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DIFFERENTIATION AND COMPOSITION ON THE HARDY AND BERGMAN SPACES

BY

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DISSERTATION

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in

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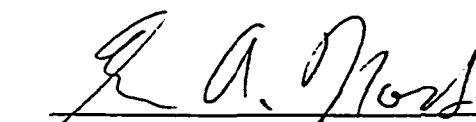
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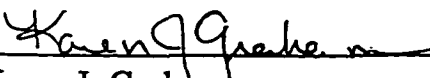
This dissertation has been examined and approved.

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Dedication

To all of my family
and all of my friends
who have helped me fly.

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Acknowledgments

There are many people at the university that deserve acknowledgment for their efforts toward and support of my journey through graduate school, but three really stand out. My acknowledgments to these people are for far more than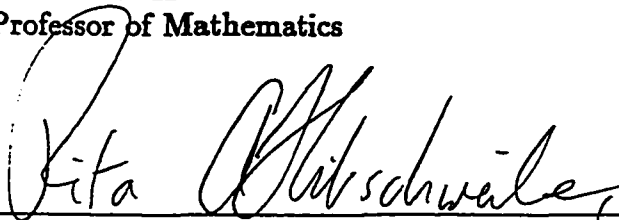 their contributions to my graduate program. I include my appreciation for their friendship, their interest in me as a person, and their wisdom about life. I will speak first about my thesis advisor, Professor Eric Nordgren. For some, Eric Nordgren would be the perfect hiking companion because of his physical height. From his vantage point he could clearly see the way, he could choose the easiest path, he could scale the large boulders and easily bound over other obstacles.

For me, however, Eric Nordgren has been the perfect hiking companion for a completely different set of reasons. If he saw the way, he never described it, preserving for me the joy of discovery. If I encountered large boulders, he didn't take a giant step to the top and pull me up; rather, he let me find the toeholds and finger holds, or even take a completely different path, all the while sharing with me the enjoyment of the climb, reveling in the beauty of the path, no matter which path I chose. For it has always been my choice, and I'm extremely happy that Professor Nordgren was my first. We have enjoyed many side trails on the way to this summit; talking of art, music, plays, books, movies, families, New York City, and even of hiking. He is a great friend, an excellent teacher, and, to use his own words, "a truly decent human being."

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some difficult parts of life. She has been supportive of my work, not only in mathematics, but also in mathematics education. From her, I have learned about teaching, studying, and engaging in mathematics. Rita, like Eric, is a person with whom I've been able to talk about life in general. She is also a "truly decent human being."

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ABSTRACT

DIFFERENTIATION AND COMPOSITION ON THE HARDY AND BERGMAN SPACES

by

Neil Portnoy
University of New Hampshire, May, 1998

Banach spaces of analytic functions are defined by norming a collection of these functions defined on a set X. Among the most studied are the Hardy and Bergman spaces of analytic functions on the unit disc in the complex plane. This is likely due to the richness of these spaces.

An analytic self-map of the unit disc induces a *composition operator* on these spaces in the natural way. Beginning with independent papers by E. Nordgren and J. V. Ryff in the 1960's, much work has been done to relate the properties of the composition operator to the characteristics of the inducing map. Every composition operator induced by an analytic self-map of the unit disc is bounded on the Hardy and Bergman spaces.

Differentiation is another linear operation which is natural on spaces of analytic functions. Unlike the composition operator, the *differentiation operator* is poorly behaved on the Hardy and Bergman spaces; that is, it is not a bounded operator.

We define a linear operator, possibly unbounded, by applying composition followed by differentiation; that is, for f in a Hardy or Bergman space and an analytic self-map of the disk, ϕ,

$$DC_\phi(f) = (f \circ \phi)'.$$

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We have found a characterization for the boundedness of this operator on the Hardy space in terms of the inducing map. The operator is bounded exactly when the image of the self-map of the disc is contained in a compact subset of the disc.

In contrast, we have found a self-map of the disc with supremum norm equal to one that induces a bounded operator on the Bergman spaces. In this setting we have found conditions necessary for boundedness, and conditions sufficient to imply boundedness. These conditions are closely related.

The techniques used involve Carleson-type measures on the unit disc. A very general question arising out of this work involves relating boundedness of the differentiation operator to characteristics of these measures.

Chapter 1

PRELIMINARIES

This chapter presents background information, including definitions and theorems, for the setting of this research. The proofs of most theorems have been omitted since the theorems are well known.

The setting for the work presented in this thesis is the Hardy and Bergman spaces of functions analytic on the unit disk. For $p \geq 1$, these are particular cases of functional Banach spaces. Functional Banach spaces are rich enough to separate the points of the underlying set, and the underlying set is rich enough to separate the functions of the Banach space. The following definition is from Cowen and MacCluer [CoM].

Definition. A Banach space of complex valued functions on a set X is called a *functional Banach space on X* if the vector operations are the pointwise operations, $f(x) = f(y)$ for each function in the space implies $x = y$, $f(x) = g(x)$ for each x in X implies $f = g$, and for each x in X, the linear functional $f \mapsto f(x)$ is continuous.

A functional Banach space whose functions are analytic on the underlying set is called a *Banach space of analytic functions.*

1.1 Subharmonic Functions

It is useful to approach the Hardy and Bergman spaces from the viewpoint of subharmonic functions.

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Definition. A function u defined on an open set S in the plane is said to be *subharmonic* if it has the following four properties:

1. $-\infty \leq u(z) < \infty$ for all $z \in S$,

2. u is upper semicontinuous on S,

3. whenever a closed disc with center a and radius r is contained in S, then

$$u(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta})d\theta,$$

and

4. none of the above integrals is $-\infty$.

It is a fact that the pth power $(0 < p < \infty)$ of the modulus of an analytic function on the unit disc is subharmonic.

THEOREM 1.1.1 (Rudin 17.3). *If R is a region in $\mathbb{C}$ and f is analytic on R and not identically zero, then $\log|f|$ and $|f|^p$, $(0 < p < \infty)$, are subharmonic in R.*

The norm of a function f in the Hardy spaces is achieved via the integral means of the modulus of the function restricted to a circle of radius $r < 1$.

Notation For any continuous function f on $\mathbb{D}$ define f_r on $\partial\mathbb{D}$ by

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

for $0 \leq r < 1$.

Notation Denote by σ normalized Lebesgue measure on $\partial\mathbb{D}$ that is,

$$\sigma(\partial\mathbb{D}) = 1.$$

It is a fact that the integral mean of a subharmonic function is a non-decreasing function of the radius.

THEOREM 1.1.2 (Rudin 17.5). *Suppose u is a continuous subharmonic function in $\mathbb{D}$, and*

$$m(r) = \int_{\partial\mathbb{D}} u_r\, d\sigma$$

for $0 \le r < 1$. If $r_1 < r_2$, then $m(r_1) \le m(r_2)$.

1.2 The Hardy Spaces $H^p(\mathbb{D})$

The classical Hardy spaces (named for G.H. Hardy) have been studied extensively. These spaces consist of those functions, analytic on the unit disk, that meet a certain restricted growth condition. Following are definitions and theorems that are related to this research.

Notation Following the notation of Rudin [Rud], if f is continuous on $\mathbb{D}$ and $0 < p < \infty$, denote the L^p norm of f_r by

$$\| f_r \|_p = \left\{ \int_{\partial\mathbb{D}} |f_r|^p d\sigma \right\}^{1/p}.$$

For $p = \infty$ denote

$$\| f_r \|_\infty = \sup_\theta |f_r(e^{i\theta})|.$$

Definition. For a function f analytic in $\mathbb{D}$ and $0 < p \leq \infty$, define

$$\| f \|_p = \sup\{\| f_r \|_p : \ 0 \leq r < 1\}.$$

The Hardy space H^p is defined to be the class of analytic functions f on $\mathbb{D}$ for which $\| f \|_p < \infty$.

Some important properties are summarized in the following theorem.

THEOREM 1.2.1. *(Rudin 17.11) If $0 < p < \infty$ and $f \in H^p$, then*

- *the nontangential limits $f^*(e^{i\theta})$ exist a.e. on $\partial\mathbb{D}$,*

- *$f^* \in L^p(\partial\mathbb{D})$,*

- *$\lim_{r\to1} \| f^* - f_r \|_p = 0$, and*

- *$\| f^* \|_p = \| f \|_p$.*

Notation f^* will denote both the non-tangential limit and the radial limit, since they are equivalent for $f \in H^p$. If ϕ is an analytic self-map of the disk, then for those points $\phi^*(e^{i\theta})$ which are in the disk, $f^* \circ \phi^*$ will be taken to mean $f \circ \phi^*$.

Note that H^2 is a Hilbert space with inner product $< f, g > = \int_{\partial\mathbb{D}} f^* \overline{g^*} d\sigma$. The repro-

ducing kernel for point evaluation at w in $\mathbb{D}$ for H^2 is given by

$$K_w(z) = \frac{1}{1 - \overline{w}z}$$

and its norm is $\| K_w \|_{H^2} = \frac{1}{\sqrt{1-|w|^2}}$.

Functions in H^2 also have a useful characterization in terms of their power series coefficients. This characterization is a consequence of Parseval's identity.

THEOREM 1.2.2. *(Rudin 17.12) Suppose f is analytic on $\mathbb{D}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$.*

Then $f \in H^2$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

The following result of J. Ryff [Ryf] will be quite useful in the chapter about DC_ϕ on the Hardy spaces.

THEOREM 1.2.3. *(Ryff) If f is in H^p where $p > 0$ and ϕ is an analytic self-map of the unit disk, then $(f \circ \phi)^* = f^* \circ \phi^*$ almost everywhere on $\partial\mathbb{D}$.*

Recall that for $1 \le p \le \infty$, H^p is a Banach space. If $1 \le p \le q \le \infty$, then $H^q \subset H^p$. In fact H^q is a closed subspace of H^p under these conditions. Therefore, the following fact, due to Duren [Dur1], about H^1, the largest of the H^p Banach spaces, is quite important.

THEOREM 1.2.4. *(Duren 3.11) A function f analytic in $\mathbb{D}$ is continuous in $\overline{\mathbb{D}}$ and absolutely continuous on $\partial\mathbb{D}$ if and only if $f' \in H^1$. If $f' \in H^1$, then*

$$\frac{d}{d\theta} f(e^{i\theta}) = ie^{i\theta} \lim_{r \to 1} f'(re^{i\theta}) \quad a.e.$$

1.3 The Bergman Spaces $A^p(\mathbb{D})$

Denote by λ normalized Lebesgue area measure on the unit disk. That is,

$$d\lambda = dA/\pi.$$

The Bergman space A^p $(0 < p < \infty)$ is the set of functions analytic on the unit disk for which

$$\int_{\mathbb{D}} |f|^p d\lambda < \infty.$$

The norm of such a function, $\| f \|_{A^p}$, is the pth root of the above integral. The space A^∞ is taken to be the same space as H^∞ with the same norm (the supremum norm). For $p \geq 1$, A^p is a Banach space. As in the case of the Hardy spaces, we have the following inclusion: for $0 < p \leq q \leq \infty$, $A^q \subset A^p$. It is also true that for any $p > 0$, $H^p \subset A^p$.

The space A^2 is a Hilbert space with inner product

$$< f, g > = \int_{\mathbb{D}} f(z)\overline{g(z)}d\lambda(z).$$

The reproducing kernel for point evaluation at $w \in \mathbb{D}$ in A^2 is given by

$$K_w(z) = \frac{1}{(1 - \overline{w}z)^2}$$

and its norm is $\| K_w \|_{A^2} = \frac{1}{1-|w|^2}$.

The Bergman spaces A^p for $1 \leq p < \infty$ are closed subspaces of the Banach spaces of

Lebesgue measurable functions on $\mathbb{D}$, $L^p(\mathbb{D}, d\lambda)$. The relationship between A^p and $L^p(\mathbb{D}, d\lambda)$ will be significant in finding sufficient conditions for boundedness of DC_ϕ on the Bergman spaces. The following theorem [Zhu] makes this relationship explicit.

THEOREM 1.3.1. *(Zhu) Suppose n is a positive integer, $p \geq 1$, and f is analytic on $\mathbb{D}$. Then f is in A^p if and only if $(1 - |z|^2)^n f^{(n)}$ is in $L^p(\mathbb{D}, d\lambda)$.*

Chapter 2

THE PLAYERS

The operators of composition and differentiation are combined to obtain the operator DC_ϕ that is the focus of this work. The idea to combine these operators came while considering the operator ranges of composition operators. The question asked was "what characteristics of an analytic self map of the disk would induce a composition operator whose range would consist of functions 'nice enough' so that their derivatives would belong to the original space?" A technique used to investigate the properties of DC_ϕ depends on Carleson measures. This chapter gives background information on these three major players; the differentiation operator, the composition operator, and Carleson measures.

2.1 Differentiation

In the setting of spaces of analytic functions it is natural to investigate the linear operator of differentiation, D, given by

$$Df = f'$$

for functions f in the space. Usually the differentiation operator is quite poorly behaved. It is not a bounded operator on H^p or A^p for any $p > 0$. Following are examples that show how poorly behaved the differentiation operator can be.

Example. Set $f(z) = (1 - z)^b$ for $0 < b < 1/2$. Then f is bounded on $\mathbb{D}$. That is, f is in

H^∞. But $f'(z) = b(1-z)^{b-1}$, so f' is not in H^p for any $p \geq \frac{1}{1-b}$, and f' is not in A^q for any $q \geq \frac{2}{1-b}$.

Example. [Dur2] Duren constructed a function f whose power series coefficients were defined in terms of a Rademacher function. The example f is analytic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$, but its derivative f' has radial limit almost nowhere, and thus f' is not in H^p for any $p > 0$. The given function f is in the Zygmund class Λ^* which consists of all continuous functions $F(\theta)$ periodic with period 2π, such that

$$|F(\theta + h) - 2F(\theta) + F(\theta - h)| \leq Ah$$

for some constant A and all $h > 0$. Thus Λ^* contains the Lipschitz class Λ_1. Duren notes that if f were required to belong to the Lipschitz class Λ_1, then the derivative f' would be bounded and thus have radial limit almost everywhere.

2.2 Composition Operators

The study of composition operators began with independent papers by E. A. Nordgren and J. V. Ryff in the 1960's. Composition is a natural mathematical operation and the operator is defined as follows.

Definition. If X is a space of real or complex valued functions on a set S and ϕ maps S into itself, then the *composition operator* induced by ϕ is defined by

$$C_\phi f = f \circ \phi$$

for functions f in the space X.

The composition operator is clearly linear. By an early result of Littlewood, composition operators leave the Hardy and Bergman spaces invariant.

THEOREM 2.2.1. *(Littlewood's subordination theorem) Suppose $\phi : \mathbb{D} \to \mathbb{D}$ is analytic, $\phi(0) = 0$, and G is subharmonic on $\mathbb{D}$. Then for $0 \le r < 1$,*

$$\int_0^{2\pi} G(\phi(re^{i\theta}))d\theta \le \int_0^{2\pi} G(re^{i\theta})d\theta.$$

If ϕ is a self map of the disk, and $\phi(0) \ne 0$, then the change of variable

$$w = \frac{\phi(0) - z}{1 - \overline{\phi(0)}z}$$

will give the result that H^p is invariant under C_ϕ. Invariance of A^p under C_ϕ is a result of an additional integration with respect to rdr.

This work is concerned with the operator DC_ϕ on the analytic Banach spaces H^p and A^p ($p \ge 1$). Note that since the functionals for point-evaluation are continuous, norm convergence of functions in functional Banach spaces guarantees pointwise convergence.

The fact that C_ϕ leaves the Hardy and Bergman spaces invariant, combined with the structure of analytic Banach spaces, results in a very well-behaved operator.

THEOREM 2.2.2. *If ϕ is an analytic self map of the disk, then the composition operator induced by ϕ, C_ϕ, is bounded onH^p and A^p.*

Proof. The proof is an application of the Closed Graph Theorem. Suppose that X is H^p or A^p, that $f_n \to f$ in X and $C_\phi f_n \to g$ in X. Since norm convergence guarantees pointwise

convergence, and since $f_n \to f$,

$$(f_n \circ \phi)(z) \to (f \circ \phi)(z).$$

Also, since $f_n \circ \phi \to g$,

$$(f_n \circ \phi)(z) \to g(z)$$

for all z in $\mathbb{D}$, so $C_\phi f = g$. This shows that Graph(C_ϕ) is closed and thus C_ϕ is bounded.

2.3 Carleson Measures

Another way to see that composition operators are bounded on the Hardy and Bergman spaces is to note that the pull-back measure induced by the self-map of the disk, ϕ, is a Carleson measure. Here are two definitions.

Definition. A *Carleson set* in the open disk $\mathbb{D}$ is a set of the form

$$S(b, h) = \{z \in \mathbb{D} : |z - b| < h\},$$

where $|b| = 1$ and $0 < h < 2$.

Definition. Suppose μ is a positive, finite Borel measure on $\mathbb{D}$. Then μ is called a *Carleson measure* in case there is a constant K so that

$$\mu(S(b, h)) \le Kh$$

for every Carleson set $S(b, h)$.

2.3.1 Carleson Measures and the Hardy Space

In 1962, in connection with his work on the corona problem, L. Carleson related characteristics of a measure μ on the disk to the continuity of the inclusion map from H^p into $L^p(\mathbb{D}, \mu)$.

THEOREM 2.3.1. *(Carleson Measure Theorem) Suppose μ is a measure on $\mathbb{D}$ and $0 < p < \infty$. Then the inclusion map*

$$I : H^p \hookrightarrow L^p(\mathbb{D}, \mu)$$

is bounded if and only if μ is a Carleson measure.

A useful variation of the Carleson Measure Theorem is given in Cowen and MacCluer. The Carleson sets in this setting are subsets of the closed disk;

$$S(\zeta, \delta) = \{z \in \overline{\mathbb{D}} : |z - \zeta| < \delta\}.$$

THEOREM 2.3.2. *Suppose μ is a finite, positive Borel measure on $\overline{\mathbb{D}}$ and $0 < p < \infty$. The following are equivalent:*

1. There is a constant K so that $\mu(S(\zeta, \delta)) < K\delta$ for $|\zeta| = 1$ and $0 < \delta < 1$.

2. There is a constant C so that

$$\int_{\overline{\mathbb{D}}} |f^p| d\mu \leq C \parallel f \parallel_p^p$$

for all f in H^p.

Carleson measures are useful in characterizing both boundedness and compactness of a composition operator. (A linear operator on a Banach space is compact if the image of the unit ball under the operator has compact closure.) The measure of interest is defined below.

Definition. For an analytic self-map of the disk, ϕ, define the pull-back measure μ on $\overline{\mathbb{D}}$ by

$$\mu(E) = \sigma((\phi^*)^{-1} E)$$

for $E \subset \overline{\mathbb{D}}$, where σ is normalized Lebesgue measure on $\partial \mathbb{D}$.

The following theorem from Cowen and MacCluer [CoM] gives characterizations for boundedness and compactness in terms of the pull-back measure defined above.

THEOREM 2.3.3. *Suppose $0 < p < \infty$ and ϕ is an analytic self-map of the disk.*

1. C_ϕ is bounded on H^p if and only if there is a constant K so that

$$\mu(S(\zeta, \delta)) \leq K\delta$$

for all ζ in $\partial \mathbb{D}$ and $0 < \delta < 1$.

2. C_ϕ is compact on H^p if and only if

$$\frac{\mu(S(\zeta,\delta))}{\delta} \to 0$$

as $\delta \to 0$ uniformly in ζ in $\partial\mathbb{D}$.

The first condition is says that the size of the pull-back measure of the Carleson sets is "big O" of δ. The second condition says that the size of the pull-back measure of the Carleson sets is "little O" of δ. These "big O" and "little O" conditions on related measures will be used in showing sufficient and necessary conditions for boundedness of DC_ϕ on both the Hardy and Bergman spaces.

Two facts will be central to the proof of the theorem. The first is the fact that the radial limit function of a composition is the composition of the radial limit functions [Ryf]. The second is a simple measure-theoretic change of variables. Thus,

$$\int_{\partial\mathbb{D}} |(f \circ \phi)^*|^p d\sigma = \int_{\partial\mathbb{D}} |f^* \circ \phi^*|^p d\sigma = \int_{\overline{\mathbb{D}}} |f^*|^p d\mu.$$

There are analytic self maps of the disk ϕ for which C_ϕ is compact on the Hardy space, yet DC_ϕ is unbounded. An easy example is the map ϕ defined by

$$\phi(z) = 1 - \sqrt{1-z}.$$

By a direct calculation, the pull-back measure μ of Carleson sets $S(1,\delta)$ is "little O" of δ. Note that the function f defined by $f(z) = (1-z)^{1/4}$ belongs to H^∞, but the image of f under DC_ϕ is not in H^p for any $p \geq 8/7$.

2.3.2 Carleson Measures and the Bergman Space

An analog of the last theorem is true for the Bergman spaces A^p. It has been generalized by MacCluer and Shapiro [MSh] for weighted Bergman spaces as well. The analog depends on the following Carleson Measure Theorem for the Bergman spaces.

THEOREM 2.3.4. *Suppose* $0 < p < \infty$ *and* μ *is a finite positive Borel measure on* $\mathbb{D}$. *Then*

1. *the inclusion map* $I : A^p \hookrightarrow L^p(\mathbb{D}, \mu)$ *is bounded if and only if there is a constant* K *so that*

$$\mu(S(\zeta, \delta)) \leq K\delta^2$$

for all ζ *in* $\partial\mathbb{D}$ *and* $0 < \delta \leq 2$, *and*

2. *the inclusion map* $I : A^p \hookrightarrow L^p(\mathbb{D}, \mu)$ *is compact if and only if*

$$\frac{\mu(S(\zeta, \delta))}{\delta^2} \to 0$$

as $\delta \to 0$ *uniformly in* ζ *in* $\partial\mathbb{D}$.

The first part of the above theorem was initially proved by Hastings. The second part has occurred in the work of Cima and Wogen, McDonald and Sundberg, and Voas. The following corollary gives the results about C_ϕ in terms of the pull-back measure induced by ϕ. Recall that λ is normalized Lebesgue area measure on $\mathbb{D}$.

COROLLARY 2.3.5. *(MacCluer and Shapiro)* *Suppose* $0 < p < \infty$ *and* ϕ *is an analytic*

self-map of the disk. Then

1. C_ϕ is bounded on A^p if and only if there is a constant K so that

$$\lambda\phi^{-1}(S(\zeta,\delta)) \leq K\delta^2$$

for all ζ in $\partial\mathbb{D}$ and $0 < \delta \leq 2$, and

2. C_ϕ is compact on A^p if and only if

$$\frac{\lambda\phi^{-1}(S(\zeta,\delta))}{\delta^2} \to 0$$

as $\delta \to 0$ uniformly in ζ in $\partial\mathbb{D}$.

In order to characterize Borel measures μ on $\mathbb{D}$ for which A^p is contained in $L^p(\mathbb{D},\mu)$, Axler [Axl] presents a Carleson Measure Theorem that employs pseudo-hyperbolic disks instead of Carleson sets. For points w and z in $\mathbb{D}$, the pseudo-hyperbolic distance between w and z is defined as

$$d(w,z) = |\phi_w(z)|,$$

where ϕ_w is the Möbius map given by

$$\phi_w(z) = \frac{w-z}{1-\overline{w}z}.$$

Note that the pseudo-hyperbolic distance is actually a metric on $\mathbb{D}$.

For a point w in $\mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk with radius r and center

w is defined by

$$D(w, r) = \{z \in \mathbb{D} : d(w, z) < r\}.$$

Two points about following theorem are notable. The first is that the quantity B does not involve p. Therefore, if μ is a positive Borel measure on $\mathbb{D}$, then A^p is contained in $L^p(\mathbb{D}, \mu)$ for some $p \in [1, \infty)$ if and only if A^p is contained in $L^p(\mathbb{D}, \mu)$ for every $p \in [1, \infty)$.

The second point of note is that the quantity A does not involve r. Thus, if μ is a positive Borel measure on $\mathbb{D}$, then the supremum in the second quantity is finite for some $r \in (0, 1)$ if and only if it is finite for all $r \in (0, 1)$.

THEOREM 2.3.6. *(Axler) Suppose $0 < r < 1$. There are constants $c(r)$ and $C(r)$, dependent only upon r, such that for all p, $1 \leq p < \infty$, and all positive Borel measures μ on $\mathbb{D}$, the following two quantities are equivalent; that is $c(r)A \leq B \leq C(r)A$.*

$A= \sup\{\int_{\mathbb{D}} |f|^p d\mu / \int_{\mathbb{D}} |f|^p d\lambda : f \in A^p, \ f \neq 0\};$

$B= \sup\{\mu(D(w, r))/\lambda(D(w, r)) : w \in \mathbb{D}\}.$

A technique similar to the above will be employed when considering conditions sufficient for DC_ϕ to be bounded on the Bergman space.

Chapter 3

DC_ϕ ON THE HARDY SPACE

If ϕ is an analytic self map of $\mathbb{D}$, then ϕ induces a possibly unbounded linear operator DC_ϕ on the Hardy spaces H^p, defined by

$$DC_\phi f = (f \circ \phi)' = f'(\phi)\phi'$$

for functions f in H^p. This chapter gives a characterization of the boundedness of DC_ϕ on the Hardy spaces in terms of the supremum norm of the inducing map ϕ.

The necessary and sufficient conditions for boundedness will be given in terms of a Carleson-type condition on a measure on $\mathbb{D}$ related to the inducing map ϕ. This technique has been used previously. In the course of proving Theorem 2.3 of her paper *Composition Operators on S^p*, MacCluer [Mac] shows that if $||\phi||_\infty = 1$, then a certain measure related to ϕ does not obey a vanishing Carleson condition. In the case of DC_ϕ on the Hardy spaces, the vanishing condition is necessary in order that $DC_\phi : H^p \to H^1$ be bounded. Therefore, the condition that $||\phi||_\infty < 1$ is necessary for boundedness.

If $f(z) = z$, then $f \in H^p$, $1 \leq p \leq \infty$. Since $DC_\phi(f) = \phi'$, it follows that the condition $\phi' \in H^1$ is necessary for $DC_\phi : H^p \to H^1$ to be bounded. If $\phi' \in H^1$, then it is known that ϕ extends continuously to $\overline{\mathbb{D}}$ [Dur1], so ϕ can be taken to be defined on the closed disc.

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Recall that the Carleson sets can be defined by

$$S(\zeta, \delta) = \{z \in \overline{\mathbb{D}} : |z - \zeta| < \delta\}$$

for $|\zeta| = 1$ and $0 < \delta < 1$. Now, suppose that ϕ' is in H^1, and define a finite positive measure μ on Borel sets $E \subset \partial \mathbb{D}$ by

$$\mu(E) = \int_E |\phi'| d\sigma,$$

where σ is normalized Lebesgue measure on $\partial \mathbb{D}$.

Notation: We shall say for two quantities A and B that $A \approx B$ if there are positive constants c and C such that $cA \leq B \leq CB$.

The following lemma is pivotal to the characterization of DC_ϕ on the Hardy spaces.

LEMMA 3.0.7. *Suppose* $\phi : \mathbb{D} \to \mathbb{D}$ *is analytic,* $\phi' \in H^1$, *and* $1 \leq p \leq \infty$. *Then for* $DC_\phi : H^p \to H^1$ *to be bounded it is necessary that*

$$\mu(\phi^{-1} S(\zeta, \delta)) = o(\delta)$$

as $\delta \to 0$ *uniformly in* ζ.

Proof. We prove the contrapositive of the implication. Suppose that $\mu(\phi^{-1} S(\zeta, \delta)) \neq o(\delta)$. Then there is a decreasing sequence $\delta_n \to 0$, a sequence ζ_n in $\partial \mathbb{D}$, and a $\beta > 0$ such that

$$\mu(\phi^{-1} S(\zeta_n, \delta_n)) \geq \beta \delta_n.$$

We construct a sequence $\{g_n\}$ in H^p such that $\|g_n\|_p \to 0$ and $\|DC_\phi g_n\|_1$ is bounded away from zero. The functions g_n are closely related to the reproducing kernel functions for points in the disk related to the sequences $\{\delta_n\}$ and $\{\zeta_n\}$.

First consider the case $1 \leq p < \infty$. Set $\alpha_n = (1 - \delta_n)\zeta_n$. Define f_n by $f_n(z) = (1 - \overline{\alpha_n}z)^{-2/p}$. Thus,

$$\|f_n\|_p^p = \int_{\partial D} |1 - \overline{\alpha_n}e^{i\theta}|^{-2}d\sigma \approx \frac{1}{\delta_n},$$

for $|\alpha_n|$ close to 1. Hence, $\|f_n\|_p \approx \delta_n^{-1/p}$.

Now set $g_n = f_n/\|f_n\|_p^2$. Since $\|f_n\|_p \to \infty$, $\|g_n\|_p \to 0$. Needed next is an estimate for $|f_n'|$. There is a natural number N such that for $n \geq N$, $\delta_n < 1/2$. For such N and for $z \in S(\zeta_n, \delta_n)$,

$$\begin{aligned}
|f_n'(z)| &= \frac{2}{p}|\overline{\alpha_n}|\,|1 - \overline{\alpha_n}z|^{-[(2/p)+1]} \\
&> C\delta_n^{-[(2/p)+1]}.
\end{aligned}$$

Since $\phi' \in H^1$, Ryff's result [Ryf] implies that $(g_n' \circ \phi)^*(e^{i\theta}) = (g_n')^*(\phi(e^{i\theta}))$ a.e.. Using the standard transformation formula we get

$$\begin{aligned}
\|DC_\phi g_n\|_1 &= \int_{\partial D} |g_n'(\phi(e^{i\theta}))|\,|\phi'(e^{i\theta})|d\sigma \\
&= \int_{\partial D} |g_n'(\phi(e^{i\theta}))|d\mu \\
&= \int_{\overline{D}} |g_n'|d(\mu\phi^{-1}) \\
&\geq \|f_n\|_p^{-2} \int_{S(\zeta_n, \delta_n)} |f_n'|d(\mu\phi^{-1}).
\end{aligned}$$

The above estimate on f_n' and the assumption on $\mu(\phi^{-1}S(\zeta_n, \delta_n))$ gives

$$\|DC_\phi g_n\|_1 > C\beta,$$

concluding the argument.

For the case $p = \infty$, define f_n by $f_n(z) = \frac{1}{1-\overline{\alpha_n}z}$. Thus we have $\|f_n\|_\infty = \delta_n^{-1}$, and we define $g_n = f_n/\|f_n\|_\infty^2$. The calculations are similar to those above.

Next is the result from MacCluer's proof of Theorem 2.3 [Mac].

LEMMA 3.0.8. *Suppose that ϕ is a non-constant analytic self-map of the disk, $\phi' \in H^1$ and that $\|\phi\|_\infty = 1$. Then $\mu(\phi^{-1}S(\zeta, \delta)) \neq o(\delta)$ as $\delta \to 0$ uniformly in ζ.*

Proof. Assume the hypothesis and recall that ϕ extends continuously to $\overline{\mathbb{D}}$. Without loss of generality suppose that $\phi(1) = 1$. Define $A_\delta = \phi^{-1}S(1,\delta) \cap \partial\mathbb{D}$. Then A_δ is a collection of intervals of $\partial\mathbb{D}$ which map under ϕ to curves in $S(1,\delta)$. It is known that since $\phi' \in H^1$, ϕ is absolutely continuous on $\partial\mathbb{D}$ [Dur1]. Thus the arc length of the image of A_δ under ϕ is given by the integral $\int_{A_\delta} |\phi'|d\sigma$. Since ϕ is continuous on $\partial\mathbb{D}$ and $1 \in A_\delta$ with $\phi(1) = 1$, the arc length of the image of A_δ under ϕ must be at least 2δ. Therefore,

$$\int_{A_\delta} |\phi'|d\sigma \neq o(\delta), \delta \to 0$$

proving the lemma.

The previous two results yield the next theorem.

THEOREM 3.0.9. *Suppose $\phi : \mathbb{D} \to \mathbb{D}$ is analytic, $\phi' \in H^1$, and $1 \leq p \leq \infty$. Then the operator $DC_\phi:H^p \to H^1$ is bounded if and only if $\|\phi\|_\infty < 1$.*

Proof. First suppose $\phi(\mathbb{D}) \subset r\mathbb{D}$ where $r < 1$, $f \in H^p$, and $\phi' \in H^1$. Then there exists an $M < \infty$ such that $|f'(\phi(z))| \leq M$ for all $z \in \overline{\mathbb{D}}$. Thus we have

$$\|(f \circ \phi)'\|_1 = \int_{\partial \mathbb{D}} |(f' \circ \phi)(e^{i\theta})| \, |\phi'(e^{i\theta})| d\sigma$$
$$\leq M \| \phi' \|_1 .$$

Thus, $DC_\phi(H^p) \subset H^1$. An application of the Closed Graph Theorem, similar to that used in Theorem 2.2.2, shows that $DC_\phi \colon H^p \to H^1$ is bounded.

Conversely, suppose $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and $\phi' \in H^1$ and $\|\phi\|_\infty = 1$. By the first two lemmas in this chapter, DC_ϕ is not a bounded operator from H^p to H^1. Therefore, if DC_ϕ is bounded, then $\| \phi \|_\infty < 1$.

Now we give the main result.

THEOREM 3.0.10. *If $1 \leq p, q \leq \infty$ and $\phi' \in H^p$, then $DC_\phi \colon H^q \to H^p$ is bounded if and only if $\|\phi\|_\infty < 1$.*

Proof. Suppose that $DC_\phi \colon H^q \to H^p$ is bounded. An application of the Closed Graph Theorem shows that the inclusion map $I : H^p \hookrightarrow H^1$ is bounded. Thus the hypothesis implies that DC_ϕ is bounded from H^q into H^1. Theorem 3.0.9 shows that $\|\phi\|_\infty < 1$.

Conversely, suppose that $\|\phi\|_\infty < 1$. Then an argument similar to that in Theorem 3.0.9 gives the result that DC_ϕ bounded from H^q into H^p.

There are two corollaries that follow from this main theorem.

COROLLARY 3.0.11. *If $1 \leq p, q \leq \infty$ and $\phi' \in H^p$, then $DC_\phi \colon H^q \to H^p$ is compact if and only if $\| \phi \|_\infty < 1$.*

Proof. If DC_ϕ is compact, then it is bounded and the previous theorem shows that $\| \phi \|_\infty$ must be less than one.

Now, suppose that $\| \phi \|_\infty = r < 1$. A linear operator T is compact if and only if T takes a weakly convergent sequence to a strongly convergent sequence [Nor2]. Suppose that $f_n \to 0$ weakly. Then f_n converges to zero pointwise. Therefore f_n' converges to zero pointwise on $\mathbb{D}$, and thus $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\phi(\mathbb{D}) \subset r\overline{\mathbb{D}}$ and $\phi' \in H^p$, we get

$$\| DC_\phi f_n \|_p^p = \int_{\partial \mathbb{D}} |(f_n' \circ \phi)(e^{i\theta})|^p \, |\phi'(e^{i\theta})|^p d\sigma \to 0$$

as $n \to \infty$. Thus the image of f_n under DC_ϕ is a strongly convergent sequence, and hence $DC_\phi \colon H^q \to H^p$ is a compact operator. $\quad\square$

COROLLARY 3.0.12. *Suppose that $\phi' \in H^1$. Then $DC_\phi \colon H^2 \to H^2$ is trace class if and only if $\|\phi\|_\infty < 1$.*

Proof. If DC_ϕ is trace class, then it is bounded, and hence, $\|\phi\|_\infty < 1$. Conversely, suppose that $\|\phi\|_\infty = r < 1$. Set $s = \frac{1+r}{2}$. Then $s < 1$ and

$$DC_\phi = DC_{\frac{1}{s}\phi}C_{sz},$$

the product of two Hilbert-Schmidt (compact) operators. Therefore DC_ϕ is a trace class operator on H^2.

The issue of boundedness of DC_ϕ is much more subtle in the Bergman spaces, as is seen in the next chapter.

Chapter 4

DC_ϕ ON THE BERGMAN SPACE

As in the case of the Hardy spaces, if ϕ is an analytic self map of the disk, then ϕ induces a possibly unbounded linear operator DC_ϕ on the Bergman spaces A^p by

$$DC_\phi f = (f \circ \phi)' = f'(\phi)\phi'$$

for functions f in A^p. The previous chapter showed that $DC_\phi : H^p \to H^q$ $(1 \le p, q \le \infty)$ is bounded if and only if $||\phi||_\infty < 1$. That is, DC_ϕ is bounded if and only if the closure of the image of $\mathbb{D}$ under ϕ is properly contained in $\mathbb{D}$.

The situation in the Bergman spaces A^p is quite different. In this chapter, necessary conditions and sufficient conditions for DC_ϕ to be bounded on A^p will be presented. In addition, we will construct a self map of $\mathbb{D}$ such that $||\phi||_\infty = 1$ and the induced operator $DC_\phi : A^p \to A^1$ is bounded.

4.1 Necessary Conditions

Once again, if $f(z) = z$, then $f \in A^p$, $1 \le p \le \infty$. Thus a necessary condition for $DC_\phi : A^p \to A^q$ to be bounded is that $\phi' \in A^q$. Another necessary condition is stated in terms of a Carleson-type measure on $\mathbb{D}$; for $E \subset \mathbb{D}$ and $1 \le q < \infty$, define

$$\mu_q(E) = \int_E |\phi'|^q d\lambda.$$

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(Recall that λ is normalized Lebesgue area measure on $\mathbb{D}$.)

Instead of the traditional Carleson sets, pseudo-hyperbolic disks, $D(w, r)$, are used in this setting.

THEOREM 4.1.1. *If $DC_\phi : A^p \to A^q$ is bounded for $1 \le p, q < \infty$, then the quantity*

$$\mu_q \phi^{-1}(D(w, r)) = o((1 - |w|)^b)$$

as $|w| \to 1$ for all $b < 2 + q$ and any fixed positive $r < 1$.

Proof. We prove the contrapositive of the asserted implication. Suppose, therefore, that there is a positive $r < 1$, a sequence w_n in $\mathbb{D}$ where $|w_n| \to 1$, and a $\beta > 0$ such that

$$\mu_q \phi^{-1}(D(w_n, r)) > \beta(1 - |w_n|)^b$$

for some $b < 2 + q$.

Define $f_n = \frac{1}{(1 - \overline{w_n}z)^{4/q}}$. Note that

$$
\begin{aligned}
\|f_n\|_{A^q}^q &= \int_{\mathbb{D}} |f_n|^q d\lambda \\
&= \int_{\mathbb{D}} \frac{1}{|1 - \overline{w_n}z|^4} d\lambda \\
&= \frac{1}{(1 - |w_n|^2)^2}.
\end{aligned}
$$

For an estimate on the modulus of the derivative of f_n on $D(w_n, r)$, note that $\overline{w_n}D(w, r)$ is a disk with the image of the w on a diameter on the real axis. Thus, the minimum of $|f_n'|$ on $\overline{D(w, r)}$ occurs at the point of $\overline{D(w, r)}$ closest to the origin. This point is $\frac{|w_n| - r}{1 - r|w_n|} \frac{w_n}{|w_n|}$.

Calculating the value of the derivative at this point gives the estimate

$$\inf\{|f_n'(z)| : z \in D(w_n, r)\} \geq \frac{K}{(1 - |w_n|^2)^{1+4/q}},$$

where K depends only on r.

Now define $g_n = \frac{f_n}{\|f_n\|_{A^q}^{1+a}}$ for $a = (2 + q - b)/2$. Since $b < 2 + q$, a is positive. It follows that $g_n \to 0$ in the A^q norm. However

$$
\begin{aligned}
\|DC_\phi g_n\|_{A^q}^q &= \int_{\mathbb{D}} |g_n'(\phi(z))|^q \, |\phi'(z)|^q d\lambda \\
&= \frac{1}{\|f_n\|_{A^q}^{q(1+a)}} \int_{\mathbb{D}} |f_n'(z)|^q d\mu_q \phi^{-1} \\
&\geq (1 - |w_n|)^{2+2a} \int_{D(w_n,r)} |f_n'(z)|^q d\mu_q \phi^{-1} \\
&> K\beta(1 - |w_n|^2)^{-2-q+2a+b}.
\end{aligned}
$$

But $2a = 2 + q - b$, so $\|DC_\phi g_n\|_{A^q}^q > K\beta$ which is strictly greater than 0. This gives the result. $\square$

The use of pseudo-hyperbolic disks in lieu of Carleson sets results in simpler proofs of the theorems about DC_ϕ on the Bergman spaces. However, the theorems can be stated and proved using the Carleson sets. The previous theorem, in combination with results by MacCluer and Shapiro [MSh], yields the following corollaries.

COROLLARY 4.1.2. *If $DC_\phi : A^p \to A^q$ is bounded, then ϕ has no finite angular derivative at any point of $\partial\mathbb{D}$.*

COROLLARY 4.1.3. *If $DC_\phi : A^p \to A^q$ is bounded, then $|\phi^*(e^{i\theta})| < 1 a.e.[\theta]$.*

ε

4.2   A Sufficient Condition

The measure $\mu_q \phi^{-1}$ can also be used to yield a sufficient condition which isvery close to the necessary condition. This will be accomplished by comparing the measure to a finite rotation-invariant measure, ν_p, defined for $p \geq 1$ and Borel sets $E \subset \mathbb{D}$ by

$$\nu_p(E) = \int_E (1 - |z|^2)^p d\lambda.$$

The last theorem in Chapter One, due to Kehe Zhu [Zhu], now can be stated as follows: *Suppose n is a positive integer, $n \geq 1$, $1 \leq p < \infty$, and f is analytic on $\mathbb{D}$. Then f is in A^p if and only if $f^{(n)}$ is in $L^p(\mathbb{D}, d\nu_p)$.*

The pseudo-hyperbolic disk $D(w, r)$ is a Euclidean disk with center c_e and radius r_e given by

$$c_e = \frac{w(1 - r^2)}{1 - r^2|w|^2}, \quad r_e = \frac{r(1 - |w|^2)}{1 - r^2|w|^2}.$$

For two positive quantities A and B (possibly infinite) we will say $A \approx B$ if there are positive constants c and C such that $cA \leq B \leq CA$.

LEMMA 4.2.1. *For a fixed positive $r < 1$ and $1 \leq p < \infty$,*

$$\nu_p(D(w, r)) \approx (1 - |w|^2)^{2+p}.$$

Proof. Note that the function $(1 - |z|^2)^p$ has its maximum and minimum on $\overline{D(w, r)}$ at the extremes of the diameter through w. Thus an estimate of $\nu_p(D(w, r))$ can be obtained

by evaluating the function $(1 - |z|^2)^p$ at the Euclidean center of $D(w, r)$ and multiplying by the area of $D(w, r)$; that is

$$\nu_p(D(w, r)) \approx (1 - |\frac{w(1 - r^2)}{1 - r^2|w|^2}|^2)^p \pi (\frac{r(1 - |w|^2)}{1 - r^2|w|^2})^2.$$

For fixed $r < 1$, we see that $\nu_p(D(w, r)) \approx (1 - |w|^2)^{2+p}$. $\square$

Also needed is a change of variable formula.

LEMMA 4.2.2. *(Axler) For each Lebesgue-integrable function* h,

$$\int_{D(w,r)} h(z) d\lambda(z) = (1 - |w|^2)^2 \int_{D(0,r)} (h \circ \phi_w)(\gamma)|1 - \overline{w}\gamma|^{-4} d\lambda.$$

The next lemma gives an estimate on point-evaluation of an analytic function in terms of the integral of the function with respect to the measure ν_p.

LEMMA 4.2.3. *Suppose* $0 < r < t < 1$ *and* f *is analytic on* $\mathbb{D}$. *Then there is a constant* K *such that*

$$|f(z)|^p \leq \frac{K}{\nu_p(D(w, t))} \int_{D(w,t)} |f|^p d\nu_p$$

for all w *in* $\mathbb{D}$ *and for all* z *in the pseudo-hyperbolic disk* $D(w, r)$. *The constant* K *is dependent on* p, r, *and* t.

Proof. Let $z \in D(w, r)$ and define E to be the Euclidean disk with center $\phi_w(z)$ and radius $t - r$. Note that $E \subset D(0, t)$. To see this, take $x \in E$. Then $|\phi_w(z) - x| < t - r$. But $|\phi_w(z)| < r$, thus $|x| < t$, and so $x \in D(0, t)$.

Since $f \circ \phi_w$ is analytic, $|f \circ \phi_w|^p$ is subharmonic. Noting that ϕ_w is its own inverse, for

any $s < t - r$, we have

$$|f(z)|^p = |(f \circ \phi_w)(\phi_w(z))|^p \leq \int_0^{2\pi} |(f \circ \phi_w)(\phi_w(z) + se^{i\theta})|^p d\theta/2\pi.$$

Integrating both sides of the above inequality from 0 to $t - r$ with respect to $(1 - s^2)^p s \, ds$

we get

$$\frac{1 - (1 - (t - r)^2)^{p+1}}{2(p + 1)} |f(z)|^p \leq \int_E |f \circ \phi_w(\gamma)|^p (1 - |\gamma|^2)^p d\lambda(\gamma),$$

where $\gamma = \phi_w(z)$. Since $E \subset D(0, t)$, we have the inequality

$$|f(z)|^p \leq \frac{2(p + 1)}{1 - (1 - (t - r)^2)^{p+1}} \int_{D(0,t)} |f \circ \phi_w(\gamma)|^p (1 - |\gamma|^2)^p d\lambda(\gamma).$$

Application of the change of variable in the preceding lemma and the estimate on

$\nu_p(D(w, t))$, and noting that $|1 - \overline{w}\gamma| \leq 16$ gives

$$\begin{aligned}
|f(z)|^p &\leq \frac{32(p + 1)(1 - |w|^2)^{2+p}}{(1 - (1 - (t - r)^2)^{p+1})(1 - |w|^2)^2 \nu_p(D(w, t))} \int_{D(w,t)} |f|^p d\nu_p \\
&= \frac{K}{\nu_p(D(w, t))} \int_{D(w,t)} |f|^p d\nu_p
\end{aligned}$$

where the constant K depends on p, r, and t. $\square$

Another piece of information is needed to prove the theorem which will give the sufficient

condition for boundedness of DC_ϕ. The lemma, due to Axler [Axl], says that for a fixed

positive $r < 1$, the unit disk can be covered by pseudo-hyperbolic disks of radius r in a way

in which they do not intersect too often, even if the pseudo-hyperbolic radius is increased.

LEMMA 4.2.4. *(Axler) Suppose that* $0 < r < 1$. *Then there is a sequence* $\{w_n\}$ *in* $\mathbb{D}$ *and a positive integer* M *such that*

$$\bigcup_{n=1}^{\infty} D(w_n, r) = \mathbb{D}$$

and each $z \in \mathbb{D}$ *is in at most* M *of the pseudo-hyperbolic disks* $D(w_n, \frac{r+1}{2})$.

The sufficient condition for boundedness of DC_ϕ will be a consequence of the next theorem.

THEOREM 4.2.5. *Suppose that* p, $1 \le p < \infty$, *and* $0 < r < 1$. *If the quantity*

$$\sup \left\{ \frac{\mu_p \phi^{-1}(D(w, r))}{\nu_p(D(w, r))} : w \in \mathbb{D} \right\}$$

is finite, then so is the quantity

$$\sup \left\{ \frac{\int_{\mathbb{D}} |f|^p d\mu_p \phi^{-1}}{\int_{\mathbb{D}} |f|^p d\nu_p} : f \in L^p(\mathbb{D}, d\nu_p), \ f \ne 0 \right\}.$$

Proof. Let M be the positive natural number and $\{w_n\}$ be the sequence in $\mathbb{D}$ guaranteed by the previous lemma. Suppose that $f \in L^p(\mathbb{D}, d\nu_p)$, $f \ne 0$. Then

$$\int_{\mathbb{D}} |f|^p d\mu_p \phi^{-1} \le \sum_{n=1}^{\infty} \int_{D(w_n, r)} |f|^p d\mu_p \phi^{-1}$$

$$\le \sum_{n=1}^{\infty} \sup \left\{ |f(z)|^p : z \in D(w_n, r) \right\} \mu_p \phi^{-1}(D(w_n, r))$$

$$\le K \sum_{n=1}^{\infty} \frac{\mu_p \phi^{-1}(D(w_n, r))}{\nu_p(D(w_n, (r+1)/2))} \int_{D(w_n, (r+1)/2)} |f|^p d\nu_p,$$

where the last inequality comes from Lemma 4.2.3 estimating point-evaluation with respect to the integral of f with respect to the measure ν_p. Here $t = (r+1)/2$. Since $D(w_n, r) \subset D(w_n, (r+1)/2)$, the above becomes

$$\int_{\mathbb{D}} |f|^p d\mu_p \phi^{-1} \leq K \sum_{n=1}^{\infty} \frac{\mu_p \phi^{-1}(D(w_n, r))}{\nu_p(D(w_n, r))} \int_{D(w_n,(r+1)/2)} |f|^p d\nu_p$$
$$\leq KB \sum_{n=1}^{\infty} \int_{D(w_n,(r+1)/2)} |f|^p d\nu_p$$
$$\leq KBM \int_{\mathbb{D}} |f|^p d\nu_p.$$

Thus, the quantity A is bounded by the constant KM times the quantity B.

Now the main theorem for sufficiency.

THEOREM 4.2.6. *If the quantity*

$$\sup\left\{ \frac{\mu_p \phi^{-1}(D(w, r))}{(1 - |w|^2)^{2+p}} : w \in \mathbb{D} \right\}$$

is finite, then DC_ϕ is a bounded operator on A^p.

Proof. This follows immediately from the previous theorem and the estimate on the size of $\nu_p(D(w, r))$.

4.3 An Example

Now we construct a self-map of the disk that induces a bounded operator $DC_\phi : A^p \to A^1$ $(1 \leq p < \infty)$. This self-map of the disk is conformal with $\phi^*(1) = 1$.

THEOREM 4.3.1. *There is an analytic self-map of the unit disc, ϕ, with $\|\phi\|_\infty = 1$, that*

induces a bounded operator $DC_\phi : A^p \to A^1$ for $1 \le p < \infty$.

Proof. The self-map of D.

First we define ψ, a self-map of D, as the composition of a Möbius map η from D to the upper half-plane, followed by the conformal map $\arcsin z$ from the upper half-plane onto the semi-infinite strip

$$H = \{z = x + iy : y > 0, -\pi/2 < x < \pi/2\},$$

and finally into D with η^{-1}. Explicitly,

$$\psi(z) = \frac{\arcsin(i\frac{1+z}{1-z}) - i}{\arcsin(i\frac{1+z}{1-z}) + i}.$$

To construct an appropriate ϕ we now fix any s such that $0 < s < 1$ and define

$$\phi(z) = s\psi(z) + (1 - s).$$

Note that $\phi^*(1) = 1$, and for $\zeta \in \partial D$, $\zeta \ne 1$, $|\phi^*(\zeta)| < 1$. Also, to simplify calculations, some of the required estimates will be obtained by considering the map ψ rather than ϕ.

The area of $\psi^{-1}S(1,\delta)$ is on the order of $e^{-1/\delta}$.

Note that the image $\psi(D)$ is symmetric with respect to the real axis. Also, since the Möbius maps η and η^{-1} interchange the point 1 and the point at infinity, taking lines and circles to lines and circles, the image $\psi(D)$ near the point 1 consists of two circular arcs, symmetric with respect to the real axis, whose tangents approach the real axis as the arcs

approach the point 1. Thus it is sufficient to consider the measure of pseudo-hyperbolic disks with real centers approaching the point 1.

Fix a positive $r < 1$ and approximate the set $\psi^{-1}D(w,r)$ for real $w \to 1$ with $w > s$. Recall that for w near $\partial\mathbb{D}$ and fixed r, the Euclidean radius of $D(w,r)$ is on the order of the distance from w to $\partial\mathbb{D}$. Let $\delta = 1 - |w|^2$. Thus there exists a constant c such that for all w, $s < w < 1$, the pseudo-hyperbolic disk $D(w,r)$ is contained in a Carleson set,

$$D(w,r) \subset S(1, c\delta).$$

Now,

$$\eta(D(w,r)) \subset \eta(S(1, c\delta))$$
$$= \{z \in H : |1 - \frac{z-i}{z+i}| < c\delta\}$$
$$= \{z \in H : \frac{2}{c\delta} < |z - (-i)|\}.$$

So for small δ we have the approximation

$$\eta(D(w,r)) \subset \{z \in H : \frac{2}{c\delta} < |z|\}.$$

Denote this last set by G.

Now, $\sin(\eta D(w,r)) \subset \sin G$. To find $\sin(G)$ we note that

$$2i \sin z = e^{i(x+iy)} - e^{-i(x+iy)} = e^{ix}e^{-y} - e^{-ix}e^{y}.$$

Since we are concerned with the case when δ is small, we have the approximation

$$\sin z \approx \frac{-1}{2} e^{-i(x+\pi/2)} e^y.$$

Thus we have

$$\sin(G) \approx \{z = x + iy : y > 0, \frac{1}{2} e^{2/c\delta} < |z|\}.$$

Now consider $\psi^{-1} D(w, r)$. This set is contained in $\eta^{-1}(\sin(G))$, which is in a neighborhood of the point 1 consisting of all points z in $\mathbb{D}$ such that

$$\frac{1}{2} e^{2/c\delta} < |\frac{1+z}{1-z}|.$$

Thus $|1 - z| < 4e^{-2/c\delta}$. Therefore there is a positive constant B such that

$$\text{Area}(\psi^{-1} D(w, r)) \leq Be^{-4/c\delta}.$$

Claim: $\frac{\mu_1 \phi^{-1} D(w,r)}{\delta^b} \to 0$ as $\delta \to 0$ for all $b, 1 \leq b < \infty$.

Note that the inclusion map $I : A^p \to A^1$ is bounded. Coupled with Theorem 4.2.6, the claim shows both that $\mu_1(\mathbb{D})$ is finite and that $\mu_1 \phi^{-1} D(w, r) \leq C\delta^b$ for some constant C. This will prove that $DC_\phi : A^p \to A^1$ is bounded.

First note that

$$|\phi'(z)| = \frac{4s}{|1 - z|^2 \left|\sqrt{1 + (\frac{1+z}{1-z})^2}\right| |i + \arcsin(i\frac{1+z}{1-z})|^2}.$$

But $\arcsin u = -i\log[iu + \sqrt{1 - u^2}]$, so for z near 1 there is a positive k so that

$$|\phi'(z)| = \frac{4s}{|1 - z|\,|\sqrt{2 + 2z^2}|\,\left|i - i\log[-\frac{1+z}{1-z} + \sqrt{1 - (i\frac{1+z}{1-z})^2}]\right|^2}$$

$$\leq \frac{k}{|1 - z|\,|\sqrt{2 + 2z^2}|\,\left|1 - \log[-\frac{1+z}{1-z} + \sqrt{1 + (\frac{1+z}{1-z})^2}]\right|^2},$$

using the principal branches of both the square root and the logarithm.

An application of l'Hospital's rule shows that

$$\left|-\frac{1+z}{1-z} + \sqrt{1 + (\frac{1+z}{1-z})^2}\right| = O(|1 - z|)^{-1},$$

so we get

$$|\phi'(z)| \approx \frac{K}{|1 - z|(-\ln|1 - z|)^2},$$

for some constant K.

Putting $t = |1 - z|$ and integrating over the bigger set $S(1, c\delta)$, we have

$$\mu_1 \phi^{-1} D(w, r) \leq \int_{\phi^{-1}S(1,c\delta)} |\phi'(z)|\,d\lambda$$

$$\leq \int_{-\pi/2}^{\pi/2} \int_0^{e^{-2/c\delta}} \frac{K}{t(-\ln t)^2} t\,dt\,d\theta/\pi$$

$$= K \int_0^{e^{-2/c\delta}} (-\ln t)^{-2}\,dt$$

$$\leq \frac{L\delta^2}{e^{2/c\delta}}$$

for some constant L, proving the claim and the theorem as well.

COROLLARY 4.3.2. $DC_\phi : A^p \to A^q$ *is bounded for* $q \leq p$ *and for* $1 \leq q < 2$.

Proof. Note that for the specified q we have

$$\frac{1}{t^{q-1}(-\ln t)^{2q}} \leq \frac{1}{t^{q-1}},$$

which is integrable. In fact,

$$\int_0^{e^{-2/c\delta}} \frac{1}{t^{q-1}} dt \approx \left(\frac{1}{e^{2/c\delta}}\right)^{2-q}.$$

Since $2 - q > 0$, this shows that the sufficient condition on $\mu_q \phi^{-1}$ is met.

Chapter 5

FURTHER QUESTIONS

This work has raised many questions, some about to the operator DC_ϕ and others beyond it. First, those about the operator.

- In the setting of the Bergman space, necessary conditions and sufficient conditions have been obtained for boundedness of DC_ϕ. These conditions are closely related. However, the question remains, can the boundedness of DC_ϕ on the Bergman space be characterized completely?

- The question of compactness of DC_ϕ on the Hardy spaces is answered along with the question of boundedness. Will there be a similar correlation on the Bergman space, or as in the case of boundedness, will there be interesting differences? The question remains, when is DC_ϕ a compact operator on the Bergman spaces?

- What is the spectrum of the operator on the Hardy and Bergman Spaces? On H^2 DC_ϕ is a trace class operator. Thus the spectrum is just the point spectrum union $\{0\}$. What are the eigenvalues of DC_ϕ on H^2?

- Can estimates be made for the norm of DC_ϕ on both the Hardy and Bergman spaces?

There are two question that are raised by this study that are not about the operator DC_ϕ. The first was posed by Professor Hadwin.

- If the operators of differentiation and composition are applied in the opposite order,

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we get a new operator. That is, for a function in H^p or A^p and an analytic self-map of the disk ϕ, define

$$C_\phi D(f) = f' \circ \phi.$$

When is $C_\phi D$ bounded and compact on the Hardy and Bergman spaces?

Another question, posed by Professor Nordgren, is perhaps the "largest" question emerging from this study.

• Can a "Carleson measure theorem" be crafted for the differentiation operator? That is, if X is H^p or A^p and μ is a measure on the disk, can boundedness of

$$D : X \to L^p(\mathbb{D}, \mu)$$

be determined completely by conditions on the measure μ?

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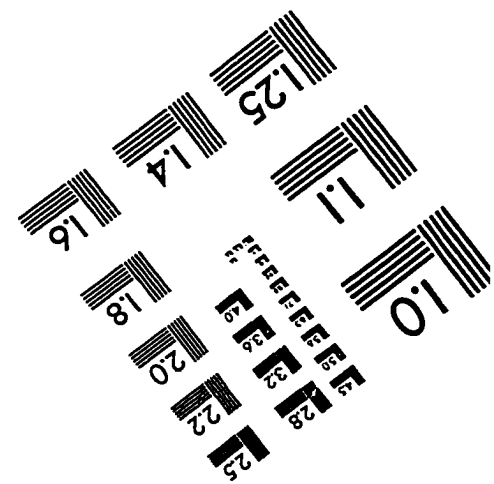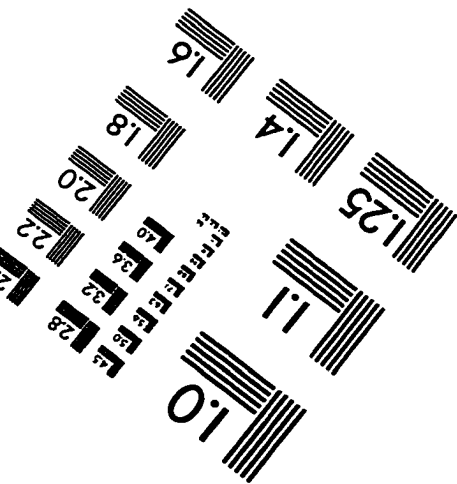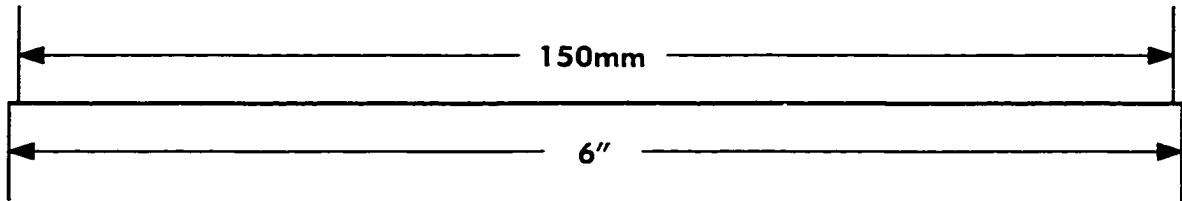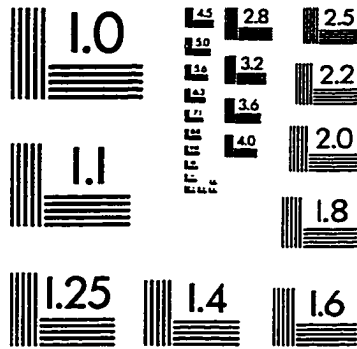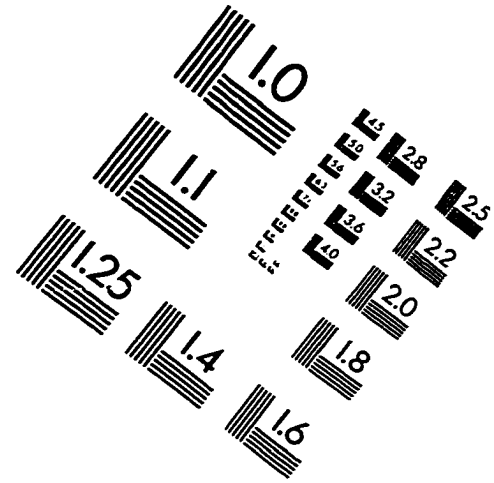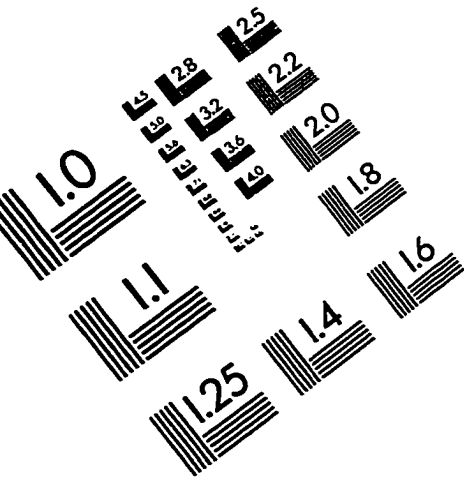
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