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A STUDY AND ANALYSIS OF THE STOCHASTIC MODEL AND ASSUMPTIONS USED IN PERT

BY

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1966

A STUDY AND ANALYSIS OF THE STOCHASTIC MODEL AND ASSUMPTIONS USED IN PERT

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser Date

Head, Mechanical Engineering Department Date

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CHAPTER I

INTRODUCTION

The Evolution of Management and PERT

In recent years we have seen significant advances in technology. The present period has often been called the Age of Massive Engineering by many authorities. This high rate of advancement has been initiated and supported by the space race as well as the demands of industry. New technologies provide the backbone for new and complex development projects which in turn end up as management's responsibility for administration.

The importance of time and efficiency in controlling a project is at the top of management's list. Many times the success of a project depends as much on the time of completion as it does on how well the finished product performs its function. Rapid technological change has decreased profit margins, increased competition, and produced a shorter lifespan for new products. Factors such as this put a big responsibility on management's role.

The management process (or method) is the result of evolution over many generations of trial and error. The present process varies somewhat with different organizations and individuals, but there is enough similarity that it can be depicted in a general form. This is best illustrated by Figure 1.

In the management process cycle, the establishing of objectives is the first and most important step. This provides the base of direction for all future effort as well as the yardstick against which accomplishments are evaluated. Given the objectives, a plan is developed which shows the interrelationships and sequence of the objectives. This plan forms a guide and shows the approach which will be taken to achieve the over-all objective. After the plan is laid out, the scheduling of resources and time is implemented. The scheduling function considers competition for materials and produces a completion date for the assigned objectives. This schedule then provides a basis for evaluation of progress. Progress is reported periodically and should enable detection of problems early enough for management to find a solution. As the project progresses changes may be desirable because of a change in the objectives, plan or schedule. After a careful analysis of the course of action to be taken, the change is incorporated by recycling through the management process. The cycle is continuously repeated until the objectives are accomplished.[1]

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[1] PERT Coordinating Group, PERT Guide For Management Use, U. S. Government Printing Office, Washington, D. C., June 1963, pp. 1-30.

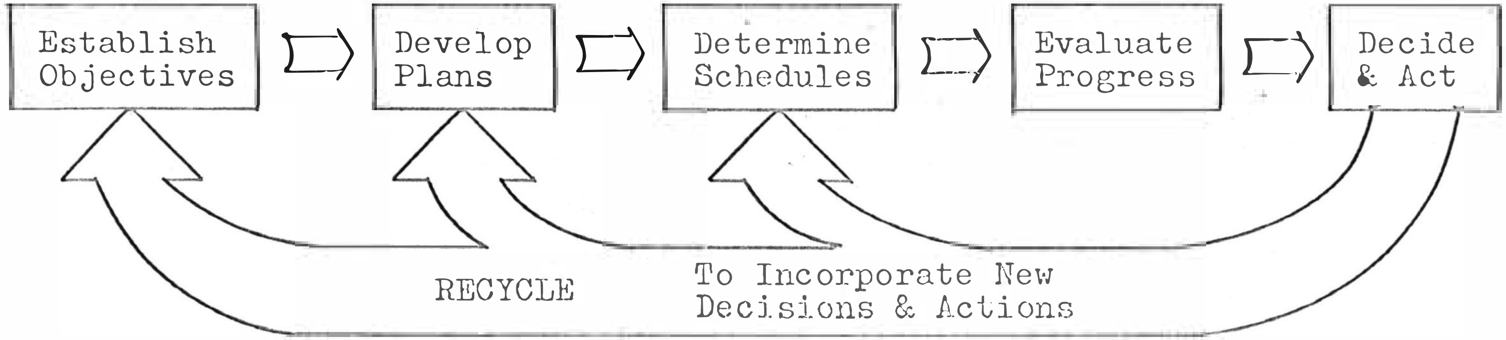

Figure 1. The Management Process Cycle*

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*Source of Data: PERT Coordinating Group, PERT Guide for Management Use, U. S. Government Printing Office, Washington, D. C., June 1963, p. 1.

The management process becomes very complex and difficult to execute for large projects. Often a project is divided among several companies, which magnifies the problem of coordinating the total program. This is pointed out by Klass.

> Today, a weapon system requires many parallel subsystem developments, each usually carried on by a different contractor, all of which must closely mesh at various stages in the program if it is to meet its timetable.[2]

The vast quantity of significant and/or finite events required in the development of these complex systems is breath-taking without saying anything for the managing of them. Of particular significance, is the fact that the planning activity must be done today to identify all the activities necessary to meet the end objective, perhaps five years in advance. The difficulty of this task is made evident by considering the nature of research and development projects.

The programming and evaluation of research and development projects are distinguished from other programs by three factors. First, research and development programs contain considerable intellectual and creative activity which is difficult to schedule or predict. Second, past

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[2] Klass, P. J., "PERT/PEP Management Tool Use Grows," Aviation Week, Vol. 73, November 28, 1960, p. 88.

experience upon which to base schedules is relatively unavailable for "one of a kind" or "one time only" projects. Third, frequent schedule changes are necessary because of the unpredictability of specific research results.[3]

Faced with these problems, management continually has sought more effective methods of fulfilling its responsibilities. The most recent means is through network analysis. This network is defined as a system, with subsystems, where the various parts are interrelated. A network may be illustrated by a flow chart or diagram. The network analysis technique is used to recognize and identify all the interconnecting links in a system or a series of subsystems. This technique can also be used as a tool of analysis to identify the performance of any subsystem for the purpose of design, coordination, and control. The most revolutionary and sophisticated variations of the network analysis techniques of management are the Program Evaluation and Review Technique and the Critical Path Method.[4]

The Program Evaluation and Review Technique commonly known as PERT was developed in 1958 by a team

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[3] PERT-Summary Report Phase 1, Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D. C., July 1958, p. 1.

[4] Johnson, R. A.; F. E. Kast; and J. E. Rosenzweig, The Theory and Management of Systems, McGraw-Hill Book Company Inc., New York, 1963.

representing the Special Projects Office of the United States Navy, in cooperation with representatives of the Lockheed Missile System Division and the management consulting firm of Booz, Allen, and Hamilton. The research team under the direction of Admiral Rayborn set out to develop an improved method for measuring and controlling development progress for the Polaris Fleet Ballistic Missile Program. This program was filled with complexities, not only because of its size, but also because of its pioneering nature which is characteristic of research and development projects.

The research team recognized the inability of conventional production management tools, such as Gantt charts, to adequately plan and schedule the complex interrelationships associated with research and development projects. Without underestimating the magnitude of the technical difficulties to overcome, these men correctly assumed that the resources, human talent, and production facilities already existed in more than sufficient quantity and quality. The one remaining ingredient necessary for success at a reasonable cost and a completion in time to make the weapon usable, was effective management. A way had to be found to communicate with, and direct, 250 prime contractors and over 9,000 subcontractors, so that all efforts would contribute to the advance of the total project.[5] The PERT approach to project management was the outcome of their efforts and the answer to the problem they faced.

This approach as outlined in the phase one summary report involves:[6]

1. The selection of specific, identifiable events which must occur along the way to successful conclusion of the project.
2. The sequencing of these events and establishing of interdependencies between events so that a project network is developed.
3. The estimate of time required to achieve these events together with a measurement of the uncertainties involved.
4. The design of an analysis or evaluation procedure to process and manipulate this data.
5. The establishing of information channels to bring actual achievement data and revision data to the evaluation point.
6. The application of electronic data processing equipment to the analysis procedure.

The success of PERT as a management tool was evident when the Navy accredited PERT as the instrument which brought the Polaris Fleet Ballistic Missile Program to

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[5] Stires, D. M., and M. M. Murphy, Modern Management Methods PERT and CPM, Materials Management Institute, Boston, 1962, p. 4.

[6] PERT-Summary Report Phase 1, op. cit., p. 1.

combat readiness nearly two years ahead of schedule. Adding to its popularity are developments and additions to the basic PERT which have extended and broadened its usefulness. These will be discussed in a later chapter.

Concurrently and independently of the development of PERT by the Special Projects Office, the Critical Path Method (CPM) of project management was developed by industry. The Integrated Engineering Control Group of E. I. duPont de Nemours and Company in collaboration with the Univac Applications Research Center developed the Critical Path Method of planning and scheduling. The desired purpose of their effort was to overcome the deficiencies of conventional methods of management. The first major test of their theoretical effort was the construction of a 10 million dollar chemical plant which proved the CPM successful.[7]

As it turned out, PERT and the CPM happened to be similar approaches to the same problem. The primary difference between the two methods lies in the statistical concepts incorporated in PERT which are not contained in the CPM. For this reason, PERT lends itself to research

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[7] Ashley, W. F., and M. T. Austin, "Case Studies in Network Planning, Scheduling, and Control of Research and Development Projects," Operations Research in Research and Development, ed. Burton V. Dean, New York, 1963, pp. 264-265.

and development projects where uncertainty may fatally undermine effective direction, while the applications of the CPM are very common in production and construction problems. The Critical Path Method assumes that the time required to perform any activity in a project is known or can be estimated with a high degree of accuracy. PERT associates a probability with its time estimates because frequently they may be only rough guesses.

After these methods were exposed, a rash of acronyms such as those that follow, began to appear in literature devoted to the same basic planning and scheduling area.

PEP--Program Evaluation Procedure

PRISM--Program Reliability Information System for Management

IMPACT--Integrated Management Planning and Control Technique

SCANS--Scheduling and Control by Automated Network Systems

CPS--Critical Path Scheduling

The differences between the approaches arise primarily as a consequence of the original job for which the method was developed. All of them share in the common procedures of the PERT and the CPM systems.

Reason and Scope of This Study

The past six years have provided a flood of publications devoted to PERT and its family of management techniques. This has exposed PERT to many potential users and as a result it is widely used today. The Department of Defense requires all of its major contractors to use PERT, and it must be used on all new projects. It has met with success in industry and is commonly used by literally thousands of large as well as small companies. This experience has shown PERT to be very valuable in management and worthy of additional analysis and study.

This study will be devoted primarily to PERT and will not include the CPM or other similar methods. The inclusion of these related methods would only result in unnecessary duplication because the understanding of the procedures and applications of one facilitates the understanding of the others. PERT was chosen over the others because it includes a statistical approach which seems to point towards future activity. The need for additional study along this line is recognized and suggested by Murray.

> It is apparent that a study should be made of the present PERT programs and techniques

> in order to update and improve the statistical aspects of the method.[8]

A specific statistical problem which is receiving relatively little attention is pointed out by Phillips.

> If further emphasis is to be placed on the statistics of the PERT Model in project management, research on methods of compensating for the merge point bias /a mathematical one of an optimistic bias in the expected project completion date/ should be pursued.[9]

This thesis will begin by providing a general review of PERT. The objectives and common uses of PERT will be discussed along with advantages and disadvantages of its application. The basic fundamentals of PERT will be developed, explained, and applied. This should lay a foundation for a more detailed and statistical approach to the various errors which may be introduced. The recognized problem areas stated above will be pursued in some detail. Possible improvements and additional uses will be suggested. A review of future activity and developments which appear to be receiving attention will be discussed.

The true value of the PERT technique of management becomes apparent only to those who understand its

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[8] Murray, J. E., "Consideration of PERT Assumptions," IEEE Transactions on Engineering Management, Vol. EM-10, September 1963, p. 94.

[9] Phillips, C. R., "Fifteen Key Features of Computer Programs for CPM and PERT," Journal of Industrial Engineering, Vol. 15, January-February 1964, p. 18.

procedures and have applied it to a project. This thesis is limited with regard to simulating actual experience in PERT usage but will, hopefully, provide the understanding of rules and procedures associated with its proper use. By following the analytical and statistical approach applied to problem areas and suggested improvements, the reader will strengthen his grasp of the involved fundamentals.

CHAPTER II

LITERATURE REVIEW

A vacuum existed in the methods of project management and PERT, along with its related techniques, moved in to fill it. PERT's effectiveness was supported by the success of its first major assignment, the Navy's Polaris project. As a result, network analysis has risen dramatically to prominence over the past eight years, and accompanying it has been an outpouring of material assigned to the subject. The interested reader can find various treatments given by reports, proceedings, bulletins, and journals. Recently, the literature has been expanded with a few complete books dealing with the generalized and instructive approach.

The government has played an important role in fostering the use of network techniques. This has been indicated by its publishing much of the early work devoted to PERT. The number of published items during a period from 1957 to 1962 is tabulated in Table 1 according to the responsible agency. Inspection of the table reveals that 96 per cent of the items appearing within the period were issued by military agencies. The Air Force (The National Aeronautics and Space Administration and The Rand Corporation are included in this category since they work on

Table 1. Number of Network Items Published by Government Agencies 1957-1962*

	1957	1958	1959	1960	1961	1962	No Date	Total
Military								
Air Force	0	0	0	7	25	46	12	90
Army	0	0	0	1	1	2	0	4
Navy	0	3	2	11	14	7	2	39
Total	0	3	2	19	40	55	14	133
Non-Military	0	0	0	0	3	3	0	6
Total	0	3	2	19	43	58	14	139

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*Source of data: Poletti, G., "The Diffusion of Network Techniques Throughout Government Publications," *IEEE Transactions on Engineering Management*, Vol. EM-11, March 1964, p. 44.

Air Force type projects) contributed the greatest number with 65 per cent of the total. The non-military government agencies contributing in 1961 and 1962 were the Atomic Energy Commission and the U. S. Congress.[1]

Other attempts have been made to measure the explosive growth of the literature over the first five years, including the contributions of industry and individuals. Bigelow identifies 63 major works (those dealing with the subject on a journalistic level were neglected) in the area of network analysis in his "Bibliography on Project Planning and Control of Network Analysis: 1959-1961."[2] By mid-1963 the U. S. Air Force PERT Orientation and Training Center, in its Bibliography: PERT and Other Management Systems and Techniques, cited 702 works in the field.[3] Martino lists over 600 references in the bibliography of his book Project Management and Control.[4] The

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[1] Poletti, G., "The Diffusion of Network Techniques Throughout Government Publications," IEEE Transactions on Engineering Management, Vol. EM-11, March 1964, p. 44.

[2] Bigelow, C. G., "Bibliography on Project Planning and Control by Network Analysis: 1959-1961," Operations Research, Vol. 10, September-October 1962, p. 728.

[3] Bibliography: PERT and Other Management Systems and Techniques, U. S. Air Force, Bolling Air Force Base, Washington, D. C., June 1963.

[4] Martino, R. L., Project Management and Control, Vol. 2, American Management Association, New York, 1964, pp. 163-184.

majority of these are proceedings, monographs, papers, transactions and reports which may not be readily available to an interested reader.

An inspection of these bibliographies reveals that there is an enormous amount of duplication and listing of techniques which are specialized for a particular need. Much of this redundancy is in a large part due to, and a reflection of, the rapid pace at which the field of network analysis has grown. Many organizations were, unknowingly, merely duplicating work already done by others. They were designing procedures applicable to their particular needs and could not assume that the users of the technique had any prior knowledge of similar works. Therefore, all the fundamentals and basic concepts were repeated in many publications with very little new work added.

In this mass of written work there is not a single, detailed publication which completely covers the subject area of PERT, even after ignoring all the related techniques. At least, there is none which does more than just make reference to the different factions. However, most of the works in this field can be accommodated by three groupings:

1. Publications which give an original account of the pioneering contributions toward the development of PERT and other techniques in the field.

2. Publications which focus on the descriptive, both generalized and detailed, treatment of the PERT model.
3. Publications which deal with the evaluation and application of PERT techniques.

The boundaries separating the above groups are not distinct, but overlap, so that a particular publication may fall into more than one classification. Generally, the major emphasis of a publication will direct itself to only one of the groups listed, and in this way the majority can be categorized.

The three divisions which follow will discuss each of the above groupings. However, only the third grouping, applications and their evaluation, will receive an elaboration on its contents, because the introduction of this thesis contained an account of how PERT came about (first grouping) and a forthcoming chapter on the stochastic model will cover a complete descriptive treatment of the PERT model (second grouping).

Report of Pioneering Work

The early literature devoted to PERT deals with the subject in its original form. In this, the accent is on the calculation of an expected completion date for a given project. Particular attention is not given to other variables such as cost, manpower requirements, equipment needs, etc. These are factors which more recent publications have been devoted to and possibly point the way to what can be expected in the future.

The terms PERT/Time or Basic PERT are employed when referring to PERT in its original context. The more complex approaches which have evolved are referred to by their respective labels, such as PERT/Cost and PERT/Reliability.

The first article to attract attention to the basic PERT technique is "Application of a Technique for Research and Development Program Evaluation," by Malcolm, Roseboom, Clark and Fazar.[5] This paper describes the PERT approach to project management. It is a fairly extensive description of the technique as applied to the Polaris project. The work was done while the authors were in the active employ of the Special Projects Office of the U. S. Navy and the management consulting firm of Booz, Allen, and Hamilton.

An equally comprehensive report summarizing the work and results of the first phase of the project, PERT,

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[5] Malcolm, D. G., and others, "Application of a Technique for Research and Development Program Evaluation," Operations Research, Vol. 7, September-October 1959, pp. 646-669.

was written by the Special Projects Office.[6] This report outlines the basic PERT procedures as they were developed and used on the Fleet Ballistic Missile Program. It presents an analysis of the activity time estimates and the mathematical computations related to the PERT model. This article and the preceding one rank among the most useful and frequently cited works in the PERT literature.

As was mentioned earlier, the CPM is a network technique closely allied with PERT. The CPM presupposes the existence of sufficient familiarity (or prior experience) with the processes involved to permit the use of single (or deterministic) estimates of time as opposed to the probabilistic, three time estimates employed in PERT. The CPM may also employ estimates of the cost of activities under both normal and crash operating conditions. These two points differentiate the CPM from PERT and explain why the CPM is less adaptable to complex research and development projects involving uncertainty. A paper discussing the CPM is authored by Kelley.[7] This work was

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[6] PERT-Summary Report Phase 1, Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D. C., July 1958.

[7] Kelley, J. E., Jr., "Critical Path Planning and Scheduling: Mathematical Basis," Operations Research, Vol. 9, May-June 1961, pp. 296-320.

done while the author was with the Univac Division of Remington Rand and working with the Du Pont Company on developing a procedure to cut time and cost off construction projects.

The above literature, although relatively complete, does not present the basic concepts of PERT in an introductory manner. The reader interested in securing a basic understanding of PERT should acquaint himself with literature devoted to explaining the fundamentals of PERT. This type of literature, although written chronologically subsequent to the work mentioned above, assumes no prior knowledge of PERT methodology and therefore lays the groundwork for a more detailed study. A citing of this type of literature can be found in the next grouping under the subheading of Generalized Descriptions.

Descriptive Treatments of the PERT Model

A sizable body of the material devoted to PERT will fall into this grouping. A quick inspection shows that the material falls into two divisions: that which concerns itself with a generalized description of the PERT methodology and that which presents an intensive and detailed description of PERT or a portion thereof. The format of two subgroups will be followed in order to emphasize the two levels of detail.

Generalized Descriptions. This area is undoubtedly not lacking in receiving its share of the PERT publications. Instead it is plagued by most of the duplication discussed earlier. Although each item may be useful in its own right, the coverage and approach taken by many is sufficiently similar that even the meticulous reader is unlikely to find that a complete study is justifiable. With this in mind, the author of this thesis will provide a severely screened review of literature in this generalized area. The actual description and explanation of the PERT model will be contained in the next chapter.

Recent editions of several management text books contain either a chapter or section on PERT. Two typical examples of this can be found in texts by Moore[8] and Starr.[9] Both give a brief introduction to PERT by illustrating how it is used. The presentation is very general and condensed but it is sufficient to serve as the introductory cause for which it was intended.

Numerous accounts can be found in periodicals. The parent publications range from weekly news reports to

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[8] Moore, F. G., Manufacturing Management, Richard D. Irwin, Inc., Fourth Edition, 1965, pp. 587-598.

[9] Starr, M. K., Production Management Systems and Synthesis, Prentice-Hall, Inc., 1964, pp. 115-128.

technical journals with the variance in article content being just as wide.

An article which presents a step-by-step primer of the basic elements of critical path (PERT) network planning is authored by VanKrugel.[10] He outlines the basis for drawing the network and identifying the activities in an ordered sequence of relationships along with presenting a method of network evaluation.

Boehm gives an account of initial efforts devoted to PERT and the CPM.[11] The essential steps in the method are demonstrated by an example which should give the reader at least a nodding acquaintance.

A general article dealing with the leading features, current progress, and basic requirements of the PERT model is presented by Miller.[12] In addition, the article points out difficulties yet to overcome in advancing PERT as a new management control tool.

There are many more published items which treat the description of PERT in the same generalized manner. None

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[10] VanKrugel, E., "Introduction to CPM," Architectural Record, Vol. 136, September 1964, pp. 337-344.

[11] Boehm, G. A. W., "Helping the Executive to Make up his Mind," Fortune, Vol. 65, April 1962, pp. 128-131.

[12] Miller, R. W., "How to Plan and Control with PERT," Harvard Business Review, Vol. 40, March-April 1962, pp. 93-104.

of these will satisfy the reader whose interest is more than at an introductory level. All of the above provide only a broad treatment, either omitting or covering at only a shallow level such topics as the probabilistic time element. Most rely to a considerable extent on graphic presentation to highlight basic features of the network technique. They do, however, serve to provide an individual with a brief indoctrination into the subject.

Detailed Descriptions. In their handbook, Stires and Murphy present a very detailed approach to the basic techniques involved in PERT.[13] The PERT method is applied to simulated planning and scheduling problems to illustrate its use. Therefore the reader will not only familiarize himself with the concepts and methodology of the technique but also gain a little proficiency in its use.

A two-volume book authored by Martino explains what PERT and the CPM are and how to use them profitably to solve problems.[14] The first volume is devoted to explaining the fundamentals and detail work of both PERT and the CPM. The second volume deals with the use of these basics

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[13] Stires, D. M., and M. M. Murphy, Modern Management Methods PERT and CPM, Materials Management Institute, Boston, 1962.

[14] Martino, R. L., Project Management and Control, Vols. 1 and 2, American Management Association, New York, 1964.

in project management and control, extending the subject into PERT/Cost and resource allocation and scheduling. The statistical aspects of PERT are not discussed because the activity times are assumed to be deterministic, i.e., contain no variance.

Moder and Phillips, in their text, provide a broad comprehensive approach to critical path technology and include an explanation of the efficient use of both PERT and the CPM.[15] The statistical approach to PERT is discussed and an introduction to network cost control is given.

In his book, Evarts presents in detail the PERT fundamentals at a level requiring only an elementary understanding of statistics.[16] Evarts extends his discussion into the use of computers to facilitate the calculations. This publication is a good place to start for an initial inquiry into the details of PERT.

The U. S. government PERT Orientation and Training Center has prepared a self-instructional course in the fundamentals of PERT.[17] It is presented in programmed-

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[15] Moder, J. J., and C. R. Phillips, Project Management with CPM and PERT, Reinhold Publishing Corporation, New York, 1964.

[16] Evarts, H. F., Introduction to PERT, Allyn and Bacon, Inc., Boston, 1964.

[17] PERT Fundamentals, Vols. 1, 2, and 3, PERT Orientation and Training Center, Washington 25, D. C., 1963.

instruction format to enable the student to proceed at his own pace without the aid of an instructor. The material requires approximately six hours for completion.

Another source which concerns itself with specific instruction on how to use and implement PERT is the Navy's PERT Data Processing Lesson Plan Handbook for Technicians.[18] It presents an outline for a plan of study along with the suggested time to be spent on each section. The total required time is approximately 17 hours. Some of the lessons are set up to be used with a computer in running problems.

There is also a certain amount of specialized work on the concepts and methodology of network analysis. Much of this is comprised of short papers which have attacked and defended PERT's distinguishing feature, the probabilistic time estimates. All agree that there are some statistical assumptions made which may introduce error, but the significance or amount of this error is difficult to determine exactly.

The discussion covers primarily two areas. One deals with the reliability of an event time as calculated by the application of the Central Limit Theorem. The

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[18] PERT Data Processing Lesson Plan Handbook for Technicians, revised November 1960, Special Projects Office, Dept. of the Navy, Washington, D. C.

other area of discussion centers around the assumption that the beta distribution can be employed to calculate a valid "expected time" for an activity based on estimates of the "optimistic," "pessimistic," and "most likely" times that an activity will take for completion.

Examples of the literature devoted to these problem areas include articles such as "Consideration of PERT Assumptions," by Murray;[19] "Attempts to Validate Certain PERT Statistics or 'Picking on PERT,'" by Grubbs;[20] "Activity Subdivision and PERT Probability Statements," by Healy;[21] "Expected Critical Path Lengths in PERT Networks," by Fulkerson;[22] and An Analytical Study of the PERT Assumptions, by MacCrimmon and Ryavec.[23]

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[19] Murray, J. E., "Consideration of PERT Assumptions," IEEE Transactions on Engineering Management, Vol. EM-10, September 1963, pp. 94-99.

[20] Grubbs, F. E., "Attempts to Validate Certain PERT Statistics or 'Picking on PERT,'" Operations Research, Vol. 10, November-December 1962, pp. 912-915.

[21] Healy, T. L., "Activity Subdivision and PERT Probability Statements," Operations Research, Vol. 9, May-June 1961, pp. 341-348.

[22] Fulkerson, D. R., "Expected Critical Path Lengths in PERT Networks," Operations Research, Vol. 10, September-October 1962, pp. 808-817.

[23] MacCrimmon, K. R., and C. A. Ryavec, An Analytical Study of the PERT Assumptions, Research Memorandum Rm-3408-PR, The Rand Corporation, Santa Monica, California, December 1962.

Some authors, such as Donaldson, have even presented an alternate method of calculating the mean and variance of a PERT activity time.[24] However, these alternate methods are also based on assumptions and it is difficult to see the significant advantage, if any, that would result in their adaptation.

Many of these articles have received published comments, both pro and con, which point out additional considerations and areas of conflict. Thus, the reader must assess the material and analyze the result with his particular interests in mind.

Applications and Their Evaluation

The need for a better management tool on the Polaris Project was pointed out in the introduction. This same requirement exists in many projects. Traditional techniques, although highly successful in many instances, rely largely on the skills of individual managers and do not assure that the many complex interrelationships existing in large projects are adequately considered. Furthermore, bar charts and Gantt charts, the traditional methods of planning and scheduling do not reflect a thorough analysis of

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[24] Donaldson, W. A., "The Estimation of the Mean and Variance of a 'PERT' Activity Time," Operations Research, Vol. 13, May-June 1965, pp. 382-385.

a project and do not contain the capability for evaluation of project status. Thus, not only may the details of initial planning be incomplete, but unexpected delays or changes may not be handled effectively.

The inability of conventional management techniques to direct and control complex programs has been recognized. A published study conducted by Peck and Scherer of the Harvard Business School indicates that overruns in time and cost are not uncommon.[25] An analysis of twelve major weapons programs shows that the average cost overrun is 3.2 times original estimate and the average time span is 1.36 times original estimate. Peck and Scherer also studied the cost and time variance factors for commercial programs that involve a significant amount of development effort. The results are summarized in Table 2. The average cost factor increase is found to be 1.7 and the time factor increase is 1.4.

PERT represents a significant step toward an improved management system which is capable of handling the variables of time, resources, and technical performance. This brings original estimates of project duration and cost closer to the actual case. Consequently, the

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[25] Peck, M. J., and F. M. Scherer, The Weapons Acquisition Process: An Economic Analysis, Division of Research, Graduate School of Business Administration, Harvard University, 1962, p. 22.

Table 2. Development Cost and Time Variance Factors in Five Advanced Commercial Developments#

Program Code	Development Cost Factor**	Development Time Factor***
Q	2.0	1.3
R	1.8	1.9
S	1.1	1.1
U	1.8*	1.3*
V	1.9*	1.3*
Average	1.7	1.4

*Estimate before project was completed and therefore may have increased.

**Actual cost ÷ original estimate.

***Actual time ÷ original estimate.

#Source of data: See footnote 25.

benefits of the PERT system have been stated in numerous publications, and its use is gaining in momentum. Correspondingly the application of PERT principles are many and varied, extending into all sorts of areas.

The Merrimack Valley Works of The Western Electric Company first used PERT in March of 1962. The first candidate for an application of PERT was the receiver portion of a data transmission system. This project was chosen because the "ship date" was less than two months away and a problem of parts shortage had set in. The work was kept under close surveillance by updating the project network every two weeks. The data receiving system was shipped on the scheduled date.[26]

More recently PERT has been applied to Western Electric's projects in earlier stages of completion, with similar success. An example of this was the microwave project while it was still in development. The network analysis directed attention to several design decisions which had been previously overlooked. The problem areas were taken care of and engineering effort, which would otherwise have been delayed, was allowed to progress. As the project moved forward the PERT network indicated that the expected "ship date" was a year away, while the scheduled "ship date" allowed only six months. The project was reviewed and various orders were expedited along with some plan changes. The actual "ship date" proved to be within one week of the PERT prediction, after updating, and the scheduled date was met.[27]

PERT was used as the prime management tool by the Air Force to closely control the installation of

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[26] Viehmann, N. J., "PERT As a Means for Control in a Multi-Project System," The Western Electric Engineer, October 1963, pp. 43-44.

[27] Ibid., p. 44.

engineering and "state of the art" changes on Atlas "E" and "F" series missiles. PERT was very effective in this application because it showed relationships between activities, acted as a scheduling device and status indicator, provided a communication tool by having identical PERT networks maintained with one in the field and one in the main office, and helped eliminate later changes with its preplanning capabilities.[28]

Another Air Force application was the C-141 subsonic aircraft program. Upon integration of the efforts of three contractors working on the propulsion system, it became apparent that the scheduled completion date would be delayed 36 weeks. Network analysis indicated that in the near future one contractor would have to wait for receipt of a production engine from a second contractor before he could proceed with the design of the engine covering. A mockup engine was supplied and constant monitoring of the interdependent effort reduced the delay from 36 to

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[28] Newnham, D. E., and others, "A PERT Control Center for Management of Major Ballistic Missile Modification Programs," Journal of Industrial Engineering, Vol. 16, July-August 1965, pp. 274-276.

8 weeks, representing a substantial reduction in extra expense.[29]

A new project scheduled for completion this year is the Stanford Linear Accelerator project at Palo Alto, California. Network planning and scheduling techniques are being used in all phases of the project which involves a building two miles long and a $114 million budget.[30]

The network technique was used by the Army Corps of Engineers to interface the assistance of the Navy, Air Force, Coast Guard, U. S. Public Health Service, Red Cross and the Mississippi National Guard in locating and removing a sunken barge. The barge which contained four large cylinders of deadly chlorine gas had sunk into the mud of the Mississippi River.[31]

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[29] Miller, R. W., Schedule, Cost, and Profit Control with PERT, McGraw-Hill, Inc., 1963, p. 167 (citing Cost Reduction through Better Management in the Federal Government, Executive Office of the President, Bureau of the Budget, April 1963).

[30] PERT Coordinating Group, PERT Guide For Management Use, U. S. Government Printing Office, Washington, D. C., June 1963, p. A.2.

[31] Ibid., p. A.4.

Automobile manufacturers now use PERT for constructing new plants as well as for the tooling up of new models.[32] In fact, industry in general is using it more since the Department of Defense requires all of its major contractors to use it. The major contractors, in turn, make similar demands on their various subcontractors, thereby increasing the use of PERT. An explanation for this requirement is pointed out in the following statement.

> In the determination of contract awards for today's development projects, the management capability of the contractor is just as important a consideration as his technical capability. This is particularly true when no advance in the technical state of the art is required and when competing contractors have essentially equal technical capability. Emphasis then shifts to the contractors ability to manage the project, in terms of achieving the objectives on time and within reasonable cost. By employing PERT, the contractor can demonstrate . . . [his] management capability.[33]

In a recent survey, it was revealed that approximately 67 per cent of all companies using PERT were using it on defense projects, 19 per cent were using it on private commercial work, and 15 per cent were using it on a

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[32] Moore, op. cit., p. 596.

[33] PERT Coordinating Group, PERT Can Benefit the Industrial Contractor, Technical Paper #2, Special Projects Office, Dept. of Navy, October 1, 1964, p. 3.

combination of defense and private commercial jobs.[34] This study indicated the effect of the defense department's requirement on industry. A summary of the commercial applications is presented in Table 3.

Table 3. Distribution of PERT Uses*

Area of Use	Per cent
Research and development	25
Construction programs	24
Programming of computers	12
Preparation of bids and proposals	12
Maintenance planning	12
Installation of computer systems	8
Distribution planning	5
Cost reduction programs	5
Miscellaneous	4

*Source of data: See footnote 34.

The table presents the percentage of the companies surveyed who used PERT in the various areas. The totals do not add up to 100 per cent because some of the companies use PERT techniques in several of the areas. We

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[34] Pocock, J. W., "PERT As an Analytical Aid for Program Planning--Its Payoffs and Problems," <u>Operations Research</u>, Vol. 10, November-December 1962, pp. 893-904.

should remember, however, that this breakdown will change as new developments in PERT's applicability add to its versatility and increase its use.

The present trend seems to indicate that the potential applications of PERT techniques are virtually limitless and its use will continue to grow with an increased awareness of areas in which this tool can make a contribution. Some potential uses being considered include marketing programs, merger or acquisition programs, introduction of new products, advertising programs and staffing of plants.[35] In these potential uses most are one-time ventures requiring the integration of the efforts of many departments. Historically, this has been the type of task environment where PERT functions best.

The benefits of PERT have been shared by many. Statements such as the inspection that used to take 56 hours now requires only 24.5 hours; PERT assisted greatly in achieving ahead of schedule completion records; a time reduction of 25 per cent; are not uncommon in studies of project improvements and savings attributable to PERT.[36]

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[35] _Ibid.,_ pp. 893-904.

[36] PERT Coordinating Group, _PERT Guide For Management Use, op. cit._, pp. A.1-A.4.

Several general benefits and advantages are common to most successful applications of PERT. From a study of various reports, the following points stand out as being the most frequently cited and indicative of the possible avail in the prudent use of PERT.

1. The provision of an effective method of communicating plans and their substance.
2. The simulation of alternate plans and schedules.
3. The prediction of time and uncertainties of performance.
4. The provision of thorough and frequent project status reports.
5. The early detection and warning of trouble spots and critical jobs.
6. The identification of areas where resources may be reallocated.
7. The identification of the objectives of the project and their interrelationships.
8. The provision of visible proof that a planning job has been done.
9. The easy adaptability to a variety of jobs.
10. The simplicity with which complex jobs can be displayed.

The literature relating to PERT is filled with success stories. These clearly indicate the enthusiasm many users have for the network technique. It is very difficult to find reports of failure or situations where PERT did not benefit the user. Perhaps, its sudden popularity can be compared to the plight which computers faced in the last decade. A circumstance where some uses were unjustified and menial, but because the computer was a symbol of progress and advancement, its use was shown to be beneficial. In view of this, some problems may arise which are due primarily to the unwarranted use of PERT. E. D. Dwyer, Chief of the Navy Management Office, issued some words of caution relating to the indiscreet use of PERT.

> We just can't go applying PERT to every management problem in Navy. We have to make intelligent application of these techniques where they can and should be used, and only there. Again we have to be cautious. Private groups are springing up all around trying to sell training sessions and applications of these tools. I am sure this can be very useful. But we have some smooth salesmen, and whether it may be electronic computers, PERT or patent medicine, we ought to be sure that the proposed remedy is really going to help us.[37]

Considering the haste with which PERT came into use, there are bound to be situations where it has met with misfortune. The problem may very well have been its misuse. But the fact remains that it did not work, at least not as effectively as it was expected. Some of the problems which have been encountered in varying degrees include:

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[37] Poletti, op. cit., p. 46 (citing "Navy Must Continue to Advance in Scientific Management," p. 9, by E. D. Dwyer, Navy Management Review, April 1962).

1. Difficulty in obtaining valid time estimates, upon which the quality of the technique depends.
2. Difficulty in constructing a network which accurately depicts the project, especially when there are alternate times for accomplishing an activity or where there is little interconnection between the different activities pursued.
3. Difficulty in determining the appropriate level of detail for project control.
4. Difficulty in receiving frequent and accurate updating.
5. Difficulty in providing the appropriate means of processing data for a variety of projects.

Another objection which may arise occasionally is the additional cost involved in PERTing a project. The cost of managing by PERT is frequently said to be twice the cost associated with conventional methods. More specifically the cost is dependent on several factors which include the degree of detail required, the degree of planning capability already available, the present effectiveness and homogeneity of the organization, and the amount and quality of PERT indoctrination given.[38] Those that have used PERT effectively are quick to point out that the resulting savings offset the increased cost.

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[38] Miller, R. W., "How to Plan and Control with PERT," op. cit., p. 98.

The subscribing factor which brought recognition to PERT was the credit attributed to it in the success of the Polaris development program. The enthusiastic reception of PERT by the military services is evident in its current requirement for PERT type planning and control as a prerequisite for the granting of new contracts. If no other justification for the adoption and written report of the system existed, this would exert sufficient pressure for the general adoption of the concept.

As an addition to knowledge, the Program Evaluation and Review Technique represents an important contribution in the area of management science. In fact, it is one of the most widely used methods of managing "one-time-through" programs. Unfortunately, the model has occasionally been misused, but this can be expected considering the rapid pace with which it has come into existence. Considering all aspects, it has been received well and based on present extensions it is very likely to continue receiving its share of published literature.

CHAPTER III

THE STOCHASTIC MODEL

This chapter deals with presenting the fundamentals of the PERT model. The approach taken exemplifies the significance of the basic terms and the supporting mathematics. Graphical illustrations are used during the presentation to help explain certain aspects. An example problem finalizes the discussion of the stochastic model.

The material is presented in a fashion which presupposes no past exposure to the PERT model and therefore may seem quite elementary at times.

Network Terminology

The network is the foundation of PERT. It represents a graphical description of a particular project and is used as a common communication tool for all interested parties. Simply stated, it represents the plan of attack.

A network consists of various activities with distinguishable milestones marking the beginning and end of the activity. An activity is a specific job which requires the expenditure of resources such as time, labor, and material.[1] It is represented in the network by an

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[1] See Appendix A for a definition of activity and all network terms.

arrow as exemplified in Figure 2. The arrow need not be of any particular length and the direction in which it points is also immaterial. The only rule accompanying this designation of an activity is that the tail of the arrow must represent the beginning of the job and the head must signify the completion.

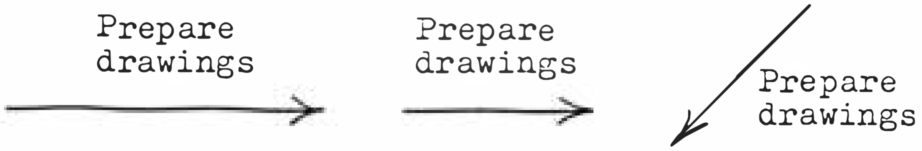

Figure 2. Equivalent Representation of the Activity "prepare drawings."

The activity is preceeded and succeeded by a particular point in time which is recognizable. In other words, we can determine or detect the time at which a particular activity, such as prepare drawings, begins and ends. It is much more difficult to identify the status of the activity once action has begun and the job is in the process of being accomplished. If we are concerned with an intermediate point of completion, say the halfway point, the proper activity representation is given by breaking the activity down into two more detailed activities. For example, the two activities could be: prepare detailed component drawings and prepare assembly drawings. By this additional breakdown we can detect the completion of the detailed drawings and the start of the preparation of the assembly drawings.

In a PERT network, the point in time between the completion of one activity and the beginning point of the next is called an event. The event is graphically represented by a circle.[2] It merely represents a junction point and consumes no time or resources.

A combination of many events and activities, constructed in a manner which shows their relationship to each other, is called a network. An entire job can be graphically shown in this form. Figure 3 shows the form of a small network.

In practice, the word "event" is not written in the circle; instead, a number is generally placed there. The events are numbered in a sequencial manner. The beginning event is labeled "number one" and each succeeding event contains a number larger than its predecessor. The numbers are used only for reference and serve no other purpose. The advantages in numbering the events in an

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[2] Some authors represent an event by a square or rectangle, however, the circle is by far the most common designation.

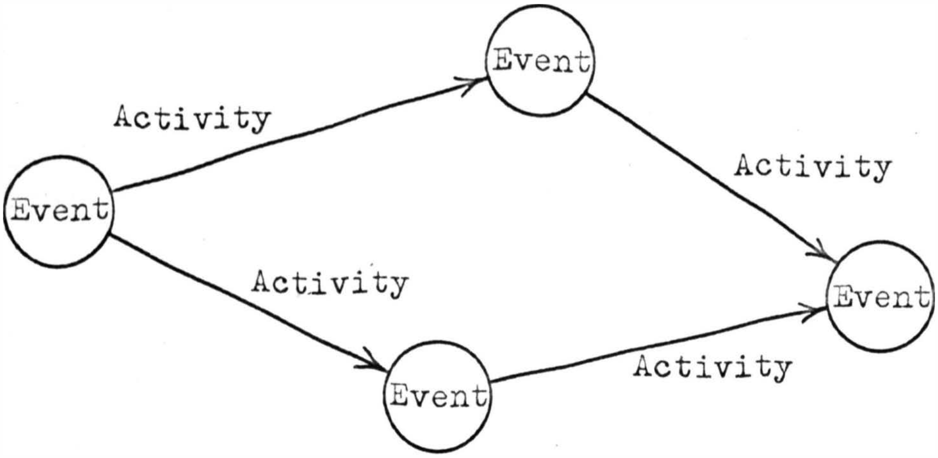


Figure 3. The General Form of a PERT Network.

increasing succession are significant. First, a particular activity can be referred to in an abbreviated form as activity (5,6) rather than "assemble test unit" or whatever the description may be. Secondly, the sequence is evident. For instance, it is easy to recognize that activity (4,5) precedes (5,6). Finally, the communication is simplified. It is easier to find an activity labeled job (35,36) than it is to pick out the descriptive title, especially in a large network.

The worded description of the individual activity may be deleted from the network to avoid cluttering. For reference purposes, a record of each activity could be kept in an attached legend which would associate the literary description with the numeric one obtained from the network. This procedure keeps the graphic representation within limits of visual clarity and leaves room for the insertion of activity time estimates. The estimates are discussed later in this chapter.

Network Development

The event-activity relationship is the fundamental component of a PERT network. This relationship is determined in the process of developing the network. Network development and construction begins with the identification of objectives. The over-all objectives of the program must be carefully determined. After these primary objectives are clearly resolved, the identifying of secondary or sub-objectives must take place. The secondary objectives are obtained from breaking down the program objectives into successively lower levels of detail. Complex development programs may require a considerable amount of detail or activity break down; therefore, a table or outline may be useful in helping to identify and organize the objectives before the network is started. The result is commonly known as the work breakdown structure and may appear similar to the one shown in Figure 4.

The level of detail required depends on the complexity of the project. The detail represented in Figure 4 is not nearly enough for a project of its scope and is

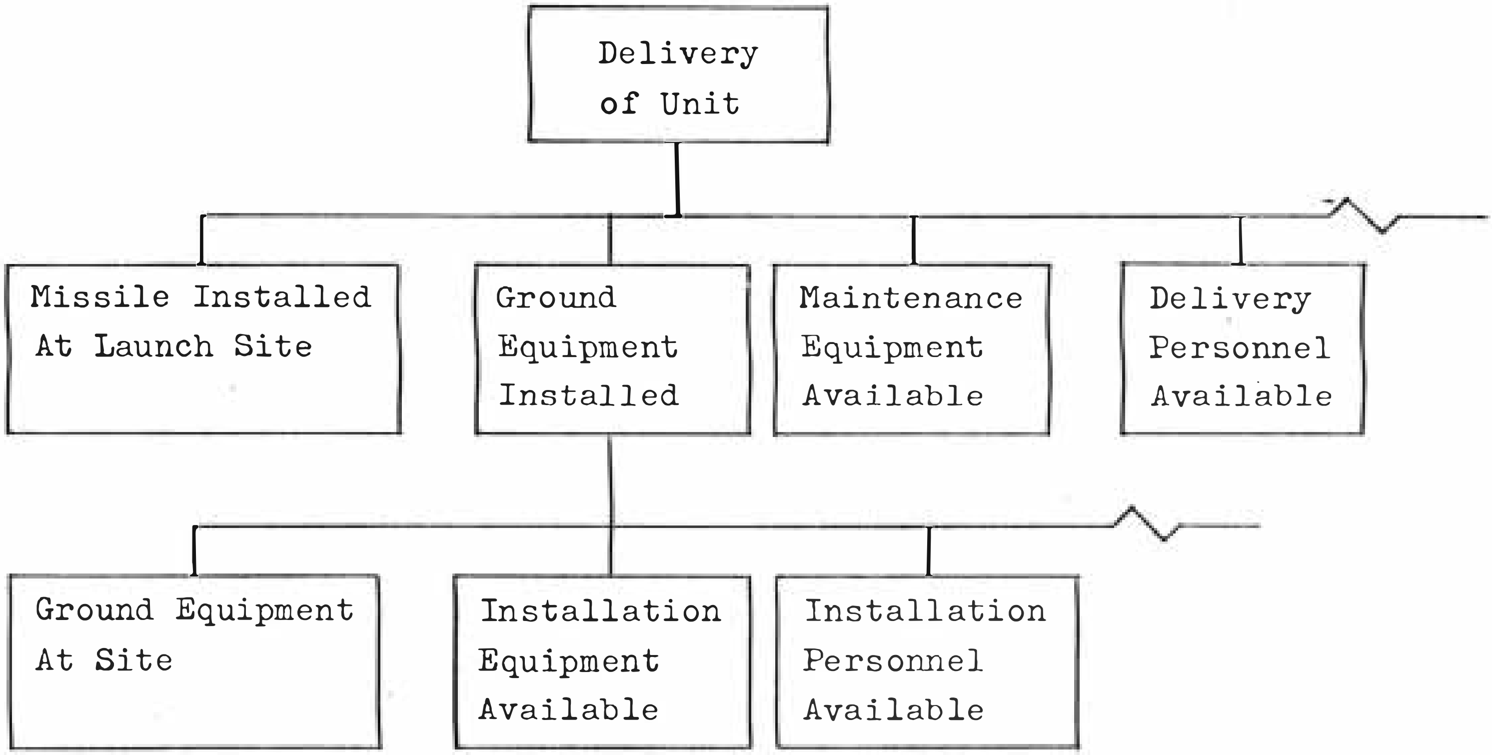


Figure 4. Work Break Down Structure of a Missile Program.

intended only to illustrate the graphical procedure involved. A large program may have to be broken down into several smaller projects to keep the size within reason. The particular nature of each item determines the number of levels to be subdivided. For any item, the subdivision of the work break down structure is continued until it reaches a level at which the lowest item in the subdivision becomes a manageable unit.

Once the objectives have been defined and organized, the PERT network is constructed. The defining and listing of activities is the first step. This is similar to defining the project objectives except that it takes place at a much more detailed level. Next, the sequence and organization of the activities are determined and put in a graphical form. This results in the network. The above procedure is only one of many general methods of network construction. Any logical, workable procedure may be employed.

To keep the network at a controllable level, it should be recognized that moderate size projects may be properly represented by one large network, whereas complex projects, such as the one introduced in Figure 4, require separate networks for each subdivision. These networks are then joined together at interface events in representing the total program.

The construction of the PERT network involves certain technicalities or rules which must be conformed to in order to obtain a network which can be processed and analyzed correctly. These regulations are explained in the next section on network logic.

Network Logic

PERT networks conventionally start at the left and proceed to the right. Therefore, in constructing the network, we should try to keep the head of the activity arrow to the right of its tail. The network begins with an initial event commonly called the network beginning event. There must be only one such event with all succeeding events stemming from it. We then proceed from this starting point.

Activities beginning at a particular event can either be in a series, series-parallel, or parallel arrangement. The parallel condition represents activities which occur concurrently. An illustration of this is the construction of a house where the interior is being decorated concurrently to the painting of the exterior. This concurrent activity should not be represented graphically as two activities between the same two events. A fundamental rule of arrow networking states that between two events there can be only one activity. To handle a situation where two activities begin and end at similar times, we use a dummy activity represented by a dashed arrow. Its proper use is demonstrated in Figure 5.

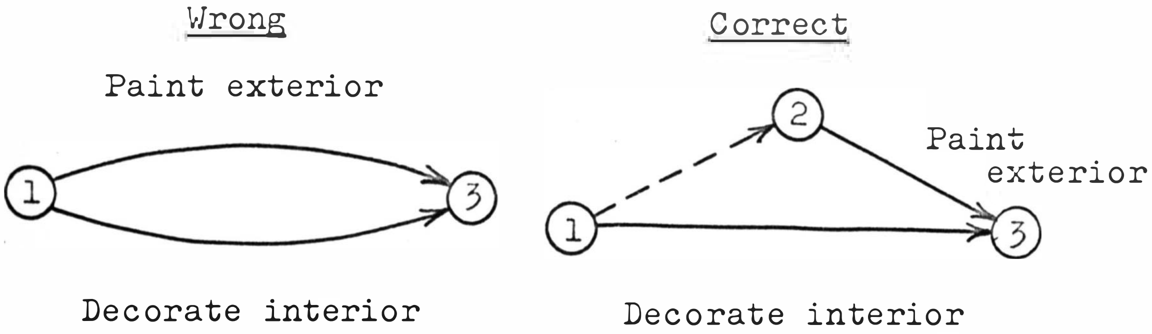


Figure 5. The Use of a Dummy Activity for Concurrent Jobs.

The dummy activity represents no time or resource expenditure but is used to keep the network logic in order. The reason for not allowing more than one activity between events should be apparent. Without the dummy activity, any reference to job (1,3) would not necessarily mean the interior decoration but could also represent the painting of the exterior. This situation could become very acute if there were several activities beginning and ending at the same events. Another purpose for this rule is realized when the procedure for using a computer to make the network calculations is considered.

Activities in series bring out the rule that no activity can begin until its predecessor or predecessors have occurred. This requires clear event and activity definition and, in addition, a depth of analysis to uncover and portray on the network the restraints of the program. Its effect is best realized during the consideration of the network portion illustrated in Figure 6.

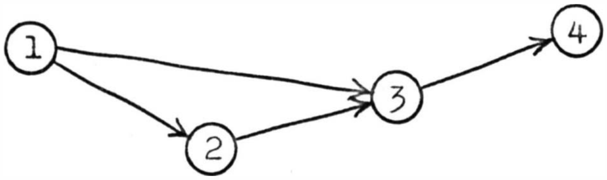
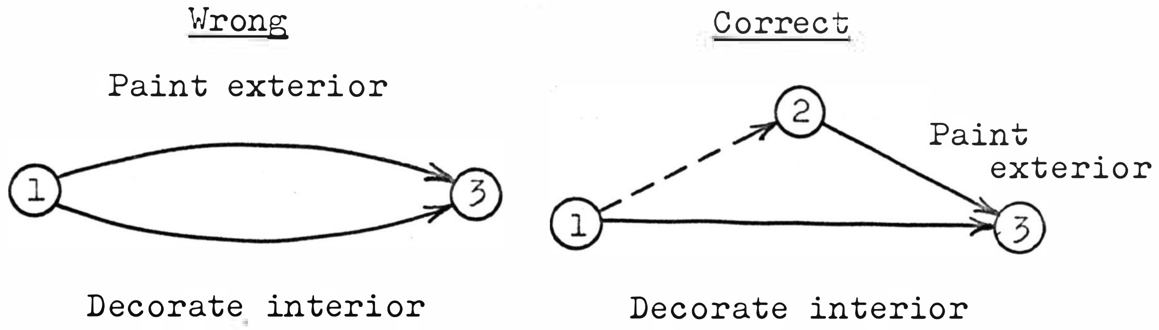


Figure 6. A Simplified Portion of a Network.

We can see from Figure 6 that the project will be completed when event 4 is reached, i.e., activity (3,4) has been accomplished. However, activity (3,4) cannot be started until both (1,3) and (2,3) have been completed. And, of course, activity (2,3) cannot start until job (1,2) is finished. Thus, we can see how the start of an activity depends on its predecessor being accomplished.

The various events and activities of a PERT network form paths that lead from the network beginning event to any specified event. A number of different paths may situation where two activities begin and end at similar times, we use a dummy activity represented by a dashed arrow. Its proper use is demonstrated in Figure 5.



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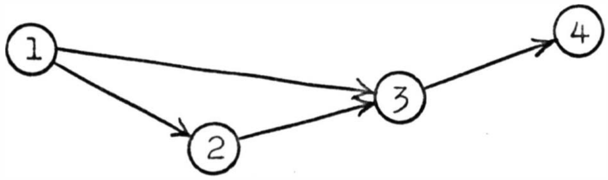

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The various events and activities of a PERT network form paths that lead from the network beginning event to any specified event. A number of different paths may be possible in reaching a desired event. In Figure 6 there are two paths which one may take in going from event 1 to event 4 (1-3-4 or 1-2-3-4). An event may occur along more than one path, but for any given path an event can occur only once. This means that no path of activities can return to one of its events or that no closed loops are permitted. A closed loop refers to the situation where an activity is both a predecessor and successor to another activity. This is both illogical and impossible.

To continue the discussion of network paths, we must consider the time required for the completion of an activity. This follows in the next section.

Activity Distribution

It was mentioned earlier that an activity requires a certain amount of time for its accomplishment. After a network has been constructed for a particular project, all the necessary activities are identified. If we can then get an estimate of the time each activity will require, the time estimate will provide us with more information to help control the project. Fortunately, a PERT network includes the capability of incorporating time estimates and uses them to supply additional information to the manager.

The estimating of the time required for the accomplishment of the various activities can be difficult in jobs for which little previous experience exists. In fact, some work depends primarily on intellectual activity which is difficult to measure. To obtain a fairly accurate indication of the time required for the execution of a specific job, the PERT methodology uses three time estimates.

The three time estimates are obtained from engineers or technicians responsible for the job in question. Time estimates are based on the assumption that existing resources will remain constant throughout the duration of the activity. The amount of manpower, machines, and material available should be known and should remain stable throughout the period of time spent on the job. An optimistic, most likely, and pessimistic time estimate is obtained for each activity.

The optimistic time is that amount of time which would be required if everything went exceptionally well and no difficulties were encountered. The chance of finishing the job in less time than this is approximately one in a hundred.

Another more realistic time estimate is the most likely time. It represents an estimate of the time required under normal conditions. If the activity were to be repeated under exactly the same conditions, this is the length of time the job would require most often.

A pessimistic estimate of the time required is perhaps self explanatory by now. It is the amount of time required if conditions were such that all things delayed the completion of the job. A fresh start may be necessary after initial failure, and this is considered in determining the time estimate. The effect of unexpected, unforeseen events such as strikes, floods, and storms are not to be weighed in the pessimistic estimate. These factors are classified as "acts of God." There should be about one chance in a hundred that the time required for completion will be more than estimated.

Based on these three estimates and various assumptions, a statistical description of the activity time can be obtained. This analysis will be beneficial in determining the expected time required for an activity or even the complete project. The statistical assumptions made in this thesis are consistent with those generally held in the original and historic PERT literature.

To accommodate the three time estimates, an assumption regarding the distribution of the activity time is made. A beta distribution is considered to be representative of the actual situation. The most likely time estimate is considered the mode of the beta distribution while the range is the interval between the optimistic and pessimistic times. A mathematical derivation of the activity expected completion time will follow. The derived expected time has special significance in that it represents the mean rather than the mode of the time distribution. There is about a 50-50 chance that the activity will require more or less than the calculated expected time.

The three time estimates for an activity performance time are indicated in Figure 7 which is shown below. The points a, m, and b correspond respectively to the optimistic, most likely, and pessimistic estimates.

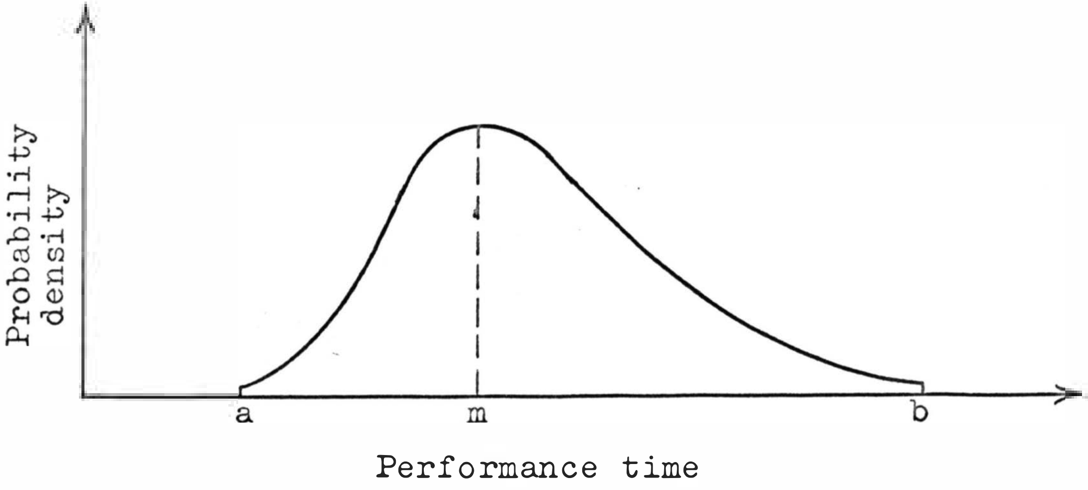

Figure 7. Distribution of Estimated Performance Time

The curve represents the frequency of occurrence of various times which it is assumed would occur if the activity were to be performed a large number of times. No assumption is made about the position of the point m relative to a and b. It may take any position between the two extremes, depending entirely on the estimator's judgment.

The beta density curve is defined as follows:

$$f(x) = Kx^p(1-x)^q \quad \text{when } 0 < x < 1 \qquad (3.1)$$

$$= 0 \quad \text{elsewhere}$$

Where K = a constant (to be determined).

The function represents a two parameter family of distributions where the parameters p and q must both be greater than minus one. A few examples of possible beta distributions with different parameter combinations are shown in Figure 8.

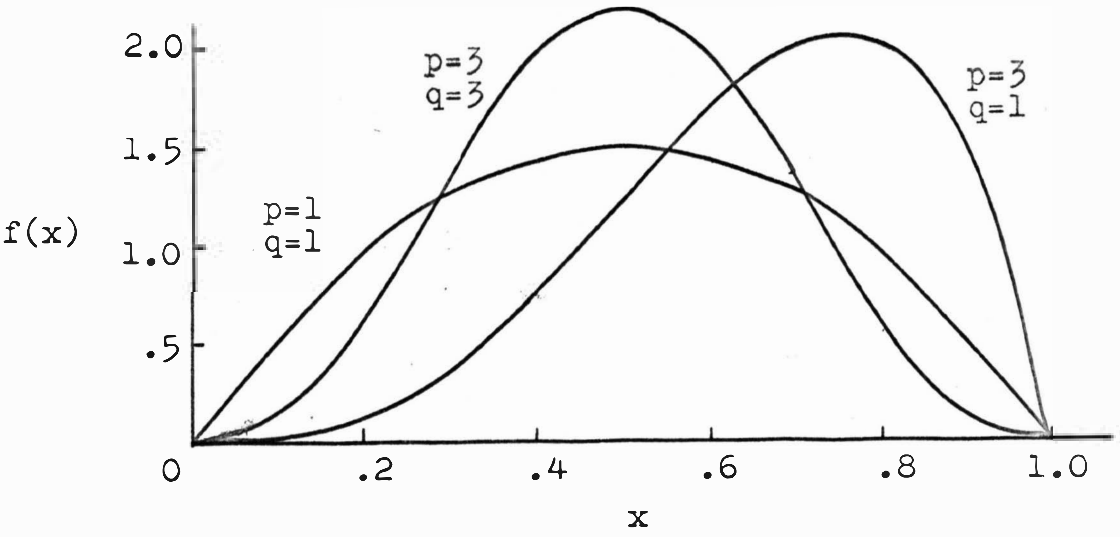

Figure 8. Examples of the Beta Distribution.

A value was used for the constant K that would make the relative size of the three curves approximately equal. This was done so that it is readily seen that the beta distribution can assume nearly any shape. With equal facility, it can portray cases where the most likely time is midway between the extremes or close to either end of the range (skewed).

We know that the probability of x falling between the boundaries of the density function is one, i.e., x must occur. This can be expressed as

$$Pr(0 < x < 1) = 1.$$

Also, the probability of x occurring between two arbitrary points a and b is given by the area under the curve (not by the value of the function). This can be represented by

$$Pr(a < x < b) = \int_a^b f(x)dx.$$

From this it follows that

$$Pr(0 < x < 1) = \int_0^1 f(x)dx = 1$$

and after substituting for f(x), obtained from equation (3.1), we get

$$Pr(0 < x < 1) = K\int_0^1 x^p(1-x)^q dx = 1. \qquad (3.2)$$

The integral in equation (3.2) is called the beta cumulative function and can be expressed in the following notation:

$$B(p+1,\ q+1) = \int_0^1 x^p(1-x)^q dx. \qquad (3.3)$$

To determine the value of the constant in the beta density, we use a function known as the gamma function, written as follows:

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^n dx. \qquad (3.4)$$

Utilizing the Laplace transform, it is possible to evaluate the gamma function and show that

$$\Gamma(n+1) = n! \quad \text{when } n \geq 0. \qquad (3.5)$$

Appendix B shows the detailed derivation of this equation. Also, a relationship between the beta and gamma functions is calculated in Appendix C. The relationship is

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\ . \qquad (3.6)$$

By substituting equation (3.3) into equation (3.2) and solving for K, we find

$$K = \frac{1}{B(p+1,\ q+1)}\ .$$

After replacing p and q by p+1 and q+1 respectively in equation (3.6) and substituting into the above equation, we come up with

$$K = \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} .$$

Evaluation of this based on equation (3.5) yields

$$K = \frac{(p+q+1)!}{p!\ q!} \qquad \text{when } p,q \geq 0.$$

Therefore, the beta density can be written as

$$f(x) = \frac{(p+q+1)!}{p!\ q!} x^p(1-x)^q \quad \text{where } p,q \geq 0. \qquad (3.7)$$

To represent the time distribution of an activity by the beta distribution, we must find the appropriate values of the parameters. We can obtain the values of p and q for any particular activity time by setting the most likely time estimate (signified by the symbol m) equal to the mode of the beta distribution, and by assuming that one-sixth of the range is the standard deviation. This assumption is reasonable for most unimodal frequency distributions. It follows from Tchebycheff's inequality that at least 89 per cent of *any* distribution lies within three standard deviations of the mean and for unimodal frequency distributions where the mean is the same as the mode, at least 95 per cent of the distribution will fall within the six standard deviation range.[3] Of course, if the distribution approaches that of the normal distribution, this

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[3] Grant, Eugene L., *Statistical Quality Control,* Third Edition, McGraw-Hill Book Company, 1964, pp. 59-60.

percentage is 99.7 per cent. Following the above course we then proceed with the determination of p and q.

If f(x) denotes the density function, the first moment about the origin which is also known as the expected value, mean or average, is defined as

$$\mu = E(x) = \int_{-\infty}^{\infty} xf(x)dx \qquad (3.8)$$

where x is continuous in the entire interval. The second moment about the origin by definition is

$$E(x)^2 = \int_{-\infty}^{\infty} x^2 f(x)dx. \qquad (3.9)$$

Equation (3.9) gives us the second moment about the origin. However, we are interested in knowing the second moment about the mean since this represents the variance and thereby the standard deviation.

Because of the analogy between bodies of unit mass and density functions, we can apply the parallel axis theorem of mechanics to our cause. Following the theorem the second moment about the mean, the moment of inertia, is equal to the second moment about any other point minus the square of the distance between the two points. From this we can postulate that

$$\sigma^2 = E(x)^2 - \mu^2 \qquad (3.10)$$

where σ^2 is the second moment about the mean. The value of the mean or μ is also the distance between the two points in consideration, the origin and the mean, because μ (one point) minus zero (the other point) equals μ.

Now using the above equations for the mean and the variance in addition to an equation which represents the mode of the beta distribution, we can find general equations in terms of p and q which represent the same things. We start with the mean, equation (3.8), but integrate only over the interval from zero to one since our density function is zero everywhere else. Substituting equation (3.7) into equation (3.8) results in

$$E(x) = \frac{(p+q+1)!}{p!\ q!} \int_0^1 x^{p+1}(1-x)^q dx.$$

After replacing the integral by its equivalent in beta notation we obtain

$$E(x) = \frac{(p+q+1)!}{p!\ q!} B(p+2,q+1).$$

After evaluation of the above representation of the mean by using the beta-gamma relationship, equation (3.6), then subsequent gamma function evaluation yields

$$E(x) = \frac{(p+q+1)!\ (p+1)!\ q!}{p!\ q!\ \ (p+q+2)!} .$$

After simplification the result is

$$E(x) = \frac{p+1}{p+q+2} . \qquad (3.11)$$

A similar procedure applied to equation (3.9) yields

$$E(x)^2 = \frac{(p+1)\ (p+2)}{(p+q+2)\ (p+q+3)} \ .$$

A combination of the above equation and the value obtained in equation (3.11) used with the relationship presented by equation (3.10) will result in a determination of the variance. This calculation is shown below:

$$\sigma^2 = E(x)^2 - \mu^2 = E(x)^2 - \left[E(x)\right]^2$$

$$= \frac{(p+1)\ (p+2)}{(p+q+2)\ (p+q+3)} - \frac{(p+1)^2}{(p+q+2)^2}$$

$$= \frac{(p+1)\ (q+1)}{(p+q+2)^2\ (p+q+3)} \ . \qquad (3.12)$$

To find an analytical representation of the mode for the beta distribution, we must first recognize that the density function will be a maximum at this point. The maximum is found by setting the first derivative of the function equal to zero and solving for the variable x. Following this procedure, the derivative of equation (3.7) is

$$f'(x) = \frac{(p+q+1)!}{p!\ q!}\left[-qx^p(1-x)^{q-1} + px^{p-1}(1-x)^q\right] dx.$$

Setting f'(x) equal to zero and solving for x yields

$$x = \frac{p}{p+q}$$

or using the notation for the mode, we have

$$m = \frac{p}{p+q} . \qquad (3.13)$$

Since we know the most likely time or the mode and have made the assumption that the standard deviation (σ) is one sixth of the range, we can determine the parameters p and q for any distribution from equations (3.12) and (3.13). These parameters when inserted in equation (3.11) enable us to calculate the expected completion date for an activity which was what we set out to do.

In order to further study the relationship between E(x) and the three time estimates, we will consider the range to be equal to that of the standardized beta distribution, 1. This does not restrict the generality of the analysis since any beta distribution can be reduced to a corresponding one containing a range of 1. Each point can be transferred from one system to the other by the linear relationship

$$x = \frac{t-a}{b-a} ; \qquad (3.14)$$

where:

x = a variable in the system comprising the standard form of the beta distribution.

t = a variable in a system which contains an arbitrary range.

a = the left-hand end of the arbitrary range.

b = the right-hand end of the arbitrary range.

Proceeding with the analysis, we set the expression for the variance or standard deviation squared equal to the square of the assumed standard deviation. Thus,

$$\sigma^2 = \frac{(p+1)\ (q+1)}{(p+q+2)^2\ (p+q+3)} = (\tfrac{1}{6})^2 = \frac{1}{36}\ .$$

Since we can obtain the numeric value for the mode (most likely time estimate) which in equation (3.13) is expressed in terms of p and q, we now have two equations in two unknowns, the unknowns being the parameters p and q. Solving for p, we proceed by eliminating q from one of the two equations. From equation (3.13) we obtain

$$q = \frac{p(1-m)}{m}\ .$$

After substituting for q in the equation representing the standard deviation squared, we obtain a cubic equation as follows:

$$\frac{\left[p+1\right]\ \left[\frac{p(1-m)}{m} + 1\right]}{\left[p + \frac{p(1-m)}{m} + 2\right]^2\ \left[p + \frac{p(1-m)}{m} + 3\right]} = \frac{1}{36}$$

$$36(p+1)\ (p-pm+m)m^2 = (p+2m)^2\ (p+3m)$$

$$(36m^2-36m^3)p^2+36m^2p+36m^3 = p^3+7mp^2+16m^2p+12m^3$$

$$p^3+(36m^3-36m^2+7m)p^2-20m^2p-24m^3 = 0.$$

To study the relation between E(x) and m, the parameter p in the above equation should be replaced by its equivalent in terms of E(x) and m. We proceed by first replacing the parameter q in equation (3.11) and then rearranging the terms in the resulting equation to find p. A sketch of the procedure follows:

$$E(x) = \frac{p+1}{p + \frac{p(1-m)}{m} + 2} = \frac{m(p+1)}{pm+p(1-m)+2m}$$

$$= \frac{m(p+1)}{p+2m}$$

Rearranging terms yields

$$p = \frac{m(1-2E(x))}{E(x)-m}$$

Next, we substitute this value for p in the previous cubic equation. A conglomerate of algebraic steps is involved in the substitution. A brief outline of the general method followed in obtaining an equation containing E(x) and m is presented below.

Substituting for p yields

$$\frac{m^3(1-2E(x))^3}{(E(x)-m)^3} + \left[36m^3-36m^2+7m\right]\left[\frac{m^2(1-2E(x))^2}{(E(x)-m)^2}\right]$$

$$-20m^2\left[\frac{m(1-2E(x))}{E(x)-m}\right] - 24m^3 = 0.$$

After expanding and dividing through by m^3, we get

$$\frac{1-6E(x)+12E(x)^2-8E(x)^3}{(E(x)-m)^3} + \frac{A}{(E(x)-m)^2}$$

$$- \frac{20(1-2E(x))}{E(x)-m} - 24 = 0$$

where $A = 144E(x)^2m^2-144E(x)^2m-144E(x)m^2+28E(x)^2+36m^2$
$+144E(x)m-28E(x)-36m+7.$

Multiplication by $(E(x)-m)^3$ to get rid of the denominator in all terms results in

$$1-6E(x)+12E(x)^2-8E(x)^3+A(E(x)-m)-20(1-2E(x))\ (E(x)-m)^2-24(E(x)-m)^3 = 0$$

Expansion and collection of terms according to powers of $E(x)$ yields

$$(144m^2-144m+36)E(x)^3+(-144m^3+108m-36)E(x)^2+(144m^3-\\-140m^2+32m+1)E(x)+(-12m^3+16m^2-7m+1) = 0.$$

After dividing the above equation by the coefficient of $E(x)^3$, factoring and simplification results in

$$E(x)^3-(m+1)E(x)^2+(m+\frac{1}{36})E(x)-(\frac{m}{12} - \frac{1}{36}) = 0. \quad (3.15)$$

Given the mode, equation (3.15) enables us to calculate $E(x)$. This expected activity time, symbolized by $E(x)$, is the quantity we have been seeking, but in calculating its numeric value a cubic equation must be solved. The necessity of going through this laborious calculation is questionable since the method is based on various assumptions.

To determine if a more simple connection exists between the mode and the mean, a study is made of the relationship between the two. Numerical calculation with the use of equation (3.15) gives the value of the mean for various modes. Appendix D contains the computer program and procedure which was developed to handle the ponderous calculations. A graphical presentation of the resulting data is contained in Figure 9. Here the mean of the distribution is plotted as a function of the mode to show any relationship which may exist between the two.

Inspection of Figure 9 reveals that the relation between these variables is approximately linear. To determine the straight line which approximates the curved, cubic relation, we may employ the method of least squares approximation. This procedure determines the linear relation in such a manner that the sum of the squared deviations between the $E(x)$ values for the true curve and the approximated line are as small as possible. A straight line located by the equation $E(x) = .183 + .635\ m$ is obtained by this method. However, PERT methodology uses the less accurate linear approximation of $E(x) = .167 + .667m$ enabling it to be written in the simple fractional form of

$$E(x) = \frac{(4m+1)}{6} .$$

This simplified relationship between $E(x)$ and m should contain sufficient accuracy and be acceptable after considering that prior computation is based on assumptions of distribution and variance, and requires among other operations the solution of a cubic equation.

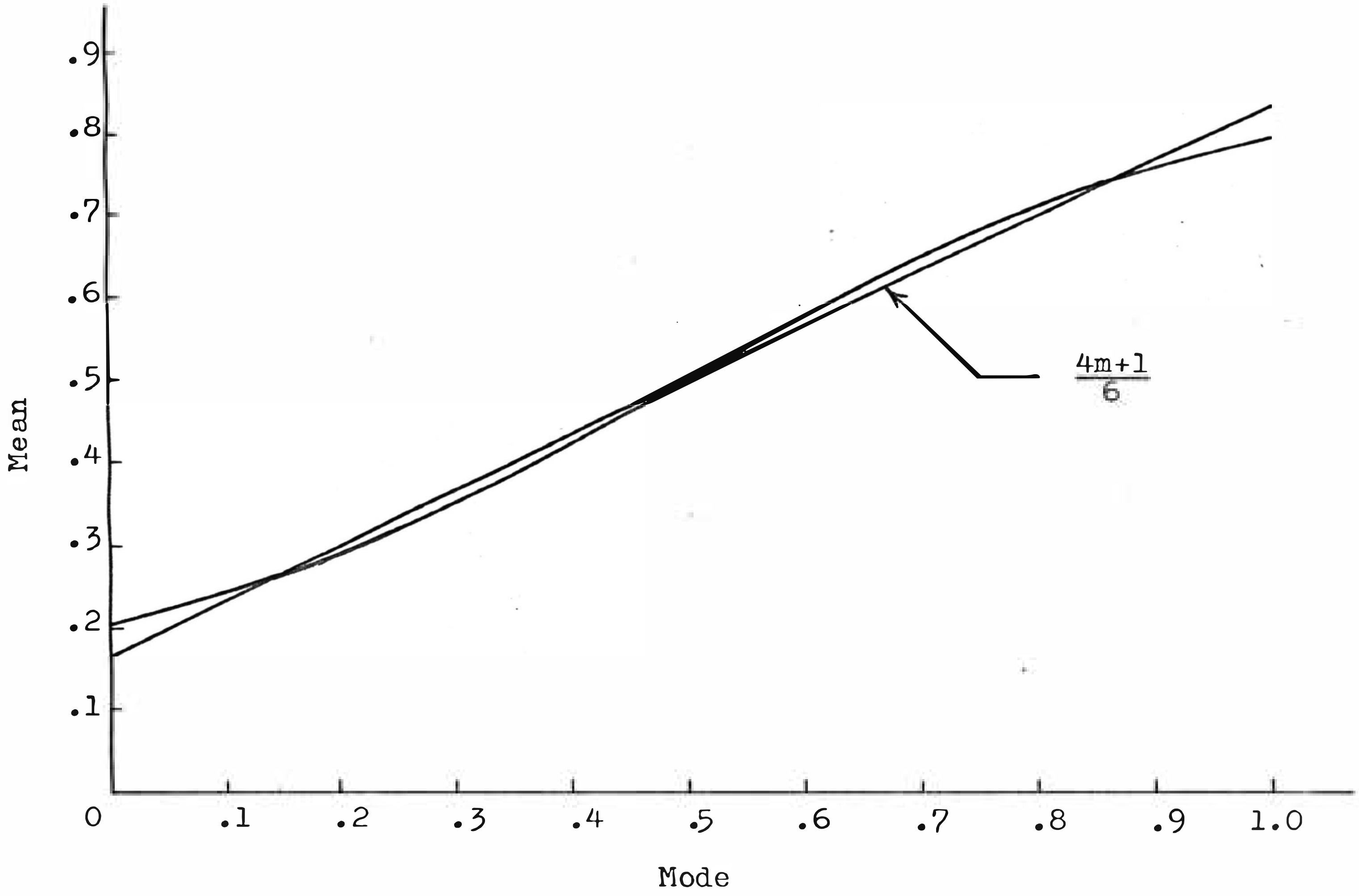

Figure 9. A Linear Approximation of a Plot of the Mode Vs. Mean For the Beta Distribution.

Using the linear relation $x = \frac{t-a}{b-a}$, the range of the beta distribution can be transformed from (0,1) to (a,b). Transforming the linear approximation for the mean from the standardized x system to the general t system entails the following procedure. Using the transformations

$$E(x) = \frac{E(t)-a}{b-a}$$

and

$$m_x = \frac{m_t-a}{b-a} ,$$

We get after substitution

$$\frac{E(t)-a}{b-a} = \frac{4(\frac{m_t-a}{b-a})+1}{6} .$$

Simplifying yields the following equation which represents the linear approximation of the expected elapsed time of an activity with a time range between a and b:

$$E(t) = \frac{a+4m+b}{6} . \tag{3.16}$$

The equation is conveniently referred to in PERT methodology by the symbol t_e.

Critical Path Analysis

Based on formula (3.16), the expected elapsed time for each activity in the network can be calculated from the time estimates. The value obtained represents the mean of the activity distribution. It is used when making network calculations to determine the entire project completion date. The earliest expected completion date for any particular event in the network is calculated by summing expected elapsed times between the network beginning event and the event in question. The determined date can either refer to a time period measured in hours, days, weeks, etc., or a calendar date. A conversion from one representation to the other can be made by preparing a special calendar containing only a numbering of the working days.

If more than one path exists in determining the earliest expected date, the largest value is the correct one for that particular event. An analogy will help in understanding this point.

Suppose that a reunion is planned among several individuals. They have planned to meet on a Friday evening after work at a location convenient for all. Because they live in different areas, the time of the event will be delayed until all arrive at the selected meeting place. Assume that the calculated travel time for the individuals ranges from one to four hours and that they all start at five o'clock. Based on this along with the requirement that the event cannot take place until all have arrived, the earliest expected time of the event is nine o'clock. Thus, the larger value sets the expected time.

An example of the determination of the earliest expected date of different events is presented in Figure 10. The expected time for each activity has been calculated using formula (3.16) and the three time estimates obtained from the group doing the actual work. This value is presented with its arrow representation of the activity. Time units are not included since they add nothing to the purpose of the example. T_E represents the symbol for the expected completion date of an event with the first event assigned a T_E value of 0, representing the starting time.

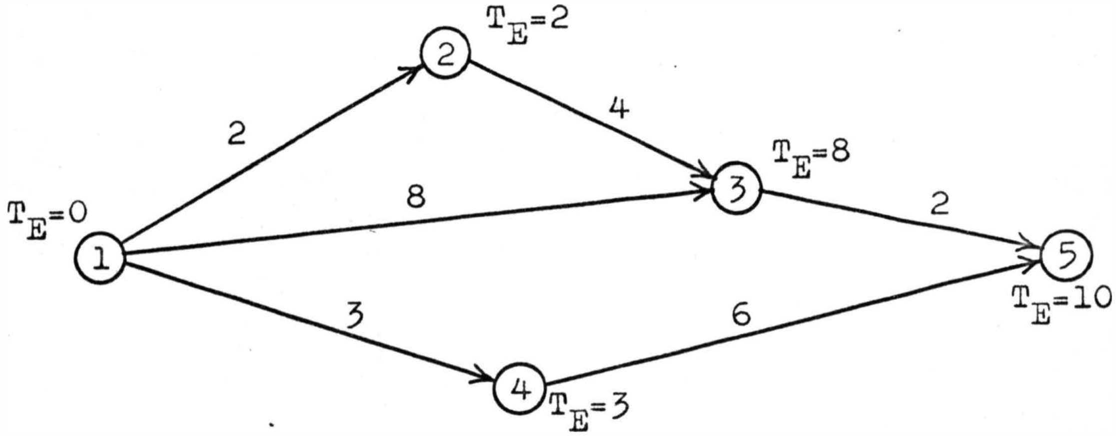

Figure 10. Event Expected Completion Dates.

We recall that T_E is calculated by summing the expected elapsed times of the appropriate activities leading to it. In referring to Figure 10, the expected completion dates of events 2 and 4 should be evident. But in determining T_E for event 3, we must consider two paths. Summation of the expected elapsed times along path 1-2-3 yields a T_E value of 6 for event 3. However, path 1-3 yields a value of 8 which is larger. Therefore, this is the correct value to assign as the expected completion time for event 3. Similarly in determining T_E for event 5, we consider three paths 1-2-3-5, 1-3-5, and 1-4-5. Summation of the expected times along these paths yields 8, 10, and 9 respectively. Again we choose the larger value, 10, as the proper T_E for event 5.

Analysis of Figure 10 shows that the activities along a path starting at the network beginning event and leading to the event in question need not be considered repeatedly in the calculations. In determining the T_E of an event, we add to the T_E value of the event immediately preceding the event in question, the expected time of the activity between these two events. All paths (preceding activities) must be considered and again the larger value chosen. For instance, the T_E of event 5 can be found by considering two values representing the immediate paths leading to it. One value is obtained by adding the t_e of activity (3,5) to the expected completion date of event 3. This gives a time requirement of 10. The other path yields a value of 9 which is found by adding to the T_E of event 4 the t_e of activity (4,5). Comparison of these two quantities results in a selection of 10 as the T_E of event 5. This condensed procedure is very time saving in larger networks and follows the procedure employed in computer programs set up to handle the calculations.

The largest T_E value, considering all events, on any network will always be associated with the network ending event. We recall that this T_E value was obtained by summing the expected elapsed times of activities on a particular path leading to the ending event. Understandably, this path represents the longest route based on the expected time required for the accomplishment of its activities. The name given to this path is descriptive of the fact that any delay along the path results in a longer required time for the completion of the project. Appropriately named, this longest path is called the critical path.

Inspection of the network in Figure 10 indicates that its critical path lies along 1-3-5. Therefore, a delay in activity (1,3) or (3,5) will affect the expected completion date of event 5 by a like amount. A severe time delay in any of the activities not on the critical path may extend the T_E of the network ending event past 10 units (a slippage of two units along path 1-2-3-5 or one unit along path 1-4-5) and thereby change the location of the critical path. This indicates the need for a frequent updating and review of the information represented by the PERT network to reflect any slippage in expected times. Replanning will be discussed in greater detail in a later section.

After determining the T_E for the network ending event, we can obtain a latest allowable date for each event in the network. Defined, the latest allowable date or time represents the longest amount of time that can elapse between the network beginning event and any given event without delaying the occurrence of the network ending event. T_L is the abbreviation used to signify this date.

The T_L value for each event is calculated by a procedure similar to that used to find T_E, except now we proceed backwards. Beginning with the network ending event, the T_L value for it is set equal to T_E. Then proceeding backward, the T_L value for each event is calculated by subtracting from the known T_L value of the succeeding event the time required for the activity between the two events under consideration. When two or more network paths yield different values for T_L, the smaller value is taken as the true T_L. The understanding of this procedure is facilitated by the consideration of an example such as the one presented in Figure 11.

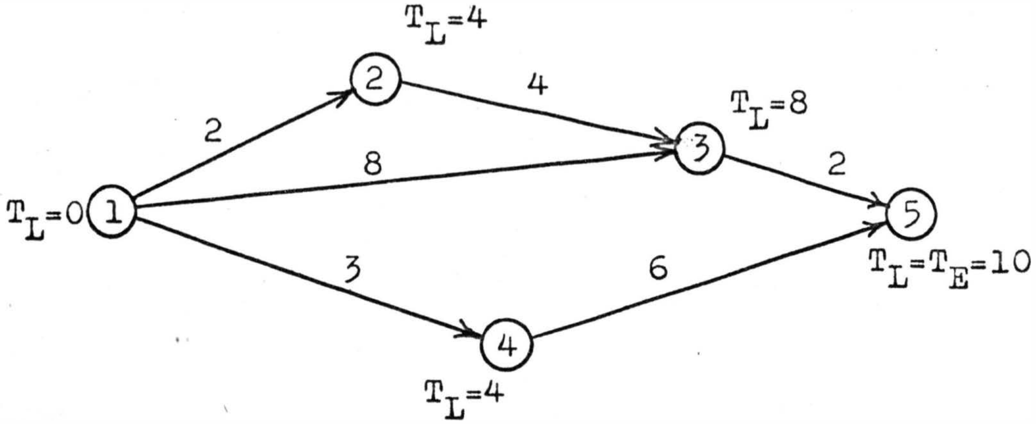

Figure 11. Event Latest Allowable Dates.

We begin by assigning the T_L value of event 5 equal to its expected date. Then working backwards, the T_L value of event 4 is calculated to be equal to 4, 10 minus 6. Similarly, the T_L of events 2 and 3 are determined. When calculating the value for event 1, we are confronted with three possible values. These values are 2, 0, and 1 and are obtained by considering the three activities stemming from event 1. The path containing activity (1,2) gives 2, activity (1,3) gives 0, and activity (1,4) gives 1 as the T_L value. However, the smallest value is the true latest allowable date and so 0 is the T_L of event 1.

The significance of using the smallest value becomes evident after assuming one of the other values as the T_L value.

Assuming T_L to be equal to 2 would in effect say that the event need not begin until two units of time have passed. This would mean that event 3 could be assured of attainment only after 10 units of time have elapsed, 2 plus 8. Event 5 would correspondingly be finished after a total time of 12 units have elapsed, resulting in the delaying of the project by two time units. Thus, the significance of using the lower value as the true T_L is illustrated, and if this true T_L value is exceeded for any reason, the date of occurrence of the network ending event will be delayed. Using 2 as the T_L value of event 1, would not have delayed the project if only path 1-2-3-5 had been considered. Therefore, some latitude may be present for the completion of some events. This brings us into a consideration of another term called slack.

For any event on a PERT network, the slack, symbolized by S, is equal to the latest allowable date minus the earliest expected date. The numeric value of the slack in time units represents the amount by which a particular event may be delayed without affecting the time of occurrence of the network ending event. Each event along the critical path has the same amount of slack which is less than the slack of all other events. Of course, this is expected to be the case when considering the nature of the critical path.

An illustration of slack values is found in Figure 12. This figure is a combination of the networks found in Figure 10 and 11 with their respective values shown together. Subtracting the two previously calculated times, we obtain the amount of slack available for each event.

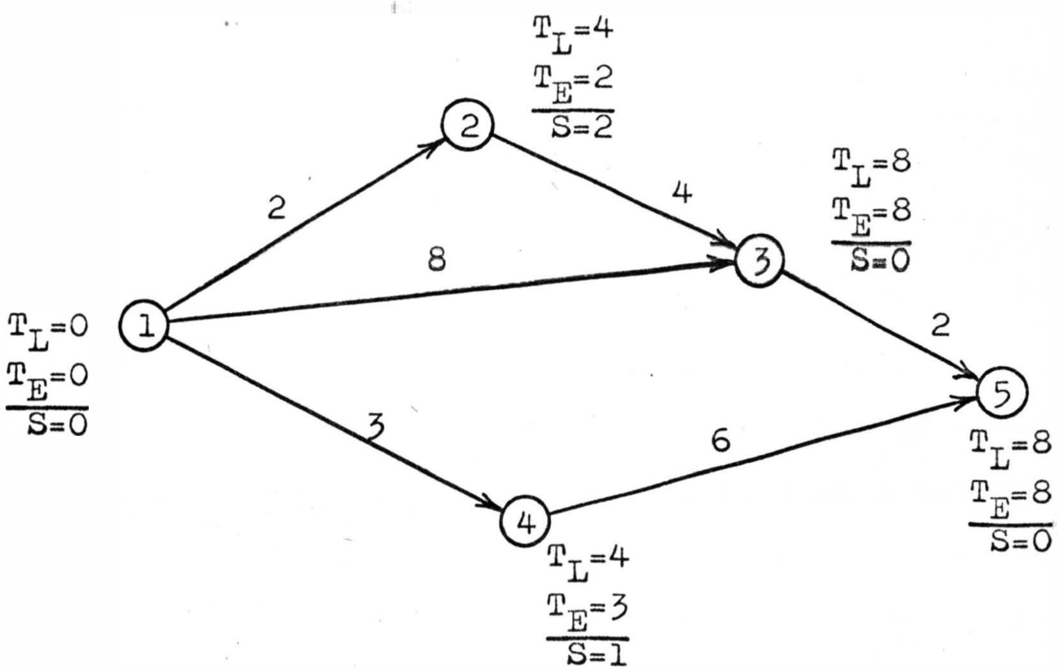

Figure 12. Event Expected Slack.

Recalling that the critical path for the network shown in Figure 12 is 1-3-5, we see that the slack along here is equal and has a value of 0. This condition should be expected since the critical path determines the required time of the project and therefore does not contain any flexibility in the time required for the accomplishment of its associated events. The remaining events do not influence the expected completion time of the project and as a result contain some slack or variation in their required completion time. For example, event 2 may be completed any time between two and four units after the start of the project and still not effect the outcome of the ending event.

The slack calculated for a particular event can be used in delaying either its preceding or succeeding activity, or it can be divided in any ratio between the two. In other words, the start of either activity (1,2) or (2,3) could be delayed by two units of time, or each could be delayed by one unit of time resulting in a total slippage of two units. Of course, if neither were delayed there would be an expected time interval of two units between the finish of activity (2,3) and the occurrence of of event 3, since event 3 is also dependent on the accomplishment of activity (1,3).

A situation may arise where a predetermined calendar date is set as the time by which an event is to occur. This date is called the scheduled completion date of the event and is symbolized by T_S. It is independent of the expected or required time to complete a project and may be either longer or shorter in duration than T_E. In making the network calculations discussed above, the latest allowable date (T_L) of the event is set equal to the scheduled completion date (T_S), since this is when the project must be done. Generally, a scheduled completion date is used only for the network ending event, but its use is not restricted to this event and can be used on any intermediate event.

When the T_L of the network ending event, obtained from the scheduled completion date, is greater than or equal to its T_E value, the slack values are positive or zero. This was the case in the example shown in Figure 12. However, if the T_L value is less than its T_E value, the slack values on the critical path are negative. A project containing negative slack values is expected to require a longer period of time for accomplishment than is allowed. This is a very undesirable situation and something must be done to alleviate the problem. Either the scheduled completion date must be extended, or the required time for the project must be reduced through replanning.

Replanning and Simulation

The procedure for reducing the expected completion date of a project is known as replanning. All individuals involved in the original construction of the PERT network should participate in the replanning process. In this way the revisions which are made will be consistent with and not result in a disruption of the project objectives.

A network may be replanned by one or more of the following methods:

1. Changing a chain of series connected activities into a series-parallel arrangement.
2. Changing resources applied to activities.
3. Changing the requirements or lowering the specifications of various activities.

The procedures involved and the resulting effects of replanning by using the above methods are illustrated by applying them to the overly simplified network in Figure 13. Let us assume that the number below the

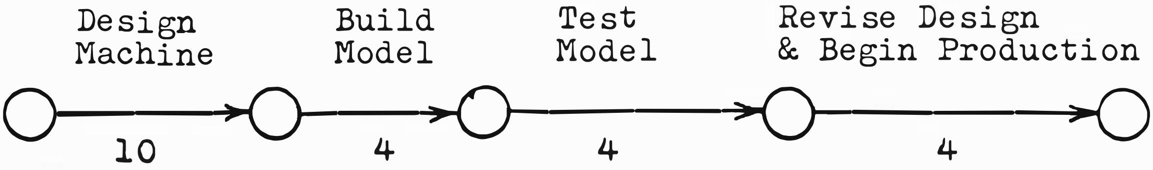

Figure 13. A Simplified PERT Network.

activity description represents the expected time (t_e), in weeks, for each activity and has been obtained from proper calculations involving valid time estimates. The network is simplified to the point that it contains no functional value for project control, but its purpose here is to serve only as a replanning example.

Inspection of Figure 13 shows that the expected completion date of the project is 22 weeks. By applying the first replanning method, we can reduce the expected completion date of the project. To do this, we shall reduce part of the series-connected activities to a series-parallel arrangement. It is very likely that the building of the model could begin before some final parts of the machine have been designed. This possible replanning is then incorporated, and the resulting network is shown in Figure 14.

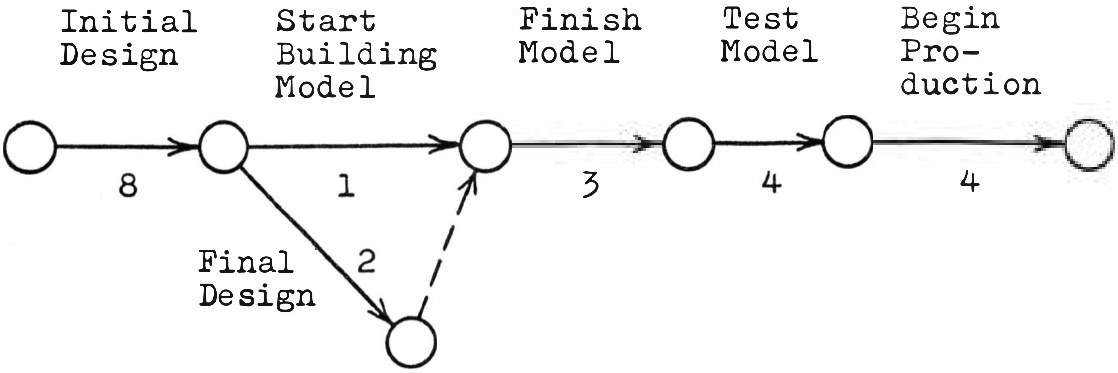

Figure 14. Network Replanning by Series-Parallel Arrangement.

Notice the use of a dummy activity in Figure 14 to maintain proper network logic. This prevents the occurrence of two activities between the same two events. The project T_E has now been reduced to 21 weeks by replanning the initial network. We can still further reduce the time requirement by using other replanning methods. If we increase the manpower at the initial design stage, the project duration will be decreased. Assuming this addition of resources, the revised network may appear similar to the one presented in Figure 15.

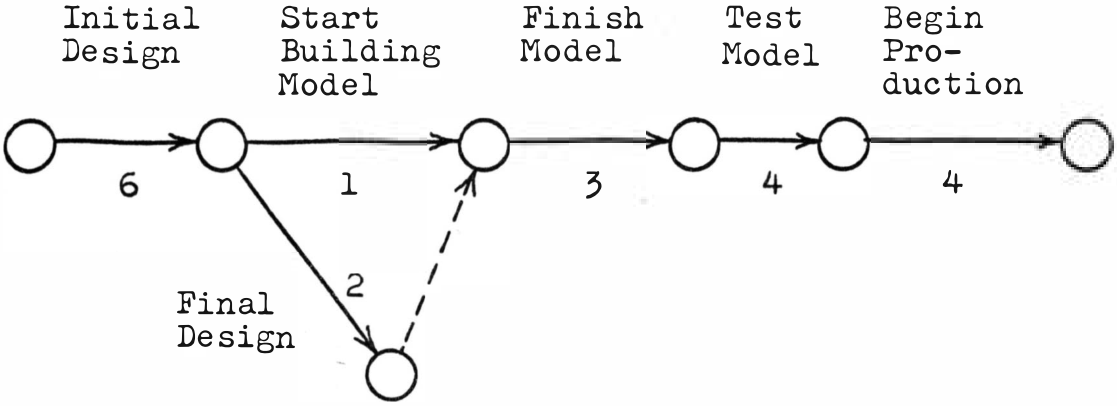

Figure 15. Network Replanning by Addition of Resources.

Further replanning may result in lowering the specifications of any activities. For illustration purposes let us assume that the testing of the model need not be as complete and therefore not require as much test time as was originally determined. Lowering the testing requirement by one week results in the revised network presented in Figure 16.

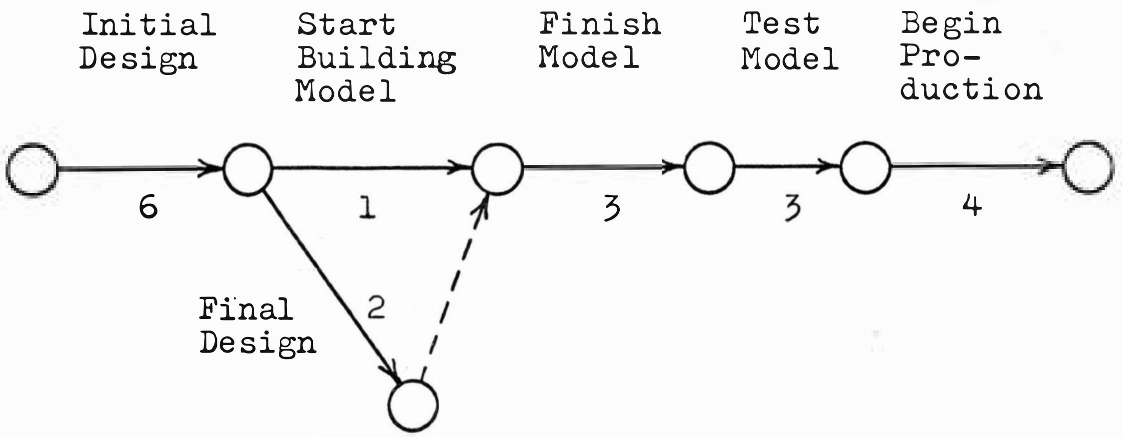

Figure 16. A Replanned Network.

This network contains all of the previous replanning changes. The resulting network has an expected completion time of 18 weeks and represents a reduction of 4 weeks over the original network in Figure 13. Thus, we can see the time reduction possible through the effective use of replanning methods. Again, the example discussed here is simplified to the point to which it has no merit as a project control technique, but serves only to illustrate the three replanning methods introduced earlier.

After replanning, the time required by the original critical path may be reduced to such an extent that some other path becomes critical. Therefore, the true effect of any replanning work can only be realized after the expected completion date for the entire project is recalculated. Determination of the expected impact from project replanning which is done prior to the time of actual implementation is called simulation. Simulation can be an effective technique for reflecting the outcome of tentative replanning proposals and alternate courses of action. This makes it a very vital aid in formulating a decision. The use of a computer is necessary for simulation work in large complex projects. A computer's speed handles the necessary calculations associated with each tentative plan with efficiency and does not hinder the use of simulation because of the additional time needed.

Probability Concepts

All time estimates involve varying degrees of uncertainty. Consequently, all the calculated times (t_e, T_E, etc.) must also be inexact and will contain the same uncertainty inherent in the original time estimates. A statistical measure of these uncertainties and the resulting probabilities is of great interest and value, for it tells the odds of completing a group of activities or an entire network within the calculated expected time. Various assumptions are made in determining probabilities. Their effect will be considered in somewhat more detail in a later chapter.

An assumption made in the derivation of the expected time for an activity is that the standard deviation (σ) of an activity is one-sixth of the range. Thus we can obtain the standard deviation for each activity in the network since the optimistic and pessimistic time estimates determine the range (see Figure 7). After finding the individual standard deviations, we may be interested in determining the standard deviation for the entire project or for a particular event. This calculation is based on the Central Limit Theorem.

The Central Limit Theorem states that the sum of a large number of random variables (in our case activity distributions) will be approximately normally distributed regardless of the distribution of the individual random variables. The mean of this resulting normal distribution will be equal to the sum of the means of the specific activities i.e., T_E, and the variance (σ^2) will be equal to the sum of the individual variances. The means and variance which are summed must be those of activities which lie on the critical path leading to the event in consideration.

Thus, for any event on the network, there is an event standard deviation (σT_E) equal to the square root of the sum of the squares of the activity standard deviations on the most critical path leading to that event. Also the distribution of the event expected time will be approximately normal with a mean equal to the sum of the t_e's along the same critical path. There should be at least four activities in the path for the Central Limit Theorem to apply. If the activity distributions are severely skewed, the minimum number of activities considered should be 10. The above statements in equation form appear as follows:

$$T_E = t_{e_1} + t_{e_2} + t_{e_3} + \cdot \cdot \cdot \cdot \cdot t_{e_n} \quad \text{when } n \geq 4$$

and,

$$\sigma_{T_E} = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \cdot \cdot \cdot \cdot \cdot \sigma_n^2)^{1/2}$$

where n is the number of activities in the critical path leading to the event in question.

Since the normal distribution has been extensively tabulated, we can obtain probability statements regarding the random variable T_E by using these tables. A condensed table of the normal distribution function is given in Appendix E.

Adopting the standardized normal random variable (Z) to the context of PERT, we obtain the equation:

$$Z = \frac{T_L - T_E}{\sigma_{T_E}} .$$

Using this formula we can obtain the probability that an event will occur by its latest allowable date or the scheduled completion date. T_L is set equal to T_S so the two can be used interchangeably. The calculated value of Z is entered in the table in Appendix E and the resulting probability of accomplishing an event by a particular time is read out. It should be noted that Appendix E does not contain negative values of Z. Such values are obtained by recognizing that the standardized normal distribution is symmetric about the value 0. Thus the probability value for a negative Z is equal to 1 minus the value read for a positive Z of the same magnitude.

In Figure 17, the shaded area represents the probability of completing a project before the time T_L. This probability is obtained by subtracting from 1 the quantity read from the tables using the calculated Z. Had the value of Z been positive, the probability (shaded area) could have been obtained directly without subtracting.

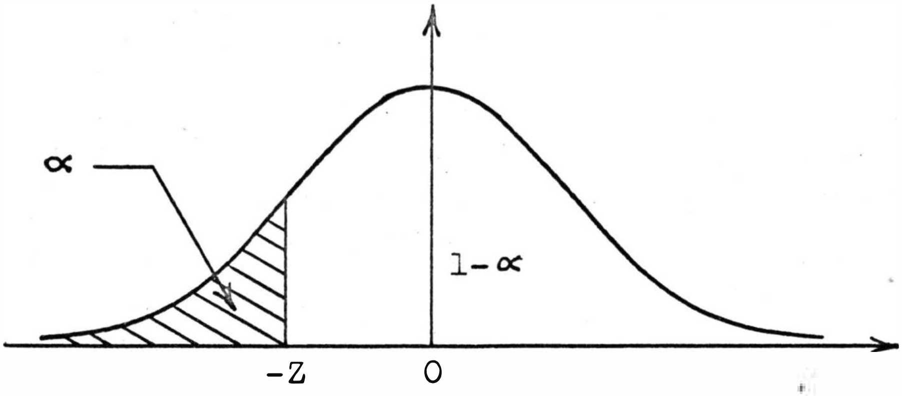

Figure 17. The Standardized Normal Distribution.

Computer Operations

All of the calculation procedures in PERT have been explained so that they can be performed by "hand." However, performing many of these calculations on realistic projects may get to be a very tedious job. For this reason, digital computers have been employed to make the necessary calculations. Without the advent of the computer the PERT technique of management would be very impractical in many projects, and its effectiveness would be severely restricted. However, speed in calculation is not the only factor to which a computer caters. The forms of computer output can be varied to suit most requirements and thereby are of value in themselves. Printouts produced by the computer depend upon the program used and, of course, the input. Another factor which determines whether computer usage is desirable is the frequency of updating computations. A network may not contain over 100 activities but if updating calculations are necessary every week then the computation becomes excessive and a computer would be beneficial.

Since PERT was first developed, over 60 computer programs have been developed.[4] Although details differ considerably, computer programs generally require certain common input data, perform a number of common calculations, and print out data in more or less standard formats. The functions which are common to all programs are indicated in the following definition of a critical path program.

> A 'critical path program' is a computer routine that performs the forward pass, backward pass, and slack computations associated with a CPM, PERT or similar network technique.[5]

Cecil R. Phillips discusses some of the comparative features and points on which programs may differ.[6]

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[4] Phillips, C. R., "Fifteen Key Features of Computer Programs for CPM and PERT," Journal of Industrial Engineering, Vol. 15, January-February 1964, p. 19.

[5] *Ibid.*, p. 14.

[6] *Ibid.*, pp. 14-17.

Several of the important features include:

1. Capacity--The capacity of critical path programs expressed in terms of the number of activities permitted ranges from a few hundred to 75,000 activities.

2. Event Numbering--A few PERT programs require that events be numbered in ascending order, which is a restriction that may inhibit flexibility and cause event-number bookkeeping problems. Therefore, a majority of the recent programs permit random numbering of events. This requires the computer to perform a renumbering function which is challenging to the best computer programmers.

3. Multiple Initial and Terminal Events--Many PERT programs permit multiple beginning and ending events. This feature is very beneficial in large projects which contain separate networks that must be merged for a single critical path computation.

4. Scheduled Dates--Most PERT programs accept scheduled completion dates for the network ending event. This means that slack figures will be related to the scheduled dates, and the critical path may have a positive, zero, or negative slack.

5. Calendar Dates--Instead of presenting the time requirements in units most PERT programs can use actual calendar dates in the input and output phases. In large projects this convenience saves a great deal of clerical effort.

6. Output Sorts--The way in which output is arranged should depend on management's requirement. Some of the current sorts include:

    a. By slack
    b. By event number

   c. By expected date
   d. By latest allowable date
   e. By responsibility identification code
   f. By key-event code

7. Updating Facility--A network may be revised by additions, deletions, and changes in any part of the input data. A few programs present a revised computation by entering only a few revision cards rather than the whole deck of cards for the network.

8. Error Detection--Critical path programs vary widely in their capability to detect network errors such as loops, nonunique activities, improper time estimates, and excessive terminal events. Some PERT programs will not detect a loop and will continue to run in a vain effort to perform the topological sort.

9. Activity versus Event Orientation--The distinction refers to the input and output data where descriptions may be associated either with activities or with events. Often the difference between the two is simply the tense of the verb in the description, for example, "perform test" versus "test performed." Activity orientation predominates the programs now available.

10. Statistical Analysis--Some programs permit the use of three time estimates in the input and calculate the expected elapsed time but do not make any summation of variance or computations of probability.

Thus, certain features of existing computer programs have been identified and defined. Some of the latest critical path programs include advanced features and more flexibility. The trend of programs follows new developments and additions to the PERT technique. Extension into the areas of cost control and resource allocation features are receiving a great deal of attention. These trends are discussed in a later chapter.

Appendix F contains a simple network problem which is solved by both "hand" computation and computer program methods. An IBM 1620 computer program (1620 Program Library No. 10.3.006) was used in running the problem. The illustration serves to point out the PERT fundamentals, network calculations, and computer output involved in the application of the PERT technique for project management. The simplicity of the network should not hinder its instructive value.

CHAPTER IV

ANALYTICAL ERRORS

Several assumptions which are made in PERT methodology have been pointed out in the previous chapter. These assumptions all contribute toward making the PERT calculated network times and probabilities vary from their actual counterparts. However, it is not the intent, nor possible, for PERT or any other technique to be exact in its projections. Improvement, though, is always possible and for this reason a study of the areas which may contribute to the error is in order. The areas investigated in this thesis will include the time estimates, the activity distribution, the linear approximation of t_e, and the network assumptions.

When the effect of a particular assumption is analyzed all other assumptions are set aside and considered independent of the one being studied. This allows one to study the error contributed by each assumption without the effect of the others interfering. However, in actual situations there will be a certain amount of error cancellation among the various assumptions so that the total network error will not be merely a sum of the individual errors.

Time Estimates

The three time estimates obtained from the technician or engineer involved in accomplishing an activity are the basis for all future analytical calculation. Here is where the human element has its entrance with an extremely important role. All further information derived by various analytical calculations is no more accurate than these estimates of time. The importance of accurate and realistic time estimates should be apparent and result in a painstaking attempt to get estimates which portray the actual situation.

In order to determine the magnitude and direction of possible errors caused by inaccurate time estimates, it is assumed that a, m, and b are the actual values of the activity distribution. Let the estimate of these values be t_a, t_m, and t_b with possible error within the limits of $0.8a \leq t_a \leq 1.1a$, $0.9m \leq t_m \leq 1.1m$, and $0.9b \leq t_b \leq 1.2b$. These intervals are based on actual experience with the PERT technique.[1]

The susceptibility of the PERT expressions for the mean and standard deviation to these incorrect time

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[1] MacCrimmon, K. R., and C. A. Ryavec, An Analytical Study of the PERT Assumptions, Research Memorandum Rm-3408-PR, The Rand Corporation, Santa Monica, California, December 1962, p. 12.

estimates can be seen from the calculated data presented in Table 4.

Table 4. Results of Imprecise Time Estimates

	Worst Possible Error--%		Net Error--%	
Mode	Mean	Standard Deviation	Mean	Standard Deviation
a	9.2	6.7	1.7	3.3
$a+\frac{b-a}{4}$	10.8	6.7	1.7	3.3
$a+\frac{b-a}{3}$	11.4	6.7	1.7	3.3
$a+\frac{b-a}{2}$	12.5	6.7	1.7	3.3
$a+\frac{2(b-a)}{3}$	13.6	6.7	1.7	3.3
$a+\frac{3(b-a)}{4}$	14.2	6.7	1.7	3.3
b	15.8	6.7	1.7	3.3

All results are expressed as a per cent of the range with the assumption that $b = 3a$. The worst absolute error of the mean was found by determining the error to be

$$= \frac{1}{b-a} \max \left[\left| \frac{(.8a+3.6m+.9b)-(a+4m+b)}{6} \right|, \left| \frac{(1.1a+4.4m+1.2b)-(a+4m+b)}{6} \right| \right]$$

$$= \frac{a+4m+2b}{60(b-a)} .$$

The worst net error of the mean (sum of worst positive and worst negative errors) was found

$$= \frac{1}{b-a}\left[\frac{-.2a-.4m-.1b+.1a+.4m+.2b}{6}\right] = \frac{1}{60} .$$

The worst absolute error of the standard deviation as a proportion of the range is

$$= \frac{1}{b-a} \max \left[\left|\frac{(.9b-1.1a)-(b-a)}{6}\right| , \left|\frac{(1.2b-.8a)-(b-a)}{6}\right|\right]$$

$$= \frac{b+a}{30(b-a)} ,$$

and the worst net error of the standard deviation is

$$= \frac{1}{b-a}\left[\frac{(.9b-1.1a)-(b-a)+(1.2b-.8a)-(b-a)}{6}\right]$$

$$= \frac{b+a}{60(b-a)} .$$

Hence, the value of the mode affects only the absolute error of the mean.

Although the worst possible error in the mean runs between 9.2 and 15.8 per cent of the range for b = 3a, some cancellation within the time estimate errors can be expected. Maximum cancellation occurs in the situation indicated by the net error. Here, the error is only 1.7 per cent of the range.

MacCrimmon and Ryavec conducted a similar analysis with b = 2a which resulted in errors that were 2 to 6 per cent higher.[2] The author of this thesis felt that a range larger than this (2a-a = a) would exist between the two endpoints of the activity distribution, and therefore, used b = 3a instead. This value was found realistic after inspecting several networks and calculating an average relation between a and b.

If the time estimates are greater in error than considered above, then the error of the mean and standard deviation will be increased. However, as was pointed out, some cancellation should occur and its effect is significant in keeping the error at a minimum.

Activity Distribution

Although the beta distribution is assumed to be representative of the form of the activity distributions, the true distributions are unknown. The major deterrent to an empirical study for determining the actual form of distributions is the nonrepetitive nature of the activities.[3] The beta distribution does contain the features of continuity, unimodality, and two non-negative abscissa intercepts which are expected properties of the actual

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[2] Ibid., pp. 12-13.

[3] Ibid., pp. 6-7.

activity distributions. So the beta distribution is at least correct with regard to its general shape.

To determine a unique beta distribution for each activity, the endpoints a and b, the mode, and the density function exponents p and q must be determined. These parameters are obtained from the three time estimates and the assumption that the standard deviation of the distribution is one-sixth of its range. This standard deviation assumption seriously restrains the inherent flexibility of the beta distribution, but is made in order to provide two equations for use in solving for the unknowns p and q.

We can see the effect of the standard deviation assumption by considering the case where the mode is midway between the distribution endpoints. A standard range of $[0,1]$ is considered since any range can be transformed into this case. From equation (3.13) we obtain

$$\frac{p}{p+q} = \frac{1}{2} .$$

Hence, p must equal q. Replacing q in equation (3.12) by its equivalent in terms of p results in

$$\frac{(p+1)^2}{(2p+2)^2(2p+3)} = \frac{1}{4(2p+3)} .$$

Setting this equal to the assumed variance of $(1/6)^2$, we get

$$\frac{1}{4(2p+3)} = \frac{1}{36}$$

or $p = 3$.

Therefore, when the mode lies in the center of the range, the parameters p and q must equal 3 in order for the standard deviation to be equal to one-sixth of the range. This restraining of the flexibility of the beta distribution occurs for all mode locations. A few examples of the density function using different modes are plotted in Figure 18. The required values of p and q are indicated for each distribution. We can see that the only combinations of p and q that are possible are those which satisfy the cubic relation resulting from setting equation (3.12) equal to 1/36. For this reason the beta distribution resulting from, for example, $q = 1$ and $p = 3$ is not possible in PERT activity distributions.

Several suggested methods of restoring the flexibility of the beta distribution have been proposed. Donaldson presents an alternate method of estimating the standard deviation (variance) of a PERT activity time given that one of the initial estimates is the mean rather than the mode.[4] His method allows considerable

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[4] Donaldson, W. A., "Estimation of the Mean and Variance of a PERT Activity Time," <u>Operations Research</u>, Vol. 13, May-June 1965, pp. 382-385.

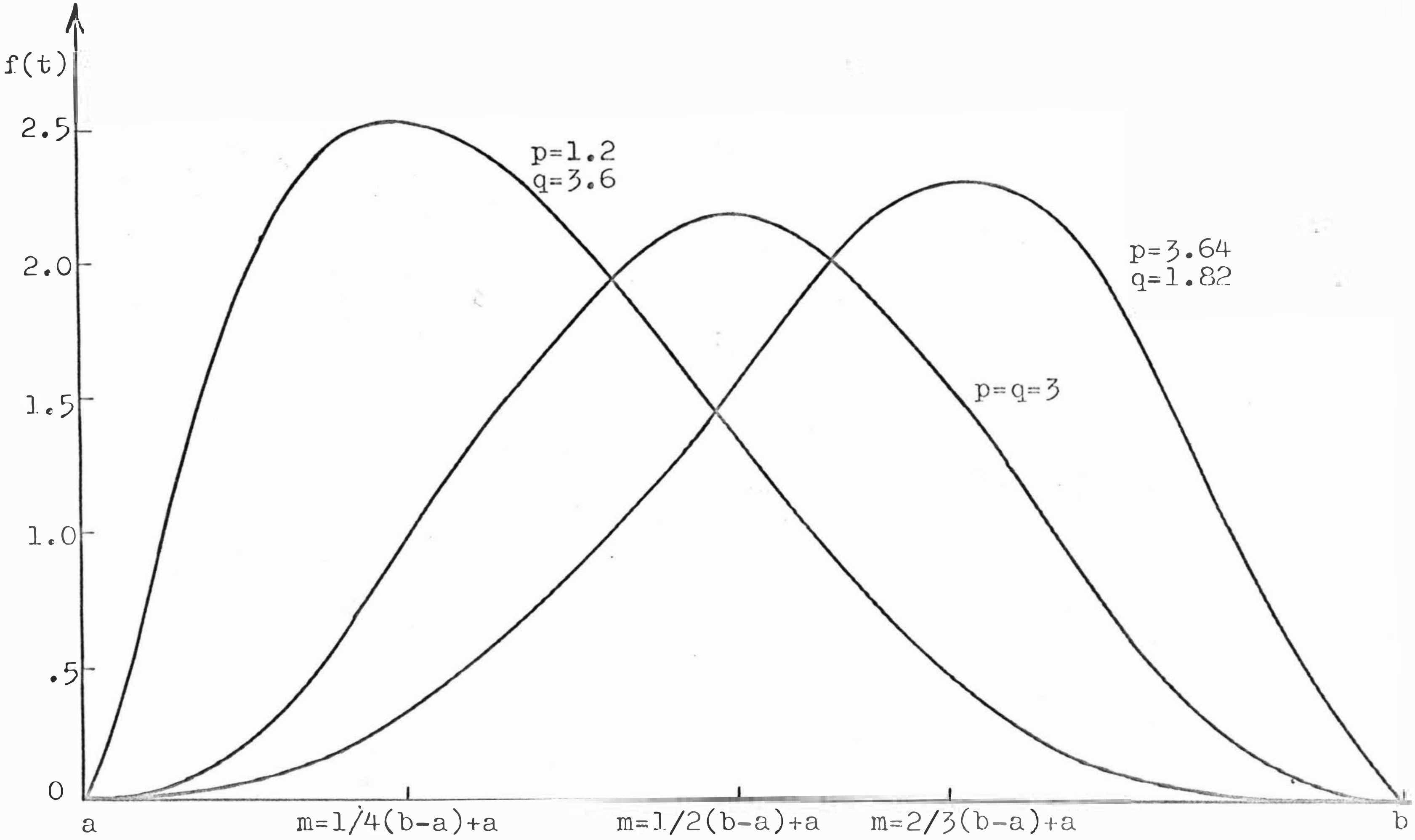

Figure 18. Examples of Restricted Beta Distributions.

variations in skew within the assumptions of a beta distribution tangential to the x-axis at each end. Murray suggests that a set of four time estimates should be considered.[5] This set is composed of the limits of the distribution, a and b, and two intermediate quantiles of the distribution, such as the quartiles or four points which are all at intermediate points on the distribution. He suggests that the solution of the resultant set of simultaneous equations be done using special-purpose nomograms developed on a digital computer. This revision would more fully utilize whatever experience the estimator has and would make it possible to remove the constraint that $\sigma = (b-a)/6$.

In addition to the restriction which the standard deviation assumption places upon the beta distribution, the assumption is false for some unusual cases of activity distribution. For example, the standard deviation of an activity distribution that approaches a rectangular form is $(b-a)/2\sqrt{3}$. Since most distributions such as this are very unlikely, the problem may not occur too frequently.

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[5] Murray, J. E., "Consideration of PERT Assumptions," IEEE Transactions on Engineering Management, Vol. EM-10, September 1963, pp. 94-99.

A problem which may occur more often was noticed by Healy.[6] He recognized that the mechanics of subdividing or summing activities in a PERT network can influence the computed probabilities for accomplishing events as scheduled. The problem is best understood by considering two activities each defined over the interval $[a, b]$. The standard deviation assumption indicates the standard deviation to be (b-a)/6. By statistical theorems, the standard deviation of the sum of these two activities is $\sqrt{2}(b-a)/6$. However, if the two activities were merged into only one activity with a new range interval from 2a to 2b, the standard deviation is assumed to be (2b-2a)/6. Hence, a discrepancy may arise depending on the amount of activity subdivision.

Thus, not only may the complete family of beta distributions be unable to represent a particular activity distribution, but also its inherent flexibility is still further restricted by the assumption that $\sigma = (b-a)/6$ in all cases. There appear to be many avenues open to improve the rigor and accuracy of the statistical aspect of the activity distribution. These problems and possible improvements will not be dealt with in any length since a

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[6] Healy, T. L., "Activity Subdivision and PERT Probability Statements," Operations Research, Vol. 9, May-June 1961, pp. 341-348.

deep and sophisticated statistical analysis is beyond the intended scope of this thesis.

Linear Approximation

Working within the restrictions on the beta distribution set by the standard deviation assumption, we can determine the difference between the actual mean and the PERT calculated mean. Cases where the parameters, p and q, define a distribution with a variance other than 1/36 will not be considered since the derivation of the linear approximation of the mean is based on a variance of 1/36. The error discovered in the other cases would stem primarily from the standard deviation assumption rather than the linear approximation. The error caused solely by the linear approximation is represented by the difference between the actual curve and the straight line in Figure 9.

Table 5 gives values of the variance for various values of p and m (q can be determined by using equation (3.13)). The variance was calculated through the use of equation (3.12). Note that .028 is the decimal equivalent of 1/36, the restricted PERT variance.

From Table 5 we can obtain the proper values of m and p corresponding to a distribution variance of 1/36. Then using equations (3.11) and (3.13) we can calculate the actual mean for the distribution. Equation (3.16)

Table 5. Variance Determined From the Mode and p

p Mode	0	.29	1/2	.70	3/4	1	1.21	1.82	2	3	6
1/2	.083		.063		.056	.050			.036	.028*	.017
1/3	.083		.054		.046	.040		.028*	.026	.019	.011
1/4	.083		.047		.038	.032	.028*		.019	.013	.007
1/6	.083		.035	.028*	.026	.021			.011	.008	.004
1/12	.083	.028*	.017		.011	.008			.004	.002	.001

*Only those combinations of the mode and p which correspond to this value of variance are possible in the PERT model.

allows us to calculate the PERT mean based on the linear approximation. Table 6 shows the PERT mean and the actual mean at each mode with the error of the linear approximation expressed as a per cent of the range.

Table 6. PERT Mean Vs Actual Mean

Mode	PERT Mean	Actual Mean	Error (% of range)
1/2	.500	.500	0.0
1/3	.389	.379	1.0
1/4	.333	.323	1.0
1/6	.278	.274	0.4
1/12	.222	.235	-1.3

Table 6 indicates that the error caused by the linear approximation averages about 1 per cent. From Figure 9 we can see that the area between the two curves, a representation of the error, is symmetrical about the midpoint, 1/2. Therefore, when the mode takes on values from 1/2 to 1 the error will be the same as that indicated by Table 6. This error is considerably less than that caused by the standard deviation assumption and the time estimates. In fact, it is zero when the mode and mean are equal to each other and many times this is nearly the actual case.

This close relation between the mode and mean was experienced in the Nike-Zeus missile application.

> A comparison of the value of m (most likely time estimate) and the calculated t_e (expected time) for a large number of activities indicated that the summation of t_e, as performed by the computer, produced a project time only 2.38 per cent greater than that indicated by summing the m values supplied by the estimates. This difference constitutes approximately six days in a one year program. It was concluded that requesting three time estimates improves the quality of the m estimate, but the use of m without the calculation of t_e would prove adequate.[7]

Hence, we are safe in concluding that the error produced by the linear approximation of the expected or mean time is negligible when compared to other error introductions.

Network Assumptions

Up to this point, only the individual activities within a PERT network have been considered. Attention will now be directed to the network as a whole. After the mean and standard deviation of each activity have been computed, they can be used to determine the effect of all activities taken together and to aid in the estimation of the project's time distribution.

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[7] Frantz, R. A., Jr., and L. B. Nothern, "An Introduction to PERT," The Western Electric Engineer, October 1963, p. 41.

As has been shown, the possible errors in the individual activities could, by themselves, cause errors in the calculation of a project mean and standard deviation. Generally, the magnitude and direction of these errors might be difficult to determine. However, even if the mean, standard deviation, and distribution obtained for each activity are correct, significant errors can still be introduced into the calculation of a network mean and standard deviation.

The main assumptions used to justify the PERT procedure for network calculations are the following:

1. A critical path is assumed to be enough longer than any other path so that the probability of another path being critical is negligible.
2. The critical path has enough activities so that the Central Limit Theorem applies.

Thus, whenever the network contains one very much longer path, the only PERT errors that can occur will be on the level of the individual activities. To illustrate what can happen when these assumptions are not satisfied, we consider the simple network shown in Figure 19.

The configuration shown in Figure 19 contains various paths in going from A to D. The PERT procedure will take as the mean and standard deviation of the project, the sum of the means and the square root of the sum of the variances along that path with the largest mean.

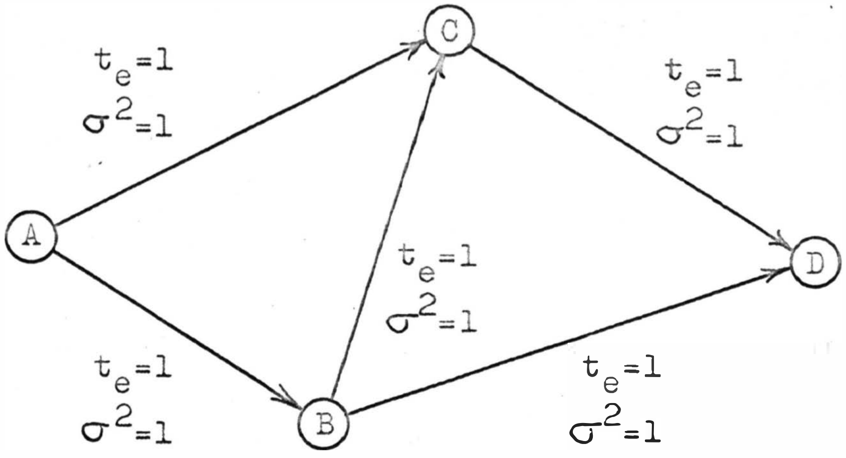

Figure 19. A Basic Network Example.

However, if the other paths have a mean very close to the first (they could be equal), the activities on these other paths will also be a major determinant of the project completion time and distribution. All activity distributions are assumed to be known exactly. This allows for the determination of the errors made by PERT on the network level alone, without confounding them with possible errors made in the activities. To get a feeling for the errors PERT makes in its network assumptions, the network in Figure 19 is analyzed both analytically and according to the PERT procedure.

Analytical calculation of project mean and variance will follow the procedure outlined by Clark.[8] Although

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[8] Clark, C. E., "The Greatest of a Finite Set of Random Variables," Operations Research, Vol. 9, March-April 1961, pp. 145-162.

the formulas presented in Clark's paper were derived from normally distributed random variables, the author (Clark) shows that they can be used with adequate accuracy in cases involving non-normal distributions. For simplicity of calculation, the mean and variance of all activities are taken to be equal to 1 and the distribution is taken as normal. It should be noticed that the critical path, ABCD, has a duration half again as long as the duration of either of the other two paths; therefore, the network considered will not present a worst case condition (all paths with equal duration) but rather will represent an intermediate and more common situation in terms of possible error. The details of the calculations (Clark's procedure) are found in Appendix G, and the results are given in Table 7.

Table 7. Summary of Results for Figure 19

	Conventional PERT	Corrected Values	Per cent Error
Mean	3.00	3.483	-14.0
Standard Deviation	1.73	1.47	+18.0

Table 7 indicates that the error caused by the network assumptions is significant, especially when the variance of the activities is large, which was the case here.

In this particular example, the variance was equal to one-half the range and played the largest role in contributing toward the error since the critical path was considerably longer than the rest. The possible error in PERT networks depends on the particular network configuration; so generalization is difficult to make, and the magnitude of the error indicated in Table 7 has little value if a different network is being considered. The basic network example does, however, indicate the potential sources of error and provide an indication of the magnitude and direction of the possible error.

The PERT-calculated mean will always be biased optimistically. This should be evident when considering that the PERT procedure favors an early completion date, by taking into account only those activities which are on the critical path and neglecting all others, some of which may have a high probability of delaying the project. The more parallel paths of approximately equal length that there are through a network, the larger will be the errors. But if the paths share a large number of common activities (high correlation), the errors will tend to be lower. The variance as calculated by PERT may be biased in either direction, depending on whether the activities considered had a high or low variance with respect to the others.

Thus, we recognize that the assumptions made in the PERT model, with regard to both individual activities and the entire network, are possible sources of error. The sources of error considered were the imprecise time estimates; the beta distribution and standard deviation assumption; the approximation formula for the mean; and the calculations underlying the project mean, standard deviation, and probability statements. The place for the largest error appears to exist in the time estimates obtained from the technical personnel involved in the project. The possible error here, especially on research and development projects, will very likely be the largest contributor to the error of the system. We have shown that the linear approximation for the mean contributes very little error, while the error caused by the remaining assumptions depends almost entirely on the particular activities and network configuration. An error of 30 to 40 per cent is possible in networks containing unusual activity distributions and parallel paths of nearly the same duration.

The following quotation from Western Electric's Zeus Project Organization brings out the problem of the inherent error in the PERT probability calculations.

> Initially the designers of the PERT technique placed a great deal of emphasis on the "probability of meeting schedule" as calculated

> by PERT. However, the many assumptions necessary to produce P_s make its use questionable from a technical point of view, and the concept proved of little value in our application.[9]

Possible modifications and improvements in the PERT model are discussed in the next chapter. The objective, of course, being to minimize the error while retaining or increasing the flexibility of the PERT model.

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[9] Frantz, R. A., *op. cit.*, pp. 41-42.

CHAPTER V

IMPROVEMENTS AND CONSIDERATIONS

The possible sources of error pointed out in the previous chapter were analytical, in that they pertained to the calculations involved in the PERT model. But in addition to these error sources, there are errors or problems of application. We should realize that even if the PERT model produced exact results, there still could be errors involved in its usage or application. Hence, the suggestions and considerations discussed in this chapter will pertain to both areas.

Network Improvements

Different means of improving PERT assumptions are suggested by various authors, but their recommendations are based on additional or other assumptions. Included are suggestions that it is easier to get estimates of 1 to 3 odds rather than end points on the frequency distribution curve and that a different number of time estimates should be obtained. This would change the PERT calculation procedures considerably, and it is difficult to tell in advance just how much, if any, improvement would result.

From an analysis of the assumption that the beta distribution represents the activity distribution, we were made aware of the seriousness of the restrictions placed on the beta distribution. The effect of the restraints was such that for an arbitrary modal value and set of end points, there is one particular beta distribution which is forced to represent the activity distribution. Considering this existing restriction along with the hypothesis that an activity approaches a normal distribution, we raise the question of whether or not the normal distribution may represent the actual activity distribution just as accurately as the beta distribution. Only empirical studies could ascertain this. We have based the hypothesis on the reasoning that since each activity may possibly be broken down into a number of smaller, more detailed activities, the original, more general activity, should approach a normal distribution according to the Central Limit Theorem. In fact, the phase one summary report issued by the Navy's Special Projects Office (the group which applied PERT to the Polaris Project) is inconsistent about whether the activity durations are normally or beta distributed.[1]

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[1] PERT-Summary Report Phase 1, Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D. C., July 1958, pp. A.1-A.4 and B.1-B.7.

Quoting from Appendix A of that report, we present the following statements: "Each activity has a time. The time is stochastic and normally distributed."[2] However, Appendix B discusses the beta distribution as a model of the distribution of the activity time. Here we discover that "the choice of the beta distribution was dictated by intuition because empirical evidence is lacking."[3]

An attempt should be made to determine the actual form of activity distributions. This is a difficult task and if, at best, only inconclusive results are obtained it still would help the PERT model considerably. By now the quantitative usage of PERT should be adequate to permit an attempt at a study of this nature. However, as was pointed out earlier, it may be nearly impossible to determine the distribution for activities associated with research and development projects. The severity of this problem is best understood after the study has progressed past the initial stage.

In the event that the normal distribution represents the actual distribution as well as, or better than, any other distribution, the PERT model could be improved upon in a number of ways. For one thing, all that would

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[2] Ibid., p. A.1.

[3] Ibid., p. B.4.

be needed to completely describe an activity distribution is an estimate of the mean and variance. This could be obtained from two estimates. For example, one value obtained could be the mean; the other estimate, signifying the interval, could be determined by plus and minus 1, 2, or 3 standard deviations. As a result, the errors and restrictions made by the standard deviation assumption and the linear approximation of the mean could be eliminated.

The network, as a whole, could be improved upon by utilizing Clark's bias correction procedure for expected time and variance calculations.[4] The procedure, outlined in Appendix G, could very possibly be incorporated in computer programs designed to handle the PERT calculations, thereby making the correction procedure very practical. Without the computer's contribution, the calculation of the linear correlation coefficients could be very time consuming.

Graphically, the corrections can be represented by the insertion of fictitious activities in the network. These correction activities require a mean and variance equivalent to the difference between the PERT calculations and the more inclusive, analytical calculations.

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[4] Clark, C. E., "The Greatest of a Finite Set of Random Variables," Operations Research, Vol. 9, March-April 1961, pp. 145-162.

The correction activity is located immediately after the event which is affected by correlation factors.

An illustration using correction activities is shown in Figure 20. The network is the same one which was analyzed in Appendix G, and the results were presented in Table 7.

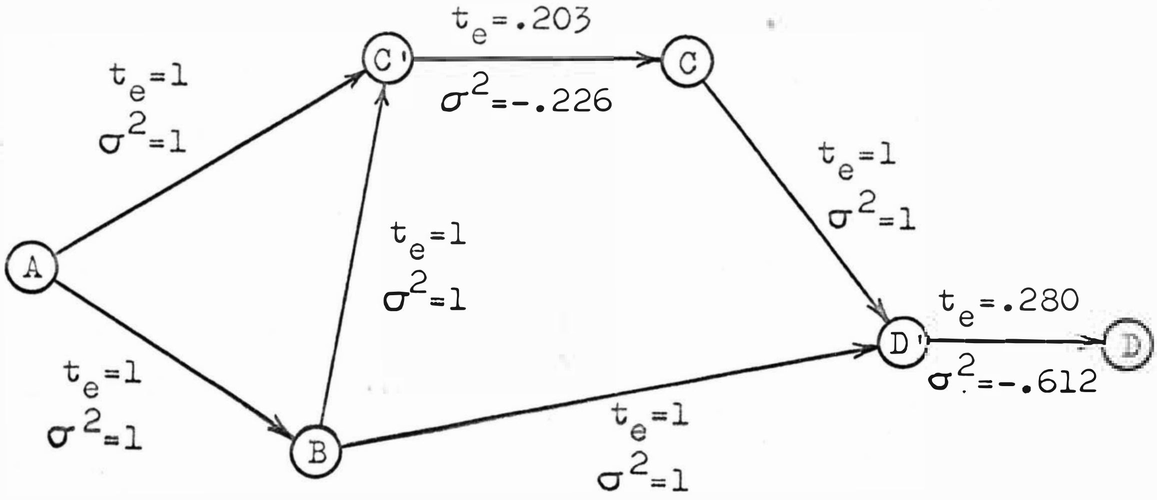

Figure 20. A Network Incorporating Correction Activities.

By incorporating the correction activities in the network, the effect on the project duration and variance is clearly shown, and the application of conventional PERT calculations to the modified network gives the corrected values for the event expected time and variance. To incorporate the correction activities, we must determine the variance, expected time, and location of each. A correction activity is not needed when the sum of the maximum times along one path is less than the sum of the minimum times along a parallel path leading to the same event. In this situation, the shorter path will not influence the terminal event calculations and can be eliminated from consideration. Of course, when there is only one activity leading to an event no bias correction is needed.

In Figure 20, the correction activity C'-C was assigned a mean time, $t_e = 2.203-2.00 = .203$, and a variance $\sigma^2 = 1.774-2.00 = -.226$. These values are merely the corrected values minus the values obtained by using conventional PERT. By a similar procedure, one obtains for the correction activity D'-D a mean time, $t_e = 3.483-3.203 = .280$, and a variance, $\sigma^2 = 2.162-2.774 = -.612$. The modified network then has an expected completion time of 3.483 and a variance of 2.162. The importance of the correction activities is realized when considering that the probability of completing the project in 3.25 time units was 56 per cent before the bias corrections were incorporated and 44 per cent after the correction activities were inserted.

Hence, the magnitude of the correction activity is the allowance that should be introduced to account for the fact that the path with the shorter duration occasionally occurs after the finish time of the path with the longest time duration. The correction activity itself does not represent any task or job, but the completion of the activities leading to it is synonymous with the occurrence of the correction activity.

Regardless of what the individual activity distributions turn out to be, we still can improve the accuracy of PERT output by paying special attention to the network critical path. After the initial network has been constructed to accurately portray the project program, then the time estimates on the critical path should be reviewed and changed if necessary. If as a result of the review the critical path changes to include other initial or first estimates, then the new critical path should be reviewed again. Also, it may be well to review the order or sequence of activities. Some times they can be organized in more than one way with a possible reduction in time resulting from a particular arrangement.

Before attempting any replanning procedures, it is best to identify paths which are nearly critical by considering the means, variances, and common activities of the various paths. This will help prevent premature withdrawal of resources from nearly critical paths when performing the replanning function. In general, any one of a number of paths could be critical, depending on the particular realization of the random activity durations that actually occurs.

Richard Van Slyke discusses in a Rand Corporation memorandum the improved computational accuracy resulting from the application of Monte Carlo, or random variable, techniques to the PERT management system.[5] He explains that the Monte Carlo technique applies the longest path algorithm to a long series of realizations, each one obtained by assigning a sample value to every activity drawn from its proper distribution. Also, in order to gain extra accuracy, the Monte Carlo approach depends on knowing the exact shape of the distribution which may be beta, normal, triangular, uniform, or discrete in form. Therefore, the method requires a distribution and standard deviation assumption similar to PERT's, but contains greater accuracy in the network calculations.

In Figure 21 a normal distribution with a mean of 66.00 and a variance of 60.27 is compared with the distribution obtained by the Monte Carlo method. The curves pertain to the same network as that presented in Appendix F. The Monte Carlo method used 10,000 realizations in

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[5] Slyke, R. V., Uses of Monte Carlo in PERT, Research Memorandum Rm-3367-PR, The Rand Corporation, Santa Monica, California, February 1963, pp. 8-21.

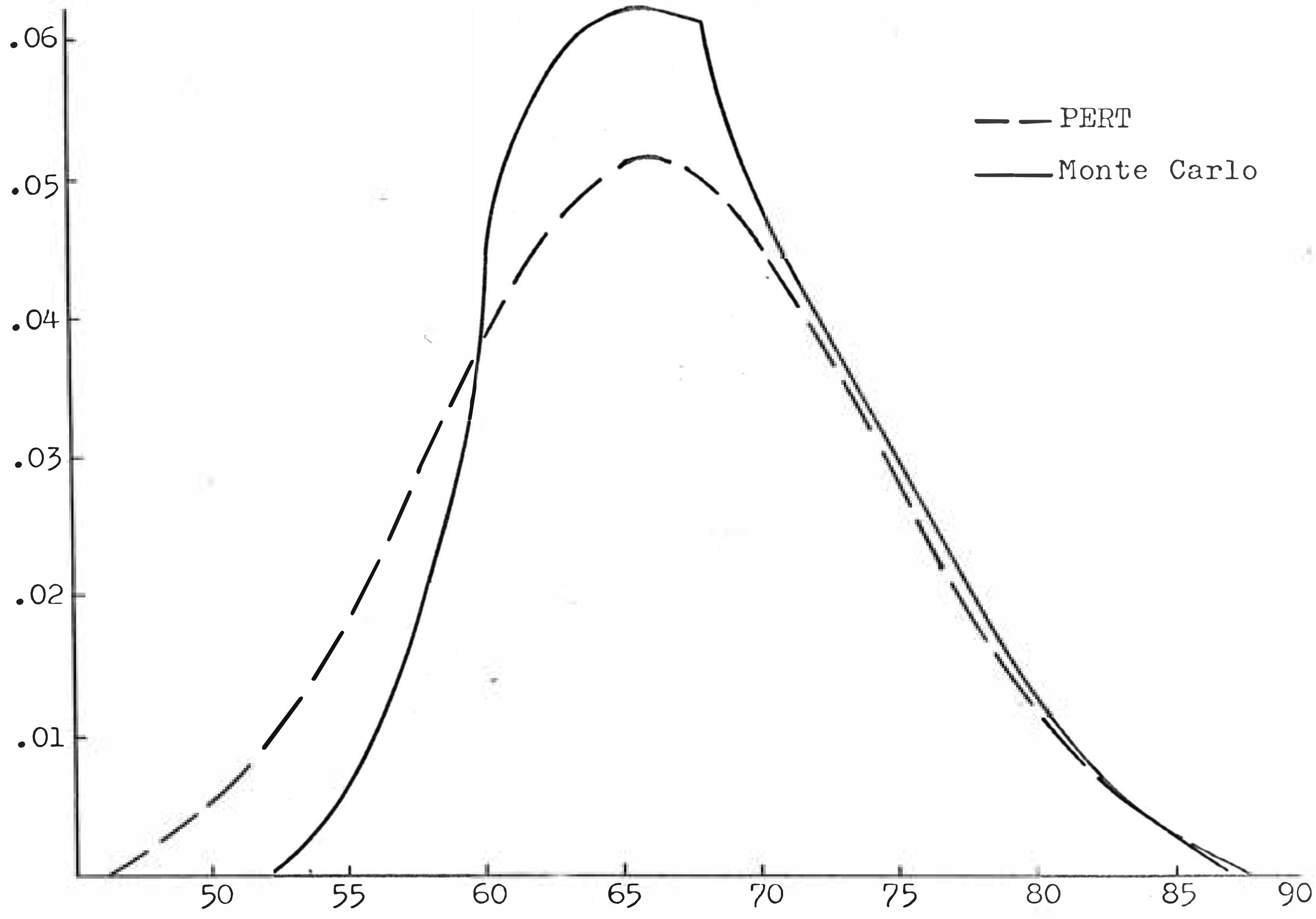

Figure 21. Probability Density Curves.*

*Source of data: See footnote 5.

with the company management, we follow the total program interests and results.

In receiving condensed PERT networks showing only the major events or milestones, upper management can follow the "management by exception" approach. They can enter the picture when an important schedule date seems to be in jeopardy. For this vertical management system to work as required, the proper level of detail must exist for each management level. The developing of networks at a different level of detail for each level of management is called the "pyramid" approach. These levels may be defined by a work breakdown structure as illustrated in Figure 4, with the networks at a higher level consisting of the condensed or integrated networks of the lower level.

There are many factors involved in determining the most appropriate level of detail. Thus far very little has been said about the required detail, and the example networks which were discussed did not lend any information, because they were extremely condensed for economy of presentation. To help in grasping a basic understanding of network detail, Moder and Phillips have presented a set of guiding questions with regard to expanding, condensing, or eliminating activities.[7] They are:

1. Who will use the network, and what are their interests and span of control?
2. Is it feasible to expand the activity into more detail?
3. Are there separate skills, facilities, or areas of responsibility involved in the activity which could be cause for more detail?
4. Will the accuracy of the logic or the time estimates be affected by more or less detail?

We can see that there are no exact rules to follow in determining the most effective level of detail. Experience is the best guide to follow because after working with two or three networks a person acquires a sense of proper network detail. Figure 22 contains an illustration of a typical PERT network chosen to help the reader get a grasp of the level of detail commonly used in reporting to upper management. A more detailed network designed for lower level control may be more informative, but space does not permit such an illustration here.

It is difficult to know when and how much PERT to use for program management. Some situations do not even lend themselves to PERTing. By this we mean that there

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[7] Moder, J. J., and C. R. Phillips, Project Management with CPM and PERT, Reinhold Publishing Corp., New York, 1964, p. 42.

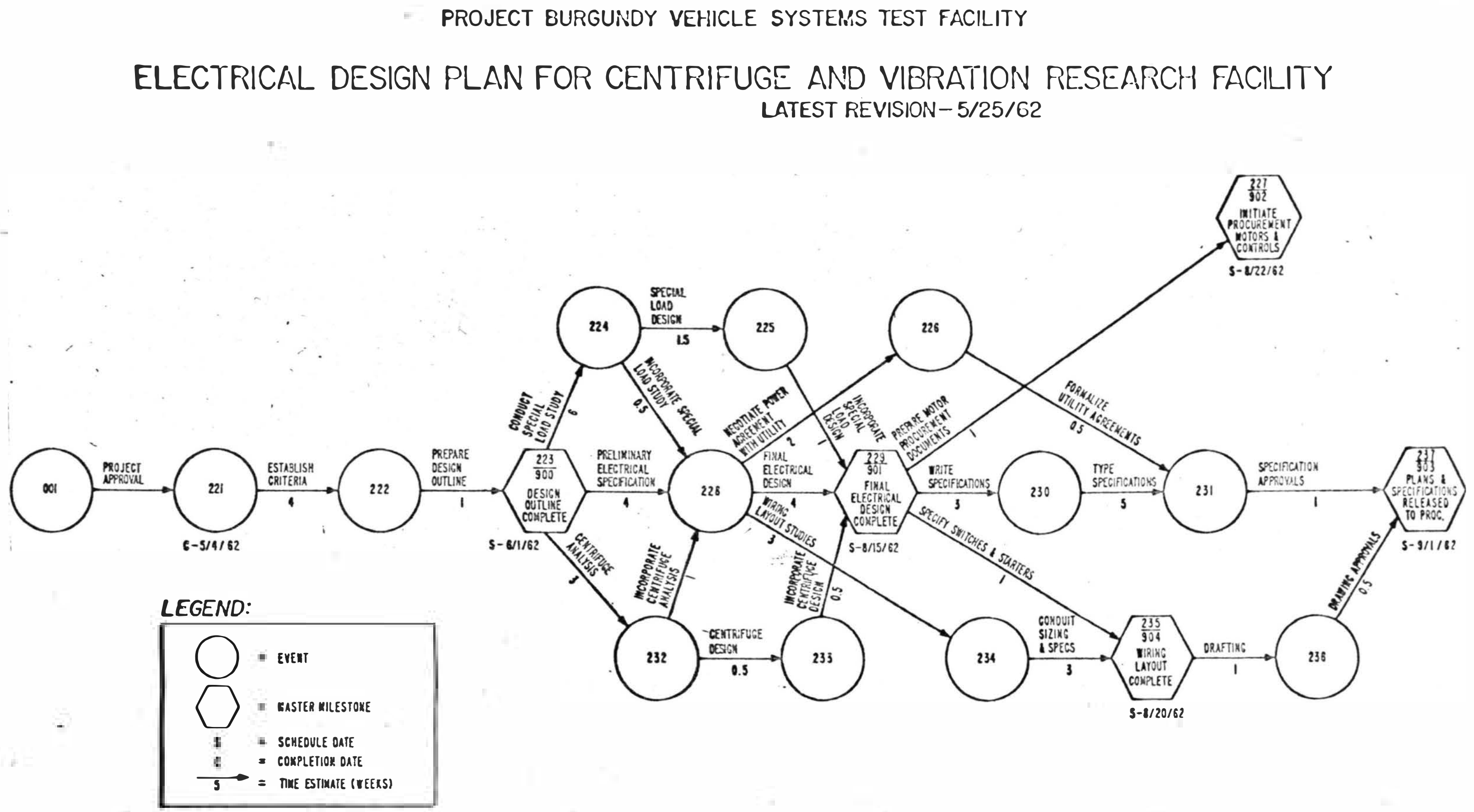

Figure 22. An Upper Management Network.*

*Source of data: <u>NASA PERT in Facilities Project Management,</u> U. S. Government Printing Office, Washington, D. C., March 1965, p. 21.

may be no one particular time or sequence when a group of activities should occur, and the sole consideration is the amount of manpower available. An area such as this should be handled outside the network if one exists for a related portion of the project. In other cases, the decision of whether to use PERT or not may not be as easy to make. For this reason, there have been some applications where a poor decision has been made and PERT has fallen short of its mark.

Based on an analysis of 22 case studies, Francis Sando finds a similar pattern of implementation in most successful projects.[8] He has formulated his analysis into two broad categories--preplanning and operations. The two categories contain the following points:

Preplanning Stage (plan before project is launched)

1. Orientation of client.
2. Thorough training of key personnel such as project managers and project coordinators.
3. Training and orientation of other project personnel who will work with the system.

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[8] Sando, F. A., "CPM: What Factors Determine its Success?" Architectural Record, Vol. 135, April 1964, pp. 211-216.

4. Arrangement for computer equipment required.

5. Establishing the proper organization to conduct network planning and control.

Operations

1. Develop accurate arrow diagrams using the logic of project manager and key personnel.

2. Implement and use only network planning, not bar charts.

3. Update regularly.

4. Hold planned meetings for project review.

5. Stick to the logic of arrow diagram if at all possible.

Sando also indicates some pitfalls to avoid in implementing network techniques. They include factors such as too general diagrams, lack of education and know how, lack of updating, and lack of computer orientation.[9]

The above discussion indicates the importance of education and feedback in using PERT. Adequate education and training for all personnel involved in the supplying of time estimates and/or the management of the project is a must. Failure to understand the relation between the three time estimates and the activity distribution could

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[9] Sando, *op. cit.*, May 1964, pp. 202-204.

result in serious network errors. The author's first exposure to PERT in industry involved the supplying of time estimates which were seriously in error. Without knowing the statistical definition of the terms optimistic, most likely, and pessimistic as they applied to PERT, the time estimates given by the author represented an arbitrary range with the most likely time being the midpoint. This resulted in frequent updating of estimates with the initial estimates in some instances being changed twice. This unfortunate experience indicates the value of feedback in changing time estimates which were initially inaccurate or have changed because of unforeseen complications.

There has been little application of PERT to manufacturing areas. This is partly due to the fact that manufacturing and production are concerned more with quantities rather than time. Nevertheless, the use of PERT in pilot model construction and early phases of production can be very effective in supplying information with regard to time per production unit. It should be noted that many programs of the space industry never leave the preliminary manufacturing stage or at least never enter into mass production. Therefore, an effort should be made to integrate the techniques of PERT with some of the methods of production control to bring in the quantity factor. Perhaps the technique could be applied in a manner similar to the one described below.

The PERT network follows the structure of the various subassemblies associated with the product. It is assumed that the product is past the development stage so that all the detailed parts have been designed and production is about to begin. We know that each part which is contained in a particular assembly must be manufactured by a certain series of operations. We also can determine the procedure for any part and from past time studies we can estimate the time required for each operation. Then based on the part's required operations, we obtain a manufacturing time requirement for the part.

Each part is represented as an activity in the PERT network with the time duration determined and set by the above procedure. All assembly and transportation operations are also represented by activities in the network logic. After the PERT network has been constructed to indicate the flow of parts and manufacturing effort for the entire product, then the expected project completion time can be calculated by conventional PERT methods. Electronic processing of data yields information pertaining to subassembly or part completion time by analysis of the proper event. Events signifying completed parts can be differentiated from other events by assigning to them a two digit number as opposed to a three digit number for events merely separating manufacturing operations. The information obtained from identifying the critical path is valuable in planning production capacities.

Experience gained through working with various PERT applications is invaluable in attempting to improve upon or extend the PERT applications. A lack of this experience has restrained the practical evaluation of the suggested improvements and modifications presented in this thesis. Their value to the PERT technique can only be determined through a testing procedure using actual applications. Also, through experience in PERT usage, new ideas which may improve the technique are generated.

CHAPTER VI

FUTURE TRENDS

A considerable amount of research is being put into extending the usefulness of PERT to include the resource and performance factors of a program. Generally speaking, the status of a development program is a function of three variables. The variables include the resources available (manpower, material, and facilities), the technical performance of systems and components, and the time required to accomplish the objectives. The ultimate objective, of course, is to enable PERT to determine trade-off relationships between the cost, time and performance objectives. Since the time variable has been the forerunner, PERT/Time has received a majority of the development effort. The present work in the cost and performance areas has been done under the labels of PERT/Cost and PERT/Reliability. A brief introduction to the basic concepts of each is presented below.

PERT/Cost involves estimating the cost associated with accomplishing an activity or group of activities. For each activity, two different estimates are made. One is the minimum time estimate and its cost, and the other is the minimum cost estimate and its time. Both estimates depend on the available or obtainable resources. The minimum time estimate represents the time required when the job is expedited as much as possible. This estimate is commonly called the crash point. The minimum cost estimate corresponds to the normal job time. This normal time-cost point is the one usually used for planning and scheduling. It does not assume the use of overtime labor or special time saving materials and equipment.

The cost estimates obtained represent only the direct cost of material, equipment and labor; therefore, in order to obtain the optimum operating point (minimum total cost) we must consider indirect costs as well. Indirect costs may include, in addition to overhead costs, the consideration of penalty costs or bonuses associated with the project completion time. The total indirect costs are assumed to increase linearly with time.

Figure 23 portrays the assumed relationship between time and costs. The "u" shaped curve representing the direct costs results from the assumption that costs will increase when the project is accelerated or delayed from a particular time point. This minimum cost point may or may not correspond to the normal operating point. Difficulties other than those associated with obtaining accurate time-cost estimates, arise in determining the shape and location of the curves. However, Miller points out that in some industrial applications data developed

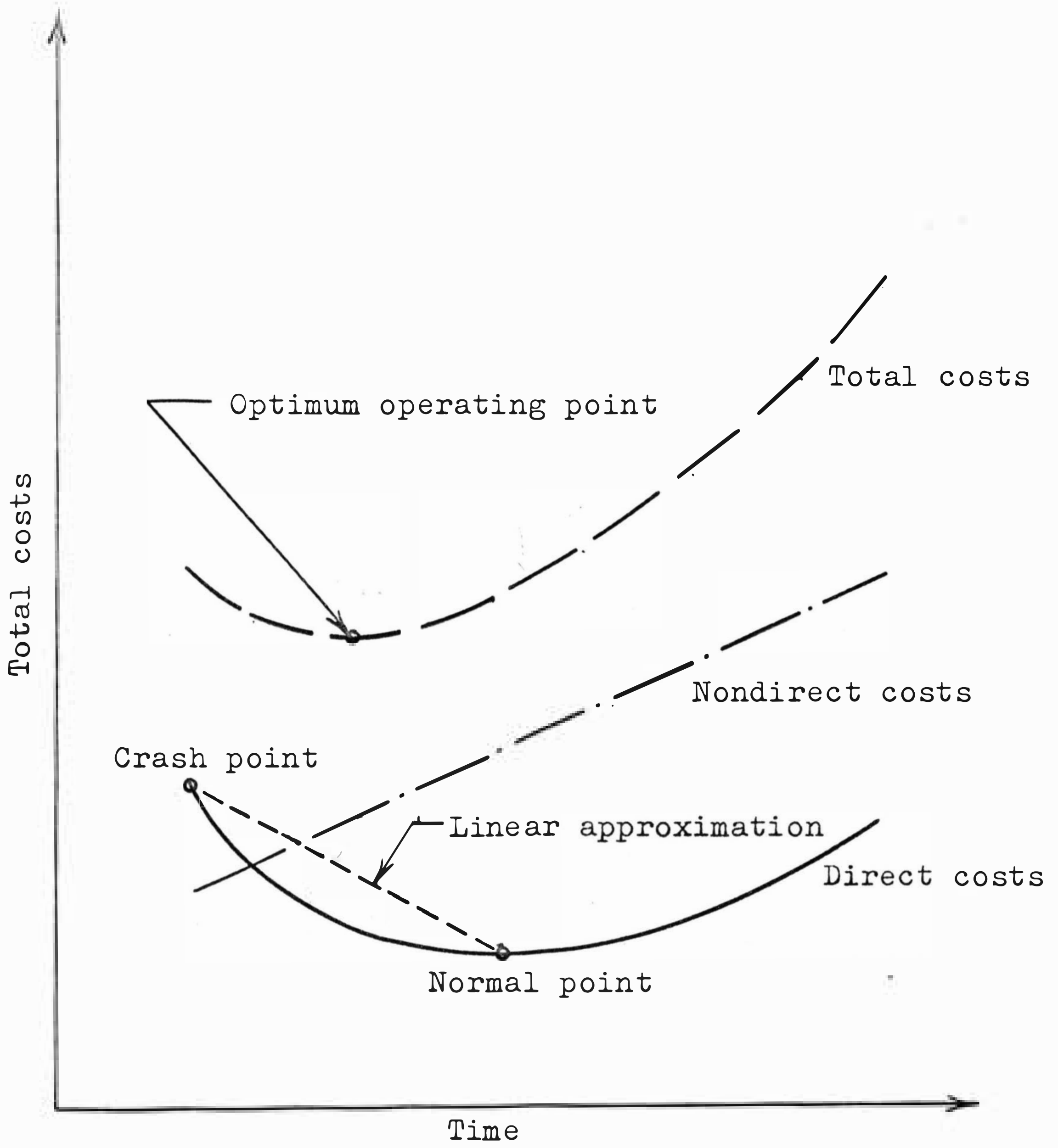

Figure 23. Assumed Relationships Between Time and Cost.

in the form of "outage" costs or loss-of-profit opportunities is presently used as the basis for improved decision making.[1] This information in conjunction with information obtained from accounting procedures should assist in making indirect cost calculations.

From Figure 23, we can see that the optimum time-cost point does not correspond to the normal project duration. To achieve this optimum operating point, activities on the critical path must be accelerated. Reducing the project duration will increase the direct cost; therefore, to keep the cost increase at a minimum, those activities which have the lowest time-cost slopes should be operated on.

The Department of Defense and the National Aeronautics and Space Administration have been very active in advancing PERT/Cost. In addition to outlining a recommended procedure to follow in using PERT/Cost, they indicate how the use of PERT/Cost eliminates unnecessary manpower costs and premium payments for expedited materials and services. To accomplish this, monthly manpower requirements are totaled by skills and examined to minimize unnecessary overtime and unnecessary hiring caused by

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[1] Miller, R. W., "How To Plan and Control with PERT," Harvard Business Review, Vol. 40, March-April 1962, p. 102.

manpower peaks followed by lay-offs. The manpower "smoothing" is accomplished by rescheduling slack activities to periods when the skills are not required by critical activities. Rescheduling slack activities can also eliminate or reduce premium payments for materials and services. Other reports can be set up to reflect the manpower load in each department.[2]

Like any other new development PERT/Cost has many problems associated with the extension of its use. A major problem is the interdependency of resource levels. For example, the decision to add machines to speed up or crash one activity may automatically affect the resource levels of other activities. This results in a false indication of the optimal operating point. Perhaps future developments and refinements will incorporate this aspect as well as other problems facing project management and PERT/Cost.

In the area of PERT/Reliability there are only a few available publications. Donald Malcolm has authored one of the first contributions in this area.[3] He

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[2] DOD and NASA Guide, PERT/Cost, U. S. Government Printing Office, Washington, D. C., June 1962, pp. 1-5 and 58-65.

[3] Malcolm, D. G., "Reliability Maturity Index (RMI) --An Extension of PERT into Reliability Management," Journal of Industrial Engineering, Vol. 14, January-February 1963, pp. 3-12.

mentions two approaches which may be taken in developing reliability information. One provides a prediction in the form of a probability statement for fraction successful of the eventual operational reliability of the end item. The second approach is discussed in greater detail. It provides a running measure of the compliance with planned reliability by analyzing information on the progress and quality of reliability programs.

The compliance to the planned reliability of a project is checked by incorporating reliability events into the network. These events indicate the start or completion of an activity resulting in the documentation of a design, a design review, a test, etc., required in the development plan in order to enhance the reliability of the end item. A reporting of information concerning both the schedule compliance and technical quality of the documentation is made to management. This assures that the reliability aspect of a project will be monitored, evaluated, and controlled according to specifications.[4]

Monitoring the compliance of the development plan to predetermined requirements does not provide a numerical forecast of the eventual operational reliability of the

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[4] *Ibid.*, pp. 6-8.

end item. Since a prediction of the eventual reliability to be experienced when the item is put into actual operation is desirable, this is the direction toward which PERT/Reliability should move. In order for PERT/Reliability to be effective, the following areas in reliability must be developed:

1. Analytical methods to predict the reject and failure rate.
2. Methods for synthesizing the effort of component reliability into the total system.
3. Means for rapid collecting, analyzing, and reporting of information to management.

The areas of PERT/Cost and PERT/Reliability will become more widely used and include many more variables as computers become more easily available and increase in capacity. The inclusion of just the early extensions of PERT into cost and performance functions has increased the data-processing load by about 20 times that associated with the basic PERT.[5] Thus, integrating the time, cost, and performance aspects of a complex project requires a computer which is high in capacity and efficiency. Present advancements in the computer industry should enhance

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[5] Malcolm, D. G., "Integrated Research and Development Management Systems," Operations Research in Research and Development, ed. Burton V. Dean, New York, 1963, p. 133.

the use and utility of the PERT extensions. This will result in a change in the management tools of the next few years.

It is difficult to forecast if the PERT extensions will receive as much published literature as the present PERT/Time has received. Norton points out that:

> The study of resource consumption and network schedules involves the conduct of engineering work, internal organization, and bookkeeping, so that a number of companies have understandably decided that publication is against their best proprietary interest, even though they may be actively working in the area.[6]

In addition to the extensions of PERT into new management areas, the author visualizes the merging of the CPM and PERT techniques. The CPM, as it exists now, could be handled by the PERT calculations with the probability concepts omitted. This general critical path technique, as it could be called after the union is made, could incorporate all parts of the CPM and PERT extensions, thereby creating a common network language familiar to all users. This would help eliminate the concurrent development of network management which is being directed from two ends.

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[6] Norton, P. V., "Resource Usage and Network Planning Techniques," Operations Research in Research and Development, ed. Burton V. Dean, New York, 1963, p. 151.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

The Project Evaluation and Review Technique, called PERT, provides a method of utilizing computers to plan and control a project. PERT's value lies in systematizing all the many steps or activities involved in the completion of the project. Those which must be completed in series and those which can proceed in parallel are distinguished and arranged in a network to show their proper relationship to each other. When time estimates are assigned to each activity, a computer can compute the series of activities, leading from beginning to end, which will require the longest total time for completion, i.e., the critical path. The time estimates also support PERT's capability to provide probabilistic information relating to the project completion.

In using PERT, we must remember that it is not a panacea for all management problems. Besides realizing that PERT does not make decisions for management but only presents information upon which to base decisions, we should bear in mind that the method is based on various assumptions. The assumption that the beta distribution represents the distribution of the activity duration is questionable. The matter is more questionable when considering that its inherent versatility is restricted by the standard deviation assumption.

Analysis of some of these assumptions shows that they can be very detrimental to the accuracy of PERT's output information. The magnitude and direction of the error is dependent on the particular network under consideration so that it is difficult to make general statements pertaining to the same. Perhaps the only generality that has any validity is the statement that the PERT calculations, by themselves, will contain an optimistic bias with regard to the project completion time.

After the analytical analysis of the errors associated with PERT, several potential improvements are suggested. These suggestions take the form of proposed changes that could simplify and/or improve the accuracy of the PERT model. Their true value can only be determined through empirical studies. An empirical investigation will also be conducive in the identification of additional improvements, whatever form they may take.

This thesis has laid the groundwork for future study of the PERT model. This has been done in part by an extensive survey of literature and the development and presentation of the existing PERT/Time methodology. Further, the identified and analytically analyzed areas of potential error should provide a starting point for additional work. Therefore, it is recommended that this study be followed by one which involves research pertaining to actual data. The participation of industry in this next stage of study is invaluable in providing a source of empirical data and the seriously needed experience in the use of PERT. Industrial contacts should be made well in advance of any contemplated work in this direction to ensure full cooperation between researcher and industry.

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APPENDIX A

Glossary of Symbols and Terms

Symbols:

a = Optimistic time estimate

b = Pessimistic time estimate

m = Most likely time estimate

t_e = Activity expected completion time

T_E = Earliest expected event completion time

T_L = Latest allowable event completion time

T_S = Event scheduled completion time

S = Event slack time

Z = Standardized normal random variable

σ = Standard deviation associated with the activity expected completion time

σ^2 = Variance associated with the activity expected completion time

σ_{T_E} = Standard deviation associated with the event expected completion time

Terms:

Activity. A time consuming element in a program plan which is represented by an arrow connecting one event to the next event. An activity cannot be started until the event preceding it has occurred and it represents work to be done or a description of a relationship between two events.

Beginning Event. An event which signifies the beginning of one or more activities of a network.

Critical Path. The sequence of interconnected events and activities extending from the network beginning event to the ending event which will require the most time to accomplish. It is the path that has the least algebraic slack.

Dummy Activity. A non-time consuming activity used to illustrate event dependency. It is not descriptive of work.

Ending Event. The event which signifies the completion of one or more activities. The ending point in time of an activity.

Event. A specific, definable accomplishment or milestone of progress in a program plan. An event is represented on a network by a circle and is not time consuming.

Expected Activity Time. A statistical mean time (t_e) which an activity is expected to require for completion. It is calculated from the three time estimates by the equation $t_e = \frac{a+4m+b}{6}$.

Expected Event Time. The earliest time (T_E) at which an event is expected to occur. The T_E value for a given event is equal to the sum of the expected activity times along the longest path from the network beginning event to the given event.

Interface Event. An event which is common to more than one related network. It signals the transfer of responsibility or the joint responsibility between networks.

Latest Allowable Event Time. The latest time (T_L) at which an event can be completed without causing the network ending event or project completion to be delayed.

Most Likely Time. The most realistic estimate of the time required to complete an activity assuming no unexpected problems will develop. This time would be expected to occur most often if the activity was to be repeated several times under exactly the same circumstances.

Network. A graphical representation of a program plan showing the sequence and interrelationship of all activities and events which must be accomplished to achieve the stated objectives of the program.

Optimistic Time. The estimated, minimum time required to complete an activity if everything goes better than expected. The activity has no more than one chance in 100 of being completed within this time.

PERT. Program Evaluation and Review Technique.

Pessimistic Time. An estimate of the longest time that an activity could require for completion if everything went exceptionally slower than expected. It has no more than one chance in 100 of being realized.

Scheduled Completion Time. A specified date or time by which the accomplishment of an event must be realized.

Slack. The difference between the latest allowable event time and the expected time ($S = T_L - T_E$). It represents the maximum time that an event may be delayed without delaying the completion of the entire project.

Standard Deviation of an Activity. A measure of variance about the expected completion time for an activity. It is computed from the formula $\sigma = \frac{(b-a)}{6}$.

Standard Deviation of an Event. A measure of variance associated with the expected event completion time. It is calculated by computing the square root of the sum of the squares of the activity standard deviation on the longest time path leading to the event under consideration.

Standardized Normal Random Variable. A calculated value (Z) associated with the standardized normal density function (which has a mean of zero and variance of one) and used to specify an area or probability determined through the equation $Z = T_L - T_E / \sigma_{T_E}$.

APPENDIX B

By definition the gamma function is

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx \qquad (n > -1).$$

Also, the Laplace transform is defined as

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt \qquad \text{when } n \geq 0.$$

Introducing a new variable of integration in the Laplace transform by setting st = x, we have

$$L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx.$$

After substituting the gamma function into the above it can be written as

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}.$$

Using Laplace transform tables, we find that

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

Whence,

$$\Gamma(n+1) = n! \qquad \text{when } n \geq 0.$$

APPENDIX C

The beta function of p and q is defined by the integral:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \qquad (p,q > 0).$$

By writing $x = \cos^2\theta$, we can obtain an equivalent form:

$$B(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta \, \sin^{2q-1}\theta \, d\theta .$$

From the definition of the gamma function we know that

$$\Gamma(k) = \int_0^\infty e^{-t}t^{k-1}dt \qquad (k > 0)$$

then by setting $t = x^2$

$$\Gamma(k) = 2\int_0^\infty e^{-x^2}x^{2k-1}dx.$$

Based on this form of the gamma functions, we can represent $\Gamma(p)\,\Gamma(q)$ as

$$\Gamma(p)\,\Gamma(q) = \left(2\int_0^\infty e^{-x^2}x^{2p-1}dx\right)\left(2\int_0^\infty e^{-y^2}y^{2q-1}dy\right)$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2p-1}y^{2q-1}dxdy.$$

To change to polar coordinates set

$x = r\cos\theta$ and $y = r\sin\theta$.

$rdrd\theta$ will be the new differential element after the change of variables.

Then,

$$\Gamma(p)\,\Gamma(q) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}(r\cos\theta)^{2p-1}(r\sin\theta)^{2q-1}\, rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r^{2p+2q-1}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\, drd\theta$$

$$= \left(2\int_0^{\infty} e^{-r^2} r^{2p+2q-1}dr\right)\left(2\int_0^{\pi/2}\cos^{2p-1}\theta\,\sin^{2q-1}\theta\, d\theta\right).$$

The first factor on the right we can recognize as $\Gamma(p+q)$, and the second is the derived equivalent form of the beta function. Therefore, we conclude that

$$\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(p,q).$$

APPENDIX D

To solve the cubic equation

$$E(x)^3-(m+1)E(x)^2+(m+\frac{1}{36})E(x)-(\frac{m}{12}-\frac{1}{36}) = 0$$

for values of m from 0 to 1 a computer program was written. The program is written in FORTRAN language and follows Newton's method for the numerical solution of the equation. The source program contains the following statements:

```
C  C    A CUBIC EQUATION SOLUTION BY NEWTONS METHOD
    1 FORMAT (3F10.5)
    2 FORMAT (2HP=,F4.2,5X,2HE=,F6.4)
      DIMENSION E(25),F(25),FD(25)
    3 READ 1,P,DP,FP
    4 J = 1
      E(1) = (4.*P+1.)/6.
    5 I = J
      F(I) = E(I)**3.-(P+1.)*E(I)**2.+(P+1./36.)
             *E(I)-(P/12.-1./36.)
      FD(I) = 3.*E(I)**2.-2.*(P+1.)*E(I)+(P+1./36.)
      J = I+1
      E(J) = E(I)-F(I)/FD(I)
      IF (E(J)-E(I)) 10,15,12
   10 IF (E(I)-E(J)-.00005) 15,15,5
   12 IF (E(J)-E(I)-.00005) 15,15,5
```

```
   15 PUNCH 2,P,E(J)
      IF (FP-P) 3,3,16
   16 P = P+DP
      GO TO 4
      END
0.0       0.05      1.00
```

The punched output is tabulated below with P representing the mode and E representing the value of E(x) or the mean.

```
C  C    A CUBIC EQUATION SOLUTION BY NEWTONS METHOD
P=0.00     E= .2052
P= .05     E= .2223
P= .10     E= .2425
P= .15     E= .2661
P= .20     E= .2929
P= .25     E= .3228
P= .30     E= .3552
P= .35     E= .3897
P= .40     E= .4256
P= .45     E= .4626
P= .50     E= .5000
P= .55     E= .5374
P= .60     E= .5744
P= .65     E= .6103
P= .70     E= .6448
P= .75     E= .6772
P= .80     E= .7071
```

P= .85 E= .7339
P= .90 E= .7575
P= .95 E= .7777
P=1.00 E= .7948

APPENDIX E

Table of Values of the Standard Normal Distribution Function

Z	0	1	2	3	4	5	6	7	8	9
.0	.500	.504	.508	.512	.516	.520	.524	.528	.532	.536
.1	.540	.544	.548	.552	.556	.560	.564	.568	.571	.575
.2	.579	.583	.587	.591	.595	.599	.603	.606	.610	.614
.3	.618	.622	.626	.629	.633	.637	.641	.644	.648	.652
.4	.655	.659	.663	.666	.670	.674	.677	.681	.684	.688
.5	.692	.695	.699	.702	.705	.709	.712	.716	.719	.722
.6	.726	.729	.732	.736	.739	.742	.745	.749	.752	.755
.7	.758	.761	.764	.767	.770	.773	.776	.779	.782	.785
.8	.788	.791	.794	.797	.800	.802	.805	.808	.811	.813
.9	.816	.819	.821	.824	.826	.829	.832	.834	.837	.839
1.0	.841	.844	.846	.849	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.893	.894	.896	.898	.900	.902
1.3	.903	.905	.907	.908	.910	.912	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.927	.928	.929	.931	.932
1.5	.933	.935	.936	.937	.938	.939	.941	.942	.943	.944
1.6	.945	.946	.947	.948	.950	.951	.952	.953	.954	.955
1.7	.955	.956	.957	.958	.959	.960	.961	.962	.963	.963
1.8	.964	.965	.956	.966	.967	.968	.969	.969	.970	.971
1.9	.971	.972	.973	.973	.974	.974	.975	.976	.976	.977
2.0	.977	.978	.978	.979	.979	.980	.980	.981	.981	.982
2.1	.982	.983	.983	.983	.984	.984	.985	.985	.985	.986
2.2	.986	.986	.987	.987	.987	.988	.988	.988	.989	.989

Table of Values . . . (Continued)

Z	0	1	2	3	4	5	6	7	8	9
2.3	.989	.990	.990	.990	.990	.991	.991	.991	.991	.992
2.4	.992	.992	.992	.993	.993	.993	.993	.993	.993	.994
2.5	.994		.994		.995		.995		.995	
2.6	.995		.996		.996		.996		.996	
2.7	.997		.997		.997		.997		.997	
2.8	.997		.998		.998		.998		.998	
2.9	.998		.998		.998		.999		.999	
3.0	.999		.999		.999		.999		1.000	

APPENDIX F

A simplified network problem is presented below. The network logic shown is assumed to be representative of the actual program. The three time estimates are presented above their respective activities in the order of optimistic (a), mostly likely (m), and pessimistic (b).

The expected time for each activity is calculated by the equation.

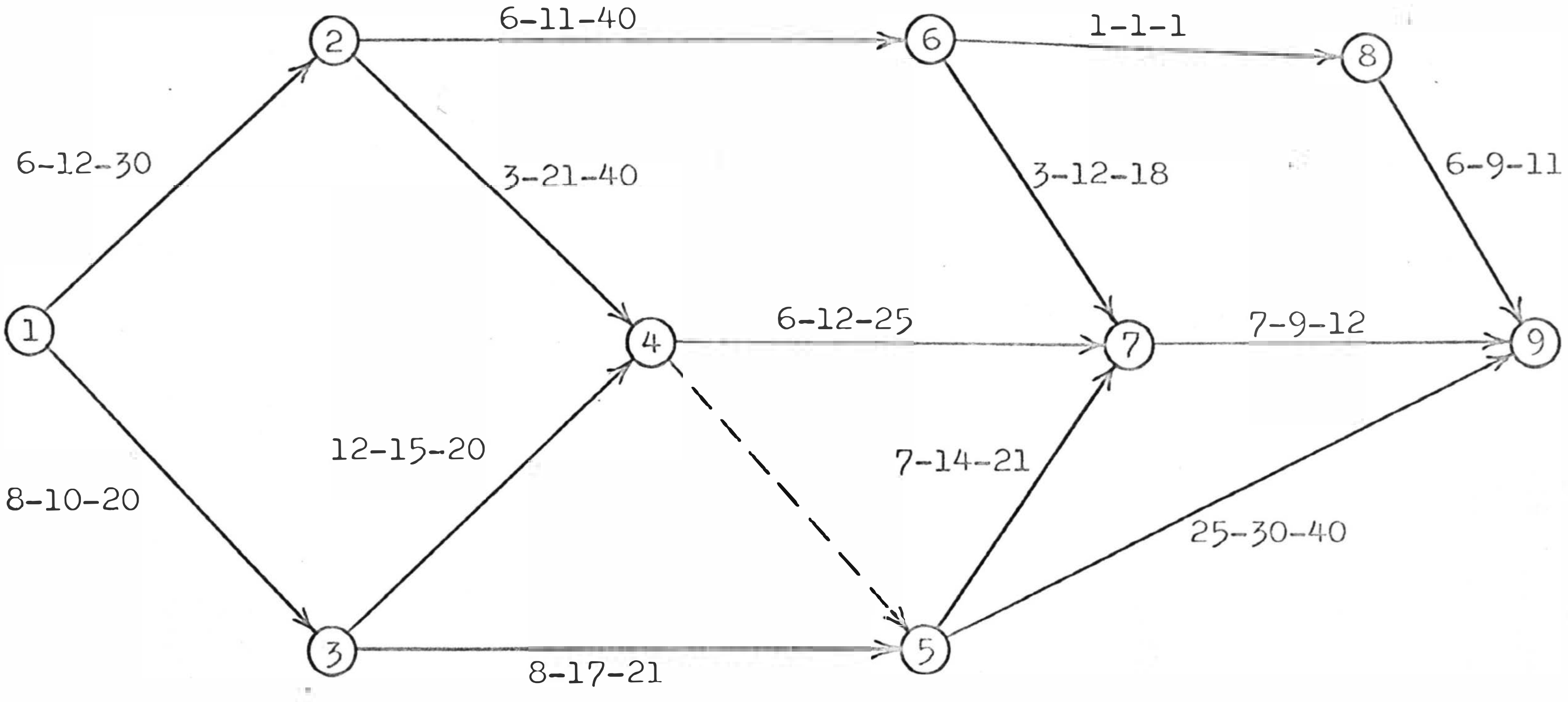
$$t_e = \frac{a+4m+b}{6}$$

and the standard deviation by the relation

$$\sigma = \frac{b-a}{6} .$$

The calculated values for expected time, standard deviation and variance are tabulated below.

Activity	t_e	σ	σ^2
1-2	14.00	4.00	16.0
1-3	11.33	2.00	4.0
2-4	21.17	6.17	38.0
3-4	15.33	1.33	1.8
3-5	16.17	2.17	4.7
4-5	0.00	0.00	0.0
2-6	15.00	5.67	32.1
4-7	13.17	3.17	10.0
5-7	14.00	2.33	5.4



Example Network Problem.

Activity	t_e	σ	σ^2
6-7	11.50	2.50	6.3
6-8	1.00	0.00	0.0
5-9	30.83	2.50	6.3
7-9	9.17	0.83	0.7
8-9	8.83	0.83	0.7

Next, the expected event completion time and the latest allowable event completion time are found by making a forward and backward pass through the network and using the above calculated values of the activity expected time. The slack is also determined from T_L-T_E. The corresponding values for each event are:

Event	T_E	T_L	S
1	0	0	0
2	14.00	14.00	0
3	11.33	19.00	7.67
4	35.17	35.17	0
5	35.17	35.17	0
6	29.00	45.33	16.33
7	49.17	56.83	7.66
8	30.00	57.17	27.17
9	66.00	66.00	0

Note that the latest allowable time of event 9 was set equal to its expected completion time, i.e., the scheduled date was assumed to be equal to T_E.

The critical path or path with minimum slack lies along 1-2-4-5-9. We use the activities and events along this path and the formula

$$Z = \frac{T_L - T_E}{\sigma_{T_E}}$$

in determining the probability of completing the project in the scheduled or latest allowable time. We find the calculated value of σ_{T_E} to be 7.8. Therefore,

$$Z = \frac{66.0-66.0}{7.8} = 0$$

and from the table in Appendix E the probability of accomplishing the project as scheduled is read as 50 per cent.

This same network was run on a 1620 computer utilizing a program (10.3.006) supplied by the International Business Machines Corporation. The form of the input and output is shown below. The output data shown contains only that pertinent to the problem. Omitted were: the T_S value for every event which the computer assigns equal to T_E if not instructed to do otherwise, and the resulting probability of accomplishment (.500); the calcúlated values for T_S-T_E; the program's capability of including a cost estimate for each activity.

Sample Problem Input

Activity	a	m	b
1-2	6.00	12.00	30.00
1-3	8.00	10.00	20.00
3-4	12.00	15.00	20.00
2-4	3.00	21.00	40.00
2-6	6.00	11.00	40.00
4-7	6.00	12.00	25.00
4-5			
3-5	8.00	17.00	21.00
6-8	1.00	1.00	1.00
6-7	3.00	12.00	18.00
5-7	7.00	14.00	21.00
8-9	6.00	9.00	11.00
7-9	7.00	9.00	12.00
5-9	25.00	30.00	40.00

70 (First assigned project completion time)

60 (Second assigned project completion time)

Sample Problem Output (Assigned project completion time of 70.00)

Activity	t_e	a	m	b	T_S	P_r	Var	T_E	T_L	S
1 2	14.00	6.00	12.00	30.00			16.00	14.00	18.00	4.00
1 3	11.33	8.00	10.00	20.00			4.00	11.33	23.00	11.67
3 4	15.33	12.00	15.00	20.00			1.77	26.66	39.17	
2 4	21.17	3.00	21.00	40.00			37.95	35.17	39.17	4.00
2 6	15.00	6.00	11.00	40.00			32.04	29.00	49.33	20.33
4 7	13.17	6.00	12.00	25.00			9.99	48.34	60.83	
4 5								35.17	39.17	4.00
3 5	16.17	8.00	17.00	21.00			4.67	27.50	39.17	
6 8	1.00	1.00	1.00	1.00			0.00	30.00	61.17	31.17
6 7	11.50	3.00	12.00	18.00			6.25	40.50	60.83	
5 7	14.00	7.00	14.00	21.00			5.43	49.17	60.83	11.66
8 9	8.83	6.00	9.00	11.00			0.69	38.83	70.00	
7 9	9.17	7.00	9.00	12.00			0.69	58.34	70.00	
5 9	30.83	25.00	30.00	40.00	70.00	.692	6.25	66.00	70.00	4.00

PROJECT COMPLETION 66.00 PROJECT SLACK 4.00

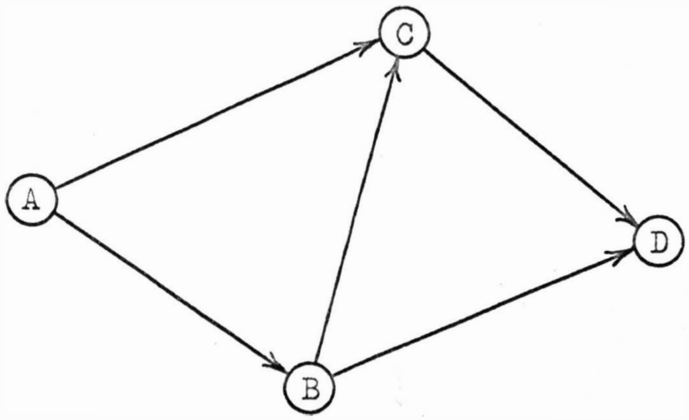
Sample Problem Output (Assigned project completion time of 60.00)

Activity	t_e	a	m	b	T_S	P_r	Var	T_E	T_L	S
1 2	14.00	6.00	12.00	30.00			16.00	14.00	8.00	-6.00
1 3	11.33	8.00	10.00	20.00			4.00	11.33	13.00	1.67
3 4	15.33	12.00	15.00	20.00			1.77	26.66	29.17	
2 4	21.17	3.00	21.00	40.00			37.95	35.17	29.17	-6.00
2 6	15.00	6.00	11.00	40.00			32.04	29.00	39.33	10.33
4 7	13.17	6.00	12.00	25.00			9.99	48.34	50.83	
4 5								35.17	29.17	-6.00
3 5	16.17	8.00	17.00	21.00			4.67	27.50	29.17	
6 8	1.00	1.00	1.00	1.00			0.00	30.00	51.17	21.17
6 7	11.50	3.00	12.00	18.00			6.25	40.50	50.83	
5 7	14.00	7.00	14.00	21.00			5.43	49.17	50.83	1.66
8 9	8.83	6.00	9.00	11.00			0.69	38.83	60.00	
7 9	9.17	7.00	9.00	12.00			0.69	58.34	60.00	
5 9	30.83	25.00	30.00	40.00	60.00	.274	6.25	66.00	60.00	-6.00

PROJECT COMPLETION 66.00 PROJECT SLACK -6.00

APPENDIX G

Analytical calculation of project mean and variance for the network below, all distributions being normally distributed, by Clark's procedure.



$E(a)=E(b)=E(c)=E(d)=E(e)=1$
$V(a)=V(b)=V(c)=V(d)=V(e)=1$

Event A:

$$E(A) = \mu_A = 0$$

$V(A) = \sigma_A^2 = 0$, by definition for the initial event of any network.

Event B:

$$E(B) = \mu_B = E(A+b) = E(A) + E(b) = 0 + 1 = 1$$

$$V(B) = \sigma_B^2 = V(A+b) = V(A) + V(b) = 0 + 1 = 1$$

Event C:

Time of C = max(a,B+c) = a:B+c

and,

$E(a) = \mu_a = 1 \qquad E(B+c) = \mu_{B+c} = E(B) + E(c) = 2$

$V(a) = \sigma_a^2 = 1 \qquad V(B+c) = \sigma_{B+c}^2 = V(B) + V(c) = 2$

The correlation coefficient $\varphi_{a,B+c} = 0$, since these two paths are independent of each other.

$$g^2 = \sigma_a^2 + \sigma_{B+c}^2 - 2\sigma_a\sigma_{B+c}\varphi_{a,B+c}$$
$$= 1 + 2 - 0$$
$$= 3$$

$$\alpha = \frac{(\mu_a - \mu_{B+c})}{g} = \frac{1-2}{\sqrt{3}} = -.577$$

$$E\left[\max(a,B+c)\right] = (\mu_{a:B+c})_1 = \mu_a\Phi(\alpha) + \mu_{B+c}\Phi(-\alpha) + g\phi(\alpha)$$

Where:

Normal density function $= \phi(\alpha) = \exp(-\alpha^2/2)/\sqrt{2\pi}$

Normal distribution function $= \Phi(\alpha) = \int_{-\infty}^{Z} \phi(t)dt$

$$(\mu_{a:B+c})_1 = \Phi(-.577) + 2\Phi(.577) + \sqrt{3}\phi(-.577)$$
$$= (.2820) + 2(.7180) + \sqrt{3}\,(.3377)$$
$$= 2.203$$

$$(\mu_{a:B+c})_2 = (\mu_a^2 + \sigma_a^2)\Phi(\alpha) + (\mu_{B+c}^2 + \sigma_{B+c}^2)\Phi(-\alpha) + (\mu_a + \mu_{B+c})\ g\phi(\alpha)$$
$$= 6.627$$

Then,

$$E(C) = (\mu_{a:B+c})_1 = 2.203$$

$$V(C) = (\mu_{a:B+c})_2 - (\mu_{a:B+c})_1^2 = 6.627 - (2.203)^2$$

$$= 1.774$$

Event D:

Time of D = max(B+e, C+d) = B+e:C+d

and,

$$E(B+e) = E(B) + E(e) = 2$$

$$E(C+d) = E(c) + E(d) = 3.203$$

$$V(B+e) = V(B) + V(e) = 2$$

$$V(C+d) = V(C) + V(d) = 2.774$$

$$\rho_{B+e,\ C+d} = \frac{\sigma_B \sigma_C \rho_{B,C}}{\sigma_{B+e} \sigma_{C+d}}$$

where:

$$\rho_{B,C} = \rho_{b,\ \max(a,b+c)} = \rho_{b,a:b+c}$$

$$\rho_{b,a} = 0,\ \rho_{b,b+c} = \frac{\sigma_b \sigma_b \rho_{bb}}{\sigma_b \sigma_{b+c}} = \frac{\sqrt{1}\sqrt{1}\ (1)}{\sqrt{1}\sqrt{2}}$$

$$= .702$$

$$\rho_{b,a:b+c} = \frac{\sigma_a \rho_{a,b} \Phi(\alpha) + \sigma_{b+c} \rho_{b+c,b} \Phi(-\alpha)}{\sigma_{a:b+c}}$$

$$= \frac{\sqrt{1}(0)\Phi(-.577) + \sqrt{2}(.702)\Phi(.577)}{\sqrt{1.774}}$$

$$= .539$$

$$\rho_{B+e,\ C+d} = \frac{\sqrt{1}\ \sqrt{1.774}(.539)}{\sqrt{2}\ \sqrt{2.774}} = .3055$$

$$g^2 = \sigma_{B+e}^2 + \sigma_{C+d}^2 - 2\sigma_{B+e}\sigma_{C+d}\rho_{B+e,\ C+d}$$

$$= 2 + 2.774 - 2\sqrt{2}\sqrt{2.774}(.3055)$$

$$= 3.338$$

$$\alpha = \frac{\mu_{B+e} - \mu_{C+d}}{g} = \frac{2 - 3.203}{\sqrt{3.338}} = -.658$$

$$(\mu_{B+e:C+d})_1 = \mu_{B+e}\Phi(\alpha) + \mu_{C+d}\Phi(-\alpha) + g\phi(\alpha)$$

$$= 2\Phi(-.658) + 3.203\Phi(.658) + \sqrt{3.338}\phi(-.658)$$

$$= 3.4832$$

$$(\mu_{B+e:C+d})_2 = (\mu_{B+e}^2 + \sigma_{B+e}^2)\Phi(\alpha) + (\mu_{C+d}^2 + \sigma_{C+d}^2)\Phi(-\alpha)$$

$$+ (\mu_{B+e} + \mu_{C+d})\ g\phi(\alpha)$$

$$= 14.293$$

Then,

$$E(d) = (\mu_{B+e:C+d})_1 = 3.483$$

$$V(D) = (\mu_{B+e:C+d})_2 - (\mu_{B+e:C+d})_1^2 = 14.293 -$$

$$(3.483)^2 = 2.162$$

By conventional PERT procedures the critical path ABCD would yield a mean of 3, and a variance of 3 or standard deviation of 1.73 ($\sqrt{3}$). Hence, we recognize a difference in the results of the two methods.

yielding a mean of 67.00 and a variance of 42.39. In both cases (the normal curve was obtained from applying the Central Limit Theorem in PERT procedures) activity distributions were assumed to be beta distributed with the standard deviation taken as one-sixth of the range.[6]

Application Considerations

A large factor which determines the success of the PERT model is the application. It should be realized that the application sets the pace by setting the output requirements from which the effect of PERT can be measured. The analytical areas discussed above actually play a back-seat role to the implementation of PERT and its effect on the final judgment. We should realize that even without any time estimates, PERT still contributes to the management and control of a project.

To be effective, PERT should be administered by the group directly responsible for the project schedule. Viewing this responsibility from management's point of view and further considering the project structure, we find that it follows a vertical slice of the total company organization. By keeping the PERT control consistent

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[6] *Ibid.*, pp. 18-19.