spaces and composition of morphisms is k-bilinear.

(b) Parts (a), (b), and (c) of Proposition 3.1 hold if $g : B \to C$ is an irreducible morphism with C indecomposable, and parts (d), (e), and (f) hold if $f : A \to B$ is an irreducible morphism with A indecomposable. This follows from the observation after Definition 2.1 that the right omnipresent maximal rank property of a linear transformation is inherited by its appropriate restrictions, and from the dual statement.

(c) Parts (c), (f), and (g) of Proposition 3.1 imply that for a fixed number of nonisomorphic indecomposable summands of the middle term of an almost split sequence, the summands cannot be much smaller than the end terms of the sequence, i.e., the multiplicities of the summands cannot be too large, and that there is a balance between the sizes of the end terms. Part (c) is false if the morphism g is not surjective, and part (f) is false if the morphism f is not injective.

We will apply this in particular to the preprojective algebra where the grading introduced in [5] allows us to interpret the multiplication-by-arrow maps into (from) a fixed homogeneous component as a minimal right (left) almost split morphism of modules over the path algebra of the quiver. We recall some facts from the latter paper.

For the remainder of this paper we fix a finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ without oriented cycles with the set of vertices Γ_0 and the set of arrows Γ_1. Let $\bar{\Gamma} = (\bar{\Gamma}_0, \bar{\Gamma}_1)$ be a new quiver with $\bar{\Gamma}_0 = \Gamma_0$ and $\bar{\Gamma}_1 = \Gamma_1 \cup \Gamma_1^*$, where $\Gamma_1 \cap \Gamma_1^* = \emptyset$ and the elements of Γ_1^* are in the following one-to-one correspondence with the elements of Γ_1: for each $\gamma : t \to v$ in Γ_1, there is a unique element $\gamma^* : v \to t$ in Γ_1^*. To turn the path algebra $k\bar{\Gamma}$ of $\bar{\Gamma}$ over a field k into a graded k-algebra, we assign degree 0 to each trivial path $e_t, t \in \Gamma_0$, and each arrow $\gamma \in \Gamma_1$; degree 1 to each arrow $\gamma^* \in \Gamma_1^*$; and compute the degree of a nontrivial path $q = \delta_1 \ldots \delta_r$ as $\deg q = \sum_{i=1}^r \deg \delta_i$. Clearly, $k\Gamma$ is the k-subalgebra of $k\bar{\Gamma}$ comprising the elements of degree 0.

Let $\mathbb{N}$ be the set of nonnegative integers. For all $t \in \Gamma_0,\ d \in \mathbb{N}$, let W_d^t be the span of all those paths in $\bar{\Gamma}$ of degree d that start at t. Note that $W_d^t \in \operatorname{mod} k\Gamma$ so

$$k\bar{\Gamma} = \underset{d\in\mathbb{N}}{\oplus}\ \underset{t\in\Gamma_0}{\oplus} W_d^t \tag{3.4}$$

is a decomposition of $k\bar{\Gamma}$ as a direct sum of its left $k\Gamma$-submodules

Let now a and b be any two functions $\Gamma_1 \to k$ satisfying $a(\gamma) \neq 0$ and $b(\gamma) \neq 0$ for all $\gamma \in \Gamma_1$. If $s(\gamma)$ is the starting point and $e(\gamma)$ is the end point of $\gamma \in \Gamma_1$, for each $t \in \Gamma_0$ set

$$m_t = \sum_{\substack{\gamma\in\Gamma_1\\ s(\gamma)=t}} a(\gamma)\gamma^*\gamma - \sum_{\substack{\gamma\in\Gamma_1\\ e(\gamma)=t}} b(\gamma)\gamma\gamma^*$$

and denote by J the two-sided ideal of $k\bar{\Gamma}$ generated by the element

$$\sum_{t\in\Gamma_0} m_t = \sum_{\gamma\in\Gamma_1} [\gamma^*, \gamma]_{a,b}$$

where $[\gamma^*, \gamma]_{a,b} = a(\gamma)\gamma^*\gamma - b(\gamma)\gamma\gamma^* *$ is the (a,b)-commutator of γ^* and γ. The factor algebra $\mathcal{P}_k(\Gamma)_{a,b} = k\bar{\Gamma}/J$ is the (a,b)-preprojective algebra of Γ.

Since the elements m_t are homogeneous of degree 1, J is a homogeneous ideal containing no nonzero elements of degree 0. Hence $\mathcal{P}_k(\Gamma)_{a,b}$ is a graded k-algebra, and the restriction to $k\Gamma$ of the natural projection $\pi : k\bar{\Gamma} \to \mathcal{P}_k(\Gamma)_{a,b}$ is an isomorphism of $k\Gamma$ with the subalgebra of $\mathcal{P}_k(\Gamma)_{a,b}$ comprising the elements of degree 0; we view the isomorphism as identification. From (3.4) we get

$$\mathcal{P}_k(\Gamma)_{a,b} = \underset{d\in\mathbb{N}}{\oplus}\ \underset{t\in\Gamma_0}{\oplus} V_d^t$$

where $V_d^t = \pi(W_d^t) \in \operatorname{mod} k\Gamma$. If $\gamma \in \Gamma_1$ we write $\beta = \pi(\gamma)$ and $\beta^* = \pi(\gamma^*)$. If q is a path in $\bar{\Gamma}$ starting at t and ending at v, we call $\pi(q)$ a path in $\mathcal{P}_k(\Gamma)_{a,b}$ starting at t and ending at v. Then V_d^t is the span of all paths of degree d in $\mathcal{P}_k(\Gamma)_{a,b}$ starting at t. Since we identify $k\Gamma$ with $\pi(k\Gamma)$, we in particular identify e_t with $\pi(e_t),\ t \in \Gamma_0$; γ with $\beta = \pi(\gamma),\ \gamma \in \Gamma_1$; W_0^t with V_0^t; and we set $W_{-1}^t = V_{-1}^t = 0$.

We need the following statement. When appropriate, the map $(c) : X \to Y$ denotes the right multiplication by c.

Theorem 3.2. *Suppose $V_d^t \neq 0$ where $t \in \Gamma_0$, $d \in \mathbb{N}$.*

(a) *V_d^t is indecomposable in* $\operatorname{mod} k\Gamma$, *and $V_d^t \cong V_c^s$ in* $\operatorname{mod} k\Gamma$, *$s \in \Gamma_0$, $c \in \mathbb{N}$, if and only if $t = s$ and $d = c$.*

(b) *The map $g_d^t : \left(\underset{s(\gamma)=t}{\oplus} V_d^{e(\gamma)}\right) \oplus \left(\underset{e(\gamma)=t}{\oplus} V_{d-1}^{s(\gamma)}\right) \longrightarrow V_d^t$ induced by the right multiplications $(a(\gamma)\beta) : V_d^{e(\gamma)} \to V_d^t$, $s(\gamma) = t$, and $(-b(\gamma)\beta^*) : V_{d-1}^{s(\gamma)} \to V_d^t$, $e(\gamma) = t$, where $\gamma \in \Gamma_1$, is a minimal right almost split morphism in* $\operatorname{mod} k\Gamma$.

(c) *The map $f_d^t : V_d^t \longrightarrow \left(\underset{s(\gamma)=t}{\oplus} V_{d+1}^{e(\gamma)}\right) \oplus \left(\underset{e(\gamma)=t}{\oplus} V_d^{s(\gamma)}\right)$ induced by the right multiplications $(\beta^*) : V_d^t \to V_{d+1}^{e(\gamma)}$, $s(\gamma) = t$, and $(\beta) : V_d^t \to V_d^{s(\gamma)}$, $e(\gamma) = t$, where $\gamma \in \Gamma_1$, is a minimal left almost split morphism in* $\operatorname{mod} k\Gamma$.

(d) *If $V_{d+1}^t \neq 0$ then $0 \to V_d^t \xrightarrow{f_d^t} \left(\underset{s(\gamma)=t}{\oplus} V_{d+1}^{e(\gamma)}\right) \oplus \left(\underset{e(\gamma)=t}{\oplus} V_d^{s(\gamma)}\right) \xrightarrow{g_{d+1}^t} V_{d+1}^t \to 0$ is an almost split sequence in* $\operatorname{mod} k\Gamma$.

Proof. These are parts of [5, Theorem 1.1 and Corollary 1.3] combined with well-known properties of preprojective modules, see [1, VIII.1]. □

Applying parts (b) and (d) of Proposition 3.1 to Theorem 3.2, we obtain the following statement.

Corollary 3.3. (a) *In the setting of Theorem 3.2(b), for a general choice of k-subspaces $U_d^{e(\gamma)} \subset V_d^{e(\gamma)}$ and $U_{d-1}^{s(\gamma)} \subset V_{d-1}^{s(\gamma)}$, the restriction of g_d^t to $(\underset{s(\gamma)=t}{\oplus} U_d^{e(\gamma)}) \oplus (\underset{e(\gamma)=t}{\oplus} U_{d-1}^{s(\gamma)})$ has maximal rank.*

(b) *In the setting of Theorem 3.2(c), for a general choice of k-subspaces $U_{d+1}^{e(\gamma)} \subset V_{d+1}^{e(\gamma)}$ and $U_d^{s(\gamma)} \subset V_d^{s(\gamma)}$, denote by $\sigma_{d+1}^{e(\gamma)} : V_{d+1}^{e(\gamma)} \to V_{d+1}^{e(\gamma)}/U_{d+1}^{e(\gamma)}$ and $\sigma_d^{s(\gamma)} : V_d^{s(\gamma)} \to V_d^{s(\gamma)}/U_d^{s(\gamma)}$ the natural projections. Then the linear transformation $((\underset{s(\gamma)=t}{\oplus} \sigma_{d+1}^{e(\gamma)}) \oplus (\underset{e(\gamma)=t}{\oplus} \sigma_d^{s(\gamma)})) \circ f_d^t$ has maximal rank.*

Remark 3.2. As follows from Remark 3.1(b), if one leaves out any number of summands in the direct sum of part (b) of Theorem 3.2 and replaces the map g_d^t by its restriction to the sum of the remaining summands, Corollary 3.3(a) will still hold. Likewise, if one leaves out any number of summands in the direct sum of part (c) of Theorem 3.2 and replaces the map f_d^t by its composition with the projection onto the sum of the remaining summands, Corollary 3.3(b) will still hold.

The results of this section have dealt with left modules over a k-algebra Λ and with the right multiplication-by-arrow maps in the preprojective algebra. One may ask if analogous results are true for right Λ-modules and for the left mulitplication-by-arrow maps. We leave it to the reader to state the analog of Proposition 3.1, and note that [5, Theorem 1.1 and Corollary 1.3] address left multiplication by arrows in $\mathcal{P}_k(\Gamma)_{a,b}$ by replacing W_d^t and V_d^t with $W_{t,d}$, the span of all those paths in $\bar{\Gamma}$ of degree d that end at t, and $V_{t,d} = \pi(W_{t,d})$, respectively. Since $V_{t,d}$ is a finite dimensional right $k\Gamma$-module for all t and d, with the appropriate replacements the analogs of Theorem 3.2 and Corollary 3.3 hold. These remarks also apply to the considerations of Section 4.

4. Corollaries and Examples

In this section we strengthen Corollary 3.3 in a form that is analogous to the result of Hochster and Laksov [4]. To help the reader see the analogy we shall first state their result.

Set $R = k[x_1, x_2, ..., x_r]$, the commutative polynomial ring graded by degree, and denote by R_d its homogeneous piece of degree d. Let $N(r, d)$ be the dimension of R_d as a vector space over k. The following is then the result of Hochster and Laksov [4].

Theorem 4.1. *Given an integer $d \geq 2$, we determine an integer n by the inequalities*

$$(n-1)r < N(r, d+1) \leq nr$$

and let $s = N(r, d+1) - (n-1)r$. Then if $F_1, F_2, ..., F_n$ are n general forms in R_d we have that the $(n-1)r$ forms x_jF_i for $j = 1, ..., r$ and $i = 1, 2, ..., n-1$ together with the s forms x_jF_n for $j = 1, 2, ...s$ (in total $N(r, d+1)$ forms) are a k-vector space basis for R_{d+1}.

By "general forms" they mean that there exists a dense Zariski open subset of the affine space $(R_d)^n$ such that if the n-tuple $(F_1, F_2, ..., F_n)$ is chosen from that open set, then the conclusion follows.

We wish to apply Corollary 2.7 to the maps g_d^t of Theorem 3.2, which is possible according to Proposition 3.1(a). To make the result clearly analogous to the result of Hochster and Laksov we must set up our notation properly.

Fix a vertex $t \in \Gamma_0$ and a nonnegative integer d. Let $s_1, s_2, ..., s_m$ be the distinct vertices that have in Γ_1 arrows from them to t, and let $u_1, u_2, ...u_n$ be the distinct vertices with arrows in Γ_1 going from t to them. To match things up with the set up in Section 2, for $i = 1, 2, ..., m$

let $V_i = V_{d-1}^{s_i}$, let W_i be the k-linear span of the arrows $\beta^*_{i,j}$ in Γ^*_1 going from t to s_i, and set $w_{i,j} = -b(\beta_{i,j})\beta^*_{i,j}$. For $i = m+1, m+2, ..., m+n = l$ let $V_i = V_{d+1}^{u_{i-m}}$, let W_i be the k-linear span of the arrows $\beta_{i,j}$ in Γ_1 going from t to u_{i-m}, and set $w_{i,j} = a(\beta_{i,j})\beta_{i,j}$. For all i, we choose $\{w_{i,j}\}$ as a basis for W_i and put the $w_{i,j}$'s in a column vector x_i. Set $U = V^t_d$. Let M_i be the vector space of $\dim V_i \times \dim W_i$ matrices with elements in k. Let B' be the affine space $\prod_{i=1}^l V_i^{\dim V_i}$. An element b' of B' is an l-tuple $[b'_1, b'_2, ..., b'_l]$ where each b'_i is a $\dim V_i$-tuple of elements of V_i, written as a row vector. For $b' \in B'$ and $m(i) \in M_i$, using ordinary matrix multiplication and the multiplication and addition in the preprojective algebra, we see that $b'_i m(i) x_i$ is an element of $U = V^t_d$.

Corollary 4.2. *Let $d > 0$ and $V^t_d \neq 0$. Fix integers a_i satisfying $0 \leq a_i \leq (\dim V_i)(\dim W_i)$, $i = 1, ..., l$, and $\sum a_i = \dim U$. For each i choose a_i linearly independent elements of M_i and call them $m(i,j)$, $1 \leq j \leq a_i$. There exists a Zariski open dense subset E of B' such that if $b' = [b'_1, b'_2, ..., b'_l] \in E$, then the elements $b'_i m(i,j) x_i$, $1 \leq i \leq l$, $1 \leq j \leq a_i$, form a basis for $U = V^t_d$.*

Proof. We already have a chosen basis for each W_i. Suppose we also choose a basis for each V_i. The pairwise tensor products of these basis elements give a basis for $V_i \otimes W_i$, so we may identify $V_i \otimes W_i$ with M_i. We may also identify $\mathrm{End}\,(V_i)$ with $V_i^{\dim V_i}$ by matching $\varphi_i \in \mathrm{End}\,(V_i)$ with the image under φ_i of the chosen basis. Under these identifications the elements $T(\varphi_i(m(i,j)))$ appearing in Corollary 2.7 become identified with the elements $b'_i m(i,j) x_i$ appearing in Corollary 4.2. Thus Corollary 4.2 is a particular case of Corollary 2.7. □

With the proper choice of the $m(i,j)$ we can get a corollary that sounds even more like the result of Hochster and Laksov.

Corollary 4.3. *Let $d > 0$ and $V^t_d \neq 0$. For each i satisfying $1 \leq i \leq m$, let $\beta^*_{i,j}$, $1 \leq j \leq \dim W_i$, be the new arrows going from t to s_i. For each i satisfying $m+1 \leq i \leq l$, let $\beta_{i,j}$, $1 \leq j \leq \dim W_i$, be the old arrows going from t to u_{i-m}. Choose positive integers n_i, $1 \leq i \leq l$, satisfying $1 \leq n_i \leq \dim V_i$ and*

$$\textstyle\sum_{i=1}^l (n_i - 1) \dim W_i < \dim V^t_d \leq \sum_{i=1}^l n_i \dim W_i,$$

and set $c = \dim V^t_d - \sum_{i=1}^l (n_i - 1)\dim W_i$. Write c as a sum of nonnegative integers $c = c_1 + c_2 + ... + c_l$, $0 \leq c_i \leq \dim W_i$. For a general choice of $\sum_{i=1}^l n_i$ elements $F_{i,k}$, $1 \leq i \leq l$, $1 \leq k \leq n_i$, where $F_{i,k} \in V_{d-1}^{s_i}$ for $1 \leq i \leq m$ and $F_{i,k} \in V_d^{u_{i-m}}$ for $m+1 \leq i \leq l$, the following $\dim V^t_d$ elements form a basis for V^t_d:

$F_{i,k}\beta^*_{i,j}$ for $1 \leq i \leq m$, $1 \leq k \leq n_i - 1$, $1 \leq j \leq \dim W_i$;
$F_{i,n_i}\beta^*_{i,j}$ for $1 \leq i \leq m$, $1 \leq j \leq c_i$;
$F_{i,k}\beta_{i,j}$ for $m+1 \leq i \leq l$, $1 \leq k \leq n_i - 1$, $1 \leq j \leq \dim W_i$;
$F_{i,n_i}\beta_{i,j}$ for $m+1 \leq i \leq l$, $1 \leq j \leq c_i$.

Here in an inequality giving the range of possible j or k, if the number on the right is less than 1, we simply mean there are no such j or k.

Proof. Choose the $m(i,j)$'s as follows. Note that $a_i = (n_i - 1)\dim W_i + c_i$. For a fixed i, the a_i elements $m(i,j)$ will be the $(n_i - 1)\dim W_i$ distinct matrices having a 1 in one place among the $(n_i - 1)\dim W_i$ positions available in the first $(n_i - 1)$ rows of the $\dim V_i \times \dim W_i$ matrices involved and zeros elsewhere. The remaining c_i elements $m(i,j)$ have a 1 in one of the first c_i places in the n_i-th rows, and zeros elsewhere. □

Corollary 4.4. (a) *If $d > 0$ and $V^t_d \neq 0$ then $\dim V^t_d < \sum_{j=1}^n \left(\dim V_d^{u_j}\right)^2 + \sum_{i=1}^m \left(\dim V_{d-1}^{s_i}\right)^2$.*

If $d \geq 0$ and $V^t_{d+1} \neq 0$ then:

(b) $0 < \dim V^t_d < \sum_{j=1}^n \left(\dim V^{u_j}_{d+1}\right)^2 + \sum_{i=1}^m \left(\dim V^{s_i}_d\right)^2$.

(c) $0 < \left(\sum\limits_{s(\gamma)=t} \dim V^{e(\gamma)}_{d+1}\right) + \left(\sum\limits_{e(\gamma)=t} \dim V^{s(\gamma)}_d\right) < 2\left(\sum_{j=1}^n \left(\dim V^{u_j}_{d+1}\right)^2 + \sum_{i=1}^m \left(\dim V^{s_i}_d\right)^2\right) - 1$

where $\gamma \in \Gamma_1$.

Proof. This is a direct consequence of Theorem 3.2 and parts (c), (f), and (g) of Proposition 3.1. □

Example 4.1. This example shows that the $F_{i,j}$ of Corollary 4.3 must be chosen generically. In other words the right omnipresent maximal rank property does not imply the left omnipresent maximal rank property. Let the quiver Γ have two vertices labeled 1 and 2 and one arrow β going from 1 to 2. $\bar{\Gamma}$ then has in addition one new arrow β^* going from 2 to 1. For any choice of nonzero functions a and b the relations become $\beta\beta^* = \beta^*\beta = 0$. In Theorem 3.2 set $d = 1$ and $t = 2$. The map becomes $V^1_0 \to V^2_1$ where V^1_0 has basis $\{e_1, \beta\}$, and V^2_1 has basis $\{\beta^*\}$. The map is multiplication by β^* so e_1 goes to β^* and β goes to 0. Consider one dimensional subspaces of V^1_0. The one spanned by β maps to 0 and so does not surject onto V^2_1, all others do surject onto V^2_1.

Example 4.2. Here we show that if in Theorem 2.1 the hypothesis that T satisfies the right omnipresent maximal rank property is weakened to the right general maximal rank property, then the conclusion might not follow. In other words the right general maximal rank property does not imply the left general maximal rank property. Let V be a vector space of dimension 3 with basis $\{v_1, v_2, v_3\}$. Let W be a vector space of dimension 2 with basis $\{w_1, w_2\}$. Let U be the quotient of $V \otimes W$ by the subspace spanned by $\{v_1 \otimes w_1, v_2 \otimes w_1\}$. Finally let $T : V \otimes W \to U$ be the quotient map. The only one-dimensional subspace W' of W such that $V \otimes W'$ has nonzero intersection with the kernel of T is the span of w_1. Thus T satisfies the right general maximal rank property. Any subspace V' of V of dimension 2 must have nonzero intersection with the span of $\{v_1, v_2\}$. Thus $V' \otimes W$ must have nonzero intersection with the span of $\{v_1 \otimes w_1, v_2 \otimes w_1\}$. This means that the restriction of T to $V' \otimes W$ cannot have maximal rank.

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Department of Mathematics, Syracuse University, Syracuse, New York 13244-1150, United States of America

E-mail address: spdiaz@syr.edu

E-mail address: mkleiner@syr.edu