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UNITARILY INVARIANT NORMS ON FINITE VON NEUMANN ALGEBRAS

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BY

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ABSTRACT

Unitarily invariant norms on finite von Neumann algebras

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John von Neumann's 1937 characterization of unitarily invariant norms on the $n \times n$ matrices in terms of symmetric gauge norms on $\mathbb{C}^n$ had a huge impact on linear algebra. In 2008 his results were extended to II_1 factor von Neumann algebras by J. Fang, D. Hadwin, E. Nordgren and J. Shen. There already have been many important applications. The factor von Neumann algebras are the atomic building blocks from which every von Neumann algebra can be built. My work, which includes a new proof of the II_1 factor case, extends von Neumann's results to an arbitrary finite von Neumann algebra on a separable Hilberts space. A major tool is the theory of direct integrals. The main idea is to associate to a von Neumann algebra $\mathcal{R}$ a measure space (Λ, λ) and a group $\mathbb{G}(\mathcal{R})$ of invertible measure-preserving transformations on $L^\infty(\Lambda, \lambda)$. Then we show that there is a one-to-one correspondence between the unitarily invariant norms on $\mathcal{R}$ and the normalized $\mathbb{G}(\mathcal{R})$-symmetric gauge norms on $L^\infty(\Lambda, \lambda)$.

CHAPTER 1

INTRODUCTION

Since John von Neumann's beautiful characterization of the unitarily invariant norms for the $n \times n$ complex matrices $\mathbb{M}_n(\mathbb{C})$, there have been over four hundred papers related to this subject. In [17] von Neumann showed that there is a natural one-to-one correspondence between the unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$ and the normalized symmetric gauge norms on $\mathbb{C}^n$. More recently, Junsheng Fang, Don Hadwin, Eric Nordgren, and Junhao Shen [10] showed that there is an analogous correspondence between the unitarily invariant norms on a II_1 factor von Neumann algebra $\mathcal{M}$ and the normalized symmetric gauge norms on $L^\infty[0,1]$. Although the proofs of both results relied on s-numbers, the proof of the latter result was different from von Neumann's proof. We provide a new proof of the II_1 factor result that more closely parallels the proof for $\mathbb{M}_n(\mathbb{C})$. The key ingredient is an "approximate" version of the Ky Fan Lemma that is used in the finite-dimensional case.

It is our goal to find a similar characterization of all the unitarily invariant norms on a finite von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H. To make these two examples look the same, we want to view $\mathbb{C}^n$ as $L^\infty(J_n,\delta_n)$, where (J_n,δ_n) is a probability space. We also want to have $J_n \subset [0,1]$. Our choice is $J_n = \left\{\frac{1}{n},\ldots,\frac{n}{n}\right\}$ and δ_n is normalized counting measure, i.e.,

$$\delta_n(E) = \frac{1}{n}\mathrm{Card}(E).$$

We define $J_\infty = [0,1]$ and δ_∞ to be Lebesgue measure. It turns out that every finite von Neumann algebra on a separable Hilbert space has a *central decomposition*, which means it can be decomposed as a direct sum of direct integrals of factor von Neumann algebras, which are either isomorphic to $\mathbb{M}_n(\mathbb{C})$ or are II_1 factors. Each finite factor von Neumann algebra has a unique tracial state. From the central decomposition we can define a tracial state τ on $\mathcal{R}$. The problem is to identify the corresponding measure space (Λ, λ). A key observation is that every maximal abelian selfadjoint subalgebra (masa) of $\mathbb{M}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n = L^\infty(J_n, \delta_n)$ and each masa in a II_1 factor is isomorphic to $L^\infty[0,1] = L^\infty(J_\infty, \delta_\infty)$. If $\mathcal{A}$ is a masa in $\mathcal{R}$, then the central decomposition of $\mathcal{R}$ decomposes $\mathcal{A}$ to a direct integral of algebras that are masas in the corresponding factor. We must analyze this decomposition carefully to see that the masas are all isomorphic, *in a very special way*, to $L^\infty(\Lambda, \lambda)$ for some measure space (Λ, λ). Once we find the measure space, we have to show how the unitarily invariant norms on $\mathcal{R}$ correspond to the normalized symmetric gauge norms on $L^\infty(\Lambda, \lambda)$. This involves defining the analogue of the "s-numbers" and proving a general approximate Ky Fan Lemma. To show that things are independent of the choices of the masas we use, we need a result on approximate unitary equivalence.

CHAPTER 2

PRELIMINARIES

2.1 Unitarily invariant norms

If $\mathcal{A}$ is a unital C*-algebra, $\mathcal{U}(\mathcal{A})$ denotes the set of all unitary elements of $\mathcal{A}$. If $T \in \mathcal{A}$ we define $|T| = (T^*T)^{1/2}$.

Lemma 1. *Suppose $\mathcal{A}$ is a unital C*-algebra and α is a norm on $\mathcal{A}$ such that $\alpha(1) = 1$. The following are equivalent.*

1. *For every $T \in \mathcal{A}$ and for every $U \in \mathcal{U}(\mathcal{A})$,*

$$\alpha(T) = \alpha(|T|) = \alpha(U^*TU).$$

2. *For all U, V in $\mathcal{U}(\mathcal{A})$,*

$$\alpha(T) = \alpha(UTV).$$

Proof. Suppose $T \in \mathcal{A}$ and for every $U \in \mathcal{U}(\mathcal{A})$, we have $\alpha(T) = \alpha(|T|) = \alpha(U^*TU)$. Then

$$\alpha(UT) = \alpha(|UT|) = \alpha\left([(UT)^*(UT)]^{1/2}\right) = \alpha\left((T^*T)^{1/2}\right) = \alpha(|T|) = \alpha(T),$$

and similarly, $\alpha(TV) = \alpha(T)$. Therefore, $\alpha(T) = \alpha(UTV)$.

Suppose $T \in \mathcal{A}$ and $\alpha(T) = \alpha(UTV)$ for every $U, V \in \mathcal{U}(\mathcal{A})$. It is clear that $\alpha(T) = \alpha(U^*TU)$. To prove $\alpha(T) = \alpha(|T|)$, the Russo-Dye Theorem [3] says the norm closed convex hull of $\mathcal{U}(\mathcal{A})$ is $\{A \in \mathcal{A} : \|A\| \leq 1\}$, and therefore we know that T is in the closed convex hull of $\{\|T\| U : U \text{ is unitary}\}$; thus $\alpha(T) \leq \|T\|$ for every $T \in \mathcal{A}$. Also suppose $T = W_1|T| = |T| W_2$, where W_1, W_2 are in the norm-closed convex hull of the set of unitaries, which implies T is in the norm closed convex hull of $\{U|T| : U \text{ is unitary}\}$ and $|T|$ is in the closed convex hull of $\{VT : V \text{ is unitary}\}$. Hence $\alpha(T) \leq \alpha(|T|)$ and $\alpha(|T|) \leq \alpha(T)$. Therefore $\alpha(|T|) = \alpha(T)$. □

Definition 2. *If $\mathcal{A}$ is a unital C^*-algebra and α is a norm on $\mathcal{A}$ satisfying $\alpha(1) = 1$ and either of the two conditions in Lemma 1, we say that α is a* unitarily invariant norm *on $\mathcal{A}$.*

Below are some properties about unitarily invariant norms.

Proposition 1. *If $\mathcal{A}$ is a unital C^*-algebra and α is a unitarily invariant norm on $\mathcal{A}$, and $T, A, B \in \mathcal{A}$, we have the following:*

1. $\alpha(T) \leq \|T\|$,
2. $\alpha(T) = \alpha(T^*)$,
3. $\alpha(ATB) \leq \|A\| \alpha(T) \|B\|$,
4. $0 \leq A \leq B$ *implies* $\alpha(A) \leq \alpha(B)$.

Note: Whenever we discuss a measure space (Ω, Σ, μ) we always assume that the space is **complete** in the sense that, whenever $E \subset F$ and $\mu(F) = 0$, we have $E \in \Sigma$.

Lemma 3. *If α is a unitarily invariant norm on a unital C*-algebra $\mathcal{R}$, $S, T \in \mathcal{R}$, and $\{U_i\}$ is a net of unitary operators in $\mathcal{R}$ such that*

$$\lim_{\iota} \|S - U_i^* T U_i\| = 0,$$

then

$$\alpha(S) = \alpha(T).$$

Proof. We have

$$0 \leq |\alpha(S) - \alpha(T)| = \lim_i |\alpha(S) - \alpha(U_i^* T U_i)|$$

$$\leq \lim_i \alpha(S - U_i^* T U_i) \leq \lim_i \|S - U_i^* T U_i\| = 0.$$

□

Definition 4. *If* (Ω, μ) *is a probability space, then* $L^\infty(\mu)$ *is a von Neumann algebra, and a unitarily invariant norm* α *on* $L^\infty(\mu)$ *is called a* normalized gauge norm on $L^\infty(\mu)$. *In this case all we require of* α *is that* $\alpha(1) = 1$ *and* $\alpha(f) = \alpha(|f|)$ *for every* $f \in L^\infty(\mu)$. *We let* $\mathbb{MP}(\Omega, \mu)$ *denote the group (under composition) of all invertible measure-preserving transformations from* Ω *to* Ω. *We say that a gauge norm* α *on* $L^\infty(\mu)$ *is* symmetric *if, for every* $\gamma \in \mathbb{MP}(\Omega, \mu)$ *and every* $f \in L^\infty(\mu)$, *we have*

$$\alpha(f \circ \gamma) = \alpha(f).$$

In [17], J. von Neumann characterized all of the unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$, which is the $n \times n$ full matrix algebra with entries in $\mathbb{C}$. In [10], J. Fang, D. Hadwin, E. A. Nordgren and J. Shen characterized the unitarily invariant norms on a II_1 factor von Neumann algebra. The goal of this thesis is to give a characterization of all unitarily invariant norms of a finite von Neumann algebra acting on a separable Hilbert space. Along the way we give a new proof of the characterization of unitarily invariant norms on a II_1 factor.

2.1.1 Unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$

Let τ_n be the normalized trace on $\mathbb{M}_n(\mathbb{C})$, i.e., $\tau_n = \frac{1}{n} Trace$.

Lemma 5. *Suppose* $T \in \mathbb{M}_n(\mathbb{C})$, *then there exists a unitary* $U \in \mathcal{U}(\mathbb{M}_n(\mathbb{C}))$, *such that*

$$U^*|T|U = \alpha \begin{pmatrix} s_T\left(\frac{1}{n}\right) & 0 & \cdots & 0 \\ 0 & s_T\left(\frac{2}{n}\right) & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_T\left(\frac{n}{n}\right) \end{pmatrix},$$

and $s_T\left(\frac{1}{n}\right) \geqslant s_T\left(\frac{2}{n}\right) \geqslant ...s_{\frac{n}{n}}(T) \geq 0$*. The numbers* $s_T\left(\frac{1}{n}\right), s_{\frac{2}{n}}..., s_T\left(\frac{n}{n}\right)$ *are unique and are called the* s-numbers *of the matrix* T*. Define* $s(T) = \left(s_T\left(\frac{1}{n}\right), s_T\left(\frac{2}{n}\right), \cdots, s_T\left(\frac{n}{n}\right)\right)$.

If α is a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$, then

$$\alpha(T) = \alpha(|T|) = \alpha(U^*|T|U) = \alpha\begin{pmatrix} s_T\left(\frac{1}{n}\right) & 0 & \cdots & 0 \\ 0 & s_T\left(\frac{2}{n}\right) & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_T\left(\frac{n}{n}\right) \end{pmatrix},$$

and thus $\alpha(T)$ depends only on the s-numbers of T.

Note that $s(T) \in \mathbb{C}^n$, and in classical matrix theory [2] the standard notation is $s_k(T)$ instead of our $s_T\left(\frac{k}{n}\right)$ for $1 \leq k \leq n$. We know that $\mathbb{C}^n$ is isomorphic to $L^\infty(\delta_n)$, where δ_n is normalized counting measure on $\left\{\frac{1}{n}, \ldots, 1\right\}$. Let $\mathbb{S}_n$ be the permutation group (i.e., all the bijective functions on $\left\{\frac{1}{n}, \ldots, 1\right\}$). It is clear that $\mathbb{S}_n = \mathbb{MP}(J_n, \delta_n)$.

In this case a normalized gauge norm β on $\mathbb{C}^n = L^\infty(\delta_n)$ is *symmetric* if, for every $f \in L^\infty(\delta_n)$ and every $\sigma \in \mathbb{S}_n$,

$$\beta(f) = \beta(f \circ \sigma),$$

that is

$$\beta((a_1, \ldots, a_n)) = \beta\left(\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right)\right).$$

We know that for each $x = (x_1, \ldots, x_n)$ in $\mathbb{C}^n$ and $|x| = (|x_1|, \ldots, |x_n|)$, there is a $\sigma \in \mathbb{S}_n$ such that

$$s(|x|) = \left(\left|x_{\sigma(1)}\right|, \cdots, \left|x_{\sigma(n)}\right|\right) \underset{\text{def}}{=} \left(s_x\left(\frac{1}{n}\right), s_x\left(\frac{2}{n}\right), \cdots, s_x\left(\frac{n}{n}\right)\right),$$

where $s_x\left(\frac{1}{n}\right) \geq s_x\left(\frac{2}{n}\right) \geq \cdots \geq s_x\left(\frac{n}{n}\right) \geq 0$. We call $s_{|x|}$ the *nonincreasing rearrangement* of $|x|$. Note that, although σ may not be unique, $s_{|x|}$ is unique.

Given a unitarily invariant norm α on $\mathbb{M}_n(\mathbb{C})$, define β_α on $\mathbb{C}^n$ by

$$\beta_\alpha(x) = \beta_\alpha(x_1, ...x_n) = \alpha \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = \alpha \begin{pmatrix} s_{|x|}\left(\frac{1}{n}\right) & & \\ & \ddots & \\ & & s_{|x|}\left(\frac{n}{n}\right) \end{pmatrix}.$$

Clearly, permutation on $\mathbb{C}^n$ corresponds to unitary conjugation by permutation matrices in $\mathbb{M}_n(\mathbb{C})$. Hence β_α is a normalized gauge norm on $L^\infty(\delta_n) = \mathbb{C}^n$.

Given a symmetric normalized gauge norm β on $\mathbb{C}^n$, we would like to define α_β on $\mathbb{M}_n(\mathbb{C})$ by

$$\alpha_\beta(T) = \beta\left(s_T\left(\frac{1}{n}\right), s_T\left(\frac{2}{n}\right), \cdots, s_T\left(\frac{n}{n}\right)\right).$$

We need to check that α_β is a norm. Clearly, $s_{\lambda T}\left(\frac{1}{n}\right) = |\lambda| s_T\left(\frac{1}{n}\right)$, so

$$\alpha_\beta(\lambda T) = \beta\left(s_{\lambda T}\left(\frac{1}{n}\right), s_{\lambda T}\left(\frac{2}{n}\right), \cdots, s_{\lambda T}\left(\frac{n}{n}\right)\right) = |\lambda| \alpha_\beta(T).$$

Also, $\alpha_\beta(T) \geq 0$ and $\alpha_\beta(T) = 0$ implies $T = 0$. The big problem is the triangle inequality: $s_{A+B}\left(\frac{k}{n}\right) \leq s_A\left(\frac{k}{n}\right) + s_B\left(\frac{k}{n}\right)$ can fail if $k > 1$. When $k = 1$, $s_T\left(\frac{k}{n}\right) = \|T\|$.

Example 1. $A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$
In this example, $s_{A+B}\left(\frac{2}{n}\right) = \frac{3}{2}, s_A\left(\frac{2}{n}\right) + s_B\left(\frac{2}{n}\right) = \frac{5}{4}$.

In order to prove the triangle inequality of α_β, *Ky Fan Norms* are involved. For $1 \leq k \leq n$ we define $KF_{\frac{k}{n}} : \mathbb{M}_n(\mathbb{C}) \to [0, \infty)$ and $KF_{\frac{k}{n}} : \mathbb{C}^n \to [0, \infty)$, by

$$KF_{\frac{k}{n}}(T) = \frac{s_T\left(\frac{1}{n}\right) + \cdots s_T\left(\frac{k}{n}\right)}{k} \text{ and } KF_{\frac{k}{n}}(x) = \frac{s_x\left(\frac{1}{n}\right) + \cdots s_x\left(\frac{k}{n}\right)}{k}.$$

To prove $KF_{\frac{k}{n}}$ is a norm on $\mathbb{M}_n(\mathbb{C})$ and on $\mathbb{C}^n$, we use the following Lemma whose proof can be found in [3]. Once we know $\alpha = KF_{\frac{k}{n}}$ is a norm on $\mathbb{M}_n(\mathbb{C})$, it easily follows that $KF_{\frac{k}{n}} = \beta_\alpha$ is a symmetric gauge norm on $\mathbb{C}^n$.

Lemma 6. *For $T \in \mathbb{M}_n(\mathbb{C})$, $KF_{\frac{k}{n}}(T) = \sup\{\mathrm{Tr}(UTP), U$ is unitary, P is a projection of rank $k\}$.*

We easily obtain the following corollary.

Corollary 7. *$\sum_{i=1}^{k} s_{A+B}(\frac{i}{n}) \le \sum_{i=1}^{k}\left[s_A(\frac{i}{n}) + s_B(\frac{i}{n})\right]$ for $A, B \in \mathbb{M}_n(\mathbb{C})$ and $1 \le k \le n$.*

The key result relates the Ky Fan norms to arbitrary unitarily invariant norms. The proof can be found in [9].

Lemma 8. *Suppose $n \in \mathbb{N}$, $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{C}^n$, $a_1 \ge a_2 \ge \cdots \ge a_n \geqslant 0, b_1 \geqslant b_2 \geqslant \cdots b_n \geqslant 0$, and if $KF_{\frac{k}{n}}(a) \le KF_{\frac{k}{n}}(b)$ for $1 \le k \le n$, then there exists $N \in \mathbb{N}$, $\sigma_1, \cdots, \sigma_N \in \mathbb{S}_n$, $0 \le t_j \le 1$, with $\sum_{j=1}^{N} t_j = 1$ such that $a \le \sum_{j=1}^{N} t_j (b \circ \sigma_j)$*

Corollary 9. *Suppose $a, b \in \mathbb{C}^n$ with $KF_{\frac{k}{n}}(a) \le KF_{\frac{k}{n}}(b)$ for $1 \le k \le n$, then, for every symmetric gauge norm β on $\mathbb{C}^n$, $\beta(a) \le \beta(b)$.*

Proof. $\beta(a) \le \beta\left(\sum_{j=1}^{N} t_j b \circ \sigma_j\right) \le \sum_{j=1}^{N} t_j \beta(b \circ \sigma_j) = \left(\sum_{j=1}^{N} t_j\right)\beta(b) = \beta(b)$. □

Lemma 10. *If β is a symmetric normalized gauge norm on $\mathbb{C}^n$, then α_β is a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$.*

Proof. We just need to prove the triangle inequality. Suppose $A, B \in \mathbb{M}_n(\mathbb{C})$.If

$$\begin{aligned} a &= \left(s_{A+B}(\frac{1}{n}), s_{A+B}(\frac{2}{n}), ..., s_{A+B}(\frac{n}{n})\right) \text{ and,} \\ b &= \left(s_A\left(\frac{1}{n}\right) + s_B\left(\frac{1}{n}\right), s_A\left(\frac{2}{n}\right) + s_B\left(\frac{2}{n}\right) \ldots, s_A\left(\frac{n}{n}\right) + s_B\left(\frac{n}{n}\right)\right), \end{aligned}$$

then, by Corollary 7, we know that $KF_{\frac{k}{n}}(a) \le KF_{\frac{k}{n}}(b)$. for $1 \le k \le n$. It follows from Corollary 9 that $\beta(a) \le \beta(b)$. However,

$$\alpha_\beta(A+B) = \beta(a) \le \beta(b) = \beta(s_A + s_B) \le \beta(s_A) + \beta(s_B) = \alpha_\beta(A) + \alpha_\beta(B).$$

□

It is easy to see that $\alpha_{\beta_\alpha} = \alpha$ and $\beta_{\alpha_\beta} = \beta$ always hold. This give us von Neumann's characterization of unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$.

Theorem 11. *[17]There is a one to one correspondence between symmetric gauge norms on $\mathbb{C}^n$ and unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$.*

2.1.2 Unitarily invariant norms on a II_1 factor

Suppose $\mathcal{M}$ is a II_1 factor von Neumann algebra. Then $\mathcal{M}$ has a unique faithful normal tracial state τ with the property that if P and Q are projections in $\mathcal{M}$, then P and Q are unitarily equivalent in $\mathcal{M}$ if and only if $\tau(P) = \tau(Q)$. In this case the measure space (J_n, δ_n) is replaced with the measure space $(J_\infty, \delta_\infty)$, where $J_\infty = [0,1]$ and δ_∞ is Lebesgue measure. A normalized gauge norm β on $L^\infty[0,1] = L^\infty(\delta_\infty)$ is *symmetric* if, for every $\gamma \in \mathbb{MP}(J_\infty, \delta_\infty)$ and every $f \in L^\infty(\delta_\infty)$, we have $\beta(f) = \beta(f \circ \gamma)$.

The main result in [10] is that there is a one-to-one correspondence between the unitarily invariant norms on $\mathcal{M}$ and the symmetric normalized gauge norms on $L^\infty(\delta_\infty)$. This looks just like von Neumann's result for $\mathbb{M}_n(\mathbb{C})$.

The definition of the s-numbers for a function in $L^\infty[0,1]$ can be obtained from nonincreasing rearrangements in measure theory. The proof in [10] doesn't use a version of the Ky Fan Lemma (Lemma 8); we present a new proof here using an "approximate" version of the Ky Fan Lemma (Theorem 20).

Lemma 12. *Suppose $f : [0,1] \to \mathbb{C}$ is measurable. Then there is a $\gamma \in \mathbb{MP}(J_\infty, \delta_\infty)$ such that $s_f \underset{\text{def}}{=} |f| \circ \gamma$ is nonincreasing on $[0,1]$. The transformation γ may not be unique, but s_f is unique (a.e.). It therefore follows that $f_1, f_2 : [0,1] \to \mathbb{C}$ are measurable, then*

$$s_{f_1} = s_{f_2} \textit{ if and only if } |f_1| = |f_2| \circ \gamma \textit{ for some } \gamma \in \mathbb{MP}(J_\infty, \delta_\infty).$$

For $0 < t \leq 1$, we define the Ky Fan norm KF_t on $L^\infty[0,1]$ by

$$KF_t(f) = \frac{1}{t}\int_0^t s_f d\delta_\infty.$$

For an operator $T \in \mathcal{M}$ and $0 \leq t \leq 1$, the t^{th} s-number of T, denoted by $s_T(t)$, was defined by Fack and Kosaki in [8] as

$$s_T(t) = \inf\{\|TE\| : E \text{ is a projection in } \mathcal{M} \text{ with } \tau(E^\perp) \leq t\}.$$

It is clear that the map $t \mapsto s_T(t)$ is nonincreasing on $[0,1]$. The t^{th} Ky Fan norm $KF_t(T)$ is defined as

$$KF_t(T) = \begin{cases} \|T\| \text{ if } t = 0 \\ \frac{1}{t}\int_0^t s_T(t)\, d\delta_\infty \text{ if } 0 < t \leq 1. \end{cases}$$

In the matrix case $|T|$ is unitarily equivalent to a diagonal matrix, which naturally corresponds to an element of $\mathbb{C}^n$. In the II_1 factor case we need a more complicated approach.

Definition 13. *A normal $*$-isomorphism $\pi : L^\infty(\delta_\infty) \to \mathcal{M}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$(\tau \circ \pi)(f) = \int_{J_\infty} f d\delta_\infty.$$

*is called a **tracial embedding.***

The following Lemma is a consequence of Hadwin-Ding in [5].

Lemma 14. *If π and ρ are tracial embeddings into a II_1 factor $\mathcal{M}$, then π and ρ are approximately unitarily equivalent in $\mathcal{M}$, i.e., there is a net $\{U_i\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\|U_i^* \pi(f) U_i - \rho(f)\| \to 0.$$

Corollary 15. *If $\pi : L^\infty(\delta_\infty) \to \mathcal{M}$ is a tracial embedding and $\gamma \in \mathbb{MP}(J_\infty, \delta_\infty)$, then $\rho : L^\infty(\delta_\infty) \to \mathcal{M}$ defined by $\rho(f) = \pi(f \circ \gamma)$ is also a tracial embedding. Hence, there is a net $\{U_i\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\|U_i^* \pi(f) U_i - \pi(f \circ \gamma)\| \to 0.$$

In the matrix case, the assertion that $|T|$ is unitarily equivalent to a diagonal matrix can be rephrased as $|T|$ is contained in a maximal abelian selfadjoint algebra (i.e., masa) of $\mathbb{M}_n(\mathbb{C})$, and every masa in $\mathbb{M}_n(\mathbb{C})$ is unitarily equivalent to the algebra of diagonal $n \times n$ matrices. Here is the analogue for a II_1 factor.

Lemma 16. *Suppose $\mathcal{A}$ is a masa in a type II_1 factor $\mathcal{M}$. Then there is a* surjective *tracial embedding $\pi : L^\infty(\delta_\infty) \to \mathcal{A}$. Moreover, if $f \in L^\infty[0,1]$ and $\pi(f) = T$, then, for almost every $t \in [0,1]$,*

$$s_f(t) = s_{\pi(f)}(t).$$

As in the matrix case we need to prove KF_t is a norm on $\mathcal{M}$ by giving an alternate characterization.

Lemma 17. *If $T \in \mathcal{M}$ and $0 < t \leq 1$, then*

$$KF_t(T) = \sup\{|\tau(UTP)| : U \in \mathcal{U}(\mathcal{M}),\, P \textit{ is a projection, } \tau(P) = t\}.$$

It was proved in Lemma 5.1 in [10].

Suppose α is a unitarily invariant norm on $\mathcal{M}$. We can choose a tracial embedding $\pi : L^\infty(J_\infty, \delta_\infty) \to \mathcal{M}$ and define a norm β_α on $L^\infty(J_\infty, \delta_\infty)$ by

$$\beta_\alpha(f) = \alpha(\pi(f)).$$

We need to show that the definition does not depend on the embedding π. If $\rho : L^{\infty}(J_{\infty}, \delta_{\infty}) \to \mathcal{M}$ is another tracial embedding, then by Lemma 14, there is a net $\{U_i\}$ of unitary operators in $\mathcal{M}$ such that, for every $f \in L^{\infty}(J_{\infty}, \delta)$

$$\|U_i^* \pi(f) U_i - \rho(f)\| \to 0.$$

Since

$$\begin{aligned}|\beta(\pi(f)) - \beta(\rho(f))| &= |\beta(U_i^* \pi(f) U_i) - \beta(\rho(f))| \\ &\leq \beta(U_i^* \pi(f) U_i - \rho(f)) \leq \|U_i^* \pi(f) U_i - \rho(f)\| \to 0,\end{aligned}$$

we see that $\beta(\pi(f)) = \beta(\rho(f))$. Moreover, it follows from Corollary 15 that, the gauge norm β_{α} is symmetric. A simple consequence is that $KF_t = \beta_{KF_t}$ is a symmetric gauge norm on $L^{\infty}(J_{\infty}, \delta_{\infty})$.

Next suppose β is a symmetric gauge norm on $L^{\infty}(J_{\infty}, \delta_{\infty})$. We want to define α_{β} on $\mathcal{M}$. If $T \in \mathcal{M}$, we can choose a masa $\mathcal{A}$ in $\mathcal{M}$ such that $|T| \in \mathcal{A}$. We then choose a surjective tracial embedding $\pi : L^{\infty}(J_{\infty}, \delta_{\infty}) \to \mathcal{A}$ and choose $f \in L^{\infty}(J_{\infty}, \delta_{\infty})$ such that $\pi(f) = |T|$ and then define

$$\alpha_{\beta}(T) = \beta(f) = \beta(s_f).$$

Since

$$s_f(t) = s_{\pi(f)}(t) = s_{|T|}(t),$$

we see that the definition is independent of $\mathcal{A}$ and π. As in the matrix case, the main difficulty is proving that α_{β} satisfies the triangle inequality. In [10] this was done using an approach that avoids proving an analogue of the matrix Ky Fan Lemma (Lemma 8). Here we prove a general version of the Ky Fan Lemma that we will need later in our paper.

Lemma 18. *Suppose $f, h \in L^{\infty}[0,1]$, and $0 \leq f, h \leq 1, \|f\|_{\infty} = 1$. Suppose f, h are non-increasing, then there exist step functions $s_f^{[m]} \geq f$ and $s_h^{[m]} \leq h$ with ranges contained in*

$\left\{\frac{k}{m} : 0 \leq k \leq m\right\}$ *such that* $\frac{1}{m} \leq s_f^{[m]} \leq 1$ *and* $0 \leq s_h^{[m]} \leq \frac{m-1}{m}$ *and* $f \leq s_f^{[m]} \leq f + \frac{1}{m}$ *and* $\max\left(h - \frac{1}{m}, 0\right) \leq s_h^{[m]} \leq h$. *It follows that* $KF_t\left(s_h^{[m]}\right) \leq KF_t(h)$ *and* $KF_t(f) \leq KF_t\left(s_f^{[m]}\right)$ *for every* $t \in (0,1]$.

Proof. For every $m \in \mathbb{N}$, let $p_i = \sup f^{-1}\left((1 - \frac{i}{m}, 1 - \frac{i-1}{m}]\right)$, $q_i = \inf h^{-1}\left((1 - \frac{i}{m}, 1 - \frac{i-1}{m}]\right)$, $i = 1, ..., m$. Let $p_0 = q_0 = 0$. Then define

$$s_f^{[m]}(x) = \sum_{i=0}^{m-1}\left(1 - \frac{i}{m}\right)\chi_{[p_i, p_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$

$$s_h^{[m]}(x) = \sum_{i=0}^{m-1}\left(1 - \frac{i+1}{m}\right)\chi_{[q_i, q_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$

It is easy to see that $f \leq s_f^{[m]} \leq f + \frac{1}{m}$; thus $\left\|f - s_f^{[m]}\right\|_\infty \leq \frac{1}{m}$. Also $\max\left(h - \frac{1}{m}, 0\right) \leq s_h^{[m]} \leq h$; so $\left\|h - s_h^{[m]}\right\|_\infty \leq \frac{1}{m}$.

Therefore, $KF_t\left(s_h^{[m]}\right) \leq KF_t(h)$ and $KF_t(f) \leq KF_t\left(s_f^{[m]}\right)$ for every $t \in (0,1]$ □

Lemma 19. *Suppose* f *is a step function on* $[a,b]$ *and* $k \in \mathbb{N}$, *then there exists an invertible measure preserving map* $\varphi_k : [a,b] \to [a,b]$ *such that*

$$\left\|\frac{1}{k}\sum_{j=1}^{k} f \circ \varphi_k^{(j)} - \frac{1}{b-a}\int_a^b f(x)\, d\delta_\infty\right\|_\infty \leq \eta \|f\|_\infty \frac{4}{k}$$

where $\eta = card\ f([a,b])$, $\varphi_k^{(j)}$ *is the composition of j* φ_k*'s, i.e.,* $\varphi_k \circ \varphi_k \circ \cdots \circ \varphi_k$.

Proof. Define $\varphi_k : [a,b] \to [a,b]$ by

$$\varphi_k(x) = \begin{cases} x + \frac{b-a}{k} & \text{if } a \leq x \leq b - \frac{b-a}{k} \\ x + \frac{b-a}{k} - b + a & \text{if } b - \frac{b-a}{k} < x \leq b \end{cases}$$

Then $\varphi_k^{(k)}$ is the identity map.

Denote $\rho_k(f) = \frac{1}{k}\sum_{j=1}^{k} f \circ \varphi_k^{(j)} - \frac{1}{b-a}\int_a^b f d\delta_\infty$, then ρ_k is linear and $\|\rho_k\| \leq 2$ (with ρ_k acting as an operator on $L^{\infty}(J_{\infty},\delta_{\infty})$. Suppose $0\le j<k$. Then $\rho_k\left(\chi_{[a+j\frac{b-a}{k},a+(j+1))\frac{b-a}{k})}\right)=0$ a.e. (δ_{∞}). Since ρ_k is linear, $\rho_k\left(\chi_{[a+j_1\frac{b-a}{k},a+(j_2))\frac{b-a}{k})}\right)=0$ whenever $0\le j_1<j_2\le k$. Suppose $a\le\alpha<\beta\le b$. We choose j_1 and j_2 such that j_1 is the largest j, $1\le j\le k$ such that $a+j_1\frac{b-a}{k}\le\alpha$ and choose j_2 to be the smallest j, $1\le j\le k$ such that $\beta\le a+j_2\frac{b-a}{k}$. Then

$$\chi_{[a+j_1\frac{b-a}{k},a+(j_2))\frac{b-a}{k})}-\chi_{[\alpha,\beta)}=\chi_{[a+j_1\frac{b-a}{k},\alpha)}-\chi_{[\beta,a+(j_2))\frac{b-a}{k})}.$$

Hence

$$\rho_k\left(\chi_{[\alpha,\beta)}\right)=\rho_k\left(\chi_{[a+j_1\frac{b-a}{k},\alpha)}\right)-\rho_k\left(\chi_{[\beta,a+(j_2))\frac{b-a}{k})}\right).$$

However, if $E\in\left\{[a+j_1\frac{b-a}{k},\alpha),[\beta,a+(j_2))\frac{b-a}{k})\right\}$ and $f=\chi_E$ then, since $f\circ\varphi_k^{(j)}=\chi_{\left(\varphi_k^{(j)}\right)^{-1}(E)}$ and the collection $\left\{\left(\varphi_k^{(j)}\right)^{-1}(E):1\le j\le k\right\}$ is disjoint, we have

$$\left\|\frac{1}{k}\sum_{j=1}^{k}f\circ\varphi_k^{(j)}\right\|_{\infty}\le\frac{1}{k}$$

and

$$\frac{1}{b-a}\int_a^b\chi_E d\delta_{\infty}\le\frac{1}{b-a}\frac{b-a}{k}=\frac{1}{k},$$

we have $\|\rho_k(E)\|_{\infty}\le\frac{2}{k}$. Hence

$$\rho_k\left(\chi_{[\alpha,\beta)}\right)\le\frac{4}{k}.$$

Suppose f is a step function, then $f=\sum_{j=1}^{n}a_j\chi_{[\alpha_j,\alpha_{j+1})}$ for some $n\in\mathbb{N}$. Denote $f_j=\chi_{[\alpha_j,\alpha_{j+1})}$, Then

$$f=\sum_{j=1}^{n}a_jf_j\int_a^b f(x)\,d\delta_{\infty}=\sum_{j=1}^{n}a_j\int_a^b\chi_{[\alpha_j,\alpha_{j+1})}d\delta_{\infty}.$$

Thus

$$\begin{aligned}\|\rho_k(f)\|_{\infty}&\le\sum_{j=1}^{n}|a_j|\,\|\rho_k(f_j)\|_{\infty}\\&\le\left(\sum_{j=1}^{n}|a_j|\right)\frac{2}{k}\le\eta\,\|f\|_{\infty}\left(\frac{4}{k}\right).\end{aligned}$$

$\square$

We call the following an *approximate Ky Fan Lemma* for $L^{\infty}(\delta_{\infty})$.

Theorem 20. *Suppose m is a positive integer. Then whenever $0 \leq f, h \leq 1$ in $L^{\infty}(\delta_{\infty})$ satisfies*

$$KF_t(h) \leq KF_t(f) \text{ for all rational numbers } 0 < t \leq 1,$$

there are, $\gamma_1, \ldots \gamma_{m^{2m}} \in \mathbb{MP}(J_{\infty}, \delta_{\infty})$, such that

$$s_h \leq \frac{1}{m^{m^2}} \sum_{i=1}^{m^{m^2}} s_f \circ \gamma_i + \frac{2}{m}$$

Hence $\beta(h) \leq \beta(f)$ for every symmetric gauge norm β on $L^{\infty}(\delta_{\infty})$.

Proof. If $f \in L^{\infty}(J_{\infty}, \delta_{\infty})$, then the map $t \mapsto KF_t(f)$ is continuous on $(0,1]$. Hence we have $KF_t(h) \leq KF_t(f)$ for all $0 < t \leq 1$. We know that $KF_t(f) = KF_t(s_f)$ and $\beta(f) = \beta(s_f)$ for every $f \in L^{\infty}(\delta_{\infty})$. We may assume that f, h are nonincreasing, and we denote u, w be the step functions as in Lemma 18. Then u, w satisfying $f \leq u \leq f + \frac{1}{m}$and $\max\left(h - \frac{1}{m}, 0\right) \leq w \leq h$. Recall that

$$u = \left(1 - \frac{i}{m}\right) \chi_{[p_i, p_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$
$$w = \left(1 - \frac{i+1}{m}\right) \chi_{[q_i, q_{i+1})}(x) \text{ for } i = 0, ..., m-1.$$

and it is easy to see that

$$\int_0^t f d\delta_{\infty} + \frac{t}{m} \geq \int_0^t u d\delta_{\infty} \geq \int_0^t w d\delta_{\infty} \geq \int_0^t h d\delta_{\infty} - \frac{t}{m},$$

for all $0 \leq t \leq 1$.

By Lemma 19, for each $m \in \mathbb{N}$, there exists a measure preserving map $\varphi_m : [0,1] \to [0,1]$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} u \circ \varphi_m^{(j)} - \int_0^1 u d\delta_\infty \right\|_\infty \leq \eta \left\| u \right\|_\infty \frac{4}{m}.$$

where $\eta = card\left(Ran\left(u\right)\right)$

Let $l(t) = \frac{1}{t}\int_0^t ud\delta_\infty$, then $l : [0,1] \longrightarrow [0,\infty)$ is a continuous function. There are 2 cases to consider:

Case 1: If $l(1) = \int_0^1 ud\delta_\infty \geq b_1 = \max\left\{w\left(t\right) : 0 < t \leq 1\right\}$, then by Lemma 19, for $\forall k = m^2 \in \mathbb{N}$, there exists $\varphi_k \in \mathbb{MP}[0,1]$ such that

$$\left\| \frac{u \circ \varphi_k^{(1)} + \cdots + u \circ \varphi_k^{\left(m^2\right)}}{m^2} - \int_0^1 u d\delta_\infty \right\|_\infty \leq \frac{4\eta \left\| u \right\|_\infty}{m^2} \leq \frac{4}{m},$$

where $\eta = \mathrm{card}\left(u\right) \leq m$.Denote $\varphi_k^{(i)}$ by γ_j Then we have

$$\frac{1}{m^2} \sum_{j=1}^{m^2} u \circ \gamma_j \geq w - \frac{6}{m}.$$

Therefore $\frac{1}{m^2}\sum\limits_{j=1}^{m^2} f \circ \varphi_{(j)} + \frac{1}{m} \geq h - \frac{4}{m}$ follows from Lemma 18. That is

$$\frac{1}{m^2} \sum_{j=1}^{m^2} f \circ \varphi_{(j)} \geq h - \frac{3}{m}$$

We can view it as

$$\frac{1}{m^{2m}} \sum_{j=1}^{m^{2m}} f \circ \varphi_{(j)} \geq h - \frac{3}{m}$$

where $\varphi_{(i+m^2t)} = \varphi_{(i)}$ for $1 \leq i \leq m^2$ and $0 \leq t \leq m^{2m-2} - 1$.

Case 2: $l(1) = \int_0^1 ud\delta_\infty < b_1$.

Then there must exist $p_1^{'} \in (0, 1)$, so that

$$l(p_1^{'}) = \frac{1}{p_1^{'}} \int_0^{p_1^{'}} ud\delta_\infty = b_1.$$

Define $u^{(1)}$ in the following way

$$u^{(1)}(x) = \begin{cases} b_1 & 0 \le x \le p_1' \\ u(x) & p_1' < x \le 1 \end{cases}$$

Then for every $t > p_1^{'}$,

$$l(t) = \int_0^t ud\delta_\infty \geqslant \int_0^t wd\delta_\infty \implies \int_0^{p_1'} ud\delta_\infty + \int_{p_1'}^t ud\delta_\infty \geqslant \int_0^{p_1'} wd\delta_\infty + \int_{p_1'}^t wd\delta_\infty.$$

Thus we have $b_1 p_1' + \int_{p_1'}^t u^{(1)} d\delta_\infty \geqslant b_1 p_1' + \int_{p_1'}^t wd\delta_\infty$, therefore $\int_{q_1}^t u^{(1)} d\delta_\infty \geqslant \int_{q_1}^t wd\delta_\infty$.

Therefore, for every $0 < t \le 1$, we have

$$u - \frac{1}{m} \le u^{(1)} \le u$$

$$KF_t\left(u^{(1)}\right) \ge KF_t(w)$$

and for every $t \le t_1$, $\left\|u^{(1)}\right\|_t = b_1 = \|h\|_t$

By Lemma 19 again, for every $k = m^2 \in \mathbb{N}$,there exist $\varphi_{(1)}, \ldots, \varphi_{(k)} : [0, 1] \longrightarrow [0, 1]$ such that

$$\left\| \frac{1}{m^2} \sum_{i=1}^{m^2} u \circ \varphi_{(i)} - \int_0^1 ud\delta_\infty \right\|_\infty \le \eta \|u\|_\infty \frac{4}{m^2} \le \frac{4}{m}.$$

Let $\varphi_{(r)}^{(1)}(t) = \begin{cases} \varphi_{(r)}(t) & t \le q_1 \\ t & t > q_1 \end{cases}$, $r = 1, \ldots, m^2$. Then $\varphi_{(r)}^{(1)} \in \mathbb{MP}[0, 1]$ for all $1 \le r \le m^2$ and

$$\left\| \frac{1}{m^2} \sum_{r=1}^{m^2} u \circ \varphi_{(r)}^{(1)} - u^{(1)} \right\|_\infty \le \frac{2}{m}.$$

That is $u^{(1)} \approx \frac{1}{m^2} \sum_{r=1}^{m^2} u \circ \varphi_{(r)}^{(1)}$ and $Ran(u^{(1)}) \subseteq \{b_1, a_2, ..., a_m\}$.

If $\frac{1}{q_1}\int_{q_1}^{1} u^{(1)} d\delta_\infty \geqslant b_2$, go to case 1.

if $\frac{1}{q_1}\int_{q_1}^{1} u^{(1)} d\delta_\infty < b_2$, do the similar process as case 2 above, we have $u^{(2)}$ and

$$\left\| \frac{1}{m^2} \sum_{i=1}^{m^2} u^{(1)} \circ \varphi_i^{(2)} - u^{(2)} \right\|_\infty \leq \frac{2}{m}$$

That is

$$\begin{aligned} u^{(2)} &\approx \frac{1}{m^2} \sum_{i=1}^{m^2} u^{(1)} \circ \varphi_i^{(2)} \\ &= \frac{1}{m^2} \sum_{i_1=1}^{m^2} \left(\sum_{i_2=1}^{m^2} \frac{1}{m^2} \left(u \circ \varphi_{i_1}^{(1)} \right) \right) \circ \varphi_{i_2}^{(2)} \\ &= \frac{1}{m^4} \sum_{i=1}^{m^2} \sum_{j=1}^{m^2} (u \circ \varphi_{i_1}^{(1)} \circ \varphi_{i_2}^{(2)}). \end{aligned}$$

and $ran \left((u)^{(2)} \right) \subseteq \{b_1, b_2, a_3, ..., a_m\}$.

Finally, after r steps(at most m), we will have

$$u^{(r)} \approx \frac{1}{m^{2r}} \sum_{i_1=1}^{m^2} \cdots \sum_{i_r=1}^{m^2} (u \circ \varphi_{i_1}^{(1)} \circ \varphi_{i_2}^{(2)} \cdots \circ \varphi_{i_r}^{(r)}),$$

and thus $u^{(r)} \geqslant w$.

since $m^{2r} | m^{m^2}$, similar as in case 1, we can view this as

$$\frac{1}{m^{m^2}} \sum_{j=1}^{m^{m^2}} u \circ \varphi_{(j)} \geq w - \frac{2}{m}.$$

In conclusion, for every m, there is an integer $N = m^{m^2}$, and there are $\gamma_1, \dots \gamma_N \in \mathbb{MP}\left(J_\infty, \delta_\infty\right)$ such that

$$\frac{1}{N} \sum_{i=1}^{N} u \circ \gamma_i \geq w - \frac{2m}{N}.$$

By Lemma 18, we know that $f \geq u - \frac{1}{m}$ and $h \leq w + \frac{1}{m}$

Thus, $\frac{1}{N} \sum_{i=1}^{N} s_f \circ \gamma_i + \frac{2m}{N} + \frac{1}{m} \geq s_h$.

Therefore, $\beta\left(f\right) \geq \beta\left(h\right)$ as $m \to \infty$. $\square$

Corollary 21. *If β is a symmetric gauge norm on $L^{\infty}(J_{\infty}, \delta_{\infty})$, then α_{β} is a norm on $\mathcal{M}$.*

Proof. We need only prove the triangle inequality. If $A, B \in \mathcal{M}$, we define $h(t) = s_{A+B}(t)$ and $f(t) = s_A(t) + s_B(t)$. Then $KF_t(h) = KF_t(A+B)$ and $KF_t(f) = KF_t(A) + KF_t(B)$, so Lemma 20 applies, and we get

$$\alpha_{\beta}(A+B) = \beta(h) \leq \beta(f) = \beta(s_A(t) + s_B(t)) \leq \beta(s_A(t)) + \beta(s_B(t)) = \alpha_{\beta}(A) + \alpha_{\beta}(B).$$

□

Since it is easily seen that $\alpha = \alpha_{\beta_{\alpha}}$ and $\beta = \beta_{\alpha_{\beta}}$, we obtain the characterization [10] of the unitarily invariant norms on a II_1 factor von Neumann algebra.

Theorem 22. *Let $\mathcal{M}$ be a type II_1 factor von Neumann algebra, then there is a one-to-one correspondence between unitarily invariant norms on $\mathcal{M}$ and symmetric gauge norms on $L^{\infty}(J_{\infty}, \delta_{\infty})$.*

2.2 Approximate Unitary Equivalence

The following is a consequence of a result of Hadwin and Ding [5]. Suppose $\mathcal{R}$ is a von Neumann algebra and $T \in \mathcal{R}$. $\mathcal{Z}(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}'$ is the center. In [4] the $\mathcal{R}$-*rank* of T was defined to be the Murray-von Neumann equivalence class of the projection P_T onto the closure of the range of T. Note that

$$P_T = \lim_{n \to \infty} (TT^*)^{1/n} \ \ (SOT),$$

so $P_T \in \mathcal{M}$.

In [5] they defined $\mathcal{R}$-rank$(S) \leq \mathcal{R}$-rank(T) to mean that P_S is Murray-von Neumann equivalent to a subprojection of P_T.

Theorem 23. *Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Let $\Phi : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ be the unique center-valued trace on $\mathcal{R}$. Suppose $\mathcal{A}$ is a unital commutative C* algebra and $\pi, \rho : \mathcal{A} \to \mathcal{R}$ are unital $*$-homomorphisms. The following are equivalent:*

1. There is a net $\{U_\lambda\}$ of unitary operators in $\mathcal{R}$ such that, for every $a \in \mathcal{A}$,

$$\|U_\lambda^* \pi(a) U_\lambda - \rho(a)\| \to 0.$$

2. $\Phi \circ \pi = \Phi \circ \rho$.

Proof. $(2) \Rightarrow (1)$. Suppose (2) is true. Suppose $x_1, \ldots, x_n \in \mathcal{A}$ and $\varepsilon > 0$. Let $\mathcal{B} = C^*(x_1, \ldots, x_n)$. Then $\mathcal{B}$ is separable and commutative. Since $\pi.\rho : \mathcal{B} \to \mathcal{R}$ are unital $*$-homomorphisms, there are weak*-weak* continuous unital $*$-homomorphisms $\hat{\pi}$ and $\hat{\rho}$ from the second dual $\mathcal{B}^{\#\#}$ of $\mathcal{B}$ into $\mathcal{R}$ such that the restrictions of $\hat{\pi}$ and $\hat{\rho}$, respectively, to $\mathcal{B}$ are π and ρ. Since Φ is weak*-weak* continuous on $\mathcal{R}$, we see that $\Phi \circ \hat{\pi} = \Phi \circ \hat{\rho}$. Suppose $x \in \mathcal{B}$. Then the range projection $P_{\pi(x)}$ equals the weak*-limit $\pi\left(|x|^{1/n}\right)$ and $P_{\rho(x)}$ is the weak*-limit of $\rho\left(|x|^{1/n}\right)$. Thus

$$\Phi\left(P_{\pi(x)}\right) = \lim_{n\to\infty} \Phi\left(\pi\left(|x|^{1/n}\right)\right) = \lim_{n\to\infty} \Phi\left(\rho\left(|x|^{1/n}\right)\right) = \Phi\left(P_{\rho(x)}\right).$$

This means, by Corollary 2.8 in Takesaki, vol 1, that $P_{\pi(x)}$ and $P_{\rho(x)}$ are Murray-von Neumann equivalent. Hence, for every $x \in \mathcal{B}$,

$$\mathcal{R}\text{-}rank(\pi(x)) = \mathcal{R}\text{-}rank(\rho(x)).$$

If follows from [5] that the restrictions of π and ρ to $\mathcal{B}$ are approximately equivalent in $\mathcal{R}$. Hence there is a unitary operator $U \in \mathcal{M}$ such that

$$\|U^* \pi(x_k) U - \rho(x_k)\| < \varepsilon$$

for $1 \le k \le n$. If we let D be the set of all pairs $d = (\mathcal{F}, \varepsilon)$, with $\mathcal{F} = \{x_1, \ldots, x_n\} \subseteq \mathcal{A}$ finite and $\varepsilon > 0$, we see that D is a directed set with respect to $\le\, = (\subseteq, \ge)$ and if we denote the U above by U_d, we obtain a net $\{U_d\}$ of unitary operators in $\mathcal{R}$ such that, for every $x \in \mathcal{A}$

$$\|U_d^* \pi(x) U_d - \rho(x)\| \to 0.$$

$(1) \Rightarrow (2)$. Suppose $a = a^* \in \mathcal{A}$ and let $\hat{\pi}$ and $\hat{\rho}$ from $\mathcal{A}^{\#\#}$ to $\mathcal{R}$ be as in the proof of $(2) \Rightarrow (1)$. The family $\left\{\chi_{\{t\}}(\pi(a)) : t \in \mathbb{R}\right\}$ is an orthogonal family of projections on the separable Hilbert space H, so, except for a countable set $E_{\pi(a)} \subseteq \mathbb{R}$ these projections must be 0. Simlarly, there is a countable subset $E_{\rho(a)} \subseteq \mathbb{R}$ such that, for $t \in \mathbb{R} \backslash E_{\rho(a)}$, we have $\chi_{\{t\}}(\rho(a)) = 0$. Suppose $-\infty < s < t < \infty$ and $s, t \notin E_{\pi(a)} \cup E_{\rho(a)}$. Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x \leq s \\ \frac{x-s}{t-s} & \text{if } s \leq x \leq t \\ 1 & \text{if } t \leq x \end{cases}$$

Then h is continuous and $P_{\pi(h(a))} = P_{g(\pi(a))} = \chi_{[s,t)}(\pi(a))$ and $P_{\rho(h(a))} = P_{g(\rho(a))} = \chi_{[s,t)}(\rho(a))$. Since π and ρ are approximately equivalent in $\mathcal{R}$, it follows from [5] that $\chi_{[s,t)}(\pi(a))$ and $\chi_{[s,t)}(\rho(a))$ are Murray-von Neumann equivalent. Thus, by Corollary 2.8 in [15],

$$\Phi\left(\chi_{[s,t)}(\pi(a))\right) = \Phi\left(\chi_{[s,t)}(\rho(a))\right).$$

Since the allowable s and t are dense in $\mathbb{R}$ and Φ is weak*-weak* continuous, we see that

$$\Phi\left(\chi_{[s,t)}(\pi(a))\right) = \Phi\left(\chi_{[s,t)}(\rho(a))\right)$$

holds for all $-\infty < s < t < \infty$. Since Φ is linear, we see that

$$\Phi(\pi(a)) = \Phi(\rho(a))$$

whenever $a = a^* \in \mathcal{A}$, and thus, for all $a \in \mathcal{A}$. Thus (2) is proved. $\square$

Remark 1. *The characterization of approximate equivalence in terms of $\mathcal{R}$-rank in [5] Ding-Hadwin holds for separable AH C*-algebras. Every nonseparable AH C*-algebra is a direct limit of separable AH C*-algebra, and the proof of Theorem 23 can easily be modified to handle the AH case.*

2.3 The Central Decomposition

We refer the reader to [11] for the theory of direct integrals and the central decomposition of a von Neumann algebra acting on a separable Hilbert space. Since we are only interested in the von Neumann algebra $\mathcal{R}$ and not how it acts on a Hilbert space, we can ignore multiplicities when using the central decomposition [11].Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space. Then we can write

$$\mathcal{R} = [\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots] \oplus \mathcal{R}_\infty$$

where $\mathcal{R}_k$ is type I_k for $1 \leq k < \infty$ and $\mathcal{R}_\infty$ is a type II_1 von Neumann algebra.

2.3.1 Measurable families

Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with a faithful tracial state acting on a separable Hilbert space $H{=}l^2_\infty$. We will associate with $\mathcal{M}$ a probability space (Ω, μ) and a unitary operator $U : H \longrightarrow L^2(\mu, H)$ that transforms $\mathcal{M}$ into a certain von Neumann algebra of operators on $L^2(\mu, H)$ that will be described next.

For each $\omega \in \Omega$, there is a type II_1 von Neumann algebra $\mathcal{M}_\omega$ in $B(H)$ that is determined by two sequences of SOT measurable functions f_n and g_n from Ω into the unit ball of $B(H)$ so that $\mathcal{M}_\omega$ is generated by the set $\{f_n(\omega) : n \in \mathbb{N}\}$, $\mathcal{M}_\omega^{\prime}$ is generated by the set $\{g_n(\omega) : n \in \mathbb{N}\}$, and each of those sets is SOT dense in the unit ball of the von Neumann algebra it generates. Suppose $\varphi : \Omega \to B(H)$ is a SOT-measurable function, and define $|\varphi| = \|\cdot\| \circ \varphi$, that is $|\varphi|(\omega) = \|\varphi(\omega)\|$ for $\omega \in \Omega$. If $|\varphi| \in L^\infty(\mu)$, then let $\|\varphi\|_\infty = \|\varphi\|_\infty$. We will assume that (Ω, μ), U, and the $f_n, g_n, \mathcal{M}_\omega$ have been chosen so that

$U^*\mathcal{M}U = \{\varphi : \Omega \longrightarrow B(H) |\ \varphi \text{ is SOT-measurable, } \varphi(\omega) \in \mathcal{M}_\omega \text{ a.e. } (\mu)\,, \text{ and } |\varphi| \in L^\infty(\mu)\}\,.$

As usual, $\varphi_1 = \varphi_2$ will mean $\varphi_1 = \varphi_2$ a.e. (μ), and each φ in $U^*\mathcal{M}U$ is the operator on $L^2(\mu, H)$ defined for $f \in L^2(\mu, H)$ by

$$(\varphi f)(\omega) = \varphi(\omega) f(\omega)\,.$$

2.3.2 Measurable cross-sections

Definition 24. *Suppose (X, d) is a metric space and $\mu : \mathrm{Bor}(X) \longrightarrow [0, \infty)$ is a finite measure. A subset B of X is called* μ-measurable *if there are $A, F \in \mathrm{Bor}(X)$ such that $B \backslash A \subset F$ and $\mu(F) = 0$. The σ-algebra of all μ-measurable sets is denoted by $\mathcal{M}_\mu$. A subset D of X is* absolutely measurable *if D is μ-measurable for every finite measure μ on $Bor(X)$. The σ-algebra of all absolutely measurable subsets of X is denoted by $\mathbb{AM}(X)$. Clearly we have*

$$\mathbb{AM}(X) = \bigcap \{\mathcal{M}_\mu : \mu \textit{ is a finite Borel measure on } X\}\,.$$

It is obvious that each $\mathcal{M}_\mu$ contains $\mathrm{Bor}(X)$, so $\mathrm{Bor}(X) \subset \mathbb{AM}(X)$. However, it is often the case that $\mathrm{Bor}(X) \neq \mathbb{AM}(X)$. If Y is another metric space, we say that a function $f : X \to Y$ is absolutely measurable *if f is $\mathbb{AM}(X)$-$\mathrm{Bor}(Y)$ measurable, i.e., for every Borel set $E \subseteq Y$, $f^{-1}(E) \in \mathbb{AM}(X)$. Recall that a finite measure space $(\Lambda, \Sigma, \lambda)$ is* complete *if, $E \in \Sigma$ whenever $E \subset F, F \in \Sigma$ and $\lambda(F) = 0$, i.e., all subsets of sets of measure 0 are in Σ. Note that statement (4) in Lemma 25 shows how, in the presence of a complete measure space, absolute measurability turns into measurability.*

Lemma 25. *Suppose X , Y and Z are metric spaces and $f : X \longrightarrow Y$, and $g : Y \to Z$. Then*

1. *f is absolutely measurable if and only if f is $\mathbb{AM}(X)$-$\mathbb{AM}(Y)$ measurable*

2. *If f and g are absolutely measurable, then $g \circ f : X \to Z$ is absolutely measurable.*

3. *For every Borel set $E \subseteq Y$, $f^{-1}(E)$ is absolutely measurable.*

4. *If* $(\Lambda, \Sigma, \lambda)$ *is a* ***complete*** *finite measure space and* $\varphi : \Lambda \to X$ *is Borel measurable, then*

   (a) φ *is* Σ-$\mathbb{AM}(X)$ *measurable, and,*

   (b) *If* f *is absolutely measurable, then* $f \circ \varphi : X \to Y$ *is measurable.*

Definition 26. *If* $f : X \to Y$ *and* $g : f(X) \longrightarrow X$ *satisfy, for every* $y \in f(X)$,

$$f(g(y)) = y,$$

then g *is called a* ***cross-section*** *for* f.

The following Theorem is from Theorem 3.4.3 in [1] and is the key to dealing with direct integrals.

Theorem 27. *Suppose* X *is a Borel subset of a complete separable metric space, and* Y *is a separable metric space.If* $f : X \longrightarrow Y$ *is a continuous function, then*

1. $f(X)$ *is an absolutely measurable subset of* Y, *and*

2. f *has an absolutely measurable cross-section* $g : f(X) \longrightarrow X$.

Here is a simple result proved using measurable cross-section.

Lemma 28. *Suppose* n *is a positive integer and* $\mathbb{M}_n(\mathbb{C})^+$ *is the set of* $n \times n$ *matrices* A *such that* $A \geq 0$. *Let* $\mathcal{U}_n$ *be the set of unitary* $n \times n$ *matrices and let* $\mathcal{D}_n$ *be the set of all diagonal* $n \times n$

matrices in $\mathbb{M}_n(\mathbb{C})^+$ *of the form* $\operatorname{diag}\left(s_{\frac{1}{n}}, \dots s_1\right)$ *with* $s_{\frac{1}{n}} \geq s_{\frac{2}{n}} \geq \cdots \geq s_1 \geq 0$. *Then there is an absolutely measurable function* $u : \mathbb{M}_n(\mathbb{C})^+ \to \mathcal{U}_n$ *such that, for every* $A \in \mathbb{M}_n(\mathbb{C})^+$,

$$u(A)^* Au(A) \in \mathcal{D}_n,$$

i.e.,

$$u(A)^* Au(A) = \begin{pmatrix} s_A\left(\frac{1}{n}\right) & & & \\ & s_A\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_A\left(\frac{n}{n}\right) \end{pmatrix}.$$

Hence, for every $T \in \mathbb{M}_n(\mathbb{C})$,

$$u(|T|)^* |T| u(|T|) = \begin{pmatrix} s_T\left(\frac{1}{n}\right) & & & \\ & s_T\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_T\left(\frac{n}{n}\right) \end{pmatrix}.$$

Proof. Let $X = \left\{(A, U_A) : A \in \mathbb{M}_n(\mathbb{C})^+, U_A \in \mathcal{U}_n, U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \dots, s_A\left(\frac{n}{n}\right)\right)\right\}$, which is a subset of $\mathbb{M}_n(\mathbb{C})^+ \times \mathcal{U}_n$. For every $(A_\lambda, U_{A_\lambda}) \in X$, and $(A_\lambda, U_{A_\lambda}) \longrightarrow (A, U_A)$, we have $A_\lambda \longrightarrow A, U_{A_\lambda} \longrightarrow U_A$, Thus

$$\|U_A^* A U_A - U_{A_\lambda} A_\lambda U_{A_\lambda}\| \to 0,$$

We also know that $\frac{1}{i}\sum_{j=1}^{i} s_A\left(\frac{j}{n}\right) = KF_i(A)$ for all $1 \leq i \leq n$ and $s_A\left(\frac{1}{n}\right) = KF_1(A) \leq \|A\|$. We can get $s_{A_\lambda}\left(\frac{i}{n}\right) \overset{\|\cdot\|}{\to} s_A\left(\frac{i}{n}\right)$ for all $1 \leq i \leq n$. Thus

$$U_{A_\lambda}^* A_\lambda U_{A_\lambda} = \operatorname{diag}\left(s_{A_\lambda}\left(\frac{1}{n}\right), \dots, s_{A_\lambda}\left(\frac{n}{n}\right)\right) \xrightarrow{\|\cdot\|} \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \dots, s_A\left(\frac{n}{n}\right)\right)$$

Therefore $U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \dots, s_A\left(\frac{n}{n}\right)\right)$, and X is a closed subset of a $\mathbb{M}_n(\mathbb{C})^+ \times \mathcal{U}_n$, which is a complete separable metric space.

Define $\pi_1 : X \longrightarrow \mathbb{M}_n(\mathbb{C})^+$ and $\pi_2 : X \longrightarrow \mathcal{U}_n$ by

$$\pi_1(A,U) = A,\ \pi_2(A,U) = U$$

It is easy to see that $\pi_1(X) = \mathbb{M}_n(\mathbb{C})^+$.

Since we know for every $A \in \mathbb{M}_n(\mathbb{C})^+$, there exists a unitary U_A such that

$$U_A^* A U_A = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right)$$

Thus by Theorem 27, there exists an absolutely measurable function $g : \mathbb{M}_n(\mathbb{C})^+ \longrightarrow X$ such that $\pi_1 \circ g = id$ on $\mathbb{M}_n(\mathbb{C})^+$, for every $A \in \mathbb{M}_n(\mathbb{C})^+$, $g(A) = (A, U_A)$. Then we define $u = \pi_2 \circ g : \mathbb{M}_n(\mathbb{C})^+ \longrightarrow \mathcal{U}_n$, it is absolutely measurable.

Therefore, for every $A \in \mathbb{M}_n(\mathbb{C})^+$,

$$u(A) = U_A \text{ and } u(A)^* A u(A) = \operatorname{diag}\left(s_A\left(\frac{1}{n}\right), \ldots, s_A\left(\frac{n}{n}\right)\right) \in \mathcal{D}_n.$$

Hence, for every $T \in \mathbb{M}_n(\mathbb{C})$,

$$u(|T|)^* |T| u(|T|) = \begin{pmatrix} s_T\left(\frac{1}{n}\right) & & & \\ & s_T\left(\frac{2}{n}\right) & & \\ & & \ddots & \\ & & & s_T\left(\frac{n}{n}\right) \end{pmatrix}.$$

□

2.3.3 Direct Integrals

Suppose $\Omega \subseteq \mathbb{R}$ is compact, μ is a probability Borel measure, H is a separable Hilbert space. Define $\int_\Omega^\oplus H d\mu = L^2(\mu, H)$ to be the set of all measurable functions $f : \Omega \to H$ such that

$$\|f\|_2^2 \underset{\text{def}}{=} \int_\Omega \|f(\omega)\|^2 d\mu(\omega) < \infty.$$

We define an inner product $\langle , \rangle$ on $L^2(\mu, H)$ by

$$\langle f, h \rangle = \int_\Omega \langle f(\omega), h(\omega) \rangle \, d\mu(\omega).$$

In this way $L^2(\mu, H)$ is a Hilbert space.

We define $L^\infty(\mu, B(H))$ to be the set of all bounded functions $\varphi : \Omega \to B(H)$ that are measurable with respect to the weak operator topology (WOT) on $B(H)$. Although the weak operator topology, strong operator topology (SOT) and $*$-strong operator topology ($*$-SOT) on $B(H)$ are different, the Borel sets with respect to these topologies are all the same. Suppose the map $\omega \mapsto T_\omega$ is in $L^\infty(\mu, B(H))$. We define an operator $T = \int_\Omega^\oplus T_\omega d\mu(\omega)$ by

$$(Tf)(\omega) = T_\omega(f(\omega)).$$

If $\varphi \in L^\infty(\mu, B(H))$ and $T_\omega = \varphi(\omega)$ for $\omega \in \Omega$, we also use the notation M_φ to denote $\int_\Omega^\oplus T_\omega d\mu(\omega)$. In this way we can view $L^\infty(\mu, B(H)) \subseteq B(L^2(\mu, H))$, and we can write $L^\infty(\mu, B(H)) = \int_\Omega^\oplus B(H) \, d\mu(\omega)$.

We have that $L^\infty(\mu)$ can be viewed as the subalgebra $\mathcal{D}$ of $L^\infty(\mu, B(H))$ of all functions φ such that $\varphi(\omega) \in \mathbb{C} \cdot 1$ a.e. (μ), that is, by identifying $h \in L^\infty(\mu)$ with the function $\omega \mapsto h(\omega) 1$. We denote $\mathcal{D}$ by

$$\mathcal{D} = \int_\Omega^\oplus \mathbb{C} \cdot 1 d\mu(\omega).$$

We have $\mathcal{D}' = L^\infty(\mu, B(H))$ and $L^\infty(\mu, B(H))' = \mathcal{D}$, therefore $\mathcal{D} = \mathcal{Z}(L^\infty(\mu, B(H)))$.

Suppose, for each $\omega \in \Omega$, $\mathcal{R}_\omega \subset B(H)$ is a von Neumann algebra. We say that the family $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ is a *measurable family* if there is a countable set $\{\varphi_1, \varphi_2, \ldots\} \subset L^\infty(\mu, B(H))$ such that

$$\text{ball}(\mathcal{R}_\omega) = \{\varphi_1(\omega), \varphi_2(\omega), \ldots\}^{-SOT} \text{ a.e. } (\mu).$$

It is known that if $\{\mathcal{R}_\omega\}_{\omega \in \Omega}$ is a measurable family, then so is $\{\mathcal{R}'_\omega\}_{\omega \in \Omega}$. Moreover, if $\{\mathcal{R}'_\omega\}_{\omega \in \Omega}$ is a measurable family, then there is a sequence $\{\psi_1, \psi_2, \ldots\} \subset L^\infty(\mu, B(H))$ such that

$$\text{ball}\left(\mathcal{R}_\omega'\right) = \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT} \text{ a.e. } (\mu).$$

If $\{\mathcal{R}_\omega\}_{\omega\in\Omega}$ is a measurable family, then we define the *direct integral* $\int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega)$ to be the set of all $T = \int_\Omega^\oplus T_\omega d\mu(\omega) \in L^\infty(\mu, B(H))$ such that

$$T_\omega \in \mathcal{R}_\omega \text{ a.e. } (\mu).$$

It is known [11] that a von Neumann algebra $\mathcal{R} \subset B\left(L^2(\mu, H)\right)$ can be written as

$$\mathcal{R} = \int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega)$$

for a measurable family $\{\mathcal{R}_\omega\}_{\omega\in\Omega}$ if and only if

$$\mathcal{D} = \int_\Omega^\oplus \mathbb{C}\cdot 1 d\mu(\omega) \subset \mathcal{R} \subset \int_\Omega^\oplus B(H)\, d\mu(\omega) = \mathcal{D}',$$

equivalently,

$$\mathcal{D} \subset \mathcal{Z}(\mathcal{R}).$$

In particular, since $\mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\mathcal{R}') = \mathcal{R}\cap\mathcal{R}'$ for every von Neumann algebra $\mathcal{R}$, we see that $\mathcal{R}$ can be decomposed as a direct integral if and only if $\mathcal{R}'$ can be decomposed as a direct integral.

Suppose $1 \le n \le \infty = \aleph_0$. We define ℓ_n^2 be the space of square summable sequences with the inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x, y \in H$ and H is a Hilbert space with dimension n.

Lemma 29. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space H. Then there are compact subsets $\Omega_n \subset \mathbb{R}$ for $1 \le n \le \infty$ and a Borel measure μ_n on Ω_n such that $\mu_n(\Omega_n) \in \{0,1\}$ and $\mathcal{A}$ is unitarily equivalent to $\sum_{1\le n\le\infty}^\oplus L^\infty(\mu_n, \mathbb{C}\cdot 1)$ acting on $\sum_{1\le n\le\infty}^\oplus L^2(\mu_n, \ell_n^2)$.*

Suppose $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space H. Then the center $\mathcal{Z}(\mathcal{R})$ of $\mathcal{R}$ is an abelian von Neumann algebra on H. From Lemma 29 we can write

$$H = \sum_{1 \le n \le \infty}^{\oplus} L^2\left(\mu_n, \ell_n^2\right)$$

and

$$\mathcal{Z}(\mathcal{R}) = \sum_{1 \le n \le \infty}^{\oplus} L^\infty\left(\mu_n, \mathbb{C} \cdot 1\right).$$

Since $\mathcal{R}$ commutes with $\mathcal{Z}(\mathcal{R})$, we can write

$$\mathcal{R} = \sum_{1 \le n \le \infty}^{\oplus} \mathcal{R}_n,$$

where $\mathcal{R}_n \subset B\left(L^2\left(\mu_n, \ell_n^2\right)\right)$. It is clear, for $1 \le n \le \infty$, that

$$\mathcal{Z}(\mathcal{R}_n) = L^\infty\left(\mu_n, \mathbb{C} \cdot 1\right),$$

which implies

$$\mathcal{R}_n \subset \mathcal{Z}(\mathcal{R}_n)' = L^\infty\left(\mu_n, \mathbb{C} \cdot 1\right)' = L^\infty\left(\mu_n, B\left(\ell_n^2\right)\right).$$

Hence, for each n, $1 \le n \le \infty$, there is a measurable family $\{\mathcal{R}_n(\omega)\}_{\omega \in \Omega_n}$ such that

$$\mathcal{R}_n = \int_{\Omega_n}^{\oplus} \mathcal{R}_n(\omega)\, d\mu_n(\omega).$$

We therefore have

$$\mathcal{R} = \sum_{1 \le n \le \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathcal{R}_n(\omega)\, d\mu_n(\omega).$$

This is called the *central decomposition* of $\mathcal{R}$.

The following Lemma is a well-known result.[11]

Lemma 30. *In the central decomposition of $\mathcal{R}$, almost every $\mathcal{R}_n(\omega)$ is a factor von Neumann algebra.*

Lemma 31. *Suppose $\mathcal{A}_n$ is a masa of a $\mathcal{R}_n$ for $1 \leq n \leq \infty$, then there is a measurable family $\{\mathcal{A}_n(\omega)\}_{\omega\in\Omega_n}$ such that*

$$\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega),$$

where $\mathcal{A}_n(\omega)$ is a masa in $\mathcal{R}_n(\omega)$.

Proof. Suppose

$$\mathcal{W} = \left(B\left(l_n^2\right)\right) \times A \times B \times C \times E \times \mathbb{N} \times \mathbb{N},$$

where $A = B = C = \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_n^2\right)\right)$ and $E = \{x \in l_n^2 : \|x\| = 1\}$. Then $\mathcal{W}$ is a complete separable metric space with product topology.

Define $\mathcal{X}_{m,k}$ to be the set of elements $(T, \{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty}, e, m, k)$ in $\mathcal{W}$ satisfying

$$TA_i = A_iT, TB_i = B_iT, \ \|(TC_m - C_mT)\, e\| \geq \frac{1}{k}, \text{ for every } i \in \mathbb{N}.$$

Then $\mathcal{X}_{m,k}$ is a closed subset of $\mathcal{W}$. We define $\mathcal{X} = \bigcup_{m,k=1}^{\infty} \mathcal{X}_{m,k}$, then $\mathcal{X}$ is a Borel subset of $\mathcal{W}$.

Let $\pi_{2,3,4} : \mathcal{X} \to A \times B \times C$ be the projection map. Then $\pi_{2,3,4}(\mathcal{X})$ consists of elements $(\{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty})$ so that there exists $T \in \text{ball}\left(B\left(l_n^2\right)\right)$ such that

$$T \in \{A_1, A_2, \ldots\}' \cap \{B_1, B_2, \ldots\}' \text{ and } T \notin \{C_1, C_2, \ldots\}'.$$

Suppose there are sequences $\{f_1, f_2, \ldots\}$, $\{\psi_1, \psi_2, \ldots\}$ and $\{g_1, g_2, \ldots\}$ contained in $L^{\infty}\left(\mu_n, B\left(l_n^2\right)\right)$ such that

$$\begin{aligned}
\text{ball}\mathcal{A}_n(\omega) &= \{f_1(\omega),\ f_2(\omega), \ldots\}^{-SOT},\\
\text{ball}\mathcal{R}_n(\omega)' &= \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT}\\
\text{ball}\mathcal{A}_n(\omega)' &= \{g_1(\omega), g_2(\omega), \ldots\}^{-SOT}.
\end{aligned}$$

By Theorem 27, we know there exists an absolutely measurable function $\Upsilon : \pi_{2,3,4}(\mathcal{X}) \longrightarrow \mathcal{X}$ such that $\pi_{2,3,4} \circ \Upsilon$ is the identity function on $\pi_{2,3,4}(\mathcal{X})$.

Define $F : \Omega_n \to A \times B \times C$ by

$$F(\omega) = \{f_i(\omega)\}_{i=1}^{\infty} \times \{\psi_i(\omega)\}_{i=1}^{\infty} \times \{g_i(\omega)\}_{i=1}^{\infty}.$$

Let

$$G = F^{-1}(\pi_{2,3,4}(\mathcal{X})) =$$
$$\left\{\omega : \text{there exists } T \in B\left(l_n^2\right) \text{ such that } T(\omega) \in \mathcal{A}_n(\omega)' \cap \mathcal{R}_n(\omega) \text{ and } T(\omega) \notin \mathcal{A}_n(\omega)\right\}.$$

We know from Lemma 25 and the completeness of (Ω_n, μ_n) that G is measurable. We need to prove $\mu_n(G^c) = 0$. Suppose not, and let $\pi_1 : \mathcal{X} \to B\left(l_n^2\right)$ be the projection map (into the first coordinate). Then, by Lemma 25, $\pi_1 \circ \Upsilon \circ F|_G$ is a measurable function from G to $B\left(l_n^2\right)$. We define T by

$$T(\omega) = \begin{cases} (\pi_1 \circ \Upsilon \circ F|_G)(\omega) & \text{if } \omega \in G \\ 0 & \text{if } \omega \notin G \end{cases}.$$

Thus

$$T = \int_G^{\oplus} T(\omega)\, d\mu_n(\omega) \oplus \int_{\Omega_n \setminus G}^{\oplus} 0 d\mu_n(\omega),$$

then $T \in \mathcal{A}_n' \cap \mathcal{R}_n$ and $T \notin \mathcal{A}_n$, which contradicts to the assumption that $\mathcal{A}_n$ is a masa. Therefore $\mu_n(G) = 0$ and

$$\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega),$$

$\mathcal{A}_n(\omega)$ is a masa a.e.(μ_n). This completes the proof. □

2.3.4 Multiplicities for Type I$_n$ factors

A type I factor von Neumann algebra is isomorphic to $B(H)$ for some Hilbert space H. However, if m is a cardinal, we can let $H^{(m)}$ denote a direct sum of m copies of H and, for each $T \in B(H)$ write $T^{(m)}$ be a direct sum of m copies of T acting on $H^{(m)}$, and let $B(H)^{(m)} = \left\{T^{(m)} : T \in B(H)\right\}$. Clearly, $B(H)^{(m)}$ is isomorphic to $B(H)$. The number m is called the ***multiplicity of*** the factor $B(H)^{(m)}$ and it is the minimal rank of a nonzero projection in $B(H)^{(m)}$. If we consider a type I von Neumann algebra acting on a separable Hilbert space as a direct integral of factors, we can change the factors so that they all have multiplicity 1. This gives another von Neumann algebra that is isomorphic to the original one. Since we are interested in finite von Neumann algebras, the type I_n algebras, with $1 \leq n < \infty$, can be written as direct integrals of copies of $\mathbb{M}_n(\mathbb{C})$, i.e., $\int_{\Omega_n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega)$ acting on $L^2(\mu_n, \ell_n^2)$ for some probability space (Ω_n, μ_n) where μ_n is a Borel measure on a compact subset Ω_n of $\mathbb{R}$. In this case, $\int_{\Omega_n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega)$ is naturally isomorphic to $\mathbb{M}_n(L^\infty(\mu_n))$ acting on $L^2(\mu_n)^{(n)}$. When we write the type I_n part of a von Neumann algebra this way, we have an isomorphic copy, but maybe not a unitarily equivalent copy of the algebra, since we changed all of the multiplicities to be 1. Note that the center $\mathcal{Z}\left(\int_{\Omega_n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega)\right) = \int_{\Omega_n}^{\oplus} \mathbb{C} \cdot 1\mu_n(\omega)$ acting on $L^2(\mu_n, \ell_n^2)$.

For example, if a von Neumann algebra is $\int_{E_1}^{\oplus} \mathbb{M}_2(\mathbb{C})\, d\eta_1(\omega) \oplus \int_{E_2}^{\oplus} \mathbb{M}_2(\mathbb{C})^{(3)}\, d\eta_2(\omega)$, then it is isomorphic to $\int_{\Omega}^{\oplus} \mathbb{M}_2(\mathbb{C})\, d\mu(\omega)$ where Ω is the disjoint union of E_1 and E_2 and $\mu(A) = \eta_1(A \cap E_1) + \eta_2(A \cap E_2)$.

Thus in the central decomposition, we can assume, for each positive integer n (i.e., $1 \leq n < \infty$), that

$$\mathcal{R}_n = \int_{\Omega_n}^{\oplus} \mathcal{R}_n(\omega)\, d\mu_n(\omega) = \int_{\Omega_n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega),$$

and

$$\mathcal{Z}(\mathcal{R}_n) = \int_{\Omega_n}^{\oplus} \mathbb{C} \cdot 1 d\mu_n.$$

For $1 \leq n < \infty$ we have that the map $\rho_n : \mathcal{R}_n \to \mathbb{C}$ defined by

$$\rho_n(T) = \int_{\Omega_n}^{\oplus} \tau_{n,\omega}(T_\omega)\, d\mu_n(\omega)$$

is a normal faithful tracial state on $\mathcal{R}_n$.

2.3.5 II_1 von Neumann algebras

Once we have changed the multiplicities of the type I_n parts of $\mathcal{R}$, we have in the decomposition

$$\mathcal{R}_\infty = \int_{\Omega_\infty}^{\oplus} \mathcal{R}_\infty(\omega)\, d\mu_\infty(\omega).$$

We have

$$\mathcal{Z}\left(\mathcal{R}_{\infty}\right)=\int_{\Omega_{\infty}}^{\oplus}\mathbb{C}\cdot 1d\mu_{\infty}\left(\omega\right).$$

we have that each $\mathcal{R}_{\infty}\left(\omega\right)$ must be an infinite dimensional finite factor, which means it must be a type II_1 factor, and we can assume it acts on ℓ^2. In this case making the multiplicity infinite can make things more convenient.

We let $\mathcal{R}_{\infty}^{(\infty)}=\left\{T^{(\infty)}=T\oplus T\oplus\cdots:T\in\mathcal{R}_{\infty}\right\}$. Clearly, $\mathcal{R}_{\infty}^{(\infty)}$ is isomorphic to $\mathcal{R}_{\infty}$, and we have

$$\mathcal{R}_{\infty}^{(\infty)}=\int_{\Omega_{\infty}}^{\oplus}\mathcal{R}_{\infty}^{(\infty)}\left(\omega\right)d\mu_{\infty}\left(\omega\right)$$

acting on $L^2\left(\mu_{\infty},\left(\ell^2\right)^{(\infty)}\right)$. The nice thing about $\mathcal{R}_{\infty}^{(\infty)}\left(\omega\right)$ is that every normal state φ on $\mathcal{R}_{\infty}^{(\infty)}\left(\omega\right)$ can be written as

$$\varphi\left(T^{(\infty)}\right)=\left\langle T^{(\infty)}e,e\right\rangle$$

for some unit vector $e\in\left(\ell^2\right)^{(\infty)}$. Since $\left(\ell^2\right)^{(\infty)}$ is isomorphic to $\ell^2=\ell^2_{\infty}$, we can, by replacing $\mathcal{R}_{\infty}$ with $\mathcal{R}_{\infty}^{(\infty)}$, assume that every normal state φ on $\mathcal{R}_{\infty}\left(\omega\right)$ can be written as

$$\varphi\left(T\right)=\left\langle Te,e\right\rangle$$

for some unit vector e. In particular, since $\mathcal{R}_{\infty}\left(\omega\right)$ is a II_1 factor, there is a unique normal tracial state $\tau_{\infty,\omega}$ on $\mathcal{R}_{\infty}\left(\omega\right)$. Hence there is a unit vector $e\left(\omega\right)\in\ell^2_{\infty}$ such that, for every $T\in\mathcal{R}_{\infty}\left(\omega\right)$,

$$\tau_{\infty,\omega}\left(T\right)=\left\langle Te\left(\omega\right),e\left(\omega\right)\right\rangle.$$

Using the measurable cross-section theorems we can choose $e\left(\omega\right)$ so that the map $e:\Omega_{\infty}\to\ell^2_{\infty}$ is absolutely measurable.

Lemma 32. *Suppose $\mathcal{R}_{\infty}$ is type II_1 von Neumann algebra with $\mathcal{R}_{\infty}=\int_{\Omega_{\infty}}^{\oplus}\mathcal{R}_{\infty}\left(\omega\right)d\mu_{\infty}\left(\omega\right)$. Then there exists a map $e\in L^2\left(\mu_{\infty},\ell^2_{\infty}\right)$ and $\left\|e\right\|_2=1$ such that for every $T=\int T_{\omega}d\mu_{\infty}\left(\omega\right)\in\mathcal{R}_{\infty}$, $\left\langle T_{\infty,\omega}e\left(\omega\right),e\left(\omega\right)\right\rangle=\tau_{\infty,\omega}\left(T_{\omega}\right)$, where $\tau_{\infty,\omega}$ is the unique normal tracial state on $\mathcal{R}_{\infty}\left(\omega\right)$.*

Proof. Suppose

$$\mathcal{W} = \text{ball}\left(B\left(l^2_\infty\right)\right) \times \prod_{i=1}^{\infty} \text{ball}\left(B\left(l^2_\infty\right)\right) \times E,$$

where $E = \{x \in l^2_\infty : \|x\| = 1\}$. Then $\mathcal{W}$ is a complete separable metric space with product topology.

Let $\mathcal{X}$ be the set of elements $(T, \{A_i\}_{i=1}^{\infty}, e)$ in $\mathcal{W}$ satisfying

$$TA_i = A_i T, \langle A_i A_j e, e\rangle = \langle A_j A_i e, e\rangle \text{ for every } i, j \in \mathbb{N}.$$

It is easy to verify that $\mathcal{X}$ is closed.

Let $\pi_2 : \mathcal{X} \to \prod_{i=1}^{\infty} \text{ball}\left(B\left(l^2_\infty\right)\right)$, $\pi_3 : \mathcal{X} \to E$ be the projection maps. Then $\pi_2(\mathcal{X})$ is the set of elements $\{A_i\}_{i=1}^{\infty}$ so that there exists $T \in \text{ball}\left(B\left(l^2_\infty\right)\right)$ such that

$$T \in \{A_1, A_2, \ldots\}' \cap \{B_1, B_2, \ldots\}' \text{ and } \langle A_i A_j e, e\rangle = \langle A_j A_i e, e\rangle \text{ for all } i, j \in \mathbb{N}.$$

There exists sequences $\{\psi_1, \psi_2, \ldots\}$ contained in $L^\infty\left(\mu_\infty, B\left(l^2_\infty\right)\right)$ such that

$$\text{ball}\mathcal{R}_\infty(\omega)' = \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT}$$

By Theorem 27, we know there exists an absolutely measurable function $\Upsilon : \pi_2(\mathcal{X}) \longrightarrow \mathcal{X}$ such that $\pi_2 \circ \Upsilon$ is the identity function on $\pi_2(\mathcal{X})$.

Define $F : \Omega_\infty \to \prod_{i=1}^{\infty} \text{ball}\left(B\left(l^2_\infty\right)\right)$ by

$$F(\omega) = \{\psi_i(\omega)\}_{i=1}^{\infty}$$

which is measurable, thus, by Lemma 25, $\pi_3 \circ \Upsilon \circ F$ is a measurable function from Ω_∞ to l^2_∞. We define e by

$$e(\omega) = (\pi_3 \circ \Upsilon \circ F)(\omega).$$

Thus e is a measurable function with $e = \int_{\Omega_\infty}^{\oplus} e(\omega)\, d\mu_\infty(\omega)$, that is $e \in L^2(\mu_\infty, \ell_\infty^2)$ and

$$\|e\|_2^2 = \int_{\Omega_\infty} \|e(\omega)\|^2 \, d\mu_\infty(\omega) = \int_{\Omega_\infty} 1 d\mu_\infty(\omega) = \mu_\infty(\Omega_\infty) = 1.$$

□

The map

$$\tau_\infty : \mathcal{R}_\infty \to \mathbb{C}$$

defined by

$$\tau_\infty(T) = \langle Te, e\rangle = \int_{\Omega_\infty} \langle T_\omega e(\omega), e(\omega)\rangle \, d\mu_\infty(\omega) \underset{\text{def}}{=} \int_{\Omega_\infty} \tau_{\infty,\omega}(T_\omega)\, d\mu_\infty(\omega)$$

is a faithful normal trace on $\mathcal{R}_\infty$. Since $\tau_{\infty,\omega}$ is a faithful normal trace on $\mathcal{R}_\infty(\omega)$ and the trace on a type II_1 factor is unique, it follows that $\tau_{\infty,\omega}$ is the usual trace.

2.3.6 The Center-valued Trace

Suppose $\mathcal{R}$ is an arbitrary finite von Neumann algebra, possibly not acting on a separable Hilbert space. There is (see [15]) a *unique* map $\Phi_{\mathcal{R}} : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ satisfying

1. $\Phi_{\mathcal{R}}$ is linear and completely positive,
2. $\Phi_{\mathcal{R}}(1) = 1$,
3. $\Phi_{\mathcal{R}}(AB) = \Phi_{\mathcal{R}}(BA)$ for all $A, B \in \mathcal{R}$,
4. $\Phi_{\mathcal{R}}$ is weak*-weak* continuous, and
5. $\Phi_{\mathcal{R}}(ATB) = A\Phi_{\mathcal{R}}(T)B$ for all $T \in \mathcal{R}$ and all $A, B \in \mathcal{Z}(\mathcal{R})$.

The map $\Phi_{\mathcal{R}}$ is called the *center-valued trace* on $\mathcal{R}$.

In the case when $\mathcal{R}$ acts on a separable Hilbert space, and we have

$$\mathcal{R} = \sum\nolimits_{1\leq n\leq\infty}^{\oplus} \mathcal{R}_n,$$

we have

$$\mathcal{Z}\left(\mathcal{R}\right) = \sum\nolimits_{1\leq n\leq\infty}^{\oplus} \mathcal{Z}\left(\mathcal{R}_n\right),$$

and we have

$$\Phi_{\mathcal{R}} = \sum\nolimits_{1\leq n\leq\infty}^{\oplus} \Phi_{\mathcal{R}_n}.$$

We can write each $\Phi_{\mathcal{R}_n}$ explicitly in terms of the central decomposition, i.e.,

$$\Phi_{\mathcal{R}_n}\left(T\right) = \int_{\Omega_n}^{\oplus} \tau_n\left(T_\omega\right)\cdot 1 d\mu_n\left(\omega\right)$$

when $1 \leq n < \infty$, and

$$\Phi_{\mathcal{R}_\infty}\left(T\right) = \int_{\Omega_n}^{\oplus} \tau_\omega\left(T_\omega\right)\cdot 1 d\mu_\infty\left(\omega\right).$$

It is clear that these maps satisfy the defining properties (1)-(5) and the uniqueness tells us that these formulas are correct.

2.3.7 Two Simple Relations

Suppose $1 \leq n \leq \infty$. There is a normal $*$-isomorphism $\gamma_n : L^\infty\left(\mu_n\right) \to \mathcal{Z}\left(\mathcal{R}_n\right)$ defined by

$$\gamma_n\left(f\right) = \int_{\Omega_n}^{\oplus} f\left(\omega\right)\cdot 1 d\mu_n\left(\omega\right).$$

Also the map $f \mapsto \int_{\Omega_n} f d\mu_n$ is a state on $L^\infty\left(\mu_n\right)$. The simple relation between this state and the $*$-isomorphism γ_n and ρ_n is given by

$$\left(\rho_n \circ \gamma_n\right)\left(f\right) = \int_{\Omega_n} f d\mu_n$$

for every $f \in L^\infty\left(\mu_n\right)$.

Another simple relationship between ρ_n and $\Phi_{\mathcal{R}_n}$ is

$$\rho_n = \rho_n \circ \Phi_{\mathcal{R}_n} .$$

2.3.8 Putting Things Together

We let Ω be the disjoint union of $\{\Omega_n : 1 \leq n \leq \infty\}$, which can be represented as a Borel subset of $\mathbb{R}$. We define a probability Borel measure μ on Ω by

$$\mu(E) = \frac{1}{2}\mu_\infty (E \cap \Omega_\infty) + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \mu_n (E \cap \Omega_n) .$$

Then the von Neumann algebra $L^\infty(\mu)$ can be written as

$$L^\infty(\mu) = L^\infty(\mu_\infty) \oplus \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} L^\infty(\mu_n)$$

We define an isomorphism

$$\gamma : L^\infty(\mu) \to \mathcal{Z}(\mathcal{R}),$$

by

$$\gamma(f_\infty \oplus f_1 \oplus f_2 \oplus \cdots) = \gamma_\infty(f_\infty) \oplus \gamma_1(f_1) \oplus \gamma_2(f_2) \cdots .$$

We can define a faithful normal tracial state $\rho : \mathcal{R} \to \mathbb{C}$ by

$$\rho\left(\sum\nolimits_{1 \leq n \leq \infty}^{\oplus} T_n\right) = \frac{1}{2}\rho_\infty(T_\infty) + \sum_{1 \leq n < \infty} \frac{1}{2^{n+1}} \rho_n(T_n) .$$

We have

1. $\rho = \rho \circ \Phi_{\mathcal{R}}$,
2. $(\rho \circ \gamma)(f) = \int_\Omega f d\mu$ for every $f \in L^\infty(\mu)$, and, as we stated above,
3. $\Phi_{\mathcal{R}}(T) = \sum\nolimits_{1 \leq n \leq \infty}^{\oplus} \Phi_{\mathcal{R}_n}(T_n)$
$= \left[\sum\nolimits_{1 \leq n < \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \tau_n(T_n(\omega)) \cdot 1 d\mu_n(\omega)\right] \oplus \int_{\Omega_\infty}^{\oplus} \tau_\omega(T_\infty(\omega)) \cdot 1 d\mu_\infty(\omega) .$

CHAPTER 3

MASAS IN FINITE VON NEUMANN ALGEBRAS

A *masa* in a C*-algebra is a maximal abelian selfadjoint subalgebra. In $B(H)$ where H is a separable infinite-dimensional Hilbert space there are many different masas. For example, the set of all diagonal operators with respect to some fixed orthonormal basis is a discrete masa. On the other hand $L^{\infty}[0,1] = L^{\infty}(\delta_{\infty})$ acting as multiplications on $L^{2}[0,1]$ with Lebesgue measure is also a masa that is not isomorphic to the diagonal masa, since it has no minimal (nonzero) projections. However, in a finite von Neumann algebra $\mathcal{R}$ with a faithful normal tracial state τ acting on a separable Hilbert space we will prove that all masas are isomorphic.

Theorem 33. *Suppose $\mathcal{A}$ is a masa in a finite von Neumann algebra $\mathcal{R}$. Then there is an tracial embedding $\pi_{\mathcal{A}} : L^{\infty}(\lambda) \to \mathcal{A}$ such that the following diagram commutes*

$$\begin{array}{ccc} L^{\infty}(\lambda) & \overset{\pi_{\mathcal{A}}}{\to} & \mathcal{A} \\ \downarrow \eta & & \downarrow \Phi_{\mathcal{R}} \\ L^{\infty}(\mu) & \overset{\gamma}{\to} & Z(\mathcal{R}) \end{array}$$

Moreover, if $\mathcal{B}$ is another masa in $\mathcal{R}$, then $\mathcal{B}$ is isomorphic to $\mathcal{A}$. In fact, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximately equivalent in $\mathcal{R}$.

We first need to prove this theorem when $\mathcal{R}$ is a finite factor. When $\mathcal{R}$ is a type I_n factor, i.e., $\mathcal{R} = \mathbb{M}_n(\mathbb{C})$, the result is obvious.

Lemma 34. *Suppose $\mathcal{A} \subset \mathbb{M}_n(\mathbb{C})$ is a masa. Then there exists a unitary $U \in \mathcal{U}(\mathbb{M}_n(\mathbb{C}))$ such that $U\mathcal{A}U = \mathcal{D}_n$, the $n \times n$ complex diagonal matrices. Hence there is a *-isomorphism $\pi_{\mathcal{A}} : L^{\infty}(\delta_n) \to \mathcal{A}$ such that, for every $f \in L^{\infty}(\delta_n)$, which is isometrically isomorphic to $\mathbb{C}^n$.*

$$\tau_n(\pi(f)) = \int_{J_n} f d\delta_n .$$

When $\mathcal{R}$ is a type II_1 factor the result is well-known [11], but we sketch a proof for completeness.

Lemma 35. *Suppose $\mathcal{M}$ is a type II_1 factor von Neumann algebra acting on a separable Hilbert space with a (unique) faithful normal tracial state τ, and suppose $\mathcal{A}$ is a masa in $\mathcal{M}$. Then there is an isomorphism $\pi_{\mathcal{A}} : L^\infty(\delta_\infty) \to \mathcal{A}$ such that, for every $f \in L^\infty(\delta_\infty)$,*

$$\tau(\pi_{\mathcal{A}}(f)) = \int_0^1 f(t)\, d\delta_\infty(t) \ .$$

Proof. Using von Neumann's theorem [11] there is an operator $A = A^*$ in $\mathcal{A}$ such that $0 \leq A \leq 1$ and $\mathcal{A} = W^*(A)$ (the von Neumann algebra generated by A). Then $\mathcal{A}$ is generated by the chain of spectral projections $\mathcal{C}_0 = \left\{\chi_{[0,s)}(A) : 0 \leq s \leq 1\right\}$. This chain is contained in a maximal chain $\mathcal{C}$ of projections in $\mathcal{R}$ Since $\mathcal{C} \subset \mathcal{C}_0' = \mathcal{A}' = \mathcal{A}$, we have $\mathcal{A} = W^*(\mathcal{C})$. Since a II_1 factor has no minimal projections and $\tau : \mathcal{C} \to [0,1]$ is injective, we can write $\mathcal{C} = \{P_t : 0 \leq t \leq 1\}$ such that, for every $t \in [0,1]$,

$$\tau(P_t) = t.$$

Since $\mathcal{C}$ is linearly independent and the linear span $sp(\mathcal{C})$ of $\mathcal{C}$ is a unital *-algebra,we know the map π

$$\chi_{[0,t)} \mapsto P_t$$

give a *-isomorphism π between $sp\left(\left\{\chi_{[0,t)}\right\}\right)$ and $sp(\mathcal{C})$ such that, for every $f \in sp\left(\left\{\chi_{[0,t)}\right\}\right)$

$$\tau(\pi(f)) = \int_0^1 f d\delta_\infty \ .$$

The map π is also a $\|\cdot\|_2$-isometry between dense subsets of $L^2(\delta_\infty)$ and $L^2(\mathcal{A},\tau)$. Thus π extends uniquely to a unitary operator from $L^2(\delta_\infty)$ to $L^2(\mathcal{A},\tau)$. Since $\lim_{n\to\infty}\|h\|_{2^n} = \|h\|_\infty$ for all $h \in L^\infty(\delta_\infty)$, this maps sends $L^\infty(\delta_\infty)$ onto $\mathcal{A}$. This is the desired map $\pi_{\mathcal{A}}$. $\square$

Corollary 36. *Suppose $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space with a faithful (tracial) state τ. The following are equivalent:*

1. There is an isomorphism $\pi : L^\infty(\delta_\infty) \to \mathcal{A}$ such that, for every $f \in L^\infty(\delta_\infty)$,

$$\tau(\pi(f)) = \int_0^1 f(t)\, d\delta_\infty(t).$$

2. There is a $T \in \mathcal{A}$ such that

   (a) $W^*(T) = \mathcal{A}$

   (b) $T = T^*$

   (c) $\tau(T^n) = \frac{1}{n+1}$ for $n \in \mathbb{N}$

   Moreover, if (2) holds, then $0 \le T \le 1$ and the map $\pi(f) = f(T)$ is the required map in (1).

Proof. $(1) \Rightarrow (2)$. Suppose π exists as in (1). Define $f(t) = t$ in $L^\infty(\delta_\infty)$ and let $T = \pi(f)$. Then $0 \le T \le 1$,

$$\mathcal{A} = \pi(L^\infty(\delta_\infty)) = \pi(W^*(f)) = W^*(\pi(f)) = W^*(T),$$

and, for each $n \in \mathbb{N}$,

$$\tau(T^n) = \tau(\pi(f^n)) = \int_0^1 t^n dt = \frac{1}{n+1}.$$

$(2) \Rightarrow (1)$. Define the state $\rho : L^\infty(\delta_\infty) \to \mathbb{C}$ by

$$\rho(f) = \int_0^1 f(t)\, d\delta_\infty(t).$$

Letting $f \in L^\infty(\delta_\infty)$ be as above, we have $\tau(T^n) = \rho(f^n) = \frac{1}{n+1}$ for each $n \in \mathbb{N}$. It follows from Lemma 1 in [16] that there is a normal (i.e., weak*-weak* continuous) $*$-isomorphism π :

$L^{\infty}(\delta_{\infty}) \to \mathcal{A}$ such that $\pi(f) = T$ and such that $\tau \circ \pi = \rho$. It is clear that, for any polynomial $p(t)$, $\pi(p) = p(T)$. Suppose $f \in L^{\infty}(\delta_{\infty})$. By changing f on a set of measure 0, we can assume that f is Borel measurable. Then there is a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ weak*. Thus

$$f(T) = (\text{weak*}) \lim_{n\to\infty} p_n(T) = (\text{weak*}) \lim_{n\to\infty} \pi(p_n) = \pi(f).$$

□

From this Lemma, we can see that $\pi(f) = f(T)$ and $\tau(T^n) = \tau(\pi_A(x^n)) = \int_0^1 x^n d\delta_{\infty} = \frac{1}{n+1}$ for $n = 1, 2, \cdots$.

Lemma 37. *Suppose $A = A^* \in B(H)$. It follows that $W^*(A) = \{p_1(A), p_2(A), \cdots\}^{-WOT}$ for a sequence of polynomials $p_1, p_2, \cdots$.*

Proof. We know that $span\{1, A, A^2, \cdots\} = \{p(A) : p \in \mathbb{C}[z]\}$, then

$$W^*(A) = W^*(p(A), p \in \mathbb{C}[z]).$$

Since $\{p(A), p \in \mathbb{C}[z]\} \subseteq \{p(A), p \in \mathbb{Q}[z]\}^{-\|\|} \subseteq \{p(A), p \in \mathbb{C}[z]\}^{-WOT}$, thus $W^*(A) = \{p_1(A), p_2(A), \cdots\}^{-WOT}$ for a sequence of polynomials. □

Lemma 38. *Suppose $\mathcal{A}_{\infty}$ is a masa of $\mathcal{R}_{\infty}$. Then there exists an operator $T = \int_{\Omega_{\infty}}^{\oplus} T_{\omega} d\mu_{\infty}(\omega)$ such that $W^*(T_{\omega}) = \mathcal{A}_{\infty}(\omega)$, and $\tau_{\omega,\infty}(T_{\omega}^n) = \langle T_{\omega}^n e(\omega), e(\omega)\rangle = \frac{1}{n+1}$ for $n \geq 1$.*

Proof. Let

$$\mathcal{Y} = B\left(l_{\infty}^2\right) \times \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_{\infty}^2\right)\right) \times \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_{\infty}^2\right)\right) \times \prod_{i=1}^{\infty} \text{ball}\left(B\left(l_{\infty}^2\right)\right) \times E,$$

where $E = \{x \in l_{\infty}^2 : \|x\| = 1\}$. It is clear that $\mathcal{Y}$ is a complete separable metric space with product topology. Let $\mathcal{X}$ be the set of tuples $(T, \{A_i\}_{i=1}^{\infty}, \{B_i\}_{i=1}^{\infty}, \{C_i\}_{i=1}^{\infty}, e)$ in $\mathcal{Y}$ satisfying

$$TA_i = A_i T, TB_i = B_i T, \langle T^n e, e\rangle = \frac{1}{n+1} \text{ for } n \geq 1.$$

From Lemma 37, we know there exists a sequence of polynomials such that $W^*(T)$ such that $W^*(T) = W^*(p_1(T), p_2(T), \cdots)$. Define $\mathcal{W}_{i,k,n}$ be the subset of $\mathcal{X}$ satisfying

$$T = T^*,\ d(A_i, p_n(T)) \geq \frac{1}{k} \text{ for } n \geq 1.$$

Let $\mathcal{W}_{i,k} = \bigcap_{n=1}^{\infty} \mathcal{W}_{i,k,n}$ and $\mathcal{W} = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{W}_{i,k}$, then $\mathcal{W}$ is a subset of $\mathcal{X}$ satisfying

$$A_i \notin W^*(p_1(T), p_2(T), \cdots),\ \text{for } i \geq 1$$

Then $\mathcal{X} \backslash \mathcal{W} = \bigcap_{i=1}^{\infty} \bigcap_{i=1}^{\infty} \mathcal{X} \backslash \mathcal{W}_{i,k}$ is a subset of $\mathcal{X}$ satisfying

$$W^*(A_1, A_2, \cdots) \subseteq W^*(p_1(T), p_2(T), \cdots),$$

which is a G_δ set. By Lemma 2.5 in [6], there exists a metric which makes $\mathcal{X} \backslash \mathcal{W}$ a complete separable space. Let $\pi_{2,3,4}$ be the projection map into second, third, fourth coordinates, then there exists an absolute measurable function $\Upsilon : \pi_{2,3,4}(\mathcal{X}) \to \mathcal{X}$ such that $\pi_2 \circ \Upsilon$ is an identity on $\pi_{2,3,4}(\mathcal{X})$.

Suppose there are sequences $\{f_1, f_2, \cdots\}, \{\psi_1, \psi_2, \cdots\}, \{\varphi_1, \varphi_2, \cdots\}$ contained in $L^\infty(\mu_\infty, B(l^2_\infty))$ such that

$$\begin{aligned} \text{ball}\mathcal{A}_\infty(\omega) &= \{f_1(\omega),\ f_2(\omega), \ldots\}^{-SOT}, \\ \text{ball}\mathcal{R}_\infty(\omega)' &= \{\psi_1(\omega), \psi_2(\omega), \ldots\}^{-SOT}, \\ \text{ball}\mathcal{R}_\infty(\omega) &= \{\varphi_1(\omega), \varphi_2(\omega), \ldots\}^{-SOT}. \end{aligned}$$

Define $F : \Omega_\infty \to \prod_{i=1}^{\infty} \text{ball}(B(l^2_\infty))$ by

$$F(\omega) = \{f_i(\omega)\}_{i=1}^\infty \times \{\psi_i(\omega)\}_{i=1}^\infty \times \{\varphi_i(\omega)\}_{i=1}^\infty$$

which is measurable, thus, by Lemma 25, $\pi_1 \circ \Upsilon \circ F : \omega \longmapsto T_\omega$ is the desired measurable function from Ω_∞ to $B\left(l_\infty^2\right)$ such that $\mathrm{ball}\mathcal{A}_\infty(\omega) = W^*(T_\omega)$, where π_1 is the projection from $\mathcal{X} \backslash \mathcal{W}$ into its first coordinate. □

Lemma 39. *Suppose $\mathcal{A}_n$ is a masa of $\mathcal{R}_n$ for every $1 \leq n \leq \infty$. Then there is an isomorphism $\pi_{\mathcal{A}_n} : L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n) \to \mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$.*

Proof. First suppose $1 \leq n < \infty$. We know that $\mathcal{R}_n$ is isomorphic to $\int_{\Omega n}^{\oplus} \mathbb{M}_n(\mathbb{C})\, d\mu_n(\omega)$, so if $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, then $\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$ where each $\mathcal{A}_n(\omega)$ is a masa in $\mathbb{M}_n(\mathbb{C})$. There is a unitary operator $U_\omega \in \mathbb{M}_n(\mathbb{C})$ such that $\mathcal{A}_n(\omega) = U_\omega^* \mathcal{D}_n(\mathbb{C})\, U_\omega$. An easy measurable cross-section proof allows us to choose the U_ω's measurably. However, $\mathcal{D}_n$ is isomorphic to $L^\infty(J_n, \delta_n)$. Define $\pi_{\mathcal{A}_n} : L^\infty(\Omega_n \times J_n) \to \int_{\Omega_n}^{\oplus} L^\infty(\delta_n)\, d\mu_n(\omega)$ by

$$\pi_{\mathcal{A}_n}(f) = \int_{\Omega_n}^{\oplus} U_\omega^* \begin{pmatrix} f\left(\omega, \frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega, \frac{n}{n}\right) \end{pmatrix} U_\omega d\mu_n(\omega).$$

Now suppose $n = \infty$. We choose $\{T_\omega\}$ as in Lemma 38, and we define

$$\pi_{\mathcal{A}_\infty}(f) = \int_{\Omega_\infty}^{\oplus} f_\omega(T_\omega)\, d\mu_\infty(\omega)\ ,$$

where $f_\omega(t) = f(\omega, t)$. □

Suppose now that $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space H,

$$\mathcal{R} = [\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots] \oplus \mathcal{R}_\infty\,.$$

For $1 \leq n < \infty$, $\mathcal{R}_n$ is a type I_n von Neumann algebra acting on H_n, $\mathcal{R}_\infty$ is a type II_1 von Neumann algebra acting on H_∞,

$$H = [H_1 \oplus H_2 \oplus \cdots] \oplus H_\infty.$$

$\mathcal{A}$ is a masa in $\mathcal{R}$. Then, we can write

$$\mathcal{A} = [\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots] \oplus \mathcal{A}_\infty,$$

where, for $1 \leq n \leq \infty$, $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$. Clearly, since $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, we know that $\mathcal{D}_n = \mathcal{Z}(\mathcal{R}_n) \subseteq \mathcal{A}_n \subseteq \mathcal{R}_n \subseteq L^\infty(\mu_n, B(H_n))$. It follows from Lemma 31 that there is a measurable family $\{\mathcal{A}_n(\omega) : \omega \in \Omega_n\}$ of von Neumann algebras such that

$$\mathcal{A}_n = \int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega).$$

If $1 \leq n < \infty$, then almost every $\mathcal{A}_n(\omega)$ must be a masa in $\mathbb{M}_n(\mathbb{C})$. If $n = \infty$, then almost every $\mathcal{A}_n(\omega)$ must be a masa in the II_1 factor $\mathcal{R}_\infty(\omega)$. Since throwing away a set of measure 0 from Ω_n doesn't change anything, we can assume that, when $1 \leq n < \infty$ every $\mathcal{A}_n(\omega)$ is a masa in $\mathbb{M}_n(\mathbb{C})$, and when $n = \infty$, every $\mathcal{A}_\infty(\omega)$ is a masa in $\mathcal{R}_\infty(\omega)$.

If $1 \leq n \leq \infty$, then each $\mathcal{A}_n(\omega)$ is isomorphic to $L^\infty(\delta_n)$ (see Lemma 34 and 35). And $\int_{\Omega_n}^{\oplus} \mathcal{A}_n(\omega)\, d\mu_n(\omega)$ is isomorphic to $\int_{\Omega_n}^{\oplus} L^\infty(\delta_n)\, d\mu_n(\omega)$, which is isomorphic to $L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n)$. The isomorphism sends a function $f(\omega, t) \in L^\infty(\Omega_n \times J_n, \mu_n \times \delta_n)$ to $\int_{\Omega_n}^{\oplus} f_\omega(t)\, d\mu_n(\omega)$, where $f_\omega(t) = f(\omega, t)$.

For each n, $1 \leq n \leq \infty$, we define $\Lambda_n = \Omega_n \times J_n$ and we define $\lambda_n = \mu_n \times \delta_n$. We let Λ denote the disjoint union of the Λ_n's for $1 \leq n \leq \infty$, and we can choose Λ to be a Borel subset of $\mathbb{R}$, and we define a probability Borel measure λ on Λ by

$$\lambda(F) = \frac{1}{2}\lambda_\infty(F \cap \Lambda_\infty) + \sum_{n=1}^{\infty} \frac{1}{2^n}\lambda_n(F \cap \Lambda_n)\ .$$

We then have

$$L^\infty(\lambda) = L^\infty(\lambda_\infty) \oplus \prod_{1 \leq n < \infty} L^\infty(\lambda_n)\ .$$

For each n, $1 \leq n \leq \infty$, there is a mapping

$$\eta_n : L^\infty(\lambda_n) = L^\infty(\mu_n \times \delta_n) \to L^\infty(\mu_n),$$

defined by

$$\eta_n(f)(\omega) = \int_{J_n}^{\oplus} f(\omega, t)\, d\delta_n(t).$$

We define $\eta : L^\infty(\lambda) \to L^\infty(\mu)$ by

$$\eta(f) = \eta(f_\infty \oplus f_1 \oplus f_2 \oplus \cdots) = \eta_\infty(f_\infty) \oplus \eta_1(f_1) \oplus \eta_2(f_2) \oplus \cdots .$$

Lemma 40. *For $1 \leq n \leq \infty$, if $\mathcal{A}_n$ is a masa in $\mathcal{R}_n$, then there exists a tracial embedding $\pi_{\mathcal{A}_n} : L^\infty(\lambda_n) = L^\infty(\mu_n \times \delta_n) \to \mathcal{A}_n$ such that the following diagram commutes*

$$\begin{array}{ccc} L^\infty(\lambda_n) & \overset{\pi_{\mathcal{A}_n}}{\to} & \mathcal{A}_n \\ \downarrow \eta_n & & \downarrow \Phi_n \\ L^\infty(\mu_n) & \overset{\gamma_n}{\to} & Z(\mathcal{R}_n) \end{array} .$$

$$\Phi_n \circ \pi_{\mathcal{A}_n} = \gamma_n \circ \eta_n.$$

where

$$\begin{aligned} \gamma_n(f) &= \int_{\Omega_n}^{\oplus} f(\omega)\, I d\mu_n(\omega), \\ \eta_n(f)(\omega, t) &= \int_{J_n} f(\omega, t)\, d\delta_n(t) \text{ and,} \\ \Phi_n\left(\int_{\Omega_n}^{\oplus} T(\omega)\, d\mu_n(\omega)\right) &= \int_{\Omega_n}^{\oplus} \tau_{\omega,n}(T(\omega))\, I d\mu_n(\omega). \end{aligned}$$

Moreover, if $\mathcal{B}_n$ is a masa in $\mathcal{R}_n$, and there is a tracial embedding $\pi_{\mathcal{B}_n} : L^\infty(\lambda_n) \to \mathcal{B}_n$ such that $\Phi_n \circ \pi_{\mathcal{B}_n} = \gamma_n \circ \eta_n$, then,

if $1 \leq n < \infty$, then there exists a unitary $U \in \mathcal{U}(\mathcal{R}_n)$ such that

$$U \pi_{\mathcal{A}_n}(L^\infty(\lambda_n))\, U^* = \pi_{\mathcal{B}_n}(L^\infty(\lambda_n)),$$

if $n = \infty$, then $\pi_{\mathcal{A}_n}$ is approximately equivalent to $\pi_{\mathcal{B}_n}$ in $\mathcal{R}_n$.

Proof. For $1 \leq n < \infty$, we have

$$\gamma_n \circ \eta_n(f)(\omega) = \gamma_n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\omega, \frac{k}{n}\right)\right) I = \frac{1}{n}\sum_{k=1}^{n}\int_{\Omega_n}^{\oplus} f\left(\omega, \frac{k}{n}\right) I d\mu_n(\omega),$$

and

$$\Phi_n\left(\pi_{\mathcal{A}_n}(f)\right) = \Phi_n\left(\int_{\Omega_n}^{\oplus} U_\omega^* \begin{pmatrix} f\left(\omega, \frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega, \frac{n}{n}\right) \end{pmatrix} U_\omega d\mu_n(\omega)\right) =$$

$$\int_{\Omega_n}^{\oplus} \tau_n\left(U_\omega^* \begin{pmatrix} f\left(\omega, \frac{1}{n}\right) & & \\ & \ddots & \\ & & f\left(\omega, \frac{n}{n}\right) \end{pmatrix} U_\omega\right) d\mu_n(\omega) = \int_{\Omega_n}^{\oplus} \frac{1}{n}\sum_{k=1}^{n} f\left(\omega, \frac{k}{n}\right) I d\mu_n(\omega).$$

Thus the diagram commutes. For $n = \infty$, by Lemma 38, we know there exists an operator $T = \int_{\Omega_\infty}^{\oplus} T_\omega d\mu_\infty(\omega)$ such that T_ω generates $\mathcal{A}_\infty(\omega)$ in weak operator topology with $0 \leq T_\omega \leq 1$ and $\tau_{\omega,\infty}\left(T_\omega^n\right) = \frac{1}{n+1}$ for $n \geq 1$. The map $\pi_{\mathcal{A}_\infty} : L^\infty(\delta_\infty) \to W^*(T) = \mathcal{A}_\infty$ is defined by $\pi_{\mathcal{A}_\infty}(f) = \int_{\Omega_\infty}^{\oplus} f_\omega(T_\omega) d\mu_\infty(\omega)$. Thus $\gamma_\infty \circ \eta_\infty(f)(\omega) = \left[\int_{J_n} f(\omega, t) d\delta_n(t)\right] I$ and $\Phi_\infty \circ \pi_{\mathcal{A}_\infty}(f)(\omega) = \tau_{\omega,\infty}\left(f_\omega(T_\omega)\right) I = \left[\int_{J_n} f(\omega, t) d\delta_n(t)\right] I$. Therefore the diagram commutes. □

Combining all of these results we obtain Theorem 33.

And we also have the following corollary.

Corollary 41. *If $\mathcal{A}$ and $\mathcal{B}$ are masas in $\mathcal{R}$, then the tracial embeddings $\pi_{\mathcal{A}}, \pi_{\mathcal{B}}$ are approximately unitarily equivalent in $\mathcal{R}$.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are masas in $\mathcal{R}$, then there are tracial embeddings $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ as in Theorem 33 . Thus $\Phi \circ \pi_{\mathcal{A}} = \Phi \circ \pi_{\mathcal{B}}$. By Theorem 23, we have $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximate unitarily equivalent. □

CHAPTER 4

MEASURE PRESERVING TRANSFORMATIONS

4.1 Basic Facts

A Borel measurable map $\sigma : [0,1] \to [0,1]$ is *measure-preserving* if and only if, for every Borel set $E \subseteq [0,1]$,

$$\delta_\infty \left(\sigma^{-1}(E)\right) = \delta_\infty (E).$$

We say that $\sigma : [0,1] \to [0,1]$ is an invertible measure-preserving map if there are measure-preserving measurable maps $\sigma_1, \sigma_2 : [0,1] \to [0,1]$ such that

$$(\sigma \circ \sigma_1)(x) = x \text{ and } (\sigma_2 \circ \sigma)(x) = x, \text{ almost everywhere } (\delta_\infty).$$

In this case, let $E = \{y \in J_\infty : \sigma \circ \sigma_1 (y) \neq y \text{ or } \sigma_2 \circ \sigma (y) \neq y\}$ and let S be the semigroup generated by $\sigma, \sigma_1, \sigma_2, id_{[0,1]}$. Then S is countable, thus denoted by $S = \{\widehat{\sigma}_n : n \in \mathbb{N}\}$. Suppose $F = \left(\underset{n\in\mathbb{N}}{\cup} \widehat{\sigma}_n (E)\right) \cup \left(\underset{n\in\mathbb{N}}{\cup} \widehat{\sigma}_n^{-1} (E)\right)$, it follows that $\delta_\infty (F) = 0$. and $\sigma(F) = \sigma_1(F) = \sigma_2(F) = F$. Therefore, on $J_\infty \backslash F$, $\sigma, \sigma_1, \sigma_2 : J_\infty \backslash F \to J_\infty \backslash F$ is bijective, also $\sigma \circ \sigma_1 = \sigma_2 \circ \sigma$. Define $\widetilde{\sigma}$ on J_∞ by

$$\widetilde{\sigma}(y) = \begin{cases} \sigma(y) & y \in J_\infty / F \\ y & y \in F \end{cases}$$

Then $\widetilde{\sigma}, \widetilde{\sigma}^{-1}$ are bijective, measurable, and $\widetilde{\sigma} = \sigma$ a.e.(δ_∞). We can change σ and σ_1, σ_2 on sets of measure 0 so that $\sigma : J_\infty \to J_\infty$ is bijective and $\sigma_1 = \sigma_2 = \sigma^{-1}$ a.e.(δ_∞). In the following sections, whenever we talk about an invertible measure-preserving transformation σ on J_∞, we will mean a bijective map $\sigma : J_\infty \to J_\infty$ such that σ and σ^{-1} are measurable and measure-preserving.

Let $\mathbb{MP}[0,1] = \{\sigma | \sigma : [0,1] \to [0,1] \text{ is an invertible measurable preserving transformation}\}$, then $(\mathbb{MP}[0,1], \circ)$ is a group.

Let $\mathcal{V}$ be all unitaries U in $\mathcal{U}(B(L^2([0,1])))$ with $U(1) = 1$, and for all $f, g \in L^\infty[0,1]$, $U(fg) = U(f)U(g)$.

Lemma 42. *$\mathcal{V}$ is $*$-SOT closed.*

Proof. Suppose $\{U_n\} \subseteq \mathcal{V}$, and $U_n \overset{SOT}{\to} U$, $U_n^* \overset{SOT}{\to} U^*$. It is easy to see $U^*U = UU^* = 1$ and $U(1) = 1$. And we know that$U_n \overset{SOT}{\to} U$ if and only if $\overline{sp}\left\{f \in L^2[0,1] : \|U_n f - Uf\|_2^2 \to 0\right\} = L^2[0,1]$. Thus there exists a subsequence $\{U_{n_k}\}$ such that for all $f, g \in L^\infty[0,1]$, $Ufg = \lim_{k\to\infty} U_{n_k}(fg) = \lim_{k\to\infty} (U_{n_k}f)(U_{n_k}g) = UfUg$, thus $U \in \mathcal{V}$. $\square$

Corollary 43. *$\mathcal{V}$ is a complete separable, metric space in the $*$-SOT.*

Proof. Since $\mathcal{V}$ is a $*$-SOT closed subalgebra of $\mathcal{U}(B(L^2[0,1]))$ and $\mathcal{U}(B(L^2[0,1]))$ is a complete separable metric space. It follows that $\mathcal{V}$ is a complete separable metric space. $\square$

Lemma 44. *There exists a group isomorphism $\sigma \to U_\sigma$ from $\mathbb{MP}[0,1]$ onto $\mathcal{V}$.*

Proof. If $\sigma \in \mathbb{MP}[0,1]$, define $U_\sigma : L^2[0,1] \to L^2[0,1]$ by $U_\sigma f = f \circ \sigma^{-1}$. Since, for every $f \in L^2[0,1]$,

$$\|U_\sigma f\|_2^2 = \int_Y \left(f \circ \sigma^{-1}\right)^2 d\delta_\infty = \int_Y |f|^2 \circ \sigma^{-1} d\delta_\infty = \int_Y |f|^2 d\delta_\infty = \|f\|_2^2,$$

U_σ is an isometry. Since $U_{\sigma^{-1}} = U_\sigma^{-1}$, U_σ is unitary. Also $U_\sigma(fg) = (fg) \circ \sigma = (f \circ \sigma)(g \circ \sigma) = (U_\sigma f)(U_\sigma g)$ when $f, g \in L^\infty[0,1]$. Thus $U_\sigma \in \mathcal{V}$.

To prove that the map $\sigma \to U_\sigma$ is onto, we suppose $U \in \mathcal{V}$. Define $x \in L^2[0,1]$ by $x(t) = t$, and define $\gamma = U(x)$. We will show that $\gamma \in \mathbb{MP}[0,1]$. Then $U(x^n) = \gamma^n$ for all $n \geq 1$. Thus

$$\begin{aligned} \|\gamma\|_\infty &= \lim \|\gamma\|_{2^n} = \lim \left[\left\|\gamma^{2^{n-1}}\right\|_2\right]^{1/2^{n-1}} = \lim \left[\left\|Ux^{2^{n-1}}\right\|_2\right] \\ &= \left[\left\|x^{2^{n-1}}\right\|_2\right]^{1/2^{n-1}} = \|x\|_\infty = 1. \end{aligned}$$

Also if $\gamma = u + iv$, then

$$\begin{aligned} 4\int v^2 d\delta_\infty &= \int \|\gamma - \bar{\gamma}\|_2^2 \, d\delta_\infty = \|\gamma\|_2 + \|\bar{\gamma}\|_2^2 - 2\mathrm{Re}\,\langle \gamma, \bar{\gamma}\rangle \\ &= 2\|\gamma\|_2^2 - 2\left\langle \gamma^2, 1\right\rangle = 2\|x\|_2^2 - 2\int x^2 d\delta_\infty = 0 \end{aligned}$$

Thus $\gamma = \bar{\gamma}$. Since

$$\int_0^1 \gamma^n d\delta_\infty = \int_0^1 x^n d\delta_\infty = \frac{1}{n+1}$$

for each $n \geq 1$. It follows from Corollary 36, using $\tau(f) = \int_0^1 f d\delta_\infty$, that $0 \leq \gamma \leq 1$. And the map $\pi(f) = f \circ \gamma$ is a weak*-continuous automorphism on $L^\infty([0,1])$ such that, for every $f \in L^\infty[0,1]$,

$$\int_0^1 f d\delta_\infty = \tau(\pi(f)) = \int_0^1 f \circ \gamma d\delta_\infty.$$

Thus

$$\delta_\infty\left(\gamma^{-1}(E)\right) = \int_0^1 \chi_E \circ \gamma d\delta_\infty = \delta_\infty(E).$$

Hence γ is a measure-preserving transformation on $[0,1]$. Furthermore, $U_\gamma f = f \circ \gamma$ is an isometry on $L^2([0,1])$ and equals U on the dense subset of polynomials. Thus $U = U_\gamma$. Since U_γ is unitary, $\gamma \in \mathbb{MP}[0,1]$.

Since $\mathcal{V}$ is closed in the $*$-strong operator topology ($*$-SOT), and the closed unit ball of $B(L^2[0,1])$ is a $*$-SOT complete metric space, we know that $\mathbb{MP}[0,1]$ is a complete separable metric space with the topology $\gamma_n \to \gamma$ if and only if $U_{\gamma_n} \to U_\gamma$ in the $*$-SOT. On $\mathbb{MP}[0,1]$ this topology is called the *weak topology.* [7] The metric for the unit ball of $B\left(L^2[0,1]\right)$ is rather complicated. For $\mathbb{MP}[0,1]$ we have a simpler metric. $\square$

Lemma 45. *$\mathbb{MP}(Y,\nu)$ is a complete separable metric space with the metric d on $\mathbb{MP}[0,1]$ defined by*

$$d(\gamma_1,\gamma_2) = \|\gamma_1 - \gamma_2\|_2 + \left\|\gamma_1^{-1} - \gamma_2^{-1}\right\|_2$$

Proof. Suppose $d(\gamma_n, r) \to 0$, then $\|\gamma_n - \gamma\|_2 \to 0$ and $\|\gamma_n^{-1} - \gamma^{-1}\|_2 \to 0$. Thus $\left\|\gamma_n^k - \gamma^k\right\|_2 \to 0$ and $\left\|\left(\gamma_n^{-1}\right)^k - \left(\gamma^{-1}\right)^k\right\|_2 \to 0$ for every $k \geq 0$. Thus $\left\|U_{\gamma_n} x^k - U_\gamma x^k\right\|_2 \to 0$ which implies $U_{\gamma_n} \to U$ in SOT and $U_{\gamma_n}^* = U_{\gamma_n^{-1}} \to U_{\gamma^{-1}} = U_\gamma^*$ in SOT. The converse is obvious. To prove completeness, a similar argument to the one above shows that if $\{\gamma_n\}$ is d-Cauchy, then $\{U_{\gamma_n}\}$ is $*$-SOT Cauchy, so there is a $\gamma \in \mathbb{MP}[0,1]$ such that $U_{g_n} \to U_\gamma$ in the $*$-SOT. Hence $\gamma_n \to \gamma$ in d. $\square$

We now turn to our measure space (Λ, λ). We want to describe a subgroup $\mathbb{G}_n(\mathcal{R})$ of $\mathbb{MP}(\Lambda,\lambda)$.

Definition 46. *Suppose $\sigma \in \mathbb{MP}(\Lambda_n, \lambda_n)$. Then $\sigma \in \mathbb{G}_n(\mathcal{R})$ if and only if, for every measurable $E \subset \Omega_n$,*

$$\sigma(E \times J_n) \subset E \times J_n, \text{ a.e.,}$$

i.e.,

$$\lambda_n(\sigma(E \times J_n) \setminus (E \times J_n)) = 0.$$

Since it is known that

$$\sigma((\Omega_n \backslash E) \times J_n) \subset (\Omega_n \backslash E) \times J_n, \text{ a.e.,}$$

it follows that

$$\sigma(E \times J_n) = E \times J_n, \text{ a.e..}$$

This implies that $\sigma^{-1} \in \mathbb{G}_n(\mathcal{R})$. Clearly, $\mathbb{G}_n(\mathcal{R})$ is a subgroup of $\mathbb{MP}(\Lambda_n, \lambda_n)$.

Definition 47. *We define $\mathbb{G}(\mathcal{R})$ to be all $\sigma \in \mathbb{MP}(\Lambda, \lambda)$ such that, for $1 \leq n \leq \infty$, $\sigma(\Lambda_n) = \Lambda_n$ and $\sigma|_{\Lambda_n} \in \mathbb{G}_n(\mathcal{R})$. We see that we can view*

$$\mathbb{G}(\mathcal{R}) = \prod_{1 \leq n \leq \infty} \mathbb{G}_n(\mathcal{R}) \ ,$$

as a product space.

We can express the following Lemma as:

$$\mathbb{G}(\mathcal{R}) = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} \mathbb{MP}(J_n, \delta_n)\, d\mu_n(\omega) \leq \mathbb{MP}(\Lambda, \lambda) .$$

Lemma 48. *Suppose $\sigma \in \mathbb{G}_n$, $1 \leq n \leq \infty$. Then there is a measurable family $\{\sigma_\omega : \omega \in \Omega_n\}$ in $\mathbb{MP}(J_n, \delta_n)$ such that, for every $f \in L^\infty(\Lambda_n,)$*

$$(f \circ \sigma)(\omega, t) = f(\omega, \sigma_\omega(t)) .$$

We write this as

$$\sigma = \int_{\Omega_n} \sigma_\omega d\mu_n(\omega) \ .$$

Proof. We can view $L^2(\Lambda_n, \lambda_n) = L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ as

$$\int_{\Omega_n}^{\oplus} L^2(J_n, \delta_n)\, d\mu_n(\omega)$$

by identifying $f \in L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ with

$$\int_{\Omega_n}^{\oplus} f_\omega d\mu_n(\omega) ,$$

where $f_\omega(t) = f(\omega, t)$. Fubini's theorem shows that this is an isomorphism, i.e.,

$$\|f\|_2^2 = \int_{\Omega_n \times J_n} |f(\omega, t)|^2\, d(\mu_n \times \delta_n) = \int_{\Omega_n} \int_{J_n} |f_\omega|^2\, d\delta_n(t) = \int_{\Omega_n} \|f_\omega\|^2\, d\mu_n(\omega) .$$

We know that $U(f) = f \circ \sigma$ is a unitary operator on $L^2(\Lambda_n, \lambda_n) = L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$. Suppose $E \subset \Omega_n$ is measurable. Then

$$P_E \underset{\text{def}}{=} \int_{\Omega_n}^{\oplus} \chi_E(\omega) 1 d\mu(\omega) \in \int_{\Omega_n}^{\oplus} B\left(L^2(J_n, \delta_n)\right) d\mu_n(\omega),$$

and the definition of $\sigma^{-1} \in \mathbb{G}_n(\mathcal{R})$ implies that $P_E U = U P_E$. Since the linear span of $\{\chi_E : E \subset \Omega_n,\ E$ measurab
is dense in $L^\infty(\Omega_n, \mu_n)$, we see that U is in the commutant of

$$\left\{ \int_{\Omega_n}^{\oplus} \varphi(\omega) 1 d\mu_n(\omega) : \varphi \in L^\infty(\Omega_n, \mu_n) \right\}.$$

Thus there is a measurable family $\{U_\omega : \omega \in \Omega_n\}$ of unitary operators in $B(L^2(J_n, \delta_n))$ such that

$$U = \int_{\Omega_n}^{\oplus} U_\omega d\mu(\omega).$$

If $h \in L^2(J_n, \delta_n)$, we define $\hat{h} \in L^2(\Omega_n \times J_n, \mu_n \times \delta_n)$ by

$$\hat{h}(\omega, t) = h(t),$$

i.e.,

$$\hat{h} = \int_{\Omega_n}^{\oplus} h d\mu_n(\omega).$$

If $h, k \in L^\infty(J_n, \delta_n)$, then $U\left(\hat{h}\hat{k}\right) = U\left(\hat{h}\right) U\left(\hat{k}\right)$, so, for almost every $\omega \in \Omega_n$,

$$U_\omega(hk) = U_\omega(h) U_\omega(k).$$

Since $L^2(J_n, \delta_n)$ is separable, there is a countable set $\mathcal{E}$ whose closure in $\|\cdot\|_2$ is

$$\{h \in L^\infty(J_n, \delta_n) : \|h\|_\infty \leq 1\}$$

(which is $\|\cdot\|_2$-closed). We now have for almost every $\omega \in \Omega_n$ and $h, k \in \mathcal{E}$,

$$U_\omega (hk) = U_\omega (h) U_\omega (k).$$

We can change U_ω on a set of measure 0 and assume that the above relation holds for all $\omega \in \Omega_n$. Suppose $h, g \in L^\infty (J_n, \delta_n)$ and $\|h\|_\infty, \|g\|_\infty \leq 1$ and suppose $\omega \in \Omega_n$. We can choose sequences $\{h_k\}$ and $\{g_k\}$ in $\mathcal{E}$ such that $\|h_k - h\|_2 \to 0$ and $\|g_k - g\|_2 \to 0$. By replacing these sequences with appropriate subsequences, we can assume that $h_k (t) \to h (t)$, $(U_\omega h_k)(t) \to (U_\omega h)(t)$, $g_k (t) \to g (t)$, $(U_\omega g_k)(t) \to (U_\omega g)(t)$ a.e. (δ_n). It follows that $\|h_k g_k - hg\|_2 \to 0$. Thus

$$U_\omega (hg)(t) = \lim_{k\to\infty} U_\omega (h_k g_k)(t) = \lim_{k\to\infty} (U_\omega h_k)(t)(U_\omega g_k)(t) = (U_\omega h)(t)(U_\omega g)(t).$$

It follows from Lemma 44 that, for each $\omega \in \Omega_n$, there is a (unique) $\sigma_\omega \in \mathbb{MP}(J_n, \delta_n)$ such that, for every $h \in L^2 (J_n, \delta_n)$,

$$U_\omega h = h \circ \sigma_\omega .$$

Our measurable cross-section theorems can be used to show that there is a measurable choice of the σ_ω's, but the uniqueness implies that $\{\sigma_\omega : \omega \in \Omega_n\}$ is measurable. □

4.2 Nonincreasing Rearrangement Functions, s-functions, and Ky Fan functions.

Theorem 49. *Suppose $f : \Lambda \to [0, \infty)$ is measurable. Then there is a $\sigma \in \mathbb{G}(\mathcal{R})$ such that, for $1 \leq n \leq \infty$, the mapping $t \mapsto (f \circ \sigma)(\omega, t)$ is nonincreasing on J_n a.e. (μ_n).*

Proof. Choose $R > \|f\|_\infty$. Suppose $1 \leq n \leq \infty$. Let

$$\mathcal{X} = \{(h, \sigma) \in L^\infty (\delta_n) \times \mathbb{MP}(J_n) : 0 \leq h \leq R, h \circ \sigma \text{ is nonincreasing on } J_n\},$$

where $\{f : 0 \leq f \leq R\}$ is given the $\|\cdot\|_{2,\delta_n}$-topology, $\mathbb{MP}(J_n)$ is given the weak topology, and $L^\infty (\delta_n) \times \mathbb{MP}(J_n)$ is given the product topology. (Note that if $n < \infty$, $\mathbb{MP}(J_n)$ corresponds to the set of $n \times n$ permutation matrices and has the discrete topology.) Since $\|\cdot\|_2$ convergence implies subsequential convergence almost everywhere, it follows that $\mathcal{X}$ is a complete separable metric space. Since every measurable h has a nonincreasing rearrangement, the map

$$\pi_1 : \mathcal{X} \to \{h : 0 \leq h \leq R\}$$

is onto, so, by Lemma 27, there is an absolutely measurable cross-section $\gamma_n : Y \to \mathcal{X}$ for π_1. Let $\eta_n = \pi_2 \circ \gamma_n : Y \to \mathbb{MP}(J_n)$.

We now define $s_n : \Omega_n \to \mathbb{MP}(J_n)$ by

$$s_n(\omega) = \eta_n(f_\omega) \in \mathbb{MP}(J_n).$$

It is clear from the construction that that $f_\omega \circ s_n(\omega)$ is a nonincreasing function of t, i.e., $f(\omega, s_n(\omega)(t))$ is a nonincreasing function of t for each $\omega \in \Omega_n$.

We define

$$\sigma_n(\omega, t) = (\omega, s_n(\omega)(t)).$$

Then $\sigma = \{\sigma_n\}_{1 \leq n \leq \infty} \in \mathbb{G}(\mathcal{R})$ has the desired properties. □

Note that the function σ is not necessarily unique, but the function $f \circ \sigma$ is unique. It is called the *nonincreasing rearrangement function* for f, and we denote it by s_f. If f and h are nonnegative measurable functions on Λ, we say that f and h are $\mathbb{G}(\mathcal{R})$-*equivalent* if and only if $s_f = s_h$ a.e. (λ). This holds if and only if there is a $\sigma_1 \in \mathbb{G}(\mathcal{R})$ such that $h = f \circ \sigma_1$.

For each $\omega \in \Omega_n$ and $t \in J_n$, $s_f(\omega, t)$ is call *the* t^{th} *s-number of* f *at* ω.

Definition 50. *Suppose $T \in \mathcal{R}$. We can write $T = \sum_{1 \leq n \leq \infty} \int_{\Omega_n}^{\oplus} T(\omega)\, d\mu_n(\omega)$. We define $s_T \in L^\infty(\Lambda, \lambda)$ by*

$$s_T(\omega, t) = s_{T(\omega)}(t)$$

when $1 \leq n \leq \infty$, $\omega \in \Omega_n$ and $t \in J_n$.

Definition 51. *Suppose $f \in L^{\infty}(\Lambda, \lambda)$ and $0 \leq f$. For each $1 \leq n \leq \infty$, and each $\omega \in \Omega_n$, we define $f_\omega \in L^{\infty}(J_n, \delta_n)$ by*

$$f_\omega(t) = f(\omega, t).$$

We view

$$f = \sum_{1 \leq n \leq \infty}^{\oplus} \int_{\Omega_n}^{\oplus} f_\omega d\mu_n(\omega).$$

We then define $s_f \in L^{\infty}(\Lambda, \lambda)$ by

$$s_f(\omega, t) = s_{f_\omega}(t).$$

Lemma 52. *Suppose $0 \leq f \in L^{\infty}(\Lambda, \lambda)$. Then there is a $\sigma \in \mathbb{G}$ such that, $f \circ \sigma = s_f$.*

Proof. For $1 \leq n \leq \infty$, the map $\omega \mapsto f_\omega$ from Ω_n to $L^{\infty}(J_n, \delta_n)$ is measurable. For each $\omega \in \Omega_n$, there is a $\sigma_\omega \in \mathbb{MP}(J_n, \delta_n)$ such that $f_\omega \circ \sigma_\omega = s_{f_\omega}$. Using measurable cross-sections, we can choose the σ_ω's so that $\{\sigma_\omega : \omega \in \Omega\}$ is measurable. Thus $\sigma = \sum_{1 \leq n \leq \infty} \int_{\Omega_n}^{\oplus} \sigma_\omega \in G$ and

$$(f \circ \sigma)(\omega, t) = f(\omega, \sigma_\omega(t)) = (f_\omega \circ \sigma_\omega)(t) = s_{f_\omega}(t) = s_f(\omega, t).$$

$\square$

Lemma 53. *Suppose $T \in \mathcal{R}$, $\mathcal{A}$ is a masa in $\mathcal{R}$, $|T| \in \mathcal{A}$, $\pi_{\mathcal{A}} : L^{\infty}(\Lambda, \lambda) \to \mathcal{A}$ is a tracial embedding as in Theorem 33, and $f \in L^{\infty}(\Lambda, \lambda)$ satisfies $\pi_{\mathcal{A}}(f) = |T|$. Then $s_T = s_f$.*

Proof. We can write

$$\mathcal{A} = \sum_{1 \leq n \leq \infty} \int_{\Omega_n}^{\oplus} \mathcal{A}_\omega d\mu_n(\omega),$$

where, for $1 \leq n \leq \infty$ and $\omega \in \Omega_n$, A_ω is a masa in $\mathcal{R}_\omega$. We can also write

$$\pi_{\mathcal{A}} = \sum_{1 \leq n \leq \infty} \int_{\Omega_n}^{\infty} \pi_\omega d\mu_n(\omega),$$

where, for each $\omega \in \Omega_n$, $\pi_\omega : L^{\infty}(J_n, \delta_n) \to \mathcal{A}_\omega$ is a tracial embedding. If $\pi_{\mathcal{A}}(f) = |T|$, then, for almost every ω,

$$\pi_\omega(f_\omega) = |T|(\omega) = |T_\omega|.$$

Thus, for almost every $\omega \in \Omega$,

$$s_{f_\omega} = s_{T_\omega} .$$

Thus $s_f = s_T$. □

Lemma 54. *Suppose $\mathcal{A}_1$, $\mathcal{A}_2$ are masas in $\mathcal{R}$, $0 \leq A_k \in \mathcal{A}_k$, $\pi_k : L^\infty(\Lambda, \lambda) \to \mathcal{A}_k$ are the isomorphisms in Theorem 33 and $f_1, f_2 \in L^\infty(\Lambda, \lambda)$ satisfy $\pi_k(f_k) = A_k$ for $k = 1, 2$. The following are equivalent:*

1. $s_{f_1} = s_{f_2}$
2. *There is a $\gamma \in \mathbb{G}(\mathcal{R})$ such that $f_2 = f_1 \circ \gamma$*
3. *There is a sequence $\{U_n\}$ of unitary operators in $\mathcal{R}$ such that*

$$\|U_n A_1 U_n^* - A_2\| \to 0.$$

4. *For every unitarily invariant norm α on $\mathcal{R}$*

$$\alpha(A_1) = \alpha(A_2)$$

5. *For every rational number $t \in (0,1]$ $KF_t(A_1) = KF_t(A_2)$.*

Proof. (1) $\Rightarrow$ (2). There are $\gamma_1, \gamma_2 \in \mathbb{G}(\mathcal{R})$ such $s_{f_k} = f_k \circ \gamma_k$ for $k = 1, 2$. By (1) we have $f_2 = f_1 \circ \left(\gamma_1 \circ \gamma_2^{-1}\right)$.

(2) $\Rightarrow$ (3). Define $\pi_3 : L^\infty(\Lambda, \lambda) \to \mathcal{A}_2$ by

$$\pi_3(f) = \pi_2(f \circ \gamma).$$

Thus $\pi_3(f_1) = A_2$. By Theorem 23, $\pi_1 \sim_a \pi_3$. Thus there is a net (sequence) $\{U_i\}$ of unitary operators in $\mathcal{R}$ such that

$$\lim_i \|U_i A_1 U_i^* - A_2\| = \lim_i \|U_i \pi_1(f_1) U_i^* - \pi_3(f_1)\| = 0.$$

Hence, for every $n \in \mathbb{N}$, there is a unitary U_n such that

$$\|U_n A_1 U_n^* - A_2\| < 1/n.$$

$(3) \Rightarrow (4)$, $(4) \Rightarrow (5)$ are trivial.

$(5) \Rightarrow (1)$. We know that $KF_t(A_1) = KF_t(s_{f_1})$ and $KF_t(s_{f_2})$. Let

$$E_t = \{\omega \in \Omega : KF_t(s_{f_1})(\omega) \neq KF_t(s_{f_2})(\omega)\},$$

and let $E = \cup E_t$, then $\lambda(E) = 0$. Therefore $\int_0^t f_1(x)\,dx = \int_0^t f_2(x)\,dx$ for every $0 < t \leq 1$. Thus $f_1(x) = f_2(x)$ except on a countable set. Therefore $f_1 = f_2$ a.e.(δ_∞). $\square$

Corollary 55. *Suppose $\mathcal{A}_1, \mathcal{A}_2$ are masas in $\mathcal{R}$, $0 \leq A \in \mathcal{A}_k$, $\pi_k : L^\infty(\Lambda, \lambda) \to \mathcal{A}_k$ are the isomorphisms in Theorem 33 and $f_1, f_2 \in L^\infty(\Lambda, \lambda)$ satisfy $\pi_k(f_k) = A$ for $k = 1, 2$. Then $s_{f_1} = s_{f_2}$.*

If $T \in \mathcal{R}$, we define

$$KF_t(T) = KF_t(s(f_T))$$

We need to define t^{th} *Ky Fan function* $KF_t(T)$ solely in terms of T and $\mathcal{R}$. (See Lemma 17)

Note that when $n = \infty$, KF_t is defined on $L^\infty(J_n, \delta_n)$ for all $0 < t \leq 1$. For $1 \leq n < \infty$, KF_t is only defined when $t \in \left\{\frac{1}{n}, \ldots, \frac{n}{n}\right\}$. The next definition extends this concept.

Definition 56. *Suppose $1 \leq n < \infty$ and $0 < t \leq 1$. We choose an integer k, $1 \leq k \leq n$ such that*

$$\frac{k-1}{n} < t \leq \frac{k}{n}.$$

We define KF_t on $L^\infty(J_n, \delta_n)$ by

$$KF_t = KF_{\frac{k}{n}}.$$

For $f \in L^{\infty}(\Lambda)$ and $1 \leq n \leq \infty$ and $\omega \in \Omega_n$ and $t \in J_n$, we define

$$KF_t(f)(\omega, t) = KF_t(s_{f_\omega}),$$

and we define, for $T \in \mathcal{R}$,

$$KF_t(T) = KF_t(s_T).$$

We easily have that for $S, T \in \mathcal{R}$

$$KF_t(S + T) \leq KF_t(S) + KF_t(T)$$

always holds.

4.3 $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norms on $L^{\infty}(\Lambda, \lambda)$

Suppose (Y, ν) is a probability space, and $\mathbb{G}$ is a subgroup of $\mathbb{MP}(Y, \nu)$. A norm β on $L^{\infty}(Y, \nu)$ is called a $\mathbb{G}$*-symmetric normalized gauge norm* if and only if

1. $\beta(1) = 1$
2. $\beta(f) = \beta(|f|)$ for every $f \in L^{\infty}(Y, \nu)$,
3. $\beta(f \circ \sigma) = \beta(f)$ for every $f \in L^{\infty}(Y, \nu)$ and every $\sigma \in \mathbb{G}$.

The examples that interest us here are for $Y = \Lambda$, $\nu = \lambda$, and $\mathbb{G} = \mathbb{G}(\mathcal{R})$, i.e., the $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norms on $L^{\infty}(\Lambda, \lambda)$.

Suppose β is a $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norms on $L^{\infty}(\Lambda, \lambda)$. For every $f \in L^{\infty}(\Lambda, \lambda)$, we see that

$$\beta(f) = \beta(s_f).$$

4.4 Approximate Ky Fan Lemma

If $T \in \mathcal{R}$, we define

$$KF_t(T) = KF_t(s(f_T))$$

We can show that KF_t satisfies the triangle inequality on $\mathcal{R}$ by describing $KF_t(T)$ directly in terms of T. The Ky Fan Lemma is more complicated. We will apply the Ky Fan Lemmas we have throughout the direct integral. However, this is impossible to do directly as the next examples show.

Example 2. *In $\mathbb{C}^n$, if $f = (1, 0, \ldots, 0)$ and $g = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, we have $KF_{\frac{k}{n}}(f) \geq KF_{\frac{k}{n}}(g)$ for $1 \leq k \leq n$, But the number N of permutations $\gamma_1, \ldots, \gamma_N$ for*

$$\sum_{j=1}^{N} f \circ \gamma_j \geq g$$

must be at least n since each $f \circ \gamma_j$ is nonzero in exactly one coordinate.

Example 3. *Suppose $\mathcal{R} = \mathcal{R}_2 = \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$ and*

$$A = \sum_{1 \leq k < \infty}^{\oplus} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$$

and

$$B = \sum_{1 \leq k < \infty}^{\oplus} \begin{pmatrix} \frac{1}{2} + \frac{1}{2^k} & \\ & \frac{1}{2} - \frac{1}{2^k} \end{pmatrix}$$

Then there are no $\sigma_1, \ldots, \sigma_N \in \mathbb{G}(\mathcal{R})$ and $t_1, \ldots, t_N \in [0, 1]$ such that

$$\sum_{k=1}^{N} t_k (s_A \circ \sigma_k) \geq s_B \, .$$

This forces us to prove an approximate version of the Ky Fan Lemma that works universally.

Theorem 57. *Suppose m is a positive integer. Then, for $1 \leq n \leq \infty$ and for all $0 \leq f, g \leq 1$ in $L^{\infty}(J_n, \delta_n)$ with*

$$KF_t(f) \geq KF_t(g) \text{ for all } t \in J_n$$

there are $\{\gamma_j : 1 \leq j \leq m^{2m}\} \subset \mathbb{MP}(J_n, \mu_n)$ such that

$$\frac{2}{m} + \frac{1}{m^{2m}} \sum_{j=1}^{m^{2m}} s_f \circ \gamma_j \geq s_g.$$

Proof. For $1 \leq n < \infty$, it follows from Lemma 9. For $n = \infty$, it is proved in Theorem 20. $\square$

Corollary 58. *For $1 \leq n \leq \infty$, if $KF_t(f) \geq KF_t(g)$ for all $t \in J_n$, then $\beta(f) \geq \beta(g)$ for all symmetric gauge norm β.*

To prove the approximate Ky Fan Lemma, we need the following Lemmas.

Lemma 59. *Suppose m, n are positive integers. $f = (f_1, \cdots, f_n), h = (h_1, \cdots, h_n)$, where $f_1, \ldots, f_n$ and $h_1, \ldots, h_n$ are integers with $1 \leq f_{i+1} \leq f_i \leq m, 1 \leq h_{i+1} \leq h_i \leq m$.and $\sum_{i=1}^{k} f_i \geq \sum_{i=1}^{k} h_i$, for $1 \leq k \leq n$.Then there exists a positive integer $N \leq m^{m^2}, \gamma_1, \cdots, \gamma_N \in \mathbb{S}_n$ such that*

$$\frac{1}{N} \sum_{i=1}^{N} f \circ \gamma_i \geq h$$

Proof. Suppose $\mathcal{S} = \left\{ \begin{pmatrix} f_k \\ h_k \end{pmatrix}, 1 \leq k \leq n \right\}$, and define an order on $\mathcal{S}$ by

$$\begin{pmatrix} f_i \\ h_i \end{pmatrix} \geq \begin{pmatrix} f_j \\ g_j \end{pmatrix} \text{ if } f_i > f_j \text{ or, } f_i = f_j \text{ and } h_i \geq h_j.$$

Then $\mathcal{S}$ is a linearly ordered set.

We say $\mathcal{S}$ is trivial if for every $\begin{pmatrix} f_k \\ h_k \end{pmatrix} \in \mathcal{S}$, $f_k \geq h_k$. If $\mathcal{S}$ is trivial, we are done, so we may assume $\mathcal{S}$ is nontrivial. Denote $\mathcal{S}_0 = \mathcal{S} \backslash \left\{ \begin{pmatrix} f_k \\ f_k \end{pmatrix}, f_k \in \{1, \cdots, m\} \right\}$. Define $p(\mathcal{S}_0) = \max(f_k)$,

$q(\mathcal{S}_0) = \max\{f_k$, with $h_k > f_k\}$, where $p(\mathcal{S}_0), q(\mathcal{S}_0) \in \{f_1, \cdots, f_n\}$, we may assume $p(\mathcal{S}_0) = f_p, q(\mathcal{S}_0) = f_q$. Then denote $l(\mathcal{S}_0) = p(\mathcal{S}_0) - q(\mathcal{S}_0)$. It is not hard to see that $f_p > h_p \geq h_q > f_q$, so $f_p - f_q \geq 2$.

Let $\gamma_{p,q}$ be the permutation that permute f_p with f_q and leave all other f_i's fixed, define $f^{(1)} = \left(f_1^{(1)}, \cdots, f_n^{(1)}\right) = \frac{1}{l(\mathcal{S}_0)}\left[(h_p - f_q) f + (f_p - h_p) f \circ \gamma_{p,q}\right]$, where $f^{(1)} \in \mathbb{N}^n$. Then denote $\mathcal{S}^{(1)} = \left\{\begin{pmatrix} f_k^{(1)} \\ h_k \end{pmatrix}, 1 \leq k \leq n\right\}$, $\mathcal{S}_0^{(1)} = \mathcal{S}^{(1)} \setminus \left\{\begin{pmatrix} f_k^{(1)} \\ f_k^{(1)} \end{pmatrix}\right\}$, we form linear convex combination of f_i's this way and update f with $f^{(1)}, \cdots, f^{(r)}$ until $l\left(\mathcal{S}_0^{(r)}\right) < l(\mathcal{S}_0)$. We can also see that $l(\mathcal{S}_0) < m$, and $r < m$, so we need at most m^m permutations to reduce $l(\mathcal{S}_0)$ for 1. Repeating this process, we need at most $\left(m^{m^2}\right)$ permutations to reduce $\mathcal{S}_0$ to a trivial set. Note that we can make the number of permutations is exactly $\left(m^{m^2}\right)!$, some permutations are duplicate. Therefore, there exists a positive integer $N = \left(m^{m^2}\right)!, \gamma_1, \cdots, \gamma_N \in \mathbb{S}_n$ such that

$$\frac{1}{N}\sum_{i=1}^{N} f \circ \gamma_i \geq h.$$

$\square$

Lemma 60. *Suppose m, n are positive integers, then there exists a positive integer $N \leq m^{m^2}$ such that for all $f = (f_1, f_2, \ldots, f_n)$ and $h = (h_1, \ldots, h_n)$ with $1 \geq f_1 \geq \cdots \geq f_n \geq 0$, $1 \geq h_1 \geq \cdots \geq h_n \geq 0$, and $\sum\limits_{i=1}^{j} f_i \geq \sum\limits_{i=i}^{j} h_i$ for all $1 \leq j \leq n$, there exist $\gamma_1, \ldots, \gamma_N \in \mathbb{S}_n$ such that*

$$\frac{1}{N}\sum_{i=1}^{N} f \circ \gamma_i + \frac{2}{m} \geq h.$$

Proof. For all $1 \leq i \leq n$, if $\frac{k-1}{m} < f_i \leq \frac{k}{m}$ for some $k \in \mathbb{N}$, then define $\widetilde{f}_i = \frac{k}{m}$ and if $\frac{k-1}{m} \leq h_i < \frac{k}{m}$ for some $k \in \mathbb{N}$, then define $\widetilde{h}_i = \frac{k-1}{m}$. Let $\widetilde{f} = \left(\widetilde{f}_1, \cdots, \widetilde{f}_n\right)$ and $\widetilde{h} = \left(\widetilde{h}_1, \cdots, \widetilde{h}_n\right)$. It is easy to check that $f_i \leq \widetilde{f}_i \leq f_i + \frac{1}{m}$ and $\max\left(h_i - \frac{1}{m}, 0\right) \leq \widetilde{h}_i \leq h_i$ for all $1 \leq i \leq n$. From Lemma 59, we know there exists a positive integer N and $\gamma_1, \cdots, \gamma_N \in \mathbb{S}_n$ such that $\frac{1}{N}\sum_{i=1}^{N}\left(m\widetilde{f}\right) \circ \gamma_i \geq \left(m\widetilde{h}\right)$. Therefore, $\frac{1}{N}\sum\limits_{j=1}^{N} f \circ \gamma_j + \frac{2}{m} \geq h$. $\square$

The following is the *Approximate Ky Fan Lemma*.

Theorem 61. *If* $f,g \in L^{\infty}(\Lambda,\lambda)$, $m \in \mathbb{N}$, $m \geq 2$, *and* $0 \leq f,g \leq 1$ *and* $KF_t(f) \geq KF_t(g)$ *a.e.* (μ) *for each rational number* $t \in (0,1]$, *then there are* $\sigma_1,\ldots,\sigma_{\left(m^{m^2}\right)!} \in \mathbb{G}(\mathcal{R})$ *such that*

$$\frac{1}{(m^{m^2})!}\sum_{k=1}^{\left(m^{m^2}\right)!} f \circ \sigma_k + \frac{1}{m} \geq g.$$

Thus, for every $\mathbb{G}(\mathcal{R})$*-symmetric normalized gauge norm* β *on* $L^{\infty}(\Lambda,\lambda)$,

$$\beta(f) \geq \beta(g).$$

Proof. Suppose $f,g \in L^{\infty}(\Lambda,\lambda)$. Since there are $\sigma_1,\sigma_2 \in \mathcal{G}(\mathcal{R})$ such that $s_f = f \circ \sigma_1$ and $s_g = g \circ \sigma_2$, we can assume $f = s_f$ and $g = s_g$. We know f,g can be viewed as $f = \sum_{1\leq n\leq\infty}^{\oplus} f_n = \sum_{1\leq n\leq\infty}^{\oplus}\int_{\Omega_n}^{\oplus} f_{n,\omega}d\mu_n(\omega)$ and $g = \sum_{1\leq n\leq\infty}^{\oplus}\int_{\Omega_n}^{\oplus} g_{n,\omega}d\mu_n(\omega)$. Suppose $m \in \mathbb{N}$ and $m \geq 2$. For $1 \leq n \leq \infty$, let $\mathcal{X}_n$ be the set of tuples $\left(F,G,\sigma_1,\sigma_2,\cdots,\sigma_{m^{m^2}}\right)$ satisfying $\frac{1}{m^{m^2}}\sum_{k=1}^{m^{m^2}} F\circ\sigma_k + \frac{1}{m} \geq G$, where $0 \leq f,g \leq 1$. Then $\mathcal{X}$ is a closed subset of ball$(L^{\infty}(J_n,\delta_n)) \times$ ball$(L^{\infty}(J_n,\delta_n)) \times \prod_{i=1}^{m^{m^2}} \mathbb{MP}(\Lambda,\lambda)$, which is a complete separable metric space with the $\|\cdot\|_2$ on ball$(L^{\infty}(J_n,\delta_n))$. Then by Theorem 27 the projection onto ball$(L^{\infty}(J_n,\delta_n))$ $\times$ball$(L^{\infty}(J_n,\delta_n))$ has an abolutely measurable range $\mathcal{Y}_n$ and an absolutely measurable cross-section ψ and we let ψ_k be the composition of projection onto the coordinate of σ_k with ψ for $1 \leq k \leq \left(m^{m^2}\right)!$. If $1 \leq n < \infty$, it follows from Lemma 60 and Theorem 20 that

$$(s_{f_\omega}, s_{g_\omega}) \in \mathcal{Y}_n$$

for almost all $\omega \in \Omega_n$. We define, for $1 \leq k \leq \left(m^{m^2}\right)!$, $\sigma_k(\omega) \in \mathbb{MP}(J_n,\delta_n)$ by

$$\sigma_k(\omega) = \psi_k(s_{f_\omega}, s_{g_\omega}).$$

This gives $\sigma_1, \dots \sigma_{\left(m^{m^2}\right)!} \in \mathbb{G}(\mathcal{R})$ such that

$$\frac{1}{(m^{m^2})!} \sum\nolimits_{k=1}^{\left(m^{m^2}\right)!} s_f \circ \sigma_k + \frac{1}{m} \geq s_g.$$

If follows that, for any $\mathbb{G}(\mathcal{R})$-symmetric normalized gauge norm β on $L^\infty(\Lambda, \lambda)$ that

$$\begin{aligned} \beta(g) &= \beta(s_g) \leq \frac{1}{(m^{m^2})!} \sum\nolimits_{k=1}^{\left(m^{m^2}\right)!} \beta(s_f \circ \sigma_k) + \beta\left(\frac{1}{m}\right) \\ &= \frac{1}{(m^{m^2})!} \sum\nolimits_{k=1}^{\left(m^{m^2}\right)!} \beta(f) + \frac{1}{m} = \beta(f) + \frac{1}{m}. \end{aligned}$$

Since $m \geq 2$ was arbitrary, it follows that $\beta(g) \leq \beta(f)$. $\square$

CHAPTER 5

MAIN THEOREM

Theorem 62. *Suppose $\mathcal{R}$ is a finite von Neumann algebra acting on a separable Hilbert space H. Let the probability space $(\Lambda, \Sigma, \lambda)$ and the group $\mathbb{G} \leq \mathbb{MP}(\Lambda, \Sigma, \lambda)$ be as above. Then there is a natural* 1-1 *correspondence between the normalized unitarily invariant norms on $\mathcal{R}$ and the normalized $\mathbb{G}$-symmetric gauge norms on $L^{\infty}(\Lambda, \lambda)$.*

Proof. Suppose α is a normalized unitarily invariant norm on $\mathcal{R}$, choose any masa $\mathcal{A}$ in $\mathcal{R}$, and choose a tracial embedding $\pi_{\mathcal{A}} : L^{\infty}(\Lambda, \lambda) \to \mathcal{A}$ as in Theorem 33. Define $\beta_{\alpha} : L^{\infty}(\lambda) \to \mathbb{R}$ by

$$\beta_{\alpha}(f) = \alpha(\pi_{\mathcal{A}}(f)),$$

If $\mathcal{B}$ is another masa in $\mathcal{R}$ and $\pi_{\mathcal{B}} : L^{\infty}(\Lambda, \lambda) \to \mathcal{B}$ is as in Theorem 33, we see from Theorem 33 that, if $\Phi : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ is the center-valued trace on $\mathcal{R}$, then

$$\Phi \circ \pi_{\mathcal{A}} = \Phi \circ \pi_{\mathcal{B}}.$$

Thus, by Theorem 33, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are approximately equivalent in $\mathcal{R}$. Hence, there is a net $\{U_i\}$ in $\mathcal{U}(\mathcal{R})$ such that, for every $f \in L^{\infty}(\Lambda, \lambda)$,

$$\|U_i^* \pi_{\mathcal{A}}(f) U_i - \pi_{\mathcal{B}}(f)\| \to 0.$$

It follows from Lemma 3 that, for every $f \in L^{\infty}(\Lambda, \lambda)$,

$$\alpha(\pi_{\mathcal{A}}(f)) = \alpha(\pi_{\mathcal{B}}(f)).$$

Thus the definition of β_α is independent of choice of the masa $\mathcal{A}$ and tracial embedding $\pi_{\mathcal{A}}$. It is easy to check that β_α is norm. To prove β_α is $\mathbb{G}$-symmetric, suppose $\sigma \in \mathbb{G}$. Then, by Lemma 48, there is a measurable family $\{\sigma_\omega : \omega \in \Omega\}$ with each $\omega \in \Omega_n$, such that $\sigma_\omega \in \mathbb{MP}\left(J_n, \mu_n\right)$ .Thus, by Theorem 33,

$$\Phi_n\left(\pi_{\mathcal{A}}\left(f \circ \sigma\right)\right) = \gamma \circ \eta\left(f \circ \sigma\right),$$

but

$$\eta\left(f \circ \sigma\right)(\omega) = \int_{J_n} \left(f \circ \sigma\right)(t, \omega)\, d\delta_n(t) = \int_{J_n} f_\omega\left(\sigma_\omega(t)\right) d\delta_n(t) = \int_{J_n} f_\omega(t)\, d\delta_n(t) = \eta(f)(\omega) \ .$$

Thus, for every $f \in L^\infty(\Lambda, \lambda)$,

$$\Phi \circ \pi_{\mathcal{A}}(f) = \Gamma\left(\eta(f)\right) = \Gamma\left(\eta\left(f \circ \sigma\right)\right) = \Phi \circ \pi_{\mathcal{A}}\left(f \circ \sigma\right).$$

Thus, $\rho(f) = \pi_{\mathcal{A}}\left(f \circ \sigma\right)$ is a tracial embedding as in Theorem 33, which implies ρ is approximately equivalent to $\pi_{\mathcal{A}}$. Hence, by Lemma 3, for every $f \in L^\infty(\Lambda, \lambda)$, we have

$$\beta_\alpha(f) = \alpha\left(\pi_{\mathcal{A}}(f)\right) = \alpha\left(\pi_{\mathcal{A}}\left(f \circ \sigma\right)\right) = \beta_\alpha\left(f \circ \sigma\right) \ .$$

Thus β_α is a normalized $\mathbb{G}$-invariant gauge norm on $L^\infty(\Lambda, \lambda)$.

Conversely, suppose β is a normalized $\mathbb{G}$-symmetric gauge norm on $L^\infty(\Lambda, \lambda)$. If $T \in \mathcal{R}$, then $W^*(|T|)$ is abelian and is contained in a masa $\mathcal{A}$ of $\mathcal{R}$. By Theorem 33 there is a tracial embedding $\pi_{\mathcal{A}} : L^\infty(\Lambda, \lambda) \to \mathcal{A}$ such that, for every $f \in L^\infty(\Omega, \mu)$,

$$\tau\left(\pi_{\mathcal{A}}(f)\right) = \int_\Omega f d\mu.$$

Choose $0 \leq f \in L^\infty(\Lambda, \lambda)$ with $\pi_{\mathcal{A}}(f) = |T|$. Then we define

$$\alpha_\beta(T) = \beta(f) = \beta\left(\pi_{\mathcal{A}}^{-1}(|T|)\right).$$

Suppose $\mathcal{B}$ is another masa in $\mathcal{R}$ with $|T| \in \mathcal{B}$. Then there is a tracial embedding $\pi_{\mathcal{B}} : L^{\infty}(\Lambda, \lambda) \rightarrow \mathcal{B}$ and an $0 \leq h \in L^{\infty}(\Lambda, \lambda)$ with $\pi_{\mathcal{B}}(h) = |T|$. It follows from Lemma 53 that

$$s_f = s_T = s_h.$$

Hence, by Lemma 54, there is a $\sigma \in \mathbb{G}$ such that

$$h = f \circ \sigma \ .$$

Thus

$$\alpha(h) = \alpha(f) = \alpha(s_T) \,.$$

Thus the definition of $\alpha_{\beta}(T) = \beta(s_T)$ is independent of the masa $\mathcal{A}$ or the tracial embedding $\pi_{\mathcal{A}}$. At this point it is easy to see that $\beta_{\alpha_{\beta}} = \beta$ holds for a $\mathbb{G}$-symmetric normalized gauge norm on $L^{\infty}(\Lambda, \lambda)$.

If U and V are unitaries in $\mathcal{R}$, then, by Lemma 53,

$$s_{UTV} = s_T \ .$$

Thus $\alpha_{\beta}(UTV) = \alpha_{\beta}(T)$ by Lemma 54. Thus α_{β} is unitarily invariant.

Clearly, $\alpha_{\beta}(1) = 1$ and $\alpha_{\beta}(zT) = |z|\, \alpha_{\beta}(T)$. To show α_{β} is a norm, we just need to check the triangle inequality. Suppose $A, B \in \mathcal{R}$. Let $h = s_A + s_B$. Since, for almost every $\omega \in \Omega$ the functions $s_A(\omega, t)$ and $s_B(\omega, t)$ are nonincreasing in t, we see that

$$s_h = h = s_A + s_B.$$

Thus, we have, if $\omega \in \Omega_n$, $n \in \mathbb{N}$, and $t = k/n$ with $1 \leq k \leq n$, or if $\omega \in \Omega_{\infty}$ and $0 < t \leq 1$ is rational, then, for almost every ω,

$$KF_t(s_h)(\omega) = KF_t(s_A + s_B)(\omega) = KF_t(s_A)(\omega) + KF_t(s_B)(\omega)$$

$$= KF_t(A)(\omega) + KF_t(B)(\omega) \geq KF_t(A+B)(\omega) = KF_t(s_{A+B})(\omega).$$

It follows from the approximate Ky Fan Lemma (Theorem 61) that

$$\beta(h) \geq \beta(s_{A+B}),$$

which means

$$\alpha_\beta(A+B) \leq \beta(h) = \beta(s_A + s_B) \leq \beta(s_A) + \beta(s_B) = \alpha_\beta(A) + \alpha_\beta(B).$$

This complete the proof. $\square$

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