View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Public Research Access Institutional Repository and Information Exchange

The Journal of Undergraduate Research

Volume 11 *Journal of Undergraduate Research, Volume 11: 2013* Article 6

2013

Theory and Applications of Correspondences

Heather Olson
South Dakota State University

Follow this and additional works at: http://openprairie.sdstate.edu/jur

Part of the Mathematics Commons

Recommended Citation

Olson, Heather (2013) "Theory and Applications of Correspondences," *The Journal of Undergraduate Research*: Vol. 11, Article 6.
Available at: http://openprairie.sdstate.edu/jur/vol11/iss1/6

This Article is brought to you for free and open access by Open PRAIRIE: Open Public Research Access Institutional Repository and Information Exchange. It has been accepted for inclusion in The Journal of Undergraduate Research by an authorized administrator of Open PRAIRIE: Open Public Research Access Institutional Repository and Information Exchange. For more information, please contact michael.biondo@sdstate.edu.

Theory and Applications of Correspondences

Author: Heather Olson

Faculty Sponsor: Donna Flint, Ph.D.

Department: Mathematics and Statistics

ABSTRACT

Throughout this paper, we will examine correspondences, also known as set-valued functions. A definition of correspondences and their graphs are given. Properties of correspondences including continuity, optimization, and existence of fixed points are considered. Examples are considered demonstrating these properties. Applications in the field of Economics are introduced. Specifically the optimization of consumer utility is examined through examples.

INTRODUCTION

Functions are typically denoted $f: X \to Y$ where $f(x) = y$ with $x \in X$ and $y \in Y$. Functions are traditionally single valued such that for any x from the domain, the output will be a single point, y, from the range. We can now expand our knowledge to correspondences, or set valued functions. Correspondences are denoted by $f: X \to\to Y$ where for any $x \in X$ the output is written $f(x) = y$, but y is some subset of Y instead of a single point. Before considering a few examples we will clearly define the graph of a correspondence as found in [5].

Definition 1: *Define $f: X \to\to Y$ to be a correspondence. The graph of $f(x)$, denoted $Gr(f)$, is defined as the set $Gr(f) \coloneqq \{(x, y) \in X \times Y \mid y \in f(x)\}$.*

Example 1: Define $f: \mathbb{R} \to\to \mathbb{R}$ such that $f(x) = \{-x, x\}$.

x	y
-2	{2,-2}
$-\frac{3}{2}$	$\{-\frac{3}{2}, \frac{3}{2}\}$
0	{0}
$\frac{1}{2}$	$\{-\frac{1}{2}, \frac{1}{2}\}$
$\frac{3}{2}$	$\{-\frac{3}{2}, \frac{3}{2}\}$
2	{-2,2}

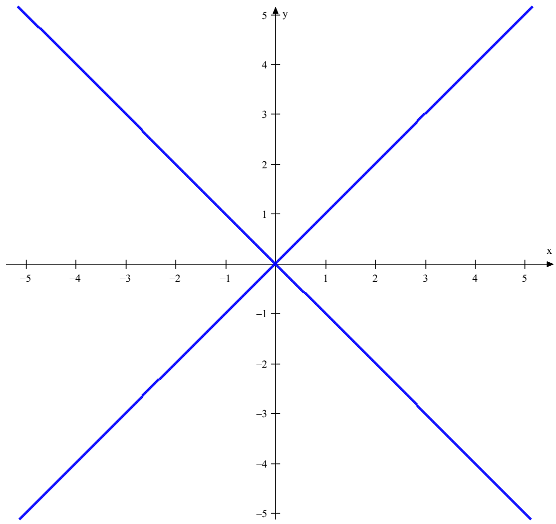


Figure 6. Table and $Gr(f)$

Example 2: Define $g: \mathbb{R}_+ \to\to \mathbb{R}_+$ such that

$$g(x) = \begin{cases} \emptyset & \text{when } x = 0 \\ \left[\frac{1}{x}, x\right] & \text{when } x < 1 \\ 2 & \text{when } x = 1 \\ \left[0, \frac{1}{2}\right] & \text{when } x > 1 \end{cases}.$$

x	y
0	$\emptyset$
$\frac{1}{2}$	$\left[\frac{1}{2}, 2\right]$
1	2
$\frac{3}{2}$	$\left[0, \frac{1}{2}\right]$
2	$\left[0, \frac{1}{2}\right]$

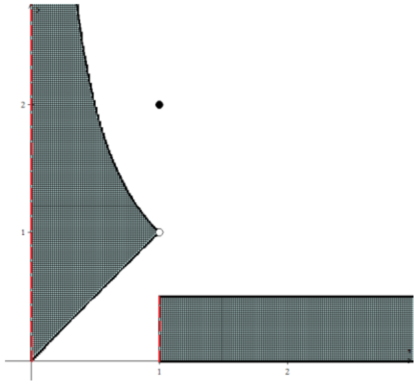


Figure 7. Table and $Gr(g)$

Example 3: Define $h\colon \mathbb{R}_+ \cup \{0\} \to\to [1,2]$ such that $h(x) = [1,2]$.

x	y
1	[1,2]
2	[1,2]
3	[1,2]
4	[1,2]
5	[1,2]

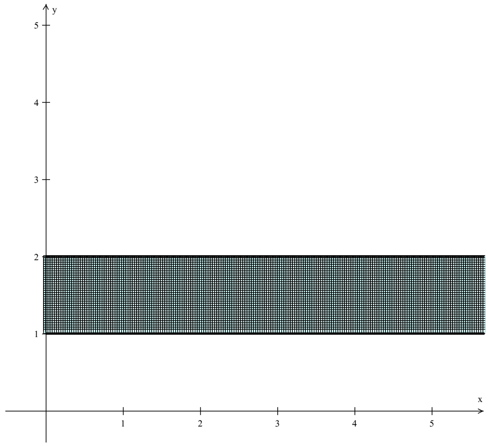


Figure 8. Table and $Gr(h)$

The examples above demonstrate three ways of representing a correspondence: definition, data values, and graphically (Figures 1, 2, and 3). The objectives of this paper are to determine what characteristics a correspondence needs in order to be continuous, when a fixed point is guaranteed to exist, where and how a correspondence is optimized, and how correspondences can be useful in other fields of study.

Considering Sets

Consider an arbitrary, nonempty set S and an arbitrary member $s \in S$. We can determine many characteristics of S by applying classifications to all of its members. Most of these classifications will be dependent upon the points that surround a particular member.

Definition 2: *Let $\epsilon > 0$, S be an arbitrary nonempty set, and $s \in S$. We define a neighborhood about s to be $N(s) = (s - \epsilon, s + \epsilon)$. A deleted neighborhood about s is defined as $N^*(s) = (s - \varepsilon, s) \cup (s, s + \varepsilon)$.*

Observing neighborhoods about s allow for determination of the characteristics of s. Member points can be classified into one or more of the following categories:

1. **Interior Point**: A member $s \in S$ is an interior point of S if there exists $N(s) \subset S$. In other words, if there exists a neighborhood about a member of the set that is completely contained in the set, then that member is an interior point.
2. **Boundary Point:** A point s is a boundary point of S if every $N(s)$ has at least one point that is in S and at least one point that is not in S. A boundary point does not have to be an a member of the set.
3. **Limit Point:** Any $x \in \mathbb{R}$ is a limit point of S if every $N^*(s)$ has at least one point in S. Note that a limit point does not have to be a member of the set.
4. **Isolated Point:** A member $s \in S$ is an isolated point of S if there exists an $N(s)$ such that the only point in $N(s)$ that is a member of S is s.

Example 4: $S = \{s \in \mathbb{R} \mid s \in (0,5)\}$

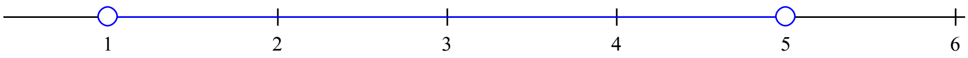


Figure 9. Graph of set S (example 4)

Interior points: Each $s \in S$ is an interior point because there is a neighborhood about each point such that every element in the neighborhood is also be a member of S (Figure 4).

Boundary points: The set $\{1, 5\}$ is the set of boundary points. Any neighborhood of these values that are not in the set have a member that is a member of S.

Limit Points: The set $\{1,5\} \cup S$ is the set of limit points of S. A deleted neighborhood about 1,5, or any member of S, results in the deleted neighborhood containing at least one element of S.

Isolated points: There are no isolated points in S because every neighborhood of every member contains other elements of S.

Example 5: $S = \{1,2,3,4,5\}$

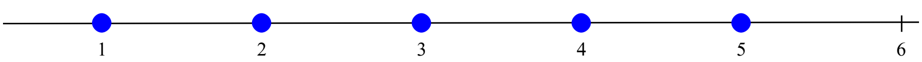


Figure 10. Graph of set S (example 5)

Interior points: There are no interior points in this example because there is always a neighborhood about each member that has elements that are not in S (Figure 5).

Boundary points: Every element of S is a boundary point because every neighborhood about every member contains one element of S which happens to be the member itself, and one point not in the set.

Limit Points: This set has no limit points because for each real number, there exists a deleted neighborhood that contains no members of S.

Isolated points: Every element in S is an isolated point by definition.

Now that we are able to classify the members within a specific set we can now use those classifications to determine its properties. Specifically, we will want to determine whether a set is an open set or a closed set.

Definition 3: *A nonempty set S is open if every $s \in S$ is an interior point. A set S is closed if its complement S^c is open or if the set contains all of its limit points.*

In Example 4, the set S is an open set since every element is an interior point. However, the set in Example 5 does not have any interior points. Consider, in example 5, $S^c = \{\mathbb{R} - \{1,2,3,4,5\}\}$. The elements in S^c are all interior points since there exists a neighborhood of each element which is contained in S^c itself. Since S^C is an open set, S is a closed set. Another way to look at this would be to reconsider the limit points of S. Since there were no limit points we determine the set of limit points to be the empty set, and since the empty set is a subset of S we can say that S contains all of its limit points. Thus S is closed.

Other necessary concepts include supremum and infimum, upper and lower bounds, and compactness. A set S is bounded above if there exists $x \in \mathbb{R}$ such that for every $s \in S$, $s \leq x$. The real number x is referred to as the upper bound of S. If this x also happens to be the smallest real number for which $s \leq x$, then x is called the supremum of S denoted $x = \sup(S)$. Note that x does not have to be a member of the set in order to be the set's supremum. A similar definition is given for the lower bound of S where for some $y \in \mathbb{R}$ and every $s \in S$, $s \geq y$. If the real number y is also the largest real number for which $s \geq y$, then y is called the infimum of S denoted $y = \inf(S)$. If a set has an upper bound and a lower bound we call the set bounded. A set is called a compact set if the set is both closed and bounded.

Referring back to Example 4, we can classify that this set is bounded with $\sup(S) = 5$ and $\inf(S) = 1$. The set is not classified as compact since we determined it to be an open set. The set in Example 5 is also bounded with $\sup(S) = 5$ and $\inf(S) = 1$. Since the set is closed and bounded we can also determine that it is compact.

Continuity of Correspondences

These preliminary ideas provide a setup for the conditions a correspondence must meet in order to be classified as a continuous correspondence. We first define the concept of continuity as found in [1].

Definition 4: *A correspondence $f: X \to\to Y$ is continuous if its codomain is compact, if every element in the domain maps to a nonempty subset of the range, and if $Gr(f)$ satisfies the following two conditions:*

1. *$Gr(f)$ is closed and*
2. *For every $(x, y) \in Gr(f)$, if a sequence $\{x_n\}$ converges to x then there must exist a sequence $\{y_n\}$ that converges to $y \in f(x)$ where $y_n \in f(x_n)$ for all n.*

When a correspondence satisfies the latter of the two conditions it is said to be lower hemicontinuous, which is commonly abbreviated LHC.

Reconsider $Gr(f)$ from Example 1 where $f(x) = \{-x, x\}$. We will consider each continuity requirement to determine if f is continuous.

Claim 1: The function has a compact codomain.

> The codomain of f is $\mathbb{R}$ which is neither closed nor bounded. Thus the codomain is not compact and we can conclude that f is not continuous.

Claim 2: Every element in the domain maps to a nonempty subset of the codomain.

> Since $f(x) = \{-x, x\}$ we can conclude that this holds true for every element in the domain.

Claim 3: The $Gr(f)$ is closed.

Recall $Gr(f) \coloneqq \{(x, y) \in X \times Y \mid y \in f(x)\}$ and note that when determining whether $Gr(f)$ is closed we consider the $(x, y) \in Gr(f)$ as a subset of all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Considering a neighborhood, in this case an infinitely small circle, about an arbitrary ordered pair in the graph, note that each point is a limit point. Consider every ordered pair in $Gr(f)^c$ and determine whether there exist limit points of $Gr(f)$ outside of $Gr(f)$. Note that each neighborhood can be made small enough so as to not contain a point in $Gr(f)$. Since $Gr(f)$ contains all of its limit points, $Gr(f)$ to be closed.

Claim 4: The $Gr(f)$ meets the LHC requirement.

Consider every sequence $\{x_n\} \to x$ and determine whether there exists a sequence $\{y_n\} \to y$ with $y_n \in f(x_n)$ for all n where $y \in f(x)$. To do this we need to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. This is basically saying that for any sequence that gets infinitely closer to an arbitrary x there exists a sequence of function values that gets infinitely closer to $f(x)$. For this example consider the two parts of the graph: $y = x$ and $y = -x$ and show that this is true for both graphs.

Define $f(x) = x$. Let $\varepsilon > 0$ and $\delta > 0$. Assume $|x - x_0| < \delta$. Then

$$|f(x) - f(x_0)| = |(x) - (x_0)| < \delta$$

Choose $\varepsilon = \delta$. Then substituting from above yields

$$|f(x) - f(x_0)| < \epsilon.$$

Next, define $f(x) = -x$. Let $\varepsilon > 0$ and $\delta > 0$. Assume $|x - x_0| < \delta$. Then

$$\begin{aligned} |f(x) - f(x_0)| &= |(-x) - (-x_0)| \\ &= |-x + x_0| \\ &= |x - x_0| < \delta \end{aligned}$$

Choose $\varepsilon = \delta$. Then substituting from above we have

$$|f(x) - f(x_0)| < \epsilon.$$

Thus, f satisfies the LHC requirement.

Next we shall reconsider Example 2. This correspondence will prove to be discontinuous for many reasons, but we will discuss each requirement.

Claim 1: The function has a compact codomain.

Consider the union of all of the subsets that make up the codomain and note that it is $\mathbb{R}_+$ which is neither closed nor bounded. Thus g does not map to a compact codomain.

Claim 2: Every element in the domain maps to a nonempty subset of the codomain.

The correspondence maps zero to the empty set and thus fails this claim.

Claim 3: The $Gr(g)$ is closed.

Consider an $(x, y) \in Gr(g)$ where $x < 1$. For this portion of the graph, note that the limit points include (0,0), every point on the curve $y = \frac{1}{x}$, every point on the line $y = x$, and (1,1). Since 0 and 1 are limit points that are not contained in $Gr(g)$, we can classify this portion to be not closed. Now consider an $(x, y) \in Gr(g)$ where $x > 1$. The limit points of this part of the graph include $(x, \frac{1}{2})$ and $(x, 0)$ which are both contained in the graph. This portion of the graph is closed. Now consider the graph as a whole. We cannot classify the graph as open because not every point on the graph is an interior point. We also cannot classify the graph as closed because it does not contain all of its limit points. Thus, we classify the graph to be neither open nor closed.

Claim 4: The $Gr(g)$ meets the LHC requirement.

Consider all of the possible sequences $\{x_n\}$ that approach 1 from the left hand side. All of the possible sequences $\{y_n\}$ are approaching 1, but $(1,1) \notin Gr(g)$. Now, consider all of the possible sequences $\{x_n\}$ that approach 1 from the right hand side. Some of the possible sequences $\{y_n\}$ are converging to values in the interval $[0, \frac{1}{2}]$. Other sequences are not converging at all. Consider the sequence where $y_n = \frac{1}{2}$ when n is even and $y_n = 0$ when n is odd. This sequence $\{y_n\}$ bounces back and forth between 0 and $\frac{1}{2}$ and does not converge at all. However, from our definition of LHC we need only there to exist at least one sequence $\{y_n\}$ that does converge. However, when $x = 1$, $y = 2$. Therefore, we still don't have a y value that is a member of the $Gr(g)$ even for the sequences that do converge from the right hand side. Thus $Gr(g)$ does not meet the LHC requirement.

Lastly, reconsider Example 3 and determine whether it is a continuous correspondence. From Example 3, we have $Gr(h)$ where, for $h: \mathbb{R}_+ \cup \{0\} \to\to [1,2]$, $h(x) = [1,2]$. For this final Example we can conclude that h is in fact a continuous correspondence. The details of each claim explaining this conclusion are left to the reader as an exercise.

Fixed Points

Next we turn to the definition of a fixed point. Recall that x is a fixed point of a single valued function $f: X \to Y$ if $f(x) = x$. The fixed point theorem, which states that if a continuous function given as $f: X \to X$ where X is a compact metric space then f has at least one fixed point, is the most common way to determine if a function has a fixed point [4]. The definition and applicable theorem differ from single valued to multi valued functions. We will define a fixed point of a correspondence and consider Kakutani's Fixed Point Theorem as found in [1].

Definition 5: *A fixed point of a correspondence $f: X \to\to Y$ is an $x \in X$ for which $x \in f(x)$.*

Definition 6: *A set A is convex if given two points contained in set A, the line segment connecting the two points is also contained in set A.*

Theorem (Kakutani's Fixed Point): *Let $f: S \to\to S$ be a correspondence. If S is nonempty, compact, and convex and if f is nonempty-valued, convex-valued, and has a closed graph then f has a fixed point.*

Proof Let f be as above and let $S = [a, b]$. Assume that f has no fixed point. Then the set $f(x)$ must not intersect the line $y = x$ for all $x \in [a. b]$. Since f is convex valued, that must mean that for all x, $f(x)$ is either all above the line or all below the line. First consider $f(a)$. Since the range of f is contained in $[a, b]$ and there is no fixed point, values in $f(a)$ must be greater than a. Thus $f(a)$ is above the line $y = x$. Let x_0 be a point in $(a, b]$ such that $f(x)$ is above the line $y = x$ for $x \in [a, x_0)$ but $f(x_0)$ is below the line $y = x$. Then $Gr(f)$ is open on the right along the vertical line $x = x_0$ which contradicts the assumption of a closed graph. Suppose no such x_0 exists. Then $f(x)$ is above the line $y = x$ for $x \in [a, b]$. This implies that $f(b) > b$, which is outside the range of the function. Therefore, f must have at least one fixed point.

Consider Example 3 where, for $h: \mathbb{R}_+ \cup \{0\} \to\to [1,2]$, $h(x) = [1,2]$. We shall restrict the domain to [1,2]. Note that [1,2] is nonempty, compact, and convex. Also note that h is nonempty-valued, convex-valued, and has a closed graph. Kakutani's Fixed Point Theorem guarantees that there is at least one fixed point in h. Every $x \in [1,2]$ will be a fixed point of h.

Recall Example 1 where $f(x) = \{-x, x\}$ but redefine the domain and range to be $f: [5,10] \to [5,10]$. Note that [5,10] is nonempty, compact, and convex. However on this interval, f is not convex-valued. Though every point in $[5,10]$is a fixed in point in f, this example does not satisfy Kakutani's fixed point theorem

With an understanding of continuity and fixed points, we now discuss optimization of correspondences. Optimization occurs when a correspondence attains an absolute maximum and/or absolute minimum within its domain. We first define a maximum and minimum as found in [4].

Definition 6: *Define $f: X \to\to Y$ to be a correspondence and*

$$A_x = \{a \in f(x) | a = sup(f(x))\}.$$

Then we define y^* as the maximum of f if $y^* = sup(A_x)$. The minimum of a correspondence is defined similarly.

In other words, the maximum of f can be found by first finding all of the suprema of each set in the codomain and constructing a set to include only those suprema that are members of the codomain. The largest member of the set will be the maximum of the function f.

The most common way to guarantee optimization of a function, whether single or multi valued, is to use the Weierstrass Theorem [1] which is stated here without proof.

Theorem (Weierstrass): *Let X be a compact metric space and $f: X \rightarrow\rightarrow R$ be continuous. Then f attains its max and min in X.*

Recall Example 3 where, for $h: \mathbb{R}_+ \rightarrow\rightarrow [1,2]$, $h(x) = [1,2]$. The correspondence has been shown to be continuous. Restricting the domain of the function to be any interval $[a, b]$ the Weierstrass theorem applies and h is guaranteed to have a maximum and minimum value on any interval $[a, b]$ since it meets the continuity and compact domain conditions. If the domain of h is restricted to be [1,2], the maximum is $A_x = \{2\}$ and $y^* = 2$.

Restricting the domain of $f(x) = \{-x, x\}$ from Example 1 to be any interval [a,b] assures the Weierstrass theorem can be applied in a similar way. Note that on a restricted domain, the graph becomes closed and thus f becomes continuous. Then the Weierstrass theorem guarantees that f will have a maximum and minimum. For the sake of example, restrict the domain to [5,10]. For the maximum, $A_x = [5,10]$ and $y^* = 10$. Simliarily for the minimum $A_x = [-10, -5]$ and $y^* = -10$.

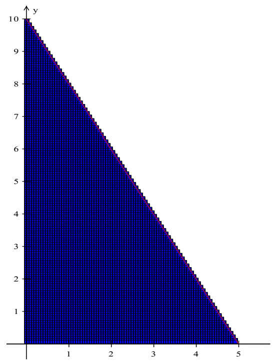
Application in Economics

Correspondences are commonly used for applications in the field of economics. We will explore one specific application here regarding a consumer's utility function. In economics, the assumption is made that whenever a consumer is faced with the choice of buying a bundle of different goods a choice can always be made that maximizes the consumer's happiness. This happiness is called utility and it differs from person to person. For each bundle of goods, a utility is assigned and we can then rank order the bundles from greatest utility to least utility. The consumer is always assumed to choose the bundle that maximizes their personal utility [3].

Example 6: Maximizing the Utility Function

Say a college student has a fixed allowance for food of $10. To simplify matters, let us assume that the only options available are pizza and pop. Furthermore, assume that one slice of pizza costs $2.00 and one bottle of pop costs $1.00. A plot of pizza along the x-axis and pop along the y-axis shows the possible combinations of pizza and pop that correspond with an allowance of \$10 (Figure 9). Note if we allow the student to exercise the option to buy fractions of slices of pizza and fractions of bottles of pop the available bundles would be the shaded region which is called the “feasible region”. In other words the student has enough money to purchase any bundle that exists within that region, including those on the border line. However, anything outside of the feasible region is not available due to the \$10 budget constraint.

Number of bottles of pop purchased



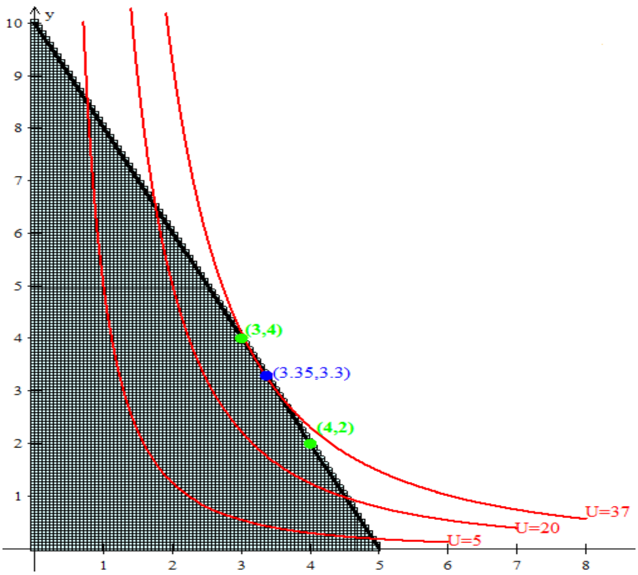
Number of pizzas purchased

Figure 11. Utility function corresponding to Example 6

How much utility does the student get from each point in the feasible region? To find out, consider the student’s utility function. A commonly used utility function is the Cobb Douglas utility function where utility is given by $U(x, y) = x^{\alpha}y^{\beta}$ where α and β are positive constants [3]. The relative size of α and β determine the importance of each good to the student. This utility function holds the property that when prices increase utility decreases [3]. Then, if we evaluate the utility function for each bundle of goods x and y, the result is a utility U. We then rank order the values of U and the bundle that gives the highest value of U is the bundle the student will choose. It is assumed that for any bundles that produce the same utility value, the student remains indifferent. In other words, having either bundle provides the student with the same amount of happiness and a bundle is chosen at random. There will exist many bundles that produce the same utility and these can be plotted along a curve. These curves are appropriately called indifference curves.

Return to Example 6 and assume that the student's utility function is given by $U(x, y) = x^2y$. Below is a plot of indifference curves for various values of U along with the feasible region (Figure 7).

Number of bottles of pop purchased



Number of pizzas purchased

Figure 12. Feasible region and indifference curves corresponding to Example 6

Recall that the student is always assumed to maximize utility. According to Figure 7, the student is able to attain a maximum utility of about $U = 37$. This indifference curve is tangent to the feasible region at (3.35,3.3). Note that if the student is only able to buy integer values of pizza and pop, the student will choose the bundle of three slices of pizza and four bottles of pop since $U(3,4) = 36$ and $U(4,2) = 32$.

In Example 6, we fixed the price of each slice of pizza and each bottle of pop. In reality, we know that prices vary from one year to the next and in some industries, such as gasoline, one day to the next. If the price of each slice of pizza and each bottle of pop were to change, how would that affect the student's choices?

Example 7: Maximizing the Utility Function with Different Prices

Suppose the college student is still constrained by a \$10 budget for food. Assume that the only options available are slices of pizza, which cost \$2 each, and bottles of pop, which now cost \$2 each.

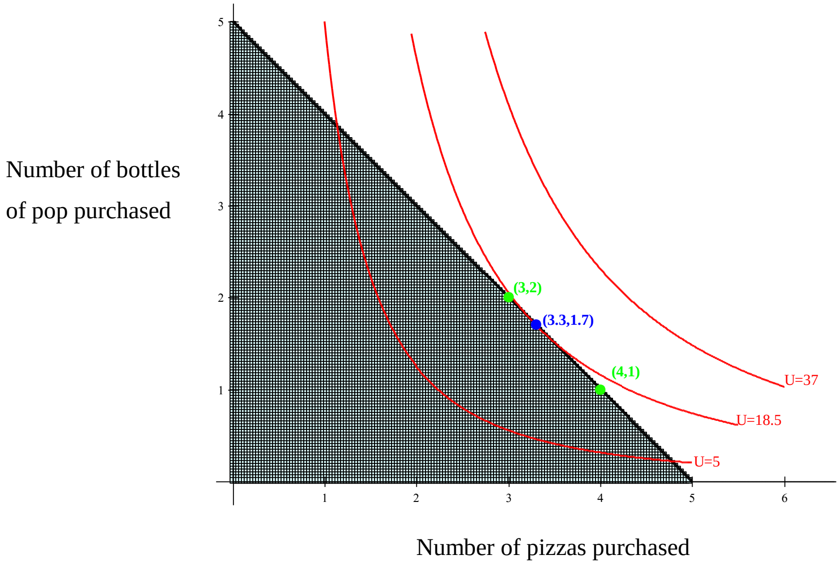


Figure 13. Feasible region and indifference curves for Example 7

The maximum utility in this market occurs at $U(3.3,1.7) \approx 18.5$. Again, since this is unrealistic we compare $U(3,2) = 18$ and $U(4,1) = 16$ and determine that the student chooses three pieces of pizza and two bottles of pop. Note that since the price of pop has increased, the feasible region has decreased. This is because the student is getting fewer products for the same amount of money. Also note that the utility value of 37 is now out of the students reach. Similarly, had the price of either commodity decreased, the feasible region would have increased and thus the maximum derived utility would have increased.

As illustrated in the examples, the feasible region for the consumer changes when the price of the goods, or commodities, change (Figure 8). Thus far we have only considered the consumer's options when we fix the price of each good. If the constraint of a fixed price for each good is removed and the prices are allowed to vary, the utility maximization problem requires the use of correspondences to determine which bundle of goods maximizes utility. We will set up the utility maximization problem for the general case and then consider a specific example.

First, define the commodity space and corresponding prices. Let $X = \{(x_1, x_2, \dots, x_L) | x_i \in \mathbb{R}_+\}$ be the commodity space, or the set of all possible consumption bundles, where L is the number of available commodities. The commodity space is defined without respect to any budget. In other words, this is all possible bundles in $\mathbb{R}_+^L$. Let $Y = \{(y_1, y_2, \dots, y_L) | y_i \in \mathbb{R}_+\}$ be the corresponding price of each commodity. Since we are allowing prices to vary, there will be a $y \in Y$ for every possible combination of prices for each commodity. Note that the price vectors will also be members of $\mathbb{R}_+^L$ [1].

From here we can define the budget correspondence, or the feasible region. Let w be the fixed budget of any given consumer. Then the set of feasible bundles, is given by $B: Y \to\to X$ where $B(y) = \{x \in X | \langle y, x \rangle \leq w\}$. Therefore B is a correspondence that inputs a specific price vector and outputs the set of all possible bundles, or the feasible region, subject to the budget, w, of the consumer at hand. Note that each $B(y) \subseteq X$ [1].

Given this information we can now reconsider the consumer's utility function U. Recall that the consumer always chooses the bundle that maximizes his utility. However, since prices are allowed to vary, the feasible region will also vary. This implies that the maximum utility will change depending upon the available bundles in the feasible region. Thus the consumer's maximum derived utility is now a function defined as $U^*: B(y) \to \mathbb{R}$ where $U^* = \max(U(y, x))$. In other words, U^* is all possible maximum derived utilities in each given feasible region [1].

Example 8: Maximizing Utility while Allowing Prices to Vary

For this example, we will combine Examples 6 and 7 into one. The student has a \$10 budget constraint, so $w = 10$. There are two commodities, pizza and pop, so $L = 2$. The commodity space is all possible bundles of pizza and pop without respect to a budget, so $X = \mathbb{R}_+^2$. In this Example we are considering two possible prices for each good, so $Y = \{(2,1), (2,2)\}$. There are two possible feasible regions, $B(2,1)$ and $B(2,2)$ which are the sets of bundles in Figures 6 and 7 respectively. The utility function is still defined as $U = x^2y$. We have two possible maximizations of derived utility, $U^*(2,1) \approx 37$ and $U^*(2,2) \approx 18.5$.

Correspondences allow us to consider the consumer's feasible region for any price that the commodities may take on. Another powerful tool is the Theorem of the Maximum which allows us to determine properties of U^* under certain conditions [1]. This theorem is stated without proof.

Theorem of the Maximum: Let X and Y be metric spaces, B a be compact-valued and continuous correspondence, and U be a continuous function. Then U^* is a continuous function.

This is powerful because knowing that the function of maximum derived utility is continuous allows us to draw valuable conclusions. If we bound prices to a reasonable range they could take on in a given period of time, the Weierstrass Theorem guarantees that the consumers maximum derived utility function will attain a maximum in the price range. This information can be used to predict consumer purchasing behavior.

The use of correspondences for this type of optimization analysis was first studied around 1959 by economist and mathematician Gerard Debreu [6]. Economist and Mathematician Nicholas C. Yannelis expanded Debreu's work to other areas of Economics and Game Theory [6]. In this work, Yannelis proves the existence of an equilibrium in a random price model. Correspondences have also been used for solving discontinuous differential equations. A. F. Filipov elaborates on their uses in his book *Differential Equations with Discontinuous Righthand Sides* [2].

REFERENCES

1. Dean, Mark. *Continuity of Correspondences*. Providence, RI: 2011. http://www.econ.brown.edu/ fac/Mark_Dean/Maths_RA6_10.pdf (accessed August 28, 2012).

2. Fillipov, A.F. *Differential Equations with Discontinuous Righthand Sides*. Dordrecht, Netherlands: Kluwer Academy Publishers, 1988.

3. Pindyck, Robert S. and Daniel L. Rubinfeld. *Microeconomics*. San Antonio, TX: Pearson Education, 2009.

4. Trench, William F. *Introduction to Real Analysis*. San Antonio, TX: Pearson Education, 2003. http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF (accessed August 28, 2012).

5. Walker, Mark. Correspondences. Tucson, AZ: 2011. http://www.u.arizona.edu/~mwalker/ 02_Equilibrium%20Existence/Correspondences.pdf (accessed April 20, 2012).

6. Yannelis, Nicholas C. *Set-Valued Functions of Two Variables in Economic Theory*. Champaign, IL: University of Illinois, 1989. http://archive.org/stream/setvaluedfunctio1601yann#page/ 3/mode/2up (accessed August 28, 2012).