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The Mathematics of Spot It

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ABSTRACT

Spot It™ is a fun, fast paced game for people of all ages. The game appears to have a rich mathematical structure, which we investigated, using combinatorial analysis. We constructed some similar *Spot It* type games with a different number of cards, and displayed that some games are impossible to complete. In particular, the original *Spot It* game has eight objects on each card; however, it is impossible to construct a complete game (a game in which every object appears with every other object) in which every card has seven objects. We have investigated the mathematics behind the game.

Keywords: mathematics, combinatorics.

INTRODUCTION

Spot It, created by Blue Orange Games, is a game for two to eight players, ages seven and up. The object of the game is to find the matching symbol between two cards the fastest. *Spot It* has a deck of fifty-five cards, each with eight objects, and every pair of cards has only one matching symbol.

In Figure 1 there are three cards from the actual *Spot It* game with eight objects on each of the three cards. A player is to find the matching objects between the cards.

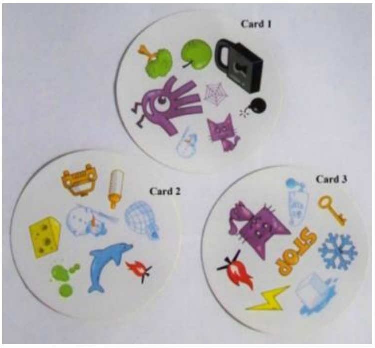

Figure 8: These are three cards from the Spot It game.

The matching symbols among the 3 card are illustrated in Figure 2.

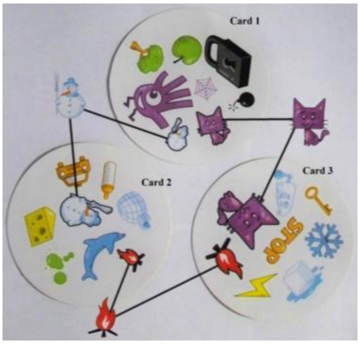

Figure 9: These are the original three cards with the matching objects highlighted.

Spot It is, in a sense, incomplete, because there are two cards that could be added to the deck. Specifically one of the cards would be created with the cactus, daisy, dinosaur, ice cube, man, maple leaf, question mark, and snow man symbols. The other card would contain a dog, exclamation mark, eye, lady bug, light bulb, skull and cross bone, snow man, and the word "stop". The matching symbol between these two cards would be the snow man. I tried contacting the Blue and Orange Games company to ask why they chose not to include these final two cards in the game; however, they did not respond. From now on in the paper we will assume the *Spot It* game has the two missing cards for a total of fifty-seven.

PROCEDURES AND ASSUMPTIONS

Some explanation of terminology is necessary in order to develop the mathematics behind *Spot It*. A symmetric game is a game in which every object appears the same amount of times. A complete game is when every object appears the same number of times, but also appears exactly once with every other object on some card. An example of a symmetric game that is not complete could be constructed as follows. Let the objects be denoted by the letters A, B, C, D, E, and F. Then the following four cards, each with three objects, form a symmetric (but not complete) *Spot It* game:

ABC ADE BDF CEF.

Each letter appears twice so the game is symmetric, but it is not complete. For example, A does not appear with F. *Spot It* would be a complete game, since it has each object appearing eight times and each object appears with every other object. This work will focus more on complete games.

Necessary variable are defined as:

N = the number of cards
r = the number of objects per card
k = the number of times each object appears
t = the number of distinct objects.

The *Spot It* game requires the following properties to be true.

Property 1: $\boldsymbol{r \geq 2}$

This must be true simply to make the game interesting. If there was only one object per card, the game would be trivial.

Property 2: $\boldsymbol{k \geq 2}$

This must be true because each object must appear twice in order for cards to have an object in common.

There are several Lemmas for which will help prove a theorem that all *Spot It* games must follow.

Lemma 1: For any symmetric game, $\boldsymbol{Nr = kt}$.

Proof: Since each of the t distinct objects appears k times, there are kt total objects. Every card has r objects and there are N cards, for a total of Nr objects. As a result, $Nr = kt$.

Lemma 2: For any complete game, $\boldsymbol{k(r-1) = t-1}$.

Proof: If object A appears k times, each card on which A appears contains $r-1$ distinct objects different from A. Thus, $k(r-1)$ objects appear with A. Since each game is complete, every object appears with A. Thus $t-1$ objects appear with A. Therefore, $k(r-1) = (t-1)$.

Lemma 3: In a symmetric game, $\boldsymbol{N = k + (r-1)(k-1)}$.

Proof: Any card will have will have objects $A_1, A_2, \ldots, A_r$. The object A_1 will then appear on $k-1$ more cards for a total of k, as shown in Figure 3 below.

The objects A_2 through A_r must then all appear $k-1$ more times and these cards must all be different, because objects A_2 through A_r have already appeared together. This creates a total of $(r-1)(k-1)$ remaining cards (Figure 3). It is known that these are all the cards because if another card was added to the deck, it would have to have a matching object with every card, and there cannot be any more cards with objects A_2 through A_r.

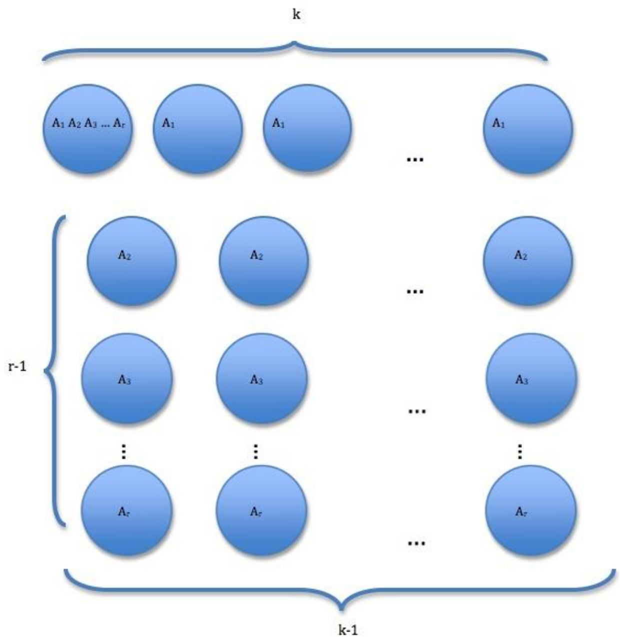

Figure 10: This figure shows the appearance of certain objects in a *Spot It* game.

Adding these cards together results in $N = k + (r-1)(k-1)$.

Theorem 1: For a complete game, $\boldsymbol{r = k}$ and $\boldsymbol{N = t = r^2 - r + 1}$.

Proof: From the Lemmas it is known that $t = k(r-1) + 1$ and $N = k + (r-1)(k-1)$. Substituting for t and N into $Nr = kt$, provides:

$$\big(k + (r-1)(k-1)\big)r = k(k(r-1) + 1)$$

$$kr + r(r-1)(k-1) = k^2(r-1) + k$$

$$kr - k + r(r-1)(k-1) - k^2(r-1) = 0$$

$$k(r-1) + r(r-1)(k-1) - k^2(r-1) = 0$$

$$(r-1)(k + r(k-1) - k^2) = 0.$$

Since $r - 1 \neq 0$,

$$k + r(k-1) - k^2 = 0$$

$$k - k^2 + rk - r = 0$$

$$k(1-k) - r(1-k) = 0$$

$$(1-k)(k-r) = 0.$$

Since $k \neq 1$, we conclude that $k - r = 0$, and therefore $k = r$. This implies $N = t$. Finally, given $t = k(r-1) + 1$, r can substitute for k to get $t = r(r-1) + 1 = r^2 - r + 1$.

To develop a new version of *Spot It*, it is necessary to determine how many objects to include per card. For example, suppose we want to create a game with four objects per card, meaning $r = 4$. Using Theorem 1, it is known that $k = 4$ and that $N = t = 4^2 - 4 + 1 = 13$; therefore, the game will have thirteen cards with four objects per card. There will be thirteen distinct objects (which we will denote as 1, 2, 3, …, 13) and each object will appear four times, as shown in Table 1.

Table 4: This is a complete *Spot It* game with 13 cards and 4 objects per card.

1 2 3 4	1 5 6 7	1 8 9 10	1 11 12 13
2 5 8 11	2 6 9 12	2 7 10 13	
3 5 9 13	3 6 10 11	3 7 8 12	
4 5 10 12	4 6 8 13	4 7 9 11	

For another example, we created a game with $r = k = 5$; therefore, $N = t = 21$.

Table 5: This is a complete *Spot It* game with 21 cards and 5 objects per card.

1 2 3 4 5	1 6 7 8 9	1 10 11 12 13	1 14 15 16 17	1 18 19 20 21
2 6 10 14 18	3 6 11 17 20	4 6 12 15 21	5 6 13 16 19	
2 7 11 15 19	3 7 10 16 21	4 7 13 14 20	5 7 12 17 18	

2 8 12 16 20	3 8 13 15 18	4 8 10 17 19	5 8 11 14 21
2 9 13 17 21	3 9 12 14 19	4 9 11 16 18	5 9 10 15 20

The *Spot It* game is analogous to balanced incomplete block designs using each card as a block. By definition, a balanced incomplete block design is an arrangement of t distinct objects into *N* blocks such that each block contains exactly r distinct objects, each object occurs in exactly k different block, and every pair of distinct objects a_i, a_j occurs together in exactly λ blocks (Hall, 1986). The following is an example of a balanced incomplete block design with $r = k = 3$ and $N = t = 7$ and with $\lambda = 1$:

$B_1: 0,1,3$ $B_2: 1,2,4$ $B_3: 2,3,5$ $B_4: 3,4,6$ $B_5: 4,5,0$ $B_6: 5,6,1$ $B_7: 6,0,2$

A *Spot It* game is a balanced incomplete block design with $\lambda = 1$ along with the additional principle of only one matching symbol between two cards/blocks. However, a balanced incomplete block design is not necessarily a *Spot It* game. There are occurrences when a balanced incomplete block design has $\lambda = 1$ and still is not a *Spot It* game. For example, the following is a balanced incomplete block design with $N = 12$, $t = 9$, $r = 4$, $k = 3$, and $\lambda = 1$, but not a *Spot It* game.

$B_1: 1,2,3$ $B_2: 4,5,6$ $B_3: 7,8,9$ $B_4: 1,4,7$ $B_5: 2,5,8$ $B_6: 3,6,9$

$B_7: 1,5,9$ $B_8: 2,6,7$ $B_9: 3,4,8$ $B_{10}: 1,6,8$ $B_{11}: 2,4,9$ $B_{12}: 3,5,7$

Note that many of the blocks have nothing in common with another block, e.g. B_1, B_2, and B_3. Balanced incomplete block designs lack the singular matching symbol principle.

RESULTS AND DISCUSSION

The initial intent of this work was to develop an algorithm for creating a complete *Spot It* game of any size, and then develop a method for creating a template of the desired game. A user could simply choose the number of objects per card and then the game would be automatically created; however, while creating *Spot It* games of various sizes in an attempt to find a pattern, one cannot create a game with $r = 7$. It is found that a *Spot It* game cannot be created for all integers, in particular, $r = 7$.

Consider a smaller game, with the values $r = k = 3$ and $N = t = 7$:

ABC ADE AFG BDF BEG CDG CEF.

A matrix can be created with the rows corresponding to the symbols (letters) and the columns corresponding to the cards; specifically, let $a_{ij} = 1$ if object i appears on card$j$, and $a_{ij} = 0$ otherwise. This matrix is shown in Table 3.

Table 6: This matrix corresponds to a complete *Spot It* game with *r* = 3 and *N* = 7.

$$\begin{array}{c c}
 & \begin{array}{ccccccc} \#1 & \#2 & \#3 & \#4 & \#5 & \#6 & \#7 \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}$$

Note that the entries in each row and column of Table 3 will add up to three (since $r = k = 3$). Also, any two columns will have exactly one entry of 1 in the same row. This is from the only matching symbol property of *Spot It*. This matrix can be used to create linear functions $L_1, L_2, \ldots, L_7$ of the variables $x_1, x_2, \ldots, x_7$ (where x_i appears in L_j if and only if object *i* appears on card *j*):

$$L_1 = x_1 + x_2 + x_3$$

$$L_2 = x_1 + x_4 + x_5$$

$$L_3 = x_1 + x_6 + x_7$$

$$\vdots$$

$$L_7 = x_3 + x_5 + x_6.$$

Note that, squaring L_1 provides:

$$L_1{}^2 = (x_1 + x_2 + x_3)^2 = x_1{}^2 + x_2{}^2 + x_3{}^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

Since each x_i appears exactly three times and also appears with every other variable exactly once, the following is derived:

$$\begin{aligned} L_1{}^2 + L_2{}^2 + \cdots + L_7{}^2 &= (x_1 + x_2 + x_3)^2 + (x_1 + x_4 + x_5)^2 + \cdots + (x_3 + x_5 + x_6)^2 \\ &= 2(x_1{}^2 + x_2{}^2 + \cdots + x_7{}^2) + (x_1 + x_2 + \cdots + x_7)^2. \end{aligned}$$

At this point, we would like to take advantage of Lagrange's Four Square Theorem to rewrite the factor $2(x_1{}^2 + x_2{}^2 + \cdots + x_7{}^2)$ as a sum of squares without the 2 in front. Specifically, we use identity 10.3.3 (Hall, 1986):

$$\left(b_1{}^2 + b_2{}^2 + b_3{}^2 + b_4{}^2\right)(x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2) = y_1{}^2 + y_2{}^2 + y_3{}^2 + y_4{}^2$$

where

$$y_1 = b_1x_1 - b_2x_2 - b_3x_3 - b_4x_4$$

$$y_2 = b_2x_1 + b_1x_2 - b_4x_3 + b_3x_4$$

$$y_3 = b_3x_1 + b_4x_2 + b_1x_3 - b_2x_4$$

$$y_4 = b_4x_1 - b_3x_2 + b_2x_3 + b_1x_4.$$

To do this, however, an eighth term is needed. So, adding $2x_8{}^2$ to each side results in

Equation 1

$$\begin{aligned} &L_1{}^2 + L_2{}^2 + \cdots + L_7{}^2 + 2x_8{}^2 \\ &\qquad = 2(x_1{}^2 + x_2{}^2 + \cdots + x_7{}^2 + x_8{}^2) + (x_1 + x_2 + \cdots + x_7)^2. \end{aligned}$$

One can now write $2(x_1{}^2 + x_2{}^2 + \cdots + x_7{}^2 + x_8{}^2)$ as a sum of eight squares $\mathrm{y}_1{}^2 + \mathrm{y}_2{}^2 + \cdots + \mathrm{y}_7{}^2 + \mathrm{y}_8{}^2$ by defining:

$$y_1 = x_1 - x_2 \qquad y_2 = x_1 + x_2 \qquad y_3 = x_3 - x_4 \qquad y_4 = x_3 + x_4$$

$$y_5 = x_5 - x_6 \qquad y_6 = x_5 + x_6 \qquad y_7 = x_7 - x_8 \qquad y_8 = x_7 + x_8.$$

Therefore squaring and adding each of these together yields,

$${y_1}^2 + {y_2}^2 + \cdots + {y_7}^2 + {y_8}^2 = 2({x_1}^2 + {x_2}^2 + \cdots + {x_7}^2 + {x_8}^2).$$

Also, for simplicity, define $w = x_1 + x_2 + \cdots + x_7$. Thus, Equation 1 becomes

$${L_1}^2 + {L_2}^2 + \cdots + {L_7}^2 + 2{x_8}^2 = {y_1}^2 + {y_2}^2 + \cdots + {y_8}^2 + w^2.$$

Note that

$$L_1 = x_1 + x_2 + x_3 = y_2 + \frac{1}{2}y_3 + \frac{1}{2}y_4.$$

Let

$$y_1 = y_2 + \frac{1}{2}y_3 + \frac{1}{2}y_4 = L_1.$$

Because $y_1 = L_1$ one can cancel to rewrite equation (1) as

$${L_2}^2 + {L_3}^2 + \cdots + {L_7}^2 + 2{x_8}^2 = {y_2}^2 + {y_3}^2 + \cdots + {y_8}^2 + w^2.$$

Repeating this one can set $y_i = L_i$ (or $y_i = -L_i$) to get the following reductions:

$${L_3}^2 + {L_4}^2 + \cdots + {L_7}^2 + 2{x_8}^2 = {y_3}^2 + {y_4}^2 + \cdots + {y_8}^2 + w^2$$

$${L_4}^2 + \cdots + {L_7}^2 + 2{x_8}^2 = {y_4}^2 + \cdots + {y_8}^2 + w^2$$

$$\vdots$$

$$2{x_8}^2 = {y_8}^2 + w^2.$$

Because of these substitutions, it is known that x_8 and w are rational multiples of y_8 and therefore $2{x_8}^2 = {y_8}^2 + w^2$ has a nontrivial integer solution.

A similar argument can be made for any existing *Spot It* game. Now looking at the game structured with $r = 7$, it can be shown why it is impossible to create.
For $r = k = 7$ and $N = t = 43$, start in a similar way by defining $L_1, L_2, \dots, L_{43}$ in terms of $x_1, x_2, \dots, x_{43}$, where x_i appears in L_j if and only if object i appears on card j. For example, without loss of generality, we could have $L_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$ (meaning the first card has objects one through seven). Then,

$$L_1{}^2 + L_2{}^2 \cdots L_{43}{}^2 = 6(x_1{}^2 + x_2{}^2 + \cdots + x_{43}{}^2) + (x_1 + x_2 + \cdots + x_{43})^2.$$

Since, each object appears seven times and also appears with every other variable exactly once. Adding $6x_{44}{}^2$ to each side yields

$$L_1{}^2 + L_2{}^2 \cdots L_{43}{}^2 + 6x_{44}{}^2 = 6(x_1{}^2 + x_2{}^2 + \cdots + x_{43}{}^2 + x_{44}{}^2) + (x_1 + x_2 + \cdots + x_{43})^2.$$

Using Hall's identity 10.3.3 again, another group of functions can be defined as $y_1, y_2, \dots, y_{44}$ by

$$y_1 = 2x_1 - x_2 - x_3 \qquad y_2 = x_1 + 2x_2 + x_4$$

$$y_3 = x_1 + 2x_3 - x_4 \qquad y_4 = -x_2 + x_3 + 2x_4$$

$$\vdots$$

$$y_{41} = 2x_{41} - x_{42} - x_{43} \qquad y_{42} = x_{41} + 2x_{42} + x_{44}$$

$$y_{43} = x_{41} + 2x_{43} - x_{44} \qquad y_{44} = -x_{42} + x_{43} + 2x_{44}.$$

With an additional definition of $w = x_1 + x_2 + \cdots + x_{43}$, one obtains:

$$L_1{}^2 + L_2{}^2 + \cdots + L_{43}{}^2 + 6x_{44}{}^2 = y_1{}^2 + y_2{}^2 + \cdots + y_{43}{}^2 + y_{44}{}^2 + w^2.$$

After repeating the same process as above, so that $L_1{}^2 = y_1{}^2, L_2{}^2 = y_2{}^2, \cdots, L_{43}{}^2 = y_{43}{}^2$, the following equations results:

$$6x_{44}{}^2 = y_{44}{}^2 + w^2.$$

As before, it is known that x_{44} and w are rational multiples of y_{44}. So, after clearing out the denominators, it is observed that the equation $6x^2 = y^2 + w^2$ has a nontrivial integer solution (meaning not all variables are zero). However, this equation can't have a nontrivial integer solution because of the following argument by contradiction: first, if this equation has a nontrivial integer solution, then we can assume *x*, *y*, and *w* have no common factor greater than 1. If *x*, *y*, and *w* had a common divisor greater than 1, then one could factor this common divisor out from both sides of the equation and cancel it from both sides. Reducing the equation modulo 3, we have

$$y^2 + w^2 \equiv 0 (\mathrm{mod}\ 3).$$

The possibilities for y are listed in Table 4 below.

Table 7: These are the only possible values for a square, modulo 3.

y	y^2 modulo 3
0	0
1	1
2	1

The same possibilities are true for w.

Because $y^2 + w^2 \equiv 0 (\mathrm{mod}\ 3)$, we must have $y \equiv 0 (\mathrm{mod}\ 3)$ and $w \equiv 0 (\mathrm{mod}\ 3)$. Thus, $3|y$ and $3|w$. Consequently, $y = 3a$ and $w = 3b$ where a and b are integers, and the following can be written:

$$(3a)^2 + (3b)^2 = 6x^2$$

$$9a^2 + 9b^2 = 6x^2$$

$$3a^2 + 3b^2 = 2x^2$$

Therefore, $3|2x^2$. Since 3 and 2 are relatively prime, we can conclude that $3|x^2$ (Gallian, 2010). This implies, by Euclid's Lemma, that $3|x$. This is a contradiction because it is known that *x*, *y*, and *w* have no common factor greater than 1. This contradiction means that no *Spot It* game is possible with these values.

When we began exploring the mathematics behind *Spot It,* we had hopes of developing an algorithm for creating various versions of the game. Mostly, the mathematics behind the game were more complicated than expected. There are still unanswered questions related to the Balanced Incomplete Block Design properties of the *Spot It* game.

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