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On Zero-Sum Rado Numbers for the Equation ax_1 + x_2 = x_3

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ON ZERO-SUM RADO NUMBERS FOR THE EQUATION

$ax_1 + x_2 = x_3$

BY

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This thesis is approved as a creditable and independent investigation by a candidate for the Master of Science in Mathematics degree and is acceptable for meeting the thesis requirements for this degree. Accptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

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ABSTRACT

ON ZERO-SUM RADO NUMBERS FOR THE EQUATION $ax_1 + x_2 = x_3$

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For every positive integer a, let $n = R_{ZS}(a)$ be the least integer, provided it exists, such that for every coloring

$$\Delta : \{1, 2, ..., n\} \rightarrow \{0, 1, 2\},$$

there exist three integers x_1, x_2, x_3 (not necessarily distinct) such that

$$\Delta(x_1) + \Delta(x_2) + \Delta(x_3) \equiv 0 \ (mod\ 3)$$

and

$$ax_1 + x_2 = x_3.$$

If such an integer does not exist, then $R_{ZS}(a) = \infty$. The main results of this paper are

$$R_{ZS}(2) = 12$$

and a lower bound is found for $R_{ZS}(a)$ where $a \geq 2$.

Introduction

We first begin by defining some important terms.

Definition 1. Let $\mathbb{N}$ denote the set of natural numbers, and let $[a, b]$ denote the set $\{n \in \mathbb{N} | \ a \leq n \leq b\}$.

Definition 2. A function $\Delta : [1, n] \rightarrow [0, t-1]$ is called a *coloring* of the set $[1, n]$ with t colors.

Definition 3. Given a coloring function Δ and a system L of linear equations in m variables, a solution $(x_1, x_2, ..., x_m)$ is said to be *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

Definition 4. If L is a system of equations in m variables, then we say that a solution $(x_1, x_2, ..., x_m)$ to L is *zero-sum* if and only if

$$\Delta(x_1) + \Delta(x_2) + \cdots + \Delta(x_m) \equiv \ 0 \ (mod \ m).$$

Zero-sum Rado numbers have arisen from numerous results in combinatorics that began with Ramsey's Theorem [1, 2] which describes monochromatic complete subgraphs. Study of the earlier Schur's Theorem [1, 3] which proved a similar result involving monochromatic solutions to the basic equation $x_1 + x_2 = x_3$, revealed a connection to Ramsey's theorem.

Definition 5. For the Schur equation, $x_1 + x_2 = x_3$, for every $t \geq 2$, let $S(t)$ be the least integer, provided it exists, such that for every t-coloring of the set $[1, S(t)]$ there exists a monochromatic solution to the Schur Equation. $S(t)$ is called the t-color *Schur Number*.

Only the 2, 3, and 4 color Schur Numbers are known while the Schur Numbers for 5 or more colors are unknown and difficult to compute, so variations were subsequently explored. Rado, a student of Schur, explored different linear equations involving monochromatic solutions [1].

Definition 6. Let L be a system of equations, then define $D_t(L)$ as the least integer, provided it exists, such that every t-coloring of the set $[1, D_t(L)]$ contains a monochromatic solution to L. $D_t(L)$ is called the t-color *Rado*

Number for L.

Schur's and Rado's work concentrated on proving that these numbers existed and the situations in which they would exist. It wasn't until years later that the focus shifted to actually finding the Rado Number for a particular equation. The first such problem was for an equation in m variables and was proven by Beutelspacher and Brestovansky [4]. Again, while some Rado numbers are known, there are a great deal of equations for which the Rado number is unknown. The initial paper that explored the concept of zero-sum is the Erdős-Ginzburg-Ziv Theorem [5], which says that every sequence of elements of $\mathbb{Z}_m$ with length at least $2m-1$ contains a subsequence of length m with a sum of zero *mod m*.

Definition 7. Let L be a system of equations in m variables and define $D_{ZS}(L)$ as the least integer, provided it exists, such that every t-coloring of the set $[1, D_{ZS}(L)]$ contains a zero-sum Solution to L. $D_{ZS}(L)$ is called the *zero-sum Rado Number* for L.

We provide proofs of Ramsey's Theorem, Schur's Theorem, and the 2-color Rado number for $L(m) : x_1 + x_2 + \cdots + x_{m-1} = x_m$ proven by Beutelspacher and Brestovansky in the Background Results section.

Background Results

Ramsey's Theorem *For all integers $l_1, l_2 \geq 2$, there exists $n = R(l_1, l_2)$ such that for every coloring of the edges on a complete graph on n vertices, there exists either a complete subgraph on l_1 vertices monochromatic in color 1, or a complete subgraph on l_2 vertices monochromatic in color 2.*

Proof. We will use a double induction on l_1, and l_2. First we note that $R(l, 2) = R(2, l) = l$ since a complete graph on l vertices must contain either a complete graph in one color or would have at least two vertices connected with the other color. Thus we complete the basis case. Now we do the inductive step. Let arbitrary integers $l_1, l_2 \geq 2$ be given. We will show that $R(l_1, l_2)$ exists. Note that our induction hypothesis is that $R(l_1 - 1, l_2)$ and $R(l_1, l_2 - 1)$ exist.

We will now show that

$$R(l_1, l_2) \leq R(l_1 - 1, l_2) + R(l_1, l_2 - 1).$$

Let $n = R(l_1 - 1, l_2) + R(l_1, l_2 - 1)$. Let G be a complete graph on n vertices and let E be the set of edges of the graph G. Let $\Delta : E \to [0, 1]$ be given. Pick an arbitrary vertex x of G and consider the $n-1$ edges eminating from x. Let Δ_0 and Δ_1 be the sets of edges adjacent to x colored 0 and 1 respectively. We now note that either

$$|\Delta_0(x)| \geq R(l_1 - 1, l_2)$$

or

$$|\Delta_1(x)| \geq R(l_1, l_2 - 1).$$

Suppose without loss of generality that

$$|\Delta_0(x)| \geq R(l_1 - 1, l_2).$$

Then we know by the induction hypothesis, that there exists either a complete graph on $l_1 - 1$ verticies in color 0, or a complete graph on l_2 verticies monochromatic in color 1. If we have a complete graph on l_2 verticies in monochromatic color 1, then we are done. If we have a complete graph on $l_1 - 1$ verticies monochromatic in color 0, then when we consider that complete graph along with our arbitrary vertex x and adjacent edges, we now have a complete graph in l_1 verticies monochromatic in color 0. Therefore we have shown that a complete graph on $n = R(l_1 - 1, l_2) + R(l_1, l_2 - 1)$ verticies must contiain either a complete graph on l_1 verticies monochromatic in color 0, or a complete graph on l_2 verticies monochromatic in color 1. Thus we have shown that

$$R(l_1, l_2) \leq R(l_1 - 1, l_2) + R(l_1, l_2 - 1)$$

which completes the proof. $\square$

Ramsey's Theorem (Multi-Color) *For all integers $l_1, l_2, \ldots, l_t \geq 2$, there exists $n = R(l_1, \ldots, l_t)$ such that for every coloring of the edges on a complete graph on n vertices, for some $i \in [1, t]$ there exists a complete subgraph on l_i vertices monochromatic in color i.*

Note while we will not prove the multi-color version, a proof would be very similar to that of the two-color proof. The significant changes are using induction on the colors, and the useful inequality would be

$$R(l_1, \ldots, l_t) \leq 2 + \sum_{i=1}^{r} (R(l_1, \ldots, l_i - 1, \ldots, l_t) - 1).$$

Schur's Theorem *If* $\mathbb{N}$ *is colored with a finite number of colors, there exist natural numbers* x_1, x_2, x_3 *having the same color such that*

$$x_1 + x_2 = x_3.$$

Proof. Suppose we have t colors. Let $n \in \mathbb{N}$ such that $n + 1 = R(3, ..., 3)$. Let G be a complete graph on $n+1$ vertices and label the vertices from 1 to $n+1$. Let (i, j) denote the edge between the ith and jth vertices. Note that if we consider the coloring $\Delta : [1, n] \to [1, t]$ then we can induce a coloring Δ' on G given by

$$\Delta'(i, j) = \Delta(|i - j|).$$

By Ramsey's theorem, there must exist a monochoromatic complete graph on 3 verticies, that is, a monochromatic triangle. Then let i, j, k with $i < j < k$ be the vertices of our monochromatic triangle so,

$$\Delta'(i, j) = \Delta'(i, k) = \Delta'(j, k).$$

Now let $x_1 = j - i, x_2 = k - j, x_3 = k - i$. We will show that the triple (x_1, x_2, x_3) is a monochromatic solution to the Schur Equation. By definition of Δ' we have $\Delta(x_1) = \Delta'(i, j), \Delta(x_2) = \Delta'(j, k), \Delta(x_3) = \Delta'(i, k)$, so

$$\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$$

and

$$x_1 + x_2 = (j - i) + (k - j) = k - i = x_3.$$

Thus we have shown that for an arbitrary finite number of colors, there exists a natural number $n = S(t)$, called the Schur Number, such that every t-coloring of the set $[1, n]$ must contain a monochromatic solution to the equation $x_1 + x_2 = x_3$.

$\square$

We now note that if we have 2 colors, then $S(2) = 5$. This leads us to the final result in the section. This theorem was proven by Beutelspacher and Brestovansky [4].

Theorem *For all $m \geq 3$, the rado number $BB(m)$ for the equation*

$$L(m) : x_1 + x_2 + \cdots + x_{m-1} = x_m$$

is $m^2 - m - 1$.

Proof. Let $m \geq 3$ be given. *Lower Bound:* We will first show

$$BB(m) \geq m^2 - m - 1$$

by exhibiting a 2-coloring of the interval $[1, m^2 - m - 2]$ that avoids a monochromatic solution.

Let $\Delta : [1, m^2 - m - 2] \to [0, 1]$ be defined by

$$\Delta(x) = \begin{cases} 0 & x \in [1, m-2] \cup [m^2 - 2m + 1, m^2 - m - 2] \\ 1 & x \in [m-1, m^2 - 2m]. \end{cases}$$

Let $(x_1, x_2, \ldots, x_m)$ be a solution to $L(m)$. We will show that $(x_1, x_2, \ldots, x_m)$ is not monochromatic. We will show that this coloring avoids a monochromatic solution to $L(m)$. Suppose $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{m-1}) = 0$. Then

$$x_1 + x_2 + \cdots + x_{m-1} \geq m - 1 > m - 2$$

so $x_m \notin [1, m-2]$. Also note that if even one of x_i for some $i \in [1, m-1]$ then

$$x_m = x_1 + x_2 + \cdots + x_{m-1} \geq m^2 - m - 1 > m^2 - m - 2$$

so then $x_m \notin [1, m^2 - m - 2]$, so $x_i \in [1, m-2]$ for all $i \in [1, m-1]$ But also note that we must have

$$x_m = [x_1 + x_2 + \cdots + x_{m-1} \leq (m-1)(m-2) = m^2 - 3m + 2 < m^2 - 2m + 1$$

so $x_m \in [m-1, m^2 - 2m]$ so $\Delta(x_m) = 1$.

Now suppose that $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_{m-1}) = 1$. Then we know that

$$x_m = x_1 + x_2 + \cdots + x_{m-1} \geq (m-1)(m-1) = m^2 - 2m + 1 > m^2 - 2m$$

so therefore $x_m \in [m^2 - 2m + 1, m^2 - m - 2]$. So $\Delta(x_m) = 0$. Thus we see that Δ avoids a monocrhomatic solution, and we have shown that

$$BB(m) \geq m^2 - m - 1.$$

Upper Bound: We will now show that

$$BB(m) \leq m^2 - m - 1$$

by showing that any 2-coloring of the interval $[1, m^2 - m - 1]$ contains a monochromatic solution.

Let $\Delta : [1, m^2 - m - 1] \to [0, 1]$, be given. We can assume without loss of generality that

$$\Delta(1) = 0.$$

Then we know that if $\Delta(m - 1) = 0$, then we have the monochromatic solution $(1, 1, \ldots, 1, m - 1)$. So assume that

$$\Delta(m - 1) = 1.$$

Now if $\Delta(m^2 - 2m + 1) = 1$, then we have the monochromatic solution $(m - 1, m - 1, \ldots, m - 1, m^2 - 2m + 1)$. So we can assume that

$$\Delta(m^2 - 2m + 1) = 0.$$

Now note that if $\Delta(m^2 - m - 1) = 0$, then we have the monochromatic solution, $(1, 1, \ldots, 1, m^2 - 2m + 1, m^2 - m - 1)$ so we can assume that

$$\Delta(m^2 - m - 1) = 0.$$

Finally note that if $\Delta(m) = 0$, then we have the monochromatic solution $(1, m, m, \ldots, m, m^2 - 2m + 1)$. But also if $\Delta(m) = 1$, then we know that we have the monochromatic solution $(m - 1, m, m, \ldots, m, m^2 - m - 1)$ so we know that no matter what m is colored, we must have a monochromatic solution. Therefore we have shown that

$$BB(m) \leq m^2 - m - 1.$$

Putting the bounds together gives us that

$$BB(m) = m^2 - m - 1$$

and we are done. $\square$

Main Results

Definition 8. Let $L(a)$ represent the equation $ax_1 + x_2 = x_3$.

Let $R_{ZS}(a)$ represent the zero-sum Rado Number for $L(a)$.

Theorem 1. *The zero-sum Rado Number for $L(2)$ is equal to 12.*

Proof. First note that our equation to consider in this case is

$$L(2) : 2x_1 + x_2 = x_3.$$

For the remainder of this proof, the term zero-sum solution will mean a zero-sum solution to $L(2)$.

Lower Bound: We will first show that

$$R_{ZS}(2) \geq 12$$

by exhibiting a 3-coloring of the interval $[1, 11]$ that avoids a zero-sum solution to $L(2)$.

Let $\Delta : [1, 11] \to \{0, 1, 2\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & x \in \{1, 4, 7, 8, 11\} \\ 1 & x \in \{2, 3, 5, 10\} \\ 2 & x \in \{6, 9\}. \end{cases}$$

We will show that no solution to $L(2)$ can be zero-sum. Note if

$$\Delta(x_1) + \Delta(x_2) + \Delta(x_3) \equiv \; 0 \; (mod \; 3)$$

then

$$\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$$

or

$$\Delta(x_i) \neq \Delta(x_j)$$

when $i \neq j$.

Let (x_1, x_2, x_3) be a solution to $L(2)$. First we will show that (x_1, x_2, x_3) is not monochromatic. Suppose $\Delta(x_1) = \Delta(x_2) = 0$. Then since $x_1, x_2 \in \{1, 4, 7, 8, 11\}$ we must have

$$x_3 \in \{3, 6, 9, 10\}$$

but then $\Delta(x_3) \neq 0$. So there are no solutions monochromatic in 0.

Now if $\Delta(x_1) = \Delta(x_2) = 1$ then since $x_1, x_2 \in \{2, 3, 5, 10\}$, we must have

$$x_3 \in \{6, 7, 8, 9, 11\}$$

so we know that $\Delta(x_3) \neq 1$. So there are no solutions monochromatic in 1.

Now note that if $\Delta(x_1) = \Delta(x_2) = 2$ then $x_3 > 11$ so there are no solutions monochromatic in 2. Thus there are no monochromatic solutions to $L(2)$.

Now to check for solutions of with the property that $\Delta(x_i) \neq \Delta(x_j)$ when $i \neq j$, suppose

$$\Delta(x_1) = 0$$

and

$$\Delta(x_2) = 1.$$

Since $x_1 \in \{1, 4, 7, 8, 11\}$ and $x_2 \in \{2, 3, 5, 10\}$ then

$$x_3 \in \{4, 5, 7, 10, 11\}$$

so $\Delta(x_3) \neq 2$.

Now suppose $\Delta(x_1) = 0$ and $\Delta(x_2) = 2$. Then

$$x_3 \in \{8, 11\}$$

but then $\Delta(x_3) \neq 1$.

If $\Delta(x_1) = 1$ and $\Delta(x_2) = 2$ then

$$x_3 = 10,$$

and so $\Delta(x_3) \neq 0$.

Now suppose that $\Delta(x_1) = 1$ and $\Delta(x_2) = 0$. Then

$$x_3 \in \{5, 7, 8, 10, 11\}$$

so $\Delta(x_3) \neq 2$.

Finally note that there are no solutions where $\Delta(x_1) = 2$ since $2(6) + x_2 > 11$ for all $x_2 \in [1, 11]$. Thus there are no solutions to $L(2)$ where $\Delta(x_i) \neq \Delta(x_j)$ when $i \neq j$. Thus this coloring is free of zero-sum solutions, so we may conclude that there exists a 3-coloring on the set $[1, 11]$ that avoids a zero-sum Solution, so

$$R_{ZS}(2) \geq 12.$$

Upper Bound: We will now show that $R_{ZS}(2) \leq 12$. Let a coloring $\Delta : [1,12] \to [0,2]$ be given. Without loss of generality, suppose

$$\Delta(1) = 0.$$

If, $\Delta(3) = 0$ we have a zero-sum solution $(1,1,3)$, so we may assume without loss of generality that

$$\Delta(3) = 1.$$

Now note that if $\Delta(5) = 2$ or $\Delta(7) = 2$ we have Zero Sum solutions $(1,3,5)$ or $(3,1,7)$, so we know that

$$\Delta(5) \neq 2$$

and

$$\Delta(7) \neq 2.$$

We now proceed by cases.

Case 1: Suppose $\Delta(5) = 0$ and $\Delta(7) = 0$. Then we have a $(1,5,7)$ as a zero-sum Solution.

Case 2: Suppose $\Delta(5) = 1$ and $\Delta(7) = 1$. Then if $\Delta(9) = 2$, we have $(1,7,9)$ as a zero-sum solution. If $\Delta(9) = 1$, then $(3,3,9)$ is a zero-sum solution. Therefore we may assume

$$\Delta(9) = 0.$$

Now, if $\Delta(11) = 0$, then $(1,9,11)$ is a zero-sum solution, so $\Delta(11) \neq 0$. Also, if $\Delta(11) = 1$, then $(3,5,11)$ is a zero-sum solution, so $\Delta(11) \neq 1$. Finally, if $\Delta(11) = 2$, we have $(5,1,11)$ as a zero-sum solution. Therefore $\Delta(11) \neq 2$. Thus we must have a zero-sum solution no matter how 11 is colored.

Case 3: Suppose $\Delta(5) = 0$ and $\Delta(7) = 1$. Then if $\Delta(9) = 2$, we have $(1,7,9)$ as a zero-sum solution. If $\Delta(9) = 1$, then $(3,3,9)$ is a zero-sum solution. Therefore we may assume

$$\Delta(9) = 0.$$

If $\Delta(11) = 0$, then $(1,9,11)$ is a zero-sum solution, and if $\Delta(11) = 2$, then $(3,5,11)$ is a zero-sum solution, so assume

$$\Delta(11) = 1.$$

If $\Delta(2) = 0$, then $(2, 1, 5)$ is a zero-sum solution, and if $\Delta(2) = 1$, then $(2, 3, 7)$ is a zero-sum solution, so assume

$$\Delta(2) = 2.$$

If $\Delta(4) = 1$, then $(1, 2, 4)$ is a zero-sum solution, and if $\Delta(4) = 0$, then $(4, 1, 9)$ is a zero-sum solution, so assume

$$\Delta(4) = 2.$$

If $\Delta(6) = 1$, then $(1, 4, 6)$ is a zero-sum solution, and if $\Delta(6) = 2$, then $(2, 2, 6)$, is a zero-sum solution, so assume

$$\Delta(6) = 0.$$

If $\Delta(8) = 0$, then $(1, 6, 8)$ is a zero-sum solution, and if $\Delta(8) = 2$, then $(2, 4, 8)$ is a zero-sum solution, so assume

$$\Delta(8) = 1.$$

Finally if $\Delta(10) = 1$, then $(2, 6, 10)$ is a zero-sum solution, and if $\Delta(10) = 0$, then $(3, 4, 10)$ is a zero-sum solution, and if $\Delta(10) = 2$, then $(1, 8, 10)$ is a zero-sum solution. Thus Δ contains a zero-sum solution no matter how 10 is colored.

Case 4: Suppose $\Delta(5) = 1$ and $\Delta(7) = 0$. Then if $\Delta(9) = 0$, then $(1, 7, 9)$ is a zero-sum solution, and if $\Delta(9) = 1$, then $(3, 3, 9)$ is a zero-sum solution, so assume

$$\Delta(9) = 2.$$

If $\Delta(11) = 1$, then $(1, 9, 11)$ is a zero-sum solution, and if $\Delta(11) = 2$, then $(5, 1, 11)$ is a zero-sum solution, so assume

$$\Delta(11) = 0.$$

If $\Delta(2) = 0$, then $(2, 5, 9)$ is a zero-sum solution, and if $\Delta(2) = 2$, then $(2, 1, 5)$ is a zero-sum solution, so assume

$$\Delta(2) = 1.$$

If $\Delta(4) = 1$, then $(4, 1, 9)$ is a zero-sum solution and if $\Delta(4) = 2$, then $(1, 2, 4)$ is a zero-sum solution, so assume

$$\Delta(4) = 0.$$

If $\Delta(6) = 0$, then $(1, 4, 6)$ is a zero-sum solution, and if $\Delta(6) = 1$, then $(2, 2, 6)$ is a zero-sum solution, so assume

$$\Delta(6) = 2.$$

If $\Delta(8) = 1$ then $(1, 6, 8)$ is a zero-sum solution, and if $\Delta(8) = 2$, then $(2, 4, 8)$ is a zero-sum solution, so assume

$$\Delta(8) = 0.$$

If $\Delta(10) = 0$, then $(1, 8, 10)$ is a zero-sum solution, and if $\Delta(10) = 2$, then $(3, 4, 10)$ is a zero-sum solution, so assume

$$\Delta(10) = 1.$$

Finally if $\Delta(12) = 2$, then $(1, 10, 12)$ is a zero-sum solution, and if $\Delta(12) = 0$, then $(3, 6, 12)$ is a zero-sum solution, and if $\Delta(12) = 1$, then $(5, 2, 12)$ is a zero-sum solution, so Δ contains a zero-sum solution no matter how 12 is colored. We note that in Case 4, we are attempting to extend the specific coloring shown as the lower bound.

Since we have shown that Δ contains a zero-sum solution in all four cases, we can conclude that every 3-coloring of the set $[1, 12]$ must contain a zero-sum Solution to $L(2)$, so

$$R_{ZS}(2) \leq 12.$$

Therefore we have proven that

$$R_{ZS}(2) = 12.$$

$\square$

We proceed to establish lower bounds for the general cases for when $a > 2$. For the rest of this paper zero-sum Solution means a zero-sum Solution to $L(a)$.

Theorem 2. *For $a > 2$ and a odd, $R_{ZS}(a) \geq 2(a^2 + 3a + 1)$.*

Proof. Let $a > 2$, a odd be given. Let $\Delta : [1, 2(a^2 + 3a + 1) - 1] \to [0, 2]$ be given by

$$\Delta(x) = \begin{cases} 0 & if\ x\ odd \\ 1 & if\ 2 \le x \le 2a,\ 2a^2 + 4a + 2 \le x \le 2(a^2 + 3a + 1) - 1\ x\ even \\ 2 & if\ 2a + 2 \le x \le 2a^2 + 4a\ x\ even. \end{cases}$$

Let (x_1, x_2, x_3) be a solution to $L(a)$. We will show that Δ avoids a zero-sum solution. First note that (x_1, x_2, x_3) cannot be monochromatic in 0 since if x_1, x_2, x_3 were all odd, then the left side of $L(a)$ would be even, and the right side of $L(a)$ would be odd.

Now consider that it is not possible for only one of x_1, x_2, x_3 to be odd since if x_1 is odd and x_2, x_3 are even, then we have

$$x_3 = ax_1 + x_2 = odd\ +\ even = odd$$

which contradicts that x_3 is even. If x_2 is odd and x_1, x_3 are even, then we have

$$x_3 = ax_1 + x_2 = even\ +\ odd = odd$$

which contradicts that x_3 is even. Finally, if x_3 is odd and x_1, x_2 are even, then we have

$$x_3 = ax_1 + x_2 = even\ +\ even = even$$

which contradicts that x_3 is odd. Thus there are no solutions to $L(a)$ where only one of (x_1, x_2, x_3) is odd.

Next note that if we have a solution (x_1, x_2, x_3) where exactly two of them are odd and one is even, then it cannot be zero-sum. Without loss of generality, suppose that x_1, x_2 are odd, and x_3 is even. Then we would have

$$\Delta(x_1) + \Delta(x_2) + \Delta(x_3) = 0 + 0 + 1 \not\equiv 0\ (mod\ 3)$$

or

$$\Delta(x_1) + \Delta(x_2) + \Delta(x_3) = 0 + 0 + 2 \not\equiv 0\ (mod\ 3).$$

Therefore, any solutions (x_1, x_2, x_3) where exactly two of them are odd, cannot be zero-sum.

Thus the only possible zero-sum solutions must involve only even numbers. Such a solution would have the form, $(x_1, x_2, x_3) = (2\alpha, 2\beta, 2\gamma)$ for some natural numbers, α, β, γ. Note that since the evens are only colored with two colors, there will only be solutions of the form $\Delta(x_1) = \Delta(x_2) = \Delta(x_3)$. We now induce a coloring $\Delta' : [1, a^2 + 3a] \to [1, 2]$ given by $\Delta'(x) = \Delta(2x)$. Note that the number a^2+3a matches up with the number m^2-m-2 if $m = a+2$ since $(a+2)^2 - (a+2) - 2 = (a^2 + 4a + 4) - (a+2) - 2 = a^2 + 3a$. Also

$$\Delta'(x) = \begin{cases} 1 & if\ x \in [1, a] \cup [a^2 + 2a + 1, a^2 + 3a] \\ 2 & if\ x \in [a+1, a^2 + 2a]. \end{cases}$$

Now substituting $m = a + 2$, we obtain

$$\Delta'(x) = \begin{cases} 1 & if\ x \in [1, m-2] \cup [m^2 - 2m + 1, m^2 - m - 2] \\ 2 & if\ x \in [m-1, m^2 - 2m]. \end{cases}$$

Note that by the proof of Beutelspacher and Brestovansky [4] that Δ' avoids a monochromatic solution to $L(m)$. Therefore, if Δ containted a zero-sum solution $(x_1, x_2, x_3) = (2\alpha, 2\beta, 2\gamma)$, then Δ' would contain a monochromatic solution $(\alpha, \ldots, \alpha, \beta, \gamma)$, but since Δ' avoids a monochromatic solution, then we know that Δ does not contain a zero-sum solution. Therefore we have shown that there exists a 3-coloring of the set $[1, 2(a^2+3a+1)-1]$ that avoids a zero-sum Solution, so

$$R_{ZS}(a) \geq 2(a^2 + 3a + 1)$$

when a is odd. □

Theorem 3: *For $a > 2$ and a even, $R_{ZS}(a) \geq a^2 + 3a + 1$.*

Proof. Let $a \geq 2,\ a$ even be given. Let $\Delta : [1, a^2 + 3a] \to [0, 2]$ be defined by

$$\Delta(x) = \begin{cases} 0 & if\ x \in [1, a] \cup [a^2 + 2a + 1, a^2 + 3a] \\ 1 & if\ x \in [a+1, a^2 + 2a]. \end{cases}$$

We will show that Δ avoids a zero-sum solution by showing that any solution to $L(a)$ is not zero-sum. Note that since we only have two colors, the only possible zero-sum solutions are monochromatic solutions. Suppose we have a solution (x_1, x_2, x_3) to $L(a)$ such that $\Delta(x_1) = \Delta(x_2) = 1$. Then

$$\begin{aligned} x_3 = ax_1 + x_2 &\geq a(a+1) + (a+1) \\ &= a^2 + 2a + 1 \\ &> a^2 + 2a \end{aligned} \tag{1}$$

Thus $x_3 \in [a^2 + 2a + 1, a^2 + 3a]$, so $\Delta(x_3) = 0$.

Now suppose we have a solution (x_1, x_2, x_3) to $L(a)$ such that $\Delta(x_1) = \Delta(x_2) = 0$. Note that $x_1,\ x_2 \in [1, a]$, because if $x_2 \in [a^2 + 2a + 1, a^2 + 3a]$, then we have

$$\begin{aligned} x_3 = ax_1 + x_2 &\geq a(1) + a^2 + 2a + 1 \\ &= a^2 + 3a + 1 \\ &> a^2 + 3a \end{aligned} \tag{2}$$

so $x_3 \notin [1, a^2 + 3a]$. Therefore, $x_1,\ x_2 \in [1, a]$. Now finally, note that

$$\begin{aligned} x_3 = ax_1 + x_2 &\geq a(1) + 1 \\ &= a + 1 \\ &> a \end{aligned} \tag{3}$$

and

$$\begin{aligned} x_3 = ax_1 + x_2 &\leq a(a) + a \\ &= a^2 + a \\ &< a^2 + 2a \end{aligned} \tag{4}$$

Therefore $x_3 \in [a+1, a^2+2a]$, so $\Delta(x_3) = 1$. Therefore Δ avoids a monochromatic solution, so Δ must avoid a zero-sum solution.

Therefore we have shown that

$$R_{ZS}(a) \geq a^2 + 3a + 1$$

when $a > 2,\ a$ even.

$\square$

Note that there does exist a lower bound using 3 colors as well when a is even.

Further Research

Areas of further research based on these results will be to prove the upper bounds given in Theorems 2 and 3 are sharp upper bounds for both the even and odd case. Initial attempts have suggested that a set required to show the upper bounds is not a finite set, but is indexable with respect to the parameter a. Once the upper bounds are completed, the research would continue onto other variations of this problem. Some variations might include other coefficients, $ax_1 + bx_2 = cx_3$, adding constants, $ax_1 + x_2 + c = x_3$, or changing the number of variables and considering zero-sum problems over more colors with the equation $x_1 + x_2 + x_3 = x_4$.

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