2019

Almost Periodic Functions on Topological Groups

Yihan Zhu
University of Windsor

ALMOST PERIODIC FUNCTIONS ON TOPOLOGICAL GROUPS

by

Yihan Zhu

A Thesis
Submitted to the Faculty of Graduate Studies
through the Department of Mathematics and Statistics
in Partial Fulfillment of the Requirements for
the Degree of Master of Science at the
University of Windsor

Windsor, Ontario, Canada

2019

Almost Periodic Functions

by

Yihan Zhu

APPROVED BY:

L. Rueda
School of Computer Science

D. Yang
Department of Mathematics and Statistics

M. S. Monfared, Advisor
Department of Mathematics and Statistics

April 23, 2019

Author's Declaration of Originality

I hereby certify that I am the sole author of this major paper and that no part of this these has been published or submitted for publication.

I certify that to the best of my knowledge, my thesis does not infringe upon anyone's copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my thesis, published or otherwise, are fully acknowledged in accordance with the standard referencing practices. Furthermore, to the extent that I have included copyrighted materials that surpasses the bounds of fair dealing within the meaning of the Canada Copyright Act, I certify that I have obtained a written permission of the copyright owner(s) to include such materials in my thesis and have included copies of such copyright clearances to my appendix.

I declare that this is a true copy of my thesis, including any final revisions, as approved by committee and the Graduate Studies Office, and that this thesis has not been submitted for a higher degree to any other University or Institution.

Abstract

We study the definition and properties of almost periodic functions on topological groups. We show the equivalence between Bochner's and Bohr's definitions of almost periodicity. We discuss Weil's construction of Bohr compactification $b(G)$ and study its properties. Using Peter-Weyl's density theorem we show that a function f in $C^b(G)$ is almost periodic if and only if it is the uniform limit of linear combinations of coefficients of the finite-dimensional irreducible unitary representations of G. We show the existence of a unique invariant mean on the space of almost periodic functions. We investigate the Fourier series of almost periodic functions, and show that it extends the classical Fourier series of 2π-periodic functions.

Acknowledgements

First, I would like to express my special thanks of gratitude to my supervisor Dr. Mehdi Monfared who gave me the golden opportunity to be his student. His patience and kindness helped me overcome various challenges in my studies. His encouragement is the motivation for me to pursue my goal in my study. And his wisdom and brilliance guided me to have optimistic opinions both on study and life.

I am also very grateful to all the members of my defense committee. In particular, I would like to thank Dr. Luis Rueda from the School of Computer Science for agreeing to be my external program reader. I am also very grateful to Dr. Yang for reading my thesis and giving me very valuable suggestions and to Dr. Hu for chairing the defense.

I also would like to thank Dr. Zhiguo Hu, Dr. Dilian Yang, Dr. Ilya Shapiro and Dr. Mohammad Hassanzadeh for their wonderful graduate courses of Analysis and Algebra. Their patient teaching led me to a fantastic mathematics world and developed my interest in learning more about interesting mathematics theorems.

Besides professors, I am grateful to my parents for their significant support from China. Without their support and care I could not finish my courses and thesis. Also, I want to thank my friends for giving me courage and relief to face the difficulties I met both in my study and life and for their accompany which makes me feel the warmth of the friendship.

Finally, I would like to offer my sincere thanks to all students, faculty members and staff in the department of Mathematics and Statistics for the harmonic, friendly and positive studying and working environment. In particular, I was deeply impressed by Ms. Dina Labelle and Mrs. Rose Spences' kind help and positive working attitudes.

Contents

CHAPTER 1

Introduction and Preliminaries

1.1. Introduction

We frequently encounter periodic phenomena in our lives. For instance, the recurrences of days and nights are repeated periodically every 24 hours due to the rotation of Earth on its axis. Seasons occur regularly because the period of revolution of Earth around Sun is roughly $365\frac{1}{4}$ days.

In nature, however, we also encounter motions that are made of a linear combination of two or more periodic motions. For example, the motion of Moon around Sun consists of the motion of Moon around Earth combined with the motion of Earth around Sun. From a mathematical point of view, such combined motions need not be periodic any longer.

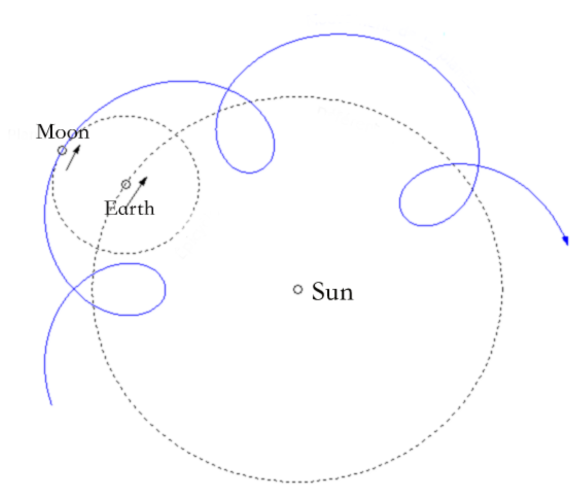

Figure 1. The motion of Moon around Sun

More precisely, if f and g are two periodic functions with periods p and p', then $f+g$ is periodic if p/p' is a rational number, otherwise $f+g$ need not be periodic. For example, both functions $\cos t$ and $\cos \sqrt{2}t$ are periodic, with periods 2π and $2\pi/\sqrt{2}$, respectively. But the function

$$f(t) = \cos t + \cos \sqrt{2}t$$

is not periodic since the equation

$$f(t) = 2$$

has a single solution $t = 0$ (see Example 2.1.6 (b)). If f was periodic, the same equation would have infinitely many solutions.

The above phenomenon leads us naturally to the concept of 'almost periodicity'. A number $\tau > 0$ is an ϵ-period if $|f(x+\tau) - f(x)| < \epsilon$, for all $x \in \mathbb{R}$. Then, although the function f above has no period, it can be shown that it has an abundance of ϵ-periods, for every $\epsilon > 0$ (Theorem 2.2.6).

Almost periodic functions are more general than periodic functions, and therefore, they form a larger class of functions than periodic ones. Unlike periodic functions, the sums, products, and limits of almost periodic functions always remain almost periodic, and therefore, the class of almost periodic functions forms a more suitable object of study from structural point of view.

This thesis is a survey of almost periodic functions and their properties. These functions were first introduced by H. Bohr in 1923 [**7**]. He further developed their properties in his papers [**8**–**10**]. The definition of almost periodicity according to Bohr depends on the notion of ϵ-periods, and is restricted to functions with a real or complex variable.

In 1927, Bochner [**4**] gave an alternative topological characterization of almost periodicity, which in some respects is easier to study than the original definition given by Bohr. Bochner's work motivated further studies and generalizations. In particular, we can mention von Neumann's paper [**37**], in which almost periodic functions were defined and studied on general groups, using Bochner's definition of almost periodicity.

In this thesis, we present a unified study of almost periodic functions in the general setting of an arbitrary topological group G. If one wishes to study almost periodicity independent of any topology on the group structure (von Neumann's point of view), then all one has to do is to assume the underlying group has discrete topology.

We remark that, in the particular case of $G = \mathbb{R}$ (the real numbers), the results discussed in this thesis cover the major theorems of the classical theory of the almost periodic functions, such as: approximation by trigonometric polynomials, existence of invariant means, expansion by Fourier series, and Parseval's identity.

What makes our study different from the usual expositions of almost periodic functions, is that we study this subject from several points of view. This allows us to have a deeper understanding of the subject, and in some cases, give multiple proofs for the results in this thesis. For example, we discuss two different constructions of the Bohr compactification $b(G)$; and we give two proofs of the existence of invariant means on the algebra $AP(G)$ of almost periodic functions.

As is often the case, in modern expositions of almost periodicity (with the notable exception of Simon [**42**]), the Bohr's point of view is abandoned in favour of Bochner's more topological definition of almost periodicity. In this thesis, we present the details of both approaches in Chapter 2. While in Chapter 3, we discuss a third approach to almost periodic functions via a compactification method due to Weil.

We now give a brief overview of the various sections of the thesis. In the remaining of Chapter 1, we recall some of the basic definitions and preliminary materials needed in the later chapters.

In Section 2.1, we introduce almost periodic functions using the topological characterization of Bochner [**4**]. Bochner's definition has the advantage of being adaptable to various generalizations, and for this reason, is commonly accepted as the basic definition of almost periodicity. The fact that almost periodic functions are stable under addition, multiplication and limits is proved in Theorem 2.1.4.

Section 2.2 is a brief introduction to P. Bohl's work on ϵ-periodicity [**6**]. This work was the original inspiration for Bohr's theory of almost periodicity.

In Section 2.3, Bohr's definition of almost periodicity is presented in the general frame work of topological groups. The equivalence of Bochner's and Bohr's definitions is proved in Theorem 2.3.6.

In Chapter 3 we present further properties of almost periodicity. Section 3.1 contains Weil's construction of a compact group $b(G)$ associated to any topological group G. The group $b(G)$ has the property that almost periodic functions on G are exactly those that can be lifted to a continuous function on $b(G)$ (Theorem 3.1.1, 3.1.4). This allows us to deduce many of the properties of $AP(G)$ from the corresponding properties of $C(b(G))$. Following Palmer [**39**], we call $b(G)$ the 'Bohr compactification' of G, although perhaps, 'Weil compactification' would have been a more appropriate name. The group $b(G)$ is also know as 'almost periodic compactification' of G [**1**].

In Section 3.2, we give an alternative construction of $b(G)$ due to Loomis [**34**]. In Loomis' approach a group structure is defined directly on the spectrum of $AP(G)$. Loomis' method has the advantage that $b(G)$ is naturally identified with the spectrum of $AP(G)$.

Section 3.3 deals with Bohr compactification in the category of Abelian topological groups. For this class of groups, the Pontrjagin duality theorem provided a simpler description of $b(G)$, which will be discussed in this section.

In Section 3.4 we establish an isometric isomorphism between the C^*-algebras $AP(G)$ and $C(b(G))$ (Theorem 3.4.1). Using this isomorphism, it follows immediately that the spectrum of $AP(G)$ can be topologically identified with $b(G)$ (Corollary 3.4.3).

The main result in Section 3.5 is the approximation Theorem 3.5.2, which is one of the central properties of almost periodic functions. This result shows that almost periodic functions are exactly those that can be uniformly approximated to arbitrary degree, by linear combinations of coefficients of finite-dimensional irreducible unitary representations of G. More specifically, in the classic setting of $G = \mathbb{R}$, almost periodic functions on $\mathbb{R}$ are exactly uniformly limits of trigonometric polynomials on $\mathbb{R}$.

Section 3.6 contains a brief study of maximally and minimally almost periodic groups, which are introduced by von Neumann [**37**] and von Neumann-Wigner [**38**].

In Chapter 4, we are concerned with proving the existence of invariant means on $AP(G)$ (the space of almost periodic functions on G), and showing some of its applications. In Section 4.1 we prove the existence of an invariant mean by using the normalized Haar measure on $b(G)$. In Section 4.2 we give an alternative proof of the existence of an invariant mean using a combinatorial technique due to W. Maak [**35**]. Uniqueness of invariant means is established in Theorem 4.2.9. Several examples of invariant means are given in Section 4.3. In the final section of Chapter 4, we show how we can use invariant means to prove the existence of Fourier series for almost periodic functions (Theorem 4.4.4). Establishing the existence of such series is another major result in the theory of almost periodic functions. Also in this section we show in details, how in the special case of continuous 2π-periodic functions, Theorem 4.4.4 gives the classical result of Fourier series expansions.

Chapter 5 contains a brief discussion on recent developments of the subject and some ideas for future work.

This thesis has three appendices. Appendix A, contains a brief discussion of uniform continuity in the context of general topological groups, which is needed in our discussion of Section 2.3.

Appendix B gives a detailed proof of the Stone-Weierstrass Theorem. This theorem is used in the proof of the Peter-Weyl density Theorem 3.5.1. This latter theorem is an important ingredient of the proof of Theorem 3.5.2.

Author's contribution. The materials in this thesis are primarily from [**17**], [**24**], [**26**], [**29**], [**34**], [**38**] and [**42**]. Additional sources are mentioned in the relevant sections. The author's main contribution has been to provide full details of the main results presented in this thesis. Among the results whose proofs have been substantially expanded, we can mention Theorem 2.1.4, Example 2.1.6, Lemma 2.2.4, Theorems 2.2.6, 2.3.4, 2.3.6 and 3.1.1, Lemma 4.2.1, Theorems 4.2.8 and 4.4.4.

1.2. Basic Definitions

DEFINITION 1.2.1. Let X be a metric space with metric d. For a positive $\epsilon > 0$, an *ϵ-mesh* in X is a finite set $\{x_1, ..., x_n\}$ in X such that for every $x \in X$,

$$\min\{d(x_1, x), ..., d(x_n, x)\} < \epsilon,$$

i.e.,

$$X = \bigcup_{i=1}^{n} B(x_i, \epsilon),$$

where $B(x_i, \epsilon)$ is the open ball in X with center x_i and radius ϵ. A metric space X is said to be *totally bounded* if it admits an ϵ-mesh for every $\epsilon > 0$.

REMARK 1.2.2. (a) A metric space X is compact if and only if it is complete and totally bounded.
(b) A subset A of a complete metric space is relatively compact (i.e., the closure of A is compact) if and only if A is totally bounded.

DEFINITION 1.2.3. Let X and Y be metric spaces, (f_α) a family of mappings from X into Y. The family (f_α) is said to be *equicontinuous* at x_0 if for every $\zeta > 0$, there exists an $\eta > 0$ such that

$$d(x, x_0) \leqslant \eta \Rightarrow d(f_\alpha(x), f_\alpha(x_0)) \leqslant \zeta$$

for all α.

The family (f_α) is said to be *equicontinuous* on X if it is equicontinuous at every point of X.

The family (f_α) is said to be *uniformly equicontinuous* on X if for every $\zeta > 0$, there exists an $\eta > 0$ such that

$$d(x, x') \leqslant \eta \Rightarrow d(f_\alpha(x), f_\alpha(x')) \leqslant \zeta$$

for all α and all $x, x' \in X$.

EXAMPLE 1.2.4. Take $X = Y = \mathbb{R}$. Let (f_α) be the family of all differentiable real-valued functions on $\mathbb{R}$ whose derivative is bounded by 1 in absolute value. Then (f_α) is uniformly equicontinuous.

LEMMA 1.2.5. *Let X and Y be metric spaces, (f_α) an equicontinuous family of mappings of X into Y. Assume that X is compact, then the family (f_α) is uniformly equicontinuous.*

THEOREM 1.2.6 (Ascoli's theorem). *Let X be a compact metric space, Y a complete metric space, $\mathscr{A}$ an equicontinuous subset of $C(X,Y)$ (the set of all continuous functions from X to Y). Assume that for each $x \in X$, the set of $f(x)$, where f runs over $\mathscr{A}$, has compact closure in Y. Then $\mathscr{A}$ has compact closure in the metric space $C(X,Y)$.*

COROLLARY 1.2.7. *Let $C^b(X)$ be the set of all bounded continuous complex-valued functions on X. A subset F of $C^b(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

DEFINITION 1.2.8. A topological space $(X, \mathscr{T})$ has the *finite intersection property* if, for any collection $\mathscr{F}$ of closed subsets of X such that $\cap\{F : F \in \mathscr{F}\} = \varnothing$, there are $F_1, \dots, F_n \in \mathscr{F}$ such that $F_1 \cap \cdots \cap F_n = \varnothing$.

THEOREM 1.2.9. *Let $(X, \mathscr{T})$ be a topological space. Then X is compact if and only if X has the finite intersection property.*

THEOREM 1.2.10 (The Inverse Mapping Theorem). *If X and Y are Banach spaces and $A : X \to Y$ is a bounded linear transformation that is bijective, then A^{-1} is bounded.*

THEOREM 1.2.11 (Urysohn's lemma). *Let X be a normal topological space, and let F and G be disjoint closed subsets of X. Then there is a continuous function $f : X \to \mathbb{R}$ such that $f(X) \subset [0,1]$, $f|_F = 0$, and $f|_G = 1$.*

THEOREM 1.2.12 (Heine-Borel theorem). *If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:*
(i) *E is closed and bounded.*
(ii) *E is compact.*
(iii) *Every infinite subset of E has a limit point in E.*

DEFINITION 1.2.13 (Partition of Unity). If X is a topological space and $E \subset X$, a *partition of unity* on E is a collection $\{h_\alpha\}_{\alpha \in A}$ of functions in $C(X, [0,1])$ such that
(i) each $x \in X$ has a neighborhood on which only finitely many h_α's are nonzero;
(ii) $\sum_{\alpha \in A} h_\alpha(x) = 1$ for $x \in E$.
A partition of unity $\{h_\alpha\}$ is *subordinate* to an open cover $\mathscr{U}$ of E if for each α there exists $U \in \mathscr{U}$ with $\operatorname{Supp}(h_\alpha) \subset U$.

THEOREM 1.2.14. *Let X be a locally compact Hausdorff space, K a compact subset of X, and $\{U_j\}_1^n$, an open cover of K. There is a partition of unity on K subordinate to $\{U_j\}_1^n$, consisting of compactly supported functions.*

For the proof of this proposition, we may refer to [**11**].

THEOREM 1.2.15 (Tychonoff's Theorem). *Let $((K_i, \tau_i))_{i \in I}$ be a nonempty family of compact topological spaces. Then their topological product is also compact.*

THEOREM 1.2.16 (Hahn-Banach Theorem). *Suppose (a) M is a linear subspace of a real vector space X,*
(b) *$p : X \longrightarrow \mathbb{R}$ satisfies $p(x+y) \leqslant p(x) + p(y)$ and $p(tx) = tp(x)$ if $x \in X, y \in X$, $t \geqslant 0$,*
(c) *$f : M \longrightarrow \mathbb{R}$ is linear and $f(x) \leqslant p(x)$ on M.*
Then there exists a linear function $F : X \longrightarrow \mathbb{R}$ such that $F(x) = f(x)$ for $x \in M$ and $-p(-x) \leqslant F(x) \leqslant p(x)$ for $x \in X$.

DEFINITION 1.2.17. Let X be a vector space and $A \subseteq X$. The *convex hull* of A, denoted by $co(A)$, is the intersection of all convex sets that contain A.

If X is a normed space, then the *closed convex hull* of A is the intersection of all closed convex subsets of X that contain A; it is denoted by $\overline{co}(A)$.

1.3. Banach Algebras and Hilbert Spaces

Definition 1.3.1. Let A be an algebra over $\mathbb{C}$. Then A is a *Banach algebra* if A is a Banach space with a norm $\|\cdot\|$ satisfying $\|ab\| \leqslant \|a\|\|b\|$, for all $a, b \in A$.

Definition 1.3.2. An *involution* on an algebra A is a map $* : A \longrightarrow A,\ a \mapsto a^*$, that satisfies

$$(x+y)^* = x^* + y^*,$$
$$(\lambda x)^* = \overline{\lambda} x^*,$$
$$(xy)^* = y^* x^*,$$
$$x^{**} = x$$

for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

If A is a Banach algebra, we shall also assume that $\|x^*\| = \|x\|$ for all $x \in A$. A Banach algebra equipped with an involution is called an involutive Banach algebra or a $*$-Banach algebra.

Definition 1.3.3. A *C^*-algebra* is a Banach algebra A equipped with an involution which satisfies

$$\|x^* x\| = \|x\|^2 \qquad \text{for all } x \in A.$$

Example 1.3.4. If X is a topological space, then the set $C^b(X)$ of all continuous complex-valued bounded functions from X, with pointwise operations and supremum norm, is a C^*-algebra.

Definition 1.3.5. If X is a vector space over $\mathbb{C}$, an *inner product* on X is a function $(\cdot|\cdot) : X \times X \to \mathbb{C}$ such that for all α, β in $\mathbb{C}$, and x, y, z in X, the following are satisfied:

(a) $(\alpha x + \beta y|z) = \alpha(x|z) + \beta(y|z)$,

(b) $(x|\alpha y + \beta z) = \bar{\alpha}(x|y) + \bar{\beta}(x|z)$,

(c) $(x|x) \geqslant 0$,

(d) $(x|y) = \overline{(y|x)}$,

(e) if $(x|x) = 0$, then $x = 0$.

An inner product induces a norm on X by the relation

$$\|x\| = (x|x)^{\frac{1}{2}}. \tag{1.3.1}$$

Definition 1.3.6. A *Hilbert space* is a vector space $\mathscr{H}$ over $\mathbb{C}$ together with an inner product $(\cdot|\cdot)$ such that relative to the metric $d(x,y) = \|x-y\|$ induced by the norm (1.3.1), $\mathscr{H}$ is a complete metric space.

Example 1.3.7. $(\mathbb{C}^n, (\cdot|\cdot))$ with inner product

$$(x|y) = \sum_{i=1}^{n} x_i \overline{y_i}$$

is a Hilbert space.

Definition 1.3.8. Let $\mathscr{H}$,$\mathscr{K}$ be Hilbert spaces. If $A \in \mathscr{L}(\mathscr{H},\mathscr{K})$, then the unique operator B in $\mathscr{L}(\mathscr{K},\mathscr{H})$ satisfying $(Ah|k) = (h|Bk)$ for all h in $\mathscr{H}$ and all k in $\mathscr{K}$ is called the *adjoint* of A and is denoted by $B = A^*$.

Example 1.3.9. If $\mathscr{H}$ is a Hilbert space, $A = \mathscr{L}(\mathscr{H})$ is a C^*-algebra where for each A in $\mathscr{L}(\mathscr{H})$, A^* is the adjoint of A.

1.4. Topological Groups

Definition 1.4.1. A *topological group* G is a topological space which is also a group such that the maps

$$G \times G \to G, \qquad (x,y) \mapsto xy$$

and

$$G \to G, \qquad x \mapsto x^{-1}$$

are continuous.

LEMMA 1.4.2. *If H is a subgroup or normal subgroup of a topological group G, then its closure $\overline{H}$ is also a subgroup, or normal subgroup of G.*

A *locally compact group* is a topological group G for which the underlying topology is locally compact and Hausdorff.

DEFINITION 1.4.3. Let X be a locally compact space and let $\mathscr{B}_X$ be the smallest σ-algebra of X that contains the open sets. Sets in $\mathscr{B}_X$ are called *Borel sets*.

A positive measure μ on $(X, \mathscr{B}_X)$ is a *regular Borel measure* if

(a) $\mu(K) < \infty$ for every compact subset K of X;

(b) for any open set $U \in X$ in $\mathscr{B}_X$, $\mu(U) = \sup\{\mu(K) : K \subseteq U$ and K is compact$\}$;

(c) for any E in $\mathscr{B}_X$, $\mu(E) = \inf\{\mu(W) : W \supseteq E$ and W is open$\}$.

If μ is a complex-valued measure on $(X, \mathscr{B}_X)$, μ is a *regular Borel measure* if $|\mu|$ is a regular Borel measure.

Let $M(X)$ be all of the complex-valued regular Borel measures on X. Note that $M(X)$ is a vector space over $\mathbb{C}$. For μ in $M(X)$, let $\|\mu\| \equiv |\mu|(X)$.

DEFINITION 1.4.4. Let G be a locally compact group. By a *left Haar measure* on G, we mean a nonzero positive regular Borel measure m on G such that $m(xE) = m(E)$ for all $x \in G, E \in \mathscr{B}_X$.

The measure m is called the *Haar measure* for G.

It can be shown that every locally compact group has left Haar measure m which is unique up to a multiplicative constant. If G is compact then $m(G) < \infty$, and so by dividing m with $m(G)$, we may assume m is normalized, that is, $m(G) = 1$.

EXAMPLE 1.4.5 (The Group Algebra). Let G be a locally compact group. The space $L^1(G)$ of all integrable functions on G (i.e., all measurable functions f on G satisfying $\|f\|_1 = \int_G |f(s)| ds < \infty$) is a Banach algebra, called the *group algebra* of G, with the convolution product $*$ defined by

$$(f * g)(t) = \int_G f(s) g(s^{-1}t) ds \qquad (f, g \in L^1(G), s, t \in G).$$

To show that $\|fg\|_1 \leqslant \|f\|_1\|g\|_1$, we observe that

$$\begin{aligned}
\|fg\|_1 &= \int_G |f * g(s)| ds \\
&= \int_G \left| \int_G f(t) g(t^{-1}s) dt \right| ds \\
&\leqslant \int_G \int_G |f(t) g(t^{-1}s)| dt ds \\
\text{(Fubini's theorem)} \quad &= \int_G \int_G |f(t) g(t^{-1}s)| ds dt \\
s \mapsto ts \quad &= \int_G \int_G |f(t) g(s)| ds dt \\
&= \int_G |f(t)| dt \int_G |g(s)| ds \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

1.5. Weak Topology

Definition 1.5.1. A *topological vector space* is a vector space E together with a topology such that with respect to this topology

(a) the map of $E \times E \to E$ defined by $(x, y) \mapsto x + y$ is continuous;

(b) the map of $\mathbb{C} \times E \to E$ defined by $(\alpha, x) \mapsto \alpha x$ is continuous.

Definition 1.5.2. A *locally convex space* is a topological vector space whose topology is defined by a family of seminorms $\mathscr{P}$ such that $\bigcap_{p \in \mathscr{P}} \{x : p(x) = 0\} = \{0\}$.

Theorem 1.5.3. *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and to each positive integer n the set*

$$V(p, n) = \{x : p(x) < \frac{1}{n}\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p, n)$. Then $\mathscr{B}$ is a convex balanced local base at 0 for a topology τ on X, which turns X into a locally convex space such that

(i) every $p \in \mathscr{P}$ is continuous, and

(ii) a set $E \subset X$ is bounded if and only if every $p \in \mathscr{P}$ is bounded on E.

THEOREM 1.5.4. *A topological vector space X is normable if and only if its origin has a convex bounded neighborhood.*

Two important examples of locally convex spaces are as following:

Let X be a normed space. For each x^* in X^*, define $p_{x^*}(x) = |x^*(x)|$. Then p_{x^*} is a seminorm and if $\mathscr{P} = \{p_{x^*} : x^* \in X^*\}$, $\mathscr{P}$ makes X into a locally convex space. The topology defined on X by these seminorms is called the *weak topology*, and is denoted by w-topology.

Let X be a normed space. For each x in X, define $p_x : X^* \to [0, \infty)$ by $p_x(x^*) = |x^*(x)|$. Then p_x is a seminorm and if $\mathscr{P} = \{p_x : x \in X\}$, $\mathscr{P}$ makes X^* into a locally convex space. The topology defined on X^* by these seminorms is called the *weak-star topology* and denoted by w^*-topology.

DEFINITION 1.5.5. Let E be a Banach space. The *strong operator topology* on $\mathscr{L}(E)$ is the topology defined by the family of seminorms $\{p_x : x \in E\}$, where $p_x(A) = \|Ax\|$. Thus if $\{T_\alpha\}_\alpha$ is a net of operators in $\mathscr{L}(E)$, then $T_\alpha \to T$ in the strong operator topology if and only if $\|T_\alpha x - Tx\| \to 0$ for all $x \in E$.

1.6. Representations of Topological Groups

DEFINITION 1.6.1. Let H be a Hilbert space, T a bounded operator on H. Then T is said to be *unitary* if T is a linear isometry of H onto H. Equivalently, T is unitary if $T^* = T^{-1}$.

DEFINITION 1.6.2. Let G be a topological group and H a Hilbert space. A *continuous unitary representation* of G in H is a homomorphism π of the group G into the unitary group $U(H)$ of H which is continuous with respect to the strong operator topology. The dimension of H is called the *dimension* of π and is denoted by $\dim \pi$. And H is called the *representation space* of π, and is denoted by H_π.

REMARK 1.6.3. (a) It follows from the definition that a continuous unitary representation of G in H is a mapping π of G into the unitary group $U(H)$ of H such that $\pi(st) = \pi(s)\pi(t)$ for any $s, t \in G$, and for every $\xi \in H$, the mapping $s \mapsto \pi(s)\xi$

of G into H is continuous (for the norm topology of H). We may also note that the condition $\pi(st) = \pi(s)\pi(t)$ implies that $\pi(e) = 1$ and $\pi(s^{-1}) = \pi(s)^{-1} = \pi(s)^*$.

(b)The strong and weak operator topologies coincide on the unitary group of H [**17**]. However, we do not require a representation π to be continuous with respect to the norm topology because norm continuity is too restrictive to be of much interest.

Definition 1.6.4. Let $\pi : G \to \mathscr{L}(H)$ be a continuous unitary representation. The functions $\pi_{\xi,\eta} : G \longrightarrow \mathbb{C}$, $s \mapsto \pi_{\xi,\eta}(s) = (\pi(s)\xi|\eta)$, where η, ξ are fixed elements in H, are called the *coefficients* of π.

We denote the linear span of all the coefficient functions of π by $\mathscr{E}_\pi$.

Definition 1.6.5. Let G be a locally compact group. If π_1, π_2 are unitary representations of G, an *interwining operator* for π_1 and π_2 is a bounded linear map $T : H_{\pi_1} \longrightarrow H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The set of all such operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$.

Two representations π_1 and π_2 are called *(unitary) equivalent* if $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary operator. That is, if there exists a unitary operator $T : H_{\pi_1} \longrightarrow H_{\pi_2}$ such that $\pi_2(x) = T\pi_1(x)T^*$ for all $x \in G$.

$$\begin{tikzcd}
H_{\pi_1} \arrow[r, "T"] \arrow[d, "\pi_1(x)"] & H_{\pi_2} \arrow[d, "\pi_2(x)"] \\
H_{\pi_1} \arrow[r, "T"] & H_{\pi_2}
\end{tikzcd}$$

Figure 2. The diagram of equivalent representations π_1 and π_2

Definition 1.6.6. Let G be a locally compact group. If π is a unitary representation of G and $u \in H_\pi$, the closed linear span M_u of $\{\pi(x)u : x \in G\}$ in H_π is called the *cyclic subspace* generated by u. If $M_u = H_\pi$, u is called a *cyclic vector* for π. π is called a *cyclic representation* if it has a cyclic vector.

Theorem 1.6.7. *Let π be a continuous unitary representation of G. The following conditions are equivalent:* (i) *the only closed subspaces of H_π invariant under $\pi(G)$*

are 0 and H; (ii) the only operators in $\mathscr{L}(H)$ that commute with every $\pi(s)$ $(s \in G)$ are the scalar operators; (iii) every non-zero vector of H is a cyclic vector for π, i.e., if $\xi \in H$, $\xi \neq 0$, then $\pi(G)\xi$ is dense in H.

Definition 1.6.8. If the conditions in Theorem 1.6.7 are satisfied, then π is said to be *topologically irreducible.* Topologically irreducible representations are usually called irreducible, when there is no fear of confusion.

Let G be a compact group. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G. When G is Abelian, $\widehat{G}$ is a set of continuous characters on G, that is, continuous functions $\chi : G \to \mathbb{T}$.

Theorem 1.6.9 (The Gelfand-Raikov Theorem). *If G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if x and y are distinct points of G, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$.*

Theorem 1.6.10. *Let G be a compact group, and (σ_α) be the family of continuous, irreducible, finite-dimensional unitary representations of G, then $\bigcap\limits_{\alpha} Ker\sigma_\alpha = \{e\}$.*

Theorem 1.6.11. *If G is a locally compact Abelian group, then every continuous unitary irreducible representation of G is one-dimensional.*

Theorem 1.6.12. *Let G be a compact group.*
(i) Every representation in $\widehat{G}$ is finite-dimensional;
(ii) Every continuous unitary representation of G is a direct sum of elements in $\widehat{G}$.

Definition 1.6.13. Suppose $\{\pi, H\}$ and $\{\pi', H'\}$ are unitary representations of G. Let $H \otimes H'$ be the Hilbert space tensor product of H and H'. The simple tensors $x \otimes x'$, with $x \in H$, $x' \in H'$, span a dense linear subspace of $H \otimes H'$. For $s \in G$, $\pi(s) \otimes \pi'(s)$ is the unitary operator on $H_\pi \otimes H_{\pi'}$, defined by

$$\pi(s) \otimes \pi'(s)(x \otimes x') = \pi(s)x \otimes \pi'(s)x'.$$

The mapping $s \mapsto \pi(s) \otimes \pi'(s)$ is a continuous unitary representation of G on $H_\pi \otimes H_{\pi'}$, called *the tensor product of* π and π' and is denoted by $\pi \otimes \pi'$.

1.7. The Gelfand Representation

Definition 1.7.1. Let A be a commutative Banach algebra and $\Delta(A)$ the set of all nonzero algebra homomorphisms from A to $\mathbb{C}$. We endow $\Delta(A)$ with the weakest topology with respect to which all the functions

$$\Delta(A) \to \mathbb{C}, \qquad \varphi \mapsto \varphi(x) \quad (x \in A)$$

are continuous. A neighborhood basis at $\varphi_0 \in \Delta(A)$ is then given by the collection of sets $U(\varphi_0, x_1, ..., x_n, \epsilon) = \{\varphi \in \Delta(A) : |\varphi(x_i) - \varphi_0(x_i)| < \epsilon, 1 \leqslant i \leqslant n\}$, where $\epsilon > 0, n \in \mathbb{N}$, and $x_1, ..., x_n$ are arbitrary elements of A. This topology on $\Delta(A)$ is called the *Gelfand topology*. The space $\Delta(A)$, equipped with the Gelfand topology, is called the *spectrum* of A or the *Gelfand space* of A.

Definition 1.7.2. For $x \in A$, we define $\hat{x} : \Delta(A) \to \mathbb{C}, \quad \varphi \mapsto \varphi(x)$. It is easily checked that the mapping $\Gamma : A \to C(\Delta(A)), \quad x \mapsto \hat{x}$ is a homomorphism. We call Γ the *Gelfand homomorphism* or *Gelfand representation* of A and we often denote $\Gamma(A)$ by $\widehat{A}$.

Theorem 1.7.3. *For a commutative C^*-algebra A, the Gelfand homomorphism is an isometric $*$-isomorphism from A onto $C_0(\Delta(A))$ (the space of all continuous complex-valued functions on $\Delta(A)$ vanishing at infinity). If A is unital, then $\Delta(A)$ is compact and $A \cong C(\Delta(A))$.*

CHAPTER 2

Almost Periodic Functions

In this chapter, we study almost periodic functions using the Bochner's characterization of almost periodicity. We discuss the notion of ϵ-periods and study Bohr's condition of almost periodicity. Basic properties of almost periodic functions are developed and several examples are discussed.

2.1. Bochner's Definition and Its Basic Properties

In this section we discuss a definition of almost periodicity which was given by Bochner in 1927 [**4**]. Bochner's definition deals with almost periodic functions on the real line $\mathbb{R}$. We will however consider the more general case of the functions defined on the topological groups. Our main reference for this is [**26**].

For a complex function f on topological group G and $a, b \in G$, we use $D_a f$ to denote the function on $G \times G = G^2$ such that $D_a f(x, y) = f(xay)$ for all $x, y \in G$. We use ${}_a f$ (resp. f_a) to denote the function on G such that ${}_a f(x) = f(ax)$ (resp. $f_a(x) = f(xa)$) for all $x \in G$. And we use ${}_b f_a$ to denote the function on G such that ${}_b f_a(x) = f(bxa)$ for all $x \in G$.

THEOREM 2.1.1. *Let G be a topological group and f be a function in $C^b(G)$ (the space of all bounded continuous complex functions on G). Let $-$ denote the closure in the uniform topology for $C^b(G)$. Then the following properties of f are equivalent:*
(i) $\overline{\{f_a : a \in G\}}$ is compact in $C^b(G)$.
(ii) $\overline{\{{}_a f : a \in G\}}$ is compact in $C^b(G)$.
(iii) $\overline{\{{}_b f_a : a, b \in G\}}$ is compact in $C^b(G)$.
(iv) $\overline{\{D_a f : a \in G\}}$ is compact in $C^b(G^2)$.

Proof. According to Remark 1.2.2 (b) a subset A of $C^b(G)$ is relatively compact if and only if it is totally bounded. Hence we only need to prove the equivalence of total boundedness for the sets $\{f_a : a \in G\}$, $\{{}_af : a \in G\}$, $\{{}_bf_a : a, b \in G\}$ and $\{D_af : a \in G\}$.

It is obvious that (iii) implies (i) and (ii). We will prove that (iv) implies (i) and (ii), (i) implies (iv), (ii) implies (iv), (i) implies (iii), and (ii) implies (iii).

First, we prove that (iv) implies (i). Let a be an arbitrary element in G. Then according to the total boundedness of $\{D_af : a \in G\}$, for any given $\epsilon > 0$, there exists a finite set $\{a_1, \ldots, a_n\}$ in G such that for some $a_i \in \{a_1, \ldots, a_n\}$,

$$\|D_{a_i}f - D_af\|_{\sup} < \epsilon.$$

Then

$$\begin{aligned}\|f_a - f_{a_i}\|_{\sup} &= \sup\{|f(xa) - f(xa_i)| : x \in G\}\\ &\leqslant \sup\{|f(xay) - f(xa_iy)| : x, y \in G\}\\ &= \|D_{a_i}f - D_af\|_{\sup} < \epsilon,\end{aligned}$$

which proves (i). The proof that (iv) implies (ii) is very similar and we omit the details.

Then we prove that (i) implies (iv). Suppose $\{f_a : a \in G\}$ is totally bounded. Given $\epsilon > 0$, there is a finite subset $\{a_1, a_2, \ldots, a_m\}$ of G such that $\{f_{a_j}\}_{j=1}^m$ is an $\epsilon/4$-mesh in $\{f_a : a \in G\}$. For $j = 1, 2, \ldots, m$, let $A_j = \{a \in G : \|f_a - f_{a_j}\|_{\sup} < \epsilon/4\}$. Consider the family of all sets

$$(A_{l_1}a_1^{-1}) \cap (A_{l_2}a_2^{-1}) \cap \cdots \cap (A_{l_m}a_m^{-1}), \qquad \text{for} \quad l_j, j \in \{1, \ldots, m\}.$$

Write the nonempty sets in this family as $B_1, \ldots, B_n$, and choose $b_k \in B_k$ for $k = 1, \ldots, n$. Let $a \in G$, and consider $aa_1, \ldots, aa_m$. Since $G = \bigcup_{j=1}^m A_j$, we can find $l_1, \ldots, l_m \in \{1, \ldots, m\}$ so that $aa_1 \in A_{l_1}, \ldots, aa_m \in A_{l_m}$, i.e., $a \in (A_{l_1}a_1^{-1}) \cap (A_{l_2}a_2^{-1}) \cap \cdots \cap (A_{l_m}a_m^{-1})$. Therefore, $G = \bigcup_{k=1}^n B_k$. Now consider any $c \in G$; let B_{k_0} be a set in

$\{B_k\}$ such that $c \in B_{k_0}$. Let $(x, y) \in G^2$, and select j_0 ($j_0 \in \{1, 2, \ldots, m\}$) such that $y \in A_{j_0}$. Then we have

$$\begin{aligned}
&|f(xcy) - f(xb_{k_0}y)| \\
&\leqslant |f(xcy) - f(xca_{j_0})| + |f(xca_{j_0}) - f(xb_{k_0}a_{j_0})| + |f(xb_{k_0}a_{j_0}) - f(xb_{k_0}y)| \\
&\leqslant \|f_y - f_{a_{j_0}}\|_{\sup} + \|f_{ca_{j_0}} - f_{b_{k_0}a_{j_0}}\|_{\sup} + \|f_{a_{j_0}} - f_y\|_{\sup}.
\end{aligned}$$

Since $y \in A_{j_0}$, $\|f_y - f_{a_{j_0}}\|_{\sup} < \epsilon/4$. By our choice of B_{k_0}, $c \in B_{k_0} = (A_{l_1}a_1^{-1}) \cap (A_{l_2}a_2^{-1}) \cap \cdots \cap (A_{l_m}a_m^{-1})$, which means $c \in A_{l_{j_0}}a_{j_0}^{-1}$ for some l_{j_0}. Also, by our choice of b_{k_0}, we have $b_{k_0} \in B_{k_0}$. Writing $c = aa_{j_0}^{-1}$ and $b_{k_0} = a'a_{j_0}^{-1}$ where $a, a' \in A_{l_{j_0}}$, then

$$\begin{aligned}
&\|f_{ca_{j_0}} - f_{b_{k_0}a_{j_0}}\|_{\sup} = \|f_a - f_{a'}\|_{\sup} \\
&\leqslant \|f_a - f_{a_{l_{j_0}}}\|_{\sup} + \|f_{a_{l_{j_0}}} - f_{a'}\|_{\sup} < \epsilon/4 + \epsilon/4 = \epsilon/2.
\end{aligned}$$

Therefore, $|f(xcy) - f(xb_{k_0}y)| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$. Since (x, y) is arbitrary, the functions $D_{b_1}f, \ldots, D_{b_n}f$ form an ϵ-mesh in $\{D_c f : c \in G\}$. Thus (iv) holds.

The proof that (ii) implies (iv) is very similar to the proof just given and we omit the details. Thus so far we can say that (i), (ii) and (iv) are equivalent.

Finally, suppose that (i) holds, we want to show (iii) holds. Since (i) and (ii) are equivalent, (ii) also holds. Let $\{f_{a_j}\}_{j=1}^m$ and $\{{}_{b_k}f\}_{k=1}^n$ be $\epsilon/2$-meshes in $\{f_a : a \in G\}$ and $\{{}_bf : b \in G\}$, respectively. Then for all $x, a, b \in G$, we have

$$|f(bxa) - f(b_kxa_j)| \leqslant |f(bxa) - f(b_kxa)| + |f(b_kxa) - f(b_kxa_j)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

for some k and j. That is, $\{{}_{b_k}f_{a_j}\}(j = 1, 2, \ldots, m; k = 1, 2, \ldots n)$ is an ϵ-mesh in $\{{}_bf_a : a, b \in G\}$. $\square$

Definition 2.1.2. Let G be a topological group. A function f in $C^b(G)$ satisfying the equivalent conditions (i)-(iv) in Theorem 2.1.1 is said to be *almost periodic*. The set of all continuous almost periodic functions on G is denoted by $AP(G)$.

Remark 2.1.3. Almost periodic functions can be studied on arbitrary groups with no underlying topology. This is equivalent to assuming that G has the discrete topology.

The following theorem shows several basic properties of elements in $AP(G)$.

Theorem 2.1.4. *Let G be a topological group, then*

(i) every constant function is in $AP(G)$.

(ii) if f is in $AP(G)$, then also $\Re f$, $\Im f$, and $\overline{f}$ are in $AP(G)$.

(iii) if f, g are in $AP(G)$, then $f+g$ and fg are in $AP(G)$.

(iv) if $f_1, f_2, \ldots, f_n, \ldots$ are in $AP(G)$ and $\lim\limits_{n\to\infty} \|f_n - f\|_{\sup} = 0$, then $f \in AP(G)$.

(v) if f is in $AP(G)$ and a, b are in G, then ${}_af$, f_a and ${}_bf_a$ are in $AP(G)$.

Proof. The proofs of (i) and (v) are obvious. ((i) can be easily proved using total boundedness while (v) comes directly from Theorem 2.1.1).

To prove (ii), let $f \in AP(G)$. Then we have $\{f_a : a \in G\}$ is totally bounded. For given $\epsilon > 0$ and any $a \in G$, there exist a finite set $\{a_1, \ldots, a_n\}$ in G (depending only on ϵ and f) and $j_0 \in \{1, \ldots, n\}$ such that

$$\|\Re f_a - \Re f_{a_{j_0}}\|_{\sup} \leqslant \|f_a - f_{a_{j_0}}\|_{\sup} < \epsilon.$$

Thus $\{\Re f_a : a \in G\}$ is totally bounded and therefore compact. This proves that $\Re f \in AP(G)$. Similarly, $\Im f \in AP(G)$ and $\overline{f} \in AP(G)$.

To prove (iii), consider $f+g$. Let $\epsilon > 0$ be given. Let $\{f_{a_j}\}_{j=1}^m$ be an $\epsilon/4$-mesh for $\{f_a : a \in G\}$ and $\{g_{b_k}\}_{k=1}^n$ be an $\epsilon/4$-mesh for $\{g_b : b \in G\}$. Let $A_j = \{a \in G : \|f_a - f_{a_j}\|_{\sup} < \epsilon/4\}$ $(j = 1, 2, \ldots, m)$ and $B_k = \{b \in G : \|g_b - g_{b_k}\|_{\sup} < \epsilon/4\}$ $(k = 1, 2, \ldots, n)$. Let the nonempty sets $A_j \cap B_k$ be written as $C_1, \ldots, C_p$, and choose $c_l \in C_l$ $(l = 1, 2, \ldots, p)$. We want to prove $\{(f+g)_{c_l}\}_{i=1}^p$ is an ϵ-mesh for $\{(f+g)_c : c \in G\}$. For any $c \in G$, there exist j_0 and k_0 such that

$$\|f_c - f_{a_{j_0}}\|_{\sup} < \epsilon/4 \quad \text{and} \quad \|g_c - g_{b_{k_0}}\|_{\sup} < \epsilon/4.$$

Let $c_{l_0} \in C_{l_0} = A_{j_0} \cap B_{k_0}$, then

$$\|f_{c_{l_0}} - f_{a_{j_0}}\|_{\sup} < \epsilon/4 \quad \text{and} \quad \|g_{c_{l_0}} - g_{b_{k_0}}\|_{\sup} < \epsilon/4.$$

Therefore

$$\begin{aligned}
&\|(f+g)_{c_{l_0}} - (f+g)_c\|_{\sup} \leqslant \|(f_{c_{l_0}} - f_c)\|_{\sup} + \|(g_{c_{l_0}} - g_c)\|_{\sup} \\
&\leqslant \|f_{c_{l_0}} - f_{a_{j_0}}\|_{\sup} + \|f_{a_{j_0}} - f_c\|_{\sup} + \|g_{c_{l_0}} - g_{b_{k_0}}\|_{\sup} + \|g_{b_{k_0}} - g_c\|_{\sup} < \epsilon.
\end{aligned}$$

Next, we show that $fg \in AP(G)$. Suppose $\|f\|_{\sup} > 0$ and $\|g\|_{\sup} > 0$, and we form an $\epsilon/(4\|g\|_{\sup})$-mesh for $\{f_a : a \in G\}$ and an $\epsilon/(4\|f\|_{\sup})$-mesh for $\{g_b : b \in G\}$. Then, as above, the sets C_l and points $c_l \in C_l$ yield an ϵ-mesh for $\{(fg)_c : c \in G\}$.

To prove (iv), note that if $\{(f_n)_{a_1}, \ldots, (f_n)_{a_p}\}$ is an $\epsilon/3$-mesh for $\{(f_n)_a : a \in G\}$, and $\|f - f_n\|_{\sup} < \epsilon/3$, then there exists an $a_q \in \{a_1, a_2, \ldots, a_p\}$ such that

$$\begin{aligned}
\|f_{a_q} - f_a\|_{\sup} \leqslant &\|f_{a_q} - (f_n)_{a_q}\|_{\sup} + \|(f_n)_{a_q} - (f_n)_a\|_{\sup} + \|(f_n)_a - f_a\|_{\sup} \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Therefore $\{f_{a_1}, \ldots, f_{a_p}\}$ is an ϵ-mesh for $\{f_a : a \in G\}$. □

With Theorem 2.1.4, we can immediately get the following consequence.

Corollary 2.1.5. *If G is a topological group, then $AP(G)$ is a C^*-subalgebra of $C^b(G)$.*

Example 2.1.6. (a) Let G be $\mathbb{R}$. Then every periodic function on $\mathbb{R}$ is almost periodic. Let $p > 0$ be a period of f. Then $\{f_a : a \in \mathbb{R}\} = \{f_a : a \in [0, p]\}$ is a compact subset of $C^b(\mathbb{R})$ since it is the image of the compact interval $[0, p]$ under the continuous function $a \mapsto f_a$, $\mathbb{R} \longrightarrow C^b(\mathbb{R})$.

(b) By Theorem 2.1.4 and part (a) above, the sums, products and uniform limits of periodic function on $\mathbb{R}$ are all almost periodic. All linear combinations of periodic functions are almost periodic. For example, trigonometric polynomials of the form

$$f(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t} \qquad (\lambda_k \in \mathbb{R}).$$

In particular, $f(x) = \cos x + \cos\sqrt{2}x$ is an almost periodic function, since it is the sum of two periodic functions. However, $f(x)$ is not periodic. We claim that the only root of the equation $f(x) = 2$ is $x = 0$. Let $\cos x = \cos\sqrt{2}x = 1$, then $x = 2k_1\pi$ and $\sqrt{2}x = 2k_2\pi$ where $k_1, k_2 \in \mathbb{Z}$. Therefore, $2k_1\pi = \frac{2k_2}{\sqrt{2}}\pi$ and the only solution is $k_1 = k_2 = 0$, i.e., $x = 0$. Similarly, the product $g(x) = \cos x \cdot \cos\sqrt{2}x$ of two periodic functions is also not periodic since the only solution for $g(x) = 1$ is $x = 0$.

(c) If we restrict the zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$ $(s \in \mathbb{C})$ to the vertical line $R(s) = \sigma_0 > 1$, and write $s = \sigma_0 + it$, then

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-\sigma_0} n^{-it} = \sum_{n=1}^{\infty} n^{-\sigma_0} e^{-it\log n}.$$

This series consists of terms that are periodic in t with period $\frac{2\pi}{\log n}$, therefore its partial sums are almost periodic. Since $\sigma_0 > 1$, the series is uniformly convergent, and thus by Theorem 2.1.4 (iv), $\xi(s)$ is almost periodic on the vertical line $s = \sigma_0 + it$.

(d) Every uniform limit of trigonometric polynomials is almost periodic.

2.2. Functions with ϵ-Almost Periods

In this section, we discuss the concept of ϵ-almost periods, which was given for the functions on $\mathbb{R}$ by Bohl [**6**]. However, we generalize the definitions of ϵ-almost periods to the functions on general topological group as follows.

Definition 2.2.1. Let G be a topological group, f a bounded continuous complex-valued function on G and $\epsilon > 0$. An element $x \in G$ is called a *left ϵ-almost period* of f if

$$\|{}_x f - f\|_{\sup} \leqslant \epsilon.$$

An element $y \in G$ is called a *right ϵ-almost period* of f if

$$\|f_y - f\|_{\sup} \leqslant \epsilon.$$

When G is Abelian, the left ϵ-almost periods are also right ϵ-almost periods. In particular, when $G = \mathbb{R}$, we give the following definition.

DEFINITION 2.2.2. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function and $\epsilon > 0$. A number $\tau > 0$ is called an *ϵ-almost period* if for all $x \in \mathbb{R}$,

$$|f(x+\tau) - f(x)| \leqslant \epsilon.$$

EXAMPLE 2.2.3. (a) If f is periodic with period p, then p is an ϵ-period for every $\epsilon > 0$.

(b) If f is uniformly continuous, then for a given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Therefore, every τ satisfying $0 < \tau < \delta$ is an ϵ-almost period.

Bohl showed that a trigonometric polynomial of the form

$$f(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t},$$

where $\lambda_k \in \mathbb{R}$ are arbitrary, has infinitely many ϵ-periods for every $\epsilon > 0$. Note that such polynomials need not be periodic.

In the remaining of this section we verify Bohl's result for the simple case where $f(x) = e^{i\alpha x} + e^{i\beta x}$. First we observe that if $\alpha/\beta = q/p$ is rational ($p, q \in \mathbb{Z}$), then f is periodic with period $\sigma = 2\pi q \alpha^{-1}$.

$$\begin{aligned} f(x+\sigma) &= e^{i\alpha(x+\sigma)} + e^{i\beta(x+\sigma)} = e^{i\alpha x + i2\pi q} + e^{i\beta x + i2\pi q\beta/\alpha} \\ &= e^{i\alpha x} e^{i2\pi q} + e^{i\beta x} e^{i2\pi p} = e^{i\alpha x} + e^{i\beta x} = f(x). \end{aligned}$$

Therefore $\sigma = 2\pi q \alpha^{-1}$ and all of its integer multiples can serve as ϵ-almost periods for all $\epsilon > 0$.

So it remains to consider the case when α/β is irrational.

We want to show the existence of ϵ-almost periods for the function f when α/β is irrational. The idea is to find ϵ-almost periods in the form of $2\pi q/\alpha$ for suitable $q \in \mathbb{N}$. First, we need the following lemmas from the number theory.

LEMMA 2.2.4. *If x is a real number and t is a positive integer, there are integers p and q such that*

$$\left|x - \frac{p}{q}\right| \leqslant \frac{1}{q(t+1)}, \qquad 1 \leqslant q \leqslant t.$$

PROOF. The $t+1$ numbers

$$0 \cdot x - [0 \cdot x], 1 \cdot x - [1 \cdot x], \ldots, tx - [tx]$$

all lie in the interval $[0, 1)$. Call them, in increasing order of magnitude, $\alpha_0, \alpha_1, \ldots, \alpha_t$. Mark the numbers $\alpha_0, \alpha_1, \ldots, \alpha_t$ on a circle of unit circumference, that is, a unit interval on which 0 and 1 are identified. Then the $t+1$ differences

$$\alpha_1 - \alpha_0,\ \alpha_2 - \alpha_1,\ \ldots,\ \alpha_t - \alpha_{t-1},\ \alpha_0 - \alpha_t + 1$$

are the lengths of the arcs of the circle between successive α's, and so they are non-negative and

$$(\alpha_1 - 0) + (\alpha_2 - \alpha_1) + \ldots (1 - \alpha_t) = 1.$$

It follows that at least one of these $t+1$ differences does not exceed $(t+1)^{-1}$. But each difference is of the form

$$g_1 x - g_2 x - N,$$

where N is an integer and $g_1, g_2 \in \{1, 2, \ldots, t\}$.
If $g_1 > g_2$, then $N \geqslant 0$. In this case, we can take $q = g_1 - g_2$, $p = N$. Then

$$\left|x - \frac{p}{q}\right| = \frac{|(g_1 - g_2)x - N|}{g_1 - g_2} \leqslant \frac{1}{(g_1 - g_2)(t+1)} = \frac{1}{q(t+1)}.$$

If $g_1 < g_2$, then we can take $q = g_2 - g_1$, $p = -N$. Then

$$\left|x - \frac{p}{q}\right| = \frac{|(g_2 - g_1)x + N|}{g_2 - g_1} \leqslant \frac{1}{(g_2 - g_1)(t+1)} = \frac{1}{q(t+1)}.$$

□

LEMMA 2.2.5. *For any irrational number x, there are infinitely many distinct rational numbers p_n/q_n, $n = 1, 2, \ldots$, such that*

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{{q_n}^2} \quad (n \geqslant 1). \tag{2.2.2}$$

PROOF. According to the above lemma, if x is irrational, then for each positive integer t, the inequalities

$$0 < \left|x - \frac{p}{q}\right| \leqslant \frac{1}{(t+1)q} < \frac{1}{q^2}, \quad 1 \leqslant q \leqslant t \tag{2.2.3}$$

have suitable p and q. Let t_1 be any integer exceeding $|x - p/q|^{-1}$. Then by Lemma 2.2.4, there exist integers p_1, q_1 such that

$$0 < \left|x - \frac{p_1}{q_1}\right| \leqslant \frac{1}{(t_1+1)q_1} < \frac{|x - p/q|}{q_1} \leqslant \left|x - \frac{p}{q}\right|.$$

Thus, $p_1/q_1 \neq p/q$. Furthermore, $|x - p_1/q_1| < 1/{q_1}^2$. By this procedure, we can obtain infinitely many rational numbers p_n/q_n with

$$0 < \cdots < \left|x - \frac{p_n}{q_n}\right| < \left|x - \frac{p_{n-1}}{q_{n-1}}\right| < \cdots < \left|x - \frac{p_1}{q_1}\right|$$

and

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{{q_n}^2} \quad (n \geqslant 1).$$

□

With the help of the above lemmas, we can find for each $\epsilon > 0$ there are infinitely many ϵ-almost periods for the function $f(x) = e^{i\alpha x} + e^{i\beta x}$.

THEOREM 2.2.6. *Let $f(x) = e^{i\alpha x} + e^{i\beta x}$, where $\alpha, \beta \in \mathbb{R}$ are such that α/β is irrational. Let $\epsilon > 0$ be given, and let p/q be a rational number such that $q \geqslant 1$ and*

$$2\pi q \left| \frac{\beta}{\alpha} - \frac{p}{q} \right| < \epsilon. \tag{2.2.4}$$

Then $\tau = 2\pi q/\alpha$ is an ϵ-almost period for f.

PROOF. The existence of rational numbers p/q satisfying (2.2.4) follows from Lemma 2.2.5. For given $\epsilon > 0$, we only need to choose n in the lemma large enough so that $1/q_n < \epsilon/2\pi$. Then we have

$$\left| \frac{\beta}{\alpha} - \frac{p_n}{q_n} \right| < \frac{1}{{q_n}^2} < \frac{\epsilon}{2\pi q_n},$$

which means $2\pi q_n \left|\beta/\alpha - p_n/q_n\right| < \epsilon$. In fact, the lemma shows the existence of infinitely many such rationals.

Then for every $x \in \mathbb{R}$, we have

$$\begin{aligned}
\left| f(x + \frac{2\pi q}{\alpha}) - f(x) \right| &= \left| e^{i\alpha x} + e^{i\beta x} e^{i2\pi q\beta/\alpha} - e^{i\alpha x} - e^{i\beta x} \right| \\
&= |e^{i\beta x}(e^{i2\pi q\beta/\alpha} - 1)| = |e^{i2\pi q\beta/\alpha} - 1| \\
&= \sqrt{(\cos(2\pi q\beta/\alpha) - 1)^2 + \sin^2(2\pi q\beta/\alpha)} \\
&= \sqrt{2(1 - \cos(2\pi q\beta/\alpha))} \\
&= 2 \left| \sin(\frac{\pi q\beta}{\alpha}) \right| \\
&= 2 \left| \sin(\pi q(\frac{\beta}{\alpha} - \frac{p}{q})) \right| \\
&\leqslant 2\pi q \left| \frac{\beta}{\alpha} - \frac{p}{q} \right| < \epsilon.
\end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, it follows that $\|f_\tau - f\|_{\sup} \leqslant \epsilon$. Therefore $\tau = 2\pi q/\alpha$ is an ϵ-almost period. □

2.3. Bohr's Definition and Its Generalization

In this section we discuss Bohr's original definition of almost periodicity and its generalization to arbitrary topological groups.

Starting in 1923, H. Bohr adopted the results of Bohl on the existence of ϵ-periods' discussion and used it as the definition of a class of functions which he called *almost periodic functions*. He developed the basic properties of almost periodic functions in [**8**], [**9**] and [**10**].

Definition 2.3.1 (Bohr). A function $f : \mathbb{R} \to \mathbb{C}$ is called *Bohr almost periodic* if it is continuous and if for every $\epsilon > 0$, there exists a number $l > 0$ such that every closed interval of length l contains an ϵ-almost period.

Thus for a function to be Bohr almost periodic, for each ϵ, the number of its ϵ-almost periods must occur with some regularity in the real line. In other words, the set of ϵ-periods must be relatively dense in $\mathbb{R}$ for every $\epsilon > 0$.

Next we extend the notion of ϵ-periods and Bohr's condition of almost periodicity to general topological groups.

Definition 2.3.2. Let G be a topological group, f a complex-valued function on G and $\epsilon > 0$. We say that f satisfies the *left Bohr's condition* (LBC) if f is continuous and for every $\epsilon > 0$, there is a compact set $K \subset G$ so that yK contains a left ϵ-almost period for all $y \in G$.

Right Bohr's condition is defined similarly. We say that f satisfies the *right Bohr's condition* (RBC) if f is continuous and for every $\epsilon > 0$, there is a compact set $K \subset G$ so that for all $y \in G$, Ky contains a right ϵ-almost period.

We say that f satisfies the *Bohr's conditions*, if it satisfies both LBC and RBC.

If G is Abelian, left and right Bohr's conditions are equivalent. A discussion of Bohr's conditions on Abelian group can be found in [**42**].

Remark 2.3.3. (a) It is easy to check that when $G = \mathbb{R}$, Bohr's conditions are consistent with the Definition 2.3.1. Because if the Bohr's conditions hold, choose

$l > 0$ large enough so that $K \subset [-l, l]$, then every interval of length $2l$ contains an ϵ-almost period. In fact, if I is such an interval and a is the center of the interval, then $I - a = [-l, l] \supset K$, and hence by assumption there must exist an ϵ-period in $a + K \subset a + (I - a) = I$. As for the converse, if every interval of length $l > 0$ contains an ϵ-almost period, then we may take $K = [0, l]$.

(b) Let us note that the inequality $\|_x f - f\|_{\sup} \leqslant \epsilon$ is equivalent to $\|_{x^{-1}} f - f\|_{\sup} \leqslant \epsilon$ where $x^{-1} \in K^{-1}y^{-1}$, and K^{-1} is compact. Thus the LBC can be rephrased as follows: for every $\epsilon > 0$, there is a compact set $K \subset G$ so that for all $y \in G$, Ky contains a left ϵ-almost period. Also, the RBC can be similarly rephrased.

For a discussion of uniform continuity on topological groups we refer to Appendix A.

THEOREM 2.3.4. (a) *Every function satisfying the LBC is bounded and right uniformly continuous.*
(b) *Every function satisfying the RBC is bounded and left uniformly continuous.*

PROOF. (a) Let f satisfy the left Bohr's condition, and $x \in G$. For $\epsilon = 1$, there exists a compact set K_0 such that $K_0 x^{-1}$ contains a left 1-almost period, say $k_0 x^{-1}$. Thus $\|_{k_0 x^{-1}} f - f\|_{\sup} \leqslant 1$, and therefore $|_{k_0 x^{-1}} f(x) - f(x)| \leqslant 1$, from which it follows that

$$|f(x)| \leqslant 1 + |f(k_0)| \leqslant 1 + \sup_{k \in K_0} |f(k)|.$$

Since K_0 is independent of x, f is bounded.

Next, we prove that f is right uniformly continuous. Given $\epsilon > 0$, we need to find a neighborhood U of the identity e such that if $y^{-1}x \in U$, then $|f(x) - f(y)| < \epsilon$. For each $y \in G$, ${}_y f$ is continuous at e, so there exists a neighborhood V_y of e such that if $z \in V_y$, then $|_y f(z) - {}_y f(e)| < \epsilon/4$. Therefore if $z, z' \in V_y$, then

$$|_y f(z) - {}_y f(z')| < \frac{\epsilon}{2}.$$

Choose a neighborhood U_y of e such that $U_y^2 \subset V_y$, then for all $z, z', z'' \in U_y$,

$$|{}_{yz''}f(z) - {}_{yz''}f(z')| < \frac{\epsilon}{2}. \tag{2.3.5}$$

Next, using the assumption that f satisfies the LBC, let a compact set K be chosen so that for all $y \in G$, there exists $w \in K$ such that $wy^{-1} \in Ky^{-1}$ with $\|{}_{wy^{-1}}f - f\|_{\sup} \leqslant \epsilon/4$. Then we have

$$\|{}_wf - {}_yf\|_{\sup} = \|{}_y({}_{wy^{-1}}f - f)\|_{\sup} \leqslant \frac{\epsilon}{4}. \tag{2.3.6}$$

Since the family $\{yU_y\}_{y\in K}$ is a covering of K, there exist $y_1, ..., y_l$ in K so that $K \subset \bigcup_{j=1}^{l} y_jU_{y_j}$. Put $U = \bigcap_{j=1}^{l} U_{y_j}$. If $y \in K$, then $y = y_jz_j$ for some $z_j \in U_{y_j}$, hence for all $z, z' \in U$, it follows from (2.3.5) that

$$|{}_yf(z) - {}_yf(z')| = |{}_{y_jz_j}f(z) - {}_{y_jz_j}f(z')| < \frac{\epsilon}{2}. \tag{2.3.7}$$

Now suppose x, y are such that $y^{-1}x \in U$, then with $w \in K$ chosen as above, it follows from (2.3.6) and (2.3.7) that

$$\begin{aligned}
|f(y) - f(x)| &= |{}_yf(e) - {}_yf(y^{-1}x)| \\
&\leqslant |{}_yf(e) - {}_wf(e)| + |{}_wf(e) - {}_wf(y^{-1}x)| \\
&+ |{}_wf(y^{-1}x) - {}_yf(y^{-1}x)| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

(b) The proof of (b) is similar to (a). We give the details for completeness. Let f satisfy the RBC. First we prove that f is bounded. For $\epsilon = 1$, there exists a compact set ${K_0}'$ such that for every $x \in G$, $x^{-1}{K_0}'$ contains a right 1-almost period, say $x^{-1}{k_0}'$. Thus $\|f_{x^{-1}{k_0}'} - f\|_{\sup} \leqslant 1$, and therefore $|f_{x^{-1}{k_0}'}(x) - f(x)| \leqslant 1$, from which it follows that

$$|f(x)| \leqslant 1 + |f(k_0')| \leqslant 1 + \sup_{k\in {K_0}'} |f(k)|.$$

Since ${K_0}'$ is independent of x, f is bounded.

Given $\epsilon > 0$, now we want to find a neighborhood V' of e such that if $xy^{-1} \in V'$, then $|f(x) - f(y)| < \epsilon$. For each $y \in G$, f_y is continuous at e, so there exists a neighborhood U'_y of e such that if $z \in U'_y$, then $|f_y(z) - f_y(e)| < \epsilon/4$. Therefore, if $z, z' \in U'_y$, then $|f_y(z) - f_y(z')| < \epsilon/2$.
Choose a neighborhood V'_y of e such that $(V'_y)^2 \subset U'_y$, then for all $z, z', z'' \in V'_y$,

$$|f_{z''y}(z) - f_{z''y}(z')| < \epsilon/2. \tag{2.3.8}$$

Since f satisfies the RBC, let the compact set K' be chosen so that for all $y \in G$, there exists $x \in y^{-1}K'$ with $\|f_x - f\|_{\sup} \leqslant \epsilon/4$. Since the family $\{V'_y y\}_{y \in K'}$ is a covering of K', there exist $y_1, ...y_l \in K'$ so that $K' \subset \bigcup_{j=1}^{l} V'_{y_j} y_j$. Put $V = \bigcap_{j=1}^{l} V'_{y_j}$. If $y \in K'$, then $y = z_j y_j$ for some $z_j \in V'_{y_j}$, hence for all $z, z' \in V$, it follows from (2.3.8) that

$$|f_y(z) - f_y(z')| = |f_{z_j y_j}(z) - f_{z_j y_j}(z')| < \epsilon/2. \tag{2.3.9}$$

Now, suppose x, y are such that $xy^{-1} \in V'$. Since K' is compact, the RBC implies that we can find $w' \in K'$ such that $\|f_{y^{-1}w'} - f\|_{\sup} \leqslant \epsilon/4$, and hence

$$\|f_{w'} - f_y\|_{\sup} = \|(f_{y^{-1}w'} - f)_y\|_{\sup} \leqslant \epsilon/4. \tag{2.3.10}$$

Hence it follows from (2.3.9) and (2.3.10) that

$$\begin{aligned}
|f(y) - f(x)| &= |f_y(e) - f_y(xy^{-1})| \\
&\leqslant |f_y(e) - f_{w'}(e)| + |f_{w'}(e) - f_{w'}(xy^{-1})| + |f_{w'}(xy^{-1}) - f_y(xy^{-1})| \\
&< \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.
\end{aligned}$$

□

Corollary 2.3.5. *Every function satisfying Bohr's conditions is uniformly continuous.*

Next we show the equivalence between Bochner's definition of almost periodicity and Bohr's conditions.

THEOREM 2.3.6. *Let G be a topological group and $f \in C^b(G)$. Then $f \in AP(G)$ if and only if f satisfies Bohr's conditions.*

PROOF. First, suppose $f \in AP(G)$. We need to show that f satisfies both LBC and RBC. To prove f satisfies LBC, given $\epsilon > 0$, by total boundedness of $\{{}_yf : y \in G\}$, we can pick $y_1, ..., y_l \in G$ so that every ${}_yf$ is within ϵ of some ${}_{y_j}f$ $(j \in \{1, 2, ..., l\})$. Let $K = \{y_1^{-1}, ..., y_l^{-1}\}$. For a given $x \in G$, pick y_j so that $\|{}_xf - {}_{y_j}f\|_{\sup} < \epsilon$. Then

$$\|{}_{xy_j^{-1}}f - f\|_{\sup} = \|{}_xf - {}_{y_j}f\|_{\sup} < \epsilon.$$

Thus xK contains a left ϵ-almost period. Since x is arbitrary, f satisfies LBC.

To prove f satisfies RBC, given $\epsilon > 0$, by total boundedness of $\{f_y : y \in G\}$, we can pick $y_1, ..., y_l \in G$ so that every f_y is within ϵ of some f_{y_j} $(j \in \{1, 2, ..., l'\})$. Let $K = \{y_1^{-1}, ..., y_{l'}^{-1}\}$. For a given $x \in G$, pick y_j so that $\|f_x - f_{y_j}\|_{\sup} < \epsilon$. Then

$$\|f_{y_j^{-1}x} - f\|_{\sup} = \|f_x - f_{y_j}\|_{\sup} < \epsilon.$$

Thus Kx contains a right ϵ-almost period. Since x is arbitrary, f satisfies RBC.

Conversely, suppose f satisfies both left and right Bohr's conditions, and we will show that $\{{}_yf : y \in G\}$ is totally bounded. Given $\epsilon > 0$, we want to find finitely many y_j in G so that every ${}_yf$, is within ϵ of some ${}_{y_j}f$. Since f satisfies LBC, there exists a compact subset K of G (depending only on ϵ), such that for all $y \in G$, there is some $x \in K$ so that

$$\|{}_{xy^{-1}}f - f\|_{\sup} \leqslant \epsilon/2.$$

Since f satisfies RBC, it is left uniformly continuous, that is, the map, $G \to C^b(G)$, $x \mapsto {}_xf$ is continuous. Therefore, $\{{}_xf : x \in K\}$ is compact in $C^b(G)$. Thus we can find $y_1, \ldots, y_n \in K$, so that every ${}_xf$ with $x \in K$, is within $\epsilon/2$ of some ${}_{y_j}f$. We note that the choices of $y_1, \ldots, y_n$ only depends on ϵ and f, since K depends on ϵ. Then we have

$$\|{}_yf - {}_{y_j}f\|_{\sup} \leqslant \|{}_yf - {}_xf\|_{\sup} + \|{}_xf - {}_{y_j}f\|_{\sup}$$

$$\leqslant \|_y(f - {}_{xy^{-1}}f)\|_{\sup} + \|_x f - {}_{y_j} f\|_{\sup}$$
$$\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $f \in AP(G)$.

□

COROLLARY 2.3.7. *If G is a topological group, then $AP(G) \subset UC(G)$.*

PROOF. This follows from Corollary 2.3.5 and Theorem 2.3.6. □

THEOREM 2.3.8. *Let G be a locally compact group and $C_0(G)$ be the space of continuous functions on G vanishing at infinity.*
(i) If G is compact, then $AP(G) = C(G)$.
(ii) If G is noncompact, then $AP(G) \cap C_0(G) = \{0\}$.

PROOF. (i) The statement in (i) is immediate from the definition of LBC and RBC. In fact, for each $\epsilon > 0$, we can take $K = G$ and choose the ϵ-almost period to be $\tau = e$.
(ii) Suppose G is noncompact, and there exists $f \in AP(G) \cap C_0(G)$ with $f(x_0) \neq 0$ for some $x_0 \in G$. Choose ϵ such that $0 < \epsilon < |f(x_0)|/2$. Since $f \in C_0(G)$, there exits a compact subset K_0 of G such that $|f(x)| \leqslant \epsilon$ whenever $x \notin K_0$. Since f satisfies LBC, there exists a compact set K such that Ky contains a left ϵ-almost period for all $y \in G$. Since G is not compact and $K^{-1}K_0x_0^{-1}$ is compact, we can choose a point $y_0 \in G$ such that $y_0 \notin K^{-1}K_0x_0^{-1}$, and let $\tau = ky_0 \in Ky_0$, $(k \in K)$, to be a left ϵ-almost period. Then by our choice of y_0, $\tau x_0 \notin K_0$, therefore $|f(\tau x_0)| \leqslant \epsilon$. However, the relation $\|_\tau f - f\|_{\sup} \leqslant \epsilon$ implies that $|f(\tau x_0) - f(x_0)| \leqslant \epsilon$ and hence

$$|f(x_0)| \leqslant \epsilon + |f(\tau x_0)| \leqslant 2\epsilon,$$

i.e., $|f(x_0)|/2 \leqslant \epsilon$ contradicting the choice of ϵ. □

EXAMPLE 2.3.9. (a) On the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, all continuous functions are almost periodic since $\mathbb{T}$ is compact.
(b) The function $f = e^{-x^2}$ $(x \in \mathbb{R})$ is in $C_0(\mathbb{R})$, and hence f is not almost periodic.

EXAMPLE 2.3.10. On the complex plane $\mathbb{C}$, a periodic function need not be almost periodic. For example, the function $f(s) = n^{-s} = e^{-s\log n}(n \geqslant 2)$ is periodic on $\mathbb{C}$ with a period $\frac{2\pi i}{\log n}$ because

$$f\left(s + \frac{2\pi i}{\log n}\right) = e^{-(s+\frac{2\pi i}{\log n})\log n} = e^{-s\log n - 2\pi i} = e^{-s\log n} = f(s).$$

But f is not bounded on $\mathbb{C}$, in fact, for $s = x \in \mathbb{R}$, $f(x) \to \infty$ if $x \to -\infty$, and according to Theorem 2.3.4 and 2.3.6 it is not almost periodic.

REMARK 2.3.11. Let $G = \mathbb{R}$ be the real line. Consider the following three classes functions:

(i) periodic functions;

(ii) almost periodic functions;

(iii) functions with ϵ-periods for every $\epsilon > 0$.

Then in this case, each of the above classes of functions is more general than the proceeding one. In other words, every periodic function is almost periodic. And every almost periodic function has ϵ-periods for every ϵ. However, the function $f(x) = e^{-x^2}$ is in the third class (Example 2.2.3 (b)) but is not almost periodic (Example 2.3.9 (b)). And the function $f(x) = \cos x + \cos\sqrt{2}x$ is almost periodic but is not periodic.

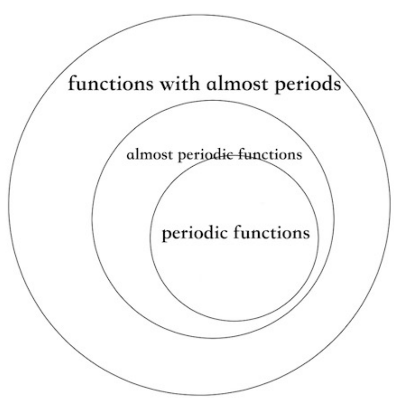

FIGURE 3. The relationship between three classes of functions

CHAPTER 3

Further Properties of Almost Periodic Functions

Almost periodic functions on topological groups can be studied from the point of view of theory of continuous functions on compact groups. This point of view is particularly useful when investigating the approximation properties of almost periodic functions.

The central idea is to associate a compact group $b(G)$ to a given topological group G with the property that almost periodic functions on G are exactly those continuous functions on G which have a unique extension to $b(G)$.

In this chapter, we will present two different constructions of the Bohr compactification $b(G)$.

In the first section, we will show a construction given by Weil [**45**], which uses the finite-dimensional continuous unitary representations of topological group G. Weil's construction is also discussed by Dixmier [**17**], which is our main reference for this section. In Section 3.2, we discuss an alternative construction due to Loomis.

3.1. Weil's Construction of the Bohr Compactification

THEOREM 3.1.1. *Let G be a topological group, then there exist a compact group Σ and a continuous homomorphism $\alpha : G \to \Sigma$ satisfying the following properties:*
(i) $\alpha(G)$ is dense in Σ;
(ii) for every compact group Σ' and every continuous homomorphism $\alpha' : G \to \Sigma'$, there exists a unique continuous homomorphism $\beta : \Sigma \to \Sigma'$ such that $\alpha' = \beta \circ \alpha$.
The pair (α, Σ) is determined up to isomorphism by these properties.

PROOF. Let $(\pi_i)_{i \in I}$ be the family of all finite-dimensional, continuous, unitary representations of G. We denote the representation space of π_i by H_{π_i}. For every

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & \Sigma \arrow[d, "\beta"] \\ & \Sigma' \end{tikzcd}$$

FIGURE 4. Universal property

dimension n, there exists at least one such representation given by $\pi(s) = I_{\mathbb{C}^n}$ for every $s \in G$.

Let U_i be the unitary group of H_{π_i}, that is, the group of all unitary operators on H_{π_i}. Since H_{π_i} is finite-dimensional, we may put $\dim H_{\pi_i} = n_i$. Here we claim that $\mathscr{L}(H_{\pi_i})$ can be identified with $\mathbb{C}^{{n_i}^2}$ and the strong operator topology can be identified with norm topology. To prove the claim, we will show that $\mathscr{L}(H_{\pi_i})$ with the strong operator topology is normable if $\dim H = n < \infty$. According to Theorem 1.5.4, we only need to show the zero operator of $\mathscr{L}(H_{\pi_i})$ has a convex bounded neighborhood. We consider the open set $V = \{T \in \mathscr{L}(H_{\pi_i}) : \|Te_i\| < 1 : i = 1, 2, \ldots, n\}$, where $\{e_1, \ldots, e_n\}$ is basis of H. It is easy to check that V is a convex neighborhood of the zero operator. Next we show it is bounded. By Theorem 1.5.3 (*ii*), we only need to prove that every seminorm $p_\xi(\xi \in H)$, $p_\xi(T) = \|T\xi\|$, is a bounded function on U. Letting $\xi = \sum\limits_{i=1}^{n} \alpha_i e_i$, then for every $T \in U$, we have $|p_\xi(T)| = \|T\xi\| \leqslant \sum\limits_{i=1}^{n} |\alpha_i| \|Te_i\| \leqslant n \sum\limits_{i=1}^{n} |\alpha_i| < \infty$. Therefore, $\mathscr{L}(H_{\pi_i})$ with the strong operator topology is isomorphic to $\mathbb{C}^{{n_i}^2}$ with the norm topology.

According to Heine-Borel theorem (Theorem 1.2.12), U_i is compact in $\mathscr{L}(H_{\pi_i})$ with respect to strong operator topology since U_i is a closed bounded subset of $\mathscr{L}(H_{\pi_i})$. Let $U = \prod\limits_{i \in I} U_i$ be the product of U_i, then U is a compact topological group (compactness follows from Tychonoff's Theorem (Theorem 1.2.15)). Let α be the continuous homomorphism (the continuity follows from ([**31**], page 91))

$$\alpha : G \longrightarrow U, \quad s \mapsto \prod_{i \in I} \pi_i(s),$$

and let Σ be the closure of $\alpha(G)$ in U, which is a compact subgroup of U. We will show that (α, Σ) satisfies the properties of the theorem.

Let $\varSigma'$ be a compact group and $\alpha' : G \to \varSigma'$ be a continuous homomorphism. Since $\alpha(G)$ is dense in $\varSigma$, any continuous homomorphism $\beta : \varSigma \to \varSigma'$ such that $\alpha' = \beta \circ \alpha$ is unique. The reason is as follows: suppose there exists another continuous homomorphism $\beta' : \varSigma \to \varSigma'$ such that $\alpha' = \beta' \circ \alpha$. Since $\varSigma = \overline{\alpha(G)}$, for any $t \in \varSigma$, there exists a net $(s_k)_k \in G$ such that $\lim \alpha(s_k) = t$. Then

$$\beta'(t) = \lim \beta' \circ \alpha(s_k) = \lim \beta \circ \alpha(s_k) = \beta(t),$$

which means $\beta' = \beta$. This proves the uniqueness of the continuous homomorphism β.

Next we prove the existence of β. We recall that the intersection of kernels of continuous unitary representations of finite-dimension of $\varSigma'$ is equal to e (Gelfand-Raikov Theorem 1.6.10). Hence $\varSigma'$ can be identified, as in the case of Σ, with a subgroup of the group of $\Pi_j V_j$, where each V_j is the unitary group of a finite-dimensional Hilbert space. More precisely, let $(\sigma_j)_{j\in J}$ be the family of continuous, unitary, finite-dimensional representations of Σ', then the identification of Σ' with a subgroup of $\Pi_j V_j$ is given by

$$\sigma : \Sigma' \to \sigma(\Sigma'), \quad y \mapsto \Pi_j \sigma_j(y).$$

Since Σ' is compact, $\sigma_j(\Sigma')$ is compact for each j, and hence by Tychonoff's theorem, $\sigma(\Sigma')$ is also compact. Since $\ker \sigma = \bigcap\limits_j \ker \sigma_j = \{e\}$, σ is injective. It is trivial that σ is surjective onto its image. Therefore, σ is a homeomorphism. Thus we may put $\alpha' = \rho$ where $\rho : s \to \prod_j \rho_j(s)$, where each ρ_j is a continuous homomorphism of G into V_j.

Since ρ_j is equivalent to one of the representations π_i, there exists a continuous homomorphism $T_{ji} : U_i \to V_j$ such that $\rho_j = T_{ji} \circ \pi_i$. Let $pr_i : \varSigma \to U_i$ be the canonical projection, and define

$$\beta_j = T_{ji} \circ pr_i : \Sigma \to V_j.$$

$$\begin{tikzcd} G \arrow[r, "\pi_i"] \arrow[dr, "\rho_j"'] & U_i \arrow[d, "T_{ji}"] \\ & V_j \end{tikzcd}$$

FIGURE 5. T_{ji} satisfying $\rho_j = T_{ji} \circ \pi_i$

Then

$$(\beta_j \circ \alpha)(s) = \beta_j(\alpha(s)) = T_{ji} \circ pr_i(\alpha(s)) = T_{ji}(\pi_i(s)) = \rho_j(s),$$

for $s \in G$.

We define $\beta : \Sigma \longrightarrow \prod_j V_j$, by $\beta(t) = \prod_j \beta_j(t)$. Since each β_j is a continuous homomorphism, β is a continuous homomorphism. Moreover,

$$\beta \circ \alpha(s) = \beta(\alpha(s)) = \prod_j \beta_j(\alpha(s)) = \prod_j \rho_j(s) = \alpha'(s), \quad \text{for all } s \in G,$$

and hence $\alpha' = \beta \circ \alpha$ and $\beta(\Sigma) \subset \Sigma'$.

The uniqueness up to isomorphism of the pair $(\varSigma, \alpha)$ is an easy consequence of the universality property. In fact, let $(\varSigma_1, \alpha_1)$ be another solution, then there exist continuous homomorphisms

$$\beta : \varSigma \to \varSigma_1, \quad \beta_1 : \varSigma_1 \to \varSigma$$

such that $\alpha_1 = \beta \circ \alpha$, $\alpha = \beta_1 \circ \alpha_1$, then $\alpha = (\beta_1 \circ \beta) \circ \alpha$, hence $\beta_1 \circ \beta$ is the identity map on $\varSigma$. Similarly, $\beta \circ \beta_1$ is the identity map on $\varSigma'$, thus $\beta : \varSigma \to \varSigma_1$ is an isomorphism which transforms α to α_1. $\square$

DEFINITION 3.1.2. The group $\varSigma$ in the above theorem is called the *Bohr compactification* of G, and will be denoted by $b(G)$. The mapping α is called the *canonical homomorphism of* G *into* $b(G)$.

The group $b(G)$ is also called the *almost periodic compactification* of G.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\pi"'] & b(G) \arrow[d, "\rho"] \\ & \mathscr{L}(H_\pi) \end{tikzcd}$$

FIGURE 6. ρ satisfying $\pi = \rho \circ \alpha$

THEOREM 3.1.3. *For every finite-dimensional, continuous, unitary representation ρ of $b(G)$, let $\rho' = \rho \circ \alpha$, which is a finite-dimensional, continuous, unitary representation of G. Then*
(i) the map $\rho \to \rho'$ is a bijection between the set of all equivalence classes of finite-dimensional, continuous, unitary representations of $b(G)$ and the set of all equivalence classes of finite-dimensional, continuous, unitary representations of G;
(ii) ρ is topologically irreducible if and only if ρ' is so;
(iii) if $\xi, \eta \in H_\rho = H_{\rho'}$, then $\rho'_{\xi,\eta} = (\rho \circ \alpha)_{\xi,\eta} = \rho_{\xi,\eta} \circ \alpha$.

PROOF. (i) Let ρ' be equivalent to ρ'_1 so that for some unitary operator T, $\rho'(s) = T^*\rho'_1(s)T$ for all $s \in G$. Since $\overline{\alpha(G)} = b(G)$, it follows that $\rho(x) = T^*\rho_1(x)T$ for all $x \in b(G)$. Thus ρ is equivalent to ρ_1. Therefore, the map $\rho \to \rho'$ is injective.

If π is a finite-dimensional, continuous, unitary representation of G, then we can view π as a continuous homomorphism of G into the compact group of unitary operators on H_π, and it follows from Theorem 3.1.1 (ii) that there exists a continuous unitary representation ρ of $b(G)$ on H_π such that $\pi = \rho \circ \alpha$, thus the map $\rho \to \rho'$ is surjective. (ii) If ρ is irreducible, then there are no proper subspace $W \subset H$ which is invariant under the action of $\{\rho(s) : s \in b(G)\}$. Therefore $\rho' = \rho \circ \alpha : G \longrightarrow \mathscr{L}(H)$ is also irreducible since $\alpha(G)$ is dense in $b(G)$. If ρ' is irreducible, then there are no proper subspace $V \subset H$ that is invariant under the action of $\{\rho'(t) : t \in G\}$. Therefore $\rho : G \longrightarrow \mathscr{L}(H)$ is also irreducible since $\alpha(G) \subset b(G)$.
(iii) For any $s \in G$, we have

$$\rho'_{\xi,\eta}(s) = (\rho \circ \alpha)_{\xi,\eta}(s) = (\rho \circ \alpha(s)\xi|\eta) = (\rho(\alpha(s))\xi|\eta) = \rho_{\xi,\eta} \circ \alpha(s).$$

Therefore, $(\rho \circ \alpha)_{\xi,\eta} = \rho_{\xi,\eta} \circ \alpha$. $\square$

In Theorem 2.1.1, we used total boundedness and ϵ-meshes to prove the equivalence of compactness of left and right translations. Now with the useful tool of "Bohr compactification", we have an another proof of this equivalence. The theorem also characterizes almost periodic functions as those continuous functions which can be lifted to a continuous function on $b(G)$. This latter property is the main reason behind the usefulness of the Bohr compactification.

THEOREM 3.1.4. *Let G be a topological group, $b(G)$ the Bohr compactification, α the canonical homomorphism of G into $b(G)$, and $f \in C^b(G)$. Then the following conditions are equivalent:*
(i) The set of ${}_sf$ $(s \in G)$ is relatively compact in $C^b(G)$.
(ii) The set of f_t $(t \in G)$ is relatively compact in $C^b(G)$.
(iii) The set of ${}_sf_t$ $(s, t \in G)$ is relatively compact in $C^b(G)$.
(iv) There exists a unique complex-valued continuous function g on $b(G)$ such that $f = g \circ \alpha$.

PROOF. $(iv) \Rightarrow (iii)$: Let $g \in C(b(G))$ and $f = g \circ \alpha$. Since $b(G)$ is compact, g is uniformly continuous, and hence the map $(\sigma, \iota) \to {}_\sigma g_\iota$ of $b(G) \times b(G)$ into the Banach space $C(b(G))$ is continuous (Theorem A.0.6). Hence the set of all ${}_\sigma g_\iota$ is compact since $b(G) \times b(G)$ is compact. The image of this set in $C^b(G)$ by the isometric map $h \to h \circ \alpha$ is compact and contains the set of ${}_sf_t$ $(s, t \in G)$ because for $x \in G$, we have

$$\begin{aligned} &{}_{\alpha(s)}g_{\alpha(t)} \circ \alpha(x) = {}_{\alpha(s)}g_{\alpha(t)}(\alpha(x)) \\ =&g(\alpha(s)\alpha(x)\alpha(t)) = g(\alpha(sxt)) = f(sxt) = {}_sf_t(x). \end{aligned}$$

$(iii) \Rightarrow (i)$ and (ii) are obvious.
$(i) \Rightarrow (iv)$ ($(ii) \Rightarrow (iv)$ is similar): For every $g \in C^b(G)$ and every $s \in G$, let $\psi(s)g =_{s^{-1}} g$. Then $\psi(s)$ is a linear isometric map of $C^b(G)$ onto $C^b(G)$, and $\psi(ss') = \psi(s)\psi(s')$. Let A be the set of all ${}_sf$ $(s \in G)$, it is clear that every $\psi(s)$ induces a bijection from A onto A, hence also a bijection $\varphi(s)$ of $\overline{A}$ onto $\overline{A}$. Each $\varphi(s)$ is isometric, and $\overline{A}$ is compact if we assume (i) holds. Hence $\varphi(G)$ is relatively compact

in the set $C(\overline{A}, \overline{A})$ of continuous functions from $\overline{A}$ into itself, equipped with the topology of uniform convergence, according to Ascoli's Theorem (Theorem 1.2.6). Hence the closure of $\varphi(G)$ in $C(\overline{A}, \overline{A})$ is a compact group $\mathit{\Gamma}$ of homeomorphisms of $\overline{A}$.

The mapping φ is a homomorphism of G into $\mathit{\Gamma}$, next we show that it is continuous. It suffices to prove that, when $s \to e$, $\varphi(s)$ converges to $\varphi(e)$ in the topology of uniform convergence on $\overline{A}$. Let $g \in \overline{A}$ and V be an open neighborhood of g in $\overline{A}$, then $\overline{A} - V$ is compact. If $h \in \overline{A} - V$, we have $|g(u) - h(u)| > 0$ at at least one point $u \in G$.

We claim that there exist an open neighborhood V_h of h in $\overline{A}$ and a neighborhood W_h of e in G such that $\varphi(s)g \notin V_h$ for $s \in W_h$. Let $r = |g(u) - h(u)| > 0$. Define $V_h = B_{r/2}(h)$, the open ball with center at h and radius $r/2$ in $\overline{A}$. In other words,

$$V_h = \{k \in \overline{A} : \|k - h\|_{\sup} < r/2\}.$$

Let W_h be a symmetric neighborhood of e such that if $s \in W_h$ then

$$|g(s^{-1}u) - g(u)| < r/2$$

(such a neighborhood exists since g is continuous at u). We claim that $\varphi(s)g \notin V_h$ for all $s \in W_h$. In fact, if $\varphi(s)g \in V_h$ for some $s \in W_h$, then $\|\varphi(s)g - h\|_{\sup} < r/2$, and in particular, $|g(s^{-1}u) - h(u)| < r/2$. Thus,

$$|g(u) - h(u)| \leqslant |g(u) - g(s^{-1}u)| + |g(s^{-1}u) - h(u)| < r/2 + r/2 = r,$$

which contradicts to the definition of r.

We may cover $\overline{A} - V$ by a finite number of neighborhoods $V_{h_1}, ..., V_{h_n}$, then for $s \in W_{h_1} \cap \cdots \cap W_{h_n}$, we have $\varphi(s)g \notin V_{h_1} \cup \cdots \cup V_{h_n}$, hence $\varphi(s)g \in V$ and thus we have shown that φ is a continuous homomorphism from G into $\mathit{\Gamma}$.

By Theorem 3.1.1 (ii), there exists a continuous homomorphism $\beta : b(G) \to \mathit{\Gamma}$ such that $\varphi = \beta \circ \alpha$. The complex-valued function $g : \sigma \mapsto (\beta(\sigma^{-1})f)(e)$ is continuous on $b(G)$, and

$$f(s) = (\varphi(s^{-1})f)(e) = (\beta(\alpha(s^{-1}))f)(e) = (\beta(\alpha(s)^{-1})f)(e) = g \circ \alpha(s).$$

This proves (iv). $\square$

Combining Theorem 2.1.1, Definition 2.1.2, Theorem 3.1.3 (iii) and Theorem 3.1.4, it is easy to get the following corollary.

COROLLARY 3.1.5. *Let G be a topological group and $f \in C^b(G)$. Then*
(i) $f \in AP(G)$ if and only if there exists a unique complex-valued continuous function g on $b(G)$ such that $f = g \circ \alpha$.
(ii) The coordinate functions of finite-dimensional continuous unitary representations of G are almost periodic.

3.2. Loomis' Alternative Construction of Bohr Compactification

In this section, we discuss another construction of Bohr compactification given by Loomis [**34**]. We follow the exposition given by Kaniuth [**29**].

Let $\Delta(AP(G))$ denote the spectrum of $AP(G)$. The idea is to define a group structure on $\Delta(AP(G))$ and show that the resulted compact group can serve as a model for Bohr compactification.

According to Theorem 1.7.3, the Gelfand homomorphism is an isometric $*$-isomorphism from $AP(G)$ onto $C(\Delta(AP(G)))$. For each $x \in G$, we define an element $\varphi_x \in \Delta(AP(G))$ by $\varphi_x(f) = f(x)$ where $f \in AP(G)$.

LEMMA 3.2.1. *The mapping $\phi : x \to \varphi_x$ from G into $\Delta(AP(G))$ is continuous and has dense range.*

PROOF. First we show ϕ is continuous. Let $x_\alpha \to x$. We need to show $\varphi_{x_\alpha} \to \varphi_x$ in $\Delta(AP(G))$. Since $\Delta(AP(G))$ carries the w^*-topology, it suffices to show that $\varphi_{x_\alpha}(f) \to \varphi_x(f)$ for each $f \in AP(G)$. This follows immediately from continuity of f, since $\varphi_{x_\alpha}(f) = f(x_\alpha) \to f(x) = \varphi_x(f)$.

Suppose that there exists a nonempty open subset U of $\Delta(AP(G))$ such that $U \bigcap \phi(G) = \varnothing$. Then by Urysohn's Lemma (Theorem 1.2.11), there exists a continuous function g on $\Delta(AP(G))$ such that $g \neq 0$ and $g|_{\Delta(AP(G))\setminus U} = 0$. Since the Gelfand homomorphism is an isometric $*$-isomorphism from $AP(G)$ onto $C(\Delta(AP(G)))$, we have $g = \hat{f}$ for some $f \in AP(G)$. However, $f(x) = \varphi_x(f) = \hat{f}(\varphi_x) = g(\varphi_x) = 0$ for all $x \in G$, contradicting $f \neq 0$. Thus $\phi(G)$ is dense in $\Delta(AP(G))$. $\square$

Remark 3.2.2. It is worth to point out that the second part of the proof works for any unital Banach subalgebra $A \subset C^b(X)$, where X is any topological space. The argument shows that the map $G \longrightarrow \Delta(A)$, $x \mapsto \varphi_x$ has dense range in $\Delta(A)$.

Let $\varphi, \psi \in \Delta(AP(G))$. For neighborhoods U of φ and V of ψ in $\Delta(AP(G))$, let

$$\Delta_{U,V} = \{\varphi_{xy} : x, y \in G \,\text{such that}\, \varphi_x \in U \,\text{and}\, \varphi_y \in V\}.$$

Then $\Delta_{U,V} \neq \varnothing$ since $\phi(G)$ is dense in $\Delta(AP(G))$. Let $\mathcal{U}$ and $\mathcal{V}$ be the set of all neighborhoods of φ and ψ, respectively. Because $\Delta_{U_1,V_1} \subseteq \Delta_{U_2,V_2}$ whenever $U_1 \subseteq U_2, V_1 \subseteq V_2$, the collection of all closed subsets $\overline{\Delta_{U,V}}$ of $\Delta(AP(G))$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$, has the finite intersection property (Definition 1.2.8). Since $\Delta(AP(G))$ is compact, according to Theorem 1.2.9 the set

$$\Delta_{\varphi,\psi} = \bigcap \{\overline{\Delta_{U,V}} : U \in \mathcal{U}, V \in \mathcal{V}\}$$

is nonempty.

Lemma 3.2.3. *Let $C_f = \{{}_af : a \in G\}$, $D_f = \{f_a : a \in G\}$, $\alpha, \beta \in \Delta(AP(G))$ and $f \in AP(G)$. Let $\epsilon > 0$ and let $\{{}_{x_1}f, \ldots, {}_{x_n}f\}$ be an ϵ-mesh for C_f and $\{f_{y_1}, \ldots, f_{y_m}\}$ an ϵ-mesh for D_f. Define neighborhoods U and V of α and β, respectively, by $U = U(\alpha, f_{y_1}, \ldots, f_{y_m}, \epsilon) = \{\gamma \in \Delta(AP(G)) : |\langle \alpha - \gamma, f_{y_j}\rangle| < \epsilon, \text{for } j = 1, \ldots, m\}$ and $V = U(\beta, {}_{x_1}f, \ldots, {}_{x_n}f, \epsilon) = \{\gamma \in \Delta(AP(G)) : |\langle \beta - \gamma, {}_{x_i}f\rangle| < \epsilon, \text{for } i = 1, \ldots, n\}$. If $x, a, y, b \in G$ are such that $\varphi_x, \varphi_a \in U$ and $\varphi_y, \varphi_b \in V$, then $|\varphi_{xy}(f) - \varphi_{ab}(f)| < 8\epsilon$.*

PROOF. Choose $j \in \{1, ..., n\}$ and $k \in \{1, ..., m\}$ such that $\|_x f - {}_{x_j} f\|_{\sup} < \epsilon$ and $\|f_b - f_{y_k}\|_{\sup} < \epsilon$. Then we have

$$
\begin{aligned}
&|\varphi_{xy}(f) - \varphi_{ab}(f)| \\
&\leqslant |f(xy) - f(xb)| + |f(xb) - f(ab)| \\
&= |_x f(y) - {}_x f(b)| + |f_b(x) - f_b(a)| \\
&\leqslant |_x f(y) - {}_{x_j} f(y)| + |_{x_j} f(y) - {}_{x_j} f(b)| + |_{x_j} f(b) - {}_x f(b)| + \\
&|f_b(x) - f_{y_k}(x)| + |f_{y_k}(x) - f_{y_k}(a)| + |f_{y_k}(a) - f_b(a)| \\
&\leqslant 2\|_x f - {}_{x_j} f\|_{\sup} + |_{x_j} f(y) - {}_{x_j} f(b)| + 2\|f_b - f_{y_k}\|_{\sup} + \\
&|f_{y_k}(x) - f_{y_k}(a)| \\
&\leqslant 4\epsilon + |\varphi_y({}_{x_j} f) - \varphi_b({}_{x_j} f)| + |\varphi_x(f_{y_k}) - \varphi_a(f_{y_k})|.
\end{aligned}
$$

Now, since φ_x, $\varphi_a \in U$ and $\varphi_y, \varphi_b \in V$,

$$|\varphi_x(f_{y_k}) - \varphi_a(f_{y_k})| < 2\epsilon \quad \text{and} \quad |\varphi_y({}_{x_j} f) - \varphi_b({}_{x_j} f)| < 2\epsilon.$$

□

COROLLARY 3.2.4. *For each pair of elements φ, ψ of $\Delta(AP(G))$, $\Delta_{\varphi,\psi}$ is a singleton.*

PROOF. Let $\alpha, \beta \in \Delta_{\varphi,\psi}$ and $f \in AP(G)$, we want to show that $|\alpha(f) - \beta(f)| < \delta$ for each $\delta > 0$. Fix δ and let $\epsilon = \delta/24$. Let U and V be the neighborhoods of α and β defined in Lemma 3.2.3. By definition of $\Delta_{\varphi,\psi}$, $\alpha, \beta \in \overline{\Delta_{U,V}}$, so there exist $x, a, y, b \in G$ such that $\varphi_x, \varphi_a \in U, \varphi_y, \varphi_b \in V$, and

$$|\alpha(f) - \varphi_{xy}(f)| < \delta/3 \quad \text{and} \quad |\beta(f) - \varphi_{ab}(f)| < \delta/3.$$

Now, by Lemma 3.2.3 we have

$$
\begin{aligned}
&|\alpha(f) - \beta(f)| \leqslant |\alpha(f) - \varphi_{xy}(f)| + |\varphi_{xy}(f) - \varphi_{ab}(f)| + |\varphi_{ab}(f) - \beta(f)| \\
&\leqslant \delta/3 + \delta/3 + 8\epsilon = \delta.
\end{aligned}
$$

□

The group structure of $\Delta(AP(G))$ is given by the following theorem.

THEOREM 3.2.5. *Let G be a topological group. For $\varphi, \psi \in \Delta(AP(G))$, let $\varphi\psi$ denote the unique element of $\Delta_{\varphi,\psi}$. Then the map*

$$\Delta(AP(G)) \times \Delta(AP(G)) \longrightarrow \Delta(AP(G)), \qquad (\varphi, \psi) \mapsto \varphi\psi$$

turns $\Delta(AP(G))$ into a compact group. Furthermore, $\varphi_x\varphi_y = \varphi_{xy}$ for $x, y \in G$.

PROOF. It follows from definitions that $\varphi_{xy} \in \Delta_{\varphi_x,\varphi_y}$. According to Corollary 3.2.4, we have $\Delta_{\varphi_x,\varphi_y} = \{\varphi_{xy}\}$, and thus $\varphi_x\varphi_y = \varphi_{xy}$ for $x, y \in G$.

We show that multiplication on $\Delta(AP(G))$ is continuous. Let α and β be two elements of $\Delta(AP(G))$. It suffices to show that given $\delta > 0$ and $f_1, \dots, f_n \in AP(G)$, there exists neighborhoods U of α and V of β in $\Delta(AP(G))$, respectively, such that $|\varphi\psi(f_j) - \alpha\beta(f_j)| < \delta$ for all $\varphi \in U$ and $\psi \in V$ and $j = 1, 2, \dots, n$. Let $\epsilon = \delta/10$ and for any $\rho \in \Delta(AP(G))$, let

$$W_\rho = \{\gamma \in \Delta(AP(G)) : |\gamma(f_j) - \rho(f_j)| < \epsilon \text{ for } 1 \leqslant j \leqslant n\}.$$

For each $j = 1, 2, ..., n$, Lemma 3.2.3 provides neighborhoods U_j of α and V_j of β such that $|\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| < 8\epsilon$ whenever $x, a, y, b \in G$ are such that $\varphi_x, \varphi_a \in U_j$ and $\varphi_y, \varphi_b \in V_j$. Let $U = \bigcap_{j=1}^{n} U_j$ and $V = \bigcap_{j=1}^{n} V_j$, since $\alpha\beta \in \overline{\Delta_{U,V}}$, we have $\Delta_{U,V} \bigcap W_{\alpha\beta} \neq \varnothing$. Now, let $\varphi \in U$ and $\psi \in V$ be arbitrary. Then $\Delta_{U,V} \bigcap W_{\varphi\psi} \neq \varnothing$ and hence there exist $x, y \in G$ such that $\varphi_x \in U$, $\varphi_y \in V$, and $\varphi_{xy} \in W_{\varphi\psi}$. Therefore we have

$$|\varphi_{ab}(f_j) - \alpha\beta(f_j)| < \epsilon \quad \text{and} \quad |\varphi_{xy}(f_j) - \varphi\psi(f_j)| < \epsilon \quad \text{for} \quad j = 1, \dots, n.$$

Because $\varphi_x, \varphi_a \in U$ and $\varphi_y, \varphi_b \in V$, $|\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| < 8\epsilon$ for $j = 1, 2, \dots, n$. Combining these inequalities gives

$$\begin{aligned} &|\varphi\psi(f_j) - \alpha\beta(f_j)| \\ &\leqslant |\varphi\psi(f_j) - \varphi_{xy}(f_j)| + |\varphi_{xy}(f_j) - \varphi_{ab}(f_j)| + |\varphi_{ab}(f_j) - \alpha\beta(f_j)| \end{aligned}$$

$$< 10\epsilon = \delta.$$

Thus the multiplication on $\Delta(AP(G))$ is continuous.

Next, we want to show the existence and continuity of inverses in $\Delta(AP(G))$. Let $\varphi \in AP(G)$ and let $(x_\alpha)_\alpha$ be a net in G such that $\varphi_{x_\alpha} \to \varphi$ in $\Delta(AP(G))$. We show that the limit does not depend on the choice of the net (x_α) in G but only on the fact that $\varphi_{x_\alpha} \to \varphi$. Let $f \in AP(G)$ and $\epsilon > 0$. Then there exist $a_1, \ldots, a_n \in G$ such that the functions ${}_{a_i}f_{a_j}$, $1 \leqslant i, j \leqslant n$, form an $\epsilon/3$ -mesh for the set of all two-sided translates ${}_af_b, a, b \in G$. Define a neighborhood U of φ in $\Delta(AP(G))$ by

$$U = \{\psi \in \Delta(AP(G)) : |\psi({}_{a_i}f_{a_j}) - \varphi({}_{a_i}f_{a_j})| < \epsilon/3, 1 \leqslant i, j \leqslant n\}.$$

If x and y are elements of G such that $\varphi_x, \varphi_y \in U$, then $|f(a_ixa_j) - f(a_iya_j)| < \epsilon/3$ $(1 \leqslant i, j \leqslant n)$, and hence $|f(axb) - f(ayb)| < \epsilon$ for all $a, b \in G$. Taking $a = x^{-1}$ and $b = y^{-1}$, this becomes $|f(y^{-1}) - f(x^{-1})| < \epsilon$. This shows that, for each $f \in AP(G)$, the net $(\varphi_{x_\alpha^{-1}}(f))_\alpha = (f(x_\alpha^{-1}))_\alpha$ forms a Cauchy net in $\mathbb{C}$ and that $\lim\limits_\alpha \varphi_{x_\alpha^{-1}}(f) = \lim\limits_\beta \varphi_{y_\beta^{-1}}(f)$, where $(y_\beta)_\beta$ is another net in G such that $\varphi_{y_\beta} \to \varphi$ in $\Delta(AP(G))$. Thus we can define a map $\varphi^{-1} : AP(G) \to \mathbb{C}$ by $\varphi^{-1}(f) = \lim\limits_\alpha \varphi_{x_\alpha^{-1}}(f)$ $(f \in AP(G))$ by taking $(x_\alpha)_\alpha$ to be any net in G such that $\varphi_{x_\alpha} \to \varphi$. It is clear that $\varphi^{-1} \in \Delta(AP(G))$ and that $\varphi_x^{-1} = \varphi_{x^{-1}}$ for every $x \in G$. Since multiplication in $\Delta(AP(G))$ is continuous and $\varphi_{ab} = \varphi_a\varphi_b$ for all $a, b \in G$, it follows that

$$\varphi\varphi^{-1} = \lim_\alpha \varphi_{x_\alpha} \cdot \lim_\alpha \varphi_{x_\alpha}^{-1} = \lim_\alpha (\varphi_{x_\alpha x_\alpha^{-1}}) = \varphi_e.$$

Consequently, $\Delta(AP(G))$ is a group and φ^{-1} is the inverse of φ.

Finally, the map $\varphi \to \varphi^{-1}$ from $\Delta(AP(G))$ into $\Delta(AP(G))$ is continuous. Let $\psi \in \Delta(AP(G))$, $f \in AP(G)$ and $\delta > 0$. Define $g \in AP(G)$ by $g(x) = f(x^{-1})$. If $\varphi \in \Delta(AP(G))$ and $x, y \in G$ are such that

$$|\varphi(g) - \psi(g)| < \delta, \quad |\varphi_x(g) - \varphi(g)| < \delta \quad \text{and} \quad |\varphi_y(g) - \psi(g)| < \delta,$$

then $|\varphi_{x^{-1}}(f) - \varphi_{y^{-1}}(f)| = |\varphi_x(g) - \varphi_y(g)| < 3\delta$ and hence

$$|\varphi^{-1}(f) - \psi^{-1}(f)| \leqslant |\varphi^{-1}(f) - \varphi_{x^{-1}}(f)| + |\varphi_{y^{-1}}(f) - \psi^{-1}(f)| + 3\delta.$$

As we have shown above, $\varphi_{x^{-1}}$ and $\varphi_{y^{-1}} \to \psi^{-1}$ whenever $\varphi_x \to \varphi$ and $\varphi_y \to \psi$. Hence it follow that $\varphi \to \varphi^{-1}$ is continuous. $\square$

We have thus achieved making $\Delta(AP(G))$ a compact group having the following properties:
(i) The map $\phi : G \to \Delta(AP(G))$ is a continuous homomorphism with dense range.
(ii) A bounded continuous function f on G is almost periodic if and only if there exists a function $\hat{f} \in C(\Delta(AP(G)))$ such that $f(x) = \hat{f}(\phi(x))$ for all $x \in G$.

Remark 3.2.6. Let $\Delta = \Delta(AP(G))$ and suppose that Δ' is a second compact group and ϕ' is a homomorphism satisfying the analogous properties (i) and (ii). Then $\hat{f}' \to \hat{f}$ is an algebraic isomorphism of $C(\Delta')$ onto $C(\Delta)$. Let $\delta : \Delta \to \Delta'$ be the associated homeomorphism; that is, $\delta(\varphi)(\hat{f}') = \varphi(\hat{f})$ for $\varphi \in \Delta$ and $f \in AP(G)$. Then

$$\delta(\phi(x))(\hat{f}') = \phi(x)(\hat{f}) = f(x) = \phi'(x)(\hat{f}') \quad \text{for all } x \in G \text{ and } f \in AP(G).$$

Thus δ extends the homomorphism $\phi' \circ \phi^{-1} : \phi(G) \to \phi'(G)$. Because δ is a homeomorphism and $\phi(G)$ is dense in Δ, it follows that δ is a topological isomorphism.

It follows from the universal property of the Bohr compactification that $\Delta(AP(G))$ is exactly the group $b(G)$ we defined in Section 3.1, and the homomorphism $\phi : G \to \Delta(AP(G))$ with dense range is the canonical homomorphism of G into $b(G)$.

3.3. Bohr Compactification of Abelian Groups

If G is a locally compact Abelian group, then according to Theorem 1.6.11 every continuous, unitary, irreducible representation of G is one-dimensional. In other words, it is a continuous homomorphism $\gamma : G \longrightarrow \mathbb{T}$. This leads to the following definition.

Definition 3.3.1. Let G be a locally compact Abelian group, a *character* of G is a continuous group homomorphism from G into the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set of all characters of G is denoted by $\widehat{G}$.

Remark 3.3.2. This definition of $\widehat{G}$ is in agreement with the one in Chapter 1 (see Definition 1.6.8) in the case when G is Abelian, since a character of an Abelian group is indeed a continuous unitary irreducible representation of the group G on $\mathbb{C}$.

If γ and $\gamma' \in \widehat{G}$, then $\gamma\gamma' \in \widehat{G}$ and $\gamma\overline{\gamma} = 1_G$ where 1_G is the identity character. Thus $\widehat{G}$ forms a group under the pointwise product of characters. If we equip $\widehat{G}$ with the topology of convergence on compact sets, then $\widehat{G}$ turns into a locally compact Abelian group. We call $\widehat{G}$ the *dual group* of G. For the proof of this result, and the following additional results on the dual group, we refer to sections 4.1 and 4.3 of Folland [**24**].

If $x \in G$ and $\xi \in \widehat{G}$, we use the notation

$$\langle x, \xi \rangle = \xi(x)$$

for the reasons of symmetry. The elements of $\widehat{G}$ are characters on G, but we can equally regard elements of G as characters on $\widehat{G}$. In other words, each $x \in G$ defines a character $\Phi(x)$ on $\widehat{G}$ by

$$\langle \xi, \Phi(x) \rangle = \langle x, \xi \rangle.$$

The following famous result states that G can be identified with $\widehat{\widehat{G}}$:

Theorem 3.3.3 (The Pontrjagin Duality). *The map $\Phi : G \longrightarrow \widehat{\widehat{G}}$ defined above is an isomorphism of topological groups.*

Using the Pontrjagin duality theorem, one can prove the following result, relating the topologies of G and $\widehat{G}$:

Theorem 3.3.4. *Let G be a locally compact Abelian group. Then G is compact if and only if $\widehat{G}$ is discrete. And G is discrete if and only if $\widehat{G}$ is compact.*

If H is a subgroup of G, we define the *annihilator* of H by

$$H^{\perp} = \{\xi \in \widehat{G} : \langle x, \xi \rangle = 1 \text{ for all } x \in H\}.$$

LEMMA 3.3.5. *$H^{\perp}$ is a closed subgroup of $\widehat{G}$.*

PROOF. For any ξ_1, $\xi_2 \in H^{\perp}$, we have $\langle x, \xi_1 \rangle = 1$ and $\langle x, \xi_2 \rangle = 1$ for all $x \in H$, then $\langle x, \xi_1\xi_2 \rangle = \xi_1(x)\xi_2(x) = \langle x, \xi_1 \rangle \cdot \langle x, \xi_2 \rangle = 1$, i.e., $\xi_1\xi_2 \in H^{\perp}$. Let $\{\xi_\alpha\}$ be a net in $H^{\perp}$ such that $\xi_\alpha \to \xi$ for some ξ in $\widehat{G}$, then for every $x \in H$, $\xi_\alpha(x) \to \xi(x)$. However, $\xi_\alpha(x) = 1$ for all α, so $\xi(x) = 1$, i.e., $\xi \in H^{\perp}$. If $\xi \in H^{\perp}$, then clearly $\xi^{-1} = \overline{\xi} \in H^{\perp}$. Therefore, $H^{\perp}$ is a closed subgroup of G. $\square$

The following theorem identifies the dual of a subgroup H and its quotient group G/H.

THEOREM 3.3.6. *Let G be a locally compact Abelian group. Suppose H is a closed subgroup of G. Define $\Phi : (G/H)\widehat{\ } \longrightarrow H^{\perp}$ and $\Psi : \widehat{G}/H^{\perp} \longrightarrow \widehat{H}$ by*

$$\Phi(\eta) = \eta \circ q, \qquad \Psi(\xi H^{\perp}) = \xi|_H,$$

where $q : G \longrightarrow G/H$, $s \mapsto sH$ is the canonical quotient map, and $\xi|_H$ is the restriction of ξ to H. Then Φ and Ψ are isomorphisms of topological groups. In other words, $\widehat{H} \cong \widehat{G}/H^{\perp}$ and $(G/H)\widehat{\ } \cong H^{\perp}$.

COROLLARY 3.3.7. *If H is a subgroup of G such that $H^{\perp} = \{1_G\}$ (the identity element of $\widehat{G}$), then $\overline{H} = G$.*

PROOF. Suppose on the contrary that $\overline{H} \neq G$. Then $\overline{H}$ is a proper closed subgroup of G, and thus $G/\overline{H}$ is a nontrivial group. By Theorem 3.3.6, $\overline{H}^{\perp} \cong (G/\overline{H})\widehat{\ }$. But since $G/\overline{H}$ has more than one elements, so does $(G/\overline{H})\widehat{\ }$ and $\overline{H}^{\perp}$ according to Pontrjagin Duality. But this is impossible since $H^{\perp} = \{1_G\}$ implies $\overline{H}^{\perp} = \{1_G\}$. $\square$

DEFINITION 3.3.8. Let G, G' be two locally compact groups, and let φ be a continuous homomorphism from G to G', then we define $\widehat{\varphi} : \widehat{G'} \to \widehat{G}$ by $\gamma \mapsto \gamma \circ \varphi$ and we call $\widehat{\varphi}$ the *dual map* of φ.

In fact, $\widehat{\varphi}$ is also a continuous homomorphism. Since

$$\widehat{\varphi}(\gamma_1\gamma_2) = (\gamma_1\gamma_2)\circ\varphi = (\gamma_1\circ\varphi)(\gamma_2\circ\varphi) = \widehat{\varphi}(\gamma_1)\widehat{\varphi}(\gamma_2),$$

$\widehat{\varphi}$ is a homomorphism. If $\gamma_\alpha \to \gamma$ in $\widehat{G'}$, then $\gamma_\alpha \to \gamma$ on compact subsets of G'. Now we want to show that $\widehat{\varphi}(\gamma_\alpha) = \gamma_\alpha\circ\varphi \to \gamma\circ\varphi = \widehat{\varphi}(\gamma)$ on compact subsets of G. If $K \subset G$ is compact, then $\varphi(K) \subset G'$ is compact, so $\gamma_\alpha \to \gamma$ uniformly on $\varphi(K)$, i.e., $\gamma_\alpha\circ\varphi \to \gamma\circ\varphi$ uniformly on K.

Now we are ready to give an alternative description of $b(G)$ for a locally compact Abelian group G.

THEOREM 3.3.9. *Let G be a locally compact Abelian group, $\widehat{G}$ be its dual group. Let G' be the compact Abelian group whose dual $\widehat{G'}$ is $\widehat{G}_d$, ($\widehat{G}$ equipped with the discrete topology). Let $\varphi : G \to G'$ be the continuous homomorphism whose dual $\widehat{\varphi} : \widehat{G'} = \widehat{G}_d \to \widehat{G}$ is the identity map. Then G' can be identified with the Bohr compactification $b(G)$, and φ with the canonical homomorphism from G to G'.*

REMARK 3.3.10. With the help of Pontrjagin duality theorem, we can specify G' and φ more directly. In fact, $G' = (\widehat{G}_d)\widehat{}$, and $\varphi = \widehat{I}$ where $I : \widehat{G}_d \to \widehat{G}$ is the identity mapping.

PROOF. To prove this theorem, we will use the uniqueness result in Theorem 3.1.1. To prove that $\overline{\varphi(G)} = G'$, we observe that if $x' \in \widehat{G'} = \widehat{G}_d$ is trivial on $\varphi(G)$, then $\widehat{\varphi}(x') = 1_G$, because for all $s \in G$,

$$\widehat{\varphi}(x')(s) := x'(\varphi(s)) = 1,$$

and thus $x' = 1_G$, since $\widehat{\varphi} = I$ is the identity map. It follows from Corollary 3.3.7 that $\overline{\varphi(G)} = G'$.

Next, suppose H is a compact group and $\psi : G \to H$ is a continuous homomorphism.

We need to prove that there exists a unique continuous homomorphism $\beta : G' \to H$ such that $\psi = \beta\circ\varphi$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & G' = (\widehat{G}_d)\widehat{\ } \arrow[d, "\beta"] \\ & H \end{tikzcd}$$

FIGURE 7. β satisfying $\psi = \beta \circ \varphi$

Replacing H with $\overline{\psi(G)}$, we may assume H is a compact commutative group. Then

$$\widehat{\psi} : \widehat{H} \to \widehat{G},\ \gamma \mapsto \gamma \circ \psi$$

is a continuous homomorphism. Since $\widehat{H}$ is discrete, $\widehat{\psi} : \widehat{H} \to \widehat{G}_d$ is also a continuous homomorphism.

It remains to show there exists a unique homomorphism

$$\widehat{\beta} : \widehat{H} \longrightarrow \widehat{G}_d$$

such that $\widehat{\psi} = \widehat{\varphi} \circ \widehat{\beta}$. However, this is evident since $\widehat{\varphi}$ is the identity map from $\widehat{G}_d$ to $\widehat{G}$, and so we may define $\widehat{\beta} = \widehat{\psi}$. $\square$

So we have shown that the pair (φ, G') satisfies the universal property in Theorem 3.1.1, and therefore $(\varphi, G') \cong (\alpha, b(G))$.

3.4. Some Applications of Bohr Compactification

Let G be a topological group. For $f \in AP(G)$, let $\widehat{f} \in C(b(G))$ be the unique function such that $\widehat{f} \circ \alpha = f$ (Theorem 3.1.4).

THEOREM 3.4.1. *The C^*-algebra $AP(G)$ is isometrically isomorphic to $C(b(G))$, under the map*

$$\kappa : AP(G) \longrightarrow C(b(G)), \qquad f \mapsto \widehat{f}.$$

PROOF. κ is an algebra homomorphism, since

$$(\widehat{f}_1 \widehat{f}_2) \circ \alpha = (\widehat{f}_1 \circ \alpha)(\widehat{f}_2 \circ \alpha) = f_1 f_2 = \widehat{f_1 f_2} \circ \alpha,$$

and since α has a dense range in $b(G)$, it follows that $\widehat{f_1 f_2} = \widehat{f_1}\widehat{f_2}$. This map is injective since $\widehat{f} = 0$ implies that $\widehat{f} \circ \alpha = 0$ and hence $f = 0$. The map is surjective since if $g \in C(b(G))$ then $f = g \circ \alpha \in C^b(G)$, and moreover, $f \in AP(G)$ and $\widehat{f} = g$ by Theorem 3.1.4. Finally, κ is isometric since

$$\|\widehat{f}\|_{\sup} = \sup_{x \in b(G)} |\widehat{f}(x)| = \sup_{t \in G} |\widehat{f}(\alpha(t))| = \sup_{t \in G} |f(t)| = \|f\|_{\sup}.$$

$\square$

The following is a general result on Banach algebras.

THEOREM 3.4.2. *Let A and B be two Banach algebras and $\kappa : A \to B$ be an isomorphism. Then mapping*

$$\theta : \Delta(B) \longrightarrow \Delta(A), \qquad \psi \mapsto \kappa^*(\psi) = \psi \circ \kappa$$

is a homeomorphism between the spectrum of the algebras.

We recall that an isomorphism between two Banach algebras is a continuous bijective algebra homomorphism. The inverse of such a map is automatically continuous by the inverse mapping theorem (Theorem 1.2.10). The map $\kappa^* : B^* \longrightarrow A^*$ is the adjoint map, and so θ is just the restriction of κ^* to $\Delta(B)$.

PROOF. By the definition of κ^*, we have $\kappa^*(\psi) \in A^*$. To show that $\kappa^*(\psi)$ is in $\Delta(A)$, let $aa' \in A$, then

$$\begin{aligned}
\langle \theta(\psi), aa' \rangle = \langle \kappa^*(\psi), aa' \rangle &= \langle \psi, \kappa(aa') \rangle \\
&= \langle \psi, \kappa(a)\kappa(a') \rangle = \langle \psi, \kappa(a) \rangle \langle \psi, \kappa(a') \rangle \\
&= \langle \kappa^*(\psi), a \rangle \langle \kappa^*(\psi), a' \rangle = \langle \theta(\psi), a \rangle \langle \theta(\psi), a' \rangle.
\end{aligned}$$

Thus $\theta(\psi)$ is multiplicative on A and hence $\theta(\psi) \in \Delta(A)$. Since κ is a bijection, it follows easily that θ induces a bijection between $\Delta(B)$ and $\Delta(A)$. It remains to show that θ and θ^{-1} are continuous with respect to the induced w^*-topologies on $\Delta(B)$ and

$\Delta(A)$. If $\psi_\alpha \to \psi$ in $\Delta(B)$, then for each $a \in A$, we have

$$\begin{aligned}\lim_\alpha \langle \theta(\psi_\alpha), a\rangle &= \lim_\alpha \langle \kappa^*(\psi_\alpha), a\rangle = \lim_\alpha \langle \psi_\alpha, \kappa(a)\rangle \\ &= \langle \psi, \kappa(a)\rangle = \langle \kappa^*(\psi), a\rangle = \langle \theta(\psi), a\rangle.\end{aligned}$$

Thus we have shown that $\theta(\psi_\alpha) \to \theta(\psi)$, proving the continuity of θ. The continuity of θ^{-1} is shown similarly. Let $\phi_\beta \longrightarrow \phi$ in $\Delta(A)$, then for each $b \in B$, we have

$$\begin{aligned}\lim_\beta \langle \theta^{-1}(\phi_\beta), b\rangle &= \lim_\beta \langle (\kappa^*)^{-1}(\phi_\beta), b\rangle = \lim_\beta \langle \phi_\beta, \kappa^{-1}(b)\rangle \\ &= \langle \phi, \kappa^{-1}(b)\rangle = \langle (\kappa^{-1})^*(\phi), b\rangle = \langle \theta^{-1}(\phi), b\rangle.\end{aligned}$$

Thus we have $\theta^{-1}(\phi_\beta) \to \theta^{-1}(\phi)$. □

Corollary 3.4.3. *If G is a topological group, then $\Delta(AP(G))$ is homeomorphic to $b(G)$, in other words, $\Delta(AP(G)) \cong b(G)$.*

Proof. It is well-known [**24**, Theorem 1.16] that for any compact set X, the spectrum of $C(X)$ is homeomorphic to X, under the mapping

$$\tau : X \longrightarrow \Delta(C(X)), \qquad x \to \tau_x,$$

where τ is the evaluation functional defined by $\tau_x f = f(x)$. By Theorem 3.4.1 the mapping

$$\kappa : AP(G) \longrightarrow C(b(G)), \qquad f \mapsto \widehat{f}$$

is an isometric isomorphism. It follows from Theorem 3.4.2 that $AP(G)$ and $C(b(G))$ have homeomorphic spectrum and this homeomorphism θ is given by the restriction of the adjoint map κ^* to $\Delta(C(b(G)))$, that is,

$$\theta = \kappa^*|_{\Delta(C(b(G)))} : \Delta(C(b(G))) \longrightarrow \Delta(AP(G)).$$

Since for $x \in b(G)$ and every $f \in AP(G)$:

$$\langle \theta(\tau_x), f\rangle = \langle \tau_x, \kappa(f)\rangle = \widehat{f}(x) = \langle \tau_x \circ \kappa, f\rangle,$$

it follows that $\theta(\tau_x) = \tau_x \circ \kappa$. Thus $b(G) \cong \Delta(AP(G))$ via the map $x \mapsto \tau_x \circ \kappa$. $\square$

REMARK 3.4.4. Since $b(G)$ can be identified with the spectrum of $AP(G)$, the map

$$\kappa : AP(G) \longrightarrow C(b(G)), \qquad f \mapsto \widehat{f}$$

can be identified with the Gelfand transform of $AP(G)$.

3.5. Approximation Theorem for Almost Periodic Functions

In this section we will prove that almost periodic functions can be approximated uniformly by linear combinations of coordinate functions of finite-dimensional representation (Theorem 3.5.2).

The proof of the result depends on the Peter-Weyl density theorem, which we shall prove first.

Recall if G is a topological group, then $\widehat{G}$ denotes the set of all (equivalence classes) of topologically irreducible continuous unitary representations of G.

THEOREM 3.5.1 (Peter-Weyl). *Let G be a compact group and $\mathscr{E}_{\widehat{G}}$ be the linear subspace of $C(G)$ spanned by the coordinate functions of representations in $\widehat{G}$. Then $\mathscr{E}_{\widehat{G}}$ is norm dense in $C(G)$.*

Elements of $\mathscr{E}_{\widehat{G}}$ are sometimes called trigonometric polynomials on G. Thus, in analogy with Weierstrass' theorem B.0.7, we may rephrase the theorem by saying that trigonometric polynomials are dense in $C(G)$, if G is compact.

PROOF. We prove this theorem using the Stone-Weierstrass theorem B.0.6. Since the identity representation $1_G : G \longrightarrow \mathscr{L}(\mathbb{C})$, $1_G(s) = 1$, belongs to $\widehat{G}$, it follows that $\mathscr{E}_{\widehat{G}}$ contains all the constants. If $s \neq t$, by Gelfand-Raikov Theorem (Theorem 1.6.9), there exists $\pi \in \widehat{G}$ such that $\pi(s) \neq \pi(t)$, it follows that for suitable vectors x, y, we have $\pi_{x,y}(s) \neq \pi_{x,y}(t)$. Therefore $\mathscr{E}_{\widehat{G}}$ is a linear subspace of $C(G)$ and the relation $\overline{\pi_{x,y}} = \overline{\pi}_{x,y}$, where $\overline{\pi} \in \widehat{G}$ is the conjugate representation of π, shows that $\mathscr{E}_{\widehat{G}}$ is closed under complex conjugation.

So all that remains to apply Stone-Weierstrass theorem is to show that the product of the coordinates is again a coordinate function. Let $\pi_{x,y}$ and $\pi'_{x',y'}$ be two coordinate functions. The tensor product

$$\pi \otimes \pi' : G \longrightarrow \mathscr{L}(H \otimes H')$$

is a continuous unitary representation of G. The product $\pi_{x,y}\pi'_{x',y'}$ is a coordinate function of $\pi \otimes \pi'$, in fact for all $s \in G$:

$$\begin{aligned}(3.5.11)\qquad &(\pi \otimes \pi'(s)(x \otimes x')|(y \otimes y'))\\ =&(\pi(s)x \otimes \pi'(s)x'|(y \otimes y'))\\ =&(\pi(s)x|y)(\pi'(s)x'|y').\end{aligned}$$

If $\pi \otimes \pi'$ was topologically irreducible, we had nothing left to show since then $\pi_{x,y}\pi'_{x',y'} \in \mathscr{E}_{\widehat{G}}$. But $\pi \otimes \pi'$ may not be irreducible, however, since G is compact, we can write

$$\pi \otimes \pi' = \sigma_1 \oplus \cdots \oplus \sigma_n \quad \text{and} \quad H \otimes H' = H_1 \oplus \cdots \oplus H_n,$$

with $\{\sigma_k, H_k\} \in \widehat{G}$ for $k = 1, \ldots, n$. The vectors $x \otimes x'$ and $y \otimes y'$ in $H \otimes H'$ can be decomposed into corresponding direct sums

$$x \otimes x' = z_1 \oplus \cdots \oplus z_n, \qquad y \otimes y' = w_1 \oplus \cdots \oplus w_n.$$

Then for $s \in G$,

$$\begin{aligned}&(\pi \otimes \pi'(s)(x \otimes x')|(y \otimes y'))\\ =&(\sigma_1(s)z_1 \oplus \cdots \oplus \sigma_n(s)z_n|w_1 \oplus \cdots \oplus w_n)\\ =&(\sigma_1(s)z_1|w_1) + \cdots + (\sigma_n(s)z_n|w_n).\end{aligned}$$

Thus by (3.5.11),

$$(\pi(s)x|y)(\pi'(s)x'|y')$$

$$=(\sigma_1(s)z_1|w_1) + \cdots + (\sigma_n(s)z_n|w_n).$$

That is, $\pi_{x,y}\pi'_{x',y'}$ is a sum of coordinate functions of representation in $\widehat{G}$, and hence $\pi_{x,y}\pi'_{x',y'} \in \mathscr{E}_{\widehat{G}}$. $\square$

THEOREM 3.5.2. *Let G be a topological group. Then $f \in AP(G)$ if and only if f is the uniform limit on G of linear combinations of coefficients of finite-dimensional representations in $\widehat{G}$.*

PROOF. First, We assume that $f \in AP(G)$. Then according to Theorem 3.1.4 $f = g \circ \alpha$, where $g \in C(b(G))$. It follows from Peter-Weyl Theorem that g is a uniform limit on $b(G)$ of linear combinations of coefficients of continuous topologically irreducible unitary representations of $b(G)$. Such representations are finite dimensional since $b(G)$ is compact. If $\pi : b(G) \longrightarrow U(H)$ is topologically irreducible, then so is $\pi' = \pi \circ \alpha : G \longrightarrow U(H)$ since $\alpha(G)$ is dense in $b(G)$. If $\xi, \eta \in H$, then $\pi'_{\xi,\eta} = \pi_{\xi,\eta} \circ \alpha$ since for $s \in G$,

$$\pi'_{\xi,\eta}(s) = (\pi'(s)\xi|\eta) = (\pi(\alpha(s))\xi|\eta) = \pi_{\xi,\eta}(\alpha(s)).$$

Thus if u_κ is a linear combination of coefficients of representations in $\widehat{b(G)}$, then $u_\kappa \circ \alpha$ is a linear combination of coefficients of finite-dimensional representations in $\widehat{G}$. Also, if $u_\kappa \to g$ uniformly on $b(G)$, then $u_\kappa \circ \alpha \to g \circ \alpha$ uniformly on G. Hence f is a uniform limit on G of linear combinations of coefficients of finite-dimensional representations in $\widehat{G}$.

Next, we assume that f is the uniform limit on G of linear combinations of coefficients of finite-dimensional representations in $\widehat{G}$. Suppose for every n, there exists a linear combination f_n of coefficients of finite-dimensional continuous unitary topologically irreducible representations of G such that $\|f - f_n\|_{\sup} \leqslant \frac{1}{n}$. There exists a function g_n on $b(G)$, a linear combination of finite-dimensional continuous unitary topologically irreducible representation of $b(G)$ such that $f_n = g_n \circ \alpha$. One has $\|g_m - g_n\|_{\sup} = \|f_m - f_n\|_{\sup} \to 0$ when m and n tend to ∞, hence the g_n converges uniformly to a continuous function g on $b(G)$, and one has $f = g \circ \alpha$. $\square$

3.6. Maximally and Minimally Almost Periodic Groups

Now, we may have a question on how big the space $AP(G)$ is, i.e., if there are enough almost periodic functions to separate the points of G. This question is closely connected to the injectivity of the canonical homomorphism α from G into its Bohr compactification $b(G)$. First, let us see the following equivalent conditions.

THEOREM 3.6.1. *For a locally compact group G, the following conditions are equivalent.*

(a) *The algebra $AP(G)$ separates the points of G.*

(b) *The canonical homomorphism $\alpha : G \longrightarrow b(G)$ is injective.*

(c) *There is an injective continuous homomorphism of G into a compact group K.*

(d) *Continuous homomorphisms of G onto compact groups separate the points of G.*

(e) *Finite-dimensional representations in $\widehat{G}$ separate the points of G.*

PROOF. $(a) \Rightarrow (b)$: If $s \neq t$ are two elements of G, choose $f \in AP(G)$ such that $f(s) \neq f(t)$. Let $\widehat{f} \in C(b(G))$ be such that $\widehat{f} \circ \alpha = f$. Then $\widehat{f}(\alpha(s)) \neq \widehat{f}(\alpha(t))$, hence $\alpha(s) \neq \alpha(t)$.

$(b) \Rightarrow (c)$ is trivial since $b(G)$ is compact.

$(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$: Suppose $s, t \in G$ are two distinct points of G and suppose that $\gamma : G \longrightarrow K$ is a continuous homomorphism into K such that $\gamma(s) \neq \gamma(t)$. Since K is compact, according to Theorem 1.6.12 (i) all representations in $\widehat{K}$ are finite-dimensional. And by Gelfand-Raikov theorem (Theorem 1.6.9), we know that $\widehat{K}$ separates the points of K. Hence for some representation $\sigma \in \widehat{K}$, $\sigma(\gamma(s)) \neq \sigma(\gamma(t))$. The map $\sigma \circ \gamma$ is finite-dimensional continuous unitary representation of G, and according to Theorem 1.6.12 (ii) hence it is a direct sum of irreducible unitary representations, that is, $\sigma \circ \gamma = \oplus_{i=1}^{N} \pi_i$, where $\pi_i \in \widehat{G}$. It follows that for at least one i, $\pi_i(s) \neq \pi_i(t)$, as we wanted to show.

$(e) \Rightarrow (a)$ Suppose $s \neq t$ and $\pi \in \widehat{G}$ is such that $\pi(s) \neq \pi(t)$. Then for some $\xi, \eta \in H$, $(\pi(s)\xi|\eta) \neq (\pi(t)\xi|\eta)$, or $\pi_{\xi,\eta}(s) \neq \pi_{\xi,\eta}(t)$. However, since π is finite-dimensional, according to Theorem 3.5.2, we have $\pi_{\xi,\eta} \in AP(G)$. It follows that $AP(G)$ separates the points of G. □

The following definition was given by von Neumann [**37**].

Definition 3.6.2. A locally compact group is *maximally almost periodic*, or is called *MAP-group*, if the equivalent conditions in Theorem 3.6.1 hold.

Remark 3.6.3. MAP-groups are also called *groups injectable into compact groups.* If G is a topological group and N is the kernel of the canonical mapping $\alpha : G \to b(G)$, then N is a closed normal subgroup of G and G/N is a MAP. The following locally compact groups are MAP-groups:
(i) subgroups of MAP-groups;
(ii) products of MAP-groups;
(iii) Abelian groups;
(iv) compact groups;
(v) free groups.

Definition 3.6.4. A *minimally almost periodic group* is a group for which the only elements of $AP(G)$ are the constant functions.

As an example of a minimally periodic function we can consider the set of all 2×2 matrices

$$G = \left\{ \begin{bmatrix} u & v \\ w & z \end{bmatrix} : u, v, w, z \in \mathbb{Q}, uz - vw \neq 0 \right\},$$

where the group operation is the matrix multiplication. Minimally almost periodic groups were studied by von Neumann and Wigner in [**38**].

CHAPTER 4

Invariant Means on Almost Periodic Functions

In this chapter we discuss the existence of means on $AP(G)$ and study some of its applications, in particular, its application in the approximation problems.

We give two proofs of the existence of a mean. In Section 4.1 we prove the existence of a mean using Bohr compactification $b(G)$. This proof is short and consists essentially of lifting an almost periodic function to $b(G)$ and integrating with respect to the normalized Haar measure on $b(G)$. In Section 4.2 we give another proof of the existence of mean by using a combinatorial result called the Marriage Lemma.

4.1. Bohr Compactification and Invariant Means

In this section, we use $b(G)$ to introduce a mean on $AP(G)$. Our main reference for this section is [**17**]. First we define what we mean by a mean on $AP(G)$.

Definition 4.1.1. An invariant mean M on $AP(G)$ is a continuous linear functional such that

(i) $M(f) \geqslant 0$ for $f \geqslant 0$;

(ii) $M(1_G) = 1$;

(iii) $M({}_sf) = M(f_s) = M(f)$, for all $f \in AP(G)$ and all $s \in G$.

In the above definitions, properties (i) and (ii) is what defines a mean. Property (iii) is the invariant property of the mean. It follows from general properties of means that if M is a mean, then $\|M\| = 1$ ([**17**, Proposition 2.1.9]).

Theorem 4.1.2. *Let G be a topological group and $f \in AP(G)$. Let K be the closed convex hull in $C^b(G)$ of the set of the ${}_sf$, where s runs over G. Then K contains a* *unique constant* $M(f)$. *If* $\widehat{f}$ *is the function corresponding to* f *on* $b(G)$, *we have*

$$M(f) = \int_{b(G)} \widehat{f}(s)ds,$$

where ds *denotes the normalized Haar measure on* $b(G)$.

PROOF. Since $AP(G)$ is isometrically isomorphic to $C(b(G))$ (Theorem 3.4.1), we may consider the case $G = b(G)$, i.e., consider $f \in C(b(G))$, where f is identified with $\hat{f}$ in Theorem 3.4.1.

Let $f \in C(b(G))$ and $\epsilon > 0$. Since f is uniformly continuous, there exists a neighborhood V of e in $b(G)$ such that $st^{-1} \in V$ implies $|f(s) - f(t)| \leqslant \epsilon$. Let $(Vs_i)_{1\leqslant i\leqslant n}$ be a finite cover of $b(G)$ and according to Theorem 1.2.14 let $(h_i)_{1\leqslant i\leqslant n}$ be a partition of unity subordinate to this coverings.

Set

$$C_i = \int_{b(G)} h_i(s)ds,$$

and let $t \in b(G)$ be arbitrary but fixed, then because

$$\int_{b(G)} f(s)ds = \int_{b(G)} f(st)ds = \int_{b(G)} \left(\sum_{i=1}^{n} h_i(s)\right) f(st)ds,$$

we have

$$\begin{aligned}
\left|\int_{b(G)} f(s)ds - \sum_{i=1}^{n} C_i f(s_i t)\right| &= \left|\int_{b(G)} \left(\sum_{i=1}^{n} h_i(s)\right) f(st)ds - \sum_{i=1}^{n} \int_{b(G)} h_i(s) f(s_i t)ds\right| \\
&= \left|\sum_{i=1}^{n} \int_{b(G)} h_i(s)[f(st) - f(s_i t)]ds\right| \\
&\leqslant \sum_{i=1}^{n} \int_{Vs_i} h_i(s)|f(st) - f(s_i t)|ds \\
&\leqslant \varepsilon \sum_{i=1}^{n} \int_{Vs_i} h_i(s)ds \\
&= \varepsilon \sum_{i=1}^{n} \int_{b(G)} h_i(s)ds
\end{aligned}$$

$$= \varepsilon \int_{b(G)} \left(\sum_{i=1}^{n} h_i(s) \right) ds = \varepsilon \int_{b(G)} ds = \varepsilon.$$

On the other hand, $\sum_{i=1}^{n} C_i = 1$. Hence the constant $\int_{b(G)} f(s)ds$ belongs to K because $\int_{b(G)} f(s)ds$ belongs to the closure of the set of convex combinations of functions ${}_{s_i}f \in K$.

Finally, every translation of f has the same integral on $b(G)$ (translation invariance of Haar measure). Hence every function in K has the same integral on $b(G)$. If a constant is in K, its value is hence necessarily $\int_{b(G)} f(s)ds$, in other words, if $g(s) = k$ for all $s \in b(G)$, $g \in K$, then $k = \int_{b(G)} g(s)ds = \int_{b(G)} f(s)ds$. This shows that the only constant function in K is $M(f) = \int_{b(G)} f(s)ds$. $\square$

THEOREM 4.1.3. *The map*

$$M : AP(G) \longrightarrow \mathbb{C}, \qquad f \mapsto M(f)$$

is an invariant mean on $AP(G)$.

PROOF. All this follows from the equality $M(f) = \int_{b(G)} \widehat{f}(s)ds$ of the theorem. $\square$

4.2. The Marriage Lemma and Its Application

In this section, we give a second proof for the existence of mean on $AP(G)$ which does not use the Bohr compactification $b(G)$. The proof relies on the following combinatorial result. Our main reference for this section is [**26**].

LEMMA 4.2.1 (The Marriage Lemma). *Let P and Q be nonempty sets. Let ρ be a function that maps P into the family of nonempty subsets of Q. Suppose that*

$$\left| \bigcup_{p \in P_1} \rho(p) \right| \geqslant |P_1| \tag{4.2.12}$$

for all finite subsets P_1 of P. Suppose also that P is finite or that $\rho(p)$ is finite for all $p \in P$. Then there is an injective function $\sigma : P \to Q$ with the property that $\sigma(p) \in \rho(p)$ for all $p \in P$.

Proof. We consider two cases.

Case 1: P is finite. We use induction on $|P|$. Let $|P| = n$. If $n = 1$, the result is trivial. Since there is only one element p in P, we only need to let $\sigma(p)$ be any element in $\rho(p)$. Suppose, as induction hypothesis, that the lemma holds whenever $1 \leqslant |P| < n - 1$. Suppose $|P| = n$. We face two alternatives to consider.

The first one is that

$$\left|\bigcup_{p\in P_1} \rho(p)\right| > |P_1| \text{ if } P_1 \subset P \text{ and } 0 < |P_1| \leqslant n.$$

Choose any $p_0 \in P$ and let $\sigma(p_0)$ be any element of $\rho(p_0)$. Then $P - \{p_0\}$, $Q - \{\rho(p_0)\}$ and the function $p \to \rho(p) \cap \{\sigma(p_0)\}^c$ satisfy the inductive hypothesis. Here $\rho(p) \cap \{\sigma(p_0)\}^c \neq \varnothing$ since if we let $P_1 = \{p\}$, according to (4.2.12) we have $|\rho(p)| > |p| = 1$. Also, $|P - \{p_0\}| = n - 1$. Hence, σ can be constructed in this case.

The second alternative is to suppose that there is a $P_0 \subset P$ such that $0 < |P_0| < n$ and $\left|\bigcup_{p\in P_0} \rho(p)\right| = |P_0|$. Then P_0 satisfies the inductive hypothesis and so σ can be defined on P_0 so as to be injective, and such that $\sigma(p) \in \rho(p)$ for all $p \in P_0$. Note that $\sigma(P_0) = \bigcup_{p\in P_0} \rho(p)$. $\sigma(P_0) \subset \bigcup_{p\in P_0} \rho(p)$ is obvious since $\sigma(p) \in \rho(p)$ for all $p \in P_0$. Since σ is injective,

$$|\sigma(P_0)| = |P_0| = \left|\bigcup_{p\in P_0} \rho(p)\right|.$$

Since all sets are finite, $\sigma(P_0) = \bigcup_{p\in P_0} \rho(p)$. Now, let us look at the set $P \cap P_0^c$. If there is a subset P_2 of $P \cap P_0^c$ such that

$$\left|\bigcup_{p\in P_2} \rho(p) \cap \sigma(P_0)^c\right| < |P_2|,$$

then we would have

$$\left|\bigcup_{p\in P_0\cup P_2} \rho(p)\right| = \left|\sigma(P_0) \cup \bigcup_{p\in P_2} \rho(p)\right| = |P_0| + \left|\bigcup_{p\in P_2} \rho(p) \cap \sigma(P_0)^c\right| < |P_0 \cup P_2|.$$

That is, (4.2.12) would fail for $P_0 \cup P_2$. Hence we can apply the inductive hypothesis to $P \cap P_0^c$ and the mapping $\rho^*(p) = \rho(p) \cap \sigma(P_0)^c$, and define σ on $P \cap P_0^c$. We may note that $\rho^*(p) \neq \varnothing$ for all $p \in P \cap P_0^c$. In fact, if for some $p' \in P \cap P_0^c$, $\rho^*(p') = \varnothing$, we would have $\rho(p') \subset \sigma(P_0) = \bigcup_{p \in P_0} \rho(p)$, which means

$$|P_0 \cup \{p'\}| > |P_0| = \left| \bigcup_{p \in P_0} \rho(P) \right| = \left| \bigcup_{p \in P_0 \cup \{p'\}} \rho(P) \right|,$$

contradicting to (4.2.12).

Case 2: P is infinite and all $\rho(p)$ are finite. We handle this case by a compactness argument. With the discrete topology, each $\rho(p)$ is compact, and so the Cartesian product $X = \prod_{p \in P} \rho(p)$ is compact. For a finite nonempty subset F of P, let H_F be the set of all $(g_p)_{p \in P} \in X$ such that all g_p are distinct for $p \in F$. By the first case, H_F is nonempty. We claim that H_F is also closed. To prove it, we want to show that H_F^c is open. Let $(g_p)_{p \in P} \in H_F^c$, then there exist $p_1, p_2 \in F$ such that $g_{p_1} = g_{p_2}$. Let $U = \prod_{p \in P} U_p$ where

$$U_p = \begin{cases} \rho(p) & \text{if } p \notin \{p_1, p_2\} \\ \{g_{p_1}\} & \text{if } p \in \{p_1, p_2\}. \end{cases}$$

Then U is an open subset of X and $(g_p) \in U \subset H_F^c$. Therefore, H_F^c is open. Since

$$H_{F_1} \cap H_{F_2} \cap \cdots \cap H_{F_n} \supset H_{F_1 \cup F_2 \cup \cdots \cup F_n},$$

the intersection

$$\bigcap \{H_F : F \text{ is a finite nonempty subset of } P\}$$

is nonempty. Choose any point (g_p) in this intersection, and let $\sigma(p) = g_p$ for all $p \in P$. Therefore, this σ has all of the required properties. $\square$

Remark 4.2.2. Intuitively, we can view P and Q as two sets of people separated by genders, i.e., P is a set of men and Q is a set of women. For each man $p \in P$, there is a subset $\rho(p)$ of Q corresponding to him. The Marriage Lemma states that if for each group of men P_1, the number of men $|P_1|$ is less than or equal to the number of all assigned woman $|\bigcup_{p\in P_1}\rho(p)|$, then we can pair up each man p with exactly one woman chosen from the set $\rho(p)$.

The Marriage Lemma can be used to establish a useful fact about metric spaces as follows.

COROLLARY 4.2.3. *Let X be a metric space with metric d, and suppose that $\{x_1,\dots,x_n\}$ is an ϵ-mesh in X for some $\epsilon>0$, such that the number n of elements of the mesh is as small as possible for the given ϵ. Let Y be any subset of X such that for each $x\in X$ there is a $y\in Y$ such that $d(x,y)<\epsilon$. Then there are an injective mapping σ of $\{x_1,x_2,\dots,x_n\}$ into Y and a sequence $\{z_1,\dots,z_n\}$ of elements of X such that $d(x_k,z_k)<\epsilon$ and $d(z_k,\sigma(x_k))<\epsilon$ where $k=1,2,\dots,n$.*

REMARK 4.2.4. If we remove the middle points z_k, we can say that there exists an injective mapping σ from $\{x_1,\dots,x_n\}$ to Y such that $d(x_k,\sigma(x_k))<2\epsilon$ for $k=1,2,\dots,n$. As a result, if $x\in X$, then for some k, $d(x,\sigma(x_k))<3\epsilon$ because for some k, $d(x,x_k)<\epsilon$. So an ϵ-mesh $\{x_1,x_2,\dots,x_n\}$ with minimum cardinality n can be replaced with a 3ϵ-mesh $\{y_1,\dots,y_n\}$ with points $y_k\in Y$ and with the same cardinality.

PROOF. For every $k=1,2,\dots,n$, let

$$\rho(x_k)=\{y\in Y: d(x_k,z)<\epsilon \text{ and } d(y,z)<\epsilon \text{ for some } z\in X\}.$$

Note that all of $x_1,x_2,\dots,x_n$ are distinct. Consider an arbitrary nonempty subset $\{x_{j_1},\dots,x_{j_r}\}$ of $\{x_1,\dots,x_n\}$ and if there are remaining elements, we write them as $x_{j_{r+1}},\dots,x_{j_n}$. We will prove that

$$|\rho(x_{j_1})\cup\cdots\cup\rho(x_{j_r})|\geqslant r. \tag{4.2.13}$$

If the set $\rho(x_{j_1})\cup\cdots\cup\rho(x_{j_r})$ is infinite, there is nothing to prove. Suppose that it is finite, we write $\rho(x_{j_1})\cup\cdots\cup\rho(x_{j_r})=\{y_1,\dots,y_s\}$, where y_l $(1\leqslant l\leqslant s)$ are distinct. For the set $A=\{y_1,\dots,y_s,x_{j_{r+1}},\dots,x_{j_n}\}$, we have $|A|\leqslant s+n-r$. We claim that A

is also an ϵ-mesh. If $z \in X$ and $d(z, x_{j_p}) \geqslant \epsilon$ for $p = r+1, \ldots, n$, then $d(z, x_{j_p}) < \epsilon$ for some $p = 1, \ldots, r$. There is also a $y \in Y$ such that $d(z, y) < \epsilon$, and by the definition of ρ, we have $y \in \rho(x_{j_p})$, i.e., $y = y_l$ for some $l = 1, 2, \ldots, s$. This proves that A is an ϵ-mesh. By the choice of n, we have $s + n - r \geqslant |A| \geqslant n$, so that $s \geqslant r$. This proves (4.2.13).

Now apply the Marriage Lemma 4.2.1 with $P = \{x_1, \ldots, x_n\}$ and $Q = Y$. So there exists an injective mapping $\sigma : P \longrightarrow Y$, $x_k \mapsto \sigma(x_k)$ such that $\sigma(x_k) \in \rho(x_k)$. By definition of $\rho(x_k)$, we can find $z_k \in X$ such that $d(x_k, z_k) < \epsilon$ and $d(\sigma(x_k), z_k) < \epsilon$. Note that here the Marriage Lemma is only used for the case that P is finite. $\square$

To construct an invariant mean on $AP(G)$, we show a couple of preliminary results.

THEOREM 4.2.5. *Let G be a topological group, f a function in $AP(G)$, and $\epsilon > 0$. Let $\{D_{a_1} f, \ldots, D_{a_n} f\}$ and $\{D_{b_1} f, \ldots, D_{b_n} f\}$ be ϵ-meshes in $\{D_a f : a \in G\}$ both having the least cardinal number n among all ϵ-meshes. Then*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} D_{a_k} f - \frac{1}{n} \sum_{k=1}^{n} D_{b_k} f \right\|_{\sup} < 2\epsilon. \tag{4.2.14}$$

REMARK 4.2.6. Recall that $D_a f$ is the function on $G \times G = G^2$ such that $D_a f(x, y) = f(xay)$. According to Theorem 2.1.1, $f \in AP(G)$ if and only if $\overline{\{D_a f : a \in G\}}$ is compact in $C^b(G^2)$. Since for $f \in AP(G)$, $\{D_a f : a \in G\}$ is totally bounded, therefore it has a finite ϵ-mesh for every $\epsilon > 0$.

PROOF. We apply Corollary 4.2.3 to the ϵ-meshes $\{D_{a_k} f\}$ and $\{D_{b_k} f\}$ in the metric space $\{D_a f : a \in G\}$. Thus we can find $c_1, c_2, \ldots, c_n \in G$ and an injective map σ of $\{1, \ldots, n\}$ onto itself such that

$$\|D_{a_k} f - D_{c_k} f\|_{\sup} < \epsilon, \quad \|D_{c_k} f - D_{b_{\sigma(k)}} f\|_{\sup} < \epsilon \qquad (k = 1, 2, \ldots, n).$$

Therefore we have

$$\|D_{a_k} f - D_{b_{\sigma(k)}} f\|_{\sup} < 2\epsilon. \tag{4.2.15}$$

Adding the inequalities (4.2.15) from 1 to n, dividing by n, using elementary properties of the norm, and noting that σ carries $\{1, \dots, n\}$ onto itself, we obtain the required inequality. $\square$

THEOREM 4.2.7. *Let G be a topological group, f a function in $AP(G)$, and $\epsilon > 0$. Let $\{D_{a_1}f, \dots, D_{a_n}f\}$ be ϵ-meshes in $\{D_a f : a \in G\}$ having the least cardinal number n among all ϵ-meshes. Then*

$$\left\| \frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k} f \right\|_{\sup} \leqslant 2\epsilon.$$

PROOF. Let u, v be arbitrary elements of G. Then $\{D_{ua_1v}f, \dots, D_{ua_nv}f\}$ is an ϵ-mesh in $\{D_a f : a \in G\}$. The reason is that given $a \in G$, we have

$$\|D_{u^{-1}av^{-1}}f - D_{a_k}f\|_{\sup} < \epsilon$$

for some $k \in \{1, 2, \dots, n\}$, so

$$\begin{aligned} \|D_a f - D_{ua_kv}f\|_{\sup} &= \sup\{|f(xay) - f(xua_kvy)| : x, y \in G\} \\ (\text{let } x = xu^{-1},\ y = v^{-1}y) \quad &= \sup\{|f(xu^{-1}av^{-1}y) - f(xa_ky)| : x, y \in G\} \\ &= \sup\{|D_{u^{-1}av^{-1}}f(x, y) - D_{a_k}f(x, y)| : x, y \in G\} \\ &= \|D_{u^{-1}av^{-1}}f - D_{a_k}f\|_{\sup} < \epsilon. \end{aligned}$$

For the two ϵ-meshes $\{D_{a_k}f\}$ and $\{D_{b_k}f\}$ with $b_k = ua_kv$, we consider the inequality (4.2.14) at the point $(e, e) \in G^2$. We have

$$\begin{aligned} &\left| \frac{1}{n}\sum_{k=1}^{n} D_{a_k}f(e, e) - \frac{1}{n}\sum_{k=1}^{n} D_{b_k}f(e, e) \right| \\ = &\left| \frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} f(ua_kv) \right| \\ = &\left| \frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k}f(u, v) \right| < 2\epsilon \end{aligned}$$

Since $(u,v) \in G^2$ was arbitrary, by taking supremum over G^2, we obtain

$$\left\| \frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k} f \right\|_{\sup} \leqslant 2\epsilon.$$

□

Now we are ready to give our second proof of the existence of a mean on $AP(G)$.

THEOREM 4.2.8. *Let G be a topological group. There is a complex linear functional M on $AP(G)$ such that*

(i) $M({}_b f_a) = M(f)$ for $f \in AP(G)$ and $a, b \in G$;

(ii) $M(f) > 0$ if $f \in AP(G)$, $f \geqslant 0$ and $f \neq 0$;

(iii) $M(1_G) = 1$;

(iv) $M(\overline{f}) = \overline{M(f)}$.

In particular, M is an invariant mean on $AP(G)$.

PROOF. Let $f \in AP(G)$ and let $\epsilon > 0$. Let E_ϵ be the set of all complex numbers z such that for some sequence $\{c_1, c_2, \dots, c_p\}$ of elements of G, the inequality

$$\left\| z - \frac{1}{p}\sum_{j=1}^{p} D_{c_j} f \right\|_{\sup} < \epsilon \tag{4.2.16}$$

holds. If $\epsilon_1 \leqslant \epsilon_2$, let $z_0 \in E_{\epsilon_1}$, then we have

$$\left\| z_0 - \frac{1}{m}\sum_{n=1}^{m} D_{c_n} f \right\|_{\sup} < \epsilon_1 \leqslant \epsilon_2 \quad \text{for some sequence } \{c_1, c_2, \dots c_m\} \text{ in } G.$$

Therefore, $z_0 \in E_{\epsilon_2}$. This indicates that $E_{\epsilon_1} \subseteq E_{\epsilon_2}$. Theorem 4.2.7 shows that E_ϵ is nonempty. Suppose that z_1 and z_2 are in E_ϵ. Then for all $(x,y) \in G^2$, we have

$$\left| z_1 - \frac{1}{p}\sum_{j=1}^{p} f(xc_jy) \right| < \epsilon, \tag{4.2.17}$$

and

$$\left| z_2 - \frac{1}{q}\sum_{k=1}^{q} f(xd_ky) \right| < \epsilon. \tag{4.2.18}$$

Setting $x = e$ and $y = d_k$ in (4.2.17), adding over $k = 1, \dots, q$, and dividing by q, we obtain

$$\left| z_1 - \frac{1}{pq} \sum_{k=1}^{q} \sum_{j=1}^{p} f(c_j d_k) \right| < \epsilon,$$

and setting $y = e$ and $x = c_j$ in (4.2.18), we obtain similarly

$$\left| z_2 - \frac{1}{pq} \sum_{j=1}^{p} \sum_{k=1}^{q} f(c_j d_k) \right| < \epsilon.$$

Hence we have

$$|z_1 - z_2| < 2\epsilon. \tag{4.2.19}$$

If $z \in E_\epsilon$, then for some sequence $\{c_1, c_2, \dots, c_p\}$, we have

$$\left| z - \frac{1}{p} \sum_{j=1}^{p} f(ac_j b) \right| < \epsilon \qquad \text{for any } a, b \in G.$$

which means

$$|z| < \left| \frac{1}{p} \sum_{j=1}^{p} f(ac_j b) \right| + \epsilon \leqslant \|f\|_{\sup} + \epsilon.$$

Let $F_\epsilon = \overline{E_\epsilon}$ be the closure of E_ϵ in $\mathbb{C}$, which is compact since E_ϵ is bounded. We note that $F_{\epsilon_1} \subset F_{\epsilon_2}$ whenever $\epsilon_1 \leqslant \epsilon_2$. Since F_1 is compact and $F_\epsilon \subset F_1$ for all $\epsilon \leqslant 1$, the relation

$$F_{\epsilon_1} \cap \cdots \cap F_{\epsilon_m} \supset F_{\min\{\epsilon_1, \dots, \epsilon_m\}} \neq \varnothing$$

shows that the family $\{F_\epsilon\}_{\epsilon > 0}$ has finite intersection property. Thus by Theorem 1.2.9 $\bigcap_{\epsilon>0} F_\epsilon$ is nonempty. If $z, w \in F_\epsilon = \overline{E_\epsilon}$, then we can find $z', w' \in E_\epsilon$ such that $|z - z'| \leqslant \epsilon$, $|w - w'| \leqslant \epsilon$. Thus by (4.2.16), we have

$$|z - w| \leqslant |z - z'| + |z' - w'| + |w' - w| \leqslant \epsilon + 2\epsilon + \epsilon = 4\epsilon.$$

It follows that $\bigcap_{\epsilon>0} F_\epsilon$ contains exactly one point. This number we take as $M(f)$. Also, we have $F_\epsilon \subset E_{\epsilon'}$ for all ϵ, ϵ' satisfying $0 < \epsilon < \epsilon'$. It follows that $M(f)$ is the unique point lying in all E_ϵ. Thus we have that $M(f)$ is the unique complex number such that for every $\epsilon > 0$, there is a sequence $\{a_1, \ldots, a_n\}$ of elements in G for which

$$\left\| M(f) - \frac{1}{n} \sum_{k=1}^{n} D_{a_k} f \right\|_{\sup} < \epsilon. \tag{4.2.20}$$

Thus for all $x, y \in G$,

$$\left| M(f) - \frac{1}{n} \sum_{k=1}^{n} f(x a_k y) \right| < \epsilon. \tag{4.2.21}$$

Now, we are going to check the properties in the theorem.

First, $M(\alpha f) = \alpha M(f)$ for all $f \in AP(G)$ and all $\alpha \in \mathbb{C}$. Let $\epsilon > 0$, and choose $b_1, b_2, \ldots, b_m \in G$ such that

$$\left| M(f) - \frac{1}{m} \sum_{k=1}^{m} f(x b_k y) \right| < \epsilon/|\alpha| \qquad \text{for all } x, y \in G.$$

Then we have

$$\left| \alpha M(f) - \frac{1}{m} \sum_{k=1}^{m} \alpha f(x b_k y) \right| < \epsilon.$$

Hence we have $M(\alpha f) = \alpha M(f)$ according to the uniqueness of $M(\alpha f)$.

Second, we have $M({}_b f_a) = M(f)$ for all $a, b \in G$. Let $\epsilon > 0$ and choose $a_1, \ldots, a_n \in G$ satisfying (4.2.21), then for all $x, y \in G$,

$$\left| M(f) - \frac{1}{n} \sum_{k=1}^{n} {}_b f_a(x a_k y) \right| = \left| M(f) - \frac{1}{n} \sum_{k=1}^{n} f(b x a_k y a) \right| < \epsilon.$$

Thus, $M({}_b f_a) = M(f)$.

Third, $M(1_G) = 1$. Let $f = 1_G$ and $\epsilon > 0$. Choose an arbitrary a from G and note that for all $x, y \in G$:

$$|1 - 1_G(xay)| = |1 - 1| < \epsilon.$$

Thus, $M(1_G) = 1$.

Next we are going to prove that $M(f+g) = M(f) + M(g)$ for $f, g \in AP(G)$. For $\epsilon > 0$, apply (4.2.20) to choose sequence $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ of elements of G such that

$$\left\| M(f) - \frac{1}{m} \sum_{j=1}^{m} D_{a_j} f \right\|_{\sup} < \epsilon/2 \tag{4.2.22}$$

and

$$\left\| M(g) - \frac{1}{n} \sum_{k=1}^{n} D_{b_k} g \right\|_{\sup} < \epsilon/2. \tag{4.2.23}$$

From (4.2.22) we see that

$$\left| M(f) - \frac{1}{m} \sum_{j=1}^{m} f(x a_j b_k y) \right| < \epsilon/2 \qquad \text{for all } x, y \in G.$$

Summing over k and dividing by n, we get

$$\left| M(f) - \frac{1}{mn} \sum_{k=1}^{n} \sum_{j=1}^{m} f(x a_j b_k y) \right| < \epsilon/2 \qquad \text{for all } x, y \in G.$$

That is,

$$\left\| M(f) - \frac{1}{mn} \sum_{k=1}^{n} \sum_{j=1}^{m} D_{a_j b_k} f \right\|_{\sup} < \epsilon/2. \tag{4.2.24}$$

Similarly we find

$$\left\| M(g) - \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} D_{a_j b_k} g \right\|_{\sup} < \epsilon/2. \tag{4.2.25}$$

Adding (4.2.24) and (4.2.25), we have

$$\left\| M(f) + M(g) - \frac{1}{mn} \sum_{k=1}^{n} \sum_{j=1}^{m} D_{a_j b_k}(f+g) \right\|_{\sup} < \epsilon.$$

Since ϵ is arbitrary, the characterization of $M(f+g)$ from (4.2.20) shows that $M(f) + M(g) = M(f+g)$.

Similarly, to prove (iv), according to (4.2.20), we have

$$\left\|\overline{M(f) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k} f}\right\|_{\sup} < \epsilon, \qquad \text{for some sequence } \{a_1, \ldots, a_n\} \in G,$$

which means

$$\left\|\overline{M(f)} - \frac{1}{n}\sum_{k=1}^{n} D_{a_k} \overline{f}\right\|_{\sup} < \epsilon.$$

Therefore, $M(\overline{f}) = \overline{M(f)}$.

Next, we establish (ii). Suppose $f \neq 0$ and $f \geqslant 0$. First we show that $M(f) \geqslant 0$. To get a contradiction, we suppose $M(f) < 0$. Let $x_0 \in G$ be such that $f(x_0) > 0$. Let $0 < \epsilon < -M(f)$. Choose $a_1, a_2, \ldots a_n$ so that (4.2.21) holds. Let $y = e$ and choose x such that $xa_1 = x_0$, so that $f(xa_1 y) = f(xa_1) = f(x_0) > 0$. Then

$$\left|M(f) - \frac{1}{n}\sum_{k=1}^{n} f(xa_k y)\right| = -M(f) + \frac{1}{n}\sum_{k=1}^{n} f(xa_k) > -M(f) > \epsilon,$$

which contradicts the assumption. Therefore $M(f) \geqslant 0$.

It remains to show that $M(f) > 0$. Let $\{D_{a_1} f, \ldots, D_{a_n} f\}$ be an $f(x_0)/2$ -mesh in $\{D_a f : a \in G\}$. Then for all $x, y, a \in G$, we have

$$f(xa_1 y) + \cdots + f(xa_n y) \geqslant \max[f(xa_1 y), \ldots, f(xa_n y)] > f(xay) - f(x_0)/2.$$

Setting $y = e$ and $a = x^{-1}x_0$, we obtain

$$f(xa_1) + \cdots + f(xa_n) > f(x_0)/2 \qquad \text{for all } x \in G.$$

Hence we have

$$M(f_{a_1} + \cdots + f_{a_n}) = nM(f) \geqslant f(x_0)/2 > 0,$$

as we wanted to show. $\square$

The linear functional M constructed in Theorem 4.2.8 is uniquely determined by properties (i)-(iv) in Theorem 4.2.8.

THEOREM 4.2.9 (Uniqueness of Invariant Mean). *Suppose that G is a topological group and that M' is any complex linear functional on $AP(G)$ such that*
(i) $M'({}_af) = M'(f)$ for all $a \in G$ and $f \in AP(G)$
or
(i)′ $M'(f_a) = M'(f)$ for all $a \in G$ and $f \in AP(G)$,
(ii) $M'(f) \geqslant 0$ for all $f \in AP(G)^+$,
(iii) $M'(1_G) = 1$.
Then $M'(f) = M(f)$ for all $f \in AP(G)$.

PROOF. Since M and M' are linear on $AP(G)$, we only need to prove $M'(f) = M(f)$ for real-valued $f \in AP(G)$. By (4.2.20),

$$-\epsilon < M(f) - \frac{1}{n}\sum_{k=1}^{n} f(xa_ky) < \epsilon \qquad \text{for all } x, y \in G, \tag{4.2.26}$$

where $\epsilon > 0$ is arbitrary and $a_1, \ldots, a_n$ are appropriate elements of G. If M' satisfies (i), we set $x = e$ in (4.2.26), and obtain

$$-\epsilon < M(f) - \frac{1}{n}\sum_{k=1}^{n} {}_{a_k}f < \epsilon. \tag{4.2.27}$$

Applying M' to (4.2.27), and using property (i), (ii) and (iii) of M', we get

$$M'(-\epsilon) < M'(M(f)) - \frac{1}{n}\sum_{k=1}^{n} M'({}_{a_k}f) < M'(\epsilon),$$

which means

$$-\epsilon \leqslant M(f) - \frac{1}{n}\sum_{k=1}^{n} M'(f) \leqslant \epsilon \quad \text{or} \quad -\epsilon \leqslant M(f) - M'(f) \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $M'(f) = M(f)$ for $f \in AP(G)$. The case in $(i)'$ is similar. $\square$

REMARK 4.2.10. As we saw in Theorem 4.1.3 that the mean $M(f) = \int_{b(G)} \widehat{f}(s)ds$ given in Section 4.1 satisfies the conditions (i)-(iii) in Theorem 4.2.9. Therefore the mean constructed in Section 4.1 is identified with the mean constructed in this section.

4.3. Examples of Invariant Means

In some cases, the mean value M on $AP(G)$ has a particularly simple form.

THEOREM 4.3.1. *Let G be a locally compact group with a left Haar measure λ. Suppose there is a sequence $\{K_n\}_{n=1}^{\infty}$ of subsets of G such that $0 < \lambda(K_n) < \infty$ for all n and such that for each $x \in G$,*

$$\lim_{n\to\infty} \frac{\lambda((xK_n) \cap {K_n}^c)}{\lambda(K_n)} = 0. \tag{4.3.28}$$

Then for every $f \in AP(G)$, we have

$$M(f) = \lim_{n\to\infty} \frac{1}{\lambda(K_n)} \int_{K_n} f(x)dx. \tag{4.3.29}$$

Before we give a proof let us give an example.

EXAMPLE 4.3.2. Consider the additive group $\mathbb{R}$. For $f \in AP(\mathbb{R})$, we have

$$M(f) = \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f(x)dx.$$

This follows from Theorem 4.3.1, by taking $K_n = [-n, n]$; note that if $x \in \mathbb{R}$, $x > 0$, then

$$(x + K_n) \cap K_n^c = [-n + x, n + x] \cap ((-\infty, -n) \cup (n, \infty)) = (n, n + x],$$

and thus

$$\frac{\lambda((x + K_n) \cap K_n^c)}{\lambda(K_n)} = \frac{x}{2n} \to 0 \qquad \text{as } n \to \infty.$$

So the condition of Theorem 4.3.1 are satisfied.

Now, we will prove Theorem 4.3.1.

PROOF. First, we prove that for $f \in C(G)$ and $b \in G$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda(K_n)} \int_{K_n} ({}_bf(x) - f(x))dx = 0. \tag{4.3.30}$$

Clearly, we have

$$\begin{aligned}
&\left|\int_{K_n} {}_bf(x)dx - \int_{K_n} f(x)dx\right| = \left|\int_{bK_n} f(x)dx - \int_{K_n} f(x)dx\right| \\
&\leqslant \int_{(bK_n)\triangle K_n} |f(x)|dx \qquad (\text{because } |1_{bK_n} - 1_{K_n}| = 1_{bK_n \triangle K_n}) \\
&\leqslant \|f\|_{\sup}[\lambda(bK_n \cap K_n^c) + \lambda(K_n \cap (bK_n)^c)].
\end{aligned} \tag{4.3.31}$$

We also have

$$\lambda(K_n \cap (bK_n)^c) = \lambda(b^{-1}(K_n \cap (bK_n)^c)) = \lambda((b^{-1}K_n) \cap K_n^c). \tag{4.3.32}$$

Substituting (4.3.32) in (4.3.31), dividing the result by $\lambda(K_n)$ and applying (4.3.28), we obtain (4.3.30).

We shall use (4.3.28) to establish (4.3.29). It obviously suffices to prove (4.3.29) for real-valued $f \in AP(G)$. For such f, let

$$p(f) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda(K_n)} \int_{K_n} f(x)dx.$$

It is evident that $p(f+g) \leqslant p(f) + p(g)$ and $p(\alpha f) = \alpha p(f)$ for $\alpha \geqslant 0$, in other words, p is a sublinear functional. By Hahn-Banach Theorem 1.2.16, there is a linear functional M_0 on $AP(G)_{\mathbb{R}}$ (real almost periodic continuous functions) such that

$$-p(-f) \leqslant M_0(f) \leqslant p(f) \qquad \text{for all } f \in AP(G)_{\mathbb{R}}. \tag{4.3.33}$$

By (4.3.30) we have

$$p({}_bf - f) = -p(-{}_bf + f) = 0 \qquad \text{for all } f \in AP(G)_{\mathbb{R}}, b \in G.$$

This and (4.3.33) shows that $M_0(f) = M_0({}_bf)$. It is also clear that $M_0 \geqslant 0$ and $M_0(1_G) = 1$. Then from Theorem 4.2.9 we infer that $M_0(f) = M(f)$ for $f \in AP(G)_{\mathbb{R}}$. Now if there were a function $f \in AP(G)_{\mathbb{R}}$ such that

$$-p(-f) = \underline{\lim_{n\to\infty}} \frac{1}{\lambda(K_n)} \int_{K_n} f(x)dx < \overline{\lim_{n\to\infty}} \frac{1}{\lambda(K_n)} \int_{K_n} f(x)dx = p(f),$$

then the proof of the Hahn-Banach theorem shows the existence of two distinct functionals M_0 and M_1 both satisfying (4.3.33). They would both be equal to M on $AP(G)$, and a contradiction would result. Consequently the limit in (4.3.29) exists and (4.3.29) is established. $\square$

EXAMPLE 4.3.3. (a) Consider the additive group $\mathbb{R}$. For $f \in AP(\mathbb{R})$, we have

$$M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)dx = \lim_{T\to\infty} \frac{1}{T-a} \int_{a}^{T} f(x)dx = \lim_{T\to\infty} \frac{1}{T+b} \int_{-T}^{b} f(x)dx,$$

where a and b are arbitrary real numbers.
(b) Consider the additive group $\mathbb{Z}$. For $f \in AP(\mathbb{Z})$, we have

$$M(f) = \lim_{m\to\infty} \frac{1}{2m+1} \sum_{j=-m}^{m} f(j) = \lim_{m\to\infty} \frac{1}{m-a+1} \sum_{j=a}^{m} f(j) = \lim_{m\to\infty} \frac{1}{m+b+1} \sum_{j=-m}^{b} f(j).$$

(c) Let G be a compact group. For $f \in AP(G) = C(G)$, we have $M(f) = \int_G f(x)dx$, where dx is the Haar measure.

4.4. Fourier Series of Almost Periodic Functions

A central result in the theory of Fourier series is that every continuous 2π-periodic function (more generally, every square integrable 2π-periodic function) on $\mathbb{R}$ has a Fourier series expansion

$$f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx}$$

where the convergence is in the mean space norm $\|\cdot\|_2$ and

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Moreover, the coefficients $\widehat{f}(n)$ satisfy the Parseval's identity

$$\int_{-\pi}^{\pi} |f(x)|^2dx = \|f\|_2^2 = 2\pi \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2.$$

In this section, we will discuss the existence of a similar 'Fourier Series' expansion for the case of almost periodic functions on topological groups. We will derive the

Fourier series expansions of almost periodic functions using the isomorphism $AP(G) \cong C(b(G))$ and the fact that coordinate functions of representations in $\widehat{b(G)}$ form an orthonormal basis for $C(b(G))$.

We start with some general results in theory of Hilbert spaces. Let E be a vector space with inner product $(\cdot|\cdot)$. We call E an *inner product space*, or a *pre-Hilbert space*. An orthonormal family $\{e_i\}_{i\in I}$ in E is called an *orthonormal basis* for E if the set of all linear combinations of the e_i is dense in E. The following is a standard result on orthonormal basis.

THEOREM 4.4.1. *Suppose that $\{e_i\}_{i\in I}$ is an orthonormal basis for an inner product space E.*
(i) If $x \in E$ and $\lambda_i = (x|e_i)$, then

$$x = \sum_{i\in I} \lambda_i e_i, \qquad \textit{(Fourier series)}$$

and

$$\|x\|^2 = \sum_{i\in I} |\lambda_i|^2. \qquad \textit{(Parseval's identity)}$$

(ii) If $x, y \in E$ and $\lambda_i = (x|e_i)$, $\mu_i = (y|e_i)$, then

$$(x|y) = \sum_{i\in I} \lambda_i \overline{\mu_i}.$$

REMARK 4.4.2. The above result is usually stated for Hilbert spaces, however, as the theorem shows, the completeness of E is not needed in this result. Since the index set I need not be countable, the sums in the above theorem are not 'series' in the traditional sense. Consider the case $x = \sum_{i\in I} \lambda_i e_i$. Here the equality means that for every $\epsilon > 0$, there exists a finite subset J_ϵ of I such that whenever J is a finite subset of I containing J_ϵ, then $\|x - \sum_{i\in J} \lambda_i e_i\| \leqslant \epsilon$.

Other sums in the above theorem are to be interpreted similarly. Although these sums are not series as they stand, the Parseval's identity $\|x\|^2 = \sum_{i\in I} |\lambda_i|^2$ implies that all but a countable number of λ_i must be zero. So if we are willing to discard the zero terms and change the index set accordingly, the above sums will turn into series. The countable index set thus obtained would of course depend on x. For a proof of the above theorem and a general discussion of uncountable sums, we refer to Chapter IX of [**18**].

The proof of the existence of Fourier series for almost periodic functions given in this section relies on an important result in the representation theory of compact groups known as Peter-Weyl Theorem.

THEOREM 4.4.3. *Let G be a compact group. For $\pi \in \widehat{G}$, let $\{e_1, \dots, e_{d_\pi}\}$ be an orthonormal basis of H_π, and let $\pi_{ij}(x) = (\pi(x)e_j|e_i)$ be the corresponding coordinate functions, where $i, j \in \{1, 2, \dots, d_\pi\}$. Then the set*

$$\{\sqrt{d_\pi}\pi_{ij} : \pi \in \widehat{G}, 1 \leqslant i, j \leqslant d_\pi\}$$

is an orthonormal basis for $L^2(G)$.

It follows from this theorem that the coordinate functions $\sqrt{d_\pi}\pi_{ij}$ satisfy the orthonormality relations

$$\sqrt{d_\pi}\sqrt{d_\sigma}\int_G \pi_{ij}(x)\overline{\sigma_{kl}}(x)dx = \delta_{\pi\sigma}\delta_{ik}\delta_{jl}.$$

And if $f \in L^2(G)$ (in particular, if $f \in C(G)$), then

$$f = \sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi} d_\pi(f|\pi_{ij})\pi_{ij},$$

where $(f|\pi_{ij}) = \int_G f(x)\overline{\pi_{ij}}(x)dx$ and the convergence of the sum is in L^2-norm. Moreover,

$$\|f\|_2^2 = \sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi} d_\pi|(f|\pi_{ij})|^2.$$

We know from Theorem 3.4.1 that the map

$$\kappa : AP(G) \longrightarrow L^2(b(G)), \qquad \kappa(f) = \widehat{f},$$

is an isometric isomorphism. Recall that for each $f \in AP(G)$, $\widehat{f}$ is the unique continuous function on $b(G)$ such that $f = \widehat{f} \circ \alpha$. We can use the isomorphism κ to transfer the canonical inner product of $L^2(b(G))$ to the space $AP(G)$. In other words, we define an inner product on $AP(G)$ by putting

$$(f|g) = (\kappa(f)|\kappa(g)) = \int_{b(G)} \widehat{f}(x)\overline{\widehat{g}}(x)dx = M(f\overline{g}) \qquad (f, g \in AP(G)),$$

where M is the mean on $AP(G)$ (see Theorem 4.1.2). With this inner product on $AP(G)$, we may view κ as an isomorphism between $AP(G)$ and $L^2(b(G))$ preserving inner products.

Let $\widehat{G}\,'$ be the set of all (equivalence classes) of finite-dimensional, continuous, unitary irreducible representations of G (thus $\widehat{G}\,' \subset \widehat{G}$; this inclusion may be strict since $\widehat{G}$ may contain infinite-dimensional representations). We know from Theorem 3.1.3 that the map

$$\widehat{b(G)} \longrightarrow \widehat{G}\,', \qquad \rho \mapsto \rho \circ \alpha = \rho'$$

is a bijection and moreover if $\xi, \eta \in H_\rho = H_{\rho'}$, then

$$\rho'_{\xi,\eta} = \rho_{\xi,\eta} \circ \alpha.$$

In particular, if $\{e_1, \ldots, e_d\}$ is an orthonormal basis of $H_\rho = H_{\rho'}$ and

$$\rho_{ij}(x) = (\rho(x)e_j|e_i), \quad \rho'_{ij}(x) = (\rho'(x)e_j|e_i),$$

then $\rho'_{ij} = \rho_{ij} \circ \alpha$.

The following is the main theorem of this section.

Theorem 4.4.4. *Let G be a topological group and $\widehat{G}\,'$ be the set of all finite-dimensional, continuous, unitary, irreducible representations of G. If for every $\sigma' \in \widehat{G}\,'$ we choose an orthonormal basis $\{e_1, \ldots, e_{d_{\sigma'}}\}$ in $H_{\sigma'}$ and put $\sigma'_{ij}(s) = (\sigma'(s)e_j|e_i)$ $(s \in G)$, then the set*

$$\{d_{\sigma'}^{1/2}\sigma'_{ij} : \sigma' \in \widehat{G}\,', 1 \leqslant i, j \leqslant d_{\sigma'}\}$$

is an orthonormal basis in $AP(G)$. In particular, every $f \in AP(G)$ has a Fourier series expansion

$$f = \sum_{\sigma' \in \widehat{G}'} \sum_{i,j=1}^{d_{\sigma'}} d_{\sigma'} (f|\sigma'_{ij})\sigma'_{ij},$$

where $(f|\sigma'_{ij}) = M(f\overline{\sigma_{ij}})$ and the convergence is in L^2-norm. Moreover, the Parseval's identity

$$\|f\|_2^2 = M(|f|^2) = \sum_{\sigma' \in \widehat{G}'} \sum_{i,j=1}^{d_{\sigma'}} d_{\sigma'}^2 |(f|\sigma'_{ij})|^2$$

holds.

Proof. Let $\sigma \in \widehat{b(G)}$ be such that $\sigma \circ \alpha = \sigma'$. Then by Theorem 4.4.3 we know that

$$\{d_\sigma^{1/2}\sigma_{ij} : \sigma \in \widehat{b(G)}, 1 \leqslant i,j \leqslant d_\sigma\}$$

is an orthonormal basis for $L^2(b(G))$. Since each $\sigma_{ij} \in C(b(G)) \subset L^2(b(G))$, it follows that the above set is also an orthonormal basis of $C(b(G))$. Using the equality $d_\sigma = d_{\sigma'}$, we have

$$\kappa(d_{\sigma'}^{1/2}\sigma'_{ij}) = d_{\sigma'}^{1/2}\kappa(\sigma_{ij} \circ \alpha) = d_\sigma^{1/2}\sigma_{ij}.$$

Therefore, by the definition of the inner product of $AP(G)$:

$$(d_{\sigma'}^{1/2}\sigma'_{ij}|d_{\pi'}^{1/2}\pi'_{kl}) = d_{\sigma'}^{1/2}d_{\pi'}^{1/2}(\kappa(\sigma'_{ij})|\kappa(\pi'_{kl})) = d_\sigma^{1/2}d_\pi^{1/2}(\sigma_{ij}|\pi_{kl}) = \delta_{\sigma\pi}\delta_{ik}\delta_{jl}.$$

This proves the orthonormality of the functions $d_{\sigma'}^{1/2}\sigma'_{ij}$.

So it remains to show that $\{d_{\sigma'}^{1/2}\sigma'_{ij} : \sigma' \in \widehat{G}', 1 \leqslant i,j \leqslant d_{\sigma'}\}$ spans a dense linear subspace of $AP(G)$. Let $f \in AP(G)$ and $\epsilon > 0$. Then $\kappa(f) = \widehat{f} \in C(b(G))$ and hence there exist functions $\phi_1, \ldots, \phi_n \in \{d_\sigma^{1/2}\sigma_{ij} : \sigma \in \widehat{b(G)}, 1 \leqslant i,j \leqslant d_\sigma\}$ and scalars

$\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that

$$\left\| \widehat{f} - \sum_{i=1}^{n} \lambda_i \phi_i \right\|_2 < \epsilon.$$

We choose $\phi_i' \in \{d_{\sigma'}^{1/2} \sigma_{ij}' : \sigma' \in \widehat{G}\,', 1 \leqslant i, j \leqslant d_{\sigma'}\}$ such that $\phi_i = \widehat{\phi_{i'}}$. Then

$$\left\| f - \sum_{i=1}^{n} \lambda_i \phi_i' \right\|_2 = \left\| \kappa(f - \sum_{i=1}^{n} \lambda_i \phi_i') \right\|_2 = \left\| \kappa(f) - \sum_{i=1}^{n} \lambda_i \kappa(\phi_i') \right\|_2 = \left\| \widehat{f} - \sum_{i=1}^{n} \lambda_i \phi_i \right\|_2 < \epsilon.$$

This completes the proof of the theorem. □

Next we apply the above theorem to almost periodic function on $\mathbb{R}$. Note that $\widehat{\mathbb{R}}$ consists of 1-dimensional representations $x \mapsto e^{i\xi x}$, $\mathbb{R} \to \mathbb{T}$, where $\xi \in \mathbb{R}$. Therefore in this case, the coordinate functions are exactly $\{e^{i\xi x} : \xi \in \mathbb{R}\}$. We recall from Example 4.3.3 (a) that the mean on $AP(\mathbb{R})$ is given by

$$M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) dx.$$

Now we can state the following corollary.

Corollary 4.4.5. *Every almost periodic function on $\mathbb{R}$ has a Fourier series expansion*

$$f = \sum_{\xi \in \mathbb{R}} \widehat{f}(\xi) e^{i\xi x}$$

where

$$\widehat{f}(\xi) = M(f e^{-i\xi x}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\xi x} dx,$$

and moreover,

$$M(|f|^2) = \sum_{\xi \in \mathbb{R}} |\widehat{f}(\xi)|^2.$$

In the special case that $G = \mathbb{T}$, Theorem 4.4.4 reduces to the classical result on the Fourier series expansion of 2π-periodic functions on $\mathbb{R}$. We recall that functions on $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ can be identified with 2π-periodic functions on $\mathbb{R}$. And since $\mathbb{T}$ is compact and $AP(\mathbb{T}) = C(\mathbb{T})$, it follows that almost periodic function on $\mathbb{T}$ are just continuous 2π-periodic functions on $\mathbb{R}$. We know $\widehat{\mathbb{T}} \cong \mathbb{Z}$, which means that irreducible unitary representations of $\mathbb{T}$ are exactly the homomorphisms

$$\phi_n : \mathbb{T} \longrightarrow \mathbb{T}, \quad \phi_n(e^{it}) = (e^{it})^n = e^{int},$$

where $n \in \mathbb{Z}$. The coordinate function of each ϕ_n is equal to ϕ_n itself, and if we identify ϕ_n with a 2π-periodic function on $\mathbb{R}$, we can view ϕ_n as the function

$$\phi_n : \mathbb{R} \longrightarrow \mathbb{T}, \quad \phi_n(x) = e^{inx}.$$

The usual inner product on $C(\mathbb{T})$ induces the following inner product on the space of continuous 2π-periodic functions

$$(f|g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx,$$

where the factor $1/2\pi$ is from the assumption that Haar measure on $\mathbb{T}$ is normalized. It follows that for every continuous 2π-periodic function on $\mathbb{R}$,

$$(f|\phi_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \widehat{f}(n).$$

Now we can state the following result which is the classical Fourier series expansion of continuous 2π-periodic function.

COROLLARY 4.4.6. *If f is a 2π-periodic continuous function on $\mathbb{R}$, then*

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\phi_n,$$

where the convergence is in $\|\cdot\|_2$ and

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2.$$

CHAPTER 5

Further Developments and Future Work

The theory of almost periodicity was started by the papers of Bohr [**8**–**10**] and continued with works of many researchers including Bochner [**4**], von Neumann [**37**] and Weil [**45**]. This phase of the theory has been discussed in details in this thesis.

The second phase can be identified with various generalizations and modifications of the definition of almost periodicity. Here we can mention Stepanoff's generalization of almost periodic functions $\mathcal{S}^p$ [**43**,**44**], Weyl's almost periodic functions W^p [**46**] and Besicovitch almost periodic functions B^p $(p \geqslant 1)$ [**2**,**3**]. In all these variations, almost periodic functions are defined as the closure of trigonometric polynomials under a particular norm. For example, Stepanoff's norm is defined by

$$\|f\|_{S,r,p} = \sup_{x\in\mathbb{R}} \left(\frac{1}{r} \int_x^{x+r} |f(s)^p| ds \right)^{1/p}.$$

For different values of r these norms give the same topology and so the same space $\mathcal{S}^p$ is obtained.

A more interesting development is Weyl's paper [**47**], in which he introduced the notion of almost periodic vectors for unitary representations. In this paper Weyl uses Bochner's characterization to define almost periodic vectors, and uses his method of 'integral equations' (or in modern terminology, spectral theory of compact operators) to study almost periodic vectors. In this paper Weyl develops a Fourier series expansion for almost periodic vectors and establishes the Parseval's identity.

Almost periodicity has been studied in topologies other than the norm topology. For example, one can consider functions whose left translations form a relatively compact set in the weak topology of $C^b(G)$. Such functions are called weakly almost periodic (WAP). Since weak topology is weaker (coarser) than the norm topology,

$WAP(G)$ is a larger class of functions than $AP(G)$. For a detailed study of WAP-functions we refer to Burckel [**12**]. Almost periodic functions on semigroups were studied by Maak [**36**].

Almost periodicity has also been studied for vector-valued functions, an early such study goes back to von Neumann and Bochner [**5**].

More recently, almost periodicity has been studied for functionals on Banach algebras, or more generally, for elements of a Banach module. If A is a Banach algebra, then a continuous linear functional $f \in A^*$ is called almost periodic if its orbit $\{a \cdot f : a \in A, \|a\| \leqslant 1\}$ is relatively compact in A^*. Weakly almost periodic functionals are defined similarly. Some of the early properties of such functionals were developed by Young [**48**], Lau-Wong [**32**], Duncan-Ülger [**19**]. For additional developments we can mention Chou [**13**–**15**], Dunkl-Ramirez [**20**], Eberlein [**21**], Hartman [**25**], Kaijser [**28**]. And almost periodic functionals on Fourier algebras of locally compact group G has been studied by Hu [**27**].

A complete characterization of weakly almost periodic functionals on Banach algebras is given in Filali, Neufang and Monfared [**22**]. More precisely, it is shown that weakly almost periodic functionals on a Banach algebra A with a bounded approximate identity, are exactly the coordinate functionals of representations of A on reflexive Banach spaces.

Recently Filali and Monfared [**23**] introduced an analogue of Bohr compactification in the category of Banach algebras. Starting with a Banach algebra A, one can find a Banach algebra $U(A)$ with the property that $U(A)$ has 'sufficiently many' irreducible finite-dimensional representations, in the sense that such representations separate the points of A. In addition, $U(A)$ has a universal type property, as well as a lifting property for almost periodic functionals. These characteristics of $U(A)$ make it a natural analogue of the Bohr compactification $b(G)$, for almost periodic functionals.

A future project of study is the investigation of conditions under which almost periodic functionals are exactly the norm closure of the coordinate functionals of finite-dimensional representations. This amounts to finding natural conditions under which the analogue of approximation theorem by trigonometric polynomials holds for $AP(A)$. Another interesting question is the study of invariant means on $AP(A)$. As in the group case, invariant means are important tools in the study of various properties of $AP(A)$, including the approximation problem mentioned above.

APPENDIX A

Uniform Continuity

For a set X, we denote by $\mathscr{F}^b(X)$ the set of all bounded complex-valued functions on X. Equipped with the supremum norm, $\mathscr{F}^b(X)$ is a Banach algebra, under the usual pointwise operations.

Definition A.0.1. Let f be a function on the topological group G, and $y \in G$, we define the left and right translates of f by

$$L_y f(x) = f(y^{-1}x), \quad R_y f(x) = f(xy).$$

Theorem A.0.2. *Let G be a topological group and $f \in \mathscr{F}^b(G)$. Then the following are equivalent:*
(i) For every $\epsilon > 0$, there exists a neighborhood U of e such that if $s, t \in G$ and $ts^{-1} \in U$, then $|f(s) - f(t)| < \epsilon$.
(ii) The map $G \longrightarrow \mathscr{F}^b(G)$, $s \mapsto L_s f$ is continuous.

A function satisfying these conditions is called *left uniformly continuous*.

Proof. Suppose (i) holds. For a given $\epsilon > 0$, let U be a neighborhood of e satisfying the condition in (i). Then for $s \in U$ and $t \in G$, we have

$$|L_s f(t) - f(t)| = |f(s^{-1}t) - f(t)| < \epsilon.$$

Since $t \in G$ is arbitrary, it follows that $\|L_s f - f\|_{\sup} \leqslant \epsilon$. Therefore the map $s \mapsto L_s f$ is continuous at e. Then we prove the map is continuous at an arbitrary $a \in G$. The set $V = Ua$ is a neighborhood of a, which means every $t \in V$ is of the form $t = sa$ for some $s \in U$. Thus for $t \in V$, we can write

$$\|L_t f - L_a f\|_{\sup} = \|L_a(L_{ta^{-1}} f - f)\|_{\sup} = \|L_s f - f\|_{\sup} \leqslant \epsilon.$$

This proves (ii).

Next, suppose that (ii) holds. For given $\epsilon > 0$, let U be the neighborhood of e such that $a \in U$ implies that $\|L_a f - f\|_{\sup} < \epsilon$. If $s, t \in G$ are such that $ts^{-1} \in U$, we have $st^{-1} = (ts^{-1})^{-1} \in U$, then

$$|f(s) - f(t)| = |f(s) - (L_{st^{-1}} f)(s)| \leqslant \|f - L_{st^{-1}} f\|_{\sup} < \epsilon.$$

This proves (i). $\square$

Claim A.0.3. Every left uniformly continuous function is continuous.

Proof. Let $a \in G$ and $\epsilon > 0$. Let U be a neighborhood of e such that $ts^{-1} \in U$ implies that $|f(s) - f(t)| < \epsilon$. Then $V = Ua$ is a neighborhood of a and for $s = ta \in V$, the relation $(ta)a^{-1} = t \in U$ implies that

$$|f(s) - f(a)| = |f(ta) - f(a)| < \epsilon.$$

Thus f is continuous at a. $\square$

Theorem A.0.4. *Let G be a topological group and $f \in \mathscr{F}^b(G)$. Then the following are equivalent:*

(i) *For every $\epsilon > 0$, there exists a neighborhood U of e such that if $s, t \in G$ and $s^{-1}t \in U$, then $|f(s) - f(t)| < \epsilon$.*

(ii) *The map $G \longrightarrow \mathscr{F}^b(G)$, $s \mapsto R_s f$ is continuous.*

A function satisfying these conditions is called *right uniformly continuous.*

Proof. Suppose (i) holds. For a given $\epsilon > 0$, let U be a neighborhood of e satisfying the condition in (i). Then for $s \in U$ and $t \in G$, we have

$$|R_s f(t) - f(t)| = |f(ts) - f(t)| < \epsilon.$$

Since $t \in G$ is arbitrary, it follows that $\|R_s f - f\|_{\sup} \leqslant \epsilon$. This proves that the map $s \mapsto R_s f$ is continuous at e. Next, we prove that the map is continuous at an arbitrary $a \in G$. The set $V = aU$ is a neighborhood of a, then every $t \in V$ is of the form $t = as$

for some $s \in U$. Thus for $t \in V$ we can write

$$\|R_t f - R_a f\|_{\sup} = \|R_a(R_{a^{-1}t} f - f)\|_{\sup} = \|R_s f - f\|_{\sup} \leqslant \epsilon.$$

This proves (ii).

Next, suppose that (ii) holds. For given $\epsilon > 0$, let U be the neighborhood of e such that $a \in U$ implies that $\|R_a f - f\|_{\sup} < \epsilon$. Thus if $s, t \in G$ are such that $s^{-1}t \in U$, then

$$|f(s) - f(t)| = |f(s) - (R_{s^{-1}t} f)(s)| \leqslant \|f - L_{s^{-1}t} f\|_{\sup} < \epsilon.$$

This proves (i). □

CLAIM A.0.5. Every right uniformly continuous function is continuous.

PROOF. Let $a \in G$ and $\epsilon > 0$. Let U be a neighborhood of e such that $s^{-1}t \in U$ implies that $|f(s) - f(t)| < \epsilon$. Then $V = aU$ is a neighborhood of a and for $s = at \in V$, then relation $a^{-1}(at) = t \in U$ implies that

$$|f(s) - f(a)| = |f(at) - f(a)| < \epsilon.$$

Thus f is continuous at a. □

We denote the set of all left (right) uniformly continuous functions by $LUC(G)$ ($RUC(G)$). We call

$$UC(G) = LUC(G) \bigcap RUC(G)$$

the set of *uniformly continuous functions.* All three spaces $LUC(G)$, $RUC(G)$ and $UC(G)$ are Banach subalgebras of $C^b(G)$.

THEOREM A.0.6. *If $f \in UC(G)$, the map $G \times G \longrightarrow C^b(G)$, $(s,t) \mapsto L_s R_t f$, is continuous.*

PROOF. Suppose (s_α, t_α) is a net in $G \times G$ such that $(s_\alpha, t_\alpha) \to (s,t)$. Then

$$\|L_{s_\alpha} R_{t_\alpha} f - L_s R_t f\|_{\sup} \leqslant \|L_{s_\alpha} R_{t_\alpha} f - L_s R_{t_\alpha} f\|_{\sup} + \|L_s R_{t_\alpha} f - L_s R_t f\|_{\sup}$$

$$= \|R_{t_\alpha}(L_{s_\alpha} f - L_s f)\|_{\sup} + \|L_s(R_{t_\alpha} f - R_t f)\|_{\sup}$$
$$= \|L_{s_\alpha} f - L_s f\|_{\sup} + \|R_{t_\alpha} f - R_t f\|_{\sup}.$$

Since f is both left and right uniformly continuous, it follows that the right hand side tends to 0 as $(s_\alpha, t_\alpha) \to (s, t)$. $\square$

APPENDIX B

Stone-Weierstrass Theorem

In this appendix we state and prove the Stone-Weierstrass approximation theorem. This result is used in Section 3.5 of this thesis. Our reference for this appendix is Rudin [**41**].

LEMMA B.0.1. *If f is a continuous complex function on $[a,b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a,b]$. If f is real, the P_n may be taken real.

PROOF. Without loss of generality, we assume that $[a,b] = [0,1]$ and $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad (0 \leqslant x \leqslant 1).$$

It is trivial that $g(0) = g(1) = 0$. We define $f(x)$ to be zero outside $[0,1]$. Then f is uniformly continuous on the whole line. We put

$$Q_n(x) = c_n(1-x^2)^n \qquad (n = 1, 2, 3, \ldots)$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)dx = 1 \qquad (n = 1, 2, 3, \ldots). \tag{B.0.34}$$

Since

$$\int_{-1}^{1} (1-x^2)^n dx = 2\int_{0}^{1} (1-x^2)^n dx$$

$$
\begin{aligned}
&\geqslant 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \\
&\geqslant 2\int_0^{1/\sqrt{n}} (1-nx^2) dx \\
&= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},
\end{aligned}
$$

it follows from (B.0.34) that

$$c_n < \sqrt{n}. \tag{B.0.35}$$

For any $\delta > 0$, (B.0.35) implies

$$Q_n(x) \leqslant \sqrt{n}(1-\delta^2)^n \qquad (\delta \leqslant |x| \leqslant 1), \tag{B.0.36}$$

so that $Q_n \to 0$ uniformly in $\delta \leqslant |x| \leqslant 1$. Now, set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt \qquad (0 \leqslant x \leqslant 1). \tag{B.0.37}$$

Given $\epsilon > 0$, we choose $\delta > 0$ such that $|y-x| < \delta$ implies

$$|f(y) - f(x)| < \epsilon/2.$$

Let $M = \sup|f(x)|$. Using (B.0.34) and (B.0.36), and the fact that $Q_n(x) \geqslant 0$. We see that for $0 \leqslant x \leqslant 1$,

$$
\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt \right| \\
&\leqslant \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt \\
&\leqslant 2M\int_{-1}^{-\delta} Q_n(t)dt + \epsilon/2\int_{-\delta}^{\delta} Q_n(t)dt + 2M\int_{\delta}^{1} Q_n(t)dt \\
&\leqslant 4M\sqrt{n}(1-\delta^2)^n + \epsilon/2 < \epsilon
\end{aligned}
$$

for all large enough n, which proves the theorem. □

Corollary B.0.2. *For every interval* $[-a, a]$*, there is a sequence of real polynomials* P_n *such that* $P_n(0) = 0$ *and such that*

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$*.*

Proof. By Lemma B.0.1, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, ...)$$

satisfy the conditions. □

Definition B.0.3. Let A be a family of functions on a set E. Then A is said to *separate points* in E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in A$ such that $g(x) \neq 0$, we say that A vanishes at no point of E.

If A is an algebra and has the property that $f \in A$ whenever $f_n \in A$ $(n = 1, 2, 3, \ldots)$ and $f_n \to f$ uniformly on E, then A is said to be *uniformly closed*. Let B be the set of all functions which are limits of uniformly convergent sequences of A. Then B is called the *uniform closure* of A.

Theorem B.0.4. *Suppose* A *is an algebra of functions on a set* E*,* A *separates points on* E*, and* A *vanishes at no point of* E*. Suppose* x_1, x_2 *are distinct points of* E*, and* c_1, c_2 *are constants (real if* A *is a real algebra). Then* A *contains a function* f *such that*

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

Proof. the assumptions show that A contains functions g, h and k such that

$$g(x_1) = g(x_2), \quad h(x_1) \neq 0, \quad k(x_2) \neq 0.$$

Let

$$u = gk - g(x_1)k, \quad v = gh - g(x_2)h.$$

Then $u, v \in A$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

satisfies the properties. □

Theorem B.0.5. *Let A be an algebra of real continuous functions on a compact set K. If A separates points on K and if A vanishes at no point of K, then the uniform closure B of A consists of all real continuous functions on K.*

Proof. The proof has four steps.
Step 1: If $f \in B$, then $f \in |B|$.

Let

$$a = \sup |f(x)| \qquad (x \in K) \tag{B.0.38}$$

and let $\epsilon > 0$ be given. By Corollary (B.0.1) there exist real numbers $c_1, c_2, \ldots, c_n$ such that

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \epsilon \qquad (-a \leqslant y \leqslant a). \tag{B.0.39}$$

Since B is an algebra, then function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of B. By (B.0.38) and (B.0.39), we have

$$|g(x) - |f(x)|| < \epsilon \qquad (x \in K).$$

Since B is uniformly closed, this shows that $|f| \in B$.

Step 2: If $f \in B$ and $g \in B$, then $\max(f, g) \in B$ and $\min(f, g) \in B$.

As we know,

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2},$$

Therefore if $f_1, \ldots, f_n \in B$, then

$$\max(f_1, \ldots, f_n) \in B \quad \text{and} \quad \min(f_1, \ldots, f_n) \in B.$$

Step 3: Given a real function f, continuous on K, a point $x \in K$, and $\epsilon > 0$, there exists a function $g_x \in B$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \epsilon \qquad (t \in K). \tag{B.0.40}$$

Since $A \subset B$ and A satisfies the hypotheses of Theorem B.0.4 so does B. Hence, for every $y \in K$, we can find a function $h_y \in B$ such that

$$h_y(x) = f(x), \quad h_y(y) = f(y). \tag{B.0.41}$$

By the continuity of h_y there exists an open set J_y, containing y, such that

$$h_y(t) > f(t) - \epsilon \qquad (t \in J_y). \tag{B.0.42}$$

Since K is compact, there is a finite set of points such that

$$K \subset J_{y_1} \cup \cdots \cup J_{y_n}. \tag{B.0.43}$$

Let

$$g_x = \max(h_{y_1}, \ldots h_{y_n}).$$

$g_x \in B$, and g_x has the other properties according to Step 2 and (B.0.41) to (B.0.43).

Step 4: Given a real function f, continuous on K, and $\epsilon > 0$, there exists a function

$h \in B$ such that

$$|h(x) - f(x)| < \epsilon \qquad (x \in K). \tag{B.0.44}$$

Since B is uniformly closed, this statement is equivalent to the conclusion that B consists of all real continuous functions on K. Let us consider the functions g_x, for each $x \in K$, constructed in step 3. By the continuity of g_x, there exist open sets V_x containing x, such that

$$g_x(t) < f(t) + \epsilon \qquad (t \in V_x). \tag{B.0.45}$$

Since K is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

$$K \subset V_{x_1} \cup \cdots \cup V_{x_m}. \tag{B.0.46}$$

Let

$$h = \min(g_{x_1}, \ldots, g_{x_m}).$$

By step 2, $h \in B$, and (B.0.40) implies

$$h(t) > f(t) - \epsilon \qquad (t \in K), \tag{B.0.47}$$

whereas (B.0.45) and (B.0.46) imply

$$h(t) < f(t) + \epsilon \qquad (t \in K). \tag{B.0.48}$$

Finally, (B.0.44) follows from (B.0.47) and (B.0.48). □

We say that an algebra A is *self-disjoint* if for every $f \in A$, its complex conjugate $\bar{f}$ also belongs to A, where $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

Theorem B.0.6 (Stone-Weierstrass). *Suppose A is a self-adjoint algebra of complex continuous functions on a compact set K, A separates points on K and A vanishes at no point of K. Then the uniform closure B of A consists of all complex continuous functions on K. In other words, A is dense in $C(K)$.*

PROOF. Let A_R be the set of all real functions on K which belong to A. If $f \in A$ and $f = u + iv$, with u, v real, then $2u = f + \overline{f}$, and since A is self-adjoint, we see that $u \in A_R$. If $x_1 \neq x_2$, there exists $f \in A$ such that $f(x_1) = 1$, $f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that A_R separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in A$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g$, $f = u + iv$, it follows that $u(x) > 0$; hence A_R vanishes at no point of K.

Thus A_R satisfies the hypotheses of Theorem B.0.5. It follows that every real continuous function on K lies in the uniform closure of A_R, hence lies in B. If f is complex continuous function on K, $f = u + iv$, then $u \in B, v \in B$, hence $f \in B$. This completes the proof. □

Stone-Weierstrass theorem implies the classical Weierstrass trigonometric approximation theorem. Recall that a trigonometric polynomial on $\mathbb{R}$ is a function of the form

$$\sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx),$$

where a_k, b_k are complex numbers. Using the identity $e^{ikx} = \cos kx + i \sin kx$, a trigonometric polynomial can be written as

$$\sum_{k=-n}^{n} c_k e^{ikx},$$

where $c_k \in \mathbb{C}$.

With the help of the Stone-Weierstrass theorem, it is easy to drive the following trigonometric approximation theorem due to Weierstrass.

THEOREM B.0.7 (Weierstrass). *Trigonometric polynomials are dense in $C(\mathbb{T})$.*

Since elements of $C(\mathbb{T})$ are identified with continuous 2π-periodic functions on $\mathbb{R}$, we may also say that every such function can be uniformly approximated by trigonometric polynomials.

Proof. Trigonometric polynomials $\sum_{k=-n}^{n} c_k e^{ikx}$ form a subalgebra of $C(\mathbb{T})$ containing the constants and closed under complex conjunction. Also, the function e^{ix} separates the points of $\mathbb{T}$. Thus Stone-Weierstrass theorem applies the result. $\square$

Since $\widehat{\mathbb{T}} \cong \mathbb{Z}$ consists of the 1-dimensional representations $e^{ix} \mapsto e^{ikx}$, $k \in \mathbb{Z}$, the trigonometric polynomials $\sum_{k=-n}^{n} c_k e^{ikx}$ can be viewed as the coordinate functions of representations in $\widehat{\mathbb{T}}$. From this point of view, we may regard Peter-Weyl theorem as generalization of the trigonometric approximation theorem of Weierstrass.

Bibliography

[1] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups, Function Spaces, Compactifications, Representations*, John Wiley & Sons, New York, 1989.

[2] A. S. Besicovitch, *On generalized almost periodic functions*, Proc. London Math. Soc. **25** (1926), 495–512.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Cambridge University Press, Cambridge, 1932.

[4] S. Bochner, *Beiträge zur Theorie der fastperiodischen Funktionen*, Math. Ann. **96** (1927), 119–147.

[5] S. Bochner and J. von Neumann, *Almost periodic functions in groups, II*, Trans. Amer. Math. Soc. **37** (1935), 21–50.

[6] P. Bohl, *Über die Darstellung von Funktionen einer Variablen durch trigonemetrische Reihen mit mehreren einer Variablen proportionalen Argumenten*, Dorpat, 1893.

[7] H. Bohr, *Sur les fonctions presque périodiques*, Acad. Sci. Paris **177** (1923), 737–739.

[8] H. Bohr, *Zur Theorie der fastperiodischen Funktionen. I. Teil: Eine Verallgemeinerung der Theorie der Fourierreihen*, Acta. Math. **45** (1925), 29–127.

[9] H. Bohr, *Zur Theorie der fastperiodischen Funktionen. II. Teil: Zusammenhang der fastperiodischen Funktionen mit Funktionen von unendlich vielen Variablen; gleichmäbige Approximation durch trigonometrische Summen*, Acta. Math. **46** (1925), 101–214.

[10] H. Bohr, *Zur Theorie der fastperiodischen Funktionen. III . Teil: Dirichletentwicklung analytischer Funktionen*, Acta. Math. **7** (1926), 237–281.

[11] G. E. Bredon, *Topology and Geometry*, Springer, New York, 1993.

[12] R. B. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.

[13] C. Chou, *Weakly almost periodic functions with zero mean*, Bull. Amer. Math. Soc. **80** (1974), 297–299.

[14] C. Chou, *Minimally Weakly almost periodic groups*, J. Funct. Anal. **36** (1980), 1–17.

[15] C. Chou, *Weakly almost periodic functions and Fourier-Stieltjes algebras of locally compact groups*, Trans. Amer. Math. Soc. **274** (1982), 141–157.

[16] J. B. Conway, *A Course in Functional Analysis*, Second Edition, Springer, New York, 1990.

[17] J. Dixmier, *C^*-Algebra*, North-Holland Publishing Company, New York, 1977.

[18] J. Dixmier, *General Topology*, Springer-Verlas, New York, 1984.

[19] J. Duncan and A. Ülger, *Almost periodic functionals on Banach algebras*, Rocky Mountain J. Math. **22** (1992), 837–848.

[20] C. F. Dunkl and D. E. Ramirez, *Weakly almost periodic functionals on the Fourier algebra*, Trans. Amer. Math. Soc. **185** (1973), 501–514.

[21] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240.

[22] M. Filali and M. Neufang and M. Sangani Monfared, *Representations of Banach algebras subordinate to topologically introverted spaces*, Trans. Amer. Math. Soc. **367** (2015), 8033–8050.

[23] M. Filali and M. Sangani Monfared, *Almost periodic functionals and finite-dimensional representations*, Studia Math. **243** (2018), 329–344.

[24] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, 1995.

[25] P. Hartman, *Mean motions and almost periodic functions*, Trans. Amer. Math. Soc. **46** (1939), 66–81.

[26] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1, Second Edition, Springer-Verlag, Berlin, 1979.

[27] Z. Hu, *Open subgroups of G and almost periodic functionals on $A(G)$*, Proc. Amer. Math. Soc. **128** (2000), 2473–2478.

[28] S. Kaijser, *On Banach modules I*, Math. Proc. Camb. Phil. Soc. **90** (1981), 423–444.

[29] E. Kaniuth, *A Course in Commutative Banach Algebras*, Springer, New York, 2009.

[30] Y. Katznelson, *An Introduction to Harmonic Analysis*, Third Edition, Cambridge University Press, Cambridge, 2004.

[31] J. L. Kelly, *General Topology*, Springer, New York, 1991.

[32] A. T.-M. Lau and J. C. S. Wong, *Weakly almost periodic elements in $L^\infty(G)$ of a locally compact group*, Proc. Amer. Math. Soc. **107** (1989), 1031–1036.

[33] W. J. Leveque, *Topics in Number Theory*, Dover Publications, New York, 1984.

[34] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, D. Van Nostrand Company, Inc. New York, 1953.

[35] W. Maak, *Fastperiodische Funktionen*, Grundlehren der Mathematischen Wissenschsften, Bd. LXI, Berlin, Springer, 1950.

[36] W. Maak, *Fastperiodische funktionen auf halbgruppen*, Acta Math. **87** (1952), 33–58.

[37] J. von Neumann, *Almost periodic functions in a group. I*, Trans. Amer. Math. Soc. **36** (1934), 445–492.

[38] J. von Neumann and E. P. Wigner, *Minimally almost periodic groups*, Ann. Math. **41** (1940), 746–750.

[39] T. W. Palmer, *Banach Algebras and the General Theory of $*$-Algebras*, Volume I, Cambridge University Press, Cambridge, 2004.

[40] C. E. Rickart, *General Theory of Banach Algebras*, R. E. Krieger Publishing Company, New York, 1960.

[41] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, New York, 1976.

[42] B. Simon, *Operator Theory: A Comprehensive Course in Analysis, Part 4*, American Mathematical Society, Providence, 2015.

[43] W. Stepanoff, *Sur quelques généralisations des fonctions presque périodques*, C. R. Acad. Sci. Paris **181** (1925), 90–92.

[44] W. Stepanoff, *Üeber einige Verallgemeinerungen der fastperiodischen Funktionen*, Math. Ann. **45** (1925), 473–498.

[45] A. Weil, *L'intégration dans les Groupes Topologiques et ses Applications*, Hermann, Paris, 1940.

[46] H. Weyl, *Integralgleichungen und fastperiodischen Funktionen*, Math. Ann. **97** (1927), 338–356.

[47] H. Weyl, *Almost periodic invariant vectors in a metric space*, Amer. J. Math. **71** (1949), 178–205.

[48] N. J. Young, *Periodicity of functionals and representations of normed algebras on reflexive spaces*, Proc. Edinburgh Math. Soc. **20** (1976), 99–120.

Vita Auctoris

Yihan Zhu was born in Jiangsu, China in 1995. She graduated from University of Nanjing Normal University in 2017 in Nanjing, and obtained her B.Sc. in Mathematics and Education. She is currently a master candidate in the department of Mathematics and Statistics at the University of Windsor and hopes to graduate in 2019.