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CENTRAL SEQUENCES AND C*-ALGEBRAS

BY

HEMANT PENDHARKAR
M.S. Mathematics, University of New Hampshire, 1996

DISSERTATION

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July 27, 1999.

Date

Dedication

To
my father-in-law V. Ananthnarayan
Richard and Torene
and
my Jyoti

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Foreword

This paper is designed to be the author's Ph.D. dissertation. It is organized as follows:

Chapters 1, and 2 contain a short presentation of the specific area of C^* -algebras in which the subject of this paper is placed. Chapter 3 contains a description of central sequences in certain C^* -algebras. Chapter 4 discusses the relationship between central sequences and multiplicity free representations. This chapter also provides a description of certain C^* -subalgebras of Continuous trace C^* -algebras.

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Abstract

Central Sequences and C*-algebras

By

Hemant Pendharkar

University of New Hampshire, September, 1999

We study central sequences of C*-algebras. We find connections of the central sequences of a C*-algebra and its representations. More specifically, we prove the following results:

- Characterization of central sequences in certain C*-subalgebras of $C(X, M_n)$, where X is a compact Hausdorff space. We also state the conditions under which central sequences are trivial/hypercentral.
- A representation of the C*-algebra is in the point norm closure of the set of all equivalence classes of irreducible representation if and only if it is multiplicity free.
- For a C*-algebra A , all of whose representations are bounded by some fixed number, the following are equivalent:
 1. A is a continuous trace C*-algebra..
 2. Every central sequence in A is trivial.
 3. $\text{Irr}(A, M_n)$ is point norm closed in $\text{Rep}(A, M_n)$.
 4. A can be written as a finite direct sum of C*-algebras of the form $C(X, M_n, \sim, \beta)$

where \sim is an equivalence relation on X and $\beta : \sim \rightarrow U_n$. U_n is the set of $n \times n$ unitary matrices.

Chapter 1

Preliminaries

1.1 Introduction

Classification of C^* -algebras and finding C^* -algebraic invariants have always interested operator algebraists. In the past, some connections between central sequences and derivations, and automorphism groups of C^* -algebras have been established. It turns out that the notion of central sequences is a useful tool for classifying certain C^* -algebras.

S. Sakai [13, 14] showed that every derivation of a simple unital C^* -algebra is inner. G. Elliott [4] has studied C^* -algebras all of whose derivations are inner. An important step towards obtaining a classification theorem for C^* -algebras using central sequences was taken by C. A. Akemann and G. K. Pedersen [1]. They established that a separable C^* -algebra is a continuous trace C^* -algebra if and only if every central sequence in it is trivial. They further showed that a derivation of a separable C^* -algebra is implemented by a multiplier if and only if every summable central sequence is trivial. They went on to describe these C^* -algebras as the direct sum of a continuous trace C^* -algebra and a reduced sum of simple C^* -algebras.

J. Phillips [11] established that a certain C^* algebra, which is not a continuous trace C^* -algebra, has an uncountable outer automorphism group. He also studied the relationship

between the automorphism group of a separable unital C^* -algebra and central sequences.

Continuous trace C^* -algebras were first described by Fell [3]. His definition is not easy to check; for that matter it is complicated to remember. Furthermore, it is also difficult to determine whether every central sequence is trivial for a given C^* -algebra. Our primary aim is to understand separable unital C^* -algebras whose central sequences are hypercentral. The goal is also to establish the relationship between central sequences and certain representations of the C^* -algebra.

The collection of all equivalence classes of irreducible representation of a C^* -algebra is called the *spectrum*. It is natural to ask, “when are two (irreducible) representations geometrically the same?”. Multiplicity theory came along addressing this question. Answering this question thus got reduced to classifying multiplicity free representation in terms of a suitable set of invariants.

In this work, we establish a connection between central sequences and multiplicity free representations. The C^* -algebras in question are unital, separable, and have a bound on the dimension of all representations. This class of C^* -algebras is a subclass of “completely continuous representation” C^* -algebras.

We describe central sequences in certain C^* -algebras. We also determine when central sequences are trivial and when they are hypercentral. We study the spectrum, topologized by the point norm topology, of a separable unital C^* -algebra, whose representations are bounded

by some fixed number. Using central sequences, we establish that for such C^* -algebras, a representation is in the point norm closure of the set of irreducible representations if and only if it is multiplicity free. We also show that the set of irreducible representations is point norm closed in the spectrum if and only if every central sequence is trivial. Consequently, for any continuous trace C^* -algebra, whose representations are bounded by a fixed number, the set of irreducible representations is point norm closed in the spectrum. We give a description of continuous trace C^* -algebras whose representations are bounded by a fixed number.

1.2 Definitions and Examples

Let H be a complex Hilbert Space, and $B(H)$ be the algebra of all bounded linear operators on H .

Definition 1.2.1 A C^* -algebra is a norm closed self adjoint subalgebra of $B(H)$.

Equivalently, it is a Banach algebra A with an *involution* $*$: $A \rightarrow A$, satisfying

$$1. (a^*)^* = a$$

$$2. (a + b)^* = a^* + b^*$$

$$3. (\lambda a)^* = \bar{\lambda} a^*$$

$$4. (ab)^* = b^* a^*$$

$$5. \|a^* a\| = \|a\|^2$$

for every $a, b \in A$, and for every $\lambda \in \mathbb{C}$.

Throughout this paper, C^* -algebras will be separable and unital with unit 1.

Examples:

1. If H is a Hilbert space then $B(H)$ is a C^* -algebra, where the $*$ operation is the adjoint operation. If dimension of $H = n < \infty$, then $B(H)$ is the C^* -algebra of all $n \times n$ matrices.

2 If X is a compact Hausdorff space then $C(X)$ the set of all continuous complex valued functions on X , is a C^* -algebra. The $*$ operation is given by complex conjugation. This is a commutative C^* -algebra.

3 If X is a compact Hausdorff space then $C(X, M_n)$ is a C^* -algebra.

Definition 1.2.2 A representation of a C^* -algebra A is a unital $*$ -homomorphism $\pi : A \rightarrow B(H)$ for some Hilbert space H , i.e. π is an algebra homomorphism and $\pi(a^*) = (\pi(a))^*$.

Definition 1.2.3 Central Sequence.

A bounded sequence $\{a_n\}$ in a C^* -algebra A , is called a *Central Sequence*, provided $\|a_n x - x a_n\| \rightarrow 0$, as $n \rightarrow \infty$, for each $x \in A$.

Example:

Suppose $\{a_n\} \subset A$ is a sequence that converges to 0. Then clearly $\{a_n\}$ is a central sequence.

Another example of a central sequence is a bounded sequence $\{a_n\}$ in $Z(A)$, where $Z(A)$ is the center of A . Observe that if $\{a_n\}$ and $\{b_n\}$ are central sequences then $\{a_n + b_n\}$ is a central sequence. In particular, a sequence $\{z_n + w_n\}$ which is the sum of a null-convergent sequence $\{z_n\}$ and a sequence $\{w_n\}$ of elements in the center of A , is also a central sequence. For suppose, $x \in A$. Then

$$\|x(z_n + w_n) - (z_n + w_n)x\| \leq \|xz_n - z_nx\| + \|xw_n - w_nx\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Definition 1.2.4 Trivial central sequence.

A central sequence in a C*-algebra A is called a *trivial* central sequence provided, it can be written as the sum of a null sequence and a sequence of elements in the center of A .

Definition 1.2.5 Hypercentral sequence

A central sequence $\{a_n\}$ in a C*-algebra A is said to be *hypercentral* provided, for any central sequence $\{b_n\}$, $\|a_nb_n - b_na_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Remark: Clearly for a C*-algebra A , if all central sequences are trivial, then they are also hypercentral.

Proof. Suppose every central sequence in A is trivial. Let $\{a_n\}$ be such a sequence. To show that $\{a_n\}$ is hypercentral. Let $\{b_n\}$ be any central sequence. Since $\{a_n\}$ is trivial, we can write $\{a_n\} = \{z_n + w_n\}$, the sum of a null sequence $\{z_n\}$ and a sequence $\{w_n\}$ of elements in the center of the C*-algebra. Now,

$$\|b_n(z_n + w_n) - (z_n + w_n)b_n\| \leq \|b_nz_n - z_nb_n\| + \|b_nw_n - w_nb_n\|.$$

The second summand on the right hand side is clearly zero for each n , and

$$\|b_nz_n - z_nb_n\| \leq \|b_nz_n\| + \|z_nb_n\|.$$

Now since $\{b_n\}$ is bounded and $\{z_n\}$ is null convergent, both the summands on the right hand side converge to zero as $n \rightarrow \infty$. This implies that $\{a_n\}$ is hypercentral.

Questions. Suppose A is a C^* -algebra.

1. When is every central sequence in A trivial?
2. When is every central sequence in A hypercentral?

We will discuss the above questions for a certain class of C^* -algebras. We now introduce examples that provide the motivation for some of the ideas in this paper.

Example 1.2.6 Let $A = C([0,1], M_2)$, the C^* -algebra of all 2×2 complex matrix valued continuous functions on the closed interval $[0,1]$. The norm of an element $f \in A$ is given by $\|f\| = \sup_{t \in [0,1]} \|f(t)\|$.

Remark: Every central sequence in A is trivial.

Proof: Suppose $\{f_n\} = \left\{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\}$ is a central sequence in A . Then $\{f_n\}$ must

asymptotically commute with every element of A . In particular, $\{f_n\}$ must asymptotically commute with the constant functions,

$$p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } q(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

i.e. we have,

$$0 = \lim_{n \rightarrow \infty} \|f_n p - p f_n\|$$

$$= \lim_{n \rightarrow \infty} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\| = \lim_{n \rightarrow \infty} \left\| \begin{pmatrix} 0 & -b_n \\ c_n & 0 \end{pmatrix} \right\|$$

and

$$0 = \lim_{n \rightarrow \infty} \| f_n q - q f_n \|$$

$$= \lim_{n \rightarrow \infty} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\| = \lim_{n \rightarrow \infty} \left\| \begin{pmatrix} -c_n & a_n - d_n \\ 0 & c_n \end{pmatrix} \right\|.$$

This implies $b_n \rightarrow 0$, $c_n \rightarrow 0$, and $d_n - a_n \rightarrow 0$, as $n \rightarrow \infty$.

Thus we can write,

$$f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 0 & b_n \\ c_n & d_n - a_n \end{pmatrix} + \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix},$$

which is the sum of a null sequence and a sequence of elements of the center. Thus every central sequence in A is trivial.

We can get interesting examples by looking at certain elementary subalgebras of A .

Example 1.2.7 Let D be a unital C^* -subalgebra consisting of diagonal elements in M_2 , and let $B = \{f \in A : f(0) \in D\}$, where A is the algebra in example 1.2.6.

For each n , define the mapping $f_n : [0,1] \rightarrow [0,1]$ given by,

$$f_n(t) = 0 \quad 0 \leq t \leq \frac{1}{4n}$$

$$f_n(t) = 4nt - 1 \quad \frac{1}{4n} \leq t \leq \frac{1}{2n}$$

$$f_n(t) = -2nt + 2 \quad \frac{1}{2n} \leq t \leq \frac{1}{n}$$

$$f_n(t) = 0 \quad \frac{1}{n} \leq t \leq 1.$$

Now for any element A in D , if $\alpha_n(t) = f_n(t)A$ for $t \in [0,1]$, then the sequence $\{\alpha_n\}$ is a central sequence. It is enough to check in the neighborhood of 0. And in the neighborhood of 0, functions take values that are close to the algebra D . Now by the definition of α_n , the elements in the sequence take values close to D in the neighborhood of 0 (since A is an element of D). Since diagonal matrices commute, it follows that the sequence $\{\alpha_n\}$ asymptotically commutes with every element of A . Also

observe that the constant function $p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in B . Taking $A = p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\|f_n(\frac{1}{2n})p(\frac{1}{2n}) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\| = \|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\| > 0 \text{ as } n \rightarrow \infty.$$

By definition, a sequence $\{\beta_n\}$ in A , is trivial if and only if it can be written as a sum of a null convergent sequence $\{z_n\}$ and a sequence $\{w_n\}$ of elements in the center of A , i.e., $\{\beta_n\}$ is trivial if and only if $\beta_n - w_n = z_n$. In the above example, the elements in the center of B are functions that take values in the center of M_2 . These are 2×2 scalar matrix valued continuous functions on $[0, 1]$. It follows that $\{\alpha_n\}$ is a non-trivial central sequence.

It turns out, (and follows from the theorem in this paper, which will characterize central sequences) that every central sequence in this algebra is hypercentral.

Chapter 2

Representations

2.1 Spectrum

Definition 2.1.1 Let $\pi : A \rightarrow B(H)$, be a representation of A on a Hilbert space H . Then $\pi(A)$ is a C^* -subalgebra of $B(H)$. A representation π is called *nondegenerate*, provided the norm closed linear span of $[\pi(A)H] = H$.

Definition 2.1.2. Let π and σ be two representations of A on Hilbert spaces H and K respectively. Then, π and σ are said to be equivalent if there is a unitary operator $U : H \rightarrow K$ such that $\sigma(x) = U\pi(x)U^*$ for all $x \in A$. This equivalence is denoted by $\pi \sim \sigma$.

Definition 2.1.3. A non-zero representation π is said to be *irreducible* provided, the C^* -algebra $\pi(A)$ is irreducible, i.e. $\pi(A)$ has no non-trivial invariant subspace.

Definition 2.1.4 An invariant subspace M for the C^* -algebra $\pi(A)$ is said to be *cyclic*, provided there exists a vector ξ in H such that the closed linear span of the

vectors of the form $\pi(x)\xi$, $x \in A$, is all of M . A representation π is called a *cyclic representation* of A on a Hilbert space H provided, H is a cyclic subspace for π .

Note: A cyclic representation is nondegenerate.

Some notations:

- $\text{Rep}(A)$ will denote the class of all representations of the C*-algebra A .
- $\text{Irr}(A)$ will denote the class of all irreducible representations of A .
- $\text{Irr}(A, M_n)/\sim$ will denote the set of equivalence classes of irreducible representations into M_n for some fixed n .

Definition 2.1.5 A linear functional f on a C*-algebra A is said to be *positive* if $f(z^*z) \geq 0, \forall z \in A$. If $f(1) = 1$, then f is called a *state*.

Theorem 2.1.6 (Gelfand-Naimark-Segal)

Every C*-algebra A is isometrically *-isomorphic to a C*-algebra of operators on some Hilbert space H .

When we write an irreducible representation π , we mean the unitary equivalence class $[\pi]$.

2.2 Topologies

Definition 2.2.1 Let A be a C^* -algebra. Then the set of all equivalence classes of irreducible representations of A (denoted by \hat{A}) is called the *spectrum* of A .

Definition 2.2.2 An ideal J in A is called a *primitive* ideal provided there exists an irreducible representation π of A such that $J = \ker \pi$.

Definition 2.2.3 Let $\text{prim}(A)$ denote the set of all primitive ideals in A . Let S be a subset of the set of $\text{prim}(A)$. We define the closure of S to be the set

$$\{ m : m \supseteq \bigcap_{j \in S} j \}.$$

This defines a topology on the set of all primitive ideals of A called the hull-kernel topology.

Consider the mapping,

$$\alpha : \hat{A} \rightarrow \{ \ker \pi : \pi \in \hat{A} \} \text{ given by } \alpha(\pi) = \ker \pi.$$

Remark 2.2.3.1 The range of α is given the relative hull-kernel topology, and \hat{A} is topologized by pulling back this topology using the map α . Thus a subset of \hat{A} is open if and only if it has the form $\alpha^{-1}(S)$ for some open subset S of $\text{prim}(A)$.

Definition 2.2.4 Point-norm topology on \hat{A}

A sequence of representations $\{\pi_n\}$ of A , converges to a representation π in the point-norm topology provided $\pi_n(a) \rightarrow \pi(a)$, for every $a \in A$.

Chapter 3

Some Results on Central Sequences

In this chapter we characterize central sequences in certain C^* -algebras. These results will be used later to establish a relationship between central sequences and multiplicity free representations.

Definition 3.1 Let A be a C^* -algebra. Let S be a subset of A . We define the *commutant* of S in A , denoted by S' , to be the set

$$S' = \{a \in A : as = sa, \forall s \in S\}.$$

In particular the commutant of A in A is the center of A .

Theorem 3.2 Let A be a unital C^* -subalgebra of M_k . Let X be a compact Hausdorff space, let x_0 be a limit point of X , and let B be the C^* -subalgebra of $C(X, M_k)$ where given by,

$$B = \{f \in C(X, M_k) \mid f(x_0) \in A\}.$$

Let $\{f_n\}$ be a bounded sequence in B . Then $\{f_n\}$ is a central sequence in B if and only if

1. for some sequence of scalar-valued functions λ_n , $\|f_n - \lambda_n I_k\| \rightarrow 0$ uniformly on compact subsets of $X \setminus \{x_0\}$, and

2. distance $(f_n, A') \rightarrow 0$ uniformly on X .

Proof: Let $\{f_n\}$ be a central sequence. Then,

$\sup_{t \in X} \|f_n(t)g(t) - g(t)f_n(t)\| \rightarrow 0$ for every $g \in B$. Suppose distance (f_n, A') does not go to zero. This implies that $\sup_{t \in X} \inf_{a \in A'} \|f_n(t) - a\|$ does not go to zero. Thus there exists an $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that, for every k ,

$$\sup_{t \in X} \inf_{a \in A'} \|f_{n_k}(t) - a\| > \varepsilon,$$

i.e. $\exists \{t_k\} \subset X, k = 1, 2, 3, \dots$ such that, $\inf_{a \in A'} \|f_{n_k}(t_k) - a\| > \varepsilon$.

Therefore $\exists \varepsilon' > \varepsilon$ such that, $\forall a \in A', \|f_{n_k}(t_k) - a\| \geq \varepsilon'$. Using the compactness argument, we can find a convergent subsequence $\{f_{n_k}(t_k)\} \subset M_k$ that converges to some $w \in M_k$. Then distance $(w, A') = \lim_{k \rightarrow \infty} \text{distance}(f_{n_k}(t_k), A') \geq \varepsilon'$.

This implies that $w \notin A'$. Therefore we can find an $a \in A$ such that, $wa \neq aw$,

i.e. $a \lim_{k \rightarrow \infty} f_{n_k}(t_k) \neq \lim_{k \rightarrow \infty} f_{n_k}(t_k)a$. This implies that $\|f_{n_k}(t_k)a - af_{n_k}(t_k)\|$ does not go to zero. Define $g \in B$ such that, $g(t) = a, t \in X$. Then clearly,

$$\sup_{t \in X} \|f_{n_k}(t)g(t) - g(t)f_{n_k}(t)\| \geq \|f_{n_k}(t_k)g(t_k) - g(t_k)f_{n_k}(t_k)\|$$

which is bounded away from zero. This is a contradiction since $\{f_n\}$ is a central sequence. Hence condition (2) of the theorem is true.

It remains to show that condition (1) is true. We want to show that distance $(f_n(t), M_k') \rightarrow 0$ uniformly on compact subsets of $X \setminus \{x_0\}$. Suppose not. This implies that there exists a neighborhood U_{x_0} of x_0 such that on $X \setminus U_{x_0}$, distance $(f_n(t), C \cdot I_k)$ does not go to zero uniformly. i.e. $\sup_{t \in X \setminus U_{x_0}} \inf_{a \in M_k(C)} \|f_n(t) - a\|$ does not go to zero.

Therefore there exists an $\varepsilon > 0$, and a subsequence $\{n_j\}$ such that for each k ,

$$\sup_{t_j \in X \setminus U_{x_0}} \inf_{a \in M_k(\mathbb{C})} \|f_{n_j}(t_k) - a\| > \varepsilon,$$

$$\text{i.e. distance } (f_{n_j}(t_j), I_k) > \varepsilon.$$

Again using the compactness argument, we can find a convergent subsequence which we will still call $\{f_{n_j}(t)\}$, that converges to some $w \in M_k$. Then,

distance $(w, C \cdot I_k) = \lim_{j \rightarrow \infty} \text{distance } (f_{n_j}(t_j), I_k) > \varepsilon$. Thus $w \notin M_k'$. We can find an

$a \in M_k$ such that $aw \neq wa$. This implies that $\|f_{n_j}(t_j)a - af_{n_j}(t_j)\|$ does not go to zero.

Define using Urysohn's lemma, a continuous function $h : X \rightarrow [0,1]$ such that,

$h(t) = 1, t \in X \setminus U_{x_0}$ and $h(x_0) = 0$, where U_{x_0} is a neighborhood of x_0 . And now define,

$g : X \rightarrow M_k$ given by $g(t) = h(t)a$. Then clearly $g \in B$, and

$$\sup_{t \in X} \|f_{n_j}(t)g(t) - g(t)f_{n_j}(t)\| \geq \|f_{n_j}(t_j)g(t_j) - g(t_j)f_{n_j}(t_j)\| \geq \varepsilon.$$

This is a contradiction, since $\{f_n\}$ is a central sequence.

Conversely, let us suppose that conditions (1) and (2) hold. We will show that $\{f_n\}$ is a central sequence. Let $g \in B$, and let $\varepsilon > 0$. Then using continuity we can find a

neighborhood U of x_0 such that, whenever $t \in U$, we have $\|g(t) - g(x_0)\| < \varepsilon$. Also by

condition (1), for $t \in X \setminus U$, we have $\|f_n(t)g(t) - g(t)f_n(t)\| \rightarrow 0$ uniformly. Now,

$$\|f_n(t)g(t) - g(t)f_n(t)\| = \|f_n(t)(g(t) - g(x_0) + g(x_0)) - (g(t) - g(x_0) + g(x_0))f_n(t)\|$$

Using boundedness of the sequence $\{f_n\}$, we can find some positive number M such that

$\|f_n\| \leq M, \forall n$. By condition (2), we know that the sequence takes values close to the

commutant of A . It follows now from triangle inequality that $\{f_n\}$ is a central sequence.

Theorem 3.3 Let A be a unital C^* -subalgebra of M_k . Let x_0 be a limit point of X . Also let $B = \{f \in C(X, M_k) : f(x_0) \in A\}$. Then every central sequence in B is hypercentral if and only if A is abelian.

Proof: Suppose A is abelian. Let $\{f_n\}$ and $\{g_n\}$ be two central sequences. Let U be a neighborhood of x_0 . We know that, $\{f_n\}$ and $\{g_n\}$ are close to scalars when restricted to $X \setminus U$ (since $X \setminus U$ is a compact subset in X). Hence they commute asymptotically when restricted to the compact subset $X \setminus U$. In the neighborhood U of x_0 , the functions take values close to the commutant of A , i.e.,

$$\|f_n g_n - g_n f_n\| \leq \sup_{t \in X \setminus U} \|f_n(t)g_n(t) - g_n(t)f_n(t)\| + \sup_{t \in U} \|f_n(t)g_n(t) - g_n(t)f_n(t)\|.$$

The two summands are uniformly small. This gives that the sequence is hypercentral.

Conversely, suppose that the commutant of A is not abelian. Let $\{t_n\}$ be a sequence in X such that $t_n \rightarrow x_0$. and let F_{t_n} be the neighborhoods of $t_n, n = 1, 2, 3, \dots$ that do not contain x_0 . Choose a, b in the commutant of A , such that $ab \neq ba$. Also, using Urysohn's lemma, define continuous functions

$$f_n : X \rightarrow [0, 1] \text{ such that } f_n|_{X \setminus F_{t_n}} = 0 \text{ and } f_n(t_n) = 1.$$

Then the sequences $\alpha_n(t) = f_n(t)a$ and $\beta_n(t) = f_n(t)b$ are central sequences that do not commute asymptotically. For,

$$\|\alpha_n \beta_n(t_n) - \beta_n \alpha_n(t_n)\| = \|ab - ba\| \quad n = 1, 2, 3, \dots$$

which is clearly not zero. Hence if commutant of A is abelian, then every central sequence is hypercentral.

Theorem 3.4 Let A be a unital C^* -subalgebra of M_k . Let x_0 be a limit point of X . Also let $B = \{f \in C(X, M_k) : f(x_0) \in A\}$. Then every central sequence in B is trivial if and only if A' consists of scalar multiples of the identity. (In other words; The algebra B has nontrivial central sequences precisely when A is a proper subalgebra of M_k .)

Proof: Suppose A is not all of M_k . We can use the functions defined above in the second half of the proof of Theorem 3.4 to construct a non-trivial central sequence. Choose a non scalar matrix $a \in A'$ and define $\alpha_n(t) = f_n(t)a$. Then $\{\alpha_n(t)\}$ is a central sequence. This sequence is not uniformly close to the scalars since,

$$\|\alpha_n(t_n) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\| = \|a - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\| > 0, n = 1, 2, 3, \dots \quad (\text{as } a \text{ is a non-scalar matrix}).$$

Conversely, suppose every central sequence is trivial. We will show that A is all of M_k . Using the Theorem 3.2, we know that if $\{f_n\}$ is a central sequence, then

1. $\|f_n(t) - \lambda_n(t)I_n\| \rightarrow 0$ uniformly on compact subsets of $X \setminus \{x_0\}$ and
2. $\text{distance}(f_n(t), A') \rightarrow 0$ uniformly on X .

But by the hypothesis, we get that $\text{distance}(A', \text{Scalars})$ goes to zero. This implies that A' is the scalar multiples of the identity.

Corollary 3.5 (To Theorem 3.2) Let A be a unital C^* -subalgebra of M_k . Let C

be the C^* -subalgebra of $C(\{1 - \frac{1}{j}\} \cup \{1\}, M_k)$, $j = 1, 2, 3, \dots$ given by

$$C = \{f : \{1 - \frac{1}{j}\} \cup \{1\} \rightarrow M_k \mid f(1) \in A\}.$$

Let $\{f_n\}$ be a bounded sequences in C . Then $\{f_n\}$ is a central sequence in C if and only if there exists a sequence of scalar valued functions λ_n such that

1. $\|f_n(t) - \lambda_n(t)I_k\| \rightarrow 0$ uniformly on compact subsets of $\{1 - \frac{1}{j}\}$, and
2. $\text{distance}(f_n(t), A') \rightarrow 0$ uniformly on $\{1 - \frac{1}{j}\} \cup \{1\}$.

Remark 3.5.1 When $X = \{1 - \frac{1}{j} : j = 1, 2, 3, \dots\} \cup \{1\}$ and $x_0 = 1$, then

$X \setminus \{x_0\} = \{1 - \frac{1}{j} : j = 1, 2, 3, \dots\}$, and compact subsets of $X \setminus \{x_0\}$ are just finite sets.

Uniform convergence on finite sets is the same as pointwise convergence. Therefore condition (1) above can be written as follow:

1. There exists scalar valued functions $\{\lambda_n(j)\}_{j=1}^{\infty}$ such that

$$\|f_n(1 - \frac{1}{j}) - \lambda_n(j)\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for every } j.$$

Corollary 3.6 Every central sequence in C is hypercentral if and only if A' is abelian.

Corollary 3.7 Every central sequence in C is trivial if and only if A' consists of scalar multiples of the identity.

Definition 3.8 Let A be any C^* -algebra. Let $\pi, \pi_1, \pi_2, \pi_3, \dots : A \rightarrow M_k$ be disjoint irreducible representations such that $\pi_n(a) \rightarrow \pi(a) \forall a \in A$.

Then we define $\rho(A) = \{\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots : a \in A\}$.

Theorem 3.9 Let $\rho(A)$ be as defined above in 3.8. Then $\rho(A)$ is $*$ -isomorphic to the C^* -algebra C in Corollary 3.5.

Proof. Define the mapping $\phi : \rho(A) = \{\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots\} \rightarrow C =$

$\{f : \{1 - \frac{1}{n}\} \cup \{1\} \rightarrow M_k \mid f(1) \in \pi(A)\}$, given by

$\phi(\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots) = f_a$ where, $f_a(1 - \frac{1}{n}) = \pi_n(a)$, $n = 1, 2, 3, 4, \dots$ and

$f_a(1) = \pi(a)$. Then the map is clearly well defined and is a C^* -homomorphism. Also the

map is one to one. For suppose, $f_a = f_b$, $a, b \in A$. Then

$f_a(1 - \frac{1}{n}) = f_b(1 - \frac{1}{n})$, for each n . This implies that $\pi_n(a) = \pi_n(b)$, for each n . Hence

$(\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots) = (\pi(b) \oplus \pi_1(b) \oplus \pi_2(b) \oplus \pi_3(b) \oplus \dots)$.

Thus the C^* -algebra $\phi(\rho(A))$ is the C^* -subalgebra

$\{f_a : \{1 - \frac{1}{n}\} \cup \{1\} \rightarrow M_k \text{ such that } f_a(1) \in \pi(A) : a \in A\}$ of C .

We will now show that $\rho(A) \supseteq C$.

Since π_i is disjoint from π , there exists an $a \in A$ such that $\pi(a) = 0$ and $\pi_i(a) = 1$. Also since $\pi_n(a) \rightarrow \pi(a) = 0$ there exists an N such that $n \geq N$ implies that

$\|\pi_n(a)\| < \frac{1}{2}$. Let g be a continuous function on the reals such that $g|_{[0, \frac{1}{4}]} = 0$ and

$$g|_{[\frac{1}{4} + \varepsilon, \infty)} = 1, \varepsilon > 0 \text{ (small)}.$$

Let $b = g(a^*a)$. Then we get that $\pi_i(b) = 1$. And for $n \geq N$ we have $\|\pi_n(a^*a)\| < \frac{1}{4}$.

Therefore

$$\|\pi_n(b)\| = \|\pi_n(g(a^*a))\| = \|g(\pi_n(a^*a))\| = 0.$$

Using disjointness again for $2 \leq j \leq N$, we can find a $c \in A$ such that $\pi_i(c) = 1$ and $\pi_j(c) = 0$. Let $x = bc$. Then $\pi_i(x) = 1$ and $\pi_j(x) = 0$, $j = 2, 3, 4, \dots$. Now using the x above we get that

$$\begin{aligned} \rho(x)\rho(A) &= \{0 \oplus \pi_i(a) \oplus 0 \oplus 0 \oplus \dots \mid a \in A\}. \\ &= \{0 \oplus T \oplus 0 \oplus \dots \mid T \in M_k\}. \\ &= B_1 \text{ (say)}. \end{aligned}$$

Thus $\rho(A) \supseteq B_1$. Continuing the same way for each π_j , $j = 2, 3, 4, \dots$ we get that the set $B_j = \{0 \oplus 0 \dots \oplus T \oplus 0 \oplus \dots \mid T \in M_k\} \subseteq \rho(A)$. Now since $\rho(A)$ is a C*-algebra, we get that $\rho(A)$ contains the norm closed span of $D = \{0 \oplus T_1 \oplus T_2 \oplus \dots \mid T_n \rightarrow 0\}$.

Now suppose $f : \{1 - \frac{1}{n}\} \cup \{1\} \rightarrow M_k$ such that $f(1) \in \pi(A)$. Write

$f = \{s, s_1, s_2, \dots\}$ where $s_n \rightarrow s \in \pi(A)$. Choose a $w \in A$ such that $\pi(w) = s$. Then

clearly $f - \rho(w) \in D \subseteq \rho(A)$. This implies that $f \in D \subseteq \rho(w) + \rho(A) \subseteq \rho(A)$. Thus $\rho(A)$ is isomorphic to C .

Chapter 4

Some Applications

4.1 Hypercentral Sequences

Definition 4.1.1 A representation π of a C*-algebra A is said to be multiplicity free provided $\pi(A)'$ is abelian.

Definition 4.1.2 Suppose π is a representation of a C*-algebra A on a Hilbert space H . Then the dimension of the representation π is the dimension of the Hilbert space $\pi(A)H$.

Theorem 4.1.3 Let A be a unital separable C*-algebra. Let N be a fixed natural number such that, $\dim \pi \leq N$ for every irreducible representation π of A . Then the following are equivalent:

- (1) Every central sequence in A is hypercentral.
- (2) If a representation π is in the point norm closure of $\text{Irr}(A, M_k)$, $k \leq N$, in the set of all representations of A , then π is multiplicity free.

Proof: Suppose every central sequence is hypercentral. Let π be a representation of

A , in the point norm closure of $\text{Irr}(A, M_k)$ for some $k \leq N$. If π belongs to $\text{Irr}(A, M_k)$, then there is nothing to prove, so suppose it does not. Then we can let $\{\pi_n\}$ be a sequence of pairwise disjoint irreducible representations in $\text{Irr}(A, M_k)$ such that, $\pi_n(a) \rightarrow \pi(a), \forall a \in A$. To show that π is multiplicity free, by Definition 4.1.1, we must show that $\pi(A)'$ is abelian.

Consider the C^* -algebra

$$\rho(A) = \{\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots : a \in A\}.$$

Then by Theorem 3.8, $\rho(A)$ is $*$ -isomorphic to a C^* -subalgebra of

$$C = \{f : \{1 - \frac{1}{n}\} \cup \{1\} \rightarrow M_k \mid f(1) \in \pi(A)\}.$$

But then, every central sequence in $\rho(A)$ is hypercentral. Using Theorem 3.9 and Corollary 3.6, we get that $\pi(A)'$ is abelian.

Conversely, let us suppose that not every central sequence is hypercentral.

We will show that $\pi(A)'$ is not abelian for some π . Let $\{a_n\}$ be a central sequence that is not hypercentral. Then there exists a central sequence $\{b_n\}$ and an $\varepsilon > 0$ be such that, after restricting to a subsequence if necessary, $\|a_n b_n - b_n a_n\| > \varepsilon$.

Using GNS construction, for each n we can find an irreducible representation π_n such that

$$\|\pi_n(a_n b_n - b_n a_n)\| = \|a_n b_n - b_n a_n\| > \varepsilon$$

where $\pi_n : A \rightarrow M_{k(n)}, 1 \leq k(n) \leq N$. $\{k(n)\}$ is a sequence in $\{1, 2, 3, \dots, N\}$. Let $\{k\}$ be the constant subsequence. Passing to the corresponding subsequence, we get that

$$\pi_n : A \rightarrow M_k, n = 1, 2, 3, \dots$$

By using the compactness argument we get that (after passing to a subsequence if required) $\pi_n(a) \rightarrow \pi(a)$ for some π and for each $a \in A$. As before, we can define the C*-algebra $\rho(A) = \{\pi(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \pi_3(a) \oplus \dots : a \in A\}$. Also note, the central sequences $\{\rho(a_n)\}, \{\rho(b_n)\}$ are not hypercentral. Now once again using the Theorem 3.9, we get that $\pi(A)'$ is not abelian.

4.2 Trivial central sequences

In this section, we give the description of a certain subclass of continuous trace C^* -algebras. Akemann and Pedersen [1] have characterized continuous trace C^* -algebras using central sequences. Their result states that a separable C^* -algebra is continuous trace if and only if every central sequence in it is trivial. The C^* -algebra need not be unital.

Lemma 4.2.1 Suppose A is a separable unital C^* -algebra, and suppose π is a finite dimensional irreducible representation of A . Let $E \subseteq \text{Irr}(A)$ such that $\pi \notin E^-$ (closure in the spectrum, see remark 2.2.3.1). Then there is an $a \in A$ such that $\pi(a) = 1$ and for each $\rho \in E$, $\rho(a) = 0$.

Proof. Since $\pi \notin E^-$, it follows from definition 2.2.3 that $\bigcap_{\rho \in E} \ker \rho \not\subset \ker \pi$.

Hence there exists a $b \in A$, with $\pi(b) \neq 0$ and for every $\rho \in E$, $\rho(b) = 0$. However via unitary equivalence, we can assume that $\pi : A \rightarrow M_n$. Since π is irreducible, we must have $\pi(A) = M_n$. Since $0 \neq \pi(b) \in M_n$, $\exists s_1, t_1, \dots, s_n, t_n \in M_n$ such that,

$\sum_{k=1}^n s_k \pi(b) t_k = 1$. Since π is onto, there exists $c_1, d_1, \dots, c_n, d_n \in A$ such that,

$\pi(c_k) = s_k, \pi(d_k) = t_k, 1 \leq k \leq n$. Let $a = \sum_{k=1}^n c_k b d_k$. Then $\pi(a) = 1$. And for each

$\rho \in E$, $\rho(a) = 0$. We can get $0 \leq a \leq 1$ by replacing a with $f(a^*a)$, where

$$f(t) = \min\{t, 1\}.$$

Definition 4.2.2 Suppose X is a compact Hausdorff space, n is a natural number and \sim is an equivalence relation on X such that \sim is a closed subset of $X \times X$. Also suppose $\beta : \sim \rightarrow U_n$, where U_n is the set of unitaries in M_n , is a map such that

$$\forall (x, y), (y, z) \in \sim, \beta(x, y)\beta(y, z) \in \beta(x, z)C.$$

Then we say that (X, \sim, β, n) is an *equivalence system*. We define,

$$C(X, \sim, \beta, M_n) = \{f \in C(X, M_n) : f(x)\beta(x, y) = \beta(x, y)f(y)\}.$$

We say (X, \sim, β, n) is *regular*, if for every

$$x \in X, \text{ for every } T \in M_n, \exists f \in C(X, \sim, \beta, M_n) \text{ such that } f(x) = T.$$

Note: We do not yet know whether every equivalence system is regular.

Lemma 4.2.3 Suppose (X, \sim, β, n) is a regular equivalence system, and B a unital C^* -subalgebra of $C(X, \sim, \beta, M_n)$ such that,

1. for every $x \in X$ and every $T \in M_n$ there is a $g \in B$ such that $g(x) = T$, and
2. for every $x, y \in X$, if $x \neq y$, there exists a $g \in B$ with $g(x) = 0, g(y) = 1$.

Then $B = C(X, \sim, \beta, M_n)$.

Proof: Note that any pure state φ of $C(X, \sim, \beta, M_n)$ can be extended to a pure state of $C(X, M_n)$, and can be represented by an $x \in X$ and a unit vector $e \in C^n$ as

$$\varphi(f) = (f(x)e, e), \forall f \in C(X, M_n).$$

However, since $f \mapsto f(x)$ is an irreducible representation on $C(X, \sim, \beta, M_n)$ (because of regularity), it follows that every functional φ represented in terms of a pair (x, e) as above, is a pure state on $C(X, \sim, \beta, M_n)$. Hence the set of pure states on $C(X, \sim, \beta, M_n)$, being a continuous image of the compact set $X \times \{e \in C^n : \|e\| = 1\}$, is closed in the weak* topology. Suppose now that φ, ψ are pure states on $C(X, \sim, \beta, M_n)$ with

$$\varphi(f) = (f(x)u, u) \text{ and } \psi(f) = (f(y)v, v) \text{ for } x, y \in X \text{ and } u, v \text{ are unit vectors in } C^n.$$

Suppose $\varphi \neq \psi$ on $C(X, \sim, \beta, M_n)$. Then there exists an $f \in C(X, \sim, \beta, M_n)$ such that $\varphi(f) \neq \psi(f)$. We want to find a $g \in \mathcal{B}$ such that $\varphi(g) \neq \psi(g)$. If x is not equivalent to y , then by (2), there exists a $g \in \mathcal{B}$ with $g(x) = 0$ and $g(y) = 1$. Hence $\varphi(g) = 0$ and $\psi(g) = 1$. So $\varphi(g) \neq \psi(g)$. On the other hand, if $x \sim y$, then by (1), we can choose a $g \in \mathcal{B}$ with $g(x) = f(x)$. Since $g \in C(X, \sim, \beta, M_n)$ and $x \sim y$, It follows that $g(y) = f(y)$. Hence $\varphi(g) = \varphi(f) \neq \psi(f) = \psi(g)$. Hence \mathcal{B} separates the closure of the pure states on $C(X, \sim, \beta, M_n)$. And by the Stone-Weirstrass theorem of J. Glimm [6], $\mathcal{B} = C(X, \sim, \beta, M_n)$.

Proposition 4.2.4 Suppose, A is a unital C*-algebra and n is a natural number.

The following are equivalent:

1. Every irreducible representation of A is n -dimensional.
2. A is *-isomorphic to $C(X, \sim, \beta, M_n)$ for some regular equivalence system (X, \sim, β, n) .

Proof: (2) implies (1).

It follows from the characterization of the pure states of $C(X, \sim, \beta, M_n)$ given in the proof of Lemma 4.2.3, combined with the uniqueness part of the GNS construction that every irreducible representation of $C(X, \sim, \beta, M_n)$ is unitarily equivalent to a representation given in terms of an $x \in X$ by $f \mapsto f(x)$. Hence every irreducible representation of $C(X, \sim, \beta, M_n)$ is n -dimensional.

(1) implies (2).

Suppose every irreducible representation of A is n -dimensional. Suppose π is an n -dimensional representation of A . Then π is the direct sum of irreducible (thus n -dimensional) representations. Hence π is irreducible. Thus

$$\text{Irr}(A, M_n) = \text{Rep}(A, M_n).$$

Let $X = \text{Irr}(A, M_n)$ with the point norm topology. Since

$\text{Irr}(A, M_n) = \text{Rep}(A, M_n)$, X must be compact. Suppose $\pi, \rho \in X$. We say that

$\pi \sim \rho \Leftrightarrow \exists$ a unitary u , $\exists \pi(\cdot) = u^* \rho(\cdot) u$. Thus \sim is an equivalence relation on X . If

$\pi_\lambda \sim \rho_\lambda$ for some nets $\{\pi_\lambda\}, \{\rho_\lambda\}$ and $\rho_\lambda \rightarrow \rho$ and $\pi_\lambda \rightarrow \pi$, then there exists a net of

unitarys $\{u_\lambda\}$ such that $u_\lambda^* \pi_\lambda(\cdot) u_\lambda = \rho_\lambda$. But the set $\{u \in M_n : u \text{ is a unitary}\}$ is

compact. So there exists a convergent subnet $u_{\lambda_k} \rightarrow u$ for some unitary u . Thus

$u^* \pi(\cdot) u = \rho$. Hence \sim is closed in $X \times X$. For each $\pi, \rho \in X$, choose $\beta(\pi, \rho) = u_{\pi, \rho}$,

unitary, i.e. $u_{\pi, \rho}^* \pi(\cdot) u_{\pi, \rho} = \rho(\cdot)$. Then suppose that $\rho \sim \tau$. Then $u_{\rho, \tau}^* \rho(\cdot) u_{\rho, \tau} = \tau(\cdot)$.

Hence $u_{\rho, \tau}^* u_{\pi, \rho}^* \pi(\cdot) u_{\pi, \rho} u_{\rho, \tau} = \tau(\cdot)$. But we also have $u_{\pi, \tau}^* \pi(\cdot) u_{\pi, \tau} = \tau(\cdot)$. Thus

$$u_{\pi, \tau} u_{\rho, \tau}^* u_{\pi, \rho}^* \pi(\cdot) u_{\pi, \rho} u_{\rho, \tau} u_{\pi, \tau}^* = (u_{\pi, \rho} u_{\rho, \tau} u_{\pi, \tau}^*)^* \pi(\cdot) (u_{\pi, \rho} u_{\rho, \tau} u_{\pi, \tau}^*) = \pi(\cdot).$$

But $\text{range } \pi = M_n$ so $u_{\pi,\rho} u_{\rho,\tau} u_{\pi,\tau}^* \in C$. Hence $u_{\pi,\rho} u_{\rho,\tau} u_{\pi,\tau} \in C u_{\pi,\tau}$. Now define the mapping $\alpha : A \rightarrow C(X, M_n)$ by $\alpha(a)\pi = \pi(a)$. Then clearly $\alpha(A) \subseteq C(X, M_n, \sim, u)$. However $\alpha(A)$ satisfies the hypothesis of Lemma 4.2.3. Thus

$$\alpha(A) = C(X, \sim, \beta, M_n).$$

Now $\ker \alpha = \bigcap_{\pi \in \text{Irr}(A, M_n)} \ker \pi = \bigcap_{\pi \in \text{Irr}(A)} \ker \pi = \{0\}$. Thus α is one to one. It follows that α is a *-isomorphism from A onto $C(X, \sim, \beta, M_n)$.

Proposition 4.2.5 Suppose (X, \sim, β, n) is a regular equivalence system with X metrizable. Then every central sequence in $C(X, \sim, \beta, M_n)$ is trivial.

Proof: Suppose $\{f_k\}$ is a central sequence in $C(X, \sim, \beta, M_n)$.

Claim: $\lim_{k \rightarrow \infty} \sup_{x \in X} \text{distance}(f_k(x), C \cdot 1) = 0$.

Assume via contradiction that the claim is false. Then, by restricting to an appropriate subsequence if necessary, we can assume that there is an $\varepsilon > 0$ and a sequence $\{x_k\}$ in X such that $\text{distance}(f_k(x_k), C \cdot 1) \geq \varepsilon$, for $k = 1, 2, 3, \dots$. Using the compactness of X , we can also assume that $x_k \rightarrow x$ for some $x \in X$. Suppose $y \in X$.

Then $\{f_k(y)\}$ is a central sequence in M_n , and hence $\text{distance}(f_k(y), C \cdot 1) \rightarrow 0$.

Thus there exists a positive integer k_0 such that, $k \geq k_0 \Rightarrow \text{distance}(f_k(y), C \cdot 1) < \varepsilon$.

If $x_k \sim y$, then $f_k(x_k) \sim f_k(y)$, $f_k \in C(X, \sim, \beta, M_n)$,

so $\text{distance}(f_k(y), C \cdot 1) = \text{distance}(f_k(x_k), C \cdot 1)$. It follows that, if $x_k \sim y$, then

$k < k_0$. Thus only finitely many $x_k \sim y$ for any particular $y \in X$. It follows that we

can find a subsequence, which we will still call $\{x_k\}$ such that for all $1 \leq k < j$ we have, x is not equivalent to x_k and x_k is not equivalent to x_j . It follows from the proof of Proposition 4.2.4 that we can assume $X = \text{Irr}(C(X, \sim, \beta, M_n))$ and \sim is the same as unitary equivalence. Thus $x_j, j = 1, 2, 3, \dots$ are inequivalent representations of $C(X, \sim, \beta, M_n)$ and $x_k \rightarrow x$ in the point norm topology. It follows that no x_k is in the closure (in the spectrum of $C(X, \sim, \beta, M_n)$) of $\{x_j : j \neq k\}$. Thus, by Lemma 4.2.3, for each k there is a $g_k \in C(X, \sim, \beta, M_n)$ such that,

1. $g_k(x_k) = 1$
2. $g_k(x_j) = 0$ for $j \neq k$.

Define $\pi : (C(X, \sim, \beta, M_n) \rightarrow \prod_{k=1}^{\infty} M_n$ (C^* -product, i.e. bounded sequences)

by $\pi(f) = \{f(x_k)\}_{k=1}^{\infty}$. By considering $\pi(g_k)$ we see that the range of π contains

$\sum_{k=1}^{\infty} M_n = \{\{A_k\} \in \prod M_n : \|A_k\| \rightarrow 0\}$. On the other hand, since $x_k \rightarrow x$, it follows

that the range of π is included in $\{\{A_k\} \in \prod M_n : \lim A_k \text{ exists.}\} := B$. We will show

that the range of π is B .

Suppose $\{A_k\} \in B$, and $A_k \rightarrow A$. It follows from regularity that there is an

$f \in C(X, \sim, \beta, M_n)$ such that $f(x) = A$. $f(x_k) \rightarrow f(x) = A$. It follows that

$\{A_k\} - \pi(f) \in \sum M_n \subseteq \text{range } \pi$. Hence $\{A_k\} \in \pi(f) + \text{range } \pi = \text{range } \pi$. Hence

$\text{range } \pi = B$. However, if D is the commutative C^* -algebra of all convergent

sequences in C , then B is isomorphic to $M_n(D)$ or $M_n(C(Y))$ for some compact

Hausdorff space Y . Hence $\text{range } \pi$ has the property that every central sequence is

trivial. Also the center of B is the set of all convergent sequences of scalar multiples of the identity matrix. Since $\{f_k\}$ is a central sequence in $C(X, \sim, \beta, M_n)$, $\{\pi(f_k)\}$ must be a central sequence in B , and thus must be trivial. Hence there is a natural number k such that, for some $\{\lambda_j \cdot 1\}$ in the center of B ,

$$\|\pi(f_k) - \lambda_j \cdot 1\| < \frac{\varepsilon}{2}.$$

In particular, looking at the k^{th} coordinate gives $\|(f_k) - \lambda_j \cdot 1\| < \frac{\varepsilon}{2}$; which contradicts $\text{distance}(f_k(x_k), C \cdot 1) \geq \varepsilon$. This proves the claim, i.e.

$$\lim_{k \rightarrow \infty} \sup_{x \in X} \text{distance}(f_k(x), C \cdot 1) = 0.$$

Suppose $T \in M_n$, and choose $\lambda \in \mathbb{C}$ such that $\|T - \lambda \cdot 1\| = \text{distance}(T, C \cdot 1)$. Then if τ_n is the normalized trace on M_n ,

$$|\tau_n(T) - \lambda| \leq \|\tau_n\| \|T - \lambda \cdot 1\| = \text{distance}(T, C \cdot 1),$$

so, $\|T - \tau_n(T) \cdot 1\| \leq \|T - \lambda \cdot 1\| + |\lambda - \tau_n(T)| < 2 \text{distance}(T, C \cdot 1)$. It follows that if

$f \in C(X, \sim, \beta, M_n)$ then, the function $\tau_n f : x \mapsto \tau_n(f(x)) \cdot 1$ is also in

$C(X, \sim, \beta, M_n)$ since unitarily equivalent matrices have the same trace. Also $\tau_n f$ is

in the center of $C(X, \sim, \beta, M_n)$. Moreover, we have

$$\|f - \tau_n f\| = \sup_{x \in X} \|f(x) - \tau_n(f(x)) \cdot 1\| \leq \sup_{x \in X} \text{distance}(f(x), C \cdot 1).$$

It follows that $\{\tau_n f_k\}$ is a sequence in the center of $C(X, \sim, \beta, M_n)$ and it follows

from the claim that $\|f_k - \tau_n f_k\| \rightarrow 0$. Hence $\{f_k\}$ is a trivial sequence.

Theorem 4.2.6 Suppose N is a natural number and A is a unital separable C^* -

algebra, such that every element of $\text{Irr}(A)$ has dimension at most N . Then the following are equivalent:

- (1) A is a continuous trace C^* -algebra.
- (2) Every central sequence in A is trivial.
- (3) $\text{Irr}(A, M_k)$ is point norm closed in $\text{Rep}(A, M_k)$, $k \leq N$.
- (4) A is a finite direct sum of C^* -algebras of the form $C(X, M_k, \sim, \beta)$.

Proof: (1) is equivalent to (2). [1, Theorem 2.4].

(2) implies (3).

Let π be in the point norm closure of irreducible representations of A . Then we can find a sequence of irreducible representations $\{\pi_n\}$ such that $\pi_n(a) \rightarrow \pi(a), \forall a \in A$. Again we can define the algebra $\rho(A)$ as in the proof of 4.4. Using the hypothesis we get that every central sequence in $\rho(A)$ is trivial. Now using Theorem 3.8, we get that $\pi(A)'$ consists of scalar multiples of identity or π is onto. This implies that π is irreducible.

(3) implies (4).

Let $J = \{k : 1 \leq k \leq N, \text{Irr}(A, M_k) \neq \emptyset\}$. It follows from (3) that $\text{Irr}(A, M_k)$ is point norm compact in $\text{Rep}(A, M_k)$ for each $k \in J$, since $\text{Rep}(A, M_k)$ is point norm compact. Suppose $k, j \in J, k \neq j$. Suppose $\pi \in \text{Irr}(A, M_k)$ and $\rho \in \text{Irr}(A, M_j)$.

Then there exist an $a \in A, 0 \leq a \leq 1$ such that $\pi(a) = 0, \rho(a) = 1$, since ρ, π are

clearly disjoint representations. Let $U_\rho = \left\{ \tau \in \text{Irr}(A, M_j) : \|\rho(a) - \tau(a)\| < \frac{1}{3} \right\}$.

Since $\rho(a) = 1$ and $a \geq 0$ we have for every $\tau \in U_\rho$, $0 \leq \tau(a)$ and $1 - \tau(a) \leq \frac{1}{3}$. Thus

$\frac{2}{3} \leq \tau(a)$. If $0 \leq g \leq 1$ is a continuous function defined on the real numbers such that

$g|_{[0, 1/3]} = 0, g|_{[2/3, \infty)} = 1$. Then $\pi(g(a)) = 0$ and $\tau(g(a)) = 1, \forall \tau \in U_\rho$. Note, U_ρ is

point-norm open. Denote $g(a)$ by x_ρ . Note that $x_\rho \geq 0$. Fix π .

Then $\{U_\rho : \rho \in \text{Irr}(A, M_j)\}$ forms an open cover of $\text{Irr}(A, M_j)$. Thus there exists

$\rho_1, \rho_2, \dots, \rho_s$ such that $\text{Irr}(A, M_j) \subseteq \bigcup_{m=1}^s U_{\rho_m}$. Let $x = x_{\rho_1} + x_{\rho_2} + \dots + x_{\rho_s}$.

Then $\pi(x) = 0$ and for each $\rho \in \text{Irr}(A, M_j)$, $\rho(x) \geq 1$ (since $\rho \in U_{\rho_m}$, and $\rho(x_{\rho_m}) = 1$,

for some m). Again using our function g , we have $\pi(g(x)) = 0$ and for each

$\rho \in \text{Irr}(A, M_j)$, $\rho(g(x)) = g(\rho(x)) = 1$. Call $g(x) = y_{k,j}$. Let $y_\pi = g \sum_{j \in J, j \neq k} y_{k,j}$.

Then $y_\pi \geq 0, \pi(y_\pi) = 0$. And for every $j \in J \setminus \{k\}, \forall \rho \in \text{Irr}(A, M_j), \rho(y_\pi) = 1$.

Now let

$$V_\pi = \left\{ \sigma \in \text{Irr}(A, M_k) \mid \|\sigma(y_\pi) - \pi(y_\pi)\| \leq \frac{1}{3} \right\}.$$

Then V_π is a point norm neighborhood of π in $\text{Irr}(A, M_k)$. Since $\pi(y_\pi) = 0$, we have

$0 \leq \sigma(y_\pi) \leq \frac{1}{3}, \forall \sigma \in V_\pi$. Let $z_\pi = g(y_\pi)$. Then for every $\sigma \in V_\pi$,

$$\sigma(z_\pi) = \sigma(g(y_\pi)) = g(\sigma(y_\pi)) = 0$$

and $\forall j \in J \setminus \{k\}, \forall \rho \in \text{Irr}(A, M_j)$,

$$\rho(z_\pi) = \rho(g(y_\pi)) = g(\rho(y_\pi)) = 1.$$

Then $\{V_\pi : \pi \in \text{Irr}(A, M_k)\}$ is an open cover and has a finite sub-cover, say $V_{\pi_1}, \dots, V_{\pi_r}$.

Let $b = z_{\pi_1} \cdot z_{\pi_2} \cdot \dots \cdot z_{\pi_r}$. Then for every $\pi \in \text{Irr}(A, M_k)$, $\pi(b) = 0$. And for every

$$j \in J \setminus \{k\}, \forall \rho \in \text{Irr}(A, M_j), \rho(b) = 1.$$

Since every element of $\text{Irr}(A)$ is unitarily equivalent to an element of $\bigcup_{k \in J} \text{Irr}(A, M_k)$,

it follows that $\bigcap_{\pi \in \bigcup_{k \in J} \text{Irr}(A, M_k)} \ker \pi = \{0\}$. For each $j \in J$, define $\pi_j = \sum_{\pi \in \text{Irr}(A, M_j)}^\oplus \pi$ and let

$$\rho = \sum_{j \in J}^\oplus \pi_j. \text{ Then } \rho \text{ is one-to-one.}$$

Hence A is isomorphic to $\rho(A)$, and the preceding arguments show that

$$\rho(A) = \sum_{j \in J}^\oplus \pi_j(A). \text{ Moreover, it follows from the definition that for each } j \in J,$$

every irreducible representation of $\pi_j(A)$ is j -dimensional. Thus (4) now follow from Proposition 4.2.4.

(4) implies (2). This follows from Proposition 4.2.5.

Bibliography

- [1] Akemann, C. A. and Pedersen, G. K., *Central sequences and inner derivations of separable C^* -algebras*, Amer. J. Math., 101 (1979), 1047-1061.
- [2] Arveson, W. B., *An Invitation to C^* -algebras*, Springer Verlag, 1976.
- [3] Dixmier, J., *C^* -algebras*, North-Holland, Amsterdam, 1977.
- [4] Elliot, G. A., *Some C^* -algebras with outer derivations*, Rocky Mountain Journal of Math. 3 (1973), 501-506.
- [5] Fillimore, P. A., *A User's Guide to Operator Algebras*, Wiley-Interscience Publication, 1996.
- [6] Glimm, J., *A Stone-Weirstrass theorem for C^* -algebras*, Ann. Math. 72 (1960), 216-244.
- [7] Hadwin, D. W., *A Bishop-Stone-Weirstrass theorem for C^* -algebras*, Indiana University Mathematics Journal, Vol. 38, No. 1 (1989).

- [8] Kadison, R. V. and Ringrose, J. R., *Fundamentals of the Theory of Operator Algebras*, Vols. I and II, Academic Press Inc., 1986.
- [9] Li, Jiankui and Pendharkar, Hemant, *Derivations of certain operator algebras*, to appear in International Journal of Mathematics and Mathematical Sciences.
- [10] Pedersen, G. K., *C*-algebras and their Automorphism Groups*, Academic Press, London, 1979.
- [11] Phillips, John, *Central sequences and automorphisms of C*-algebras*, American Journal of Mathematics, 110 (1988), 1095-1118.
- [12] Raeburn, Iain and Williams D. P., *Morita Equivalence and Continuous Trace C*-algebras*, American Mathematical Society's Mathematical Surveys and Monographs, Vol. 60, 1998.
- [13] Sakai, S., *Derivations of simple C*-algebras*, J. Functional Anal. 2 (1968), 202-206.
- [14] Sakai, S., *Derivations of simple C*-algebras II*, Bull. Soc. Math. France, 99 (1971), 259-263.
- [15] Willard, S., *General Topology*, Addison-Wesley Publishing Company, 1968.