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PERFECT MATCHINGS: MODIFIED AZTEC DIAMONDS, COVERING GRAPHS AND n-MATCHINGS

BY

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DISSERTATION

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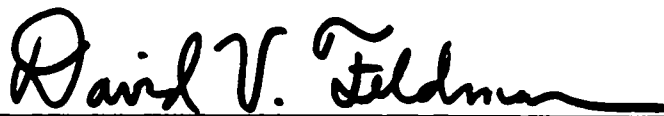
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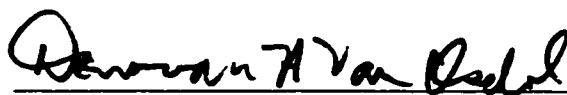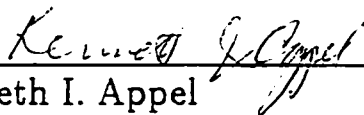
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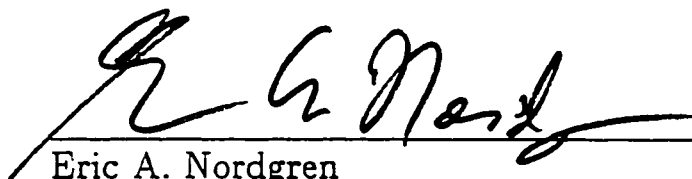
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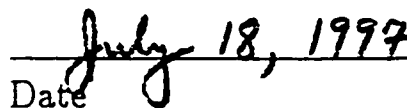
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DEDICATION

To my parents and my grandmother,
to Didier,
to all my mathematics teachers.

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My love goes to my parents who, through many years of care prepared me for this work, and kept supporting me even from far away.

Finally, to my husband Didier, for his patience, permanent availability and love. May we "discover new worlds..." together.

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ABSTRACT

PERFECT MATCHINGS: MODIFIED AZTEC

DIAMONDS, COVERING GRAPHS AND n-MATCHINGS

by

Adriana Badauta Cransac

University of New Hampshire, September, 1997

In the Introduction, we present the problems we are going to study and we establish the basic definitions, concepts and results that are used throughout.

We begin the first chapter with a presentation of the Aztec diamond and the behaviour of its random domino tilings. We introduce the dual – matching – problem and we explore the structure of the perfect matchings of modified Aztec diamonds. We show that some of these matchings can be extended to matchings of the dual Aztec diamond, pointing out a bijection between these types of matchings. We determine the number of perfect matchings for each of the modified graphs and the placement probabilities of the edges belonging to such a matching at a given location. We conclude with a theorem presenting the common asymptotic behaviour of the dual and the modified Aztec diamonds and we deduce a version of the Arctic Circle Theorem for these graphs.

The second part is dedicated to the study of non-ramified perfect n-matchings, their decomposition into perfect matchings and 2-matchings as well as their relations to the perfect matchings of covering graphs. For the n-covering graphs we use the permutation derived graph construction. We determine the number of liftings of a

given n-matching to a matching of a branched covering graph and then of an n-covering graph, together with necessary and sufficient conditions for the existence of the lifting. In particular, for the case of 2-matchings, we obtain a uniform behaviour of liftings of cycles. First, we deduce a theorem that relates the number of perfect matchings of the branched covering graph we have introduced to the number of perfect 2-matchings of the initial graph. Then we study the 2-covering graphs, their number, we determine the number of liftings of a 2-matchings (as a power of 2) and we obtain a theorem that characterizes the 2-matchings as the average of perfect matchings of 2-covering graphs. We conclude with some considerations about the maximum, minimum and the realization of this average and methods of computing it.

INTRODUCTION

DEFINITIONS, NOTATIONS AND BASIC RESULTS

This purpose of this study is to address two questions from perfect matching theory.

The first part is inspired by the results obtained in [EKLP92a], [EKLP92b], [JPS95] and [CEP95] regarding random domino tilings of a plane region called the Aztec diamond and a very interesting phenomenon called the Arctic Circle theorem. An abstract of these results can be found in section 1.1. Considering this, we introduce some modifications of the dual problem, creating non-planar graphs that are not very different from the initial ones regarding the number and structure of their perfect matchings, and which turn out to have the same asymptotic behaviour.

The second part has as a starting point by the same articles mentioned above. We want to investigate the case of a 2-layer domino tiling, or respectively, a perfect 2-matching in the dual. This leads to the study of perfect 2-matchings and their relations with perfect matchings of covering graphs. The results that we obtain hold not only for the dual Aztec diamond, but for all graphs, therefore we present them in this general form.

Illustrations for chapters I and II can be found in Appendix A, respectively in Appendix B.

Here follows a collection of basic definitions, notations and elementary results from

graph theory and topology, which will be assumed throughout the thesis. Additional terminology and theorems will be presented, as needed, in each chapter or may be found in the list of references, as indicated.

0.1 Graph Theory

The basic mathematical structures which are the object of this study are defined in the following paragraph:

DEFINITION 0.1 *A* **multigraph** $G = (\ V(G), E(G)\ )$ *consists of a set of vertices* $V(G)$ *and a set of edges* $E(G)$ *such that every edge determines a subset of* $V(G)$ *with at most two elements.*

$\square$

- *an edge and a vertex are* **incident** *if the vertex belongs to the edge;*

- *the vertices incident to an edge are called* **endpoints** *of that edge;*

- *two vertices are* **adjacent** *if there exists an edge to which they both belong;*

- *two edges are* **adjacent** *if they share a common vertex;*

- *edges with only one endpoint are called* **loops;**

- *several edges that have the same (two) endpoints form a* **multiple edge.**

DEFINITION 0.2 *A (simple) graph is a multigraph with no loops or multiple edges.*

□

Let $v \in V(G)$ and $e \in E(G)$.

- $\mathbf{E}(v) = \{e \in E(G) :\ e$ *and v are incident$\}$;*

- $\mathbf{N}(v) = \{u \in V(G) :\ u$ *and v are adjacent$\}$,*

  and the elements of this set are called the **neighbors** *of v;*

- *the* **degree** *of a vertex $\mathbf{d}_G(v) = |E(v)|$.*

DEFINITION 0.3 *A* **k-regular** *(multi)graph is a (multi)graph such that* $d_G(v) = k, \forall v \in V(G)$.

□

- *a* **walk** *is an alternating sequence of incident edges and vertices, beginning and ending with vertices;*

- *a* **path** *is a walk in which all edges are distinct and all vertices, except possibly the endpoints, are distinct;*

- *the* **length** *of a path is the number of edges in the path;*

- *a* **closed** *walk is a walk in which the endpoints coincide;*

- *a* **cycle** *is a closed path;*

- *H is a* **subgraph** *of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$;*

- *H* is a **spanning** *subgraph of G if $V(H) = V(G)$;*

- *a graph is* **connected** *if every two vertices are joined by a path;*

- *a maximal connected subgraph of a graph is a* **component** *of the graph;*

- *a* **tree** *is a connected graph without cycles;*

- *a* **rooted** *tree is a tree with a distinguished vertex called root;*

- *a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(H)$ is an* **isomorphism** *between G and H; in this case we say that the two graphs are* **isomorphic;**

- *a* **graph automorphism** *is an isomorphism from the graph to itself.*

The following basic results will be used in section 2.4.2.:

- *a tree with v vertices has $v - 1$ edges;*

- *the cycles in a graph G can be identified with the set $Z(G)$ of $\{0, 1\}$ vectors of dimension $|E(G)|$: a cycle C is represented by the characteristic function of its edge set, a vector $z_C \in Z(G)$;*

- *$Z(G)$ is a vector space over $\mathbf{Z}_2$; if $k(G)$ is the number of connected components of G the dimension of this space is $|E(G)| - |V(G)| + k(G)$, and it is called the* **cyclomatic** *number of the graph;*

- *the sum of two cycles C and C' is the cycle C'' such that $z_{C''} = (z_C + z_{C'}) \bmod 2$;*

- *the parity of the sum of two cycles is equal to the parity of the sum of their lengths;*

- *the cycles corresponding to a vector space basis for $Z(G)$ form a* **cycle base** *of the graph.*

We are going to introduce the concepts of matchings and factors in a graph. Various types of matchings will be studied in this paper.

DEFINITION 0.4 *Given an integer valued function $f : V(G) \to N$, an* **f-matching** *is an assignment of non-negative integer weights to the edges so that the sum of the weights at any vertex v is less or equal to $f(v)$.*

A f-matching is **perfect** *if we have equality for each v.*

$\square$

DEFINITION 0.5 *A* **matching** *in a graph is a set of edges such that any two edges in the set are disjoint.*

$\square$

Note that a matching is just a special type of f-matching, for $f(v) = 1$, $\forall v \in V(G)$.

- *the number of perfect matchings of a graph G is denoted by $\Phi(G)$.*

DEFINITION 0.6 *An* **f-factor** *is a spanning subgraph G' of G such that*
$d_{G'}(v) = f(v), \forall v \in V(G).$

$\square$

Sometimes it is necessary to introduce an orientation of the edges of a graph.

DEFINITION 0.7 *An **orientation** of a graph is a function which associates to each edge $\{u, v\}$ either the pair (u, v) or the pair (v, u).*

*A **plus-minus orientation** of a graph is a function which associates to each edge $e = \{u, v\}$ one of the sets: $\{e^+ = (u, v), e^- = (v, u)\}$ or $\{e^+ = (v, u), e^- = (u, v)\}$.* □

DEFINITION 0.8 *If $\vec{G}$ is an orientation of G, the **skew adjacency matrix** of $\vec{G}$, $A_s(\vec{G})$, is defined by:*

$$a_{i,j} = \begin{cases} 1 & (u_i, u_j) \in E(\vec{G}) \\ -1 & (u_j, u_i) \in E(\vec{G}) \\ 0 & otherwise \end{cases}$$
□

We will use plus-minus orientations in section 2.3. and graph orientations together with the concepts mentioned below, in section 2.5.

- *a cycle C in a graph G is **nice** if the graph obtained from G by deleting all vertices of C has a perfect matching;*

- *a nice cycle is **oddly oriented** iff it has an odd number of edges oriented in the direction of the routing chosen on the cycle (it does not matter which one);*

- *a **Pfaffian orientation** of a graph is an orientation such that*
  $$det(A_s(\vec{G})) = (\Phi(G))^2;$$

- *a graph is **Pfaffian** iff it has a Pfaffian orientation.*

0.2 Topology and Topological Graph Theory

We begin this section with a few notions about covering spaces:

DEFINITION 0.9 *A map $\rho : \bar{X} \to X$ between two topological spaces is a **covering projection** if every point x in X has a neighborhood U such that ρ maps each component of $\rho^{-1}(U)$ homeomorphically onto U.*

*A map $\rho : \bar{X} \to X$ between two topological spaces is a **branched covering projection** if there exists a finite set B of points of X (called **branch points**) such that the restricted map $\rho : \bar{X} \setminus \rho^{-1}(B) \to X \setminus B$ is a covering projection.* □

- *the space $\bar{X}$ is called a (**branched**) **covering space**;*

- *the set $\rho^{-1}(x)$ is the **fiber** above x;*

- *if the size $|\rho^{-1}(x)|$ of the fiber is n for all $x \in X$ we have an **n-covering space**.*

Often, when we talk about a graph, we have in mind some topological representation of it:

DEFINITION 0.10 *An **embedding** of a graph G on a surface S is a mapping of the vertices of G to distinct points of S and of edges of G to disjoint open arcs of S such that:*

- the image of no edge contains that of a vertex;

- the image of the edge $\{u,v\}$ joins the points corresponding to vertices u and v. □

When dealing with a topological representation of a graph, most of the times we refer to it as "the graph".

DEFINITION 0.11 *A graph is* **planar** *iff it can be embedded in the plane.*

$\square$

CHAPTER I

STATISTICS FOR RANDOM MATCHINGS OF MODIFIED AZTEC DIAMONDS

1.1 The Initial Problem

This is a brief introduction to the Aztec diamond, the problems and the terminology related to it. We present an abstract of the shuffling method and the main results from [EKLP92a], [EKLP92b], [JPS95] and [CEP95] that are used in this chapter.

DEFINITION 1.1 *The* **Aztec diamond** *of order* **n** *is the union of those lattice squares* $[a, a+1] \times [b, b+1]$, $a, b \in \mathbf{Z}$, *that lie completely inside the tilted square* $\{(x, y) : |x| + |y| \leq n + 1\}$.

$\square$

This region was studied for the particularly interesting behaviour of its domino tilings.

- *a* **domino** *is a closed* 1×2 *rectangle;*

- *a* **tiling** *of a region R by dominos is a set of dominos whose interiors are disjoint and whose union is R.*

Figure 3 presents an example of a tiled Aztec diamond.

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A first result concerns the number of domino tilings of an Aztec diamond of order n—denoted $AD(n)$. This result was obtained using various methods like alternating sign matrices ([EKLP92a]), urban renewal ([GIP96]) or the shuffling algorithm ([EKLP92b]). We describe the last method and we explain how it is used to deduce $AD(n)$.

Before presenting the shuffling algorithm we need to introduce some conventions:

- *the* **standard coloring** *of the Aztec diamond is a black-white checkerboard fashion coloring of the squares so that the line $\{(x, y) : x + y = n + 1\}$ passes only through white squares;*

- *the union of two adjacent squares is a* **domino space** *(to distinguish it from an actual domino);*

- *a horizontal domino is* **north** *or* **south-going** *according to whether its leftmost square is white or black, and a vertical domino is* **west** *or* **east-going** *according to whether its upper square is white or black;*

- *two dominos that share a side of length 2 form a* **good block** *if they point away from each other and a* **bad block** *if they point towards each other;*

- *a* **location** *is the midpoint of the bottom edge of a north-going domino space;*

- $p_{i,j}^n$ *denotes the* **placement probability** *of a north-going domino at location (i, j) in a random tiling of the Aztec diamond of order n.*

For an illustration of the standard coloring and locations see figure 1. The locations are marked by black dots.

Note that the standard coloring depends upon the size of the Aztec diamond, in the sense that for odd values of n (like the one we presented in figure 1) the square $[0,1] \times [0,1]$ is black, while for n even the same square is colored white.

For examples of north, south, west and east-going dominos, good blocks and bad blocks, see figure 2.

We mention that bad blocks can only appear on square regions formed of four squares such that the upper left one is black, while good blocks occupy similar regions but with the upper left square white.

Figure 3 displays an example of tiling of an Aztec diamond of order 5, with the directions of the dominos marked by arrows.

The shuffling algorithm allows the creation of a random tiling of the Aztec diamond of order $n + 1$ starting with a random tiling of the Aztec diamond of order n.

ALGORITHM 1.2 *(Shuffling Algorithm)*

Input: *a random tiling of Aztec diamond of order n.*

Steps *to perform:*

- **destruction***: remove all bad blocks;*

- **sliding***: move the remaining dominos according to their directions;*

- **creation***: fill the empty squares randomly with vertical or horizontal parallel dominos.*

Output: *a random tiling of the Aztec diamond of order $n + 1$.*

$\square$

An illustration of this algorithm can be found in figure 4. Part (a) represents a tiling of an Aztec diamond of order 4 with the directions of the dominos marked by arrows and with the unique bad block emphasized. Part (b) is the result of the first step of the algorithm, destruction: the bad block has been removed. Part (c) represents the result of sliding. The directions of the dominos correspond to the new coloring and the free regions are split into 2×2 blocks. Part (d) is the result of the creation stage: the free blocks have been filled with pairs of dominos.

Shuffling is introduced in [EKLP92b], where it is shown that it is an involution on the partial domino tilings of the plane with no bad blocks. We will explain now why this algorithm produces random tilings and how it can be used to deduce the number of tilings of an Aztec diamond of order n.

Let us begin with a random Aztec diamond of order n — which appears with a probability of $\frac{1}{AD(n)}$. Suppose the tiling we chose has k bad blocks. We delete all bad blocks and then all dominos that are left are free to move in their direction. As mentioned above, there are no bad blocks after sliding and dominos do not collide while sliding. It is clear that they remain inside the region of an Aztec diamond of order $n + 1$. In [EKLP92b] it is also proved that after sliding there is an unique way of organizing the free space inside this bigger Aztec diamond into 2×2 squares. Note that the free space is formed by $4k$ squares that were freed by removal of bad blocks plus $4(n + 1)$ squares obtained by increasing the region from order n to order $n + 1$. Thus the number of free 2×2 squares is $k + n + 1$. Each of them can be filled either

with a pair of horizontal dominos or a pair of vertical ones, which form good blocks in the new standard coloring (we have to change the coloring after sliding). All good blocks of the new tiling appear this way, since if there were any good blocks before the creation stage, they would have represented bad blocks in the old coloring, and we already know that this is impossible. Thus, starting with a tiling with k bad blocks, we obtain 2^{k+n+1} distinct tilings (with exactly $k + n + 1$ good blocks) of the Aztec diamond of order $n + 1$, each with a probability of $\frac{1}{2^{k+n+1}AD(n)}$.

Let us start now with a domino tiling of the Aztec diamond of order $n + 1$. In order to obtain a tiling of order n that produced it, it is enough to run the algorithm backwards, step by step. Change the coloring, remove the bad (ex-good) blocks and slide the dominos back. The result is a partial tiling of an Aztec diamond of order n with no bad blocks, and with a number of free 2×2 squares. All the domino tilings of the Aztec diamond of order n that produce, through the shuffling algorithm, the tiling we started with, must have their bad blocks exactly in these free 2×2 squares. Their positions being determined, the only thing that could vary is the orientation of the dominos: vertical or horizontal. Thus there are 2^k tilings of order n which generate each tiling of order $n + 1$ with $k + n + 1$ good blocks.

Putting the two observations together we obtain a (uniform) probability of $\frac{1}{AD(n)2^{n+1}}$ for any of the tilings of the Aztec diamond of order $n + 1$. This insures the fact that the outcome is a random tiling and it offers us a recurrence relation for $AD(n)$:
$AD(n + 1) = AD(n)2^{n+1}$. Hence, the number of domino tilings of an Aztec diamond of order n has been determined:

THEOREM 1.3 *(Elkies, Kuperberg, Larsen and Propp)*

The Aztec diamond of order n has exactly

$$AD(n) = 2^{n(n+1)/2}$$

domino tilings.

□

The shuffling algorithm appears, in the form we presented above, in [JPS95], where it is used to prove the Arctic Circle theorem. As mentioned, this algorithm produces random tilings, the order of the diamond increasing with 1 at each step. Running the shuffling until we get a large enough Aztec diamond we notice that a typical random tiling can be split into two regions. In the center the dominos seem to have no privileged direction, while close to the borders the dominos organize in layers with the same direction. The border between these regions is close to a circle. In order to formalize this result, we need the following conventions:

- *the* **north polar region** *is the union of those north-going dominos that are each connected to the boundary by a sequence of north-going adjacent dominos; the* **south**, **east** *or* **west polar region** *are defined similarly;*

- *the* **temperate zone** *is the union of those dominos that do not belong to any of the polar regions.*

In figure 5 we present the polar region of a tiling. In this example the north, south and east and west polar regions are differentiated using arrows which indicate the directions of the dominos.

The result is the following:

THEOREM 1.4 *(Arctic Circle Theorem)*

Fix $\epsilon > 0$. Then for sufficiently large n, all but an ϵ fraction of the domino tilings of the Aztec diamond of order n will have a temperate zone whose boundary stays uniformly within distance ϵn of the inscribed circle (of radius $n/\sqrt{2}$).

□

Another proof of a slightly stronger form of the Arctic Circle theorem is based on the study of placement probabilities, i.e. the probability that a domino covers a given pair of adjacent squares in a random tiling. Due to the rotational symmetry of the Aztec diamond it is enough to study only the north-going placement probabilities. For purposes of uniformity, the regions are "normalized", i.e. rescaled by a factor of n so that all locations are inside the circle of radius 1. The following asymptotic behaviour of the probabilities has been deduced in [CEP95]:

THEOREM 1.5 *(Cohn, Elkies and Propp)*

Let U be an open set containing the points $(\pm\frac{1}{2}, \frac{1}{2})$. If (x, y) is the normalized location of a north-going domino space in the Aztec diamond of order n, and $(x, y) \notin U$, then as $n \to \infty$, the placement probability at (x, y) is within $o(1)$ of $\mathcal{P}(x, y)$, where

$$
\mathcal{P}(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 \geq 1/2 \text{ and } y < 1/2 \\[2ex] 1 & \text{if } x^2 + y^2 \geq 1/2 \text{ and } y > 1/2 \\[2ex] \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{2y-1}{1-2x^2-2y^2}\right) & \text{if } x^2 + y^2 < 1/2 \end{cases}
$$

The $o(1)$ error bound is uniform in (x, y).

□

As we can see, outside the inscribed circle, in the northern region we are "sure" to find only north-going dominos, while in the remainder of the region we are sure that there are no north-going dominos (but there are, respectively, south, east or west-going dominos). This formula alone is not enough to prove entirely the Arctic Circle theorem. For a complete proof see [CEP95].

1.2 The Dual Problem

The problem of a (complete) tiling of the Aztec diamond (or any region formed of squares) with dominos is equivalent to a perfect matching problem for the dual graph. (i.e. the dual of the graph determined by the line segments which border the squares).

We work with subgraphs of the plane square grid with vertices

$$\{ (s/2, t/2) \; : \; s, t \; odd, \; s, t \in \mathbf{Z} \}$$

and edges

$$\{ \{(i,j),(k,l)\} \; : \; |i - k| + |j - l| = 1 \}$$

DEFINITION 1.6 *The* **dual Aztec diamond** *of order* **n**, $\mathbf{G_n}$, *is the subgraph of the plane grid induced by the set of vertices:*

$$\{ (s/2, t/2) \; : \; s, t \; odd, \; |s| + |t| \leq 2n \}$$

$\square$

This representation of the graph has $(0,0)$ as a center of symmetry. We consider this representation as the standard one and we will mention any exception (modification of the coordinates of the center).

- *the* **corners** *are the vertices of the dual Aztec diamond of order n with*

  $|x| = n \pm 1/2$ *or* $|y| = n \pm 1/2$;

- *the* **borders** *are the lines* $y = \pm x \pm n$.

Figure 6 presents a domino tiling of an Aztec diamond of order 5 and the corresponding matching of the dual.

For our further investigations we choose to work in terms of the dual graph. All the results of the previous section hold, by the following dictionary:

- **domino** *translates as* **edge**;

- **tiling** *translates as* **matching**;

- **north**, *(*south, **east** *or* west*)*-going *dominos become* **north**, *(*south, **east** *or* west*)*-going *edges*;

- *the* **polar regions, temperate zone** *or* **inscribed circle** *are the zones defined in the previous section*;

- *the* **placement probability** *for an edge is the same as the placement probability for the corresponding domino.*

For the dual version of the shuffling algorithm we introduce the following convention:

- *the* **standard coloring** *of the dual square lattice is a checkerboard coloring such that the line* $x + y = n$ *(the upper-right border) passes through only black squares.*

Note that with this convention, white squares of the dual correspond to 2×2 blocks of the Aztec diamond such that the upper left square is white. Thus the following are true:

- a **location** *is the center of a white square;*

- *an edge is a* **north,** *(***south, east** *or* **west***)-***going** *edge if it is situated north (respectively east, south, west) of the location in the white square adjacent to it;*

- *a* **bad block** *is formed by two parallel edges on the sides of a black square, while a* **good block** *is formed by two parallel edges on the sides of a white square.*

Figure 7 presents a dual Aztec diamond of order 5 with the standard coloring and the north-going edges represented on it.

Let us mention that 90, 180 and 270 degrees rotations are isomorphisms of the dual Aztec diamond which transform the north-going edges into east, respectively south and west-going edges. There is an isomorphism between the set of matchings that contain a north-going edge at location (i, j) and each of the sets of matchings that contain an east, south or west-going edge at its transforms: $(j, -i)$, $(-i, -j)$ or $(-j, i)$. Thus it is enough to determine the placement probabilities for the north-going edges, $p_{i,j}^n$, the other probabilities being easily determined by:

$$pe_{i,j}^n = p_{-j,i}^n$$

for the east-going edges at location (i, j),

$$ps_{i,j}^n = p_{-j,-i}^n$$

for the south-going edges at location (i, j) and

$$pw_{i,j}^n = p_{j,-i}^n$$

for the west-going edges at location (i, j).

Figure 8 presents all possible types of perfect matchings of the dual Aztec diamond of order n with respect to the way its borders were matched. We will refer to these types throughout the chapter. In all illustrations the dashed regions are free zones for which all possible matchings are accepted.

*In the dual Aztec diamond two adjacent corners may be matched with each other, and we say they are matched **together**, or each of them with their other neighbor, in which case we say that they are matched **separately**.*

In the subsequent sections we need the observations :

LEMMA 1.7 *The following are true for matchings of the dual Aztec diamond of order n:*

i) The number of perfect matchings in which two specified adjacent corners are matched separately is $2^{(n-1)n/2}$. The restriction of these matchings to the dual Aztec diamond of order $n - 1$ with the center shifted one unit in the direction opposite to the corners determines a bijection between the two sets of matchings.

ii) The number of perfect matchings in which two specified adjacent corners are matched together is $2^{n(n+1)/2} - 2^{(n-1)n/2}$.

iii) The number of perfect matchings of the graph formed by deleting two adjacent corners from the dual Aztec diamond is $2^{n(n+1)/2} - 2^{(n-1)n/2}$.

iv) The number of perfect matchings in which two pairs of adjacent corners are matched, pair by pair, separately, is equal to $2^{(n-2)(n-1)/2}$. The restriction of these matchings to the dual Aztec diamond of order $n-2$ determines a bijection between the two sets of matchings.

Proof:

i) Once two adjacent corners are matched separately, there is a unique way of matching the vertices on the border lines which start at these two corners. This can be further extended to a perfect matching by matching the remaining vertices, which form an Aztec diamond of order $n-1$. Figure 9a illustrates this situation.

ii) Follows from i).

iii) Any perfect matching of this graph, (represented in figure 9b) can be uniquely extended to a perfect matching of the dual Aztec diamond in which the two vertices that were deleted are matched together (figure 9c). Thus we obtain the same value as in part ii).

iv) Note that in this case we are talking either about matchings of type F or type G as in figure 8. It is clear that the region left to be matched is an Aztec diamond of order $n-2$.

<div align="right">

q.e.d.

</div>

The standard colorings of the graphs involved in the various types of matchings described above (the dual Aztec diamond of order $n-2$ centered at $(0,0)$, and the various dual Aztec diamonds of order $n-1$ centered at $(0,1),(0,-1),(1,0)$ and $(-1,0)$) all conform to the standard coloring of the dual Aztec diamond of order n. Thus the

positions of the north (south, east, west)-going edges coincide for all these graphs.

1.3 Modifications of the Problem

We define a series of graphs, small modifications of G_n, of higher genus or cross-cap number, and which inherit to some extent the matching properties of the dual Aztec diamond.

The idea is to take the planar representation of the dual Aztec diamond given by its definition as a subgraph of the plane grid, to form the rectangle defined by the lines $y = \pm(n - 1/2)$ and $x = \pm(n - 1/2)$ and to consider this rectangle as a representation of some other surface, therefore to paste opposite sides and, together with them, the corresponding corners and edges of the Aztec diamond.

The operation of identification that we use is described below:

- *by* **pasting** *the oriented edges* (u, v) *and* (u', v') *of a graph G we create a new graph with the set of vertices* $V(G) \backslash \{u', v'\}$ *and with set of edges*

  $E(G) \backslash \{\{u', v'\}\} \cup \{\{u, w\} \ : \ w \in N(u')\} \cup \{\{v, w\} \ : \ w \in N(v')\}.$

Thus the vertex u is identified with u', v with v' and after the identification only one representative is kept for the eventual double edge.

Using this tool we can define now the following graphs:

DEFINITION 1.8 *The* **Cylindrical Aztec diamond** *of order* n *is the graph CG_n formed from G_n by pasting*

$$((1/2 - n, 1/2), (1/2 - n, -1/2)) \text{ with } ((n - 1/2, 1/2), (n - 1/2, -1/2)).$$

Note that this graph is obviously planar, but we study it first, as an intermediate step to some more interesting constructions (the Torus Aztec diamond and the Klein Aztec diamond).

DEFINITION 1.9 *The* **Möbius Aztec diamond** *of order* **n** *is the graph* MG_n *formed from* G_n *by pasting*

$$((1/2 - n, 1/2), (1/2 - n, -1/2)) \text{ with } ((n - 1/2, -1/2), (n - 1/2, 1/2)).$$
□

This is a graph of genus 1 and cross-cap number 1.

DEFINITION 1.10 *The* **Klein Aztec diamond** *of order* **n** *is the graph* KG_n *formed from* G_n *by pasting*

$$((1/2 - n, 1/2), (1/2 - n, -1/2)) \text{ with } ((n - 1/2, -1/2), (n - 1/2, 1/2))$$

and

$$((-1/2, 1/2 - n), (1/2, 1/2 - n)) \text{ with } ((-1/2, n - 1/2), (1/2, n - 1/2)).$$
□

This graph has genus 1 and cross-cap number 2.

DEFINITION 1.11 *The* **Projective Aztec diamond** *of order* **n** *is the graph* PG_n *formed from* G_n *by pasting*

$$((1/2 - n, 1/2), (1/2 - n, -1/2)) \text{ with } ((n - 1/2, -1/2), (n - 1/2, 1/2))$$

and

$$((-1/2, 1/2 - n), (1/2, 1/2 - n)) \text{ with } ((1/2, n - 1/2), (-1/2, n - 1/2)).$$
□

This is a graph of genus 2 and cross-cap number 1.

DEFINITION 1.12 *The* **Torus Aztec diamond** *of order* **n** *is the graph* TG_n *formed from* G_n *by pasting*

$$((1/2 - n, 1/2), (1/2 - n, -1/2)) \text{ with } ((n - 1/2, 1/2), (n - 1/2, -1/2))$$

and

$$((-1/2, 1/2 - n), (1/2, 1/2 - n)) \text{ with } ((-1/2, n - 1/2), (1/2, n - 1/2)).$$

□

This graph has genus 1.

In what follows, whenever we talk about the planar representation of these graphs we refer to the one obtained by cutting apart the sides of that we have pasted together from G_n. Figure 10 shows the planar representations for each of these graphs $(n = 3)$, where the identified vertices have been marked with the same symbol. The generic name for the graphs described above will be **modified** Aztec diamonds. For them, we apply the same vocabulary as the one developed in section 2 for dual Aztec diamonds. It makes sense to talk about orientation of edges if we consider the planar representation of these graphs. We have to pay special attention to the edges that we have pasted together, and which become east and west, or south and north-going edges in the same time.

Note that the Projective and the Torus Aztec diamonds preserve the property of the dual Aztec that 90, 180 and 270 degrees rotations (of the planar representations)

are isomorphisms, while for the Cylindrical, Möbius and Klein Aztec diamonds only the 180 degrees rotation remains an isomorphism. Thus the computation of the placement probabilities for an edge in a perfect matching of one of the first two graphs reduces to the computation of the north-going ones, while for the other three graphs we have to compute the west-going probabilities as well, the other two being deduced with the help of the following relation:

$$P_{i,j}^n = P_{-j,-i}^n$$

which relates east to west and south to north.

We have to mention that in the planar representation of these graphs, the edge obtained through pasting appears twice, so if this edge belongs to some matching we represent, we have an extra apparent edge in the drawing of the matching.

Figure 11 presents a comparison between the perfect matchings of the modified graphs and those of the dual Aztec diamond, for $n = 2$. We already notice some similarities between the matchings situated in the same rows and above the line, the way they seem to "arise", in a natural way, from the first column, and the somehow "special" character of the matchings situated under the line. We prove here that there is a bijection between the group of similar matchings and then we study the special matchings in the sections dedicated to each of these graphs.

We need a way of distinguishing the two situations.

A perfect matching of a modified Aztec diamond is:

- **extendible** *(to the dual Aztec diamond) iff a planar representation of it is a*

submatching for some perfect matching of the dual Aztec diamond;

- **special** *if it is not extendible.*

Note that the only edges we can add to an extendible matching to form a perfect matching are between two adjacent corners.

THEOREM 1.13 *There is a bijection between:*

i) the perfect matchings of the dual Aztec diamond of types A-F and the extendible matchings of the Cylindrical Aztec diamond or of the Möbius Aztec diamond.

ii) the perfect matchings of the dual Aztec diamond of types A-E and the extendible matchings of the Klein, the Projective or the Torus Aztec diamond.

Proof:

We describe the bijection for the first part and then we will see that small modifications give us the second part.

To make the proof easier to follow, the bijections presented in this theorem are graphically represented in figures 12, 13, 14 and 15. The contents of the dashed zone is to be copied from one drawing to the other within the rows for figures 12, 13 and within the columns for figures 14, 15.

The two modified Aztec diamonds we are discussing here have two fewer vertices than the dual Aztec diamond. According to the definition, the deleted vertices are $(n - 1/2, 1/2)$ and $(n - 1/2, -1/2)$.

In the planar representation all vertices, except possibly the ones that were involved in the identification process, appear matched.

Suppose that $(1/2-n, 1/2)$ and $(1/2-n, -1/2)$ are matched together in a matching of one modified Aztec diamond (figures 12 A, B, C and 13 F). Then, in a planar representation of the matching, the deleted vertices appear as matched, therefore it is sufficient to add this edge and all vertices of the dual Aztec diamond are matched in a matching of type A, B, C or F. Conversely, deleting this edge from the set of edges of a matching of the dual Aztec diamond of type A, B, C or F produces a matching of the modified Aztec diamond in which $(1/2 - n, 1/2)$ and $(1/2 - n, -1/2)$ are matched together. These matchings differ, between types, by the way the non-identified corners are matched, and, within the same type, at some vertex other than the ones where we modify the matching. So far we have a one-to-one correspondence.

Suppose now that the vertices $(1/2 - n, 1/2)$ and $(1/2 - n, -1/2)$ are matched separately in a matching of a modified Aztec diamond (figure 13 D, E). In this case, in the planar representation of the graph, no edge of the matching appears twice, thus two vertices appear as unmatched. If these two vertices are not adjacent we have a special matching (figure 16 a, b). If they are adjacent then we can match them together, thus obtaining one of the matchings of the dual Aztec diamond of type D or E. Deleting this edge from a matching of type D or E gives us the matching of the modified Aztec diamond we started with.

Thus we have defined the bijection we were seeking.

The definition of our bijection for the second case is a refinement of the previous one, since we delete four vertices this time, and each pair of two could be matched together or separately in the dual Aztec diamond.

Suppose vertices $(1/2 - n, 1/2)$ and $(1/2 - n, -1/2)$ are matched together and vertices $(1/2, 1/2 - n)$ and $(-1/2, 1/2 - n)$ are matched together in a modified Aztec diamond. In the planar representation, $(n - 1/2, 1/2)$ and $(n - 1/2, -1/2)$, respectively $(1/2, n - 1/2)$ and $(-1/2, n - 1/2)$ appear matched (figure 14 A) and by adding/deleting these two edges we get a bijection between the matchings of the dual Aztec diamond of type A and the matchings of the modified Aztec diamond extendible to these.

If only two of the corners are matched together (figures 14 B and C, 15 D and E), and the two vertices that are apparently unmatched in the planar representation are adjacent, adding/deleting the edge between them as well as the apparent edge between the doubles of the first two corners determines a bijection between the remaining extendible matchings of the modified Aztec diamonds and matchings of the dual Aztec diamond of types B-E.

There is nothing extendible to matchings of types F or G because by identification two of the corners would be matched, each, with two other vertices.

<div align="right">q.e.d.</div>

It is worth mentioning that there are no perfect matchings with two consecutive pairs of corners matched separately for any of the graphs considered here. Figure 17 indicates a quick way of eliminating this case.

The bijections from theorem 1.13 will also be the source of results concerning the placement probabilities of the modified graphs, which we are able to express using the placement probabilities for the dual Aztec diamond.

1.4 Cylindrical Aztec Diamonds

We define a one-to-one function from the set of perfect matchings of CG_n to the set of perfect matchings of G_n. For that purpose we need the following partial result:

LEMMA 1.14 *There are no special matchings of the Cylindrical Aztec diamond.*

Proof:

For $n = 2$ all matchings are presented in figure 11, and there are no special matchings.

Suppose that $n \geq 3$. A special matching would fall into the following two categories, with respect to the way its corners are matched (figure 16 a):

$(1/2 - n, 1/2)$ with $(3/2 - n, 1/2)$ and $(1/2 - n, -1/2)$ with $(n - 3/2, -1/2)$

or

$(1/2 - n, 1/2)$ with $(n - 3/2, 1/2)$ and $(1/2 - n, -1/2)$ with $(3/2 - n, -1/2)$.

The two cases being symmetric we only study the first one. For any matching with these two edges there is only one way of matching the vertices on the lines $y = x \pm n$, presented in figure 18. We still have to find a matching of a subgraph of the plane grid, therefore a bipartite graph, which has $(n - 1)n$ vertices in one class and $(n + 1)(n - 2)$ vertices in the other, therefore evidently has no perfect matching.

<div align="right">q.e.d.</div>

COROLLARY 1.15 *There is a one to one correspondence between the perfect matchings of the dual Aztec diamond of order n of types A-F and the perfect matchings of the Cylindrical Aztec diamond of order n.*

Proof:

This theorem is a direct consequence of theorem 1.13 and of the previous lemma.

q.e.d.

THEOREM 1.16 *The number of perfect matchings of the Cylindrical Aztec diamond of order n is*

$$CAD(n) = 2^{\frac{(n-2)(n-1)}{2}}(2^{2n-1} - 1)$$

Proof:

From lemma 1.7 we can deduce that the number of matchings of type G is $AD(n-2)$. We apply then theorem 1.3 and we find that

$$CAD(n) = AD(n) - AD(n-2).$$

Using the fact that $AD(n) = 2^{\frac{n(n+1)}{2}}$ we obtain the result.

q.e.d.

THEOREM 1.17 *The placement probabilities for the Cylindrical Aztec diamond at location (i, j) are described by the formulas:*

$$cn_{i,j}^n = \begin{cases} \frac{p_{i,j}^n - 2^{1-2n}p_{i,j}^{n-2}}{1-2^{1-2n}} & |i| + |j| \leq n - 2 \\[3mm] \frac{p_{i,j}^n - 2^{1-2n}}{1-2^{1-2n}} & |i| + |j| > n - 2 \\ & \text{and } j \geq 0 \\[3mm] \frac{p_{i,j}^n}{1-2^{1-2n}} & |i| + |j| > n - 2 \\ & \text{and } j < 0 \end{cases}$$

for the north-going edges and

$$
cw_{i,j}^n = \begin{cases} \dfrac{p_{j,-i}^n - 2^{1-2n} p_{j,-i}^{n-2}}{1 - 2^{1-2n}} & |i| + |j| \le n - 2 \\[3em] \dfrac{p_{j,-i}^n}{1 - 2^{1-2n}} & |i| + |j| > n - 2 \\[1em] & \text{and } (i,j) \ne (1 - n, 0) \\[2em] \dfrac{p_{j,-i}^n - 2^{-n} + 2^{1-2n}}{1 - 2^{1-2n}} & (i,j) = (1 - n, 0) \end{cases}
$$

for the west-going edges.

	type E	type G
$|i| + |j| \le n - 2$	——	$AD(n-2)p_{i,j}^{n-2}$
$|i| + |j| > n - 2$ and $j \ge 0$	——	$AD(n-2)$
$|i| + |j| > n - 2$ and $j < 0$	——	——

Table 1 Matchings to be subtracted for north-going edges in the Cylindrical Aztec diamond

	type E	type G
$|i| + |j| \le n - 2$	——	$AD(n-2)p_{j,-i}^{n-2}$
$|i| + |j| > n - 2$ and $(i,j) \ne (1 - n, 0)$	——	——
$(i,j) = (1 - n, 0)$	$AD(n-1) - AD(n-2)$	——

Table 2 Matchings to be subtracted for west-going edges in the Cylindrical Aztec diamond

Proof:

The number of matchings of the Aztec diamond of order n that contain a north-going edge at location (i, j) is $AD(n)p_{i,j}^n$ and for west-going edges it is $AD(n)p_{j,-i}^n$. From theorems 1.13 and 1.15 we deduce that from this total we have to subtract the number of matchings of type G that might contain an edge at our current location. Thus we have to consider separately the locations situated inside the Aztec diamond of order $n-2$ and the ones situated outside this region. For the ones outside we have to decide whether an edge situated at such a location can belong to a matching of the forbidden type, G. Finally we have to study separately the location $(1 - n, 0)$, since here, when using the bijection from theorem 1.13, we delete a west-going edge (from type E). This is the only case when we delete a north or west-going edge. We know from lemma 1.7 that there are $AD(n-2)$ matchings of type G and $AD(n-1) - AD(n-2)$ matchings of type E.

The tables 1 and 2 present the amounts that have to be subtracted from matchings of types E or G for each of the regions.

Subtracting these and dividing through by $CAD(n)$ we obtain the placement probabilities.

<div align="right">q.e.d.</div>

1.5 Möbius Aztec Diamonds

The results of this section are somewhat similar to those referring to the Cylindrical Aztec diamond because we are using the same bijection to the perfect matchings

of the dual Aztec diamond of types $A - F$, from theorem 1.13, but in this case we have some special matchings to consider.

LEMMA 1.18 *The number of special matchings of the Möbius Aztec diamond is equal to*

$$2AD(n-1).$$

Proof:

There are two possibilities for special matchings, as presented in figure 16 b:

$$\{(1/2 - n, 1/2), (3/2 - n, 1/2)\} \text{ and } \{(1/2 - n, -1/2), (n - 3/2, 1/2)\}$$

or

$$\{(1/2 - n, 1/2), (n - 3/2, -1/2)\} \text{ and } \{(1/2 - n, -1/2), (3/2 - n, -1/2)\}$$

The two cases are symmetric with respect to the x-axis. Any matching of this type must also contain all edges that link the vertices on the borders of positive, respectively negative, y-coordinate to their unique (left or right) neighbor still available (figure 19). We notice that the vertices that are left unmatched form a dual Aztec diamond of order $n - 1$ centered either at $(0, 1)$ or at $(0, -1)$.

<div align="right">

q.e.d.

</div>

THEOREM 1.19 *The number of perfect matchings of the Möbius Aztec diamond of order n is*

$$MAD(n) = 2^{\frac{(n-2)(n-1)}{2}}(2^{2n-1} + 2^n - 1)$$

Proof:

From theorem 1.13 and lemma 1.5 we can deduce that the number of perfect matchings of the Möbius Aztec diamond is equal to the number of matchings of the dual Aztec diamond of types A-F plus the special matchings.

$$MAD(n) = AD(n) + 2AD(n-1) - AD(n-2)$$

Using the fact that $AD(n) = 2^{\frac{n(n+1)}{2}}$ we obtain our result.

<div align="right">q.e.d.</div>

THEOREM 1.20 *The placement probabilities for the Möbius Aztec diamond at location (i,j) are described by the formulas:*

$$mn_{i,j}^n = \begin{cases} \frac{p_{i,j}^n + 2^{-n}(p_{i,j-1}^{n-1}+p_{i,j+1}^{n-1}) - 2^{1-2n}p_{i,j}^{n-2}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| \le n-2 \\[4mm] \frac{p_{i,j}^n + 2^{-n}(p_{i,j-1}^{n-1}+1) - 2^{1-2n}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| > n-2 \text{ and } j > 0 \\[4mm] \frac{p_{i,j}^n + 2^{-n}p_{i,j+1}^{n-1}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| > n-2 \text{ and } j < 0 \\[4mm] \frac{p_{i,j}^n + 2^{-n} - 2^{1-2n}}{1+2^{1-n}-2^{1-2n}} & (i,j) = (\pm(n-1),0) \end{cases}$$

for the north-going edges and

$$mw_{i,j}^n = \begin{cases} \frac{p_{j,-i}^n + 2^{-n}(p_{j-1,-i}^{n-1}+p_{j+1,-i}^{n-1}) - 2^{1-2n}p_{j,-i}^{n-2}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| \le n-2 \\[4mm] \frac{p_{j,-i}^n + 2^{-n}p_{j-1,-i}^{n-1}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| > n-2 \text{ and } j > 0 \\[4mm] \frac{p_{j,-i}^n + 2^{-n}p_{j+1,-i}^{n-1}}{1+2^{1-n}-2^{1-2n}} & |i|+|j| > n-2 \text{ and } j < 0 \\[4mm] \frac{p_{j,-i}^n}{1+2^{1-n}-2^{1-2n}} & (i,j) = (n-1,0) \\[4mm] \frac{p_{j,-i}^n - 2^{-n} + 2^{1-2n}}{1+2^{1-n}-2^{1-2n}} & (i,j) = (1-n,0) \end{cases}$$

for the west-going edges.

Proof:

The number of matchings of the Aztec diamond of order n that contain a north-going edge at location (i,j) is $AD(n)p_{i,j}^n$ and for a west-going edge, $AD(n)p_{j,-i}^n$. For

	special type a	special type b
$|i| + |j| \le n - 2$	$AD(n-1)p^{n-1}_{i,j+1}$	$AD(n-1)p^{n-1}_{i,j+1}$
$|i| + |j| > n - 2$ and $j > 0$	$AD(n-1)$	$AD(n-1)p^{n-1}_{i,j-1}$
$|i| + |j| > n - 2$ and $j < 0$	$AD(n-1)p^{n-1}_{i,j+1}$	—
$(i,j) = (\pm(n-1),0)$	$AD(n-1)$	—

Table 3 Matchings to be added for north-going edges in the Möbius Aztec diamond

	special type a	special type b
$|i| + |j| \le n - 2$	$AD(n-1)p^{n-1}_{j+1,-i}$	$AD(n-1)p^{n-1}_{j-1,-i}$
$|i| + |j| > n - 2$ and $j > 0$	—	$AD(n-1)p^{n-1}_{j-1,-i}$
$|i| + |j| > n - 2$ and $j < 0$	$AD(n-1)p^{n-1}_{j+1,-i}$	—
$(i,j) = (\pm(n-1),0)$	—	—

Table 4 Matchings to be added for west-going edges in the Möbius Aztec diamond

the Möbius Aztec diamond we have to subtract from these the number of matchings of type G that might contain a north or west-going edge at the current location, and to add the number of special matchings that have such an edge. Finally we have to study separately the location $(1 - n, 0)$, since here, when using the bijection from theorem 1.13, we delete a west-going edge (from type E). This is the only case when, through the bijection, we delete a north or west-going edge.

The regions involved in the creation of these matchings can be deduced from figures 12, 13 and 19.

The number of matchings to be subtracted from matchings of types E or G is the same as for the Cylindrical Aztec diamond, presented in tables 1 and 2 from the proof of theorem 1.17.

The number of matchings to be added (the special ones) are represented in tables 3 and 4.

After combining the corresponding tables and dividing through by $MAD(n)$ we obtain the formulas.

<div align="right">q.e.d.</div>

1.6   Klein Aztec Diamonds

This graph combines the properties of the two previous ones: it behaves like a Cylindrical Aztec diamond with a horizontal axis and like a Möbius Aztec diamond with a vertical axis. As the latter has no special matchings it turns out that all the special matchings of the Klein Aztec diamond can be derived from special matchings

of the Möbius Aztec diamond.

LEMMA 1.21 *The special matchings of the Klein Aztec diamond are in one-to-one correspondence with the special matchings of the Möbius Aztec diamond.*

Proof:

If a matching of the Klein Aztec diamond is extendible to the Cylindrical Aztec diamond from which it could be obtained by pasting, then it is extendible to the dual Aztec diamond, because the Cylindrical Aztec diamond has no special matchings. Thus all special matchings must contain a pair of edges like in figure 16. These two cases extend uniquely to one of the two types of matchings presented in figure 20, i.e. all vertices on borders with the y-coordinate positive, respectively negative, are matched with the unique (left or right) neighbor which is still free, with the exception of the corners, which have both two neighbors available.

The two cases are symmetric with respect to the x-axis.

We notice that the vertices that are left unmatched form a dual Aztec diamond of order $n - 1$ and that by adding the edge between the two corners apparently unmatched we extend these matchings to the corresponding special matchings of the Möbius Aztec diamond (figure 19). Deleting the same edge gives the inverse direction.

<div align="right">q.e.d.</div>

THEOREM 1.22 *The number of perfect matchings of the Klein Aztec diamond of order n is*

$$KAD(n) = 2^{\frac{(n-2)(n-1)}{2}}(2^{2n-1} + 2^n - 2)$$

Proof:

From theorem 1.13 and lemma 1.21 we deduce that the perfect matchings Klein Aztec diamond are in one-to-one correspondence with the perfect matchings of the Möbius Aztec diamond which are not extendible to a matching of type F. Therefore

$$KAD(n) = AD(n) + 2AD(n-1) - 2AD(n-2).$$

<div align="right">

q.e.d.

</div>

THEOREM 1.23 *The placement probabilities for the Klein Aztec diamond at location (i, j) are described by the formulas:*

$$kn_{i,j}^n = \begin{cases} \frac{p_{i,j}^n + 2^{-n}(p_{i,j-1}^{n-1} + p_{i,j+1}^{n-1}) - 2^{2-2n}p_{i,j}^{n-2}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| \leq n - 2 \\[4ex] \frac{p_{i,j}^n + 2^{-n}p_{i,j-1}^{n-1} - 2^{1-2n}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } 1 < j < n - 1 \\[4ex] \frac{p_{i,j}^n + 2^{-n}p_{i,j+1}^{n-1}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } 1 - n < j < 0 \\[4ex] \frac{p_{i,j}^n + 2^{-n} - 2^{1-2n}}{1 + 2^{1-n} - 2^{2-2n}} & (i, j) = (\pm(n-1), 0) \\[4ex] \frac{p_{i,j}^n + 2^{-n}(p_{i,j-1}^{n-1} + p_{-i,-j-1}^{n-1})}{1 + 2^{1-n} - 2^{2-2n}} & (i, j) = (0, n-1) \end{cases}$$

for the north-going edges, and

$$
kw_{i,j}^n = \begin{cases}
\dfrac{p_{j,-i}^n + 2^{-n}(p_{j-1,-i}^{n-1} + p_{j+1,-i}^{n-1}) - 2^{2-2n}p_{j,-i}^{n-2}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| \leq n - 2 \\[3em]

\dfrac{p_{j,-i}^n + 2^{-n}p_{j-1,-i}^{n-1}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\
& \text{and } i, j > 0 \\[3em]

\dfrac{p_{j,-i}^n + 2^{-n}p_{j-1,-i}^{n-1} - 2^{1-2n}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\
& \text{and } j > 0, i \leq 0 \\[3em]

\dfrac{p_{j,-i}^n + 2^{-n}p_{j+1,-i}^{n-1}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\
& \text{and } j < 0, i > 0 \\[3em]

\dfrac{p_{j,-i}^n + 2^{-n}p_{j+1,-i}^{n-1} - 2^{1-2n}}{1 + 2^{1-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\
& \text{and } i \leq 0, j < 0 \\[3em]

\dfrac{p_{j,-i}^n}{1 + 2^{1-n} - 2^{2-2n}} & (i, j) = (n - 1, 0) \\[3em]

\dfrac{p_{j,-i}^n - 2^{-n} + 2^{2-2n}}{1 + 2^{1-n} - 2^{2-2n}} & (i, j) = (1 - n, 0)
\end{cases}
$$

for the west-going edges.

Proof:

According to theorem 1.13, we have to exclude from the matchings of the dual

Aztec diamond those that have of an north or west-going edge at the current location

	type B	type F	type G
$|i| + |j| \leq n - 2$	—	$AD(n-2)p_{i,j}^{n-2}$	$AD(n-2)p_{i,j}^{n-2}$
$|i| + |j| > n - 2$ and $0 \leq j < n - 1$	—	—	$AD(n-2)$
$|i| + |j| > n - 2$ and $j < 0$	—	—	—
$(i,j) = (0, n-1)$	$AD(n-1) - AD(n-2)$	—	$AD(n-2)$

Table 5 Matchings to be subtracted for north-going edges in the Klein Aztec diamond

	type E	type F	type G
$|i| + |j| \leq n - 2$	—	$AD(n-2)p_{j,-i}^{n-2}$	$AD(n-2)p_{j,-i}^{n-2}$
$|i| + |j| > n - 2$ and $1 - n < i \leq 0$	—	$AD(n-2)$	—
$|i| + |j| > n - 2$ and $i > 0$	—	—	—
$(i,j) = (1-n, 0)$	$AD(n-1) - AD(n-2)$	$AD(n-2)$	—

Table 6 Matchings to be subtracted for west-going edges in the Klein Aztec diamond

and are extendible to a matching of type F or G. We also have to pay special attention to the location $(0, n-1)$, where a north-going edge disappears from matchings of type B. Thus, tables 1 and 2 extended to include types B and F become tables 5 and 6.

As for the special matchings, it is clear from lemma 1.21 that we can use the tables 3 and 4 for the Möbius Aztec diamond, which just have to be updated at location $(0, n-1)$ for the north-going edges, that is, to add at this location the extra $AD(n-1)p_{-i,-j-1}^{n-1}$ that we get from the special matchings of type a.

After division by $KAD(n)$ we obtain the formulas.

<div align="right">q.e.d.</div>

1.7 Projective Aztec Diamonds

If we cut open any of the sides that have been pasted together we end up with a Möbius Aztec diamond with a vertical or with a horizontal axis of symmetry. Thus we prove that all the special matchings of the Projective Aztec diamond "come from" special matchings of a Möbius Aztec diamond.

LEMMA 1.24 *The special matchings of the Projective Aztec diamond are in one-to-one correspondence with the special matchings of a Möbius Aztec diamond with horizontal symmetry and a Möbius Aztec diamond with vertical symmetry.*

Proof:

All special matchings must contain a pair of edges like in figure 16 b, which can be situated either along a horizontal axis (exactly like in the figure), or along a vertical axis. The first situation produces one of the special matchings from figures 21 a or c, i.e. all vertices on borders with the y-coordinate positive, respectively negative, are matched with the unique (left or right) neighbor which is still free, with the exception of the corners, which have both two neighbors available. For the other case we get the matchings from figure 21 b and d. It is now clear that no other combination is possible, since once we have chosen which pair of opposite vertices is to be matched the special way, the other corners cannot be matched the special way anymore.

We notice that the vertices that are left unmatched form a dual Aztec diamond of order $n - 1$ and that by adding the edge between the two corners apparently unmatched, we extend these matchings to special matchings of one of the two Möbius

Aztec diamonds mentioned. Deleting the same edge gives the inverse direction.

q.e.d.

THEOREM 1.25 *The number of perfect matchings of the Projective Aztec diamond of order n is*

$$PAD(n) = 2^{\frac{(n-2)(n-1)}{2}}(2^{2n-1} + 2^{n+1} - 2)$$

Proof:

From lemma 1.24 we deduce that the Projective Aztec diamond has twice as many special matchings as the the Möbius Aztec diamond, i.e. $4AD(n-1)$. Combining this with theorem 1.13 we have

$$PAD(n) = AD(n) + 4AD(n-1) - 2AD(n-2).$$

q.e.d.

THEOREM 1.26 *The north-going placement probabilities for the Projective Aztec diamond at location (i,j) are described by the formula:*

$$ppn_{i,j}^n = \begin{cases} \dfrac{p_{i,j}^n + 2^{-n}(p_{i,j-1}^{n-1} + p_{i,j+1}^{n-1} + p_{i-1,j}^{n-1} + p_{i+1,j}^{n-1}) - 2^{2-2n} p_{i,j}^{n-2}}{1 + 2^{2-n} - 2^{2-2n}} & |i| + |j| \leq n - 2 \\[4ex] \dfrac{p_{i,j}^n + 2^{-n}(1 + p_{i,j-1}^{n-1} + p_{i-1,j}^{n-1}) - 2^{1-2n}}{1 + 2^{2-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } i > 0, j > 0 \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(1 + p_{i,j-1}^{n-1} + p_{i+1,j}^{n-1}) - 2^{1-2n}}{1 + 2^{2-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } i < 0, j > 0 \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(p_{i,j+1}^{n-1} + p_{i-1,j}^{n-1})}{1 + 2^{2-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } i > 0, j < 0 \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(p_{i,j+1}^{n-1} + p_{i+1,j}^{n-1})}{1 + 2^{2-n} - 2^{2-2n}} & |i| + |j| > n - 2 \\ & \text{and } i < 0, j < 0 \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(1 + p_{i+1,j}^{n-1}) - 2^{1-2n}}{1 + 2^{2-n} - 2^{2-2n}} & (i,j) = (1 - n, 0) \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(1 + p_{i-1,j}^{n-1}) - 2^{1-2n}}{1 + 2^{2-n} - 2^{2-2n}} & (i,j) = (n - 1, 0) \\[3ex] \dfrac{p_{i,j}^n + 2^{-n} p_{i,j+1}^{n-1}}{1 + 2^{2-n} - 2^{2-2n}} & (i,j) = (0, 1 - n) \\[3ex] \dfrac{p_{i,j}^n + 2^{-n}(p_{-i,-j-1}^{n-1} + p_{i,j-1}^{n-1} - 1)}{1 + 2^{2-n} - 2^{2-2n}} & (i,j) = (0, n - 1) \end{cases}$$

for the north-going edges.

	special type a	special type b
$|i| + |j| \leq n - 2$	$AD(n-1)p_{i,j+1}^{n-1}$	$AD(n-1)p_{i+1,j}^{n-1}$
$|i| + |j| > n - 2$ and $i > 0, j > 0$	$AD(n-1)$	—
$|i| + |j| > n - 2$ and $i < 0, j > 0$	$AD(n-1)$	$AD(n-1)p_{i+1,j}^{n-1}$
$|i| + |j| > n - 2$ and $i > 0, j < 0$	$AD(n-1)p_{i,j+1}^{n-1}$	—
$|i| + |j| > n - 2$ and $i < 0, j < 0$	$AD(n-1)p_{i,j+1}^{n-1}$	$AD(n-1)p_{i+1,j}^{n-1}$
$(i,j) = (1-n,0)$	$AD(n-1)$	$AD(n-1)p_{i+1,j}^{n-1}$
$(i,j) = (n-1,0)$	$AD(n-1)$	—
$(i,j) = (0,1-n)$	$AD(n-1)p_{i,j+1}^{n-1}$	—
$(i,j) = (0,n-1)$	$AD(n-1)p_{-i,-j-1}^{n-1}$	—
	special type c	special type d
$|i| + |j| \leq n - 2$	$AD(n-1)p_{i,j-1}^{n-1}$	$AD(n-1)p_{i-1,j}^{n-1}$
$|i| + |j| > n - 2$ and $i > 0, j > 0$	$AD(n-1)p_{i,j-1}^{n-1}$	$AD(n-1)p_{i-1,j}^{n-1}$
$|i| + |j| > n - 2$ and $i < 0, j > 0$	$AD(n-1)p_{i,j-1}^{n-1}$	—
$|i| + |j| > n - 2$ and $i > 0, j < 0$	—	$AD(n-1)p_{i-1,j}^{n-1}$
$|i| + |j| > n - 2$ and $i < 0, j < 0$	—	—
$(i,j) = (1-n,0)$	—	—
$(i,j) = (n-1,0)$	—	$AD(n-1)p_{i-1,j}^{n-1}$
$(i,j) = (0,1-n)$	—	—
$(i,j) = (0,n-1)$	$AD(n-1)p_{i,j-1}^{n-1}$	—

Table 7 Matchings to be added for north-going edges in the Projective Aztec diamond

Proof:

The special matchings are contained in table 7. As mentioned before, in is sufficient to determine the north-going probabilities. From theorem 1.13 we know that we have to exclude from the matchings of the dual Aztec diamond that have a north-going edge at the given location those of types F or G that have an edge there. To this, we have to add the matchings of one of the Möbius Aztec diamonds mentioned in lemma 1.24 that have an edge at the current location. These matchings could be obtained by extending one of the dual Aztec diamonds of order $n - 1$ and of center $(0, 1), (0, -1), (1, 0)$ or $(-1, 0)$. We also have to pay special attention to locations $(\pm(n - 1), 0)$ and $(0, \pm(n - 1))$, and to keep in mind that for type B the north-going edges at $(0, n - 1)$ disappear.

The numbers that we have to subtract are the same as for the Klein Aztec diamond and they are presented in the table from theorem 1.23.

<div align="right">q.e.d.</div>

1.8   Torus Aztec Diamonds

This category of Aztec diamonds can be of course related to the Cylindrical Aztec diamonds, but it turns out that, unlike the latter, they have special matchings, as proved in the following:

LEMMA 1.27 *There is a bijection between the special matchings of the Torus Aztec diamond and the perfect matchings of the dual Aztec diamond of types F and G.*

Proof:

Any special matching of the Torus Aztec diamond must have at least two adjacent corners matched in the special way (like in figure 16 a).

If only two corners are matched like this, the matching could be extended to a matching of one of the Cylindrical Aztec diamonds obtained by cutting apart the pair of pasted edges. Since the Cylindrical Aztec diamond has no special matchings, this situation cannot occur.

The remaining situation, with all corners matched in the special way, produces one of the two types of matchings from figure 22. After completing the matching to all positions that have just one neighbor left available we can see that the regions left to be matched are dual Aztec diamonds of order $n - 2$, like in the case of the matchings of types F and G for the dual Aztec diamond of order n. Thus we can define a bijection from one set to another, and, as no other particular relation seems to exist between these matchings, we make a choice: for every matching of type F we associate the matching from figure 22 a which has the same matching of the dual Aztec diamond of order $n - 2$, and similarly for type G with figure 22 b.

This result, combined with theorem 1.13 leads directly to the following:

THEOREM 1.28 *The number of perfect matchings of the Torus Aztec diamond of order n is*

$$TAD(n) = 2^{\frac{n(n+1)}{2}}$$

□

From what we have proved so far we expect that the placement probabilities do not change too much. This is actually the case.

THEOREM 1.29 *The north-going placement probabilities for the Torus Aztec diamond are:*

$$
tn_{i,j}^n = \begin{cases}
p_{i,j}^n & (i,j) \neq (0, n-1) \\
\\
p_{i,j}^n - 2^{-n} & (i,j) = (0, n-1)
\end{cases}
$$

Proof:

Due to the bijection we have established in theorem 1.13 all placement probabilities remain the same at locations situated in the region of the dual Aztec diamond of order $n - 2$. The same is true for the locations of negative y-coordinate situated outside this region, because north-going edges do not occur at these locations in matchings of types F or G or in the corresponding matchings of the Torus Aztec diamond.

All matchings of type G and no matching of type F have north-going edges at locations $(i,j) \neq (0, n-1), j \geq 0$ outside the dual Aztec diamond of order $n - 2$, therefore there are $AD(n-2)$ matchings with a north-going edge at such a location. On the other hand, at the same locations, a north-going edge can be found in all special matchings of type a if $i < 0$ and for all special matchings of type b if $i > 0$. The number of these matchings is in each case $AD(n-2)$. Thus the placement probabilities at these locations change by $-AD(n) + AD(n) = 0$.

For the special location however, we have a modification due to the fact that all matchings of types B and G have a north-going edge there, which is deleted when

we apply the bijection from theorem 1.13 and none of the other special matchings contain a north-going edge there.

Thus the total decreases by $AD(n-1) - AD(n-2) + AD(n-2) = AD(n-1)$.

<div align="right">**q.e.d.**</div>

Finally, as a consequence of theorem 1.13 and lemma 1.27 we notice that with a small modification, described in the lemma, we can apply the **shuffling algorithm** to generate **random matchings of the Torus Aztec diamond.**

1.9 Conclusions

From what we could see, despite the differences for some categories of matchings, these graphs are very similar with respect to the structure and the statistics of their matchings. The difference between their placement probabilities and the probabilities of our initial graph tend to zero as n goes to ∞. Thus we have a common :

COROLLARY 1.30 *(Arctangent Formula for the Modified Aztec Diamonds)*

Let U be an open set containing the points $(\pm\frac{1}{2}, \frac{1}{2})$. If (x,y) is the normalized location of a north-going edge place in the Aztec diamond of order n, and $(x,y) \notin U$, then as $n \to \infty$, the placement probability at (x,y) is within $o(1)$ of $\mathcal{P}(x,y)$, where

$$\mathcal{P}(x,y) = \begin{cases} 0 & \text{if } x^2 + y^2 \geq 1/2 \text{ and } y < 1/2 \\ \\ 1 & \text{if } x^2 + y^2 \geq 1/2 \text{ and } y > 1/2 \\ \\ \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{2y-1}{1-2x^2-2y^2}\right) & \text{if } x^2 + y^2 < 1/2 \end{cases}$$

The o(1) error bound is uniform in (x, y).

Proof:

From theorems 1.17, 1.20, 1.23, 1.26 and 1.29 it follows that the difference between the north-going probabilities of the dual Aztec diamond and those of the modified graphs is bound, in each case, by a sequence that not only tends to zero as n goes to infinity, but in addition does not depend on the location:

$$\left| p_{x,y}^n - cn_{x,y}^n \right| \leq \frac{2^{2-2n}}{1 - 2^{1-2n}}$$

$$\left| p_{x,y}^n - mn_{x,y}^n \right| \leq \frac{2^{2-n}}{1 + 2^{1-n} - 2^{1-2n}}$$

$$\left| p_{x,y}^n - kn_{x,y}^n \right| \leq \frac{2^{2-n}}{1 + 2^{1-n} - 2^{2-2n}}$$

$$\left| p_{x,y}^n - ppn_{x,y}^n \right| \leq \frac{2^{3-n}}{1 + 2^{2-n} - 2^{2-2n}}$$

$$\left| p_{x,y}^n - tn_{x,y}^n \right| \leq \frac{1}{2^n}$$

in the case of the Torus the difference being in fact zero at all locations but one.

This ensures not only the equalities:

$$\lim_{n \to +\infty} p_{x,y}^n =$$

$$\lim_{n \to +\infty} cn_{x,y}^n = \lim_{n \to +\infty} mn_{x,y}^n = \lim_{n \to +\infty} kn_{x,y}^n = \lim_{n \to +\infty} ppn_{x,y}^n = \lim_{n \to +\infty} tn_{x,y}^n$$

but also the uniform bound of the error. By theorem 1.5 we obtain the result.

Similarly

$$\lim_{n \to +\infty} pw_{x,y}^n =$$

$$\lim_{n \to +\infty} p_{y,-x}^n = \lim_{n \to +\infty} cw_{x,y}^n = \lim_{n \to +\infty} mw_{x,y}^n = \lim_{n \to +\infty} kw_{x,y}^n.$$

Thus we need not worry about the west-going probabilities of the modified graphs because they have the same limit as the west-going probabilities of the dual Aztec diamond, so they can be easily obtained from the north-going ones.

<div align="right">q.e.d.</div>

The resemblance between the Aztec diamond and its modifications does not stop here. In fact we can prove a result similar to theorem 1.4:

COROLLARY 1.31 *(Arctic Circle Theorem for the Modified Aztec Diamonds)*
For any $\epsilon > 0$, for sufficiently large n, the probability that the boundary of the temperate zone stays within distance ϵn of the arctic circle differs from 1 by an amount exponentially small in n.

Proof:

This result is based on the knowledge of the structure of matchings for modified Aztec diamonds that we aquired in the previous section. Let us fix one (any of the) modified Aztec diamond. First, when we choose a random matching of the modified Aztec diamond, the probability that this matching is extendible approaches 1 as $n \to \infty$. Second, when we choose a random matching of the dual Aztec diamond, the probability that it is the extension of a matching of the modified Aztec diamond also approaches 1 as $n \to \infty$. Thus our theorem is a direct consequence of 1.4.

<div align="right">q.e.d.</div>

CHAPTER II

MATCHINGS AND COVERING GRAPHS

2.1 Perfect Matchings, 2-Matchings and n-Matchings

In this chapter we study some special types of f-matchings: perfect n-matchings and perfect non-ramified n-matchings, the relations between the matchings, 2-matchings and n-matchings, $n \geq 3$, of the same graph G, as well as their relation with the matchings of some covering graphs of G. From that we deduce some information about the number of non-ramified perfect n-matchings for G.

- **an n-matching** *is an f-matching such that $f(x) = n, \forall x \in V(G)$;*

- *a* **non-ramified** *f-matching is an f-matching such that at each vertex there are at most two edges with non-zero weights;*

- *the* **sum** *of an f_1-matching with an f_2-matching is a $(f_1 + f_2)$-matching such that*

  $w(e) = w_1(e) + w_2(e)$ for each edge of the graph;

- *the* **support** *of an f-matching is the set of edges of non-zero weight;*

- *we say that an f-matching is* **trivial** *if its support is a matching;*

- *we say that an f-matching is* **proper** *if it is not trivial.*

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In what follows we only study perfect n-matchings.

Let us note a few elementary facts:

- *the support of a non-ramified perfect n-matching is a set of independent edges and cycles;*

- *all 2-matchings are non-ramified;*

- *if a graph has perfect n-matchings for some n then it has, perfect nk-matchings for all $k \geq 1$, obtained trivially by adding the n-matching to itself k times; in particular, if a graph has perfect matchings then it has (trivial) n-matchings for all $n \geq 1$.*

Figure 23 presents a few examples of perfect n-matchings, all but the last one being non-ramified (to be included).

We have mentioned some sufficient conditions for the existence of n-matchings in a graph. For a given graph, the existence of n-matchings is not guaranteed.

Example 2.1

Let us consider the graph represented in figure 24, with vertices $\{u, v, x, y, z\}$ and edges $\{u, x\}, \{x, v\}, \{u, y\}, \{y, v\}, \{u, z\}, \{z, v\}$.

This graph has no n-matchings, for any n. To see that this is true, let us notice that the pairs of edges adjacent to x, respectively y, z, have to be given weights of the type $\{k, n - k\}$, $\{l, n - l\}$, $\{m, n - m\}$ for some $1 \leq k, l, m \leq n - 1$. The weights add up to $k + l + m$ at u and $3n - (k + l + m)$ at v. Obviously, these two numbers cannot be both equal to n. $\qquad \square$

Some necessary and sufficient conditions for the existence of perfect f-matchings are known (see [LP86] pg 72, for example) for bipartite graphs. For general graphs, these conditions have been deduced via f-factors ([LP86], chapter 10).

We are interested in the possible decompositions of a given n-matching into sums of matchings and 2-matchings. From theorem 6.4, [HS93], we can deduce the following:

LEMMA 2.2 *All perfect n-matchings, for n even, can be decomposed into a sum of perfect 2-matchings.*

Proof:

Suppose we have a perfect n-matching for some even $n > 2$. We use the idea from [LP86] page 383, which relates f-matchings and f-factors. Replace each edge of weight k with k parallel edges. We have constructed an n-regular multigraph. Theorem 6.4, [HS93], assures us that all the connected components of this new graph have 2-factors. Take the union of these 2-factors and transform them back into 2-matchings the obvious way. Subtract then from the initial n-matching the perfect 2-matching that we have obtained, we are left with a perfect $(n-2)$-matching and we can apply an inductive argument.

<div align="right">q.e.d.</div>

Let us note that:

- *a perfect 2-matching can be written as the sum of two perfect matchings iff its support has only even length cycles.*

Thus, in certain circumstances, for example when the graph has only even cycles, we can deduce that any perfect n-matching, for n even, is the sum of n perfect matchings.

Another consequence of the lemma is:

COROLLARY 2.3 *A graph has a perfect n-matching, for n even, iff it has a perfect 2-matching.*

$\square$

The phenomenon does not repeat for odd values of n. The two graphs below constitute counterexamples to what could be a decomposition theorem for $n = 3$.

Example 2.4

In figure 25 we have a (smallest) 3-regular graph which has no perfect matching. Assign the weight 1 to each edge and we have a perfect 3-matching which is neither the sum of 3 perfect matchings nor the sum of a perfect matching with a perfect 2-matching.

$\square$

Example 2.5

The graph from the previous example has perfect 2-matchings, but it has no perfect matchings. Would the existence of both perfect matchings and 2-matchings be enough to obtain a decomposition? The answer is negative, as it can be seen from the following:

As we can see, the graph in figure 26 has a perfect 3-matching, it has perfect matchings and 2-matchings, but their sum can not be the 3-matching we started with.

□

2.2 Structure and Decomposition of Non-Ramified Perfect n-Matchings

We have already noticed that the support of a non-ramified perfect n-matching is a set of independent edges and cycles. This fact leads to a decomposition theorem, that does not apply, as we have seen before, for the general n-matching.

Let us detail the structure of a non-ramified perfect n-matching first (see figure 27 for an illustration):

- *all independent edges in the support have weight n;*

- *the edges along the cycles have weights m and $n - m$, alternating along the cycle, for some $1 \leq m \leq n - 1$;*

- *if the cycle has even length, m can take any of the $n - 1$ values mentioned above;*

- *an odd length cycle can only belong to a $n = 2k$ matching, and in this situation the unique possibility is of having all edges of weight k.*

A consequence of these observations is:

LEMMA 2.6 *Let C be a cycle of length l with a perfect proper n-matching.*

i) If l is even then the n-matching is the sum of m perfect matchings having as support

the edges of weight m, and n − m perfect matchings having as support to the edges of weight n − m.

ii) If l is odd and n = 2k then the n-matching is the sum of k perfect 2-matchings which have all as support the cycle itself.

From this we can deduce our decomposition theorem.

THEOREM 2.7 *For k ≥ 1 the following are true:*

i) Any non-ramified perfect 2k-matching is the sum of k perfect 2-matchings.

ii) Any non-ramified perfect (2k + 1)-matching is the sum of 2k + 1 perfect matchings.

Proof:

i) This part could be deduced from 2.2, but the previous lemma gives us a constructive argument: we can decompose the odd length cycles into 2-matchings, the even length cycles into matchings an then group them to obtain the perfect 2-matching we were looking for. The independent edges pose no problems.

ii) Follows from the lemma, using the fact that we have no odd- length cycles in this case.

$$q.e.d.$$

COROLLARY 2.8 *If G has no odd cycles, any non-ramified perfect n-matching is the sum of n perfect matchings of G.*

☐

We notice that there are many ways in which these decompositions could be done. We will attempt to enumerate them later on.

Now let us mention the necessary and sufficient condition for the existence of non-ramified perfect n-matchings, which results from the theorem:

COROLLARY 2.9 *For a given $k \geq 1$*

i) G has a non-ramified perfect $2k$-matching iff it has a perfect 2-matching.

ii) G has a non-ramified perfect $(2k + 1)$-matching iff it has a perfect matching.

□

Let us introduce the following notations, for $n \geq 2$:

- $N_n(G)$ *is the number of non-ramified perfect n-matchings for G;*

- $N_n^c(G)$ *is the number of non-ramified perfect n-matchings of G with c cycles in the support;*

- $N_n^{c,c'}(G)$ *is the number of non-ramified perfect n-matchings of G with c cycles in the support, exactly c' of them being of odd length.*

The following result shows that it is enough to know the structure of the set of perfect 2-matchings of a graph G to be able to determine completely all non-ramified perfect n-matchings.

LEMMA 2.10 *Let G be a graph and $n \geq 3$.*

i) The numbers of non-ramified perfect n-matchings are described by

$$N_n^{c,c'}(G) = (n-1)^{c-c'} N_2^{c,c'}(G) \quad \text{if} \quad n = 2k$$

$$N_n^c(G) = (n-1)^c N_2^{c,0}(G) \qquad \text{if} \quad n = 2k+1.$$

ii) If the graph has no odd cycles,

$$N_n(G) = \sum_{c \geq 0} (n-1)^c N_2^c(G).$$

Proof:

These equalities are obtained by counting how many n-matchings have the same support as a given 2-matching. They are based on the observation from the beginning of the section, that there are $n-1$ ways of assigning weights to an even length cycle, there is only one way of assigning weights for an odd length cycle when n is even and no way of doing it for an odd length cycle when n is odd.

<div align="right">q.e.d.</div>

2.3   Covering Graphs and Perfect n-Matchings

DEFINITION 2.11 *A graph $\bar{G}$ is a (branched) covering graph of G if it is a (branched) covering space such that the (potential) branch points are only among the vertices of G.*

<div align="right">□</div>

We begin this section with an observation, which is the source of all results presented further on.

LEMMA 2.12 *Suppose $\bar{G}$ is a (branched) covering graph of G. If M is a perfect matching of $\bar{G}$, then, for $f(x) = |p^{-1}(x)|, \forall x \in V(G)$, there exists a perfect f-matching*

of G with the weights of the edges defined by $w(e) = |E(M) \cap p^{-1}(e)|$. The support of this f-matching is $p(M)$.

□

DEFINITION 2.13 *If M is a perfect matching of a (branched) covering graph of G then the f-matching with $w(e) = |E(M) \cap p^{-1}(e)|$ is the* **projection** *of the matching M; we say that the same f-matching* **lifts** *to the matching M.*

□

2.3.1 Branched Covering Graphs

A construction similar to the following graphs can be found in [LP86], chapter 10, where theorem 10.1.1 mentions the equivalence between the existence of a perfect matching of such graphs and the existence of a perfect f-matching for the initial one. Here, for the case of non-ramified perfect n-matchings, we prove more: we find the number of matchings corresponding to a given non-ramified perfect n-matching.

DEFINITION 2.14 *Given a graph G, $\hat{G}^n$ are the graphs with:*

$$\hat{V}^n = V(G) \times \{1, \ldots, n\}$$

$$\hat{E}^n = \{ \{(x,i),(y,j)\} \ : \ \forall \{x,y\} \in E(G) \text{ and } i,j = 1 \ldots n\}$$

□

Note that all these graphs are branched covering graphs with $|p^{-1}(x)| = n$ for all the vertices $x \in V(G)$, and that the projection of a perfect matching in such a graph determines a perfect n-matching of G. In fact, the fiber above an edge is the bipartite complete graph $K_{n,n}$.

Given a non-ramified perfect n-matching of G, we may ask in how many ways does it lift to a perfect matching of $\hat{G}^n$. The next lemma gives us the answer:

LEMMA 2.15 *Let $\mathcal{E}$ be the set of independent edges in a non-ramified perfect n-matching of G and let $C_{l,m}$ be the set of cycles of length l in which there exists an edge of weight m. The number of perfect matchings of $\hat{G}^n$ this n-matching lifts to is*

$$\left(\prod_{l \text{ even}} \prod_{1 \leq m \leq \lfloor n/2 \rfloor} \left(\frac{n!}{m!} \frac{n!}{(n-m)!} \right)^{\frac{l}{2}|C_{l,m}|} \right) \left(\prod_{l \text{ odd}} \left(\frac{n!}{k!} \right)^{l \cdot |C_{l,k}|} \right)^{*} (n!)^{|\mathcal{E}|}$$

*where the * term is to be considered only for $n = 2k$.*

Proof:

The fiber above one edge is formed of n^2 edges, each vertex being incident to n of them.

On the fiber over an independent edge, there are $n!$ perfect matchings, which explains the term involving $\mathcal{E}$.

Suppose that we want to lift a cycle with weights m and $n - m$. Start at a vertex u in the cycle and let us choose m edges over the edge $\{u, v\}$ of weight m incident to our vertex. There are $\binom{n}{m}$ ways of choosing the m vertices we start with (in the fiber over u) and $\frac{n!}{m!}$ ways of choosing the m vertices we match them with (in the fiber over v), then we have $n - m$ edges left in this fiber and we can choose the vertices we match them to in $\frac{n!}{(n-m)!}$ ways, and so on, we alternate these values along the cycle till we reach the fiber over the last vertex before u. Here we have only the choice which of the vertices that are still unmatched to match together. This can be done $(n - m)!$ ways. Taking the product of these, we obtain the rest of the formula, including, for

$n = 2k$, the special case $m = k$.

<div align="right">**q.e.d.**</div>

These numbers depend highly on the weights assigned to the edges. We will see that when $n = 2$, this result takes a much nicer form.

2.3.2 Permutation Derived Graphs

We have seen in the previous section that there are many possibilities of lifting a non-ramified perfect n-matching of a graph to perfect matchings of the branched covering graphs $\hat{G}^n$. This seems natural, since the fiber above each edge has size n^2. If we want the number of perfect matchings of a covering graph to get closer to the number of non-ramified perfect n-matchings of the initial graph, it makes sense to reduce the fibers above the edges to n. This is the minimum we can take to give to each edge the possibility of participating to a n-matching as an independent edge (of weight n). We create these covering graphs by choosing n independent edges from each of the fibers of $\hat{G}^n$ over the edges of G. For any such graph, there might exist some n-matchings which lift to this covering graph only. Therefore we study all the possibilities.

The construction of such coverings can be done via voltage graphs.

DEFINITION 2.16 *Let G be a graph whose edges have been assigned plus and minus directions. A* **permutation voltage assignment** *for G is a function α from the plus-directed edges into the symmetric group S_n. The pair $< G, \alpha >_n$ is called a* **permutation voltage graph.** $\qquad\square$

DEFINITION 2.17 *To a given permutation voltage graph $< G, \alpha >_n$ we associate a* **permutation derived graph**, *G^α, which is a graph such that*

$$V(G^\alpha) = V(G) \times \{1, \ldots, n\}$$

$$E(G^\alpha) = \{ \{(x,i),(y,\pi(i))\} \ : \ i \in \{1 \ldots n\}, \alpha(x,y) = \pi \}$$

where $(x,y) = e^+$ for the edge $e = \{x,y\}$.

□

Figure 28 presents a few examples of S_3-voltage graphs and their associated permutation derived graphs for a cycle of length 4.

First we need to show that all possible covering spaces of G can be obtained through the method mentioned above. The answer to this question is given by theorem 2.4.5 from [GT]:

THEOREM 2.18 *(Gross and Tucker, 1977)*

Let the graph map $q : \tilde{G} \to G$ be a covering projection. Then there is an assignment α of permutation voltages to the base graph such that the derived graph G^α is isomorphic to $\tilde{G}$.

□

Automorphisms (homeomorphisms) of permutation derived graphs can be obtained by permuting the vertices from each of the fibers amongst themselves. The next question is how many of these are in fact homeomorphic covering spaces. We will not address this question here, we just mention a version of theorem 2.5.4 from [GT], to be used in section 2.4.2.:

THEOREM 2.19 *Let G be a graph, let α be a permutation voltage assignment for G, and let T be a spanning tree for G. There exists a voltage assignment β such that all the edges in T have voltage $\mathbf{e}$ and the derived graph G^β is isomorphic to G^α.*

□

All permutation derived graphs with permutations from S_n are subgraphs of $\hat{G}^n$. Their perfect matchings also project to perfect n-matchings of G. We are going to determine a necessary and sufficient condition for a non-ramified perfect n-matching of G to lift to a given permutation derived graph of G. For this, we define the voltage of a walk in a voltage graph:

- *the* **voltage** *of a* **minus-oriented edge** *is the group inverse of the voltage of the corresponding plus-oriented edge;*

- *the* **total voltage** *of a (oriented)* **walk** *is equal to the product of the voltages encountered in a traversal of that walk.*

Given a permutation π,

- c_i *represents the number of cycles of length i in the cycle decomposition of π;*

- $(c_1, \ldots, c_n)$ *is the* **cycle structure** *of π.*

Let us note that:

- *reversing the direction of the walk replaces the total voltage of the walk with its group inverse;*

- *the cycle structure of the voltage of a given walk does not depend on the direction of traversal;*

- *the cycle structure of the voltage of a cycle with a chosen routing does not depend on the point where we start.*

Keeping these facts in mind, we mention another result from [GT] (theorem 2.4.3).

THEOREM 2.20 *Let C be a cycle of length k in the base graph G of a permutation voltage graph $< G, \alpha >_n$ with net voltage π, and let be the cycle structure of π. Then the preimage of C in the derived graph G^α has $c_1 + \ldots + c_n$ components, including, for $j = 1 \ldots n$, exactly c_j cycles of length kj.*

$\square$

Now we can present our result about liftings of non-ramified n-matchings. Obviously, there is only one way of lifting an independent edge of weight n: we take all the edges above it in the matching. Thus it is enough to determine how many ways does a cycle lift.

THEOREM 2.21 *Let C be a cycle of length l with a proper n-matching N, let α be a voltage assignment for C, $p : C^\alpha \to C$ be the covering projection and π be the voltage of C of cycle structure $(c_1, \ldots, c_n)$.*

i) If l is even and the edges along C have weights m and $n - m$ $(m \geq 1)$, the N lifts to a perfect matching of $p^{-1}(C)$ iff $p^{-1}(C)$ is the union of an m-covering graph S_1 with an $n - m$-covering graph S_2.

The number of liftings of N is equal to the number of ways we can form S_1 and S_2 if $m \neq n - m$ and it is twice this number if $m = n - m$.

ii) If l is odd and $n = 2k$, then N lifts to a perfect matching of $p^{-1}(C)$ iff

$c_i = 0$, $\forall i$ odd (i.e. iff all components of the preimage have even length).

In this case the number of liftings of N is $2^{c_1 + \ldots + c_n}$.

Proof:

i) Let M_1, (M_2) denote the perfect matching of C formed of the edges of weight m $(n - m)$. Suppose that N lifts to a perfect matching, M. For each component of the preimage: cycle K of even length li, with the size of the fiber equal to i, $M|_K$ is a perfect trivial i-matching of the cycle, of support either M_1 or M_2. Let S_1, respectively S_2, be the union of those components that project to M_1, respectively M_2. According to definition 2.13, $w(e) = |E(M) \cap p^{-1}(e)|$, so $m = |E(M) \cap E(S_1) \cap p^{-1}(e)|$ and $n - m = |E(M) \cap E(S_2) \cap p^{-1}(e)|$. This gives us the required size of the fibers.

Suppose now that S_1 and S_2 which satisfy the conditions of the hypothesis exist. Proceed as follows: split N into a perfect m-matching N_1 and a perfect $(n - m)$-matching N_2 (both formed of independent edges), lift N_1 to S_1 and N_2 to S_2 (we have already noticed that there is no problem in lifting independent edges) and the problem is solved.

If $m \neq n - m$, for each choice S_1 and S_2 we can only lift N_1 to S_1 and N_2 to S_2, so we have a one-to-one correspondence. If $m = n - m$ then we can also lift N_1 to S_2 and N_2 to S_1, thus the correspondence is two-to-one.

ii) N is a perfect $2k$-matching and l is odd.

Suppose that N lifts to a perfect matching M. M is a union of perfect matchings for the components of $p^{-1}(C)$. The length of a component is, as we know from 2.20, equal to il, for each of the c_i components corresponding to the c_i cycles of length i in

π. Since l is odd, components with i odd have no matchings. Thus, $c_i = 0$ for all i odd.

Conversely, if $p^{-1}(C)$ has no cycles of odd length, then it has at least a perfect matching. Let us choose one such matching, M. Given of a component K with fiber size i, the projection of $M|_K$ is a $i/2$-proper matching. Adding these weights over all components of $p^{-1}(C)$ gives us the $n/2 = k$ value we were looking for.

In the argument above, we made a choice M. It is clear that any such matching would do. Each cycle in $p^{-1}(C)$ has exactly two perfect matching and we have $c_1 + \ldots + c_n$ cycles, which gives a total of $2^{c_1 + \cdots + c_n}$ liftings.

<div align="right">q.e.d.</div>

From the theorem above we can see that odd length cycles and even length cycles have a different behaviour with respect to their liftings. In particular, for the even length cycles, not only the number of components of its preimage is involved, but also the size of the fiber for each component. Thus, in the general case, there is no uniform way of describing the liftings of a non-ramified perfect n-matching. We will see later that for $n = 2$ the situation becomes more favorable.

2.4 Perfect 2-Matchings

We now apply the results of the previous section to the case $n = 2$. Our purpose is to study the enumeration of perfect 2-matchings of a graph.

In what follows, $\mathcal{CG}_2(G)$ represents the number of 2-covering graphs of a given graph G.

2.4.1 Matchings of $\hat{G}^2$

LEMMA 2.22 *A perfect 2-matching of a graph G with exactly d edges of weight 2 lifts to 2^{v-d} perfect matchings of $\hat{G}^2$, where $v = |V(G)|$.*

Proof:

We apply lemma 2.15 for the case $n = 2$. We recall the general formula:

$$\left(\prod_{l \text{ even}} \prod_{1 \le m \le \lfloor n/2 \rfloor} \left(\frac{n!}{m!} \frac{n!}{(n-m)!} \right)^{\frac{l}{2} |C_{l,m}|} \right) \left(\prod_{l \text{ odd}} \left(\frac{n!}{k!} \right)^{l \cdot |C_{l,k}|} \right) (n!)^{|\mathcal{E}|}$$

Here we can only have $m = 1$. Let $C_l = |C_{l,1}|$. Thus the formula above becomes:

$$\left(\prod_{l \text{ even}} 4^{\frac{l}{2}|C_l|} \right) \left(\prod_{l \text{ odd}} 2^{l \cdot |C_l|} \right) 2^{|\mathcal{E}|} = \left(\prod_{l} 2^{l \cdot |C_l|} \right) 2^{|\mathcal{E}|} = 2^{(\sum_l l \cdot |C_l|) + |\mathcal{E}|}$$

Since $\sum_l l \cdot |C_l| + 2|\mathcal{E}| = v$ and $|\mathcal{E}| = d$ the result follows.

q.e.d.

Thus we can express the relation between the number of perfect matchings of $\hat{G}^2$ and the number of perfect 2-matchings of the base graph G, as follows:

THEOREM 2.23 *The number of perfect matchings of $\hat{G}^2$ is*

$$\Phi(\hat{G}^2) = \sum_d n_2(G; d) 2^{v-d}$$

where $n_2(G; d)$ represents the number of perfect 2-matchings of G with d edges of weight 2.

$\square$

- the **weight** *of a perfect matching is 2^d if the projection has d edges of weight 2.*

COROLLARY 2.24 *The weighted number of perfect matchings of $\hat{G}^n$ is*

$$2^v N_2(G)$$

$\square$

2.4.2 A Mean Value Theorem for Perfect Matchings of 2-Covering Graphs

In this section we show that the average number of perfect matchings of 2-covering graphs for a given graph G is the number of perfect 2-matchings of G.

As an introduction, we present in figures 29, 30, 31 and 32, tables labeled, on the columns, by 2-covering graphs of the initial graphs, on the rows, by their 2-matchings, and at each entry in the table we have the number of liftings of the respective 2-matching to the 2-covering graph. We also include the totals by rows and by columns. We observe that all rows sum up to the same value, which is equal to the number of columns. Thus the total is the product between the number of lines and columns, i.e. the number of 2-matchings and the number of 2-covering graphs of our graph. Dividing the equality by the number of 2-covering graphs produces the result we mentioned in the beginning of the section.

The previous examples suggest the following approach for the proof of the main theorem: after computing $|\mathcal{CG}_2(G)|$ (the number of 2-covering graphs of a given graph G), show that the total number of perfect matchings of 2-covering graphs that a perfect 2-matching can lift to is equal to $|\mathcal{CG}_2(G)|$, then conclude the way we did above.

We will limit ourselves to the study of connected graphs without loss of generality. We can do so because both f-matchings and covering graphs for a given graph are a union of f-matchings, respectively covering graphs for its components.

We begin with counting the 2-covering graphs of a given (connected) graph. Let

e represent the group identity of S_n. With the convention that $|V(G)| = v$ and $|E(G)| = e$, we have the formula:

LEMMA 2.25 *The number of 2-covering graphs of a given graph G is*

$$|\mathcal{CG}_2(G)| = 2^{e-v+1}.$$

Proof:

By theorem 2.18 we know that we can think of 2-covering graphs as permutation derived graphs with permutations from S_2. The voltages of the edges are be either e or (12). We must mention that there is no need for a plus-minus orientation in this case, since all permutations of S_2 are there own inverses.

Let T be some spanning tree for G.

If $G = T$, then $e - v + 1 = 0$ and indeed there exists a unique 2-covering graph for G.

Suppose G has at least one cycle. Theorem 2.19 tells us that there are at most 2^{e-v+1} different derived graphs (up to isomorphism), since once we have fixed the voltages for all the edges of T to be e, there are only $|E(G)| - |E(T)| = e - (v - 1)$ edges left and for each we have two choices.

To complete the proof we must show that all 2^{e-v+1} choices mentioned above lead to distinct graphs: We begin by choosing such a 2-covering graph, G^α given by the set of voltages α. Let us remember that isomorphisms of permutation derived graphs are in fact permutations of the vertices within the fibers. We apply an isomorphism and we produce a new permutation derived graph, G^β, which corresponds to another choice of voltages, β. Suppose we have an edge $a = \{u, w\}$ in T such that $\beta(a) = (12)$.

This means that we have permuted one of the two sets of vertices from the fibers over the endpoints. Suppose that we have permuted u_1 with u_2. We want to keep the voltages along T equal to e. The only way to have these voltages, (except, of course, the obvious, which is to permute back u_1 and u_2), is to permute the other endpoints, w_1 and w_2, from the fiber over w, and to keep on doing so, (updating the voltages at each step), as long as we meet edges of voltage (12) in T. In fact the procedure stops when we meet endpoints of T or vertices that have already been permuted by the initial isomorphism. Thus we end up permuting all the vertices that haven't been permuted by the initial isomorphism, clearly producing back the graph G^α, and our claim is proved.

<div align="right">q.e.d.</div>

Note that the power of 2 in the previous theorem is in fact the cyclomatic number of the graph and that to construct our permutation derived graphs we choose the voltages for a cycle base of G, built with the help of a spanning tree (see for example [Bol] pg 35-37).

COROLLARY 2.26 *A planar graph G with f finite faces has 2^f double covers.*

<div align="right">□</div>

*Recall that e is an **even** permutation and (12) is an **odd** permutation.*

We are going to show how theorem 2.21 transforms in this context.

LEMMA 2.27 *Let M be a perfect 2-matching of G and choose G' a 2-permutation derived graph. Then M lifts to a perfect matching of G' iff, for each cycle in the*

support of M, the length and the voltage of the cycle have the same parity.

The number of liftings of M to perfect matchings of G' is 2^c, where c is the number

of cycles in the support of M.

Proof:

Both even and odd length cycles have the weights of the edges equal to 1. From

theorem 2.21 we deduce that a 2-matching of an even length cycle lifts iff there are

exactly two components in the preimage of the cycle. There is, evidently, exactly one

way of getting two components, corresponding to a total voltage of the cycle equal to

e, and there are two possible liftings of the 2-matchings of this cycle. On the other

hand, for odd length cycles we need exactly one component in the preimage, thus a

total voltage of **(12)**. The number of liftings of the 2-matching is again 2. Since for

the independent edges there is just one possible lifting, we take the product over all

the cycles in the support and we obtain a total of 2^c liftings.

q.e.d.

In figure 33 we have a summary of the proof, described by the two possible liftings

of a proper 2-matching for a cycle of length 3 and for a cycle of length 4.

As a consequence we have the following result regarding the number of 2-covering

graphs given perfect 2-matching can lift to:

LEMMA 2.28 *Given a perfect 2-matching M with c cycles in the support, the num-*

ber of 2-covering graphs with perfect matchings which project to M is equal to $2^{e-v-c+1}$.

Proof:

We use again theorem 2.19. Choose all but one of the edges of each cycle in the support of M and complete this to a spanning tree. Choose the voltages of edges along it to be e. Thus we have fixed the voltages for $v - 1$ edges. In lemma 2.27 we have a necessary and sufficient condition for lifting of 2-matchings, expressed as a condition on the voltages of the cycles in the support. In order to have the right voltage for any of these cycles it is enough to set the voltage of the unique edge in the cycle that does not belong to the spanning tree so that is has the same parity as the length of the cycle. Thus we have fixed the voltages for other c edges. For the remaining ones we can use any of the two permutations available and we can argue like in lemma 2.25 that all choices generate different 2-covering graphs.

<div align="right">q.e.d.</div>

The previous results lead to the main theorem of this chapter:

THEOREM 2.29 *The number of perfect matchings of 2-covering spaces of a given graph G is*

$$\sum_{H \in C\mathcal{G}_2(G)} \Phi(H) = 2^{e-v+1} N_2(G).$$

Proof:

To show that this is true we just have to put together the results from lemma 2.27 and lemma 2.28. For each 2-matching, there are $2^{e-v-c+1}$ 2-covering graphs that it can lift to, and there are 2^c different liftings to each of these, which makes a total of 2^{e-v+1}. We have to make sure that they are all different, but two perfect matchings which project to different 2-matchings can not be identical.

<div align="right">q.e.d.</div>

- $\mu_n(G)$ *represents the average number of perfect matchings of n-covering graphs for a given graph G.*

A very interesting result, based on the previous theorem and on lemma 2.25, is that:

THEOREM 2.30 *The average number of perfect matchings of 2-covering graphs for a given graph G is the number of perfect 2-matchings of G.*

$$\mu_n(G) = N_2(G)$$

Proof:

We just need to point out that the factor which multiplies $N_2(G)$ in the previous theorem is in fact the number of 2-covering graphs of G.

q.e.d.

The last theorem implies, non-constructively, that any graph G has a 2-covering that has at most as many matchings as G has 2-matchings. Of course the theorem also implies G has a 2-covering that has at least as many matchings as G has 2-matchings, but for this we can give a construction, see below.

2.5   Conclusions and Comments

First, we want to mention a slightly different approach for the results in the last subsection, avoiding the use of theorem 2.19, and implicitly the use of spanning trees.

The alternate method consists of determining the size of the isomorphism classes of permutation derived graphs from lemmas 2.25 and 2.28 directly, using group-theoretic arguments. For the first one, for example, we would show that the number of permutation derived graphs is 2^e and that the size of their isomorphism classes is 2^{v-1}, thus obtaining the number of 2^{e-v+1} different 2-covering graphs.

We would like to be able to deduce more information about the number of perfect 2-matchings, from the perfect matchings of 2-covering graphs. Based on the results obtained so far, we draw some conclusions about the maximum and minimum number of perfect matchings among 2-covering graphs of a given graph G, whether or not the average can be realized for one such covering and some methods of computing these numbers.

For the maximum, we have a positive result:

LEMMA 2.31 *For any graph G there exists a 2-covering graph $\bar{G}Max$ such that all perfect 2-matchings of G lift to perfect matchings of $\bar{G}Max$. Thus $\bar{G}Max$ has the maximum number of perfect matchings among the 2-covering graphs.*

Proof:

The construction of $\bar{G}Max$ is based again voltage graphs and uses a spanning tree T with a fixed root. Proceed as follows: assign voltage e to all edges in T, then start adding the remaining edges one by one and assign them the voltage e or $(1,2)$ depending whether the (unique) cycle formed with the new edge and which contains the root has even or odd length. Thus we built a cycle base and the total voltage

for each of its cycles has the same parity with its length, because we get the identity on the even length cycles and the transposition for the odd length ones. From this, using the observation that the parity of the length /voltage of a cycle coincides with the parity of the number of odd length/voltage basic cycles which form it, we can conclude that all cycles in the graph have the correct voltage from the point of view of liftings (see lemma 2.27).

<div align="right">**q.e.d.**</div>

If the graph has only even length cycles the proof of the lemma tells us in fact an obvious thing: $\bar{G}Max$ is the (unique) 2-covering formed of two components, each of them isomorphic to the initial graph. This graph has $(\Phi(G))^2$ perfect matchings, which is, again, the obvious upper bound for perfect 2-matchings of G, since all perfect 2-matchings can be thought of as the sum of two perfect matchings in this case.

In fact, for the 2-covering graph formed of two copies of G the number of perfect matchings remains $(\Phi(G))^2$ even if there are odd length cycles but this number might not be a maximum anymore. In addition we have:

$$(\Phi(G))^2 = \sum_{c \geq 0} 2^c N_2^{c,0}(G)$$

This is the consequence of the fact that every 2-matching with c, cycles, all even, lifts 2^c ways to this covering graph (which is another way of saying that every 2-matching with c, all even, cycles can be identified with 2^c pairs of matchings). We have already met the terms on the right-hand of the equation above (lemma 2.10), and we know it represents the number of non-ramified perfect 3-matchings. Thus:

$$N_3(G) = (\Phi(G))^2.$$

The minimum is, obviously, at least $\Phi(G)$. From examples, we can see that this minimum can be $\Phi(G)$ (figures 29, 30, 31) or bigger (figure 32), that it can be attained at one or several covering graphs.

Given a graph G, if there exists a permutation voltage assignment such that, for all cycles in the supports of all 2-matchings, the parities of the voltage and of the length the are opposite, $\Phi(G)$ is the minimum. If not, some proper 2-matching can lift to the 2-covering graph, and the problem is to find the covering which accepts the minimum number of liftings. As we can see, in both situations we need to know the structure of the 2-matchings of G.

We conjecture that there is no algorithm for finding the 2-covering graph with a minimum of perfect matchings significantly better than the one that goes through all 2-covering graphs and finds their number of perfect matchings.

By theorem 2.30, the existence of a 2-covering graph with exactly as many perfect matchings as the average would reduce our counting problem to ordinary matching theory. To help us decide whether, in general, this construction is possible or not, we give yet another version of theorem 2.29:

COROLLARY 2.32 *The average number of perfect matchings of a 2-covering graph which project to proper 2-matchings is the number of proper perfect 2-matchings.*

Proof:

This can be deduced directly from

$$\sum_{H \in C\mathcal{G}_2} \Phi(H) = 2^{e-v+1} N_2(G).$$

It is sufficient to subtract from both sides $2^{e-v+1}\Phi(G)$, since all trivial 2-matchings lift to each of the covering graphs exactly one way.

<div align="right">**q.e.d.**</div>

Unfortunately, the following example shows us that the average is not easily realized:

Example 2.33

Let H_n be the subgraphs of the plane grid $\mathbf{Z} \times \mathbf{Z}$ induced by the set of vertices $\{1,2\} \times \{1,\ldots,n\}$.

The number of perfect matchings of these graphs is described by a Fibonacci sequence:

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, \ a_2 = 2.$$

The parity for the elements of this sequence is described by *odd, even, odd, ...* repeated to infinity.

The number of perfect 2-matchings is expressed by the sequence b_n which has as recurrence relation:

$$b_n = 2b_{n-1} + b_{n-2} - b_{n-3}, \quad b_1 = 1, \ b_2 = 3, \ b_3 = 6.$$

Here we have the following parity sequence: *odd, odd, even, even, odd, even, odd,... .*

Thus the difference of the two sequences, which represents the number of proper 2-matchings, equal to the average, has infinitely many odd terms.

Each proper 2-matching lifts at a given covering graph an even (power of 2) number of ways. Therefore the number of perfect matchings of a covering graph which project to these 2-matchings must be even. If the average is odd, it can not be realized.

□

Theorem 2.30 suggests some other methods of finding the number of perfect 2-matchings of a graph.

One such method is to compute the number of perfect matchings of all its 2-covering graphs.

A less costly but less precise solution would be to generate random 2-covering graphs, then to determine their number of perfect matchings and then to take the average. Random 2-covering spaces are not hard to obtain, with the use of a spanning tree: we simply construct permutation derived graphs the way described in the proof of lemma 2.25.

Both methods require a good way of determining the number of perfect matchings of a graph. Such methods exist for planar graphs, and they use a Pfaffian orientation of the graph (see [LP86], chapter 8). Unfortunately the covering graphs are far from being planar, and they are as far from being Pfaffian graphs, as shown in the following example:

Example 2.34

Let G be the subgraph of the plane grid induced by $\{1, 2, 3, 4\} \times \{1, 2, 3\}$. Assign

the identity voltage to all edges but $\{(2,1),(1,2)\}$ and $\{(3,1),(4,1)\}$.

We will show that the permutation derived graph with the voltages defined above (figure 34) is not Pfaffian.

Let us take the following cycles:

$$C_1 : (\ (1,1); (2,1); (3,1); (3,2); (2,2); (1,2); (1,1)\ ) \times \{1\}$$

$$C_2 : (\ (1,2); (2,2); (3,2); (4,2); (4,3); (3,3); (2,3); (1,3); (1,2)\ ) \times \{1\}$$

$$C_3 = C_1 + C_2$$

C_1, C_2, C_3 are all nice cycles in the permutation derived graph and the first two have exactly two edges in common. According to theorem 8.3.2. in [LP86], $\tilde{G}$ is a Pfaffian orientation of G iff all nice cycles are oddly oriented. In our case, for any routing of the first two cycles, with k_1, respectively k_2 edges in the direction of the routing, the third cycle has exactly $k_1 + k_2 - 2$ edges oriented the same way, and this number is even. Thus the covering graph we constructed is not Pfaffian.

$\square$

This negative result shows us how easily can covering graphs can fail to be Pfaffian.

Finally let us mention that a theorem similar to 2.29 does not hold for $n \geq 3$. For this, it is enough to study the n-matchings of a cycle of length 4. Illustrations of this case, for $n = 3$ and 4, are presented in figures 35 and 36.

Example 2.35

Let C be a cycle of length 4.

The number of its n matchings is $n + 1$ for all n.

It has $P(n)$ n-covering graphs, where $P(n)$ denotes the number of integer partitions of n. An asymptotic evaluation of the number $\log P(n)$ is $\pi\sqrt{2n/3} - \log(4n\sqrt{3})$ ([Tom], pg 65).

The n-covering graph formed by n copies of C has 2^n matchings.

$$\lim_{n\to\infty} \frac{\mu_n(C)}{N_n(C)} \geq \lim_{n\to\infty} \frac{\frac{2^n}{P(n)}}{n+1} = \infty$$

Thus not only the average is bigger than the number of n-matchings, but in the difference between them increases indefinitely as n goes to ∞.

$\square$

APPENDIX A
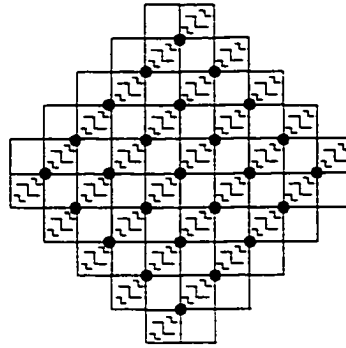
Figure 1 Standard checkerboard coloring and locations for the Aztec diamond of order 5

Figure 2 North, south, west and east-going dominos, good block and bad block
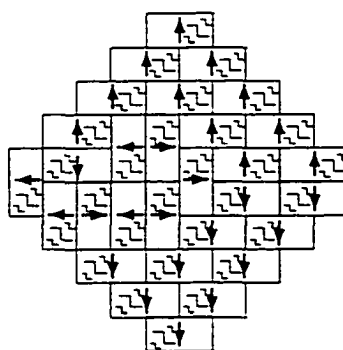
Figure 3 A tiling of the Aztec diamond of order 5 with the directions of the dominos
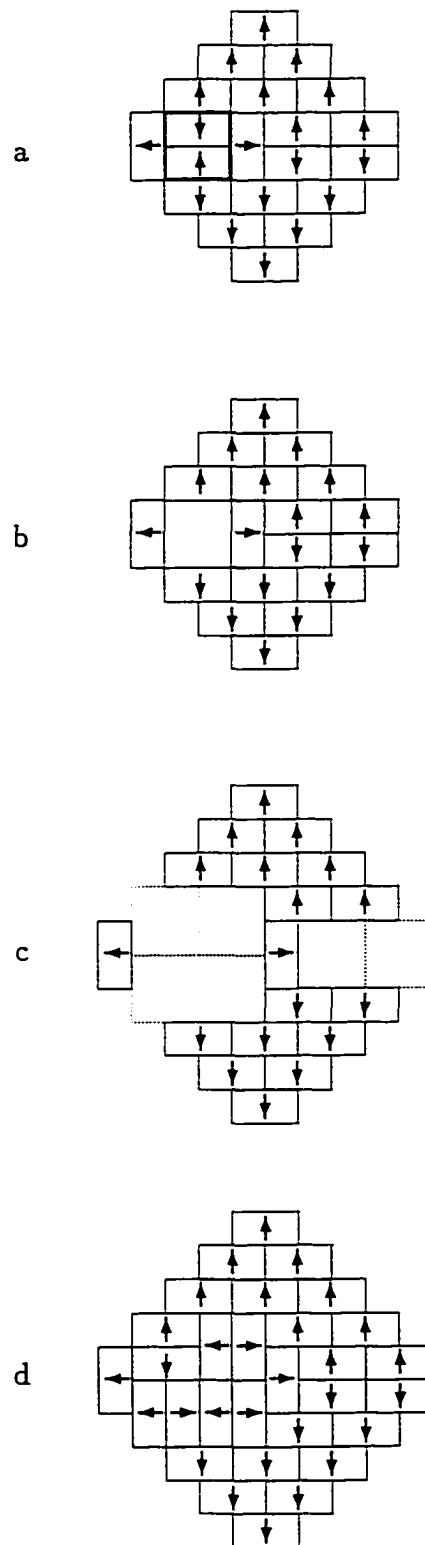
Figure 4 An illustration of the shuffling algorithm
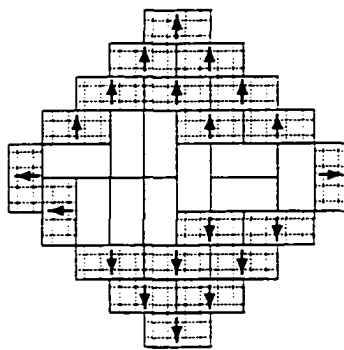
Figure 5 Polar region for a tiling of the Aztec diamond of order 5

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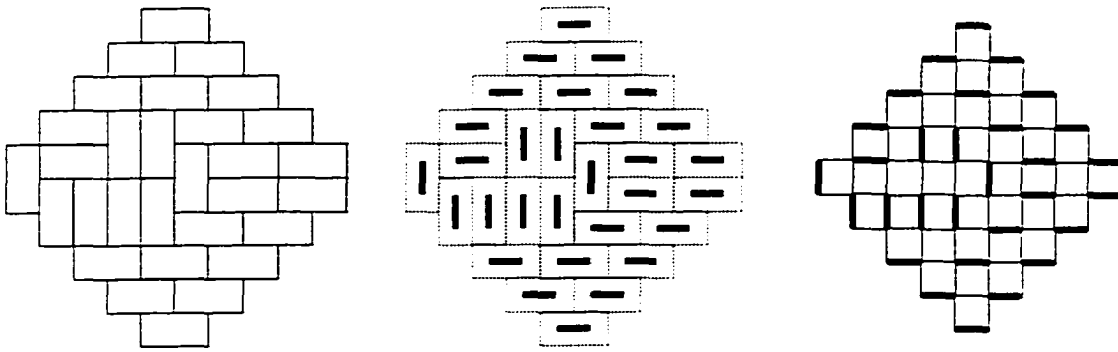
Figure 6 A tiling of the Aztec diamond of order 5 and the corresponding matching of the dual Aztec diamond, with an intermediate step
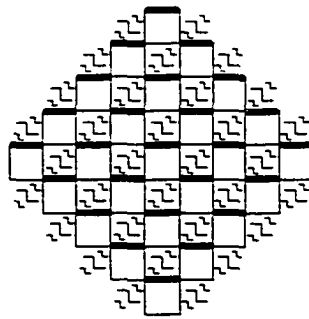
Figure 7 Standard checkerboard coloring and north-going edges
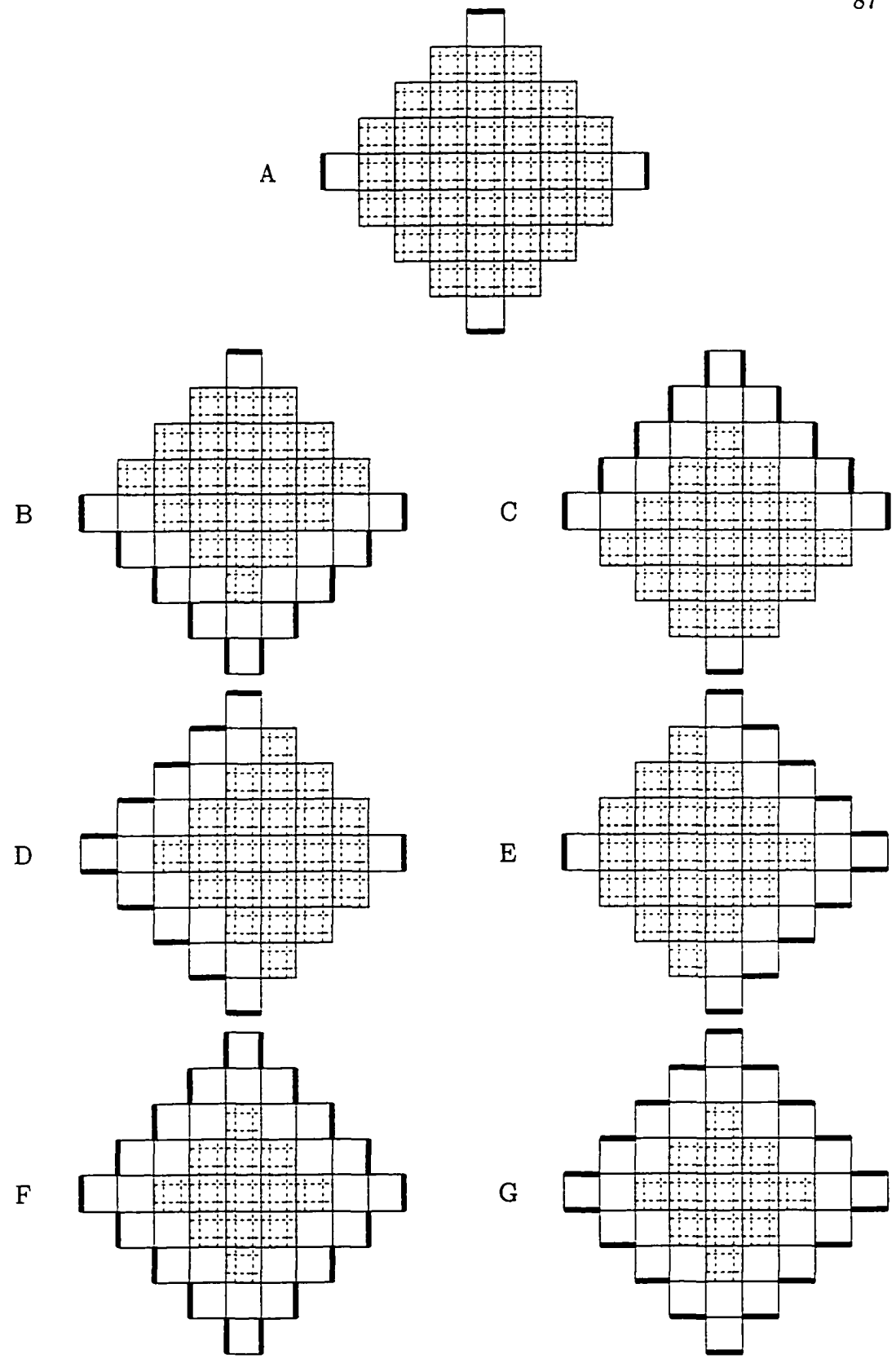
Figure 8 Types of matchings of the dual Aztec diamond

a   b   c
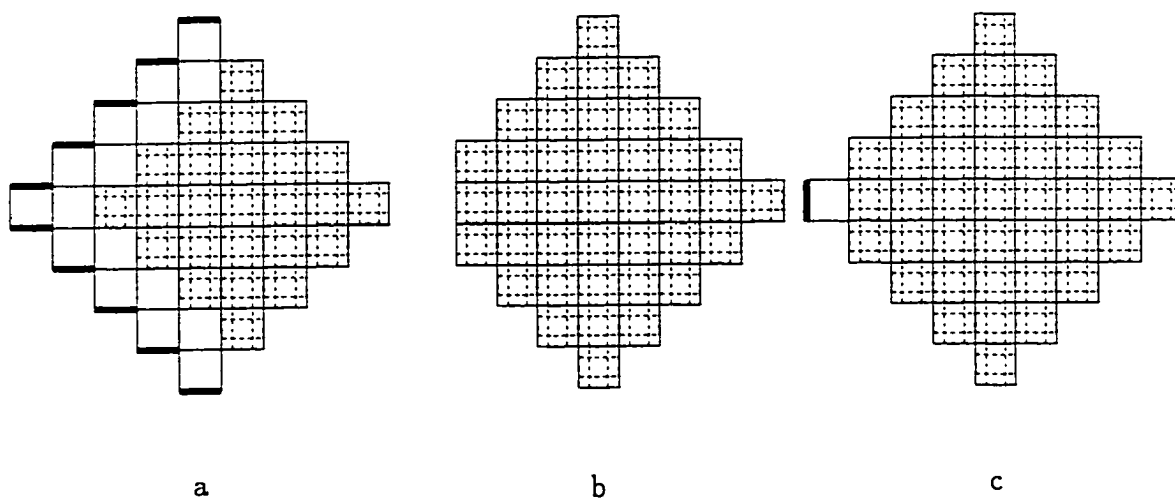
Figure 9 Illustration for Lemma 1.7
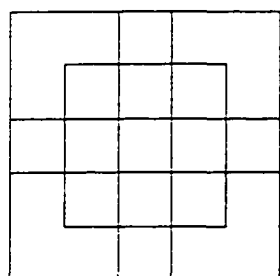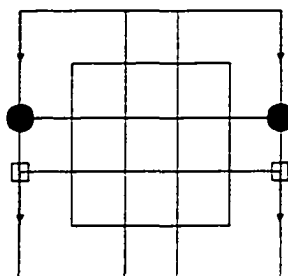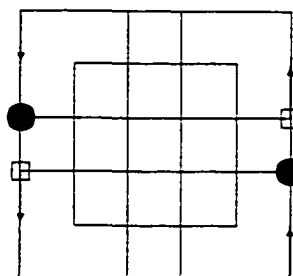
dual   Cylindrical   Möbius
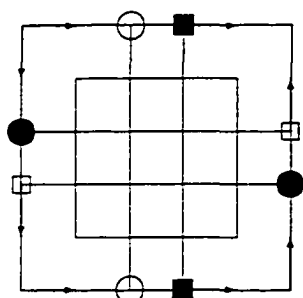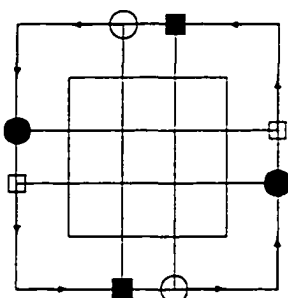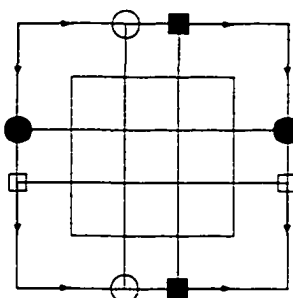
Klein   Projective   Torus

Figure 10 Planar representation of modified Aztec diamonds

Figure 11 Comparison of matchings of the modified Aztec diamonds for n = 2
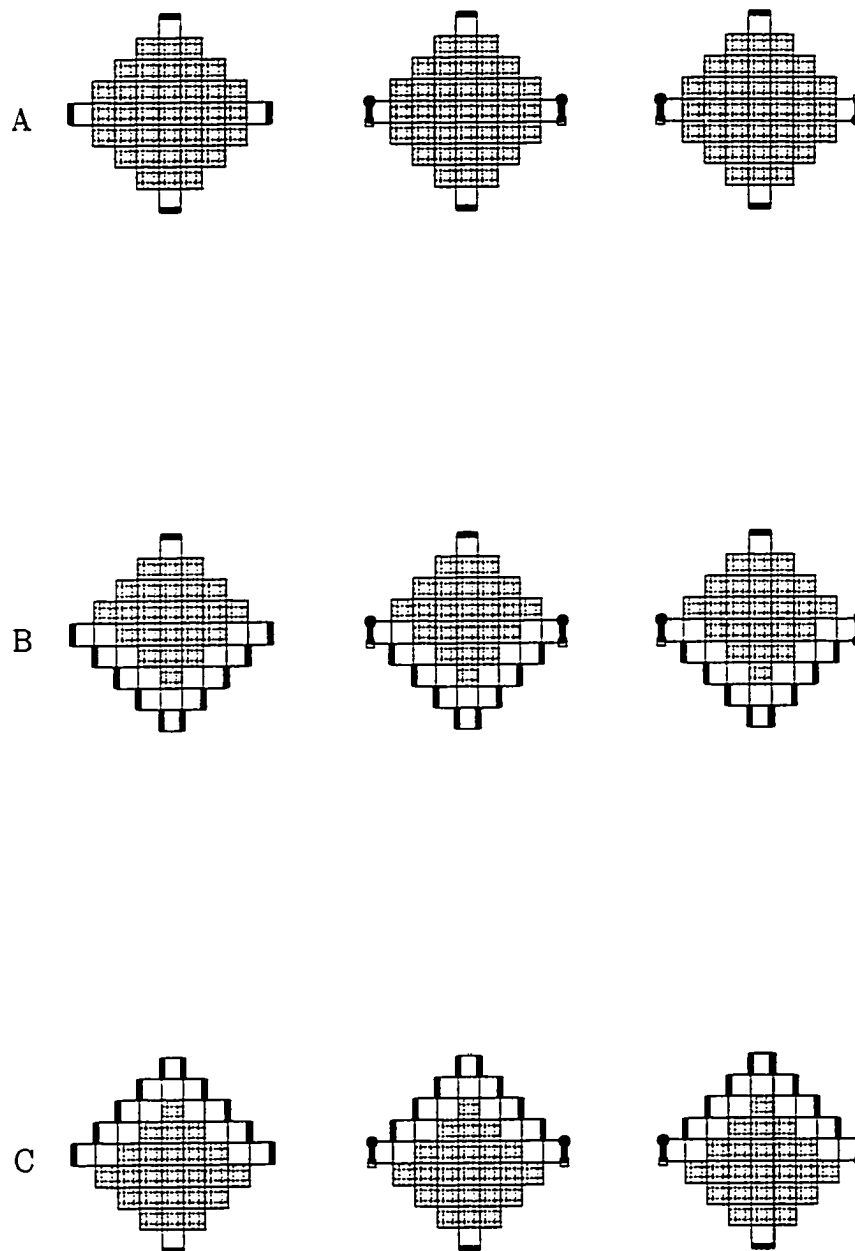
Figure 12 The bijection between matchings of the dual, the Cylindrical and the Möbius Aztec diamonds of types A, B, C
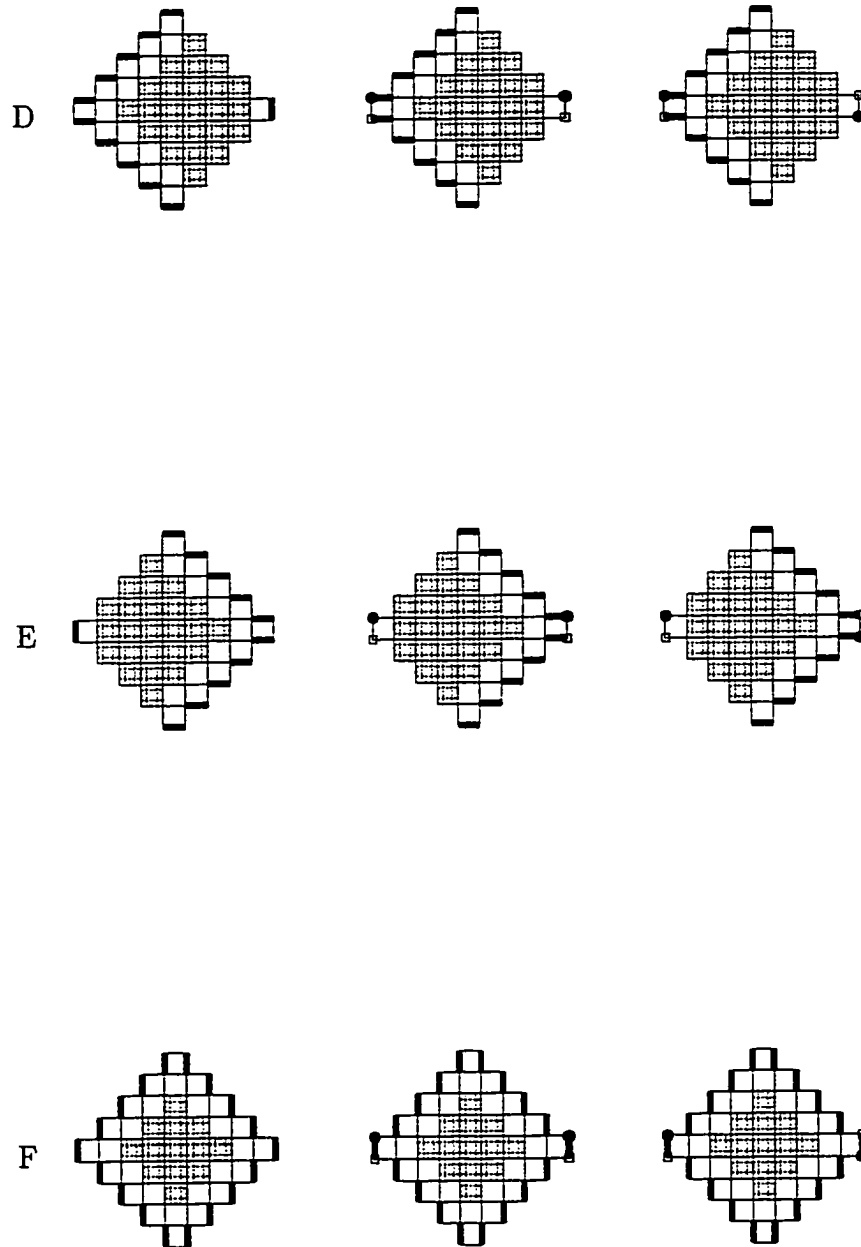
Figure 13 The bijection between matchings of the dual, the Cylindrical and the Möbius Aztec diamonds of types D, E, F
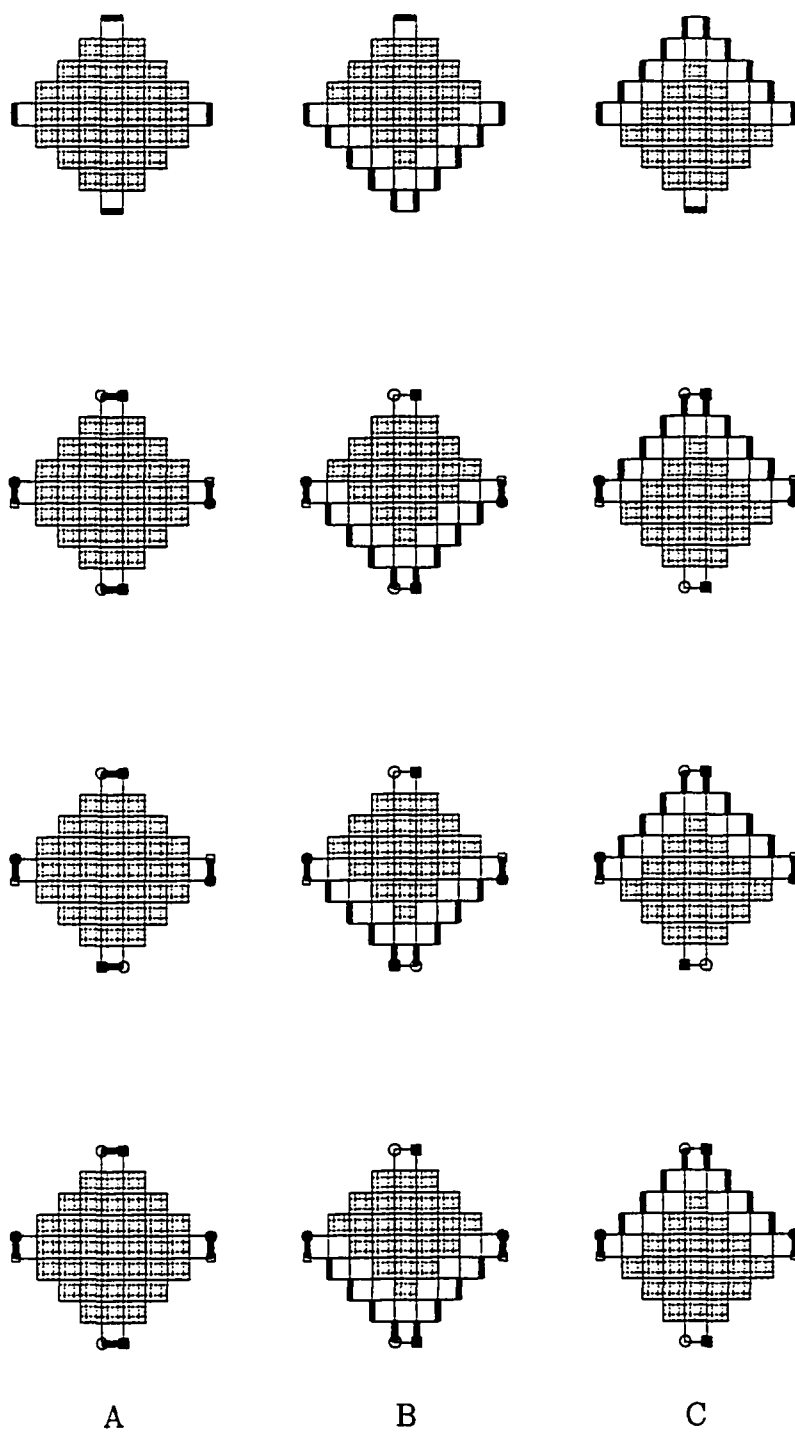
Figure 14 The bijection between matchings of the dual, the Klein, the Projective and the Torus Aztec diamonds of types A, B, C

D   E
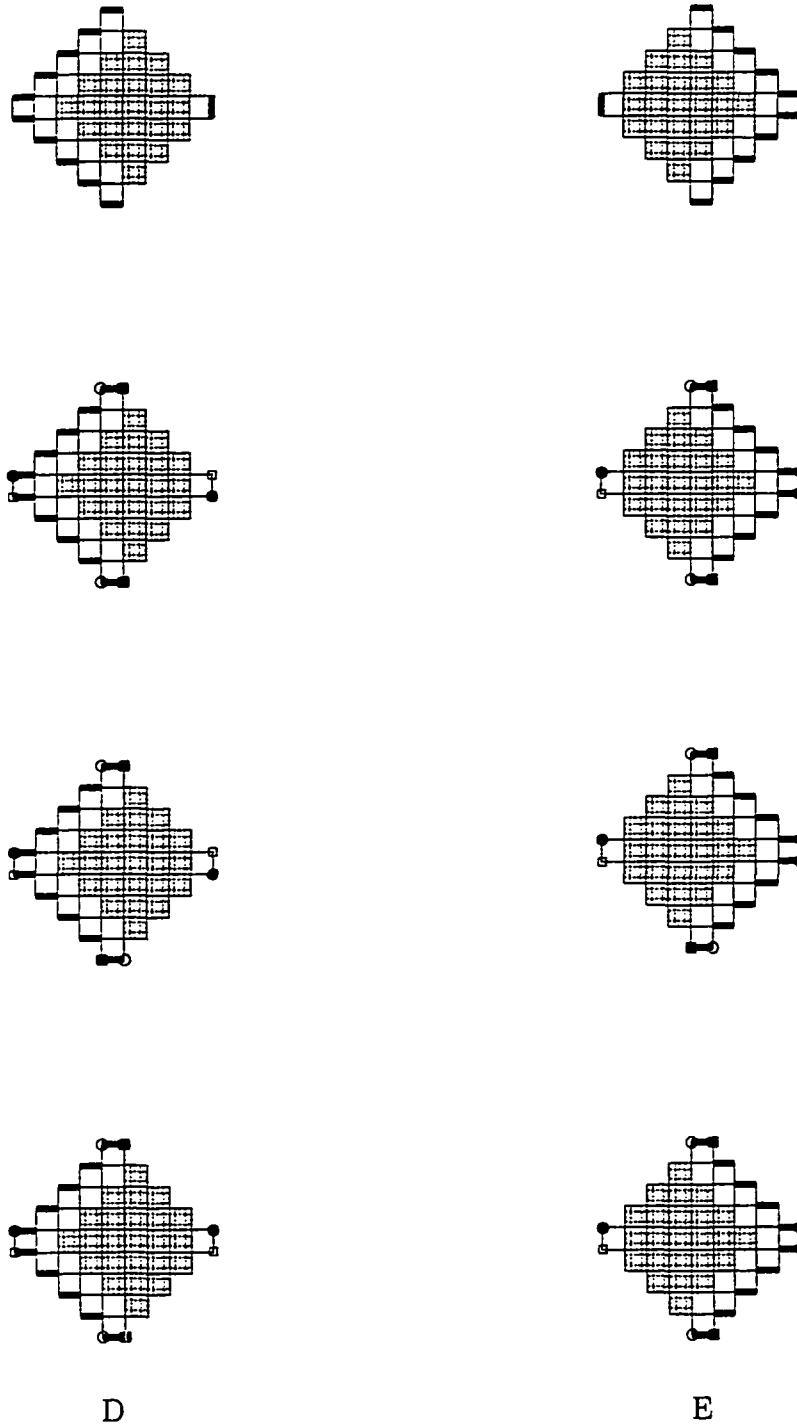
Figure 15 The bijection between matchings of the dual, the Klein, the Projective and the Torus Aztec diamonds of types D, E
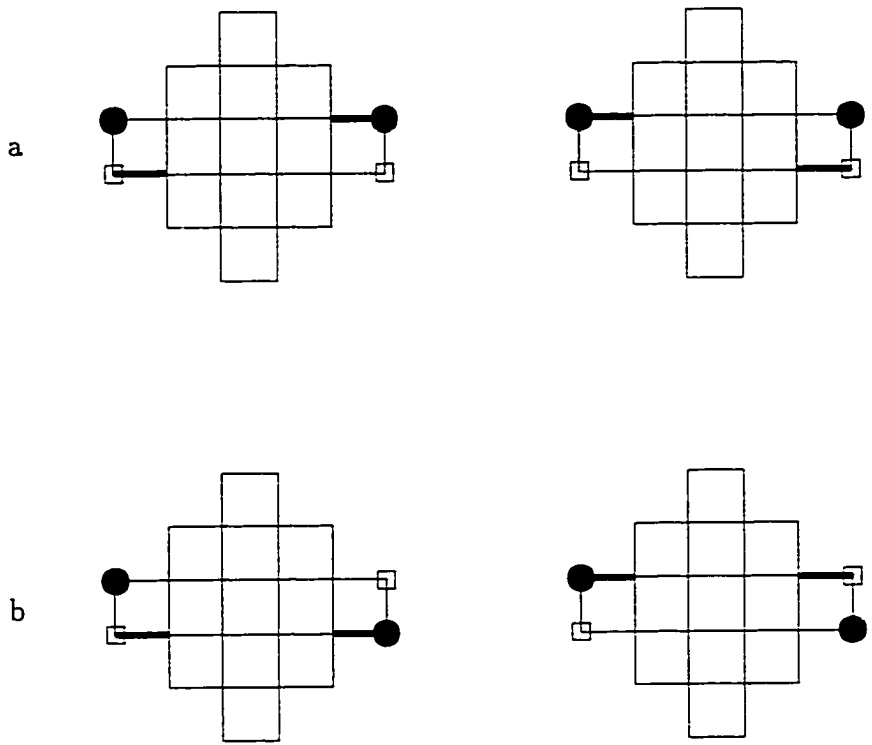
Figure 16 Types of special matchings
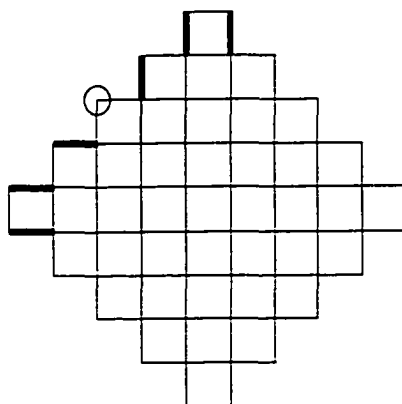
Figure 17 Matching of the corners which can not be extended to a perfect matching
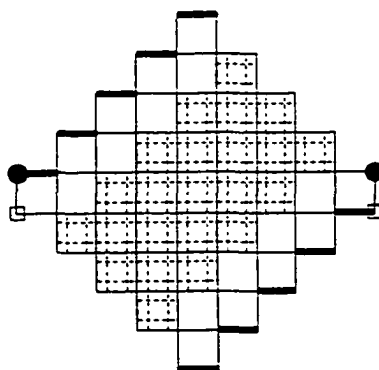
Figure 18 A hypothetical special matching of the Cylindrical Aztec diamond
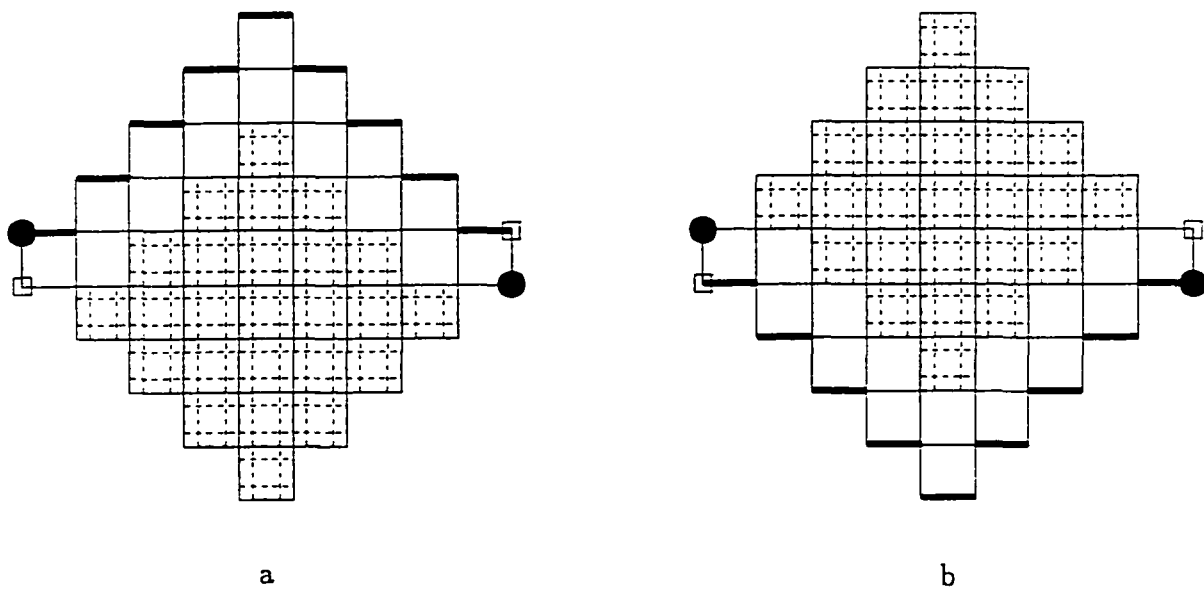
Figure 19 Special matchings of the Möbius Aztec diamond
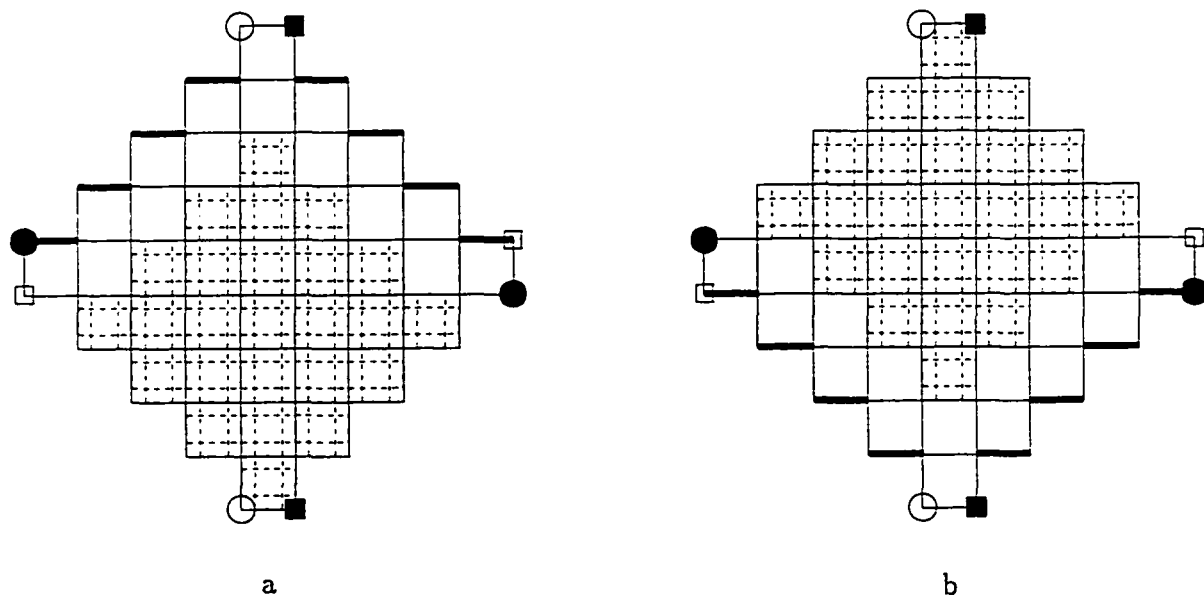
Figure 20 Special matchings of the Klein Aztec diamond
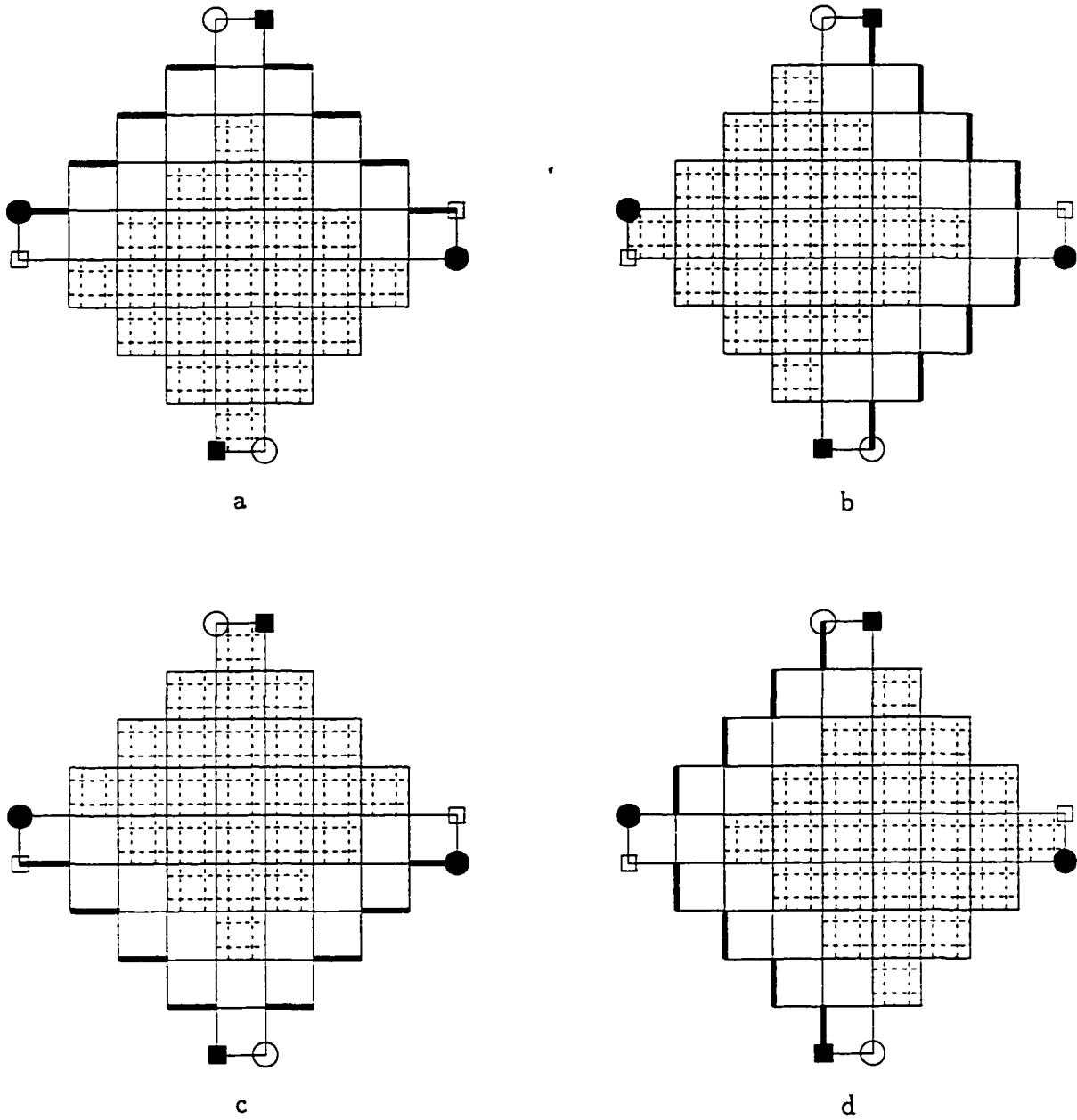
Figure 21 Special matchings of the Projective Aztec diamond

a   b
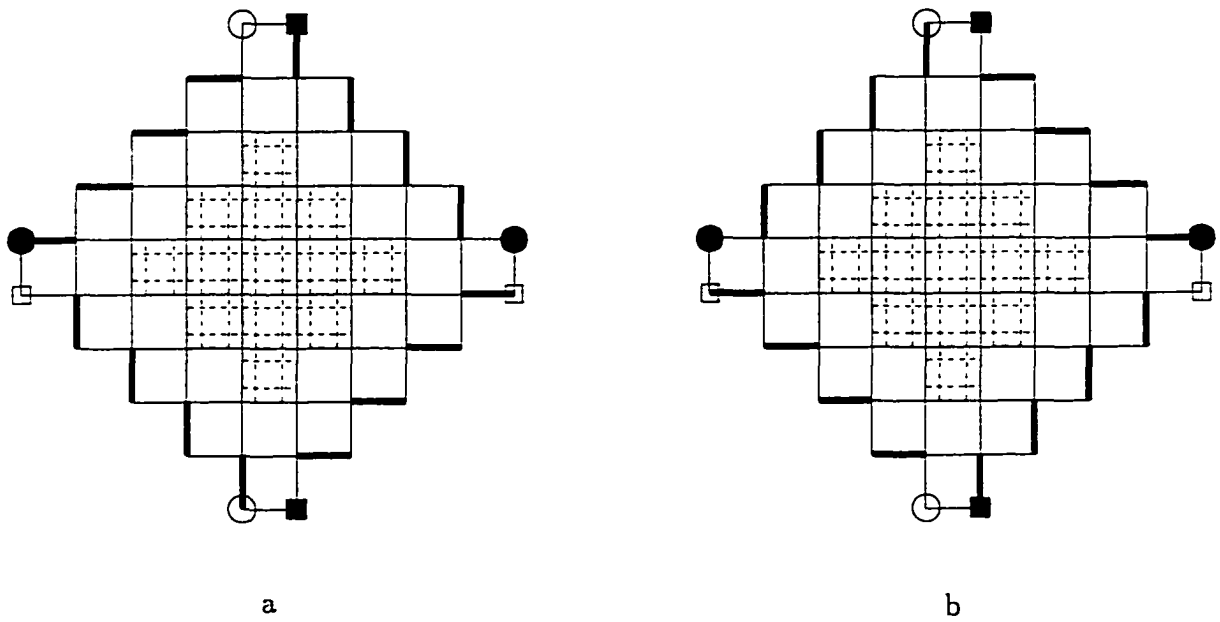
Figure 22 Special matchings of the Torus Aztec diamond

APPENDIX B
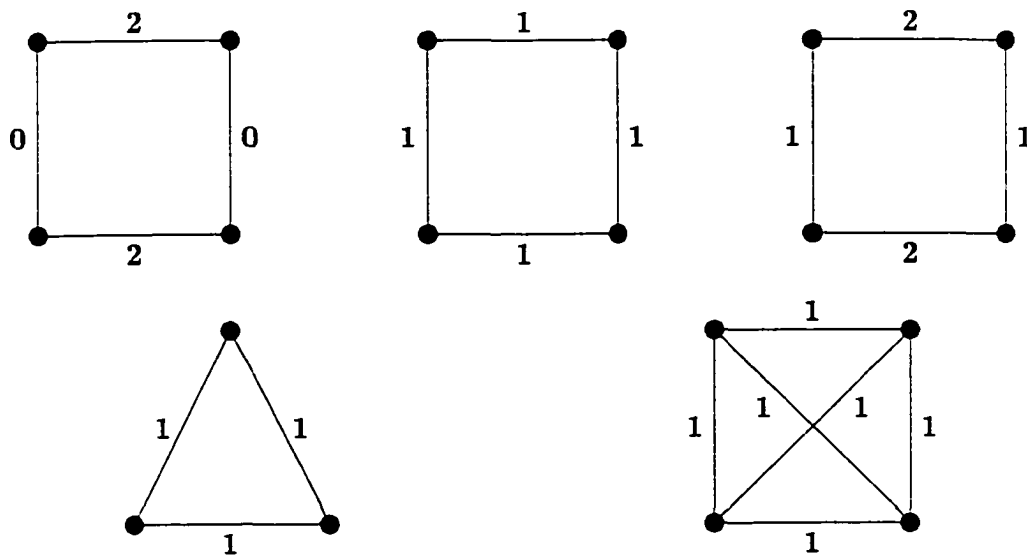
Figure 23 Examples of n-matchings
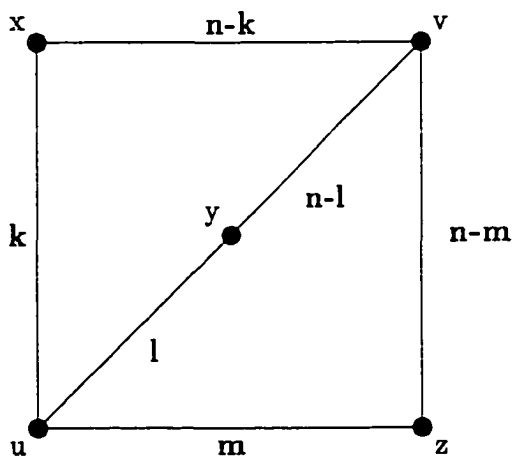
Figure 24 Example of graph with no n-matching
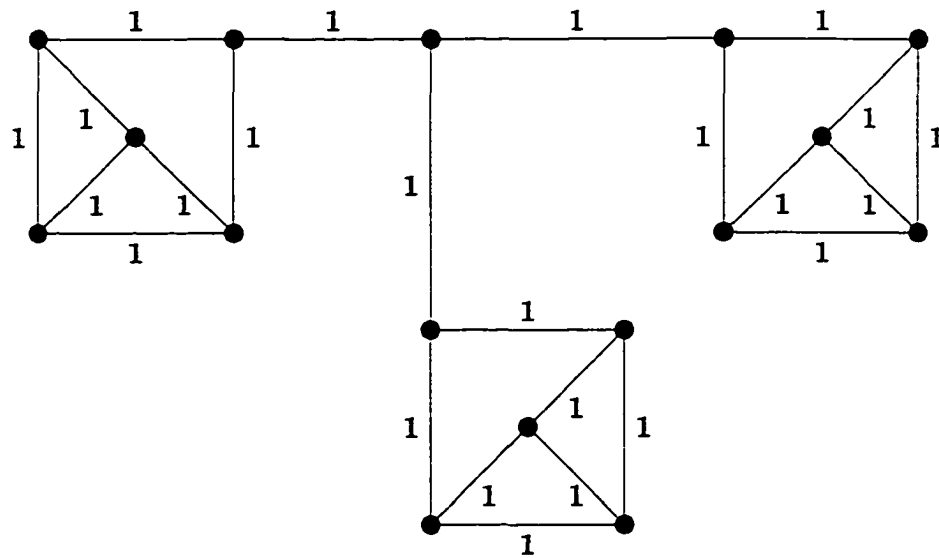
Figure 25 Example of graph with a perfect 3-matching and no perfect matching
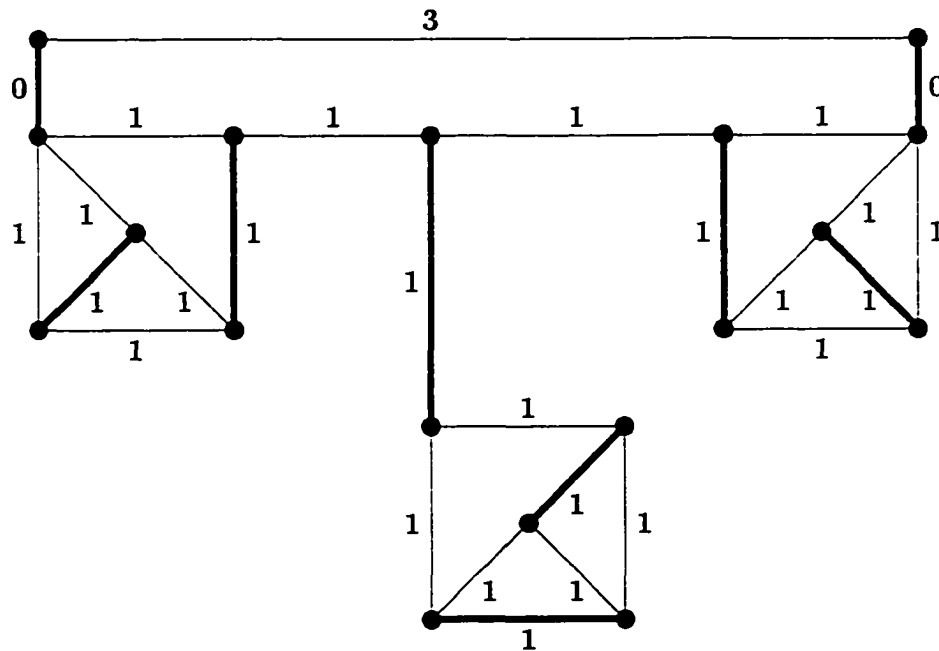
Figure 26 Example of independent perfect matching and 3-matching
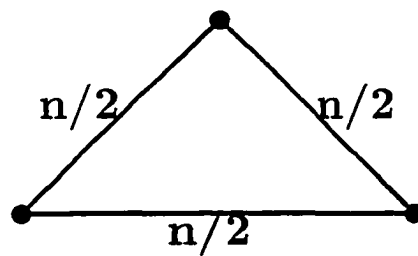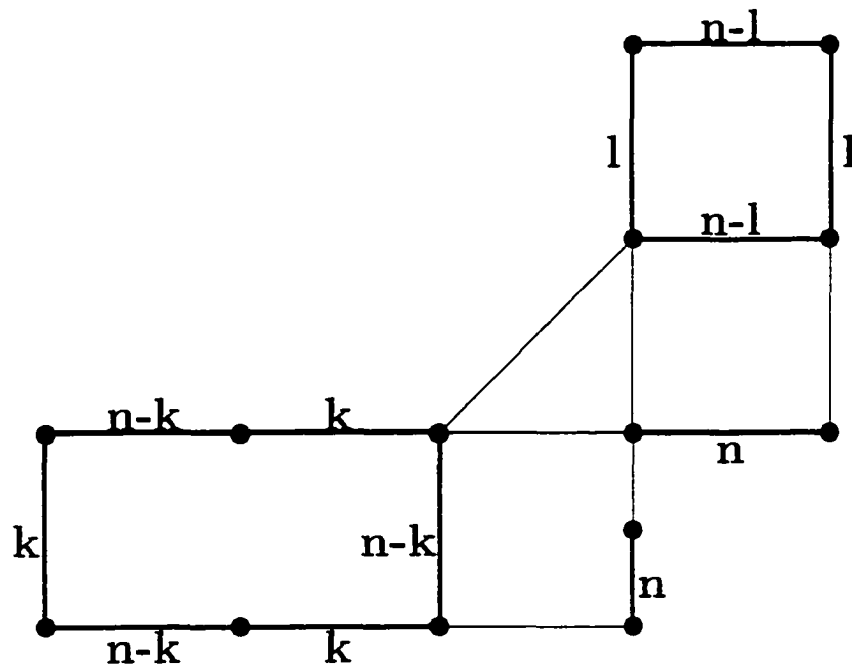
Figure 27 Illustration of the structure of non-ramified perfect n-matchings
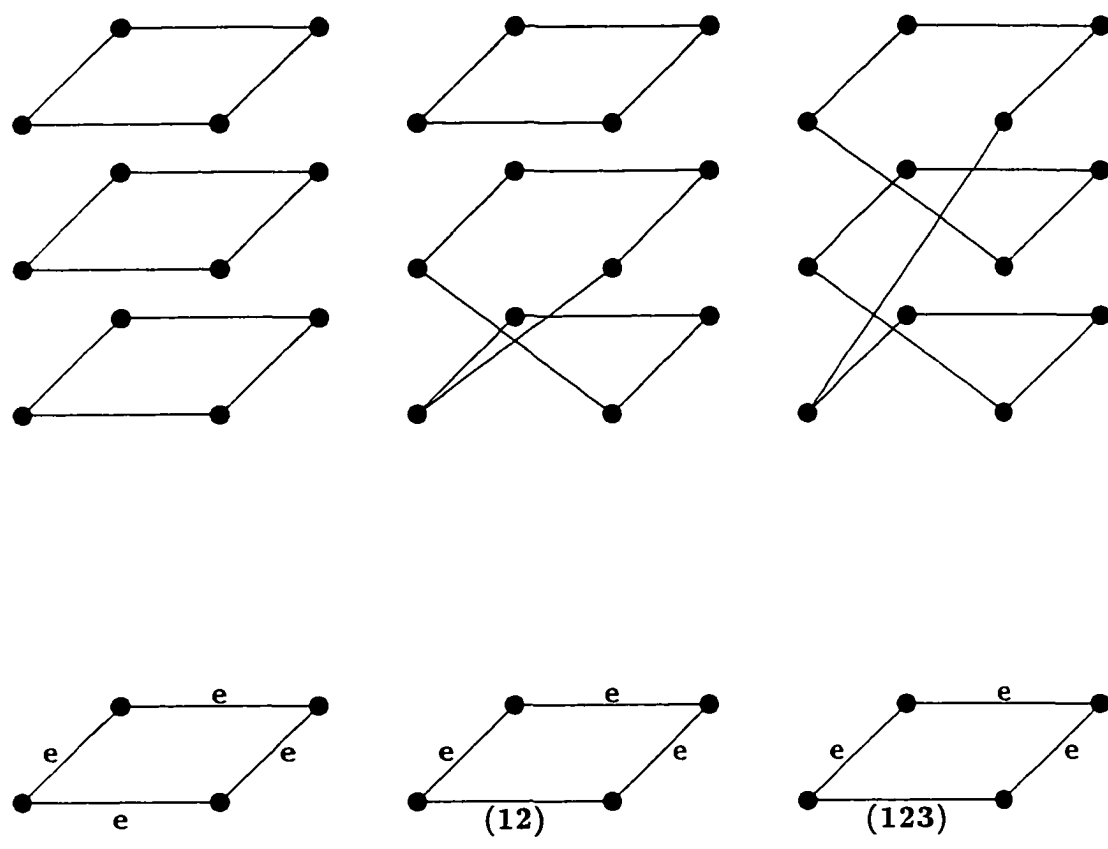
Figure 28 S_3-voltage graphs and the associated permutation derived graphs for a cycle of length 4
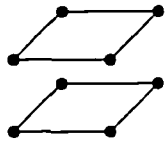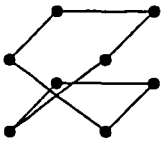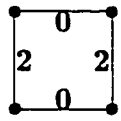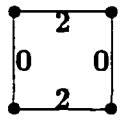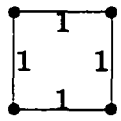
			②
square (0,2,2,0)	1	1	2
square (2,0,0,2)	1	1	2
square (1,1,1,1)	2	–	2
③	4	2	⑥

Figure 29 Liftings of 2-matchings for a cycle of length 4
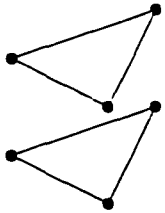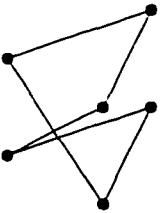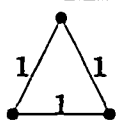
			②
triangle (1,1,1)	–	2	2
①	0	2	②

Figure 30 Liftings of 2-matchings for a cycle of length 3
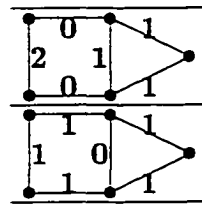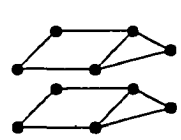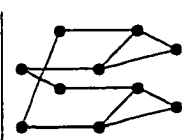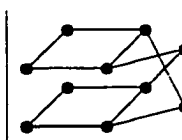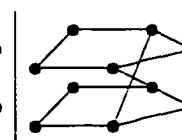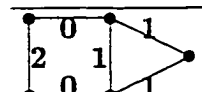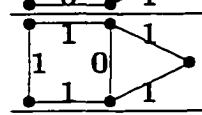
Figure 31 Liftings of 2-matchings for a graph formed of a square and a triangle adjacent at an edge

	1	1	1	1	4
	1	1	1	1	4
	1	1	1	1	4
	2	2	-	-	4
	2	-	2	-	4
	2	-	-	2	4
⑥	9	5	5	5	㉔

Figure 32 Liftings of 2-matchings for a graph formed of two squares adjacent at an edge

a

b
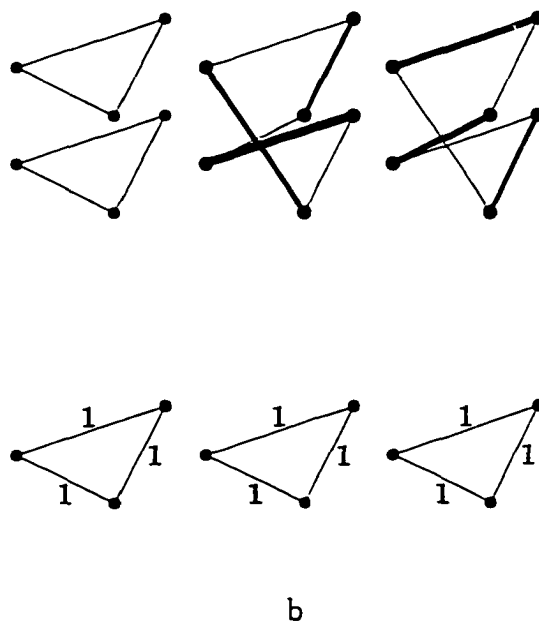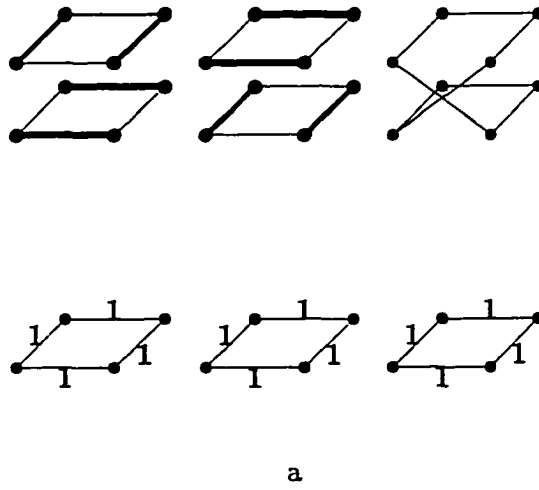
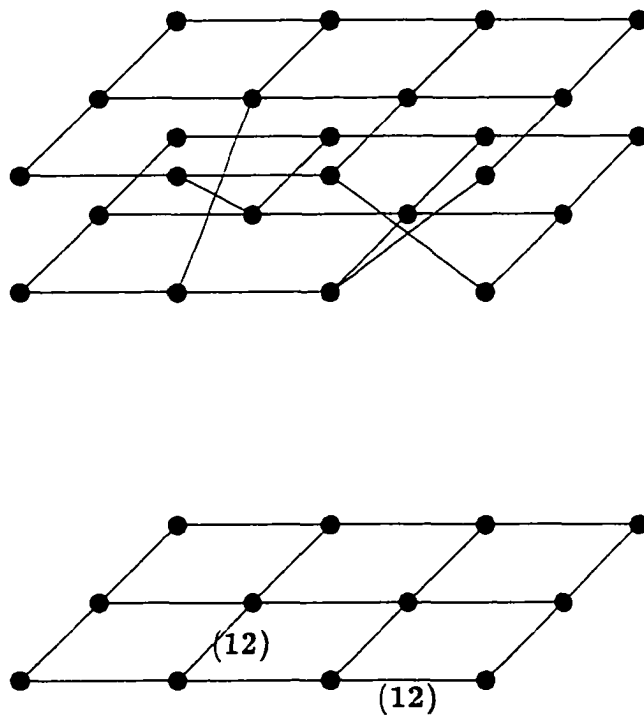Figure 33 All possiblities of lifting proper 2-matchings for an even length cycle (a) and for an odd length cycle (b)
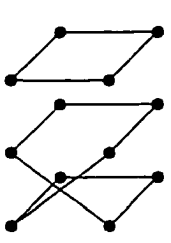
(12)

(12)

Figure 34 Example of non-Pfaffian 2-covering graph
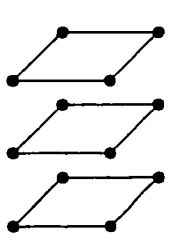
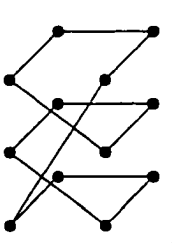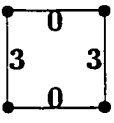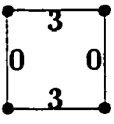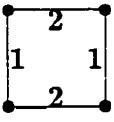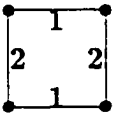|				③
	1	1	1	3
	1	1	1	3
	3	1	-	4
	3	1	-	4
④	8	4	2	⑭

Figure 35 Liftings of 3-matchings for a cycle of length 4

Figure 36 Liftings of 4-matchings for a cycle of length 4

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