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Laminar Combined Convection from a Nonisothermal Spining Cone: An Integral Approach

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LAMINAR COMBINED CONVECTION FROM A NONISOTHERMAL SPINNING CONE - AN INTEGRAL APPROACH

BY

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SPINNING CONE - AN INTEGRAL APPROACH

This thesis is approved as a creditable and independent investigation by the candidate for the degree of Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

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LAMINAR COMBINED CONVECTION FROM A NONISOTHERMAL SPINNING CONE - AN INTEGRAL APPROACH

Abstract

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Under the supervision of Associate Professor A. A. Hayday

The thesis shows that the accuracy of the integral method is retained in the solution of problems dealing with simultaneous free and forced convection from bodies of revolution. The specific example dealing with combined convection on rotating cones not only typifies such problems but strongly implies, in general, that the accuracy of the integral method depends fundamentally on the consistent use of the multilayer concept; the latter reflects the physical fact that if several generating mechanisms for a given flow property act simultaneously they give rise to correspondingly different regimes where their influence is felt. Results of the thesis are compared with available exact solutions of the similarity class and found to be in good agreement over a wide range of Prandtl numbers.

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BRKC

NOMENCLATURE

A, C_1, C_2, m, n	Constants defined in equation
C_p	Specific heat at constant pressure
g	Acceleration due to gravity
h	Local heat transfer coefficient
Nu_x	Local Nusselt number $\frac{hx}{K}$
Pr	Prandtl number
Gr_x	Local Grashoff number
Re_r	Local Reynolds number
K	Thermal conductivity
q	Local heat flux
$r(x)$	Local radius of cone
T	Temperature of fluid in boundary layer
T_w	Wall temperature
T_∞	Ambient temperature
u	Velocity component in x-direction
v	Velocity component in y-direction
w	Velocity component in z-direction
u_1	Characteristic velocity in x-direction in the boundary layer
x, y, z	Coordinate system shown in Figure I
μ	Viscosity
ρ	Density
ν	Kinematic viscosity
α	Thermal diffusivity $\frac{k}{\rho C_p}$, also coordinate

Ω	Constant angular velocity
ϕ	Semi-vertical angle of the cone
θ	$T - T_{\infty}$
θ_w	$T_w - T_{\infty}$
δ	Viscous boundary layer thickness
Δ	Thermal boundary layer thickness
ξ	δ/Δ
η	Δ/δ

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1. INTRODUCTION

A widely used method for the solution of various problems dealing with laminar boundary layer flows is that originated by T. von Karman and K. Pohlhausen some forty years ago*. The basic idea on which their so-called integral method rests is to satisfy, for a given set of boundary conditions, Prandtl's boundary layer equations on the average in contrast to what are commonly called "exact" solutions, the latter in reality being numerical integrations of Prandtl's equations. The sense of "average" is the following: The partial differential equations are integrated with respect to a coordinate locally normal to the surface of a body whose boundary layer is studied; the upper limit is any positive number, say ℓ, where ℓ is greater or equal to a local boundary layer thickness, there may be several of these. For practical computations only the strict equalities need be considered. Polynomial representations for the flow fields (i.e., velocity and temperature) are then assumed, substituted into the various integrals that finally yield ordinary differential equations with δ -- the hydrodynamic boundary layer thickness--as one dependent variable. For two dimensional flows and certain flows with rotational symmetry δ is, in fact, the only dependent variable as long as

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*The original works were published simultaneously, both appearing in 1921 in ZAMM, Vol. I. Later they were discussed and extended by many other writers. All such pertinent papers are evaluated in Schlichting's book (9), a standard reference on boundary layer theory.

dissipation effects are negligible and no transfer of heat takes place. If heat transfer does occur, it is determined by the structure of the thermal layer of thickness Δ, the latter, in general, being different from δ. It is the account of such differences that strongly affects the accuracy of the integral method and forms the basis for the thesis problem.

Our intent in solving the thesis problem is two fold: Firstly, we intend to illustrate that the accuracy of the integral method observed in treating the simpler two dimensional flow problems can be retained in solving more complicated problems characterized by the fact that several generating mechanisms for a particular flow property (velocity and temperature) act simultaneously; this is the main reason for considering combined convection on a cone spinning in a fluid at rest*. Secondly, we wish to show in the simplest possible manner that the accuracy can be retained over a wide range of Prandtl numbers of technical interest. Our main conjecture, forming the basis for the thesis, is that such accuracy depends fundamentally on a proper account of the various generating mechanism for a given flow property.

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*The velocity vector $\bar{V}$ is known to have two essential components u, w, the first parallel to the cone surface and the second in the circumferential direction. The first of these may be induced either by the spin of the cone or by the action of the buoyancy force or both; the second is due to the spin alone but, because the governing equations are interdependent, u and w are interrelated. Hence, there act simultaneously two generating mechanisms for u and w.

By this we mean that, whenever several of such mechanisms coexist, they give rise, in general, to different regimes where their influence is felt, the latter being various different boundary layer thicknesses. While such "multilayer concept" is by no means entirely new, it has apparently not been exploited in the solution of free and combined (free and forced) convection problems. Our assertion is confirmed by the accuracy of the thesis results. We consider the specific example of a cone whose surface temperature distribution is linear, because this is the only cone flow for which an "exact" solution of the similarity class exists under the combined influence of spin and the buoyancy force. The numerical solutions of the full equations are due to Hering and Grosh (10), and these together with some unpublished results of Hayday provide the standards for comparison. The limiting cases of pure forced convection are also compared with the results of Hartnett and Deland (5).

Pertinent references on integral methods of solution of free convection problems for various other geometries are (1, 2, 3, 15, 16, 18). The latter four references employ the simplifying assumption that the thermal and hydrodynamic boundary layer thicknesses are equal, undoubtedly accounting for the significant inaccuracies when the Prandtl number is different from unity. This assertion appears to be reinforced by our second set of calculations in which we too assumed that the thickness of the boundary layers were the same; these results are shown to be in considerably poorer agreement with the corresponding exact solutions than those for which the above assumption is rejected.

2. ANALYSIS

1. Basic Equations and General Remarks on the Solution Technique.

For all of the subsequent work, it is assumed that the flow is of the boundary layer type, laminar, steady, with dissipation and curvature effects negligible and physical properties of the fluid essentially constant. Under these conditions, the differential equations* expressing the principles of conservation of mass, linear momentum, energy in an orthogonal, body-oriented coordinate system (Fig. I) take the form

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial Y} = 0 \qquad (2.1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} - \frac{w^2}{r}\frac{dr}{dx} = \nu\frac{\partial^2 u}{\partial Y^2} - \frac{1}{\rho}\frac{dP}{dx} - g\cos\phi \qquad (2.2)$$

$$u\frac{\partial(rw)}{\partial x} + v\frac{\partial(rw)}{\partial Y} = \nu\frac{\partial^2(rw)}{\partial Y^2} \qquad (2.3)$$

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial Y} = \frac{\nu}{P_r}\frac{\partial^2\theta}{\partial Y^2} \qquad (2.4)$$

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*A brief derivation of these equations and the corresponding integral equations are given in Appendix A.

In (2.1)-(2.4) u, v, w are respectively the x, y, z components of the velocity field with x being measured along a meridian curve, y along the local normal and z in the circumferential direction; $r(x) = x \sin \phi$ is the radius of the cone with half angle ϕ and $\theta = T - T_\infty$ stands for a local temperature difference. A complete nomenclature is given on pages iv and v.

The appropriate boundary conditions for our problem are

$$u(x,0) = v(x,0) = 0 \qquad w(x,0) = r\Omega \qquad \theta(x,0) = \theta_w(x)$$

$$\lim_{Y\to\infty} u(x,Y) = \lim_{Y\to\infty} w(x,Y) = 0 \qquad \lim_{Y\to\infty} \theta(x,Y) = 0 \tag{2.5}$$

The set (2.5) states that at the surface the relative motion between the fluid and the body is zero, and hence the only non-vanishing velocity component there is w(x,0), induced by the uniform rotation of the cone. Moreover, the body spins in a fluid otherwise at rest, the ambient fluid being at a uniform temperature. The surface temperature distribution $\theta_w(x)$ must, of course, be specified. Together, (2.5) and (2.2) imply now that

$$-\frac{dP}{dx} = \rho_\infty g \cos\phi$$

and standard development gives

$$-\frac{1}{\rho}\frac{dP}{dx} - g\cos\phi = g\beta\cos\phi\,\theta$$

where $\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T}\Big|_p$ is the coefficient of thermal expansion*. Hence, (2.2) may be replaced with

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2}{r} \frac{dr}{dx} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta \cos\phi \, \theta \qquad (2.6)$$

It is this equation which is actually used in the computations that follow.

At this point, it is worthwhile to bring up the general character of combined (free and forced) convection problems of the type treated in the thesis. For this purpose, we consider the implications of the set (2.1), (2.3) - (2.6). Equations (2.3), (2.4), (2.6) imply that any solution of a combined convection problem where free and forced convection are of equal importance must necessarily involve a simultaneous treatment of the equations of motion and energy. This is true for both exact and approximate solutions. While the same comment applies to pure free convection problems, the solution here is more complicated because of the z-component equation of motion (2.3) coupled through w to (2.2) and through θ to (2.4). The simpler problems of pure free convection and pure forced convection afford the following simplifications: In the first case, the motion is due solely to the buoyancy force, and hence, $w \equiv 0$, i.e., in (2.6) $\frac{w^2}{r} \frac{dr}{dx} = 0$ and (2.3) does not appear; in the second problem the buoyancy force is zero, i.e.,

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*The first statement is a consequence of the fact that at the edge of the boundary layer not only the velocities but also all y derivatives vanish.

$g\beta\cos\phi\,\theta = 0$, and so the energy equation (2.4) is uncoupled from the equations of motion (2.3), (2.6), the latter remaining coupled through w. In the special case of very slow rotation, we may set $w^2 = 0$ (but not w) and, hence, when free convection is the dominant mechanism, forced convection affects, through the w equation, the surface stresses but contributes nothing to heat transfer.

We seek approximate solutions to the problems outlined above. The starting point for all of the subsequent numerical computations is provided by the integral equations corresponding respectively to (2.6), (2.3) and (2.4). These are:

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\int_0^{(\delta,\Delta)} u^2\,dy - \frac{1}{x}\int_0^{\delta} w^2\,dy = -\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} + g\beta\cos\phi\int_0^{\Delta}\theta\,dy \qquad (2.7)^*$$

$$\left(\frac{d}{dx}+\frac{2}{x}\right)\int_0^{\delta} uw\,dy = -\nu\left.\frac{\partial w}{\partial y}\right|_{y=0}. \qquad (2.8)$$

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\int_0^{(\delta,\Delta)} u\theta\,dy = -\frac{\nu}{Pr}\left.\frac{\partial\theta}{\partial y}\right|_{y=0}. \qquad (2.9)^{**}$$

where δ stands for the thickness of the velocity boundary layer and Δ denotes the thickness of the thermal layers.

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*(δ , Δ), read δ or Δ. A brief derivation of equations (2.7) and (2.8) is given in Appendix B.

**(δ , Δ), read δ or Δ.

The proper choice of the limit in the first integral of (2.7) depends fundamentally on both the Grashof and Prandtl numbers* and must be made for each particular numerical example. While it is, of course, possible to write (2.7)-(2.9) with one limit, say L, where $L > \delta$, Δ, we have not done so in order to better emphasize the Prandtl number dependence of the solutions and hence, the general "two-boundary layer" nature of the problems. The notation used in (2.7)-(2.9) indicates clearly that the buoyancy and thermal effects influence the limits on u and θ but not on w. This is further reflected in the form of the assumed profiles for the velocity and thermal fields,

$$\frac{u}{u_1} = \frac{Y}{(\delta,\Delta)}\left[1 - \frac{Y}{(\delta,\Delta)}\right]^2 \tag{2.10}$$

$$\frac{w}{r\Omega} = \left[1 - \frac{Y}{\delta}\right]^2 \tag{2.11}$$

$$\frac{\theta}{\theta_w} = \left[1 - \frac{Y}{\Delta}\right]^2 , \tag{2.12}$$

$$\theta_w = A x$$

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*These well known dimensionless parameters are of fundamental importance in heat transfer problems. Their influence on the various heat transfer calculations is made precise later. A quick qualitative assessment of their importance may be obtained independently on the basis of dimensional analysis as indicated in Appendix C.

The profiles (2.10)-(2.12) form the simplest set satisfying the boundary conditions (2.5)*. We have chosen these primarily for simplicity and, as customary, justify their use a posteriori by comparing our results with exact theory, other approximate results, and, whenever possible, with experiments. Moreover, our main aim is to calculate the Prandtl number influence on heat transfer, and for this purpose the profiles prove to be quite adequate. That the Prandtl number greatly affects the heat transfer is well known from available exact solutions; it may already be anticipated from the order of magnitude analysis yielding the differential equations (2.1)-(2.6) and showing that $\frac{\delta}{\Delta} \sim \sqrt{Pr}$. Hence, for $Pr \ll 1$, $\Delta \gg \delta$, for $Pr \gg 1$, $\Delta \ll \delta$. The required treatments for high and low Prandtl numbers are necessarily different from one another; and, therefore,

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*More precisely, (2.10) and (2.12) are the lowest order polynomials consistent with (2.5) and the "smoothing condition" at $y = \Delta$, the latter requiring that the first derivatives vanish there. It must be mentioned, however, that (2.10)-(2.12) do not satisfy all of the conditions implied by the differential equations (2.1), (2.3), (2.4), and (2.6). Thus,

$$\left(-\frac{w^2}{r}\frac{dr}{dx} - \nu\frac{\partial^2 u}{\partial Y^2}\right)_{Y=0} = g\beta\cos\phi\,\Theta_w$$

$$\left.\frac{\partial^2 (rw)}{\partial Y^2}\right|_{Y=0} = 0 \qquad \left.\frac{\partial^2 \Theta}{\partial Y^2}\right|_{Y=0} = 0$$

which are immediate consequences of (2.6), (2.3), (2.4) and the boundary conditions (2.5). While we realize this, the purpose of the thesis is to assess, in the simplest way, possible improvements in the integral method applied to combined convection problems; hence, our choice of the polynomials.

the presentation is divided accordingly. The low Prandtl number analysis requires separate consideration of pure forced convection, $Gr_x = 0$, from the more general case $Gr_x \neq 0$. While at low Prandtl numbers this split-up is dictated by the choice of the profiles, no corresponding separate treatments are required at high Prandtl numbers where the limiting cases of pure free and pure forced convection are obtained by taking appropriate limits of the more general results. All numerical results for $Pr \ll 1$ and $Pr \gg 1$ join smoothly at $Pr \approx 1$.

2. Solutions Heat Transfer at $Pr \ll 1$.

Case I. $Gr_x = 0$, Pure Forced Convection.

The buoyancy force is zero, and the velocity boundary layer is independent of the thermal layer. Hence, both u and w profiles have the same limits and (2.10)-(2.12) specialize to

$$\frac{u}{u_1} = \frac{Y}{\delta}\left(1 - \frac{Y}{\delta}\right)^2 \tag{2.13}$$

$$\frac{w}{r\Omega} = \left(1 - \frac{Y}{\delta}\right)^2 \tag{2.14}$$

$$\frac{\theta}{\theta_w} = \left(1 - \frac{Y}{\Delta}\right)^2 \tag{2.15}$$

The appropriate set corresponding to (2.7)-(2.9) is

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\int_0^\delta u^2\,dY - \frac{1}{x}\int_0^\delta w^2\,dY = -\nu\left.\frac{\partial u}{\partial Y}\right|_{Y=0} \tag{2.16}$$

$$\left(\frac{d}{dx} + \frac{2}{x}\right)\int_0^\delta u w\,dY = -\nu\left.\frac{\partial w}{\partial Y}\right|_{Y=0} \tag{2.17}$$

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\int_0^\Delta u\theta\,dY = -\frac{\nu}{P_r}\left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} \tag{2.18}$$

Since $\Delta \gg \delta$, we have

$$\int_0^\Delta u\theta\,dY = \int_0^\delta u\theta\,dY + \int_\delta^\Delta u\theta\,dY = \int_0^\delta u\theta\,dY$$

where the last equality is implied by (2.13). We substitute now (2.13)-(2.15) into (2.16)-(2.18) and, after some simplifications, obtain* the respective differential equations in x,

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\frac{u_1^2\delta}{105} - \frac{(r\Omega)^2\delta}{5x} = -\nu\frac{u_1}{\delta} \tag{2.19}$$

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*The integrals in (2.16)-(2.18) have the following values:

$$\int_0^\delta u^2\,dY = \frac{u_1^2\delta}{105} \qquad \int_0^\delta u w\,dY = \frac{u_1 r\Omega\delta}{30} \qquad \int_0^\delta w^2\,dY = \frac{(r\Omega)^2\delta}{5}$$

$$\int_0^\delta u\theta\,dY = u_1\theta_w\Delta\left[\xi/_{12} - \xi^2/_{15} + \xi^3/_{60}\right]$$

$$\xi = \delta/\Delta$$

$$\left(\frac{d}{dx}+\frac{2}{x}\right)\frac{u_1 r\Omega\delta}{30} = \frac{2\nu r\Omega}{\delta} \tag{2.20}$$

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\left[u_1\theta_w\Delta\left(\xi/_{12}-\xi^2/_{15}+\xi^3/_{60}\right)\right] = \frac{\nu}{Pr}\,\frac{2\theta_w}{\Delta} \tag{2.21}$$

The intermediate results (2.19)-(2.21) are now considered as the equations determining the three unknowns u_1, δ, and Δ. While this system appears rather complicated, the desired solutions are obtained quite easily. The reason is that the functions u_1, δ, and Δ we seek correspond to exact solutions of the similarity class; therefore, without loss of generality, we may set

$$\delta = C_1 x^n \qquad\qquad \Delta = C_2 x^m$$

and use effectively (2.19)-(2.21) to determine the exponents n, m together with the constants C_1, C_2, and u_1. The procedure in this particular case is the following: First we use the w-equation (2.20) to obtain an expression for the product $u_1\delta$, then take the energy equation (2.21) to deduce a relationship for $\xi = \frac{\delta}{\Delta}$, and finally solve for δ from the u-equation (2.19). With δ known, it is then a simple matter to give explicit formulae for u_1 and Δ*. We give now the details, with

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*This is by no means true in the subsequent cases because the simultaneous algebraic equations for u_1, δ, Δ are quite difficult to handle. Fortunately, as will be shown later, one need not solve explicitly for u_1, δ, Δ in order to compute heat transfer; hence, there is lack of elegance but in no way does this effect the results.

$$u_1 \delta = F(x) \qquad\qquad \delta = C_1 x^n$$

(2.20) reduces to a first order linear equation in F(x)

$$\frac{dF(x)}{dx} + \frac{3F(x)}{x} = \frac{60\nu}{C_1} x^{-n} \tag{2.22}$$

whose general solution is

$$F(x) = u_1 \delta = \frac{60\nu}{4-n} \frac{x^{4-n}}{C_1 x^3} + \frac{\text{Constant}}{x^3}$$

Now, $u_1 > 0$ and obviously $\delta(0) = 0$. Therefore, the constant is zero; and using again the fact that $\delta = C_1 x^n$, we obtain the required result,

$$u_1 \delta = \frac{60\nu}{4-n} \frac{x}{\delta} \tag{2.23}$$

This expression is substituted into the energy equation yielding an equation in $\xi = \frac{\delta}{\Delta}$,

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\left[\frac{60\nu}{4-n} \frac{\theta_w x}{\delta\, \xi} \left(\xi/12 - \xi^2/15 + \xi^3/60\right)\right] = \frac{2\theta_w}{\Delta} \frac{\nu}{P_r}$$

Setting

$$g(\xi) = \frac{1}{\xi}\left(\xi/12 - \xi^2/15 + \xi^3/60\right)$$

and using $\Delta = c_2 x^m$, there follows from above a first order linear equation in $G(x) \equiv g[\xi(x)]$,

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\left[\frac{60\nu}{4-n} \frac{x^{1-n}\theta_w}{c_1} G(x)\right] = \frac{2\theta_w}{c_2 x^m} \frac{\nu}{P_r} \tag{2.24}$$

The general solution is*

$$G(x) = \frac{4-n}{3-m} \frac{1}{30 P_r} \frac{c_1}{c_2} x^{n-m} + \frac{c}{x^{3-n}}$$

We take C = 0 showing a posteriori that $n < 3$ and hence, the required boundedness of G(x) at x = 0 is consistent with this choice. Since $\frac{c_1}{c_2} = x^{-n+m} \frac{\delta}{\Delta} = x^{-n+m}\xi$ and $G(x) = g[\xi(x)]$, the above equation implies

$$\frac{1}{\xi^2}\left(\xi/_{12} - \xi^2/_{15} + \xi^3/_{60}\right) = \frac{4-n}{3-m} \frac{1}{30 P_r} \tag{2.25}$$

Once n and m are specified, this statement, (2.25), becomes a basic functional relationship between $\xi = \frac{\delta}{\Delta}$ --the ratio of boundary layer thickness--and the Prandtl number Pr. As it stands,

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*It suffices to note that since $\theta_w = Ax$, (2.24) is easily expressed as

$$\frac{dG(x)}{dx} + \frac{3-n}{x} G(x) = \frac{4-n}{30} \frac{x^{n-m-1}}{P_r} \frac{c_1}{c_2}$$

obviously implying the general solution given above.

(2.25) implies only that m = n; this follows because its right hand side is independent of x; hence, ξ is independent of x and therefore, m = n. In other words, $\xi = \xi(Pr)$.

An explicit formula for δ is obtained from (2.19), together with (2.23); we set $H(x) \equiv 1/\delta^4$ and write (2.19) as a linear equation for H(x),

$$\frac{dH}{dx} + \frac{1}{x}\left[\frac{40-7n}{3}\right] H = \frac{140(4-n)^2}{(60\nu)^2} \frac{\Omega^2 \sin^2\phi}{5x} \tag{2.26}$$

The general solution is

$$H(x) = \frac{420(4-n)^2}{(40-7n)(60\nu)^2} \frac{\Omega^2 \sin^2\phi}{5} + C\, x^{\frac{7n-40}{3}}$$

but, by the same argument as above, C = 0; and, hence, the required result for $\delta(x)$ is

$$\frac{\delta}{x} = 3.217294419\ Re_r^{-1/4} \tag{2.27}$$

The boundary layer thickness δ is thus constant, and (2.27) gives the value of C_1 in $\delta(x) = C_1 x^n$ with n = 0; of course, m = n and so also m = 0.

With δ known, the explicit forms of $u_1(x)$, $\Delta(x)$ are now obtained from (2.23) and (2.25). The results are:

$$u_1 = 1.442137676\ \nu x \left(\frac{\nu^2}{\Omega^2 \sin^2\phi}\right)^{-1/2} \tag{2.28}$$

and
$$\Delta = \frac{\delta}{\xi}$$

$$= \frac{0.1072431 73\left(\frac{\nu^2}{\Omega^2 \sin\phi}\right)^{1/4}}{\frac{3Pr+2}{45Pr} - \sqrt{\frac{12Pr+7}{180Pr}}}$$

where

$$\xi = 30\left[\frac{3Pr+2}{45Pr} - \sqrt{\frac{12Pr+7}{180Pr}}\right] \qquad (2.29)$$

Calculations of local heat transfer are based on the formula

$$q = -K \frac{\partial\theta}{\partial Y}\Big|_{Y=0} \qquad (2.30)$$

It is customary to introduce the heat transfer coefficient

$$h \equiv \frac{q}{\theta_w}$$

and present the results in terms of the related Nusselt number,

$Nu_x = \frac{hx}{K}$

$$Nu_x = \frac{2x}{\Delta} = 0.92956\,\xi\left(\frac{Re_r^2}{5}\right)^{1/4} \qquad (2.31)$$

where the last equality follows from (2.29). Results based on formulae of the type (2.31) are discussed in the next section.

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*In the denominator it is actually $\pm$. Since $0 < \xi \leq 1$, this specifies the negative sign in the denominator.

Case II. $Gr_x > 0$, Combined Convection.

Herein we generalize Case I by considering the additional effect of the buoyancy force on heat transfer. This implies that u velocity field and the thermal field θ are now directly interacting with one another. The pertinent profiles are, thus,

$$\frac{u}{u_1} = \frac{Y}{\Delta}\left(1 - \frac{Y}{\Delta}\right)^2 \tag{2.32}$$

$$\frac{w}{r\Omega} = \left(1 - \frac{Y}{\delta}\right)^2 \tag{2.33}$$

$$\frac{\theta}{\theta_w} = \left(1 - \frac{Y}{\Delta}\right)^2 \tag{2.34}$$

$$\theta_w = Ax$$

Of course, (2.32)-(2.34) are appropriate specializations of (2.10)-(2.12).

The integral equations corresponding to (2.7)-(2.9) are

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\int_0^\Delta u^2 dY - \frac{1}{x}\int_0^\delta w^2 dY = -\nu \left.\frac{\partial u}{\partial Y}\right|_{Y=0} + g\beta \cos\phi \int_0^\Delta \theta\, dY \tag{2.35}$$

$$\left(\frac{d}{dx} + \frac{2}{x}\right)\int_0^\delta uw\, dY = -\nu \left.\frac{\partial w}{\partial Y}\right|_{Y=0} \tag{2.36}$$

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\int_0^\Delta u\theta\, dY = -\frac{\nu}{P_r}\left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} \tag{2.37}$$

Together (2.32)-(2.37) imply* the differential equations in x,

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{u_1^2\Delta}{105}\right)+\nu\frac{u_1}{\Delta}=\left(\frac{r^2\Omega^2\xi}{5x}+g\beta\cos\phi\,\frac{\theta_w}{3}\right)\Delta \tag{2.38}$$

$$\left(\frac{d}{dx}+\frac{2}{x}\right)\left[u_1 r\Omega\delta\left(\frac{\xi}{12}-\frac{\xi^2}{15}+\frac{\xi^3}{60}\right)\right]=\frac{2\nu r\Omega}{\delta} \tag{2.39}$$

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{u_1\theta_w\Delta}{30}\right)=\frac{2\nu}{P_r}\,\frac{\theta_w}{\Delta} \tag{2.40}$$

We follow the procedure established in Case I, seeking solutions of the form**

$$\delta = c_1 x^n \quad , \quad \Delta = c_2 x^m$$

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*The values of the integrals are the following:

$$\int_0^\Delta u^2\,dY=\frac{u_1^2\Delta}{105},\quad \int_0^\delta w^2\,dY=\frac{r^2\Omega^2\delta}{5},\quad \int_0^\Delta \theta\,dY=\frac{\theta_w\Delta}{3}$$

$$\int_0^\delta uw\,dY=u_1 r\Omega\delta\left(\frac{\xi}{12}-\frac{\xi^2}{15}+\frac{\xi^3}{60}\right)$$

$$\int_0^\Delta u\theta\,dY=\frac{u_1\theta_w\Delta}{30}$$

**The notation is the same as in Case I; but, of course, C_1, C_2 and the functions F, G, H have by no means the same values as in Case I. In fact, F, G, H have here different meanings than in Case I.

The differential equations (2.38)-(2.40) are used to determine the functions

$$F(x) \equiv u_1 \Delta \, , \qquad G(x) = \xi \, g[\xi(x)], \qquad H(x) \equiv \frac{1}{\Delta^4}$$

where

$$g[\xi(x)] = \frac{\xi}{12} - \frac{\xi^2}{15} + \frac{\xi^3}{60}$$

The system F, G, H is

$$\frac{dF}{dx} + \frac{2F}{x} = \frac{60\,\alpha}{C_2\, x^m} \tag{2.41}$$

$$\frac{dG}{dx} + (4-m)\frac{G}{x} = \frac{C_2}{C_1} P_r \frac{3-m}{30} x^{m-n-1} \tag{2.42}$$

$$\frac{dH}{dx} + \left[4 + \frac{7}{3} P_r (3-m)\right]\frac{H}{x} = \frac{140}{x} \frac{(3-m)^2}{(60\alpha)^2} \left[\frac{\Omega^2 \sin^2\phi}{5} \xi + \frac{g\beta C_8 \phi A}{3}\right] \tag{2.43}$$

where (2.41), (2.42), (2.43) correspond respectively to (2.40), (2.39), and (2.38); i.e., the θ, w, u system (in that order).

The general solutions are

$$F(x) = \frac{60\,\alpha}{3-m} \frac{x}{C_2 x^m} + \frac{C}{x^2} \tag{2.44}$$

$$G(x) = \frac{C_2}{C_1} P_r \frac{3-m}{30} \frac{x^{4-n}}{(4-n)\, x^{4-m}} + \frac{C}{x^{4-m}} \tag{2.45}$$

$$H(x) = \frac{7(3-m)^2 P_r^2}{180\, \nu^2} \frac{\Omega^2 \sin^2\phi\, \xi/5 + g\beta \cos\phi\, \Delta/3}{4 + \frac{7}{3} P_r (3-m)} + \frac{C}{x^{[4 + \frac{7}{3} P_r (3-m)]}} \tag{2.46}$$

In each equation, C stands for an arbitrary constant, but the solutions we seek require (just as in Case I) the constants to be zero.

The procedure for obtaining formulae for u_1, δ , Δ is now somewhat different from Case I. Moreover, as mentioned in a previous footnote, no explicit formulae for these quantities are given. We use F (i.e., the energy equation) to deduce $u_1\Delta$ and then the functions H and G to obtain two implicit relationships between δ and Δ. The first result is

$$u_1 \Delta = \frac{60\nu}{P_r} \frac{x}{\Delta} \tag{2.47}$$

the latter two results are

$$\frac{\Delta}{x} = \left(\frac{7}{20} \frac{P_r^2}{4 + 7P_r} \right)^{-1/4} \left[\frac{Re_r^2}{5} \xi + \frac{Gr_x}{3} \right]^{-1/4} \tag{2.48}$$

and

$$\xi\, q(\xi) = \frac{3-m}{4-n} \frac{P_r}{30} \tag{2.49}$$

To obtain (2.47), (2.48), we have set $C \equiv 0$ in (2.44) and (2.46) and also used $\Delta = C_2 x^m$. The statement (2.49), the second implicit relationship between δ and Δ, follows from the G-function after some obvious simplifications. Instead of trying to solve now for δ, Δ from (2.48) and (2.49), we proceed as follows: With the definition of $g(\xi)$ we may rewrite (2.49) as

$$P_r = \frac{4-n}{3-m} \, 30\xi \left(\frac{\xi}{12} - \frac{\xi^2}{15} + \frac{\xi^3}{60} \right) \qquad (2.50)$$

As in Case I, (2.50) implies that $m = n$ and together with (2.48) this means that $m = n = 0$. But $0 < \xi \leq 1$ and hence, for any choice of ξ between the above bounds, (2.50) yields a compatible Prandtl number. Rewriting (2.48) in the form

$$\frac{\delta}{x} = \xi \left[\frac{7}{20} \, \frac{P_r^2}{4+7P_r} \right]^{-1/4} \left[\frac{R_{er}^2}{5} \xi + \frac{Gr_x}{3} \right]^{-1/4} \qquad (2.51)$$

and substituting into this expression any two compatible values of ξ and Pr from (2.50), yields δ with Gr_x and Re_r^2 as parameters; therefore, δ is known. With ξ and δ known, Δ is determined; and by (2.47) so is u_1. In this way, we have avoided the awkward problem of solving (2.48), (2.49) for δ and Δ. Of course, the same development could have been used in Case I.

The Nusselt number relationship corresponding to (2.31) is now

$$\frac{Nu_x}{\sqrt{Re_r}} = 2 \left[\frac{7}{20} \, \frac{P_r^2}{4+7P_r} \right]^{1/4} \left[0.2\,\xi + \frac{1}{3} \, \frac{Gr_x}{Re_r^2} \right]^{1/4} \qquad (2.52)$$

Note that for $Gr_x = 0$ (2.52) is different from (2.31). In the next section we shall show that (2.52) is valid for $Gr_x \geq 1$ but is not accurate in the range $0 \leq Gr_x < 1$.

Heat Transfer at $Pr \gg 1$.

This problem is characterized by the statement $\delta > \Delta$. For this reason, it is not necessary to separate the discussion for $Gr_x = 0$ from $Gr_x > 0$. Moreover, the solution procedure is virtually the same as for $Pr \ll 1$, Case I, with the exception that again no explicit formulae for u_1, δ, Δ are given. It will suffice, therefore, to give here only the main steps of the solution.

The appropriate profiles are

$$\frac{u}{u_1} = \frac{Y}{\delta}\left(1 - \frac{Y}{\delta}\right)^2 \tag{2.53}$$

$$\frac{w}{r\Omega} = \left(1 - \frac{Y}{\delta}\right)^2 \tag{2.54}$$

$$\frac{\theta}{\theta_w} = \left(1 - \frac{Y}{\Delta}\right)^2 \tag{2.55}$$

$$\theta_w = A\,x$$

The integral equations (2.7)-(2.9) specialize to

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\int_0^{\delta} u^2 dY - \frac{1}{x}\int_0^{\delta} w^2 dY = -\nu \left.\frac{\partial u}{\partial Y}\right|_{Y=0} + g\beta\cos\phi \int_0^{\Delta} \theta \, dY \tag{2.56}$$

$$\left(\frac{d}{dx}+\frac{2}{x}\right)\int_0^\delta u w \, dY = -\nu \left.\frac{\partial w}{\partial Y}\right|_{Y=0} \tag{2.57}$$

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\int_0^\Delta u\theta \, dY = -\alpha \left.\frac{\partial \theta}{\partial Y}\right|_{Y=0} \tag{2.58}$$

where in (2.56) we have used $\int_0^\delta \theta \, dY = \int_0^\Delta \theta \, dY + \int_\Delta^\delta \theta \, dY = \int_0^\Delta \theta \, dY$

The corresponding differential equations are*

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\frac{u_1^2\delta}{105} - \frac{1}{x}\frac{(r\Omega)^2\delta}{5} = g\beta \cos\phi \frac{\theta_w \Delta}{3} - \nu\frac{u_1}{\delta} \tag{2.59}$$

$$\left(\frac{d}{dx}+\frac{2}{x}\right) u_1 r\Omega\delta = 60\nu\frac{x}{\delta} \tag{2.60}$$

$$\left(\frac{d}{dx}+\frac{1}{x}\right) u_1\theta_w\delta\left[\eta^2/12 - \eta^3/15 + \eta^4/60\right] = \frac{2\nu}{P_r}\frac{\theta_w}{\Delta} \tag{2.61}$$

By setting

$$F(x) \equiv u_1\delta \quad , \quad G(x) \equiv \frac{\eta^2}{12} - \frac{\eta^3}{15} + \frac{\eta^4}{60} \quad , \quad H(x) \equiv \frac{1}{\delta^4}$$

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*The values of the integrals are

$$\int_0^\delta u^2 dY = \frac{u_1^2\delta}{105} \qquad \int_0^\delta u w \, dY = \frac{1}{30}u_1 r\Omega\delta \, , \quad \int_0^\delta w^2 dY = \frac{(r\Omega)^2\delta}{5}$$

$$\int_0^\Delta u\theta \, dY = u_1\theta_w\delta\left[\frac{1}{12}\eta^2 - \frac{1}{15}\eta^3 + \frac{1}{60}\eta^4\right]$$

where

$$\Delta = C_1 x^m \quad , \quad \delta = C_2 x^n \qquad \eta \equiv \frac{\Delta}{\delta}$$

equations (2.59)-(2.61) are restated as

$$\frac{dF}{dx} + \frac{3F}{x} = \frac{60\nu}{C_1} x^{-n} \tag{2.62}$$

$$\frac{dG}{dx} + (3-n)\frac{G}{x} = \frac{(4-n)C_1}{30 P_r C_2} x^{n-m-1} \tag{2.63}$$

$$\frac{dH}{dx} + \left[4 + \frac{7}{3}(4-n)\right]\frac{H}{x} = \frac{7(4-n)^2}{180\nu^2}\left(\frac{\Omega^2 \sin^2\phi}{5} + \frac{g\beta\cos\phi\,A\eta}{3}\right)\frac{1}{x} \tag{2.64}$$

The particular solutions required for our boundary value problem are

$$F(x) = \frac{60\nu}{4-n}\,\frac{x}{\delta} \tag{2.65}$$

$$G(x) = \left(\frac{4-n}{3-m}\right)\frac{1}{30 P_r}\,\frac{\delta}{\Delta} \tag{2.66}$$

$$H(x) = \frac{7(4-n)^2}{180\nu^2}\,\frac{3}{40-7n}\left(\frac{\Omega^2 \sin^2\phi}{5} + \frac{g\beta\cos\phi\,A\,\eta}{3}\right) \tag{2.67}$$

From (2.65) and the fact that $\delta = C_2 x^n$, there follows

$$u_1\delta = \frac{60\nu}{4-n}\;\frac{x}{\delta} \qquad (2.68)$$

from (2.66) we obtain

$$\eta \cdot G(x) = \left(\frac{4-n}{3-m}\right)\frac{1}{30\,Pr} \qquad (2.69)$$

implying m = n. From (2.67) and the definition of H, we obtain

$$\delta^4 = \frac{60\,\nu^2\,(40-7n)}{7\,(4-n)^2}\left(\frac{\Omega^2 \sin^2\phi}{5} + g\,\frac{\beta\cos\phi\,A}{3}\,\eta\right) \qquad (2.70)$$

i.e.,

$$\frac{x}{\delta} = 0.46478\left(\frac{Re_r^2}{5} + \frac{Gr_x}{3}\right)^{1/4}$$

We complete the development in the same manner as in Case II. Using the definition of $G \equiv \frac{1}{12}\eta^2 - \frac{1}{15}\eta^3 + \frac{1}{60}\eta^4$, (2.69) is rewritten as

$$Pr = \frac{1}{30}\left(\frac{4-n}{3-m}\right)\frac{1}{\eta \cdot G} \qquad (2.71)$$

and familiar arguments imply that m = n = 0. Since $0 < \eta \leq 1$, (2.71) gives a unique Pr value for each η within the specified bounds; and equation (2.70) gives then a compatible value for δ with

Re_r , Gr_x as parameters. For each such value of δ , (2.68) yields the compatible value of $u_1(x)$.

The appropriate Nusselt number formula is

$$\frac{Nu_x}{\sqrt{Re_r}} = \frac{0.92956}{\eta}\left(0.2 + \frac{Gr_x}{Re_r^2}\,\frac{\eta}{3}\right)^{1/4} \qquad (2.72)$$

3. RESULTS

The results of main interest are the following:

a) variation of the boundary layer thickness ratios ξ, η with Prandtl number

and

b) local heat transfer.

We note immediately that while ξ, η depend (for any chosen polynomial flow field representations) on Prandtl number alone, local heat transfer depends, in general, explicitly on all three parameters, Gr_x, Re_r and Pr. Of course, our remark in reference to a) by no means implies that δ itself is solely a function of the Prandtl number; obviously it also depends, in general, on all three parameters. Moreover, it is clear that b) depends directly on a).

Quantitative evaluations of a) and b) are based respectively on formulae (2.25), (2.50), (2.71), and (2.31), (2.52), (2.72). Results depending on the former are shown in Figures II, III, and IV. In all cases we note a strong influence of the Prandtl number on the boundary layer thickness ratios. Figures II and III show ξ vs. Pr for the low Prandtl number analysis with $Gr_x \equiv 0$ and $Gr_x > 0$. Figure IV presents the Prandtl number dependence of the boundary layer thickness ratio when the Prandtl number is high.

It is clear from the graphs that what is meant by "high" Prandtl number is specified by the inequality $Pr \geq 1.333$. Correspondingly, a "low" Prandtl number is one satisfying $Pr \leq 1.333$. The particular value 1.333 is compatible with the particular profiles we have used. When these are changed, so is the value of the Prandtl modulus specifying the subdivision into the high and low Prandtl number regions.

Heat transfer results in the form of $Nu_x / \sqrt{Re_r}$ are summarized in Table 1 for the range $0.1 \leq Pr \leq 100$ and $0 \leq \frac{Gr_x}{Re_r^2} \leq 100$. Percentage errors based on a comparison of our results with the exact (numerical) solutions of Hering-Grosh (Reference 11) and Hayday (unpublished) are tabulated in column 5. The agreement is very satisfactory; that it is mainly due to our use of the two-boundary layer concept is strongly implied by the values in column 6 which show the results of an independent integral technique study but with $\delta = \Delta$. (It was necessary to perform these calculations because the problem has not been discussed elsewhere from the point of view of the integral method.) Note in particular that for

Pr = 10 our results, with the exception* of $Gr_x = 0.1$, are within 3% from the exact solutions, whereas when $\delta = \Delta$ the corresponding values deviate by at least 34%. All of the results given in Table 1 are shown in Figures V to XIII, which we think are self-explanatory.

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*This is not the only exception, the same being true, for example, when $Pr = Gr_x = 0.1$. Furthermore, observe that the larger deviations are in the small range $0 < Gr_x \leq 1$ but that at $Gr_x = 0$ the results are quite accurate. During the final stages of this work, a plausible explanation for this peculiar behavior of the solutions has been suggested to the writer by Professor Hayday. He conjectures that the relatively large errors in the small Grashof number range are primarily the result of not exploiting the "multi-layer" concept to the fullest extent and partly the result of the choice of the profiles, the latter not satisfying all of the conditions implied by the partial differential equations. Strong evidence supporting Professor Hayday's viewpoint is the following: The multilayer concept asserts, in harmony with the general boundary layer theory and known exact solutions, that different generating mechanisms for a particular flow characteristic within the boundary layer manifest themselves in correspondingly different regions of their dominance. In reference to our problem, this implies that there should, in principle, exist two velocity boundary layers, say δ_u, δ_w, the first describing the region of the u-variation and the second the region of the w-variation. The reason for this is that the u-field may be caused either by the buoyancy force alone or may be induced solely by the spin of the cone. When both act simultaneously, δ_u and δ_w (being functions of Gr_r/Re_r^2) assess then the relative importance of one generating mechanism in comparison to the other. A preliminary exploration of the validity of these statements is now well under way and the available results not only fully support them but also show further significant improvements on both heat transfer and skin friction data. In fact, it turns out that $\delta_w \approx \delta_u$, irrespective of the Grashof number, but with δ_w replacing u_1 as a new, and obviously more significant, dependent variable, the entire equation system changes and so do the results. The reason that our analysis is quite good for $Gr_x = 0$, $Pr \ll 1$ is due to the fact that it is entirely separate from the more general case $Gr_x \neq 0$. Were we to take the limiting values of the latter as $Gr_x \to 0$, the results would show the <u>largest</u> deviations but, nonetheless, smaller than in the $\delta = \Delta$ calculation. Of course, in the more general multilayer treatment it is not necessary to consider $Gr_x = 0$, $Gr_x \neq 0$, separately.

Table 1

		Nu_x / Re_r			% Error	
Pr	Gr_x / Re_r^2	Exact Solution	$\delta \neq \Delta$ Integral	$\delta = \Delta$ Method	$\delta \neq \Delta$	$\delta = \Delta$
	0	.111009	.11189	.21382	.794	9.26
	0.1	.17864	.19945	.2222	11.65	24.4
	1.0	.28025	.2604	.27324	7.08	2.5
	5.0	.41012	.38332	.37373	6.53	8.88
0.1	10.0	.48766	.4550	.43831	6.7	10.12
	20.0	.57949	.54052	.51758	6.72	10.68
	50.0	.72590	.66926	.64796	7.8	10.74
	100.0	.86764	.8076		6.92	
	0	.4295	.4351	.4844	1.3	12.8
	0.1	.46183	.4590	.50343	.61	9.0
	1.0	.61175	.5900	.61901	3.56	1.19
	5.0	.86242	.8250	.84667	4.3	1.83
0.7	10.0	1.01687	.9711	.99310	4.5	2.34
	20.0	1.20397	1.1490	1.1725	2.6	2.6
	50.0	1.5099	1.4402	1.4679	4.6	2.8
	100.0	1.7940	1.71087	1.7422	4.6	2.9
	0	.51816	.5315	.51167	2.5	1.2
	0.1	.5467	.5556	.53177	1.63	2.7
	1.0	.7000	.6965	.6539	.5	6.6
	5.0	.9755	.9624	.8944	1.34	8.3
1.0	10.0	1.1480	1.1309	1.0490	1.5	8.6
	20.0	1.3579	1.3366	1.2386	1.6	8.8
	50.0	1.7019	1.6755	1.5506	1.6	8.9
	100.0	2.0217	1.9886	1.8403	1.64	9.0
	0	1.4080	1.4128	.74187	.34	47.3
	0.1	1.4323	1.5823	.7710	10.4	46.2
	1.0		1.6211	.9481		
	5.0		2.0766	1.2967		
10.0	10.0	2.3533	2.4005	1.523	2.0	54.5
	20.0	2.7697	2.8110	1.7827	2.23	35.2
	50.0	3.4185	3.5002	2.2482	2.4	34.2
	100.0	4.0494	4.1486	2.6696	2.45	34.1
	0		3.4535	1.336		
	0.1		3.4791	1.3885		
	1.0	no exact	3.6876	1.7073		
	5.0	solutions	4.3426	2.3352		
100.0	10.0	available	4.88402	2.7423		
	20.0		5.6174	3.2103		
	50.0		6.9071	4.0486		
	100.0		8.1490	4.8075		

CONCLUSIONS

The specific results presented in the thesis strongly suggest that the consistent use of the multilayer concept is the dominant factor determining the accuracy of the integral method. While this concept, in itself, is not entirely new, its consistent use (particularly for combined convection problems) seems to appear here for the first time. A full exploitation of this idea is, of course, not given here and remains for the future.

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APPENDIX A

Remarks on the Navier-Stokes equation and a brief derivation of boundary layer equations:

The fundamental equations governing flows of viscous fluids--the continuity equation and the Navier-Stokes equations--may be expressed in vector form as

$$\frac{\partial \overline{V}}{\partial t} + \operatorname{div}\,(\rho \overline{V}) = 0, \qquad \text{(A.1)}$$

and

$$\frac{\partial \overline{V}}{\partial t} - \overline{V} \times \operatorname{curl} \overline{V} + \operatorname{grad}\,\left(\frac{1}{2}\overline{V}^2\right) = F - \frac{1}{\rho}\operatorname{grad} P - \nu \operatorname{curl}\operatorname{curl} \overline{V}. \qquad \text{(A.2)}$$

For steady, incompressible flow, $\frac{\partial}{\partial t} = 0$ and is constant. Hence, in particular $\frac{\partial \overline{V}}{\partial t} = 0$, $\operatorname{div}\,(\overline{V}) = 0$; the latter implies that

$$\operatorname{curl}\operatorname{curl} \overline{V} = \operatorname{grad}\operatorname{div}\,(\overline{V}) - \nabla^2 \overline{V} = \nabla^2 \overline{V}.$$

Thus, for steady incompressible flow, the system (A.1), (A.2) simplifies to

$$\operatorname{div}\,(\overline{V}) = 0. \qquad \text{(A.3)}$$

and

$$-\overline{V} \times \operatorname{curl} \overline{V} + \frac{1}{2}\operatorname{grad} \overline{V}^2 = F - \frac{1}{\rho}\operatorname{grad} P + \nabla^2 \overline{V}. \qquad \text{(A.4)}$$

We shall use now a general orthogonal coordinate system. Let α, β, γ denote the coordinate curves; the corresponding increments of length are then $h_1 d\alpha$, $h_2 d\beta$, and $h_3 d\gamma$, and the increment of length squared for any curve S is

$$dS^2 = h_1^2 (d\alpha)^2 + h_2^2 (d\beta)^2 + h_3^2 (d\gamma)^2.$$

Let $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ be the unit tangent vectors along the α, β, γ curves and let u, v, w stand for the corresponding components of the velocity vector $\bar{V}$. Then

$$\operatorname{div} \bar{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial\alpha}(h_2 h_3 u) + \frac{\partial}{\partial\beta}(h_3 h_1 v) + \frac{\partial}{\partial\gamma}(h_1 h_2 w) \right], \tag{A.5}$$

$$\nabla^2 \bar{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial\alpha}\left(\frac{h_2 h_3}{h_1} \frac{\partial \bar{V}}{\partial\alpha} \right) + \frac{\partial}{\partial\beta}\left(\frac{h_3 h_1}{h_2} \frac{\partial \bar{V}}{\partial\beta} \right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1 h_2}{h_3} \frac{\partial \bar{V}}{\partial\gamma} \right) \right], \tag{A.6}$$

$$\operatorname{grad} P = \frac{\bar{e}_1}{h_1}\frac{\partial P}{\partial\alpha} + \frac{\bar{e}_2}{h_2}\frac{\partial P}{\partial\beta} + \frac{\bar{e}_3}{h_3}\frac{\partial P}{\partial\gamma}, \tag{A.7}$$

$$|\bar{V}|^2 = u^2 + v^2 + w^2, \tag{A.8}$$

and

$$\operatorname{curl} \bar{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & \frac{\partial}{\partial\gamma} \\ h_1 u & h_2 v & h_3 w \end{vmatrix}$$

$$= L\bar{e}_1 + M\bar{e}_2 + N\bar{e}_3 \tag{A.9}$$

where

$$L = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial\beta}(h_3w) - \frac{\partial}{\partial\gamma}(h_2v)\right]$$

$$M = \frac{1}{h_3h_1}\left[\frac{\partial}{\partial\gamma}(h_1u) - \frac{\partial}{\partial\alpha}(h_3w)\right]$$

$$N = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial\alpha}(h_2v) - \frac{\partial}{\partial\beta}(h_1u)\right]$$

Moreover,

$$\overline{V} \times \text{curl } \overline{V} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u & v & w \\ L & M & N \end{vmatrix}$$

$$= \overline{e}_1 (vN - Mw) + \overline{e}_2 (Lw - Nu) + \overline{e}_3 (mu - Lv).$$

$$= A\overline{e}_1 + B\overline{e}_2 + C\overline{e}_3 , \qquad (A.10)$$

where

$$A = Nv - Mw ,$$

$$B = Lw - Nu ,$$

$$C = Mu - Lv .$$

Equation (A.3) is now expressed as

$$\frac{\partial}{\partial\alpha}(h_2h_3u) + \frac{\partial}{\partial\beta}(h_3h_1v) + \frac{\partial}{\partial\gamma}(h_1h_2w) = 0. \qquad (A.11)$$

Substituting (A.6)-(A.10) into (A.4), we obtain

$$- (A e_1 + B e_2 + C e_3) + \frac{1}{2}\left[\frac{\bar{e}_1}{h_1}\frac{\partial(\bar{V}^2)}{\partial\alpha} + \frac{\bar{e}_2}{h_2}\frac{\partial(\bar{V}^2)}{\partial\beta} + \frac{\bar{e}_3}{h_3}\frac{\partial(\bar{V}^2)}{\partial\gamma}\right]$$

$$= \bar{F} - \frac{1}{\rho}\left[\frac{e_1}{h_1}\frac{\partial P}{\partial\alpha} + \frac{e_2}{h_2}\frac{\partial P}{\partial\beta} + \frac{e_3}{h_3}\frac{\partial P}{\partial\gamma}\right]$$

$$+ \frac{\nu}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2 h_3}{h_1}\frac{\partial\bar{V}}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_3 h_1}{h_2}\frac{\partial\bar{V}}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1 h_2}{h_3}\frac{\partial\bar{V}}{\partial\gamma}\right)\right]$$

and the corresponding component forms are

$$- A + \frac{1}{2}\frac{1}{h_1}\frac{\partial\bar{V}^2}{\partial\alpha} = F_\alpha - \frac{1}{\rho}\frac{1}{h_1}\frac{\partial P}{\partial\alpha}$$

$$+ \frac{\nu}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_3 h_1}{h_2}\frac{\partial u}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial\gamma}\right)\right], \quad \text{(A.12)}$$

$$- B + \frac{1}{2}\frac{1}{h_2}\frac{\partial\bar{V}^2}{\partial\beta} = F_\beta - \frac{1}{\rho}\frac{1}{h_2}\frac{\partial P}{\partial\beta}$$

$$+ \frac{\nu}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2 h_3}{h_1}\frac{\partial v}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_3 h_1}{h_2}\frac{\partial v}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1 h_2}{h_3}\frac{\partial v}{\partial\gamma}\right)\right], \quad \text{(A.13)}$$

and

$$- C + \frac{1}{2}\frac{1}{h_3}\frac{\partial\bar{V}^2}{\partial\gamma} = F_\gamma - \frac{1}{\rho}\frac{1}{h_3}\frac{\partial P}{\partial\gamma}$$

$$+ \frac{\nu}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\alpha}\left(\frac{h_2 h_3}{h_1}\frac{\partial w}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_3 h_1}{h_2}\frac{\partial w}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_1 h_2}{h_3}\frac{\partial w}{\partial\gamma}\right)\right], \quad \text{(A.14)}$$

where the quantities A, B, C have been defined previously.

For flows generated by spinning bodies of revolution, it is very convenient to use a body oriented coordinate system. Thus, we set $\alpha = x$, $\beta = y$ and $\gamma = z$, where x is measured along meridian curves, y along local normals, and z in the circumferential direction. Clearly, $h_1 = h_2 = 1$ and $h_3 = r(x)$ where $r(x)$ is the local radius of the body of revolution*. The particular forms of equation (A.12)-(A.14), with $\frac{\partial}{\partial z} = 0$, are then the following:

$$\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial y} = 0 , \tag{A.15}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + w \frac{\partial w}{\partial x} - v \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w}{r} \frac{\partial (rw)}{\partial x}$$

$$= F_x - \frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\nu}{r} \left[\frac{\partial}{\partial x} \left(r \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y} \left(r \frac{\partial u}{\partial y}\right) \right] , \tag{A.16}$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial y} + u \frac{\partial v}{\partial x} - u \frac{\partial u}{\partial y} - \frac{w}{r} \frac{\partial (rw)}{\partial y}$$

$$= F_y - \frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\nu}{r} \left[\frac{\partial}{\partial x} \left(r \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y} \left(r \frac{\partial v}{\partial y}\right) \right] , \tag{A.17}$$

and

$$\frac{1}{r} \left[u \frac{\partial (rw)}{\partial x} + v \frac{\partial (rw)}{\partial y} \right] = \frac{\nu}{r} \left[\frac{\partial}{\partial x} \left(r \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y} \left(r \frac{\partial w}{\partial y}\right) \right] . \tag{A.18}$$

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*See S. Goldstein, Modern Developments in Fluid Dynamics, Vol. 1, pp 114, Clarendon Press, Oxford, (1938).

Further obvious simplifications yield the corresponding set

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \, , \tag{A.19}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r}\frac{dr}{dx} = F_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{r}\frac{dr}{dx}\frac{\partial u}{\partial x}\right], \tag{A.20}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = F_y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{r}\frac{dr}{dx}\frac{\partial v}{\partial x}\right], \tag{A.21}$$

and

$$u\frac{\partial(rw)}{\partial x} + v\frac{\partial(rw)}{\partial y} = \nu\left[\frac{\partial}{\partial x}\left(r\frac{\partial w}{\partial x}\right) + \frac{\partial^2(rw)}{\partial y^2}\right]. \tag{A.22}$$

Equations (A.19)-(A.22), once simplified on the basis of boundary layer hypothesis, are directly applicable to the thesis problem. We discuss now such simplifications. The arguments are standard.

Let x be a standard for length and u or w be standard for velocities; that is, $x \sim O(1)$ and $u,\ w \sim O(1)$. The central assumption is that $y \sim O(\delta)$ where δ is "small" in the sense that $\delta \ll 1$ physically this means that the viscosity of the fluid is small or more precisely that the Reynolds number is large.

Consider now equation (A.19) together with the above order of magnitude simplifications:

$$\underset{1}{\frac{\overset{1}{\partial(ru)}}{\partial x}} + \underset{\delta}{\frac{\partial(rv)}{\partial y}} = 0 \, . \tag{A.23}$$

The first term is obviously on the order of one and, therefore, rv is on the order of δ, but since $r(x) \sim O(1)$, $v \sim O(\delta)$; therefore, the continuity equation remains unchanged. The same arguments applied to (A.20) give now

$$u \underset{1}{\overset{1}{\frac{\partial u}{\partial x}}} + v \frac{\partial u}{\partial y} - \frac{w^2}{r} \underset{1}{\overset{1}{\frac{dr}{dx}}} = F_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\underset{1}{\overset{1}{\frac{\partial^2 u}{\partial x^2}}} + \underset{\delta^2}{\overset{1}{\frac{\partial^2 u}{\partial y^2}}} + \underset{1}{\overset{1}{\frac{1}{r}\frac{dr}{dx}\frac{\partial u}{\partial x}}}\right]$$

clearly, if the inertia terms, viscous forces and pressure forces are to be of equal importance, $\nu \sim O(\delta^2)$. Thus, boundary layer form of (A.20) is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r}\frac{dr}{dx} = F_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial^2 u}{\partial y^2}\nu \quad \text{(A.24)}$$

Of course, the pressure P, the density ρ, and the x component of the body force are all of order one.

The same procedure applied to (A.21) shows that this equation may be neglected and

$$\frac{\partial P}{\partial y} \sim F_y \ .$$

The boundary layer form of equation (A.22) may be shown to be

$$u\frac{\partial (rw)}{\partial x} + v\frac{\partial (rw)}{\partial y} = \frac{\nu \partial^2 (rw)}{\partial y^2} \ . \quad \text{(A.25)}$$

The boundary layer form of the energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{K}{\rho C_P}\frac{\partial^2 T}{\partial y^2} \ . \quad \text{(A.26)}$$

APPENDIX B

Integration of the boundary layer equations--"the integral equations."

Herein we develop the integral forms of the conservation equations corresponding to the system (2.1), (2.3), (2.4) and (2.6). The procedure is straightforward. Basically, it amounts to a formal integration of the equations of motion and energy with respect to y, the upper limit being ℓ where ℓ is greater than either the velocity boundary layer thickness δ or the thermal boundary layer thickness Δ. Implicit use is made of the continuity equation and the boundary conditions imposed on the partial differential equations.

We integrate (2.6) and obtain

$$\int_0^{\ell} \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{w^2}{r} \frac{dr}{dx}\right) dy = \int_0^{\ell} \left(\nu \frac{\partial^2 u}{\partial y^2} + g \beta \cos \emptyset \, \theta\right) dy. \quad (B.1)$$

An obvious development of the left hand side of (2.1) yields

$$\int_0^{\ell} u \frac{\partial u}{\partial x} dy + uv \Big|_0^{\ell} - \int_0^{\ell} u \frac{\partial v}{\partial y} dy - \int_0 \frac{w^2}{r} \frac{dr}{dx} dy.$$

The boundary conditions u = v = 0 at y = 0 and δ implies that the second term in the above expression is identically zero. Moreover, the continuity equation gives

$$\frac{\partial v}{\partial y} = - \frac{1}{r} \frac{\partial (ru)}{\partial x}$$

where $r(x) = x \sin \phi$. Using now this information, in connection with the expression following (B.1), we obtain the final rearrangement of the left hand side of (B.1),

$$\left(\frac{\partial}{\partial x} + \frac{1}{x}\right) \int_0^{\ell} u^2 \, dy - \frac{1}{x} \int_0^{\ell} w^2 \, dy .$$

The final expression for the integrated x-component of the momentum equation is thus,

$$\left(\frac{\partial}{\partial x} + \frac{1}{x}\right) \int_0^{\ell} u^2 \, dy - \frac{1}{x} \int_0^{\ell} w^2 \, dy = g \beta \cos \phi \int_0^{\ell} \theta \, dy - \nu \frac{\partial u}{\partial y} \Big|_{y=0}^{\ell} \qquad (B.2)$$

where the last term on the right hand side has been obtained by adopting a condition of smoothness for the velocity profile, that is

$$\frac{\partial u}{\partial y}\Big|_{y=\ell} = \frac{\partial u}{\partial y}\Big|_{y=\delta} = 0.$$

The same sort of arguments applied to equation (2.3) and (2.4) yield

$$\left(\frac{\partial}{\partial x} + \frac{2}{x}\right) \int_0^{\ell} u \, w \, dy = - \nu \frac{\partial w}{\partial y}\Big|_{y=0}, \qquad (B.3)$$

$$\left(\frac{\partial}{x} + \frac{1}{x}\right) \int_0^{\ell} u \, \theta \, dy = - \alpha \frac{\partial \theta}{\partial y}\Big|_{y=0}. \qquad (B.4)$$

In actual computations the assumed polynomial representations for the u, w, and θ fields imply that we may without loss of generality replace ℓ, appropriately with δ and Δ as discussed in the thesis.

APPENDIX C

Dimensional Analysis

Herein we use dimensional analysis to deduce the significant dimensionless parameters arising in combined convection problems. Our aim, in particular, is to deduce independently from the previous discussion the overall form of the Nusselt number formulae. The procedure, commonly attributed to Rayleigh, is very well known; and for this reason, the treatment is brief. The book by Jacob may serve as a standard reference.

For pure forced convection, dimensional analysis shows that

$$Nu = f(Pr, Re); \tag{C.1}$$

whereas, for pure free convection, one finds that

$$Nu = f(Pr, Gr). \tag{C.2}$$

To find analogous formulae for combined convection, we assume that the heat transfer coefficient depends on the physical properties of the fluid, the temperature and velocity. Hence we may state

$$h = C\, w^a\, L^b\, \mu^f\, k^j\, \rho^m\, C_p^n\, (g\beta)^p\, \theta_w^s \tag{C.3}$$

where

h	,	heat transfer coefficient
C	,	dimensionless constant
L	,	characteristic length of the body
w	,	circumferential velocity,

and a, b, f, j, m, n, p, s are exponents to be determined; the basic units are taken as

H	,	for heat energy*
T	,	for time
L	,	for length
M	,	for mass
θ	,	for temperature.

Expressed in terms of the latter, (3.3) takes the form

$$HT^{-1}L^{-2}\theta^{-1} = (T^{-1}L)^{a}L^{b}(MT^{-1}L^{-1})^{f}(HT^{-1}L^{-1}\theta^{-1})^{j}$$

$$(ML^{-3})^{m}(HM^{-1}\theta^{-1})^{n}(\theta^{-1}LT^{-2})^{p}\theta^{s}$$

Now, in any such equation, physical arguments dictate that the exponents on the basic units be the same on both sides. Collecting the exponents associated respectively with H, T, L, θ, and M, we obtain therefore

$$\begin{aligned} j + n &= 1 \\ -a - f - j - 2p &= -1 \\ a + b - f - j - 3m + p &= -2 \\ -j - n - p + s &= -1 \\ f + m - n &= 0. \end{aligned} \qquad (C.5)$$

Clearly, the number of exponents exceeds the number of equations by three. Since there are eight exponents and five

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*While H may be expressed in terms of the basic units of mass, length, time, and temperature, we follow Max Jacob and treat it as an independent quantity.

equations, we may express any five of the exponents in terms of the remaining three. Choosing a, b, s, j, f, we obtain

$$\begin{aligned} a &= m - 2p \\ b &= m + p - 1 \\ s &= p \\ j &= 1 - n \\ f &= n - m \end{aligned} \tag{C.6}$$

Together, (C.6) and (C.3) yield

$$\frac{hL}{K} = C\left(\frac{\mu C_p}{k}\right)^n \left(\frac{WL\rho}{\mu}\right)^m \left(\frac{g\beta\theta_w L^3}{W^2 L^2}\right)^p \tag{C.7}$$

where m, n and p remain undetermined. Their values may be obtained on the basis of an experiment. Defining now

$$Nu = \frac{hL}{K}$$

$$Re = \frac{WL}{\nu}$$

and

$$Gr = \frac{g\beta\theta_w L^3}{\nu^2}$$

equation (C.7) implies the relationship

$$Nu = C(Pr)^n (Re)^m \left(\frac{Gr}{Re^2}\right)^p. \tag{C.8}$$

This is the required result. Observe that the Grashof number and Reynolds number appear as a ratio that may be used to assess the relative importance of free and forced convection when both act simultaneously. An alternate form of (C.8) is

$$\frac{Nu}{Re^m} = f\left(Pr, \frac{Gr}{Re^2}\right). \tag{C.9}$$

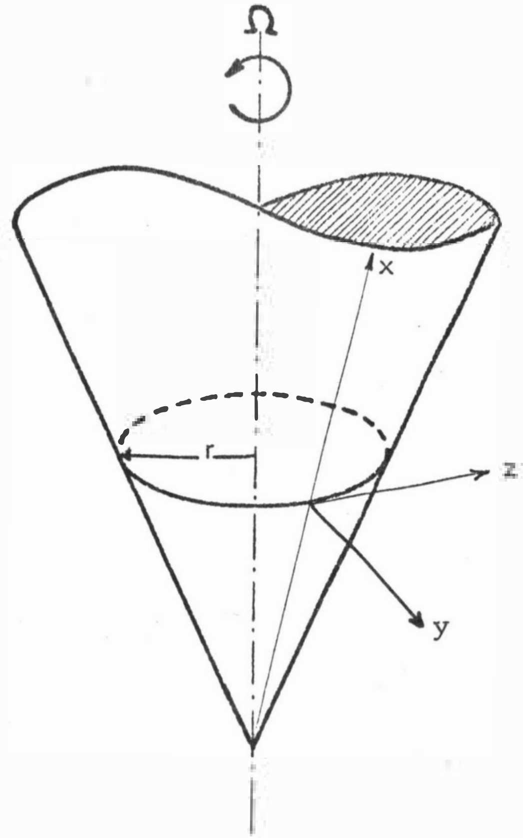
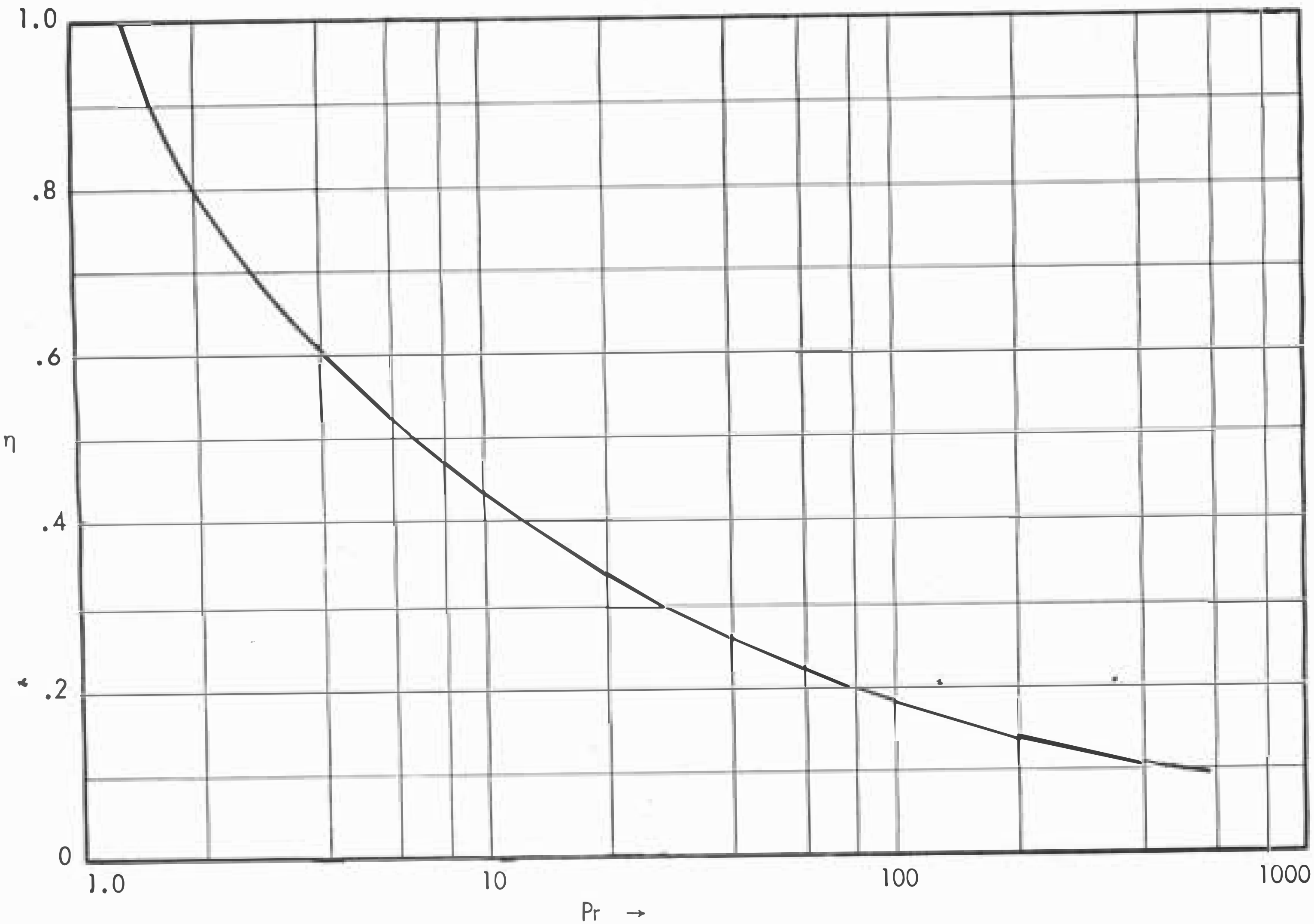
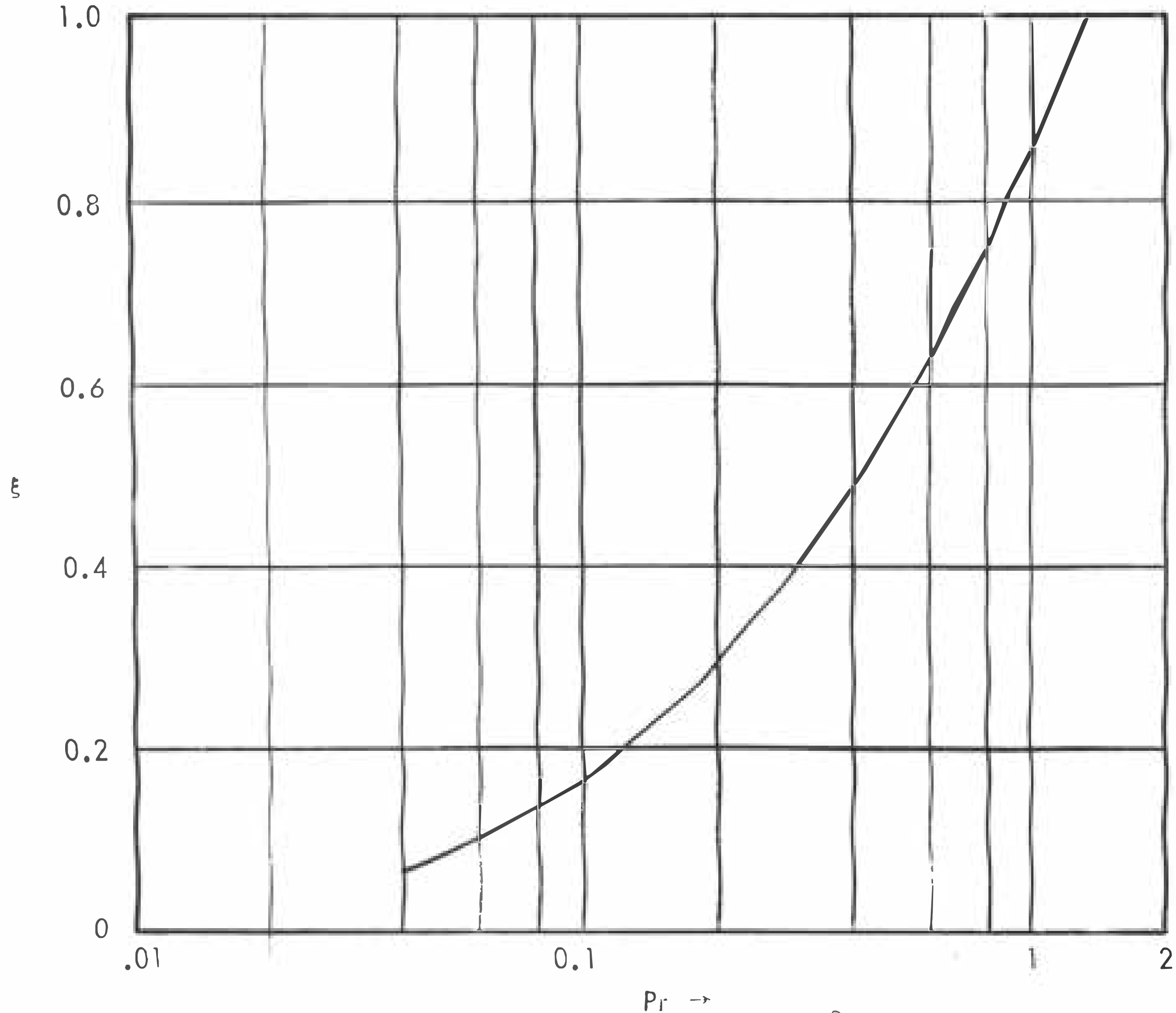


Figure 1. Coordinate System.



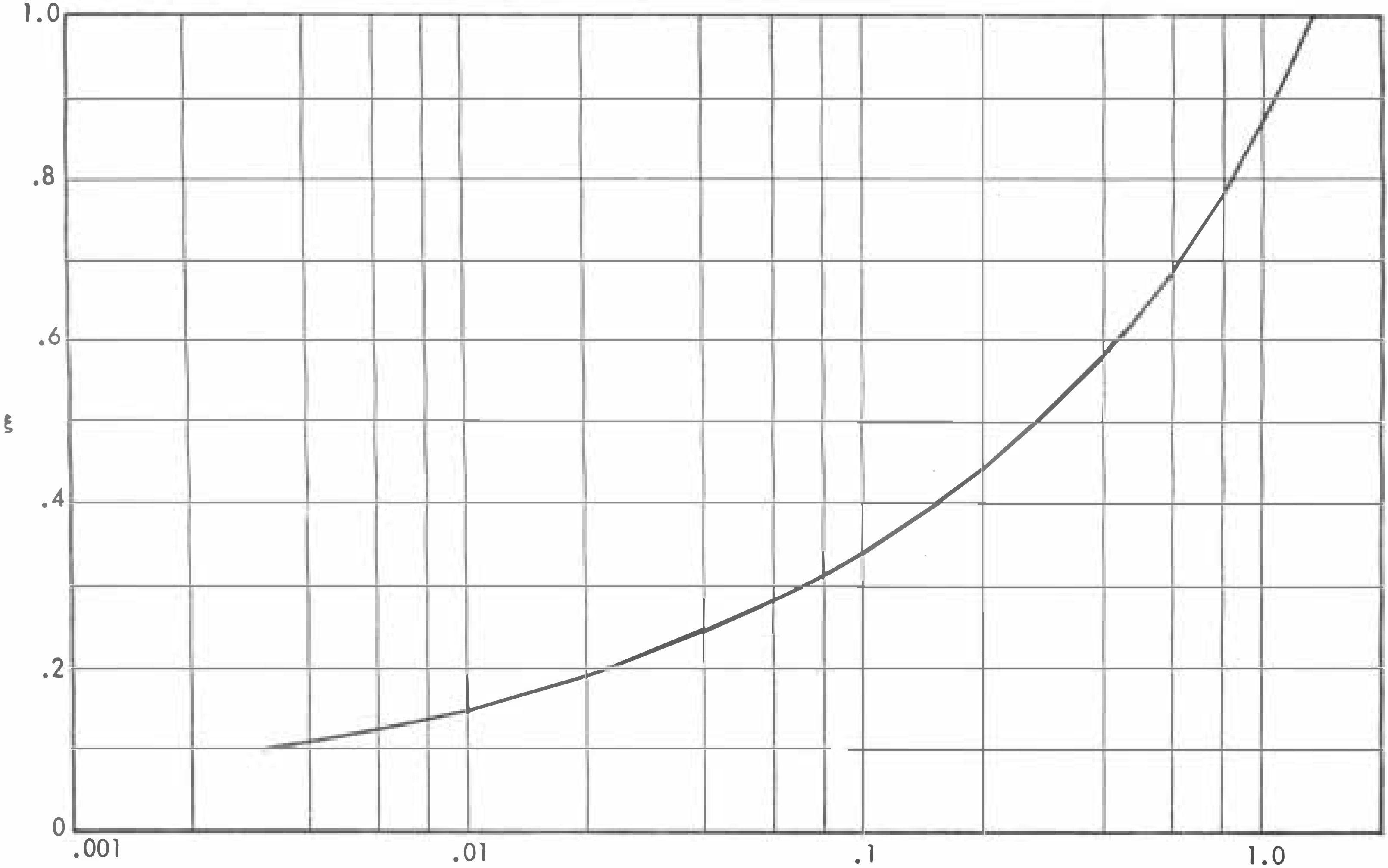
High Prandtl numbers $\eta = \frac{\Delta}{\delta}$

Figure 11. Variation of Boundary Layer Thickness Ratio with High Prandtl Numbers.



Low Prandtl numbers $\xi = \frac{\delta}{\Delta}$

Figure III. Variation of Layer Thickness Ratio with Low Prandtl Numbers. $Gr_x = 0$.



Low prandtl numbers $\xi = \frac{\delta}{\Delta}$

Figure IV. Variation of Layer Thickness Ratio with Low Prandtl Numbers. $Gr_x > 0$.

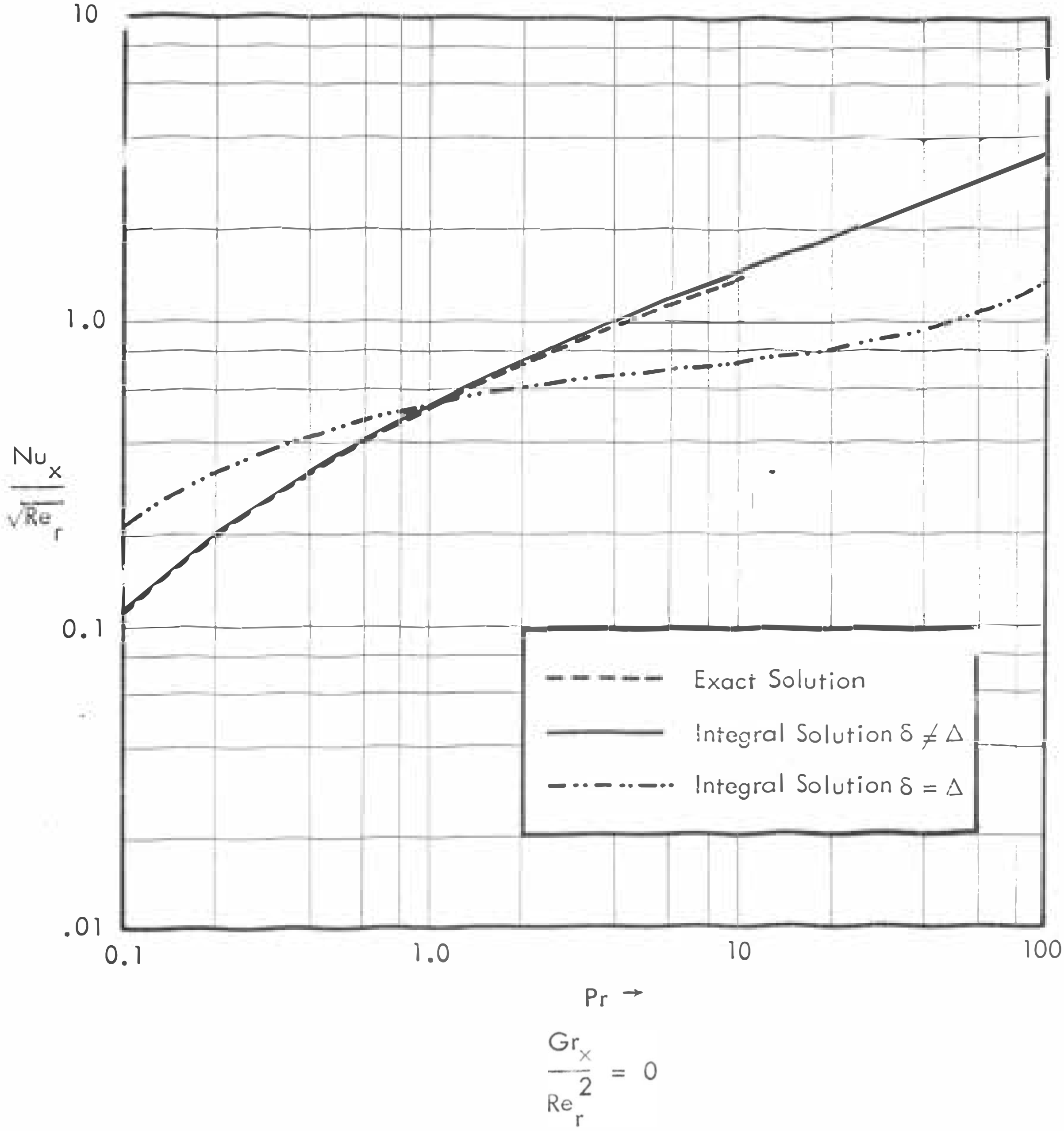
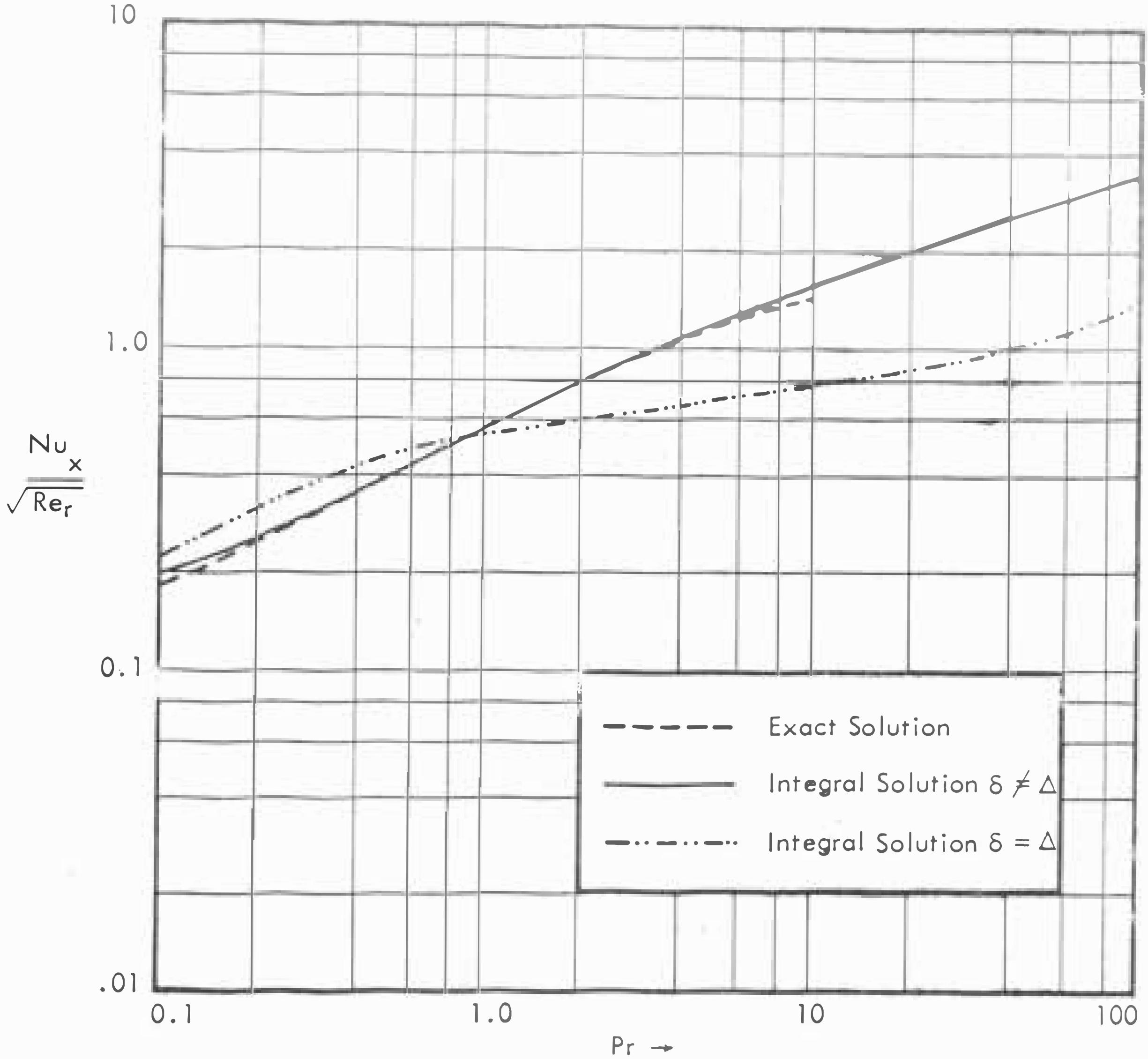


Figure V. Variation of Local $\frac{Nu_x}{\sqrt{Re_r}}$ with Prandtl Numbers for $Gr_x/Re_r^2 = 0$.



$$\frac{Gr_x}{Re_r^2} = .1$$

Figure VI. Variation of Local $Nu_x / \sqrt{Re_r}$ with Prandtl Numbers.

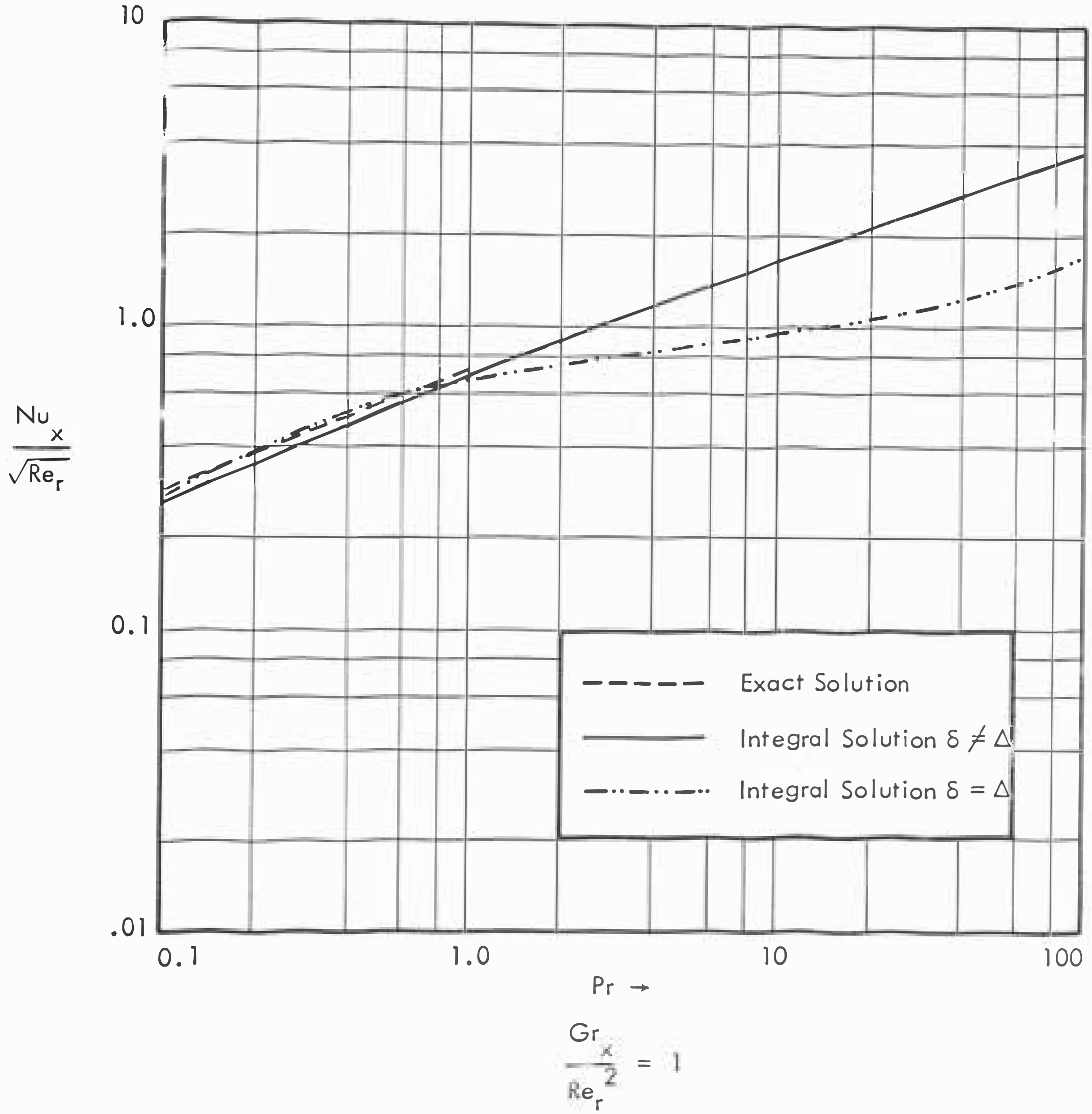


Figure VII. Variation of Local $Nu_x / \sqrt{Re_r}$ with Prandtl Numbers.

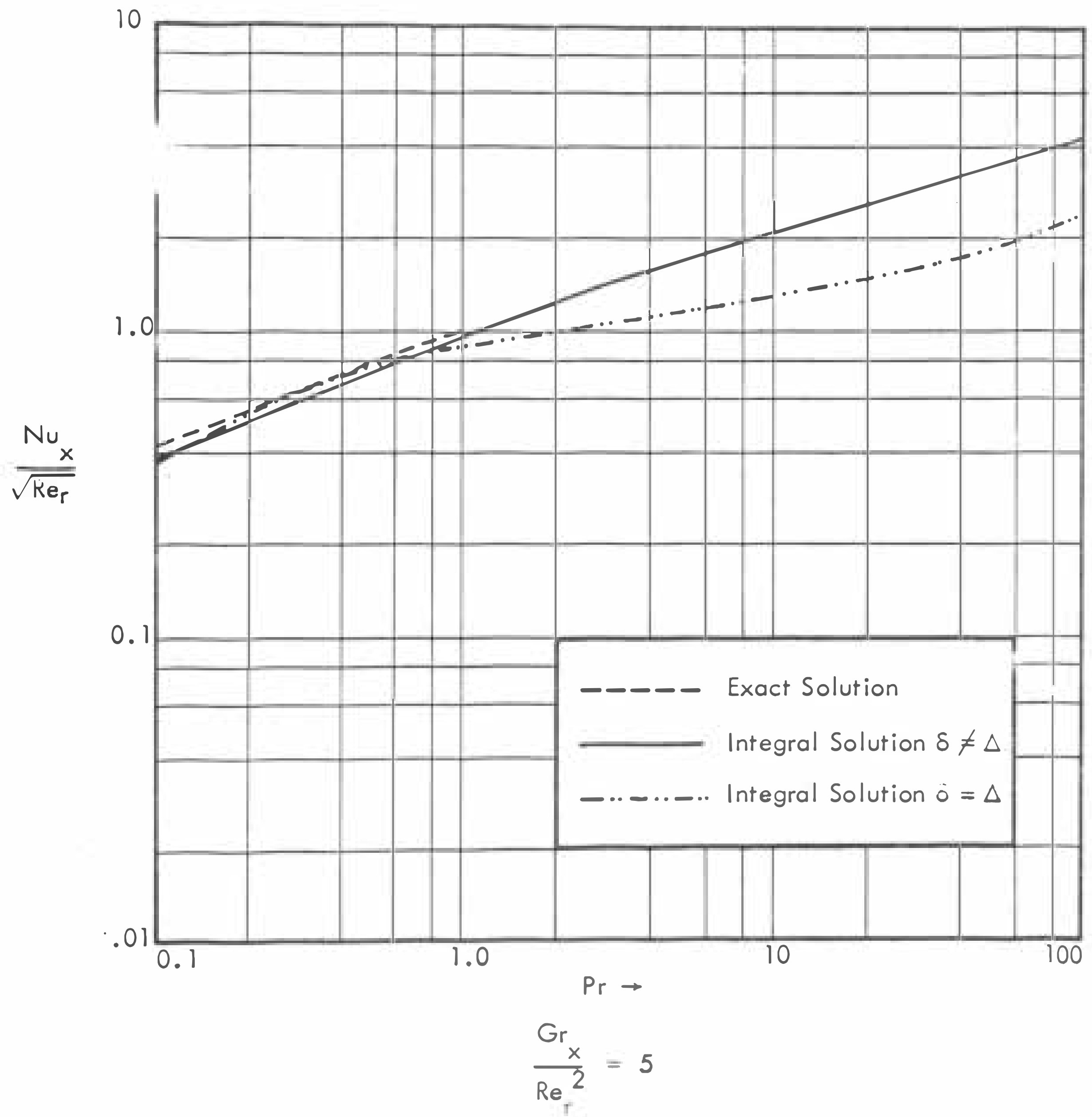
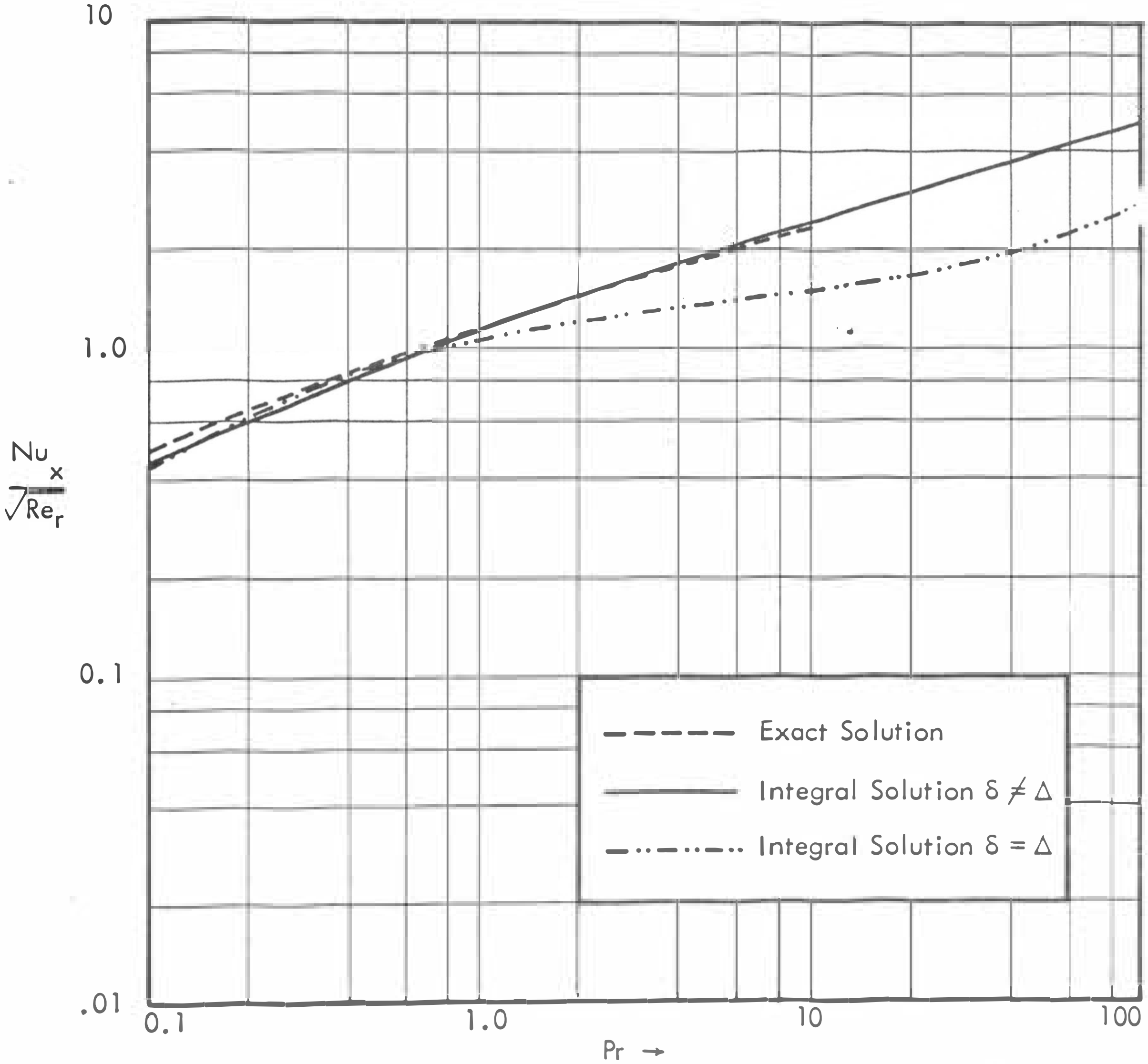


Figure VIII. Variation of Local $Nu_x / \sqrt{Re_r}$ with Prandtl Number.



$$\frac{Gr_x}{Re_r^2} = 10$$

Figure IX. Variation of Local $Nu_x / \sqrt{Re_r}$ with Prandtl Numbers.

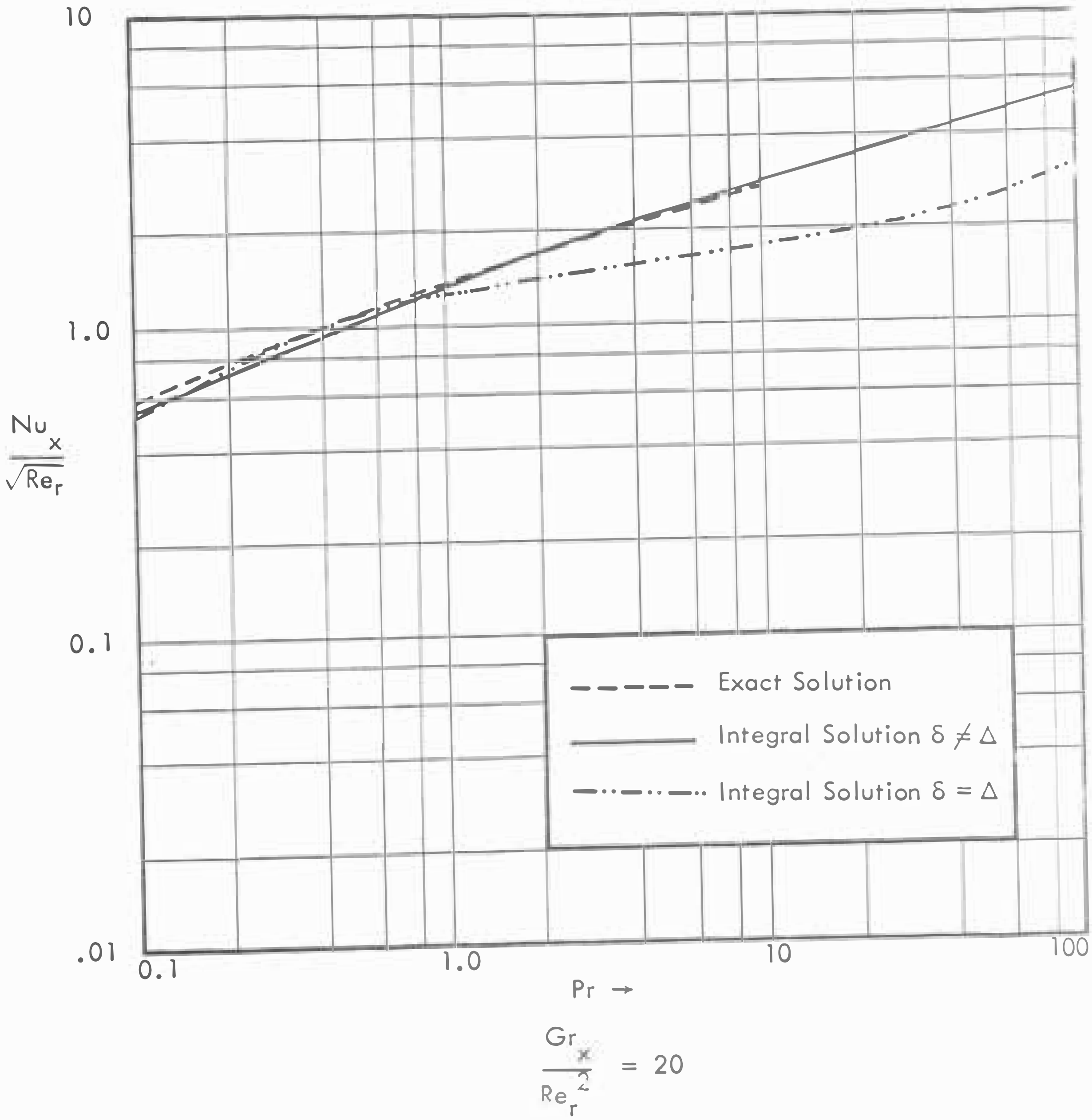


Figure X. Variation of Local $Nu_x/\sqrt{Re_r}$ with Prandtl Numbers.

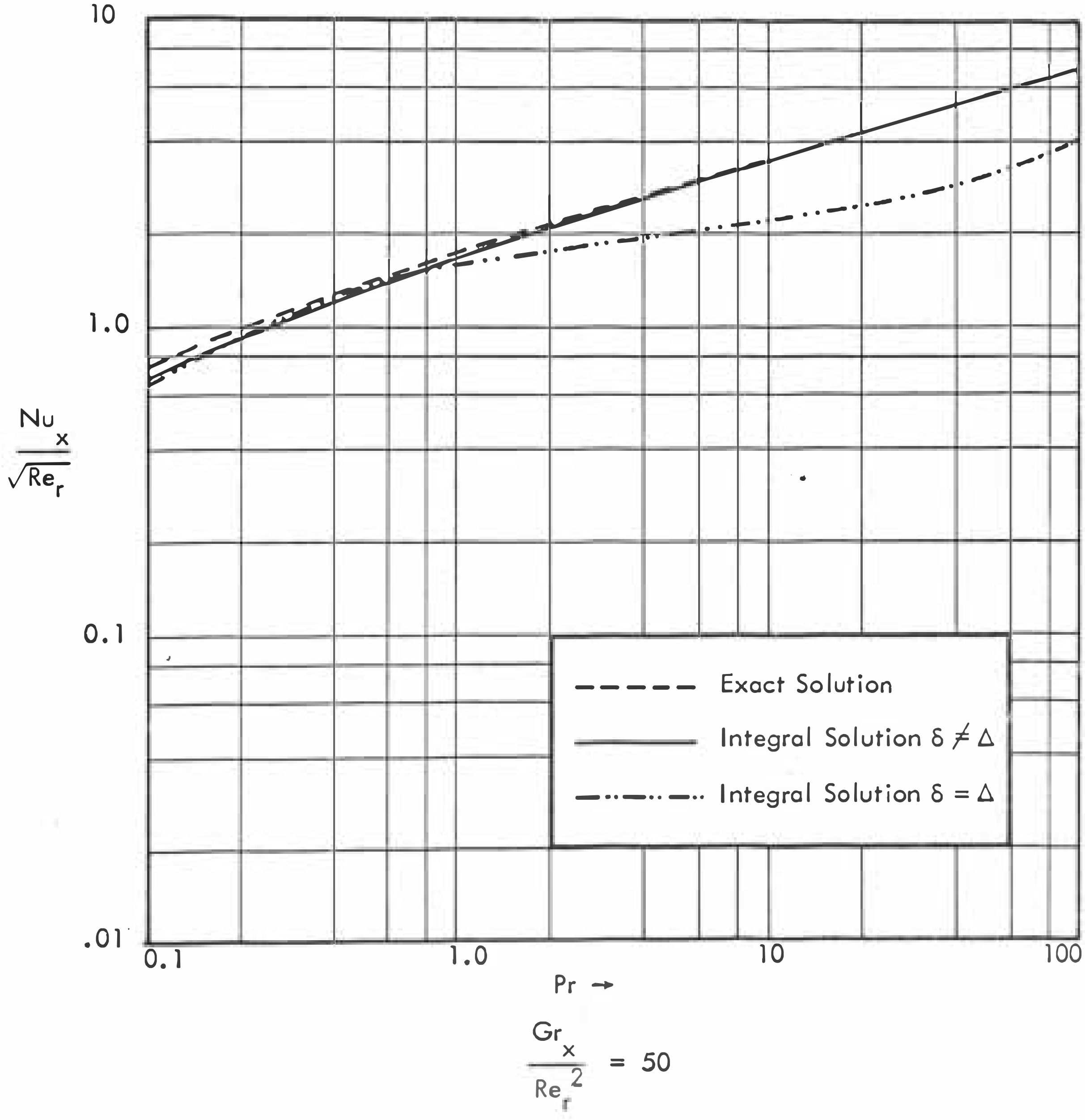


Figure XI. Variation of Local $Nu_x/\sqrt{Re_r}$ with Prandtl Numbers.

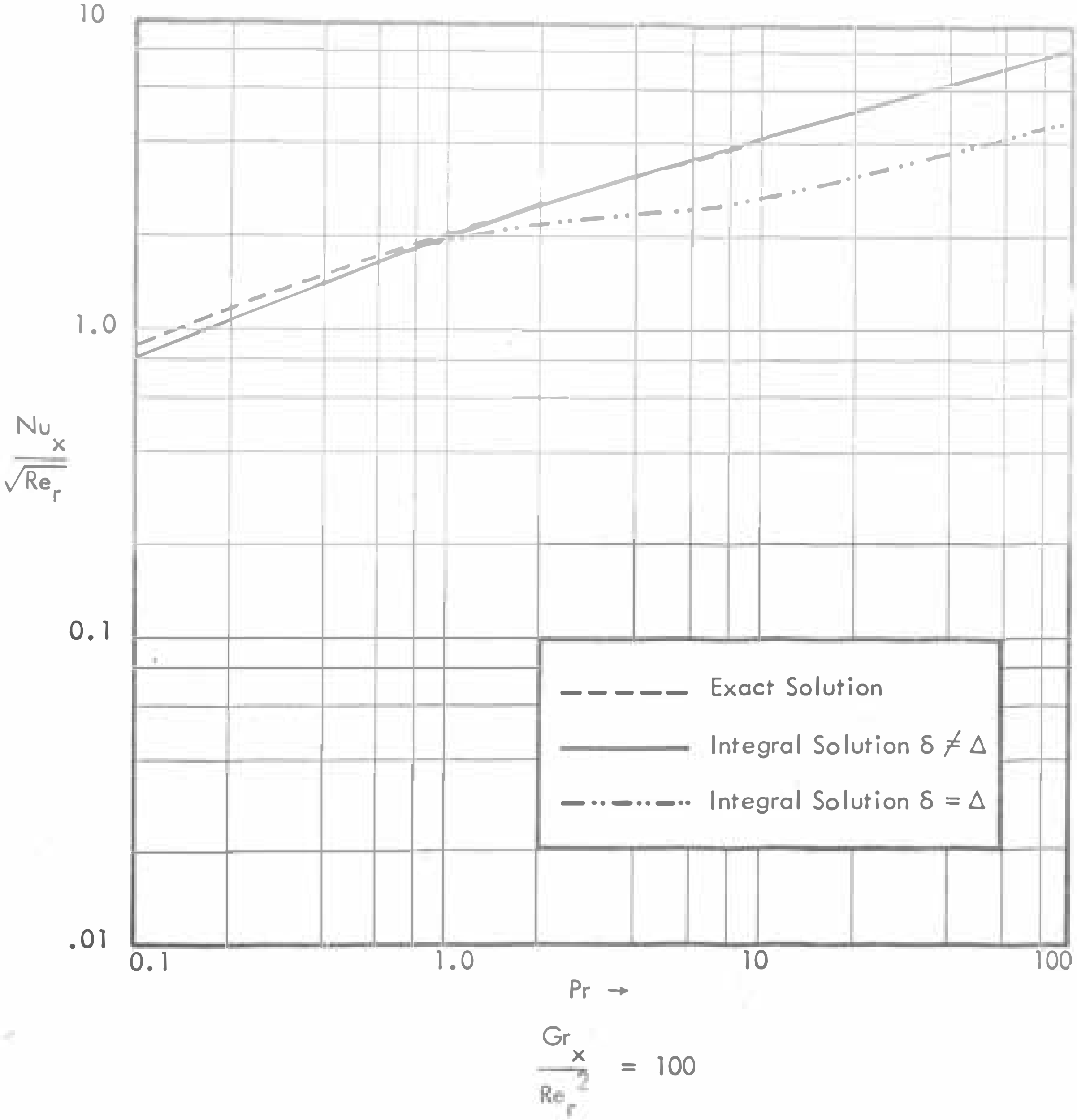


Figure XII. Variation of Local $Nu_x/\sqrt{Re_r}$ with Prandtl Numbers.

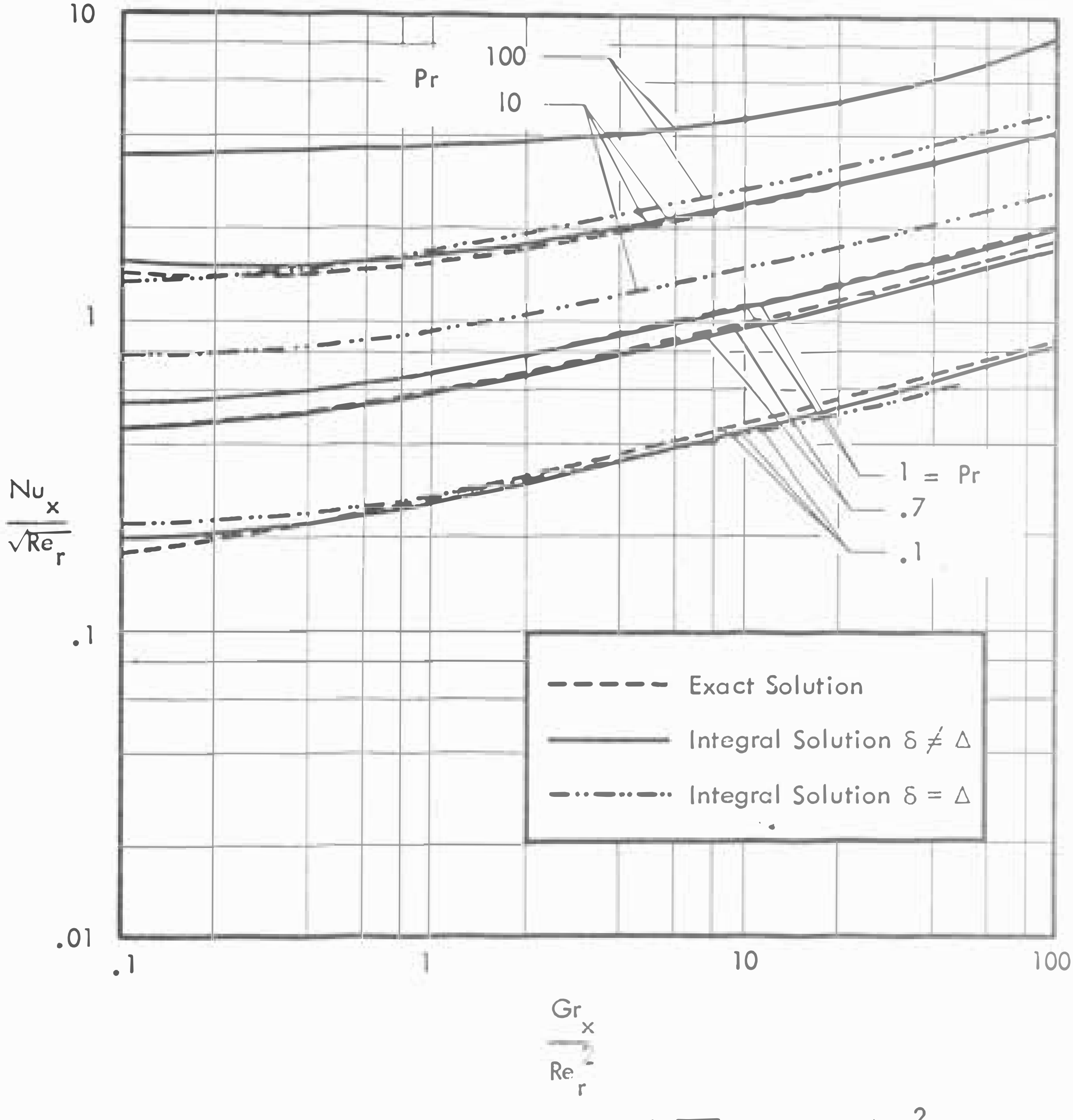


Figure XIII. Variation of Local $Nu_x/\sqrt{Re_r}$ with Gr_x/Re_r^2 for Different Prandtl Numbers.