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TRANSIENT VOLTAGE AND CURRENT AMPLITUDES IN INDUCTIVELY COUPLED CIRCUITS

by

Robert A. Heartz

December 15, 1950

Submitted to the Graduate Committee of South Dakota State College in partial fulfillment of the requirements for the degree of Master of Science.

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This is to certify that, in accordance with the requirements of South Dakota State College for Master of Science Degree, Robert A. Heartz has presented to this committee three bound copies of an acceptable thesis, done in his major field; and has satisfactorily passed a two hour oral examination on this thesis, the major field Electrical Engineering and minor field Mathematics.

Advisor

Jan. 6, 1951
Date

Head of Major Dept.

Head of Minor Dept.

Representative of Graduate Committee

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SECTION I

INTRODUCTION

When two circuits are so placed that energy may be transferred from one to the other they are said to be coupled. If the coupling is obtained by a circuit element (resistor, capacitor or coil) common to both circuits the coupling is direct. If the circuits are linked only by a magnetic field the coupling is called inductive. This paper will concern itself only with inductively coupled circuits. Figure 1 is an example of this type circuit.

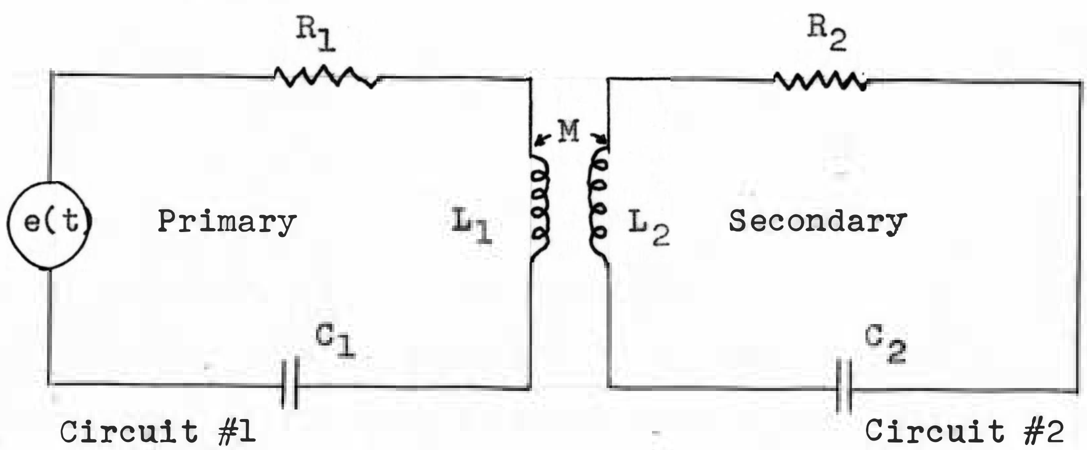

Figure 1

The steady state characteristics of this network are, a high gain and good frequency selectivity. Thus it finds extensive use in communication work. There are many texts which give excellent analyses of the steady state characteristics of this circuit. Morecrofts' text "Principles of Radio Communication" is particularly good.

Although the circuit has been used extensively for a good many years there has been no complete analysis of its transient characteristics. This analysis involves the solution of two simultaneous differential equations. This solution involves the determination of roots of a fourth degree equation. Therefore a completely general solution is impractical. If it is assumed that the resistance of the two circuits is negigible, the roots of the fourth degree equation will occur in imaginary pairs and the equations have a definite solution. The approximate damping affect of resistance may then be introduced into this solution. Again Morecroft has an excellent analysis of the circuit based on this assumption. However, in the determination of the constant terms and the frequency terms, the effect of resistance is excluded and sometimes it is necessary that the resistance be large. Frequently the condensers of the circuit are punctured by too high voltage, or the coils, resistances or instruments are burned out because this lack of knowledge of the transient characteristics. This introduces the idea that the resistance may play an important part in the determination of the magnitude of the transients.

Because there are many texts which have very good descriptions and analyses of the characteristics of this circuit it is considered necessary to give only a brief description of general transient characteristics.here.

Let it be assumed that a pulse voltage, e(t), is applied to the circuit of Fig. 1. Looking at the circuit from the primary side, the circuit may be represented as R_1, L_1, C_1 plus the impedance of the secondary circuit which is reflected by the coupling into the primary circuit. If R_1 and the reflected R_2 are not too large the energy imparted to the circuit by the pulse voltage will oscillate because the total circuit may be represented as a series R, L, and C circuit. Where R is R_1 plus the reflected R_2, L is L_1 plus the reflected L_2,etc. Now we look at the circuit from the secondary side. The secondary may be represented as R_2, L_2, C_2 plus the reflected impedance of the primary. The energy imparted to this secondary is the voltage impulse reflected through the coupling. Thus the secondary may be represented by an RLC circuit and the energy reflected into the secondary will oscillate at a frequency different from that of the primary. Therefore there are two oscillating energies coupled together by a common factor. The energy will be transferred from one circuit to the other at a rate depending on the difference of the two frequency terms. Whether the energy transfer will be complete will depend on the circuit constant. Several of the references listed in the bibliography give an excellent description of the characteristics of this circuit.

SECTION II

OBJECT - STATEMENT OF PROBLEM AND METHOD OF SOLUTION

The purpose of this paper is to establish upper limits or means by which the upper limit may be determined for the transient currents and condenser voltages in magnetically coupled circuit and to discover any dangerous transient conditions that can exist. A high ratio of transient amplitude to steady state amplitude is interpreted here as a dangerous condition.

Figure 2 shows the magnetically coupled tuned circuits which will be analysed.

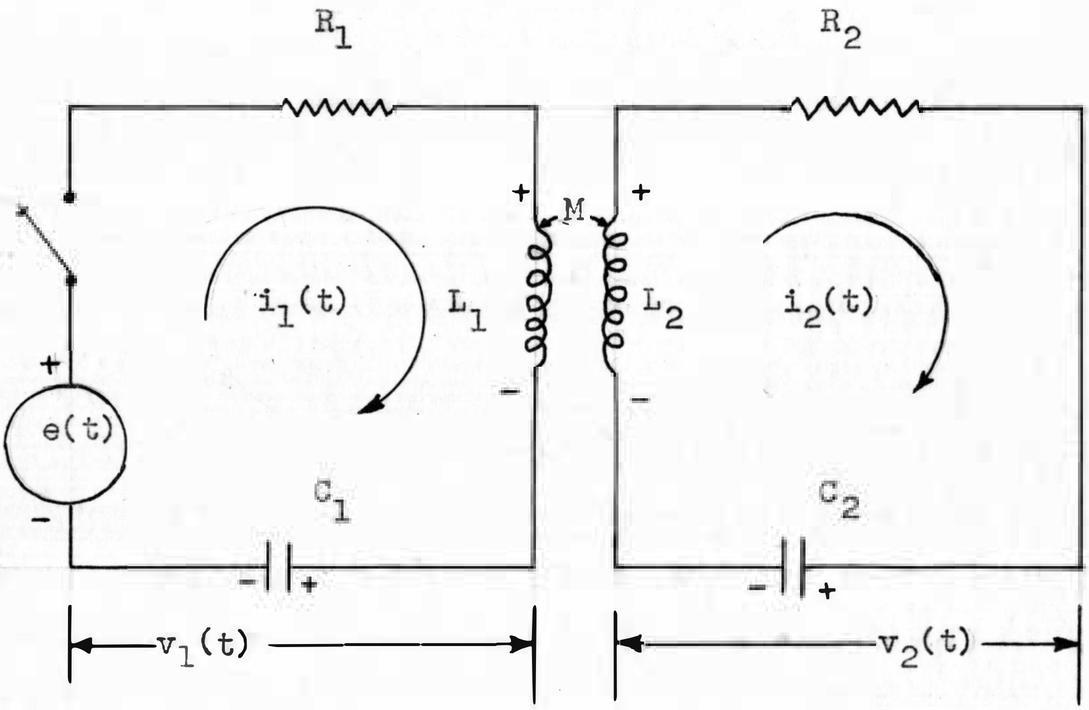

Figure 2

A descriptive type analysis is of little value in determining the transient limits. A laboratory method of analysis would be satisfactory but would require a precision of measurement that is not easily obtainable and would also require a very large number of tests. The mathematical analysis of the circuit requires the solution of a fourth degree differential equation. Fortunately this difficulty can be overcome by assuming the roots of the quartic equation and definite results can be obtained.

The equations of the network of Fig. 2 are as follows:

$$e(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + \frac{1}{C_1} \int i_1(t)\,dt + M \frac{di_2(t)}{dt}$$

$$0 = M \frac{di_1(t)}{dt} + R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} + \frac{1}{C_2} \int i_2(t)\,dt .$$

There are four available methods for solving the above equations. These methods are, the classical, the Cauchy-Heaviside, the Fourier Transform, and the Laplace Transform. If the solution were to consider one specific case there would be little choice in the method of solution. However, a general solution is wanted, a solution that has as wide a range of flexibility as possible. The Laplace Transform method is by far the best suited for this purpose. The constants are readily determined and the method of solution is simple and straight forward. This is evident in the general development in the next section. The notation used will be the same as developed by Gardner and Barnes in their text "Transient in Linear System". A brief description of the notation to be used follows.

The primary circuit will refer to circuit #1 of Fig. 1.

The secondary circuit will refer to circuit #2 of Fig. 1.

$i_1(t)$ instantaneous loop current of the primary.

$i_2(t)$ instantaneous loop current of the secondary.

$v_1(t)$ instantaneous voltage drop across C_1.

$v_2(t)$ instantaneous voltage drop across C_2.

$e(t)$ instantaneous input voltage rise. This term will also include the effect of closing the switch.

(The sign and direction of the above quantities are indicated on the circuit diagram.)

$\mathcal{L}$ indicates the procedure of taking the Laplace Transform of all that follows the symbol.

$\mathcal{L}^{-1}$ indicates the inverse Laplace Transform.

$I_1(s)$ indicates the $\mathcal{L}\, i_1(t)$, $I_2(s) = \mathcal{L}\, i_2(t)$, etc.

$Z_{11}(s)$ the loop impedance function seen by the current $I_1(s)$.

$Z_{22}(s)$ the loop impedance function seen by the current $I_2(s)$.

$Z_{12}(s)$ the mutual impedance function to loop currents $I_1(s)$ and $I_2(s)$.

(For an example of this notation see the illustration at the end of the section.)

$\mathcal{R}$ indicates the process of taking the real part of all that follows.

$\triangleq$ means equal by definition.

$\approx$ means approximately equal.

$\rightarrow$ means approaches (as an example "$A \rightarrow B$" means that A approaches in value B).

$|A|$ means the absolute value of the constant within the bars.

$\bar{A}$ means the conjugate of the term under the bar.

Condensor voltages and coil currents at the time $t = 0$, are assumed zero throughout this analysis. Other initial conditions would simply add other terms to the initial equations. These added terms can be carried through the analysis in the same manner as the principal terms. Therefore the analysis would be more complex.

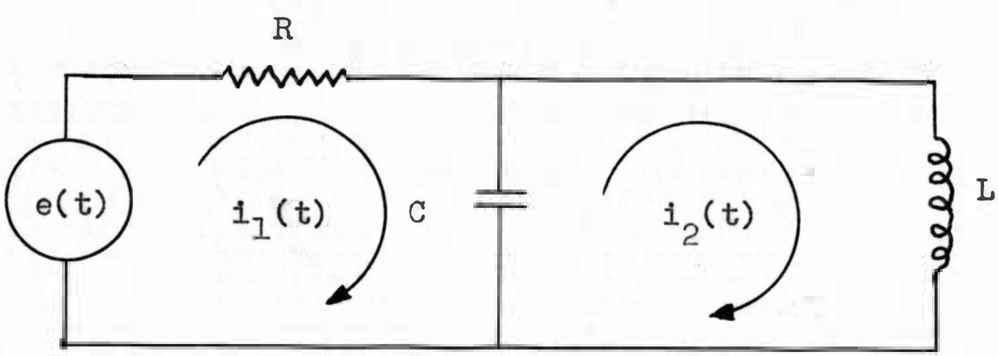

Figure 3

Little would be gained by including these terms.

The following example illustrates the notation and method of solution. Given the circuit of Figure 3, the equations are as follows.

$$e(t) = Ri_1(t) + \frac{1}{C}\int i_1(t)\,dt - \frac{1}{C}\int i_2(t)\,dt$$

$$0 = -\frac{1}{C}\int i_1(t)\,dt + \frac{1}{C}\int i_2(t)\,dt + L\frac{di_2(t)}{dt}$$

Neglecting initial conditions the equations in transform notation are:

$$E(s) = RI_1(s) + \frac{1}{Cs}I_1(s) - \frac{1}{Cs}I_2(s)$$

$$0 = -\frac{1}{Cs}I_1(s) + \frac{1}{Cs}I_2(s) + LsI_2(s)$$

These equations may be factored as follows:

$$E(s) = \left[R + \frac{1}{Cs}\right]I_1(s) - \left[\frac{1}{Cs}\right]I_2(s)$$

$$0 = -\left[\frac{1}{Cs}\right]I_1(s) + \left[\frac{1}{Cs} + Ls\right]I_2(s)$$

The impedance function of loop 1 is $Z_{11}(s) = \left[R + \frac{1}{Cs}\right]$.

The impedance function of loop 2 is $Z_{22}(s) = \left[\frac{1}{Cs} + Ls\right]$

The mutual impedance function is $Z_{12}(s) = \left[\frac{1}{Cs}\right]$

Using this notation the network equations are:

$$E(s) = Z_{11}(s)I_1(s) - Z_{12}(s)I_2(s)$$

$$0 = -Z_{12}(s)I_1(s) + Z_{22}(s)I_2(s) .$$

SECTION III

THE GENERAL SOLUTION

The general solution of a coupled network involves the determination of the roots of a quartic equation. There is no practical solution for an equation of the fourth degree. However a solution can be assumed and then it can be shown how variation of the circuit parameters affect the assumed solution. Condenser voltages and coil currents are assumed to be zero at the instant the switch, K, is closed. The effect of closing the switch will be included in the expression for e(t).

Let $Z_{11}(s)$ and $Z_{22}(s)$ be the self impedance functions to the loop currents, $i_1(t)$ and $i_2(t)$ respectively. Let $Z_{12}(s)$ be the mutual impedance function. Then, by the transform method illustrated on pages 8 and 9:

$$Z_{11}(s) = \frac{L_1}{s}\left[s^2 + 2a_1 s + w_1^2\right]$$

$$Z_{22}(s) = \frac{L_2}{s}\left[s^2 + 2a_2 s + w_2^2\right]$$

$$Z_{12}(s) = Ms$$

$$a_1 = \frac{R_1}{2L_1} \qquad w_1 = \frac{1}{\sqrt{L_1 C_1}}$$

$$a_2 = \frac{R_2}{2L_2} \qquad w_2 = \frac{1}{\sqrt{L_2 C_2}}$$

Let 'k' be the coefficient of coupling.

$$k^2 = \frac{M}{L_1 L_2}$$

The equations for the circuit (Fig.2) in the transform notation are:

$$E(s) = Z_{11}(s)I_1(s) - Z_{12}(s)I_2(s)$$

$$0 = -Z_{12}(s)I_1(s) + Z_{22}(s)I_2(s)$$

Solving these equations for $I_1(s)$ and $I_2(s)$ by the method of determinants gives;

$$I_1(s) = \frac{Z_{22}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)^2} E(s)$$

$$I_2(s) = \frac{-Z_{12}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)^2} E(s)$$

Therefore from the Laplace Transform theory:

$$i_1(t) = \mathcal{L}^{-1} I_1(s) = \mathcal{L}^{-1} \frac{Z_{22}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)^2} E(s) \quad \ldots 1(a)$$

$$i_2(t) = \mathcal{L}^{-1} I_2(s) = \mathcal{L}^{-1} \frac{-Z_{12}(s)}{Z_{11}(s)Z_{22}(s) - Z_{12}(s)^2} E(s) \quad \ldots 1(b)$$

$$v_1(t) = \mathcal{L}^{-1} V_1(s) = \frac{1}{C_1} \mathcal{L}^{-1} \frac{I_1(s)}{s} \qquad \ldots 1(c)$$

$$v_2(t) = \mathcal{L}^{-1} V_2(s) = \frac{1}{C_2} \mathcal{L}^{-1} \frac{I_2(s)}{s} \qquad \ldots 1(d)$$

Expanding the term, $Z_{11}(s)Z_{22}(s) - Z_{12}(s)^2$, gives,

$$\frac{L_1L_2(1-k^2)}{s^2}\left[s^4 + 2\frac{a_1+a_2}{1-k^2}s^3 + \frac{w_1^2 + w_2^2 + 4a_1a_2}{1-k^2}s^2 + 2\frac{a_1w_2^2 + a_2w_1^2}{(1-k^2)}s + \frac{w_1^2w_2^2}{1-k^2}\right] = \frac{L_1L_2(1-k^2)}{s^2}\left[s^4 + \ldots\right].$$

Let the above quartic expression, in brackets, be represented by the simpler notation, $[s^4 + \ldots]$. Then substituting for $Z_{11}(s)$, $Z_{22}(s)$ and $Z_{12}(s)$ in equations 1, they become;

$$i_1(t) = \frac{1}{L_1(1-k^2)} \mathcal{L}^{-1} \frac{s\left[s^2 + 2a_2s + w_2^2\right]}{\left[s^4 + \ldots\right]} E(s) \qquad \ldots 2(a)$$

$$i_2(t) = \frac{M}{L_1L_2(1-k^2)} \mathcal{L}^{-1} \frac{s^3}{\left[s^4 + \ldots\right]} E(s) \qquad \ldots 2(b)$$

$$v_1(t) = \frac{1}{C_1} \mathcal{L}^{-1} \frac{I_1(s)}{s} \qquad \ldots 2(c)$$

$$v_2(t) = \frac{1}{C_2} \mathcal{L}^{-1} \frac{I_2(s)}{s} \qquad \ldots 2(d)$$

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Equations (2) are perfectly general solutions for the network of Fig. 2. The only condition imposed is that condenser voltages and coil currents be zero at the instant the switch is closed. The effect of these initial currents or voltages may be taken into consideration by writing the network equations as follows:

$$E_1(s) = Z_{11}(s)I_1(s) - Z_{12}(s)I_2(s)$$

$$E_2(s) = -Z_{12}(s)I_1(s) + Z_{22}(s)I_2(s)$$

$E_1(s)$ and $E_2(s)$ are made to include initial voltages and currents. These terms make the solution longer but no more complex.

To carry the solution further it is necessary to determine the roots of the expression $\left[s^4 + \ldots\right]$. The roots must occur in conjugate or real pairs. Furthermore, the maximum amplitudes and the most dangerous conditions will occur when these pairs of roots are conjugate pairs. Therefore this paper will develop only the solution where the roots are conjugate pairs. The solution for the case where the roots are real pairs is very similar to this solution. In the following analysis these similarities will be pointed out.

Let $\left[s^4 + \ldots\right] = (s+\alpha_1 - j\beta_1)(s+\alpha_1 + j\beta_1)(s+\alpha_2 - j\beta_2)(s+\alpha_2 + j\beta_2)$

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Expanding the right hand side of the preceding equation and equating the coefficients of like powers of s, the following equalities are derived:

$$\alpha_1 + \alpha_2 = \frac{a_1 + a_2}{1-k^2} \qquad \ldots 3(a)$$

$$\Omega_1^2 + \Omega_2^2 + 4\alpha_1\alpha_2 = \frac{w_1^2 + w_2^2 + 4a_1a_2}{1-k^2} \qquad \ldots 3(b)$$

$$\alpha_1\Omega_2^2 + \alpha_2\Omega_1^2 = \frac{a_1w_2^2 + a_2w_1^2}{1-k^2} \qquad \ldots 3(c)$$

$$\Omega_1^2\Omega_2^2 = \frac{w_1^2w_2^2}{1-k^2} \qquad \ldots 3(d)$$

where $\Omega_1^2 = \alpha_1^2 + \beta_1^2$ and $\Omega_2^2 = \alpha_2^2 + \beta_2^2$. It is necessary to solve equations (3) simultaneously to determine the α's and β's. Dr. Harold M. Crothers, in an unpublished paper, has shown that the general solution of equations (3) is:

$$\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = \frac{a_1 + a_2 \pm \sqrt{(a_1 + a_2)^2 - (1-k^2)(4a_1a_2 + x)}}{2(1-k^2)} \qquad \ldots 4\binom{a}{b}$$

$$\begin{matrix}\Omega_1^2 \\ \Omega_2^2\end{matrix} = \frac{w_1^2 + w_2^2 - x \pm \sqrt{(w_1^2 + w_2^2 - x)^2 - 4w_1^2w_2^2(1-k^2)}}{2(1-k^2)} \qquad \ldots 4\binom{c}{d}$$

In the above equations, x is the real root, which vanishes when the constant term vanishes, of the following equations;

$$x^3 - 2(w_1^2 + w_2^2 - 2a_1a_2)\, x^2$$
$$+ \left[(w_1^2 - w_2^2)^2 + 4k^2(w_1^2w_2^2) + 4(a_1 - a_2)(a_1w_2^2 - a_2w_1^2)\right] x$$
$$-4k^2(a_1w_2^2 - a_2w_1^2)^2 = 0 \qquad \ldots 5$$

Equations similar to 4 and 5 were also derived by Pierce in his book "Electrical Oscillations and Electric Waves"(1920). Equations 4 and 5 also apply to the non-oscillatory case. In this case $\Omega_1^2 = \alpha_1^2 - \beta_1^2$ and $\Omega_2^2 = \alpha_2^2 - \beta_2^2$ and the roots are;

$$\left[s^4 + \ldots\right] = (s+\alpha_1 - \beta_1)(s+\alpha_1+\beta_1)(s+\alpha_2-\beta_2)(s+\alpha_2+\beta_2)$$

Equations 2 in terms of α and β are as follows from (2a);

$$i_1(t) = \frac{1}{L_1(1-k^2)}\, \mathcal{L}^{-1} \frac{s\left[s^2 + 2a_2 s + w_2^2\right]E(s)}{\left[(s+\alpha_1)^2 + \beta_1^2\right]\left[(s+\alpha_2)^2 + \beta_2^2\right]} \qquad \ldots 6(a)$$

The equations for $i_2(t)$, $v_1(t)$ and $v_2(t)$ may be written in a similar manner from equations (2b), (2c) and (2d).

From this point, the solution may proceed by two different methods of analysis. One method would be to specify e(t), determine its Laplace Transform and substitute this result into equations 6. Then taking the inverse transform gives the desired results. The second method of analysis is to leave E(s) undeclared and examine the transfer or system function. The transfer or system function may be defined for this particular case as the part of equation 6 whose value is dependent solely on network parameters. To be more specific, equation 6(a) may be written as follows;

$$i_1(t) = \mathcal{L}^{-1} I_1(s) = \mathcal{L}^{-1} N_1 G_1(s) \left[E(s) \right]$$

$N_1 G_1(s)$ is the system function where,

$$N_1 = \frac{1}{L_1(1-k^2)}$$

$$G_1(s) = \frac{s\left[s^2 + 2a_2 s + w_2^2\right]}{\left[(s+\alpha_1)^2 + \beta_1^2\right]\left[(s+\alpha_2)^2 + \beta_2^2\right]}$$

E(s) is called the forcing function. This system function can then be combined with the forcing function by means of the convolution integral. The formulation of the convolution integral is treated extensively in the Gardner and Barnes text "Transients in a Linear System", pages 228 through 236.

This second method of analysis has several distinct advantages. The constants determined from an analysis of the system function are a part of the constant term for any forcing function used. The damping factor and the frequency components determined from an analysis of the transfer function are of course the same for any forcing function used. The disadvantage of the second method is in the combination of the system function and forcing function. The convolution integral can be very complex.

The method which best fits this problem is a combination of the two methods. There are only three types of forcing function which are of practical concern. They are,

1. The unit pulse function (a pulse of exceedingly short duration compared to the time constants of the system)
2. The unit step function
3. The sine or cos. function

For these three forcing functions the problems of how the transfer function combines with the forcing function is best determined by solving the equation by method one and comparing this result with the analysis of the system function. The formulation of the real convolution integral will be outlined in the following so that the equations may fit a more general case. In the following only the case of $i_1(t)$ will be developed. The case for $i_2(t)$, $v_1(t)$ and $v_2(t)$ follow the same procedure as for the case $i_1(t)$. Equation 6 may be written;

$$\mathcal{L}^{-1}\frac{I_1(s)}{E(s)} = \mathcal{L}^{-1}N_1G_1(s) = \frac{1}{L_1(1-k^2)}\mathcal{L}^{-1}\frac{s\left[s^2+2a_2s+w_2^2\right]}{\left[(s+\alpha_1)^2+\beta_1^2\right]\left[(s+\alpha_2)^2+\beta_2^2\right]}$$

It should be noted that the $\mathcal{L}^{-1}\dfrac{I_1(s)}{E(s)}$ does not equal $i_1(t)/e(t)$.

$$N_1G_1(s) = \left[\frac{K_1}{s+\alpha_1-j\beta_1} + \frac{\bar{K}_1}{s+\alpha_1+j\beta_1} + \frac{K_2}{s+\alpha_2-j\beta_2} + \frac{\bar{K}_2}{s+\alpha_2+j\beta_2} \right] \frac{1}{2} N_1$$

The bar over the constant means the conjugate of the term. If the roots are real pairs then;

$$N_1G_1(s) = \left[\frac{K_1}{s+\alpha_1-\beta_1} + \frac{K_{11}}{s+\alpha_1+\beta_1} + \frac{K_2}{s+\alpha_2-\beta_2} + \frac{K_{22}}{s+\alpha_2+\beta_2} \right] \frac{1}{2} N_1$$

The solution follows in the same manner as for imaginary roots but of course the constants are different and the simplified notation of "conjugate" and "real part" cannot be applied. The following analysis will apply only for the case where the roots are imaginary. Let $\mathcal{R}$ mean the real part of all that follows in the brackets.

$$N_1G_1(s) = N_1 \mathcal{R} \left[\frac{K_1}{s+\alpha_1-j\beta_1} + \frac{K_2}{s+\alpha_2-j\beta_2} \right]$$

similarly,

$$N_2G_2(s) = N_2 \mathcal{R} \left[\frac{K'_1}{s+\alpha_1-j\beta_1} + \frac{K'_2}{s+\alpha_2-j\beta_2} \right].$$

$$\mathcal{L}^{-1} N_1G_1(s) = N_1 \mathcal{R} \left[\mathcal{L}^{-1} \left(\frac{K_1}{s+\alpha_1-j\beta_1} + \frac{K_2}{s+\alpha_2-j\beta_2} \right) \right]$$

$$= N_1 \mathcal{R} \left[K_1 e^{(-\alpha_1+j\beta_1)t} + K_2 e^{(-\alpha_2+j\beta_2)t} \right]$$

In the following let $Z_1 = -\alpha_1 + j\beta_1$ and $Z_2 = -\alpha_2 + j\beta_2$. The inverse transform of the system functions for equation 6 are as follows;

$$\mathcal{L}^{-1}\frac{I_1(s)}{E(s)} = \mathcal{L}^{-1}N_1G_1(s) = N_1\mathcal{R}\left[K_1e^{Z_1t} + K_2e^{Z_2t}\right] \qquad \ldots 7(a)$$

$$\mathcal{L}^{-1}\frac{I_2(s)}{E(s)} = \mathcal{L}^{-1}N_2G_2(s) = N_2\mathcal{R}\left[K'_1e^{Z_1t} + K'_2e^{Z_2t}\right] \qquad \ldots 7(b)$$

$$\mathcal{L}^{-1}\frac{V_1(s)}{E(s)} = \mathcal{L}^{-1}N_3G_3(s) = N_3\mathcal{R}\left[\frac{K_1}{Z_1}e^{Z_1t} + \frac{K_2}{Z_2}e^{Z_2t}\right] \qquad \ldots 7(c)$$

$$\mathcal{L}^{-1}\frac{V_2(s)}{E(s)} = \mathcal{L}^{-1}N_4G_4(s) = N_4\mathcal{R}\left[\frac{K'_1}{Z_1}e^{Z_1t} + \frac{K'_2}{Z_2}e^{Z_2t}\right] \qquad \ldots 7(d)$$

where $N_1 = \dfrac{1}{L_1(1-k^2)}$, $N_2 = \dfrac{M}{L_1L_2(1-k^2)}$, $N_3 = \dfrac{1}{L_1C_1(1-k^2)}$, $N_4 = \dfrac{M}{L_1L_2C_2(1-k^2)}$

The constant terms are:

$$K_1 = \frac{Z_1\left[Z_1^2 + 2a_2Z_1 + w_2^2\right]}{j\beta_2\left[(\alpha_2 + Z_1)^2 + \beta_2^2\right]} = |K_1|\ e^{j\psi_1} \qquad \ldots 8(a)$$

$$K_2 = \frac{Z_2\left[Z_2^2 + 2a_2Z_2 + w_2^2\right]}{j\beta_2\left[(\alpha_1 + Z_2)^2 + \beta_1^2\right]} = |K_2|\ e^{j\psi_2} \qquad \ldots 8(b)$$

$$K'_1 = \frac{Z_1^3}{j\beta_1\left[(\alpha_2 + Z_1)^2 + \beta_2^2\right]} = |K'_1|\ e^{j\psi'_1} \qquad \ldots 8(c)$$

$$K'_2 = \frac{Z_2^3}{j\beta_2\left[(\alpha_1 + Z_2)^2 + \beta_1^2\right]} = |K'_2|\ e^{j\psi'_2} \qquad \ldots 8(d)$$

The angle terms associated with these constants are ψ_1, ψ_2, ψ_1', and ψ_2' respectively.

$$\psi_1 = -\frac{\pi}{2} + \mathrm{Tan}^{-1}\frac{\beta_1}{\alpha_1} + \mathrm{Tan}^{-1}\frac{2\beta_1(a_2-\alpha_1)}{w_2^2-\beta_1^2+\alpha_1^2-2a_2\alpha_1} - \mathrm{Tan}^{-1}\frac{2(\alpha_2-\alpha_1)\beta_1}{\beta_2^2-\beta_1^2+(\alpha_2-\alpha_1)^2}$$

$$\psi_2 = -\frac{\pi}{2} + \mathrm{Tan}^{-1}\frac{\beta_2}{\alpha_2} + \mathrm{Tan}^{-1}\frac{2\beta_2(a_2-\alpha_2)}{w_2^2-\beta_2^2+\alpha_2^2-2a_2\alpha_2} - \mathrm{Tan}^{-1}\frac{2(\alpha_1-\alpha_2)\beta_2}{\beta_1^2-\beta_2^2+(\alpha_1-\alpha_2)^2}$$

$$\psi_1' = -\frac{\pi}{2} + \mathrm{Tan}^{-1} 3\frac{\beta_1}{\alpha_1} - \mathrm{Tan}^{-1}\frac{2(\alpha_2-\alpha_1)\beta_1}{\beta_2^2-\beta_1^2+(\alpha_1-\alpha_2)^2}$$

$$\psi_2' = -\frac{\pi}{2} + \mathrm{Tan}^{-1} 3\frac{\beta_2}{\alpha_2} - \mathrm{Tan}^{-1}\frac{2(\alpha_1-\alpha_2)\beta_2}{\beta_1^2-\beta_2^2+(\alpha_2-\alpha_1)^2}$$

These constant terms may be written in a manner more convenient form for calculations by substituting the following relationships, $\beta_1^2 = \Omega_1^2 - \alpha_1^2$ and $\beta_2^2 = \Omega_2^2 - \alpha_2^2$. Then, the terms in equations 8 become:

$$Z_1^2 + 2a_2Z_1 + w_2^2 = 2(a_2-\alpha_1)Z_1 + \omega_2^2 - \Omega_1^2$$

$$Z_2^2 + 2a_2Z_2 + w_2^2 = w_2^2 - \Omega_2^2 + 2(a_2-\alpha_2)Z_2$$

$$(\alpha_2 + Z_1)^2 + \beta_2^2 = \Omega_2^2 - \Omega_1^2 + 2(\alpha_2 - \alpha_1)Z_1$$

$$(\alpha_1 + Z_2)^2 + \beta_1^2 = \Omega_1^2 - \Omega_2^2 + 2(\alpha_2 - \alpha_1)Z_2$$

The next step would naturally be the analysis of equation (7). However this is also a good place to continue the general solution, to demonstrate the application of the convolution integral and to find out how these transfer functions fit into the solution for the three particular forcing functions previously mentioned.

The complex multiplication theorem, as stated and proved in the text "Transients in a Linear System", is as follows. If the functions $f_1(t)$ and $f_2(t)$ are $\mathcal{L}$ transformable and have, respectively, the $\mathcal{L}$ transforms, $F_1(s)$ and $F_2(s)$, then;

$$\mathcal{L}\left[\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau\right] = F_1(s)\, F_2(s)$$

The processes indicated within the brackets is called convolution in the real domain. This theorem states that the product of two functions in the complex Laplace domain goes over into the convolution of the two functions in the real domain. To apply this theorem for this case, Let $f_1(t) = e(t)$ and $f_2(t) = \mathcal{L}^{-1} N_1 G_1(s)$. Therefore $F_1(s) =$ $E(s)$ and $F_2(s) = N_1 G_1(s)$.

$$F_1(s)F_2(s) = N_1G_1(s)E(s) = I_1(s)$$

Then

$$i_1(t) = \mathcal{L}^{-1}F_1(s)F_2(s) = \int_0^t e(t-\tau)\left\{\Re\left[K_1e^{Z_1\tau} + K_2e^{Z_2\tau}\right]\right\} d\tau$$

Let $e(t) = \sin(wt)$ then,

$$i_1(t) = \Re\left[\int_0^t (K_1e^{Z_1\tau} + K_2e^{Z_2\tau}) \sin(wt - w\tau)\, d\tau\right].$$

It is evident that this integral can be rather difficult, but for some special forcing functions it is method of solution when the straight forward method or method one is practically impossible.

For the three particular forcing functions which have been chosen for development, the Laplace Transforms are;

1. $\mathcal{L}(\text{unit impulse at } t = 0) \triangleq \mathcal{L}\lim_{a \to 0} \dfrac{u(t)-u(t-a)}{a} = 1$

2. $\mathcal{L}(\text{unit step at } t = 0) \triangleq \mathcal{L}u(t) = \dfrac{1}{s}$

3. $\mathcal{L}(\text{unit sine or cos function}) \triangleq \mathcal{L}\sin(wt-\theta) = \dfrac{w\cos\theta - s\sin\theta}{s^2 + w^2}$

The proper selection of θ will give a cosine or sine function or any combination of cosine or sine.

For case #1, the unit pulse input, it is evident that $i_1(t) = \mathcal{L}^{-1}N_1G_1(s)$ or $i_1(t)$ is equal to the inverse transform of the system function.

For case #2, the only change in equation 7 is in the constant terms. The equations are;

$$i_1(t) = N_1 \mathcal{R}\left[K_{1s}e^{Z_1 t} + K_{2s}e^{Z_2 t}\right]$$

where,

$$K_{(1s)} = K_1/Z_1 = \left|K_{(1s)}\right| e^{j\Psi(1s)}$$

$$K_{(2s)} = K_2/Z_2 = \left|K_{(2s)}\right| e^{j\Psi(2s)}$$

The angle functions associated with these constants are;

$$\Psi_{(1s)} = \Psi_1 - \mathrm{Tan}^{-1} \frac{\beta_1}{\alpha_1}$$

$$\Psi_{(2s)} = \Psi_2 - \mathrm{Tan}^{-1} \frac{\beta_2}{\alpha_2}$$

For case #3, the constants are not only changed but there is added to the transient term a steady state term. This steady state term may be determined by use of the "Final Value Theorem" which is given in Gardner and Barnes text "Transients in a Linear System" page 267. This theorem states that;

$$\lim_{s \to 0} s\,F(s) = \lim_{t \to \infty} f(t)$$

Letting $e(t) = \sin(wt - \theta)$ and then solving equations 6 by taking the inverse transform there results;

$$i_1(t) = N_1 \mathcal{R}\left[K_{(1\ \sin)} e^{Z_1 t} + K_{(2\ \sin)} e^{Z_2 t} + K_{(3\ \sin)} e^{jwt}\right]$$

The constants are,

$$K_{(1\ \sin)} = K_1 \frac{w \cos \theta - Z_1 \sin \theta}{Z_1^2 + w^2} = \left|K_{(1\sin)}\right| e^{j\Psi(1\sin)}$$

$$K_{(2\ \sin)} = K_2 \frac{w \cos \theta - Z_2 \sin \theta}{Z_2^2 + w^2} = \left|K_{(2\sin)}\right| e^{j\Psi(2\sin)}$$

$$K_{(3\ \sin)} = \frac{w\left[w_2^2 - w^2 + 2ja_2 w\right]}{\left[(\alpha_1 + jw)^2 + \beta_1^2\right]\left[(\alpha_2 + jw)^2 + \beta_2^2\right]} = \left|K_{(3\sin)}\right| e^{j\Psi(3s}$$

The angle functions associated with these constants are,

$$\Psi_{(1\ \sin)} = \psi_1 + \mathrm{Tan}^{-1} \frac{-\beta_1 \sin \theta}{w \cos \theta + \alpha_1 \sin \theta} - \mathrm{Tan}^{-1} \frac{-2\alpha_1 \beta_1}{w_2^2 - \beta_1^2 + \alpha_1^2}$$

$$\Psi_{(2\ \sin)} = \psi_2 + \mathrm{Tan}^{-1} \frac{-\beta_2 \sin \theta}{w \cos \theta + \alpha_2 \sin \theta} - \mathrm{Tan}^{-1} \frac{-2\alpha_2 \beta_2}{w_2^2 - \beta_2^2 + \alpha_2^2}$$

These values were arrived at by solving equations 6 with the specified e(t) and then comparing these results with the respective inverse transforms of the system function rather than by the method of convolution. The method of solution for $i_2(t)$, $v_1(t)$ and $v_2(t)$ proceeds in the same manner.

The similarity in the preceding results suggests that the response equation may be written in a general equation as follows:

$$f(t) = N\,\mathcal{R}\left[Ae^{Z_1 t} + Be^{Z_2 t} + Ce^{jwt}\right] \qquad \ldots 9$$

where $f(t)$ represents $i_1(t)$, $i_2(t)$, $v_1(t)$, or $v_2(t)$ and N, A, B, and C are constant terms associated with the particular $f(t)$ and input forcing function. The following table, Table 1, gives the expression for these constant terms for $f(t)$ and any of the three forcing functions.

The transient portion of equation 9 could be written in terms of an envelope and angle function. Let $f_{(x)}$ represent this transient then;

$$f_{(x)} = N\,\mathcal{R}\left[Ae^{Z_1 t} + Be^{Z_2 t}\right]$$

$$= N\left[|A|^2 e^{-2\alpha_1 t} + 2|A||B|\, e^{-(\alpha_1+\alpha_2)t}\cos\left[(\beta_1-\beta_2)t+\psi_1-\psi_2\right] + |B|^2\, e^{-2\alpha_2 t}\right]^{\frac{1}{2}} \cos\left(\beta_2 t + \psi_2 + \varphi(t)\right) \qquad \ldots 10$$

where $\varphi(t) = \mathrm{Tan}^{-1} \dfrac{|A|e^{-\alpha_1 t}\sin\left[(\beta_1-\beta_2)t+\psi_1-\psi_2\right]}{|A|\,e^{-\alpha_2 t}\cos\left[(\beta_1-\beta_2)t+\psi_1-\psi_2\right] + |B|\,e^{-\alpha_2 t}}$

and ψ_1 and ψ_2 are the angle terms associated with A and B respectively. A and B are the same constants given in Table 1. The envelope equation is;

$$N\left[|A|^2 e^{-2\alpha_1 t} + 2|A||B|\, e^{-(\alpha_1+\alpha_2)t}\cos\left[(\beta_1-\beta_2)t+\psi_1-\psi_2\right] \;\; |B|^2\, e^{-2\alpha_2 t}\right]^{\frac{1}{2}} \qquad \ldots 11$$

Table 1 and equation 9 represent the complete general solution for the network of Fig. 2. The next section will be an analysis of the system function.

TABLE I

CONSTANTS OF RESPONSE EQUATION

Symbols:

$$u_1(t) = (\text{unit pulse at } t = 0) \triangleq \lim_{a \to 0} \frac{u(t) - u(t - a)}{a}$$

$$u_2(t) = (\text{unit step at } t = 0) \triangleq 1 \text{ or } u(t)$$

$$u_3(t) = (\text{unit sinusoid}) \triangleq \sin (wt - \theta)$$

$$Z_1 = -\alpha_1 + j\beta_1$$

$$Z_2 = -\alpha_2 + j\beta_2$$

Basic response equation:

$$f(t) = N\mathcal{R}\left[Ae^{Z_1 t} + Be^{Z_2 t} + Ce^{jwt}\right]$$

TABLE I (cont'd)

Constants:

$$K_1 = \frac{Z_1 \left[Z_1^2 + 2a_2 Z_1 + w_2^2\right]}{j\beta_1\left[(\alpha_2 + Z_1)^2 + \beta_2^2\right]}$$

$$K_2 = \frac{Z_2 \left[Z_2^2 + 2a_2 Z_2 + w_2^2\right]}{j\beta_2\left[(\alpha_1 + Z_2)^2 + \beta_1^2\right]}$$

$$K_1' = \frac{Z_1^3}{j\beta_2\left[(\alpha_2 + Z_1)^2 + \beta_2^2\right]}$$

$$K_2' = \frac{Z_2^3}{j\beta_2\left[(\alpha_1 + Z_2)^2 + \beta_1^2\right]}$$

$$K_3 = \frac{w \cos \theta - Z_1 \sin \theta}{Z_1^2 + w^2}$$

$$K_4 = \frac{w \cos \theta - Z_2 \sin \theta}{Z_2^2 + w^2}$$

$$K_5 = \frac{w\left[w_2^2 - w^2 + 2ja_2 w\right] e^{-j\theta}}{\left[(\alpha_1 + jw)^2 + \beta_1^2\right]\left[(\alpha_2 + jw)^2 + \beta_2^2\right]}$$

$$K_6 = \frac{-w^3 e^{-j\theta}}{\left[(\alpha_1 + jw)^2 + \beta_1^2\right]\left[(\alpha_2 + jw)^2 + \beta_2^2\right]}$$

TABLE I (cont'd)

e(t) (forcing function	f(t) (response function)	CONSTANTS			
		A	B	C	N
Unit Pulse $u_1(t)$	$i_1 t$	K_1	K_2	0	$\frac{1}{L_1(1-k^2)}$
	$i_2 t$	K'_1	K'_2	0	$\frac{M}{L_1 L_2(1-k^2)}$
	$v_1(t)$	$\frac{K_1}{Z_1}$	$\frac{K_2}{Z_2}$	0	$\frac{1}{L_1 C_1(1-k^2)}$
	$v_2(t)$	$\frac{K'_1}{Z_1}$	$\frac{K'_2}{Z_2}$	0	$\frac{M}{L_1 L_2 C_2(1-k^2)}$
Unit Step $u_2(t)$	$i_1(t)$	$\frac{K_1}{Z_1}$	$\frac{K_2}{Z_2}$	0	$\frac{1}{L_1(1-k^2)}$
	$i_2(t)$	$\frac{K'_1}{Z_1}$	$\frac{K'_2}{Z_2}$	0	$\frac{M}{L_1 L_2(1-k^2)}$
	$v_1(t)$	$\frac{K_1}{Z_1^2}$	$\frac{K_2}{Z_1^2}$	$\frac{w_2^2}{\Omega_1^2 \; \Omega_2^2}$	$\frac{1}{L_1 C_1(1-k^2)}$
	$v_2(t)$	$\frac{K'_1}{Z_1^2}$	$\frac{K'_2}{Z_2^2}$	0	$\frac{N}{L_1 L_2 C_2(1-k^2)}$

TABLE I (cont'd)

e(t) (forcing function)	f(t) (response function)	CONSTANTS			
		A	B	C	N
Unit sinusoid $u_3(t)$	$i_1(t)$	K_1K_3	K_2K_4	K_5	$\frac{1}{L_1(1-k^2)}$
	$i_2(t)$	$K_1'K_3$	$K_2'K_4$	K_6	$\frac{M}{L_1L_2(1-k^2)}$
	$v_1(t)$	$\frac{K_1}{Z_1}K_3$	$\frac{K_2}{Z_2}K_4$	$\frac{K_5}{jw}$	$\frac{1}{L_1C_1(1-k^2)}$
	$v_2(t)$	$\frac{K_1'}{Z_1}K_3$	$\frac{K_2'}{Z_2}K_4$	$\frac{K_6}{jw}$	$\frac{M}{L_1L_2C_2(1-k^2)}$

SECTION IV

ANALYSIS OF SYSTEM FUNCTION

In the previous section the system function was defined as the part of equation 6 which is solely dependent on the network parameters. Furthermore it was shown that the inverse Laplace Transform of this function gave a transient equation which had certain properties inherent in the transient response equations of the network for any type input function or forcing function. This system function was represented by the symbols NG(s). The $\mathcal{L}^{-1}$NG(s) are equations 7.

If this thesis were to be a complete analysis of the transient conditions in mutually coupled networks, a complete analysis of the system function would be an important contribution to the analysis. Curves could be plotted showing how $\mathcal{L}^{-1}$NG(s) varies as the circuit parameters vary. These circuit parameters could be represented by five dependent variables, a_1, a_2, w_1, w_2 and k. The variables could be reduced to four variables, a_1/a_2, w_1/w_2, a_2/w_2 and k. An analysis of this type is possible but it is a very long procedure and beyond the scope of this thesis. For any specific case, the expression for the maximum transient amplitude, or curves of the complete transient response may be readily determined by use of the equation of the envelope function, equation 10. Furthermore it is possible to determine when particularly dangerous conditions occur and the upper limits of transient amplitude without becoming involved in this extensive, complete analysis.

Because of the reasons just mentioned the following analysis will be split up roughly into three regions of operation. The first region will be where a_1 and a_2 are small compared to the frequency terms, w_1^2 and w_2^2. The second region will be where the damping is large or a_1 and a_2 are large enough so that no simplifying assumptions can be made. The third region will be where $a_1 w_2^2 \approx a_2 w_1^2$. This region covers the particular case where $a_1 = a_2$ and $w_1 = w_2$.

The steps in the procedure of solution are much more definite than these three regions. Therefore, the analysis for the three regions will be carried through in each step of the solution rather than carry the complete solution through for each of the regions.

The first step in the analysis of the system functions is the determination of α_1, α_2, β_1, and β_2 by equations 4 and 5. Equations 4 and 5 are repeated here for convenience.

$$\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = \frac{a_1 + a_2 \pm\sqrt{(a_1 + a_2)^2 - (1-k^2)(4a_1 a_2 + x)}}{2(1-k^2)} \qquad \ldots 4\binom{a}{b}$$

$$\begin{matrix}\Omega_1^2 \\ \Omega_2^2\end{matrix} = \frac{w_1^2 + w_2^2 - x \pm\sqrt{(w_1^2 + w_2^2 - x)^2 - 4w_1^2 w_2^2(1-k^2)}}{2(1-k^2)} \qquad \ldots 4\binom{c}{d}$$

Where x is the real root that vanishes when the constant term vanishes of the following equation:

$$x^3 - 2(w_1^2 + w_2^2 - 2a_1a_2)x^2 + \Big[(w_1^2 - w_2^2)^2 + 4k^2(w_1^2w_2^2) + 4(a_1 - a_2)(a_1w_2^2 - a_2w_1^2)\Big]x - 4k^2(a_1w_2^2 - a_2w_1^2)^2 = 0 \quad \ldots 5$$

For the first region, where a_1 and a_2 are small compared to w_1^2 and w_2^2, x in equation $4\binom{c}{d}$ may be neglected. Inspection of equation 5 shows that x will be small if a_1 and a_2 are small. Another way of arriving to this conclusion is by inspection of equations 3(a) and 3(d). These equations show that if a_1 and a_2 are small compared to w_1^2 and w_2^2, then α_1 and α_2 will be small compared to Ω_1^2 and Ω_2^2. Therefore, $4\alpha_1\alpha_2$ and $4\frac{a_1a_2}{1-k^2}$ may be neglected in equation 3(b). Equations 3(b) and 3(d) may be solved simultaneously for Ω_1^2 and Ω_2^2. The resultant equations for Ω_1^2 and Ω_2^2 are exactly the same as equations $4\binom{c}{d}$ if x is considered negligible.

Because x is negligible in equations $4\binom{c}{d}$ does not imply that it will be negligible in equations $4\binom{a}{b}$. Therefore, the value of x must be determined.

Sample calculations have shown that, if a_1 and a_2 are small, a very close approximation of x may be determined by using only the last three terms of equation 5. This is a quadratic equation. Its solution is;

$$x = \frac{(w_1^2 - w_2^2)^2 + 4k^2 w_1^2 w_2^2 + 4(a_1 - a_2)(a_1 w_2^2 - a_2 w_1^2)}{4(w_1^2 + w_2^2 - 2a_1 a_2)}$$

$$-\frac{1}{2}\sqrt{\left[\frac{(w_1^2 - w_2^2)^2 + 4k^2 w_1^2 w_2^2 - 4(a_1 - a_2)(a_1 w_2^2 - a_2 w_1^2)}{2(w_1^2 + w_2^2 - 2a_1 a_2)}\right]^2 - 8\frac{k^2(a_1 w_2^2 - a_2 w_1^2)^2}{w_1^2 + w_2^2 - 2a_1 a_2}}$$

Examination of this equation shows that it meets the condition that x must be the real root that vanishes when $(a_1 w_2^2 - a_2 w_1^2) \rightarrow 0$. The other solution for x does not meet this condition and therefore may be discarded. The value of x determined in this manner should be checked and adjusted by substitution in equation 5. Sample calculations have shown this method gives a very close approximation of x and generally no adjustment is needed. If this value of x needs radical adjustments to make it fit equation 5, then a_1 and a_2 cannot be considered negligible in the determination of the frequency terms, or in other words, x will be large and cannot be neglected in equation $4\binom{c}{d}$.

The above equation for x shows that x must always be positive. Equations 4 are limited to real numbers. Therefore, x must be limited to values which give real numbers in the radicals of equations 4. If a_1 and a_2 are small, then equation $4\binom{a}{b}$ give the limiting values of x. The possible range of x is,

$$0 \leqq x \leqq \frac{a_1^2 - a_2^2 + 4k^2 a_1 a_2}{1-k^2}$$

Therefore the possible range of α_1 and α_2 is,

$$\alpha_1 = \alpha_2 = \frac{a_1 + a_2}{2(1-k^2)} \quad \text{when } x = \frac{a_1^2 - a_2^2 + 4k^2 a_1 a_2}{1-k^2}$$

$$\begin{matrix}\alpha_1 \\ \alpha_2\end{matrix} = \frac{a_1 + a_2 \pm \sqrt{(a_1 + a_2)^2 - (1-k^2) a_1 a_2}}{2(1-k^2)} \quad \text{when } x = 0.$$

This shows that for any value of x neither α_1 nor α_2 may be zero. This merely checks a physical truth. If α_1 or α_2 were zero than there would be unending oscillations in the network. This could occur only if the resistances were zero. In general unending oscillations cannot occur unless the network involves feedback or "negative" resistance.

The special region where $a_1 w_2^2 - a_2 w_1^2 \rightarrow 0$ fits very well into the above analysis, because if $a_1 w_2^2 - a_2 w_1^2 = 0$, then x must be zero. If $a_1 = a_2 = a_0$ and $w_1 = w_2 = w_0$ than the equations for $\alpha_1, \alpha_2, \Omega_1^2$ and Ω_2^2 may be simplified to the following;

$$\alpha_1 = \frac{a_0}{1-k} \qquad \Omega_1^2 = \frac{w_0^2}{1-k}$$

$$\alpha_2 = \frac{a_0}{1+k} \qquad \Omega_2^2 = \frac{w_0^2}{1+k}$$

As a_1 and a_2 become larger, x will be larger for a specific ratio of $\frac{a_1 w_2^2}{a_2 w_1^2}$. It is noted that regardless of

of how large the damping becomes, x is zero when $\frac{a_1}{a_2} = \frac{w_2^2}{w_1^2}$.
Numerous calculations were made in an attempt to find a set of constants which would give a value of x that would make the radicals of equations 4 imaginary. Conditions were found which would make the radicals of equations $4\binom{a}{b}$ approach zero, but none of the conditions tried made any of the radicals imaginary. These calculations also showed that the quadratic equation for x, given on page 28, gives a good approximation of x, even when the damping was large enough to make the roots all real. However, when x is large it is very important that x be checked and adjusted by substitution into equation 5.

If critical damping is defined as the point between oscillatory conditions and non-oscillatory conditions then both β_1 and β_2 must equal zero. Therefore, $\alpha_1 = \Omega_1$ and $\alpha_2 = \Omega_2$ and $\alpha_1^2 + \alpha_2^2 = \Omega_1^2 + \Omega_2^2$. Substituting equations 4 into the latter equation and simplifying, gives the following equation:

$$\frac{(1-k^2)(w_1^2 + w_2^2)}{2(a_1^2 + a_2^2 + 2k^2 a_1 a_2)} = 1$$

Therefore if a system of parameters makes the above ratio greater than 1, then the roots of the quartic equation, $[s^4 + \ldots]$, are all real pairs. If the system of parameters makes the ratio less than 1, then at least one pair of roots is imaginary.

This particular case where the damping is large can be studied by a different method than presented here. Dr. Y. J. Liu in his paper, "Stability and Transient Analysis of Controlled Longitudinal Motion of Aircraft with Non-Ideal Automatic Control", develops a quartic chart for the solution of equations of the type, $\left[s^4 + \ldots\right]$. Dr. Liu's method has some particular advantages and therefore will be briefly outlined here.

Dr. Liu's first step is to write the equation $\left[s^4 + \ldots\right]$ in a non-dimensional form as follows;

$$\left[s^4 + \ldots.\right] = p^4 + y_3 p^3 + y_2 p^2 + y_1 p + 1 = 0$$

where

$$p = \left[\frac{1-k^2}{w_1^2 w_2^2}\right]^{\frac{1}{4}} s$$

$$y_3 = 2 \frac{a_1 + a_2}{(1-k^2)^{3/4} \, (w_1^2 w_2^2)^{\frac{1}{4}}}$$

$$y_2 = \frac{w_1^2 + w_2^2 + 4a_1 a_2}{(1-k^2)^{\frac{1}{2}} \, (w_1^2 w_2^2)^{\frac{1}{2}}}$$

$$y_1 = 2 \frac{a_1 w_2^2 + a_2 w_1^2}{(1-k^2)^{\frac{1}{4}} \, (w_1^2 w_2^2)^{3/4}}$$

Let

$$M = \frac{y_3 y_1 - 4}{y_2^2} \qquad \text{and } N = \frac{y_3^2 + y_1^2 - 4y_2}{y_2^3}$$

The values of M and N are called the stability criterions. Knowing the values of M and N it can be determined from inspection of the stability chart whether the roots of the quartic equation are two real pairs, one real pair and one imaginary pair or two imaginary pairs. Further charts presented will give the values of $\alpha_1, \alpha_2, \Omega_1$ and Ω_2.

Dr. Liu's work is of particular value where sustained oscillation can occur. In our particular problem sustained oscillations cannot occur and the method of analysis presented here is more direct and in general much simpler then the use of the quartic chart.

The next step is the analysis of the constant terms. The analysis in the region where the damping is small will be developed first. The analysis for this region will be roughly split up into three cases as follows:

Case 1, where $w_1^2 \gg w_2^2$

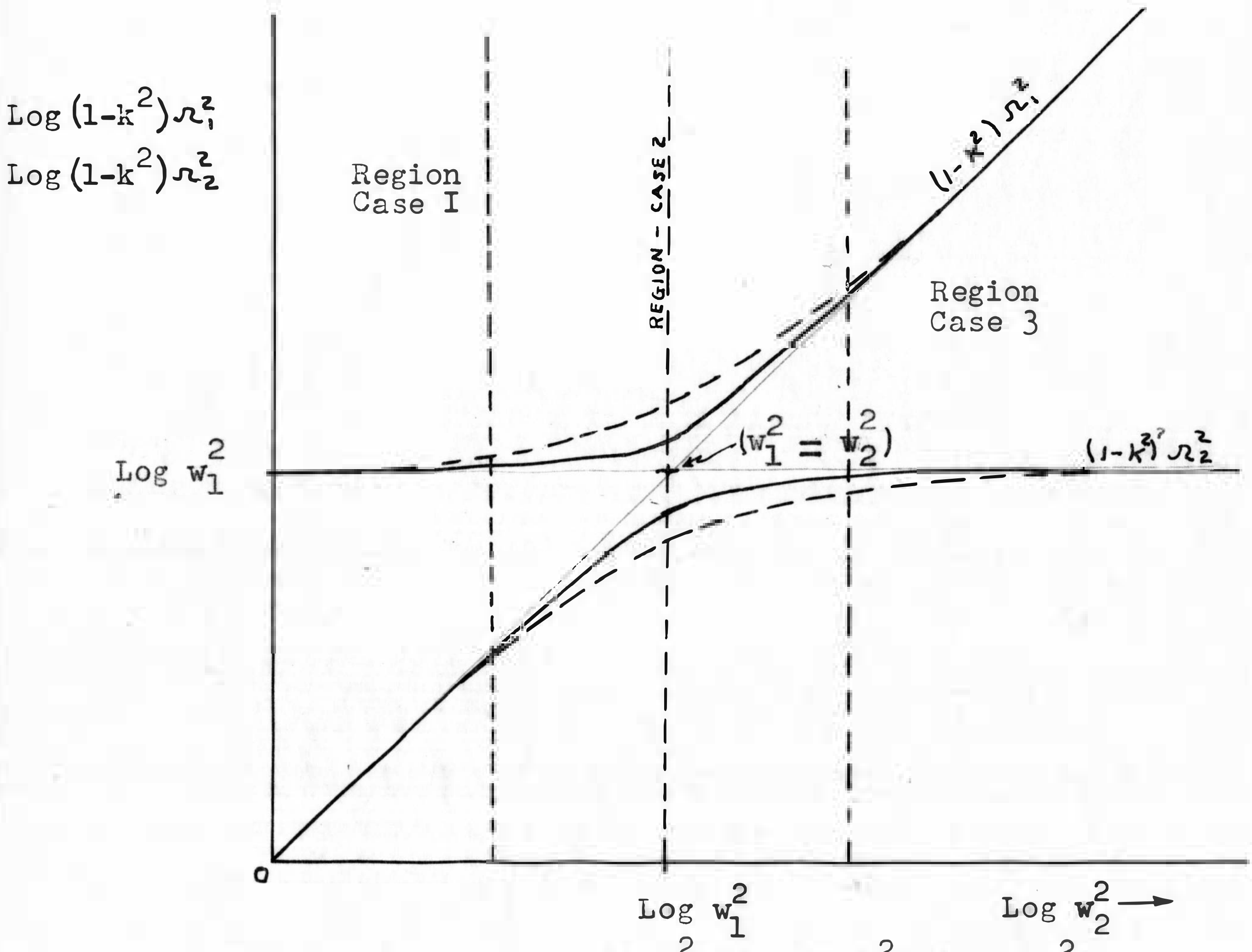
Case 2, where $w_1^2 \approx w_2^2$

Case 3, where $w_1^2 \ll w_2^2$

In the previous step it was shown that if the damping is small x could be neglected in equation $4(\frac{c}{d})$. Equation $4(\frac{c}{d})$ becomes as follows;

$$\Omega_1^2, \Omega_2^2 = \frac{w_1^2 + w_2^2 \pm \sqrt{(w_1^2 + w_2^2)^2 - 4w_1^2 w_2^2(1-k^2)}}{2(1-k^2)}$$

If w_1^2 and k^2 are held constant while w_2^2 is varied from 0 to ∞, Ω_1^2 and Ω_2^2 will vary as indicated by Fig. 4. The approximate range of the three cases are marked on the figure. The dotted curve indicates the affect of increasing K.



$(1-k^2)\Omega_1^2$ & $(1-k^2)\Omega_2^2$ vs. w_2^2

Plotted to Log Scales

FIGURE 4

From Fig. 4 it can be seen that the specific range of the three cases is dependent on the value of k. The value of x will also effect the range as α_1 and α_2 increase. In the following analysis, it should be kept in mind, that the ranges of the specific cases are not definite. Therefore, the results of the following analysis should be interpreted as a rough guide to the limiting values of the constant rather than an approximation of the value for a specific case.

If α_1 and α_2 are much smaller than β_1 and β_2, it would seem that the equations for the constants (equation 8) could be simplified by assuming that $-\alpha_1 + j\beta_1 \approx j\beta_1$ and $-\alpha_2 + j\beta_2 \approx j\beta_2$. However, the terms in equations for the constants involve differences which in some cases make α the important factor in determining the value of the constant. Furthermore, if α becomes an important factor, then β_1^2 and β_2^2 cannot be assumed approximately equal to Ω_1^2 and Ω_2^2. The terms involving differences are;

$$Z_1^2 + 2a_2Z_1 + w_2^2 = w_2^2 - \Omega_1^2 - 2\alpha_1(a_2 - \alpha_1) + 2j\beta_1(a_2 - \alpha_1)$$

$$Z_2^2 + 2a_2Z_2 + w_2^2 = w_2^2 - \Omega_2^2 - 2\alpha_2(a_2 - \alpha_2) + 2j\beta_2(a_2 - \alpha_2)$$

$$(\alpha_2 - Z_1)^2 + \beta_2^2 = \Omega_2^2 - \Omega_1^2 + 2\alpha_1(\alpha_1 - \alpha_2) - 2j\beta_1(\alpha_1 - \alpha_2)$$

$$(\alpha_1 - Z_2^2) + \beta_1^2 = \Omega_1^2 - \Omega_2^2 + 2\alpha_2(\alpha_2 - \alpha_1) - 2j\beta_2(\alpha_2 - \alpha_1)$$

In the above terms the imaginary part of the terms involve no differences, therefore Ω_1 and Ω_2 may be substituted for β_1 and β_2. Fig. 4 shows that when,

$$w_2^2 \ll w_1^2 \text{ than } \Omega_1^2 \rightarrow \frac{w_1^2}{1-k^2} \text{ and } \Omega_2^2 \rightarrow \frac{w_2^2}{1-k^2}$$

$$w_2^2 \approx w_1^2 \text{ than } \Omega_1^2 - \Omega_2^2 \rightarrow \text{a minimum}$$

$$w_2^2 \gg w_1^2 \text{ than } \Omega_1^2 \rightarrow \frac{w_2^2}{1-k^2} \text{ and } \Omega_2^2 \rightarrow \frac{w_1^2}{1-k^2}$$

Substituting these approximations into the above terms, assuming k is small, so that $(1-k^2) \approx 1$, except where differences are involved and also assuming that α_1 and α_2 are small, the constants approach the following limits;

For Case I $w_2^2 \ll w_1^2$

$$K_1 \rightarrow \frac{w_2^2(1-k^2) - w_1^2}{w_2^2 - w_1^2} \rightarrow 1$$

$$K_2 \rightarrow \frac{-k^2 w_2^2 - 2\alpha_2(a_2 - \alpha_2)(1-k^2) + 2j\beta_2(a_2 - \alpha_2)}{w_2^2 - w_1^2} \rightarrow \left[\text{a small value compared to } K_1\right]$$

$$K_1' \rightarrow -\frac{w_1^2}{w_2^2 - w_1^2}$$

$$K_2' \rightarrow -\frac{w_2^2}{w_1^2 - w_2^2}$$

For Case 2 $w_1^2 \approx w_2^2 \approx w_0^2$

$$K_1 \rightarrow \frac{1+k}{2}$$

$$K_2 \rightarrow \frac{1-k}{2}$$

$$K'_1 \rightarrow \frac{1+k}{2k}$$

$$K'_2 \rightarrow \frac{1-k}{2k}$$

For Case 3 $w_2^2 \gg w_1^2$

$K_1 \rightarrow$ [a small value compared to K_2]

$K_2 \rightarrow -1$

$$K'_1 \rightarrow - \frac{w_1^2}{w_2^2 - w_1^2}$$

$$K'_2 \rightarrow - \frac{w_2^2}{w_1^2 - w_2^2}$$

In the above approximations α_1 and α_2 are assumed negligible except where differences are involved. In Case 2 differences are involved, but these differences are generally large enough so that α_1 and α_2 may be neglected. $(1-k^2)$ was assumed approximately equal to 1. Again it should be pointed out that these approximations should not be used for any specific case but are presented as an approximate guide to the final result..

The constant terms for the regions where $a_2w_2^2 - a_1w_1^2 \rightarrow 0$ and where the damping is large cannot be simplified. For these regions the phase angle also becomes an important factor. The numerator of the constants K_1 and K_2 will be maximum when the real part and the imaginary part are equal. However, it is noted that as we increase α_1 or α_2 to obtain this condition the denominator will also increase. The effect is that the constant terms cannot become much larger than they are when α_1 and α_2 are small. Increasing α_1 or α_2 means that the damping is increased. Therefore, the vectors will be greatly attenuated when they finally are in phase. Inspection of the phase position indicates that as the damping is increased the initial phase difference between the vectors is decreased. This would indicate that the time required before the vectors are in phase is less. However, in practical cases this change in phase position is more than compensated by the increased attenuation.

The final step is the substitution of the constants and α_1, α_2, β_1 and β_2 into equation 7 or equation 15, the envelope function. This step is discussed in the next section.

SECTION V

TRANSIENT LIMITS AND DANGEROUS CONDITIONS

The general equation of the network of Fig. 2 is $f(t) = N\mathcal{R}\left[Ae^{Z_1 t} + Be^{Z_2 t} + Ce^{jwt}\right]$ where f(t), N(a real constant) and complex constants, A, B and C are defined in Table 1. $Z_1 = -\alpha_1 + j\beta_1$ and $Z_2 = -\alpha_2 + j\beta_2$.

The above equation describes three rotating vectors. The first two terms are the transient part of the equation and the third term is the steady state term. The two transient vectors are rotating at different velocities, β_1 and β_2 respectively. If there were no damping, the maximum transient amplitude will be the point where these vectors are in phase. However, these two vectors are exponentially damped by the factors $e^{-\alpha_1 t}$ and $e^{-\alpha_2 t}$. Therefore these vectors may be attenuated to a small value before they are in phase. The maximum transient amplitude will be governed by following four factors:

1. The attenuation of the vectors
2. The difference in speed of rotation
3. The initial phase position
4. The absolute magnitude of the constant

A plot of the envelope function (equation 11), of the transient term is the simplest way of determining the exact transient amplitude. The envelope equation is given in Section III, equation 11. A plot of this equation will reveal the maximum transient amplitude.

A close approximation of the maximum amplitude may be determined by letting $\cos\left[(\beta_1 - \beta_2)t + \psi_1 - \psi_2\right] = 1$, and solving this term for the minimum positive value of t. If,

$$\cos\left[(\beta_1 - \beta_2)t + \psi_1 - \psi_2\right] = 1$$

then

$$(\beta_1 - \beta_2)t + \psi_1 - \psi_2 = \cos^{-1} 1 = 0,\ 2\pi,\ 4\pi, \ldots$$

Let the minimum positive value of $t = t_o$, then the envelope equation becomes;

$$N\left[|A|\, e^{-\alpha_1 t_o} + |B|\, e^{-\alpha_2 t_o}\right]$$

Physically t_o is the time required for the two transient vectors to move from their initial phase position, (when $t = 0$), to a position where they are in phase. Therefore the accuracy of this method for determining maximum transient will depend on the ratio of $\frac{\alpha_m}{\beta_1 - \beta_2}$, where α_m is the smaller value of α_1 or α_2. This method will always give a transient amplitude slightly less than the true maximum amplitude. The true maximum amplitude may be determined by selecting several trial values of t slightly less than t_o and solving the envelope equation until the maximum point is determined.

Sample calculations have shown that if the ratio of

$\frac{\alpha m}{\beta_1 - \beta_2} > \frac{1}{10}$, the equation is very accurate. Whereas, if

$\frac{\alpha m}{\beta_1 - \beta_2} < \frac{1}{2}$, then the equations were almost 50% in error.

In this case a good sample value of t to try is equal to $1/2\ t_o$. In these calculations $\psi_1 - \psi_2$ was assumed to be 180°.

When there is a steady state term, this method may not be of any practical value, because the transient amplitude at $t = 0$ may have a value which balances out the steady state value. In such cases the maximum transient can occur when $t = 0$, but as this transient is completely balanced out by the steady state term, the actual voltage or current, at $t = 0$, is zero. This case is illustrated in the appendix.

The upper limit of the transient amplitude may be determined by assuming the damping is zero. This limit is $N(|A| + |B|)$. The current or voltage upper limit is $N(|A| + |B| + |C|)$.

The complete expression for f(t) may be written in terms of an envelope and angle function. Let m(t) equal the transient envelope equation, then;

$$f(t) = m(t)\cos(\beta_2 t + \psi_2 + \varphi(t)) + N|C|\cos(wt + \psi_3)$$

The envelope of f(t) is,

$$f(t) = \left[m(t)^2 + 2N|C|m(t)\left[\cos\,(w-\beta_2)t - \psi_2 + \psi_3 - \varphi(t)\right] + N^2|C|^2\right]^{1/2} \cos\left[\beta_2\, t + \psi_2 + \varphi(t) + p(t)\right]$$

$$p(t) = \frac{N|C|\,\sin\,\left[(w-\beta_2)t - \psi_2 + \psi_3 - \varphi(t)\right]}{m(t) + N|C|\,\cos\,\left[(w-\beta_2)t - \psi_2 + \psi_3 - \varphi(t)\right]}$$

The carrier frequency has been taken as β_2. Calculations are made easier if the carrier frequency is taken as highest frequency term.

For specific cases, assumptions can be made which simplify this equation. However it is often simpler to start with the general equation, equation 9, and derive an envelope equation for the specific case. For example, when $w_1^2 - w_2^2$ is large, one of the transient constant terms might be negligible and the envelope equation could be written in terms of the large transient and the steady state term. If one of the damping terms is large it may be possible to neglect the term with the larger damping. The example in the appendix is another illustration of how a simpler envelope equation may be derived.

The ratio of the transient amplitude to the steady state amplitude is $\frac{|A| + |B|}{|C|}$ or the actual current, or voltage amplitude, is $\frac{|A| + |B| + |C|}{|C|}$. A dangerous condition will occur when this ratio is large because quite often engineers design their circuits safely for the steady state amplitudes and assume that this will provide safety for the transients.

Inspection of the constant terms shows that $w^2 - w_2^2$ appears only in the numerator of the constants for the primary circuit (i.e. $i_1(t)$ and $v_1(t)$). This indicates that measurements or calculations of the transient or steady state term in the secondary do not indicate what the primary conditions are. This may be considered another dangerous condition.

Inspection of the constants of Table I, shows that the amplitude of the transient constants for any of the three input or forcing functions, cannot be greater than the respective constants for the system function. In the preceding section, the system function was analysed for specific regions of operation. A few simplifications were presented. In general, these simplifications applied only when several assumptions, which are not always valid, were made. In any specific case where definite values are known for the parameters, any simplifying assumptions that can be made are immediately evident. Therefore, it is felt that further analysis of this sort would be misleading and such an analysis is not essential.

SECTION VI

SUMMARY

This summary will be limited to general equations and to methods of calculation which will yield limits rather than the specific limits for any particular case or region. Any specific limits or results for a definite region presented in this thesis have been based on several assumptions, assumptions which are not valid for all cases. Therefore these specific limits will be omitted in this review. It is felt that the general equations developed have simplified the calculations to such an extent that for any specific case, limits or the complete transient response may readily be determined.

Figure 3 is an illustration of the network which is analyzed for transient conditions. In Section III, the general equation for this network was developed. This equation is,

$$f(t) = N \mathcal{R}\left[Ae^{Z_1 t} + Be^{Z_2 t} + Ce^{jwt}\right] \qquad \ldots 9$$

where f(t) is $i_1(t)$, $i_2(t)$, $v_1(t)$ or $v_2(t)$. The real constant, N, and the complex constants, A, B, and C are given in Table I, page 26, for an input voltage, e(t), of a unit pulse function, a unit step function or a unit sinusoidal or cosinusoidal function. $Z_1 = -\alpha_1 + j\beta_1$ and $Z_2 = -\alpha_2 + j\beta_2$.

The analysis of this equation was based on the analysis of the system function. The system function was defined as the $\mathcal{L}^{-1}$ NG(s), where NG(s) is the part of the general equations which is dependent only on the network parameters. The system function is part of the response for any type of input.

α_1, α_2, β_1, and β_2 are determined from equations 4 and 5. In Section IV, equations 4 and 5 were analyzed and simplified as much as possible. Also in this section, the constants for the system function were analyzed and a few limiting values of these constants were established for nebulous regions of operation. The constants A and B of equation 11 cannot be greater in value than the respective constants for the system function.

The first two terms of equation 11 are the transient portion. The transient portion can be written in terms of envelope and angle function. This is equation 10, page 25. A plot of this envelope equation will give the maximum transient amplitude. A close approximation of this maximum amplitude may be determined by letting $\cos\left[(\beta_1-\beta_2)t+\psi_1-\psi_2\right] = 1$. The envelope equation becomes;

$$N\left[|A|\, e^{-\alpha_1 t_0} + |B|\, e^{-\alpha_2 t_0}\right]$$

where t_0 is the minimum positive value of t determined by the following equation,

$$\cos^{-1}\left[(\beta_1 - \beta_2)t + \psi_1 - \psi_2\right] = 0,\ 2\pi,\ 4\pi,\ \ldots$$

This will give an amplitude slightly less than the true value. The true value may be determined by taking successive values of t less than t_o. When there is a steady state term,the maximum transient can occur at $t = 0$, but at this point the steady state will balance out the transient. Therefore this method, discussed in Section IV, may be of little use. The example in the appendix illustrates this case. The complete expression for f(t) may be written in terms of envelope and angle function. This is illustrated in Section IV and several ways this expression may be simplified are mentioned. The example in the appendix illustrates how the expression for f(t) may be simplified for the case where $w_1 = w_2 = w = w_o$ and $k \ll 1$.

The upper limit of the transient term may be determined by assuming that the damping is zero. Then the upper limit is $N\left[|A| + |B|\right]$. The upper current or voltage limit is $N\left[|A| + |B| + |C|\right]$.

Dangerous conditions are interpreted to be points where the ratio of transient amplitude to steady state amplitude is high. The upper limit of this ratio is $\frac{|A| + |B|}{|C|}$. A closer approximation of this ratio may be determined by a more exact expression for the transient term, $|A| + |B|$. The fact that the term $w^2 - w_2^2$ appears only in the numerator of the transient and steady state terms for the primary circuit,(i.e. $i_1(t)$ and $v_1(t)$), indicates that calculations or measurements in one circuit do not indicate what the conditions are in the other circuit. This may be interpreted as another dangerous condition.

In conclusion it should be pointed out that these results are limited to the circuit in Fig. 2 and oscillatory conditions have been assumed. However, similar analysis can be developed and similar results determined for the non-oscillatory case. For a circuit consisting of inductive coupling and parallel elements, the same form of analysis may be used if the differential equations are written on a node basis, a current source input assumed, and the equations solved for $v_1(t)$ and $v_2(t)$ instead of $i_1(t)$ and $i_2(t)$. This will give different equations for a_1, a_2, w_2, w_2 and k. The solution then follows in the same manner as the series circuits analysis presented here. An excellent reference on the node basis and current source method of solving circuits is Gardner and Barnes text "Transients in Linear Systems", chapter II.

Directly coupled circuits may also be analyzed in this manner. The equations for directly coupled circuits reduce to a Laplace Transform equation of the third order. Its roots can be assumed to be $(s + \alpha_1 - j\beta_1)(s + \alpha_1 + j\beta_1)(s + \alpha_2)$. The analysis can proceed in the same manner as presented in this thesis. The solution is simpler because β_2 in this case is zero. A good many filter circuits may be represented as directly coupled circuits.

APPENDIX

The procedure involved in the solution of mutually coupled circuits is illustrated by the following calculations for a special case. Fig. 3 is the circuit diagram. Let the circuit parameters be as follows:

$L_1 = L_2 = 10^{-4}$ henries $\qquad C_1 = C_2 = 10^{-8}$ farads

$R_1 = R_2 = .5$ ohms $\qquad M = 5 \times 10^{-7}$ henries

$e(t) = \sin(w_o t - \theta)$ where $\theta = \frac{\pi}{2}$

$$w_1 = w_2 = w_o = \frac{1}{\sqrt{L_1 C_1}} = 10^6 \text{ rdns/sec.}$$

$$a_1 = a_2 = a_o = \frac{R_1}{2L_1} = 2.5 \times 10^3 \text{ sec.}^{-1}$$

$$k = \frac{M}{\sqrt{L_1 L_2}} = .005$$

Therefore,

$$\alpha_1 = \frac{a_o}{1-k} = 2{,}512 \qquad \Omega_1^2 = \frac{w_o^2}{1-k} = 1.005025 \times 10^{12}$$

$$\alpha_2 = \frac{a_o}{1+k} = 2{,}488 \qquad \Omega_2^2 = \frac{w_o^2}{1+k} = .995025 \times 10^{12}$$

$$\beta_1^2 = 1.005019 \times 10^{12} \qquad \beta_1 = 1.0025 \times 10^6$$

$\beta_2^2 = .995019\text{x}10^{12}$ $\beta_2 = .9975\text{x}10^6$

$\Omega_1^2 - \Omega_2^2 = 1\text{x}10^{10}$ $\beta_1 - \beta_2 = 5\text{x}10^3$

Substituting the above values into the constants given in Table I and evaluating, gives the following values:

$K_1 = .503$ $K_3 = 1.41\text{x}10^{-4} \angle 45^\circ$

$K_2 = .500$ $K_4 = 1.41\text{x}10^{-4} \angle -45^\circ$

$K_1' = 1.005\text{x}10^2$ $K_5 = -1\text{x}10^{-4}$

$K_2' = -.995\text{x}10^2$ $K_6 = 2\text{x}10^{-2} \angle -90^\circ$

Therefore the constant terms for $i_1(t)$ are:

$A = K_1 K_3 = .710\text{x}10^{-4} \angle 45^\circ$

$B = K_2 K_4 = .704\text{x}10^{-4} \angle -45^\circ$

$C = K_5 = -1\text{x}10^{-4}$

$N = 1\text{x}10^4$

The constant terms for $i_2(t)$ are:

$A = K_1' K_3 = 1.41\text{x}10^{-2} \angle 45^\circ$

$B = K_2' K_4 = -1.40\text{x}10^{-2} \angle -45^\circ$

$C = 2\text{x}10^{-2} \angle -90^\circ$

$N = 50$

For this particular case, the calculations may be simplified by assuming that $|A| = |B|$ and that $\alpha_1 = \alpha_2 = 2{,}500$. The error introduced by these assumptions is very small. It is further noted that $(\beta_1 - w) = -(\beta_2 - w) = 2{,}500$. The general equation of Table I may be factored and written as follows:

$$f(t) = N\mathcal{R}\left[|A|\, e^{-2{,}500t}\left[e^{j(2{,}500t+\psi_1)} + e^{-j(2{,}500t+\psi_2)}\right] + C\right] e^{jwt}$$

For $i_1(t)$, $\psi_1 = -\psi_2 = 45^{\circ}$ and C is a real number. Therefore substituting the values for the constants and evaluating, $i_1(t)$ becomes:

$$i_1(t) = \mathcal{R}\left[.707e^{-2{,}500t}\left[e^{j(2{,}500t + 45^{\circ})} + e^{-j(2{,}500t + 45^{\circ})}\right] - 1\right] e^{jwt}$$

$$= \left[1.414e^{-2{,}500t}\cos(2{,}500t + 45^{\circ}) - 1\right] \cos(wt)$$

The equation for $i_2(t)$ may be written as follows:

$$i_2(t) = \left[1.414e^{-2{,}500t}\sin(2{,}500t + 45^{\circ}) - 1\right] \left[-\sin(wt)\right]$$

The equations for $i_1(t)$ and $i_2(t)$ are now in terms of envelope and frequency or carrier function. A plot of the envelope of $i_1(t)$ and $i_2(t)$ is given in Figure 6.

The above equations are in terms of a transient (i. e. and exponentially decaying term) and a steady state term. Let the envelope transient terms for $i_1(t)$ and $i_2(t)$ be represented by $i_1(x)$ and $i_2(x)$. Then:

$$i_1(x) = \left[1.414e^{-2,500t}\cos(2,500t + 45^{\circ})\right]$$

$$i_2(x) = \left[1.414e^{-2,500t}\sin(2,500t + 45^{\circ})\right]$$

It can be shown that these envelope equations are the same as derived directly by use of equation 11. The angle functions or carrier terms will be different for the two different ways of deriving the envelope equation. Figure 6 is a plot of the two envelope equations.

The upper limit of the transient term is,

$$N\left[|A| + |B|\right] = 1.414 \text{ amps, (primary)}$$

$$= 1.414 \text{ amperes, (secondary)}$$

The ratio of maximum transient limit to steady state amplitude is,

$$\frac{|A| + |B|}{|C|} = 1.414 \text{(primary)}$$

$$= 2.82 \text{(secondary)}$$

The approximate maximum transient amplitude is,

$$N\left[|A|\, e^{-\alpha_1 t_o} + |B|\, e^{-\alpha_2 t_o}\right] = .133 \text{ amps(primary)}$$

$$= .628 \text{ amps(secondary)}$$

Where

$$t_o = 3\pi \times 10^{-4} \text{sec. (primary)}$$

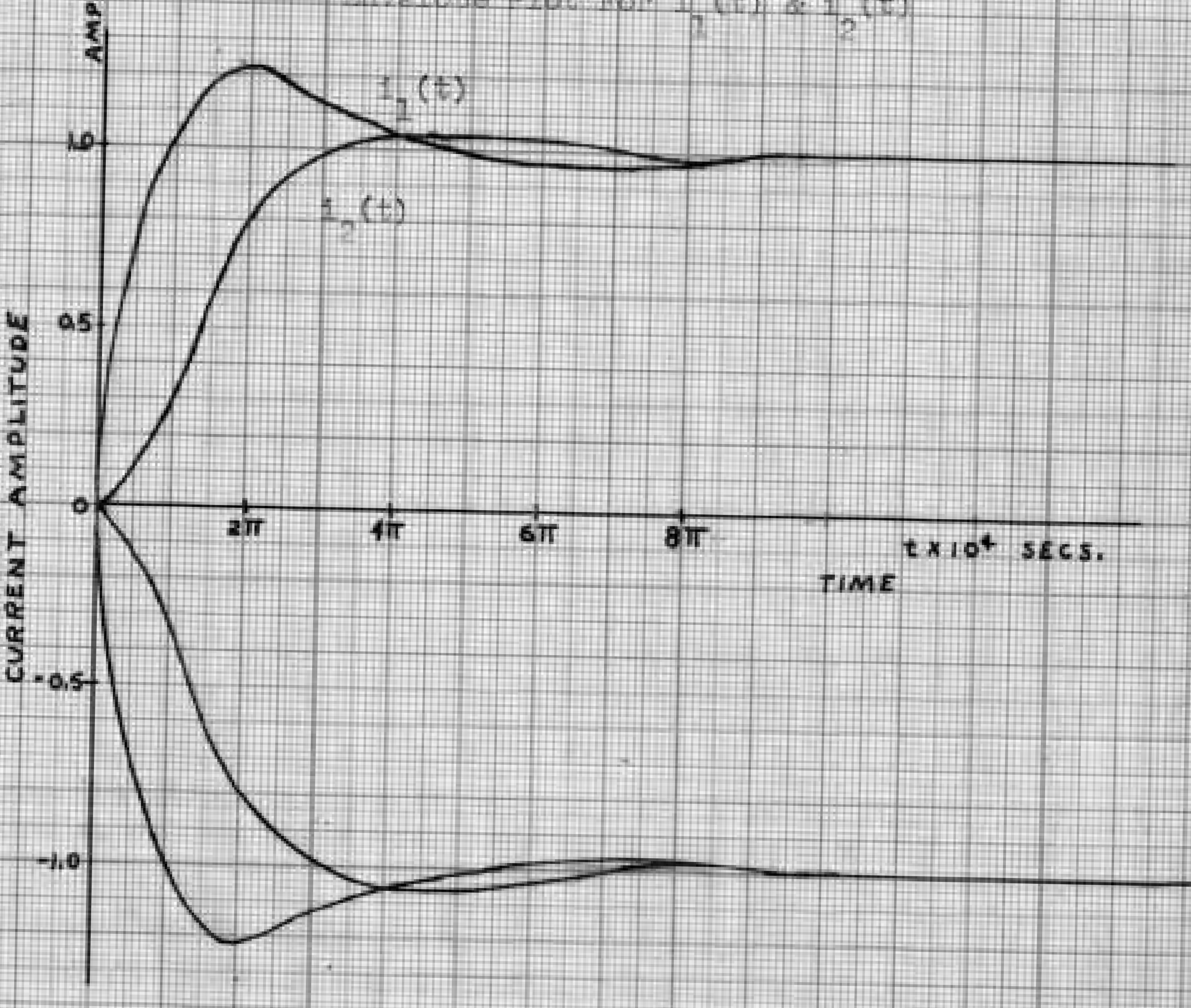
$$= \pi \times 10^{-4} \text{sec. (secondary)}$$

Inspection of the curves for the envelope equations show that these calculations are practically useless for this particular case. This is because the ratio of $\frac{\alpha m}{\beta_1 - \beta_2}$ is too large.

For this particular case the transient amplitudes were not dangerous. However, this does not imply that dangerous conditions do not exist. The reason they do not appear in this case is because the frequency terms β_1, β_2 and w, are all very close in value. Therefore the time lapse is large before the vectors of transient amplitude or the vectors of transient amplitude and steady state amplitude are in phase. Thus the attenuation is large. Consideration of the general equations show that this can occur only when $w_1 = w_2 = w = w_0$. If w were greatly different from w_0 and thus greatly different from β_1 or β_2, then the maximum current could approach $N\left[|A| \text{ or } |B| + C\right]$. If β_1 were made greatly different from β_2 by increasing the coupling or unbalancing the circuits, then the transient term could approach $N\left[|A| + |B|\right]$ as a limit. If β_1, β_2 and w were all greatly different in value then the maximum current could approach $N\left[|A| + |B| + |C|\right]$.

Figure 5

Envelope Plot For $i_1(t)$ & $i_2(t)$



Envelope Equations:

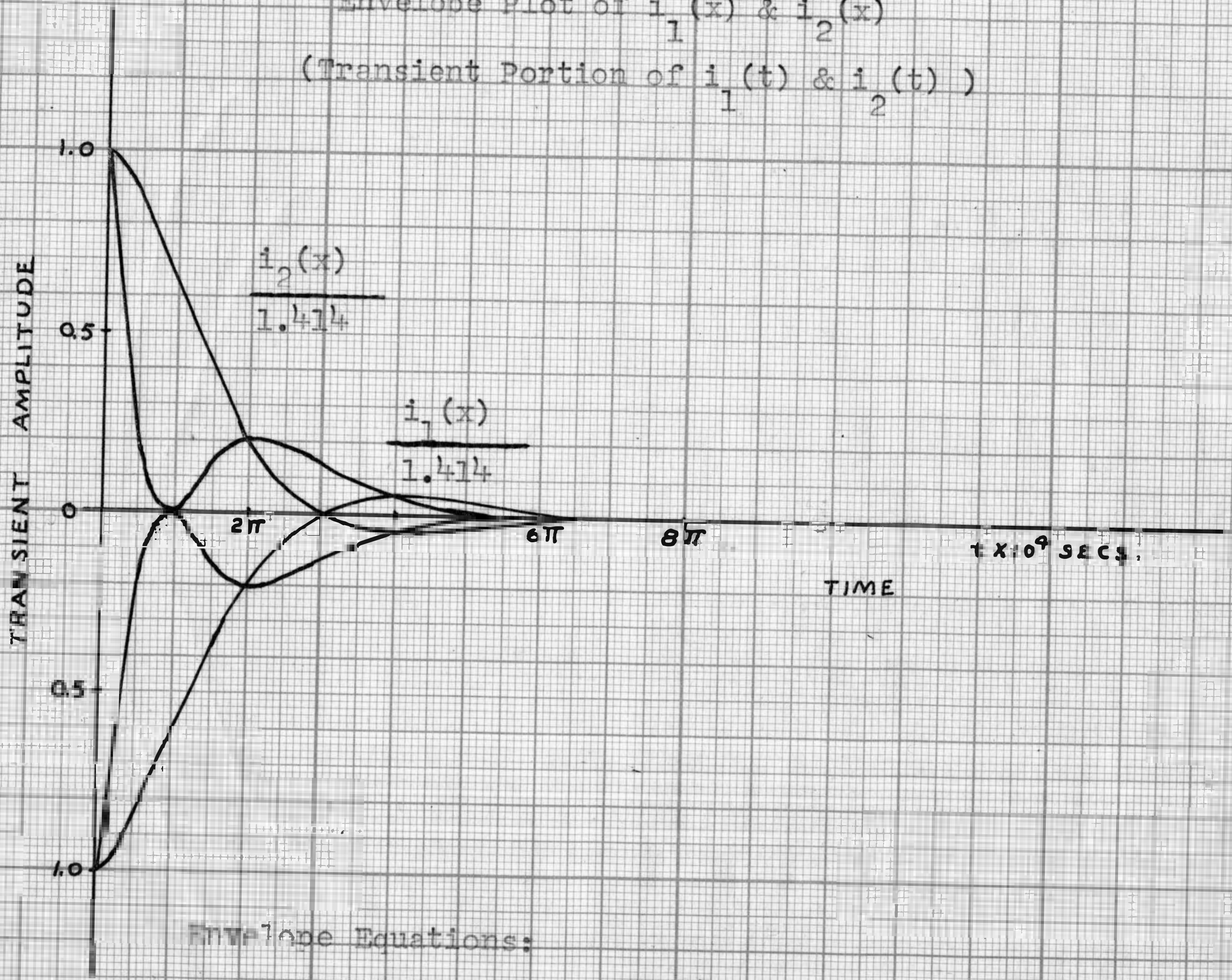
$$i_1(t) = 1.414\, e^{-2,500t} \cos(2,500t + 45^\circ) - 1$$

$$i_2(t) = 1.414\, e^{-2,500t} \sin(2,500t + 45^\circ) - 1$$

Figure 6

Envelope Plot of $i_1(x)$ & $i_2(x)$

(Transient Portion of $i_1(t)$ & $i_2(t)$)



Envelope Equations:

$$i_1(x) = (1.414)e^{-2,500t}\left[1-\sin(5\text{x}10^3t)\right]^{\frac{1}{2}}$$

$$i_2(x) = (1.414)e^{-2,500t}\left[1+\sin(5\text{x}10^3t)\right]^{\frac{1}{2}}$$

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