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Contributions to the theory of distance functions and its application in general topology

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CONTRIBUTIONS TO THE THEORY OF DISTANCE FUNCTIONS AND ITS APPLICATION IN GENERAL TOPOLOGY

BY

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Dedication

To my parents and to Kris, thanks for all the help and support.

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Many people have made this work possible with very different contributions.

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ABSTRACT

CONTRIBUTIONS TO THE THEORY OF DISTANCE FUNCTIONS AND ITS APPLICATION IN GENERAL TOPOLOGY

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University of New Hampshire, May, 1997

As non-Hausdorff spaces are becoming more important in topology, there is a need to consider new notions in topology to supplement the usual structures. This work uses distance functions to find useful generalizations (in a non-Hausdorff context) of the classes of spaces that are important in the Hausdorff setting. We begin, in the first part, with a historical overview that traces the evolution of the notion of distance and its role in the development of general topology.

In the second part of this work, we launch our study of distance functions. Using non-symmetric distance functions, called asymmetric, we generalize the class of symmetrizable spaces which itself includes Moore spaces and metrizable spaces. We also introduce generalizations of Gamma spaces, Nagata spaces and developable spaces. We conclude this work with a number of results about pseudo-metrizable and metrizable spaces.

An underlying theme of this work is that distance functions can provide intuitively-appealing proofs for known theorems that usually have more complex derivations and are often presented with explicit use of the Hausdorff property.

Overview

In his 1992 article, *On Non-Hausdorff spaces*, Ivan L. Reilly [Rei] notes that

> life without Hausdorff is not only possible but that it is imperative. Recent developments in the theory of continuous lattices and in theoretical computer science ... justify this position. [1]

Of particular interest to us is the observation that these topologies are often generated by a distance function. With this in mind we launch a study of the most fundamental properties of distance in order to find useful generalizations (in a non-Hausdorff context) of the classes of spaces that are already important in the Hausdorff setting. In many cases we take our cue from the work that took place at the beginning of this century when the concept of topological space was being developed.

The first part of this work is an historical overview that traces the evolution of the notion of distance and the development of its intrinsic properties. We observe that in an effort to generalize the concept of the linear continuum, the pioneering works of Fréchet, Hildebrandt, and Chittenden were often centered around the concept of a distance function subject to a variety of axioms. Hildebrandt, for example, considers the possibility of non-symmetric distances. This leads us to a study of the evolution of *symmetrics* and *quasimetrics* [2], as well as to an investigation into the origin of the problem of metrization *before* the topological setting was fully defined.

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[1] Reilly, *On Non-Hausdorff spaces*, p.331.

[2] This, of course, uses modern terminology. The distinctions will be made clear in the body of this dissertation.

Of particular interest in our study are the distances that satisfy the *coherent* condition introduced by Pitcher and Chittenden [PC] in 1918, and later used by Niemytzki [Ni_1] in his classical result that a topological space is metrizable if, and only if, there is a coherent, symmetric distance for the space.

In part two of this work we focus on the study of distance functions *per se*, where a non-negative function $d : X \times X \to \mathbf{R}$ is a *distance* for X iff $d(p,p) = 0$ for every $p \in X$. Asymmetrizable spaces are studied as a generalization of the class of symmetrizable spaces, which itself includes Moore spaces and metrizable spaces.

We begin in Chapter 3 by providing a foundation for our study and establish a number of connections with symmetric spaces.

In Chapter 4 we investigate the varying forms of a local triangle inequality and make connections with γ*-spaces* and *Nagata spaces*. We also introduce the notion of a γ^**-distance*. Of particular interest in this study is the ability of such distances to separate two closed sets, one of which is compact.

In Chapter 5 we study *developably asymmetrizable* spaces as a generalization of developable spaces. We conclude in Chapter 6 with some metrization results based on ideas discussed in the preceding sections.

An underlying theme in our work is the role that distances can play is providing direct, intuitively-appealing proofs for known theorems that usually have more complex derivations and that are often presented with explicit use of the Hausdorff property. We strive to identify the specific distance property that is sufficient to carry a proof and to avoid, where possible, the use of even the T_1-property.

Throughout this work **N** and **R** denote, respectively, the set of natural numbers and the set of real numbers. The reader may consult Willard's *General Topology* [Wi_1] for terms not defined in this work.

Part I

A Brief History of Distance Functions

Chapter 1

Distances and Topology: The Beginning

The mathematical climate at the end of the nineteenth century and the beginning of this century was one of axiomatization and abstraction. The investigations into the appropriate axioms for defining the linear continuum by Cantor, Veblen or Huntington, for example, reflect this trend.

Writing in the *Transactions of the American Mathematical Society* in 1914, Ralph Root states:

> The work of these writers [referring to Cantor, Veblen and Huntington] is directed toward a complete characterization of the linear continuum in terms of order alone. A set or class of elements, otherwise undefined, is assumed to fulfill conditions, stated in terms of order, sufficiently restrictive to admit of effective use of the class in the rôle of range[1] of the continuous independent variable.
>
> Meanwhile certain classes have been recognized as being, in effect, the range of an independent variable, while not fulfilling all the conditions of the linear

[1] Root uses the "range" of a variable to refer to what would now be called the "domain" of a function.

continuum. The desirability of utilizing analogies that exist between theories that pertain to these classes and the theory of the continuum is obvious. To this end it is important to generalize, as far as may be, the notion of point sets so as to be applicable to such classes.[2]

This area of investigation eventually gave rise to the area of mathematics known as General Topology, and was pursued by a number of mathematicians of the era. If Fréchet took the initiative, many would soon follow, most notably Hausdorff and Kuratowski. Each mathematician had his perspective of what should constitute the basic defining structure. The following brief survey of some of the basic primitive notions was given by Sierpiński in 1927.

> Depuis la Thèse de M. Fréchet on connait quelques essais de baser la Topologie (Analysis Situs) sur telle ou telle notion primitive, p.e. sur celle de la limite (Fréchet), du point d'accumulation (F. Riesz), de l'écart[3], du voisinage[4] (Hausdorff, Fréchet), de la fermeture (Kuratowski).[5]

Of particular interest for our work are the contributions to the emerging field of topology that involved distance functions or constructs of a similar nature. Most of those contributions emanated from the students of E.H. Moore at the University of Chicago. In 1910 E.H. Moore published *Introduction to a Form of General Analysis* [Mo]. As a result of his interest, he directed his students to this area of study, introducing them to the work of

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[2] Root, *Limits in terms of order, with example of limiting element not approachable by a sequence*, p.51.

[3] In current terminology an *écart* is a metric.

[4] The usual translation of *voisinage* is more likely to be "neighborhood", although the original definition of a *voisinage* was a distance function.

[5] Sierpiński, *La notion de dérivée comme base d'une théorie des ensembles abstraits*, p.321.

Fréchet and Riesz and directing thesis work that led to the significant contributions of T.H. Hildebrandt, A.D. Pitcher and E.W. Chittenden.

This chapter focuses on the initial contribution of Maurice Fréchet and the efforts of Moore's students.

1.1 Maurice Fréchet's initial contribution

Sur Quelques Points Du Calcul Fonctionnel [Fr_1], Maurice Fréchet's Ph.D. dissertation, was published in 1906. Drawing from the work of Volterra, Arzelà and Hadamard, Fréchet observes that by introducing the notion of an *opération fonctionnelle*[6] on a variety of sets, he can avoid a duplication of similar results that have different proofs. That is, Fréchet observes that similar theorems are given in various context; however the arguments in the proofs are based on the specific nature of the elements on which the functions are defined. Keeping with the trend toward abstraction of the era, Fréchet sets out to establish a general theory for the *Calcul Fonctionnel.*

> Considérons un ensemble E formé d'éléments quelconques (nombres, points, fonctions, lignes, surface, etc.) mais tels qu'on sache discerner les éléments distincts. A tout élément A de cet ensemble faisons correspondre un nombre déterminé U(A); nous définissons ainsi ce que nous appellerons une *opération fonctionnelle* uniforme dans E.
>
> ...
>
> Les résultats qu'on obtiendra dans de telle théorie seront d'autant plus étendus qu'on s'adressera à des ensembles plus généraux. Pour obtenir la plus

[6]An *opération fonctionnelle* can be viewed as simply a real-valued function.

grande généralité possible, il y aurait donc lieu de chercher d'abord des résultats applicables à des ensembles abstraits, c'est-à-dire dont on ne spécifie pas la nature des éléments.[7]

If Fréchet is to succeed in his goal to establish a theory of functions defined on an abstract set, he must equip that set with a minimum of structure. Yet the structure introduced must be flexible enough to encompass most, if not all, of the characteristics of the sets on which functions are usually defined. The search for this primitive structure culminates with the following observation.

Les résultats les plus connus et en fait les plus importants de la théorie des ensembles sont ceux que l'on déduit de la notion de limite d'une suite d'éléments.[8]

As his primitive notion Fréchet settles on an abstract notion of convergence of sequences that satisfy two axioms.

Dorénavant, nous nous limiterons donc à l'étude des ensembles tirés d'une classe (L) d'éléments de nature quelconque mais satisfaisant aux conditions suivantes: on sait distinguer si deux éléments de la classe (L) sont distincts ou non. De plus, on a pu donner une definition de la limite d'une suite d'éléments de la classe (L). Nous supposons donc qu'étant choisie au hasard une suite infinie d'élément (distincts ou non) de la classe (L), on puisse dire d'une façon certaine si cette suite $A_1, A_2, \ldots, A_n, \ldots$ a ou non une limite A (d'ailleur unique). Le procédé qui permettra de donner la réponse (autrement dit la définition de la

[7]Fréchet, *Sur quelque points du calcul fonctionnel*, p.4.

[8]Fréchet, *Ibid.*, p.5.

limite) est d'ailleurs absolument quelconque, assujetti seulement à satisfaire aux conditions I et II dont nous avons parlé et qui sont les suivantes:

I) Si chacun des éléments de la suite infinie $A_1, A_2, \ldots, A_n, \ldots$ est identique à un même élément A, la suite a certainement une limite qui est A.

II) Si une suite infinie $A_1, A_2, \ldots, A_n, \ldots$ a une limite A, toute suite d'éléments de la première suite pris dans le même ordre: $A_{n_1}, A_{n_2}, \ldots, A_{n_p}, \ldots$ (les nombres entiers $n_1, n_2, \ldots, n_p$ iront donc en croissant) a une limite qui est aussi A.[9]

The *Classe (L)* is the first abstraction provided by Fréchet. In the context of a *Classe (L)* Fréchet is able to introduce analogs of definitions from the theory of the linear continuum. He defines an *élément limite* of a set as any point that is the limit of a sequence of distinct points from the set; an *ensemble dérivé* of a set as the set of all limit points of the set; an *ensemble fermé* as any set that contains all of its limit points; and an *ensemble parfait* as any set which is identical with its derived set. Perhaps most importantly, a notion of compactness is introduced:

> Nous dirons qu'un ensemble est *compact* lorsqu'il ne comprend qu'un nombre fini d'éléments ou lorsque toute infinité de ses éléments donne lieu à au moins un élément limite. [10]

In this context Fréchet can also introduce a sequential definition of continuity and prove a number of results of classical analysis in this very general context of a *Classe (L)*. However, the theory of *Classe (L)* lacks the property that the derived sets are necessarily closed. This prompts Fréchet to consider other options. Thus, rather than restrict himself to a *Classe*

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[9] Fréchet, *Ibid.*, p.5–6.

[10] Fréchet, *Ibid.*, p.6.

(L) where all derived sets are closed, Fréchet considers other means by which to determine the convergence of a sequence. This leads to the introduction of his *Classe (V)*.

> Considérons une classe (V) d'éléments de nature quelconque, mais tels qu'on sache discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'à deux quelconques d'entre eux A, B, on puisse faire correspondre un nombre $(A,B) = (B,A) \geq 0$ qui jouit des propriétés suivantes: 1° La condition nécessaire et suffisante pour que (A,B) soit nul est que A et B soient identiques. 2° Il existe un fonction positive bien déterminée $f(\varepsilon)$ tendant vers zéro avec ε, telle que les inégalités $(A,B) \leq \varepsilon$, $(B,C) \leq \varepsilon$ entrainent $(A,C) \leq f(\varepsilon)$, quels que soient les éléments A, B, C. Autrement dit, il suffit que (A,B) et (B,C) soient petits pour qu'il en soit de même de (A,C). Nous appellerons *voisinage* de A et de B le nombre (A,B).
>
> Ceci étant, nous pourons dire qu'une suite d'éléments de la classe (V): $A_1, A_2, \ldots$ tend vers un élément A, si le voisinage (A_n, A) tend vers zéro avec $\frac{1}{n}$. Si une suite $A_1, A_2, \ldots$ a une limite A, elle ne peut en avoir qu'une, car si B était limite de la même suite, les nombres (A, A_n) et (B, A_n) seraient infiniment petits avec $\frac{1}{n}$, donc aussi (A,B) ($2^{ième}$ condition). Alors (A,B) serait nul et par suite les éléments A, B ne seraient pas distincts ($1^{ére}$ condition).
>
> De plus, cette définition de la limite satisfait bien aux conditions I et II que nous avons imposées en général à toute définition de la limite (n°7) et cela grâce aux conditions 1° et 2° imposées à la definition du voisinage.
>
> Néanmoins toute définition de la limite satisfaisant aux conditions I et II, ne peut être déduite de la notion de voisinage. Il nous suffira pour le prouver de démontrer le théorème suivant.

Théorème. — L'ensemble dérivé d'un ensemble d'éléments d'une classe (V) est un ensemble fermé.[11]

In the development of his theory for *Classe (V)*, Fréchet generates a primitive structure using a distance function. Moreover, the *voisinage* enables Fréchet to give an alternative formulation of continuity that can be used as well to introduce uniform continuity. The *Classe (V)* gives Fréchet enough structure to prove the majority of the results of the theory of functions of the time, while working with an abstract set for the domain of his functions.

In applying his results to concrete examples, Fréchet observes that the specific *voisinages* he uses are of a more restrictive nature. This observation leads him to introduce the *classe (E)*, which is a special case of a *Classe (V)*. The type of distance function that will generate the structure for this class will be called an *écart*.

Lorsque nous appliquerons les résultats généraux de la PREMIÈRE PARTIE à des exemples concrets, nous reconnaîtrons d'abord que, dans chaque cas, on peut faire correspondre à tout couple d'éléments A, B un nombre $(A, B) \geq 0$, que nous appellerons *l'écart des deux éléments* et qui jouit des deux propriétés suivantes: a)L'écart (A, B) n'est nul que si A et B sont identiques. b) Si A, B, C, sont trois éléments quelconques, on a toujours $(A, B) \leq (A, C) + (C, B)$.

Lorsqu'on peut définir l'écart de deux éléments quelquonques d'une certaine classe, nous dirons que celle-ci est une classe (E).

Il est facile de voir que le nombre ainsi défini satisfait aux conditions imposées à la définition du voisinage. En effet, la condition 1° du $n°27$, sera remplie d'elle-même et la condition 2° sera remplie, en prenant par exemple $f(\varepsilon) = 2\varepsilon$, si l'on

[11] Fréchet, *Ibid.*, p.17–18.

tient compte de la condition b) actuelle.

Ainsi l'écart est un voisinage qui jouit d'une propriété particulière, ou, si l'on veut, toute classe (E) est une classe (V). Dans la plupart des demonstrations des théorèmes connus, la propriété b) de l'écart intervient dans les raisonnements. Cependant la théorie développée dans ce Chapitre montre qu'elle n'est pas indispensable et qu'il suffit de se servir du voisinage sans avoir besoin pour cela de compliquer notablement le raisonnement.

Une exception doit être faite cependant pour le théorème que nous allons établir maintenant; l'hypothèse qu'on opère sur une classe (E) intervient en effet d'une maniére essentielle dans la démonstration. Malgré cela, il est pourtant vraisemblable que l'énoncé reste exact pour toutes les classes (V). ...

Théorème.— La condition nécessaire et suffisante pour que toute opération continue dans un ensemble E d'éléments d'une classe (E), 1° soit bornée dans cet ensemble, 2° y atteigne sa limite supérieure, est que cet ensemble E soit extrémal.[12]

The proof of this theorem marks the end of the first part of Fréchet's dissertation. The second part focuses on the application of the results of the first part to specific examples with most of the effort going into finding an appropriate *écart* for each example considered.

A valuable perspective on this work of Fréchet is given by the world-famous Russian topologist P. S. Alexandroff and his coauthor Fedorchuk in the following:

On the abstract side, Fréchet in his original paper of 1906 was not content with his so brilliant introduction of the very important class of metric spaces, he also

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[12]Fréchet, *Ibid.*, p.30–31.

attempted to define the concept of a topological space and gave several versions of his definition.[13]

For our study we note that Fréchet introduced the structures for two of his *classes* using a distance function, one of which we immediately recognize as a metric. The next article of Fréchet that we consider is important, since in it originates what we will refer to as the Metric Recognition Problem.

In 1910 Fréchet publishes *Les ensembles abstraits et le calcul fonctionnel* [Fr_2], which is intended as a complementary article to his 1906 thesis. The introduction indicates the nature of the paper.

> Je veux ajouter dans ce qui suit quelques compléments à ma Thése: *Sur quelques points du calcul fonctionnel* [dont on trouvera un excellent résumé dans un important ouvrage de M. Schoenflies]. J'aurai d'ailleurs soin de rappeler les définitions qui sont nécessaires pour la compréhension des nouvelles propositions que j'aurai à énoncer.
>
> Dans une première section, je donne quelques propositions nouvelles relatives aux ensembles compacts. Dans la suivantes, je complète ma généralisation du théorème de Cantor-Bendixson en introduisant les nombres transfinis. Puis j'étudie les ensembles à une infinité de dimensions. Enfin je termine par quelques observations générales.[14]

Even though the results of this publication are interesting, with a number of them being

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[13] Alexandroff and Fedorchuk, *The main aspect in the development of set-theoretical topology*, p.5.

[14] Fréchet, *Les ensembles abstraits et le calcul fonctionnel*, p.1.

proved in the context of what will be called *complete* metric spaces [15], distance functions do not play a significant role in this publication. It is a conjecture stated in one of the concluding remarks that draws our attention.

Je voudrais terminer par quelques réflexions au sujet des observations les plus intéressantes qui ont porté sur ma Thèse.

Je signale d'abord une intéressante contribution apportée par M. Hahn à l'étude des classes (V). Je rappelle que j'avais ainsi nommé toute classe où l'on peut faire correspondre à chaque couple d'éléments A, B un certain nombre [A,B] (le voisinage de A et de B), qui jouit des mêmes propriétés que l'écart mais en remplaçant la condition:

$$(A, B) \leq (A, C) + (B, C)$$

par une autre plus générale, du moins en apparence. On suppose seulement qu'il existe une fonction positive $f(\varepsilon)$ infiniment petit avec ε, telle que si

$$[A, C] < \varepsilon \text{ et } [B, C] < \varepsilon,$$

on a

$$[A, B] < f(\varepsilon).$$

Tous les théorèmes de ma Thèse énoncés pour les classes (E) étaient aussi démontrés pour les classes (V), sauf un seul (page 31). J'avais écrit que le

[15] As Fréchet puts it, in "une classe (E) admettant une généralisation du théorème de CAUCHY sur la convergence". [Fr_2, p.2]

théorème était trés vraisemblablement vrai aussi pour une classe (V). M. Hahn a démontré que cette supposition était exacte. Cette démonstration m'a confirmé dans la conviction qu'il n'existe probablement aucune différence essentielle entre les classes (V) et les classes (E). Autrement dit, je pense qu'étant donné une classe (V), on doit pouvoir donner une définition d'un écart dans cette classe, de façon que les suites convergentes et leurs limites restent telles, qu'on se serve de la définition primitive du voisinage ou de l'introduction de l'écart. J'ai donc abandonné dans le présent travail la considération des classes (V) qui introduisent des complications de raisonnements inutiles; j'ai du reste vérifié sur quelques-unes de mes nouvelles démonstrations données pour des classes (E) qu'elles s'étendent presque immédiatement aux classes (V).[16]

Féchet's goal to establish an abstract theory of Analysis was the motivating force behind his study of the *ensembles abstraits*. Unknowingly, this study marks a first step in the development of a theory that would be called general topology. It is interesting to note that distance functions are instrumental in this first effort and that metric spaces are the first type of spaces studied.

Mathematicians working in this new field of research would follow many different paths. In our next section we investigate the work of three American mathematicians who, under the guidance of E.H. Moore at the University of Chicago, would follow in the path of Fréchet with their use of distances or distance-like constructions to study abstract spaces.

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[16]Fréchet, *Ibid.*, p.22–23.

1.2 Chicago and the theory of distances

In 1912 volume of the *Transactions of the American Mathematical Society* Hildebrandt [Hi] publishes the results of his Ph. D. dissertation, which was written in 1910. In this Hildebrandt develops additional details for Fréchet's theory of abstract sets. In particular Hildebrandt determines the convergence of sequences in an abstract set by means of a generalized distance-like construct named a *K-relation* [17].

> The present paper concerns itself with the Fréchet point of view. It had its inception in an attempt to replace the distance function δ of Fréchet by a weaker condition on the class Q. The fact that in most instances the δ appears in connection with an inequality of the type
>
> $$\delta_{q_1 q_2} \leq \frac{1}{m}$$
>
> suggested the adoption of the second K-relation of Moore, $K_{q_1 q_2 m}$, in place of the δ. By stating, in the case of every theorem, the precise conditions on K sufficient to carry the argument, and extending this idea to the case in which the class Q is subjected only to the condition of the existence of a limit, a twofold result was obtained: (a) *that an unconditioned limit suffices for the theorems on sequentially continuous functions obtained by Fréchet, and* (b) *that it is possible to obtain the theory of sets of elements with a distance function δ, subjected to weaker conditions than those imposed by Fréchet.* To show that the conditions in question were weaker, the complete existential theory, of the properties of the

[17] Hildebrandt denotes the distance between two elements q_1 and q_2 by $\delta_{q_1 q_2}$.

K-relation, and as a consequence of the corresponding properties of the δ, was constructed.[18]

Hildebrandt defines his K-relations as follows:

The K-relation is a relation between pairs of elements, and positive and negative integers.

The nature of the K-relation will evidently depend upon the nature of elements considered, and the situation in which it is to be employed. Relative to a class Q, we may consider the K-relation as drawn up in the form of a table which specifies for every combination of a pair of elements q_1, q_2 with an integer m, i.e., for every $q_1 q_2 m$, whether or not the relation holds. We denote the fact that the K-relation is holding (not holding) between $q_1 q_2 m$ by

$$K_{q_1 q_2 m} \quad (-K_{q_1 q_2 m})\,.$$ [19]

Hildebrandt imposes only one restriction on his K-relations, namely,

For every pair of elements q_1, q_2 of the class Q there exists at least one integer m such that

$$K_{q_1 q_2 m}.$$ [20]

The K-relations are meant to replace the distances of Fréchet and can be conditioned by any combination of eight axioms introduced by Hildebrandt to yield a *system* (Q, K)

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[18]Hildebrandt, *A contribution to the foundations of Fréchet's calcul fonctionnel*, p.238–239.

[19]Hildebrandt, *Ibid.*, p.242–243.

[20]Hildebrandt, *Ibid.*, p.243.

that will play the role of the *classes* of Fréchet.

The K-relations form an interesting theory. However, to fully explore them here would be beyond the scope of this dissertation. Furthermore, Hildebrandt establishes an equivalence between his K-relations and the axioms that he introduced for them, and a general concept of distance itself subject to a number of axioms. We will focus on the distances and their axioms. We will not use Hildebrandt's notation as it is based on a set of symbols that are not in use nowadays. Instead we state his axioms using today's notation.

The conditions introduced by Hildebrandt for a distance, denoted by δ, are as follows:

(0) $\delta_{q_1 q_2} \geq 0$ for every pair $q_1,\ q_2$;

(2) $\delta_{q_1 q_2} = \delta_{q_2 q_1}$;[21]

(3) $\delta_{q_1 q_2} = 0 \Rightarrow q_1 = q_2$;

(4) $q_1 = q_2 \Rightarrow \delta_{q_1 q_2} = 0$;

(5) there is a function $\phi : \mathbf{R} \to \mathbf{R}$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that:

$$\text{if } \delta_{q_1 q_2} \leq \varepsilon \text{ and } \delta_{q_2 q_3} \leq \varepsilon \text{ then } \delta_{q_1 q_3} \leq \phi(\varepsilon);$$

(6) there is a function $\phi : \mathbf{R} \to \mathbf{R}$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that:

$$\text{if } \delta_{q_2 q_1} \leq \varepsilon \text{ and } \delta_{q_2 q_3} \leq \varepsilon \text{ then } \delta_{q_1 q_3} \leq \phi(\varepsilon);$$

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[21] Hildebrandt comments that, when a distance function satisfies this property, "The δ is a symmetrical function of its argument."

(7) there is a function $\phi : \mathbf{R} \to \mathbf{R}$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that:

$$\text{if } \delta_{q_1 q_2} \leq \varepsilon \text{ and } \delta_{q_3 q_2} \leq \varepsilon \text{ then } \delta_{q_1 q_3} \leq \phi(\varepsilon);$$

(8) there is a function $\phi : \mathbf{R} \to \mathbf{R}$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that:

$$\text{if } \delta_{q_2 q_1} \leq \varepsilon \text{ and } \delta_{q_3 q_2} \leq \varepsilon \text{ then } \delta_{q_1 q_3} \leq \phi(\varepsilon);$$ [22]

We note that each one of Fréchet's *classes* came equipped with a distance that satisfied a number of pre-established properties. The only condition imposed by Hildebrandt on his distances is that δ be a non-negative real-valued function (satisfies condition (0)). The systems $(\mathcal{Q}, \delta)$, which replace the *classe* of Fréchet, are more general. However, we observe that the *classes* of Fréchet can be recovered by appropriate selection of axioms for δ. A *classe (V)*, for example, is a system $(\mathcal{Q}, \delta)$ where δ satisfies conditions (2), (3), (4), (5). In this case Hildebrandt, following conventions that had been introduced by E. H. Moore, denotes such a system as being of the form $(\mathcal{Q}, \delta^{2345})$.

In the second part of the paper, Hildebrandt sets out to prove the theorems about abstract sets found in Fréchet's dissertation. However, in each case he will attempt to determine the *least conditions* that will be necessary in order to establish the given result. He is able to accomplish his task with a system $(\mathcal{Q}, K)$ which corresponds to a system $(\mathcal{Q}, \delta)$ with δ satisfying conditions (3), (6), (7). This is a weaker set of axioms than the one used by Fréchet; in particular we note that the distance function used by Hildebrandt lacks the Axiom of Symmetry, and surprisingly that the distance $\delta_{q_1 q_2}$ need not equal 0, even if

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[22] Hildebrandt, *Ibid.*, p.247.

$q_1 = q_2$.

Although Hildebrandt's paper may not be regarded as a significant contribution to the early history of topology, it is a ground-breaking contribution in the history of distance functions in that he isolates the various properties of a distance, including symmetry.

We now turn to another generalization of the work of Fréchet, due to the collaborative efforts of two other students of E.H. Moore.

This paper of A.D. Pitcher and E.W. Chittenden [PC], *On the foundations of the calcul fonctionnel of Fréchet*, was published in 1918. It is not the first contribution of either one of these authors in this area of study. In particular, we note that Chittenden was able to show in 1916 that a *voisinage* and an *écart* are equivalent[23]. This collaboration is a study of distance functions that is directed toward generalizing the results of Fréchet.

> In the present paper we follow the example of Fréchet in assuming once and for all that $\delta(qq) = 0$ and that $\delta(q_1q_2) = \delta(q_2q_1)$. In other words we assume that the distance from an element to itself is zero and that the distance between two elements is independent of the order in which they are taken. In the first part of the paper we give very simple conditions on systems $(\mathcal{Q}; \delta)$ which are sufficient for many purposes and which, in the case of compact sets, we show to be equivalent, so far as limit of a sequence is concerned, to the *voisinage* and thus to the *écart* of Fréchet. The remainder of the paper is devoted to the theory of functions on the sets $\mathcal{Q}$ of systems $(\mathcal{Q}; \delta)$.[24]

[23]This paper will be discussed in the next chapter.

[24]Pitcher and Chittenden, *On the foundations of the calcul fonctionel of Fréchet*, p.66.

We focus on the first part of the paper. Pitcher and Chittenden consider six conditions for their distances which we introduce using their notation.

We follow Fréchet in saying that q is a limit of the sequence q_n, $L_n q_n = q$, when and only when $L_n \delta(q_n q) = 0$. We shall be interested in the following properties of δ, or of systems $(Q;\delta)$.

(1) If $\delta(q_1 q_2) = 0$ then $q_1 = q_2$.

(2) If $L_n q_{1n} = q$ and $L_n \delta(q_{1n} q_{2n}) = 0$ then $L_n q_{2n} = q$.[25]

(3) If $L_n q_{1n} = q = L_n q_{2n}$ then $L_n \delta(q_{1n} q_{2n}) = 0$.

(4) If $L_n \delta(q_{1n} q_{2n}) = 0$ and $L_n \delta(q_{2n} q_{3n}) = 0$ then $L_n \delta(q_{1n} q_{3n}) = 0$.

(5) There is a function $\phi(e)$ such that $L_{e=0}\phi(e) = 0$ and such that if $\delta(q_1 q_2) \leq e$, $\delta(q_2 q_3) \leq e$ then $\delta(q_1 q_3) \leq \phi(e)$. [26]

(6) $\delta(q_1 q_2) + \delta(q_2 q_3) \geq \delta(q_1 q_3)$.

It will be seen at once that (2), (3), and (4) are important properties which are implied by (5) and (6). We will show that (2), (3), and (4) play a fundamental rôle.[27]

Their choice of terminology is also explained:

The property (2) will prove to be of fundamental importance and, for lack of a better term we venture to call a system $(Q;\delta^2)$ a *coherent* system.[28]

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[25]Here $L_n q_{1n} = q$ denotes $q_{1n} \to q$ and $L_n \delta(q_{1n} q_{2n}) = 0$ denotes $\delta(q, q_{1n}) \to 0$.

[26]The authors use $L_{e=0}\phi(e) = 0$ to mean $\lim_{\varepsilon \to 0} \phi(e) = 0$.

[27]Pitcher and Chittenden, *Ibid.*,p.67.

[28]Pitcher and Chittenden, *Ibid.*,p.68.

Here again Pitcher and Chittenden use the notation $(\mathcal{Q}; \delta^2)$ to mean that the system is determined by a distance δ that satisfies condition (2). They go on to establish their main result:

> Theorem 7. If $(\mathcal{Q}; \delta^1)$ is a coherent system then there is an L-equivalent, limited system $(\mathcal{Q}; \bar{\delta}^{13})$ such that on every compact set $\overline{\mathcal{Q}}$ of $\mathcal{Q}$, $\bar{\delta}^{13}$ is a voisinage, i.e., the system $(\overline{\mathcal{Q}}; \bar{\delta}^{13})$ is a system $(\overline{\mathcal{Q}}; \bar{\delta}^{15})$.[29]

That is, the system $(\overline{\mathcal{Q}}; \bar{\delta}^{13})$ is metrizable when $\overline{\mathcal{Q}}$ is compact.

We remark that the authors show a tremendous insight with their selection of axioms for a distance function. As we note in the next chapter, Niemytzki [Ni_1] is able to prove in 1927 that a symmetric distance satisfying the coherent condition is equivalent to a metric. We also note that, in the evolution of distance ideas, the distances that satisfy conditions (3) will be closely related to the important class of developable spaces that are introduced by R. H. Bing in 1951.

The papers considered in this chapter were all written early in the evolution of general topology when the search for an appropriate primitive notion was still underway. They focused on the primitive notion of a sequence of points with its limit which was often determined using a distance function satisfying a number of different axioms. With the publication of his book Hausdorff [Hau] successfully generalized the *classe (E)* of Fréchet using neighborhoods, and the search for a primitive construct would soon come to an end. With the maturing of topology, a number of new problems would emerge and attract the attention of the mathematicians of the time.

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[29]Pitcher and Chittenden, *Ibid.*,p.71.

In the next chapter we turn our attention to one of those problems and investigate the theory of metrization from the point of view of distances. We are especially drawn to the early history of semimetric and symmetric spaces with a brief mention of quasimetric spaces.

Chapter 2

Metrization, Semimetrization,and Symmetrization

In this chapter we will look in turn at four areas of topology closely related to the theory of distance functions. We will trace the evolution of each subject as it fits into the broad context of distances. We have already seen that at their inception metric spaces were defined by means of a distance function. This fact also holds for semimetric spaces, quasimetric spaces and symmetric spaces. We will first turn our attention to the history of metrization as it is the richest. We will then provide a brief overview of the evolution of semimetric and symmetric spaces, completing our survey with quasimetric spaces.

2.1 Metrization

In 1923 Alexandroff and Urysohn published what is often considered the first Metrization theorem[1] and promoted the Metrization Problem as one of the most important problem in General Topology. This section will focus on the early contributions to the Metrization Problem, as well as a number of investigations in what we will call the *Metric Recognition*

[1] See Hodel [Ho_2] for example.

Problem which can be considered as the precursor to the Metrization Problem. Keeping with our focus on distance function, we will consider results based on distance constructions or what is sometime referred to as *Explicit Metrization*[2].

According to Alexandroff and Urysohn:

> C'est M. Fréchet qui a le premier formulé explicitement le problème d'indiquer les conditions pour qu'une classe ($\mathcal{L}$) soit une classe ($\mathcal{D}$).[3]

In 1918, Fréchet gave a definition of the problem.

> Ainsi, l'objet de ce travail est le suivant: sachant qu'on peut toujours de bien des façons définir dans une classe d'éléments abstraits les suites convergentes et leur limites, il s'agit de déterminer à quelles conditions supplémentaires il faut assujettir ce choix pour que l'on puisse définir sur cette classe une distance telle que la convergence définie d'avance ne soit pas modifiée quand on la définit au moyen de cette distance.[4]

Fréchet is working in the context of a *classe (L)* and wants to find what extra conditions are needed for the *classe (L)* to guarantee that he actually has an abstract set of *classe (D)*[5]. This marks a change in direction from what to this point had been the usual problems about metrization, which we call the Metric Recognition Problem.

In 1927, Chittenden sums up the mathematical research that followed after the publication of Fréchet's dissertation, stating:

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[2] See the contribution of Shore and Sawyer [ShS] for more details.

[3] Alexandroff and Urysohn, *Une condition nécessaire et suffisante pour qu'une classe* ($\mathcal{L}$) *soit une classe* ($\mathcal{D}$), p.1274.

[4] Fréchet, *Relations entre les notions de limite et de distance*, p.55.

[5] At this time Fréchet continues to modify the definitions for his classes as he continues to deepen his theory. Here a *classe (D)* corresponds to a *classe (E)* from his Thesis, that is, the class of metric spaces.

Attempts have been made by Fréchet, E.R. Hedrick, A.D. Pitcher, and the writer to obtain effective generalizations of the theory of metric spaces. That is, to impose hypotheses which yield substantially the same group of of theorems about point sets and are less restrictive. It has however been established in each case that the conditions proposed imply that the resulting space is equivalent to a metric space.[6]

We define the Metric Recognition Problem as the problem of showing that a proposed generalization of a *classe (V)* or a *classe (E)* of Fréchet is in fact a metric space.

The most important result in this area of research is the proof the Fréchet's conjecture that the *classe (V)* and the *classe (E)*, defined in his dissertation, are in fact equivalent. The proof is due to Chittenden [Ch_1, 1916] who, starting with a *voisinage* [7] and using a strategy introduced by Hahn, actually constructs an *écart* (that is, a metric) that has the same convergent sequences as the original *voisinage*. This result also provides a powerful method for proving metrization results that would be used frequently in the 1920's. Chittenden's contribution leads us to recognize that Fréchet was the first mathematician to give a *metrization-like* result.

This metrization-like result of Fréchet [Fr_3, 1913] was introduced as a response to an attempt by E.R. Hedrick to introduce generalizations for the *classes* of Fréchet. Fréchet, starting with the Hedrick's conditions on a *classe(L)*, constructs a distance functions, in this case a *voisinage*, with the same convergent sequences as in the original *classe (L)*.

If what we call the Metric Recognition Problem focuses on the problem of determining when a given class is a metric space, the Metrization Problem is different in nature. The

[6] Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, p.26.

[7] See our discussion of Fréchet's thesis.

focus there is to determine what *topological* conditions must a given space also satisfy to guarantee that it is a *metrizable* space. Fréchet [Fr_5] launches this area of investigation in *Relations entre les notions de limites et de distance.* By relaxing the requirement on his *classes*, he is able to introduce a number of necessary conditions for a *classe (L)* to be a *classe (D)*. But, he fails to achieve his goal[8] as no sufficient conditions will be introduced. We note, however, that Fréchet does introduce two new *classes*: the *classe (S)* and the *classe (E)*. A *classe (S)* is a *classe (L)* in which every derived set is closed. Of interest for our future investigations, he redefines the *classe (E)* to be a *classe (L)* where the convergence of sequences is determined by a distance function satisfying only $(A, B) = (B, A) \geq 0$, $(A, B) = 0$ iff $A = B$ with the convergence of sequences being defined in the usual way $A_n \rightarrow A$ iff $(A, A_n) \rightarrow 0$. Fréchet also adjust his terminology and refers to this new distance function as an *écart* [Fr_5, p54].

With the publication of *Une condition nécessaire et suffisante pour qu'une classe (L) soit une classe (D)* [Ale, 1923] Alexandroff and Urysohn recast the Metrization Problem of Fréchet in the context of a Hausdorff topological space, giving the Metrization Problem its final form. Their result is obtained by introducing sequences of open covers that satisfy a number of conditions. A clever construction gives a *voisinage* and a direct application of Chittenden's result solves the problem. This paper ushers a new era in topology. The interest in Fréchet's *classes* is declining, while the spaces of Hausdorff become relevent due in part to the contributions of the Polish and Russian schools of topology. A quote, attributed to Urysohn by Lindenbaum in 1926, suggests that the further role of distance in topology will not be significant.

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[8]See the quote at the beginning of this section.

> La notion de *distance* étant étrangère à la Topologie, il vaudrait donc mieux de ne pas en faire usage.[9]

The Metrization Problem, however, would be a major source of research in topology during this era. We pursue our investigation of the Metrization Problem by considering the 1927 publication of Niemytzki and note an obvious similarity between the conditions studied in this publication and the research we shall undertake in this dissertation.

Niemytzki sets out to generalize the theorem of Pitcher and Chittenden[10], but first, he introduces a new type of space, which he names *symmetric space*, and defines as a *classe (E)* of Fréchet that is also *a topological space of Hausdorff.*

The principal result in this article states that:

> Theorem. A symmetric space in which the condition (Ch)[11] is satisfied is a metric space.

> I shall give two demonstrations of this theorem: one, based entirely on the method developed by Pitcher and Chittenden; another, based on the methods and results of the Russian school.[12]

Along with the introduction of a new class of space, this work serves to reinforce the significance of the contribution of Pitcher and Chittenden.

The last publication that we consider in this section is the 1937 contribution of A.H. Frink, entitled *Distance functions and the Metrization problem* [Fri]. In some ways this

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[9]Lindenbaum, *Contributions à l'étude de l'espace métrique I*, p.14.

[10]See our previous chapter.

[11]Here (Ch) is defined by $\lim \delta(y_n, x) = 0$, if $\lim \delta(x_n, x) = 0$, and $\lim \delta(x_n, y_n) = 0$.

[12]Niemytzki, *On the third axiom of metric space*, p.508.

research aligns more closely with the Metric Recognition Problem than with the Metrization Problem. In general, Frink proves that, if a space is determined by a distance function satisfying certain conditions, then the space could also have been determined by a metric.

Frink introduces the following conditions for a distance function, where the distance between two elements a and b is simply denoted by ab.

I. $ab = 0$ if and only if $a = b$;

II. $ab = ba$ (symmetry);

III. $ac \leq ab + bc$ (triangle property);

IV. if $ab < \varepsilon$ and $cb < \varepsilon$, then $ac < 2\varepsilon$ (generalized triangle property);

V. for every $\varepsilon > 0$ there exists $\phi(\varepsilon) > 0$ such that if $ab < \phi(\varepsilon)$ and $cb < \phi(\varepsilon)$, then $ac < \varepsilon$ (uniformly regular); [13]

The conditions considered by Frink are not new. However, the interest lies in the following lemma as it provides an explicit method for constructing a metric, given a distance satisfying conditions (I), (II), and (IV).

LEMMA. If a, x_1, x_2, ..., x_n, b are $n+2$ points of a space with a distance function satisfying I, II, and IV, then

$$ab \leq 2ax_1 + 4x_1x_2 + 4x_2x_3 + \ldots + 4x_{n-1}x_n + 2x_nb.$$ [14]

We also remark that Frink addresses the issue of *Unsymmetric Distance Functions* and

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[13] Frink, *Distance functions and the Metrization problem*, p.133

[14] Frink, *Ibid.*, p.134.

spaces determined by them. Following a remark about the need for a specific definition of convergence and introducing one, she establishes two metrization theorems for such space. This is the first time, as far as we know, that results of this type are published.

> THEOREM 1. A space with an unsymmetric, uniformly regular distance function is metrizable.[15]

> THEOREM 2. A space with an unsymmetric distance function ab satisfying I and VII is homeomorphic to a metric space.[16]

A number of other metrization theorems are also proved by Frink.

Historically, the solution of the Metrization Problem is attributed to Bing, Nagata, and Smirnov in works published in the early Fifties. However, this would not mark the end of metrization considerations in topology. On the contrary the concepts employed in these independent solutions would generate more interest and lead to new areas of study that would become known loosely as Generalized Metric Spaces. Although symmetric distances were almost an implicit assumption in previous studies of distance functions, it is in this setting that a systematic investigation of the symmetric and semimetric spaces would take place.

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[15]Frink, *Ibid.*, p.138.

[16]Here, Property VII is defined as:

VII. Given a point a and a number $\varepsilon > 0$, there exists a number $\phi(a, \varepsilon) > 0$ such that if $ab < \phi(a, \varepsilon)$, $cb < \phi(a, \varepsilon)$, $cd < \phi(a, \varepsilon)$, then $ad < \varepsilon$; [Fri, p.138]

See Frink, *Ibid.*, p.139.

2.2 Semimetric and Symmetric Spaces

As we have already observed, a number of researchers had considered spaces had were defined by distances that were symmetric, but might not satisfy the triangle inequality. For example, the investigations of Pitcher and Chittenden, mentioned earlier, are of this nature.

It was F.B. Jones, however, who in the Fifties initiated an area of research that would further solidify the importance of semimetric spaces. As a student of R. L. Moore at the University of Texas, Jones was well grounded in the theory of Moore spaces, which were known to be regular, Hausdorff spaces with a topology generated by a symmetric distance. Jones' belief was that many of the properties of Moore spaces could be proved in the more general class of regular semimetric spaces.

Wilson [Wi_1] in 1931 had already ventured down this path in a consideration of semimetric spaces, but according to McAuley [McA] his work is not done in the context of a topological space. This may be so, since his work dealt more with finding equivalent distance functions than dealing with topological properties of spaces. We should observe, however, that Wilson is the first author to note the importance of unique limits of a sequence in this area of investigation [Wi_1, p.362].

The work done by the students of Jones was presented in 1955 during the *Summer Institute on Set Theoretic Topology* (see [Jo], [Br], [McA]). This marks the beginning of this area of investigation in general topology. We point the reader to Gruenhage's survey of generalized metric spaces [Gr] for an overview of this type of investigation and to the more recent paper of Hodel on the history of generalized metric spaces [Ho_6]. We observe, however, that most of the work done in this field has usually been carried out in the presence of a strong separation axiom, usually regularity, and that only recently (see [Saw]) has this line of investigation been carried out without this condition.

In 1966 Arhangel'skiĭ initiated an intensive study of *symmetric spaces* as a significant generalization of semimetric spaces. Symmetric spaces differ from semimetric spaces in that the spheres generated by the distance function need not be topological neighborhoods of their centers.

Both symmetric and semimetric spaces have been studied extensively in the 60's, 70's and 80's. Thus, not requiring the triangle inequality for a distance has proved to be an interesting and valuable generalization of the metric.

We also observe that a large amount of work has been done, especially in the last fifteen years, in the theory quasimetric spaces, which are spaces defined by a metric that may lack the axiom of symmetry. The history of quasimetric distances starts briefly with Wilson in 1931 [Wi_2] with numerous contribution from mathematicians around the world since that time. Most notably, we acknowledge the contributions by Frink in 1937, Albert [Al, 1941], Ribeiro [Ri, 1943] early on, and more recently we note the work of Kofner [Ko_2, 1980] among many others.

Keeping with the approach of generalizing topological spaces generated by distances by considering various axioms that condition a distance, we devote the rest of this work to a study of spaces determined by distances that lack the Axiom of Symmetry, and satisfy a one-sided version of the coherent condition introduced by Pitcher and Chittenden.

In closing we suggest that this brief historical overview of the contributions of distance functions to topology does establish distance functions as valuable tools in this field and that Urysohn may have been pre-mature with his 1926 statement about the less significant role that distances would likely play in the future history in topological investigations.

Part II

A Contribution to the Theory of Distance

Chapter 3

Basic facts about distance functions and topology

3.1 Introduction

A nonnegative real-valued function $d : X \times X \rightarrow \mathbf{R}$ is a *distance function*, or a *distance*, for X if and only if $d(p,p) = 0$ for every $p \in X$. For any distance function d, if $S_d(p,\varepsilon) = \{x \in X \mid d(p,x) < \varepsilon\}$, the sphere centered at p of radius ε, then

$$\mathcal{T}_d = \{A \subseteq X \mid \text{for every } p \in A, \text{ there exists } \varepsilon > 0 \text{ such that } S_d(p,\varepsilon) \subseteq A\}$$

is a topology for X. We say that d is *a distance function for* $(X, \mathcal{T})$ iff $\mathcal{T} = \mathcal{T}_d$.

A distance function d for X is an *asymmetric* if and only if

$$d(p,q) = 0 \text{ iff } p = q \text{ for every } p, q \in X.$$

A topological space $(X, \mathcal{T})$ is *asymmetrizable* iff there is an asymmetric d for X such that $\mathcal{T} = \mathcal{T}_d$; d is then said to be an *asymmetric for* $(X, \mathcal{T})$.

By historical convention a distance function might be called a *pseudo*-asymmetric; however, a pseudo-asymmetric for a T_1-space must be an asymmetric. It becomes important to distinguish between spaces that are T_1 and other spaces, as demonstrated with the following result.

Lemma 3.1 *For any distance d*

(1) $(X, \mathcal{T}_d)$ *is a* T_1*-space iff d is an asymmetric, and,*

(2) *if* $(X, \mathcal{T}_d)$ *is a* T_0*-space, then d is separating (i.e., if* $d(p,q) = d(q,p) = 0$, *then* $p = q$*).* *The converse holds if* $x_n \to p \Leftrightarrow d(p, x_n) \to 0$.

Historically, distance functions were introduced with the intention that the distance convergence would coincide with the topological convergence in $(X, \mathcal{T}_d)$, that is, $d(p, x_n) \to 0$ iff $x_n \to p$ in $(X, \mathcal{T}_d)$. We remark that:

it is always the case that

$$\text{if } d(p, x_n) \to 0 \text{ then } x_n \to p \text{ in } (X, \mathcal{T}_d),$$

but the converse may fail; see Example 3.11.

In general, if $\{S_d(p,\varepsilon) | \varepsilon > 0\}$ is a neighborhood base in $(X, \mathcal{T})$ for each p, then $\mathcal{T} = \mathcal{T}_d$. However, $S_d(p,\varepsilon)$ is not always a neighborhood of p in $(X, \mathcal{T}_d)$. We say that a topological space $(X, \mathcal{T})$ is *asemimetrizable* if and only if there is an asymmetric d defined on X such that for every $p \in X$, $\{S_d(p,\varepsilon) | \varepsilon > 0\}$ is a neighborhood base for p in $(X, \mathcal{T})$; in this case we say that d is an *asemimetric for* $(X, \mathcal{T})$.

Since this investigation seeks to draw distinctions between asymmetrizability and asemimetrizability, we establish the following properties.

Lemma 3.2 *A space $(X, \mathcal{T})$ is pseudo-asemimetrizable iff it is first countable.*

Proof: An asemimetrizable space is first countable, since $\{S_d(p, 1/2^n) | n \in \mathbf{N}\}$ is a countable neighborhood base for p. Conversely, if $\{U_n(p) | n \in \mathbf{N}\}$ is a decreasing local base for p in $(X, \mathcal{T})$, then the distance d for X such that [1]:

$$d(p,q) = \begin{cases} 1/2^n, & \text{if } n = \min\{k | q \notin U_k(p)\}; \\ 0, & \text{if } q \in U_n(p) \text{ for every } k, \end{cases}$$

is a pseudo-asemimetric for $(X, \mathcal{T})$ such that $S_d(p, 1/2^n) = U_n(p)$.

Lemma 3.3 [SR] *For any distance function d the following are equivalent:*

(1) *For every p, $\{S_d(p, \varepsilon) | \varepsilon > 0\}$ is a neighborhood base for p in $(X, \mathcal{T})$;*

(2) *for every $\varepsilon > 0$ and every $p \in X$,*

$$\{x \in X | \exists \alpha > 0 \text{ such that } S_d(x, \alpha) \subseteq S_d(p, \varepsilon)\} \in \mathcal{T} \text{ and } \mathcal{T} \subseteq \mathcal{T}_d;$$

(3) $(X, \mathcal{T})$ *is first countable and $x_n \to p$ (in $(X, \mathcal{T})$)* iff $d(p, x_n) \to 0$;

(4) $(X, \mathcal{T})$ *is Fréchet and $x_n \to p$ (in $(X, \mathcal{T})$)* iff $d(p, x_n) \to 0$.

This lemma points out the important role played by first countable spaces in these considerations. One should remark that in first countable spaces, Hausdorff spaces are identical with the larger class of spaces with unique (sequential) limits, where a space $(X, \mathcal{T})$ *has unique sequential limits* iff when $x_n \to p$ and $y_n \to q$ then $p = q$. For non-Hausdorff spaces,

[1] A construction similar to this can be found in Hildebrandt's 1912 contribution [Hi].

the spaces with unique (sequential) limits take on additional significance as demonstrated in the next two lemmas.

Lemma 3.4 [SR] *For any asymmetric d, the following are equivalent:*

(1) *d has unique limits;*

(2) *if $p \neq q$, then $\exists \varepsilon > 0$, such that $S_d(p, \varepsilon) \cap S_d(q, \varepsilon) = \emptyset$;*

(3) *countably compact subsets of $(X, \mathcal{T}_d)$ are closed;*

(4) *$(X, \mathcal{T}_d)$ has unique limits.*

Lemma 3.5 *If d is a distance for a space $(X, \mathcal{T})$ with unique limits, then*

$$d \text{ is an asymmetric for}(X, \mathcal{T}) \text{ and } x_n \to p \Leftrightarrow d(p, x_n) \to 0$$

Proof: If $(X, \mathcal{T})$ has unique sequential limits, then $(X, \mathcal{T})$ is a T_1-space and, therefore, d is an asymmetric. Since $d(p, x_n) \to 0 \Rightarrow x_n \to p$, it is enough to show that if $x_n \to p$ then $d(p, x_n) \to 0$.

Suppose $x_n \to p$ and assume that there is $\varepsilon > 0$ such that $d(p, x_n) > \varepsilon$ for all n; otherwise choose a subsequence with this property. Let $A = X \setminus \{x_n | n \in \mathbb{N}\}$. Then, since $(X, \mathcal{T}_d)$ has unique limits, for every $q \in A$, there is $\delta > 0$ such that $S_d(q, \delta) \subseteq A$. Thus, $A \in \mathcal{T}_d$ so that $\{x_n | n \in \mathbb{N}\}$ is closed, contradicting that $x_n \to p$ which is A.

3.2 Lattice properties of topologies generated by distance functions.

Of interest in our study are the relationships between an existing topology and topologies generated by distance functions, as well as the possible connections between topologies generated by different distance functions defined on the same set.

Lemma 3.6 *For a space $(X, \mathcal{T})$, if d is a distance for X, then*

$$\mathcal{T} \subseteq \mathcal{T}_d \text{ iff, } x_n \to p \text{ in } (X, \mathcal{T}) \text{ whenever } d(p, x_n) \to 0.$$

Proof: Suppose that $\mathcal{T} \subseteq \mathcal{T}_d$ and that $d(p, x_n) \to 0$. Let $p \in G$, $G \in \mathcal{T}$. Since $G \in \mathcal{T}_d$, there is an $\varepsilon > 0$ such that $S_d(p, \varepsilon) \subseteq G$, and thus x_n is eventually in G.

Conversely, suppose that if $d(p, x_n) \to 0$ then $x_n \to p$ in $(X, \mathcal{T})$. Let $G \in \mathcal{T}$, but assume that $G \notin \mathcal{T}_d$; then there is a point p in G such that for every $n \in \mathbf{N}$, $S_d(p, 1/2^n) \nsubseteq G$. For each $n \in \mathbf{N}$, choose $x_n \in S_d(p, 1/2^n) \cap (X \setminus G)$. Then $d(p, x_n) \to 0$ so that $x_n \to p$ which contradictions that no x_n is in G.

A space $(X, \mathcal{T})$ is *sequential* iff, when $A \subseteq X$ contains the limit points of every convergent sequences in A, then A is closed. We note that for any distance function d, $(X, \mathcal{T}_d)$ is sequential, and that it is possible to extend Lemma 3.6 to the class of sequential topological spaces:

Lemma 3.7 *For a space $(X, \mathcal{T})$, if $\mathcal{T}_1$ is a sequential topology for X, then*

$$\mathcal{T} \subseteq \mathcal{T}_1 \text{ iff, } x_n \to p \text{ in } (X, \mathcal{T}) \text{ whenever } x_n \to p \text{ in } (X, \mathcal{T}_1).$$

Given distance functions d_1 and d_2 for X, $(d_1 \vee d_2)$, the *join* of d_1 and d_2, is given by:

$$(d_1 \vee d_2)(p,q) = \max\{d_1(p,q), d_2(p,q)\};$$

similarly, one defines $(d_1 \wedge d_2)$, the *meet* of d_1 and d_2, by:

$$(d_1 \wedge d_2)(p,q) = \min\{d_1(p,q), d_2(p,q)\}.$$

Then $S_{d_1 \vee d_2}(p,\varepsilon) = S_{d_1}(p,\varepsilon) \cap S_{d_2}(p,\varepsilon)$, and $S_{d_1 \wedge d_2}(p,\varepsilon) = S_{d_1}(p,\varepsilon) \cup S_{d_2}(p,\varepsilon)$.

The next results provide basic tools for our study.

Lemma 3.8 *For any distance functions d_1 and d_2 for X:*

$$\begin{aligned} d_1 \le d_2 \quad &\Rightarrow \quad \textit{if } d_2(p,x_n)\to 0, \textit{ then } d_1(p,x_n)\to 0; \\ &\Leftrightarrow \quad d_2(p,x_n)\to 0 \textit{ iff } (d_1 \vee d_2)(p,x_n)\to 0; \\ &\Rightarrow \quad \mathcal{T}_{d_1 \vee d_2} = \mathcal{T}_{d_2}; \\ &\Rightarrow \quad \mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}. \end{aligned}$$

Lemma 3.9 *For any distance functions d_1 and d_2 for X:*

$$\begin{aligned} \textit{if } d_2(p,x_n)\to 0, \textit{ then } d_1(p,x_n)\to 0 \quad &\Leftrightarrow \quad d_1(p,x_n)\to 0 \textit{ iff } (d_1 \wedge d_2)(p,x_n)\to 0; \\ &\Rightarrow \quad \mathcal{T}_{d_1 \wedge d_2} = \mathcal{T}_{d_1} \\ &\Leftrightarrow \quad \mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2} \end{aligned}$$

The relationship among the topologies of interest is summarized in the following theorem. In the lattice of topologies for X, we denote by $(\mathcal{T}_1 \vee \mathcal{T}_2)$ the least topology containing both $\mathcal{T}_1$ and $\mathcal{T}_2$.

Theorem 3.10 *If d_1 and d_2 are distance functions for X, then for $i = 1, 2$,*

$$\mathcal{T}_{d_1 \wedge d_2} = (\mathcal{T}_{d_1} \cap \mathcal{T}_{d_2}) \subseteq \mathcal{T}_{d_i} \subseteq (\mathcal{T}_{d_1} \vee \mathcal{T}_{d_2}) \subseteq \mathcal{T}_{d_1 \vee d_2}.$$

Proof: First note that $(d_1 \wedge d_2) \leq d_i \leq (d_1 \vee d_2)$ for $i = 1, 2$ so that

$$\mathcal{T}_{(d_1 \wedge d_2)} \subseteq \mathcal{T}_{d_i} \subseteq \mathcal{T}_{(d_1 \vee d_2)}.$$

Then $(\mathcal{T}_{d_1} \vee \mathcal{T}_{d_2}) \subseteq \mathcal{T}_{(d_1 \vee d_2)}$ by the lattice property of join. Also, $\mathcal{T}_{d_1 \wedge d_2} \subseteq (\mathcal{T}_{d_1} \cap \mathcal{T}_{d_2})$.

Finally, suppose $U \in (\mathcal{T}_{d_1} \cap \mathcal{T}_{d_2})$. Then $\forall p \in U$, $\exists \varepsilon_1$ and ε_2, such that $S_{d_1}(p, \varepsilon_1) \subseteq U$ and $S_{d_2}(p, \varepsilon_2) \subseteq U$. If $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then $S_{d_1 \wedge d_2}(p, \varepsilon) \subseteq U$, so that $U \in \mathcal{T}_{d_1 \wedge d_2}$.

Example 3.11 *$(\mathcal{T}_{d_1} \vee \mathcal{T}_{d_2})$ need not be $\mathcal{T}_{d_1 \vee d_2}$.*

Let $X = \{0\} \cup A \cup B$, where $A = \{1/3^n | n \in \mathbf{N}\}$ and $B = \{2/3^n | n \in \mathbf{N}\}$. There is a distance d_1 for X such that for $p \neq q$:

$$d_1(q,p) = d_1(p,q) = \begin{cases} q, & \text{if } p = 0 \text{ and } q \in A; \\ 1, & \text{if } p = 0 \text{ and } q \in B, \text{ or } p \in B \text{ and } q \in B; \\ \min\{p,q\}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}_{d_1}$ is the finite complement topology for X.

Similarly, there is a distance d_2 for X such that for $p \neq q$:

$$d_2(q,p) = d_2(p,q) = \begin{cases} q, & \text{if } p = 0 \text{ and } q \in B; \\ 1, & \text{if } p = 0 \text{ and } q \in A, \text{ or } p \in A \text{ and } q \in A; \\ \min\{p,q\}, & \text{otherwise.} \end{cases}$$

Again, $\mathcal{T}_{d_2}$ is the finite complement topology for X and, therefore, $(\mathcal{T}_{d_1} \vee \mathcal{T}_{d_2})$ is the finite complement topology.

On the other hand, we observe that $S_{d_1 \vee d_2}(p, \varepsilon) = \{0\}$, when $0 < \varepsilon < 1$. So, $\{0\}$ is in $\mathcal{T}_{d_1 \vee d_2}$, but it is not in the finite complement topology.

This example also shows that the topological convergence in $(X, \mathcal{T}_d)$ might differ from the distance convergence given by d. We note that the $\langle 2/3^n \rangle$ converges to 0 in $(X, \mathcal{T}_{d_1})$, while $d_1(0, 2/3^n) = 1$ for every $n \in \mathbb{N}$.

3.3 Symmetric properties for a distance

A distance d is *symmetric* iff $d(p,q) = d(q,p)$ for every $p, q \in X$. Keeping with the usual terminology, we will say that d is a *symmetric* iff it is a symmetric asymmetric. A topological space $(X, \mathcal{T})$ is *symmetrizable* [Ar] iff there is a symmetric d on X such that $\mathcal{T} = \mathcal{T}_d$. In this case, we say that d is a *symmetric for* $(X, \mathcal{T})$. A space $(X, \mathcal{T})$ is *semimetrizable* iff there is a symmetric asemimetric d for $(X, \mathcal{T})$, in which case d is a *semimetric for* $(X, \mathcal{T})$.

We will use the same terminology for other kinds of distance functions. For example, a distance function d is *locally symmetric* [CR$_2$, 1984] (or *has property* (s_1) [Ne, 1971]) iff

$$\text{for any } p \in X, \text{ if } d(p, x_n) \to 0, \text{ then } d(x_n, p) \to 0.$$

Hence, a topological space $(X, \mathcal{T})$ is *locally symmetrizable* iff there is a locally symmetric asymmetric d for $(X, \mathcal{T})$; we will in this case refer to d as a *local symmetric* for $(X, \mathcal{T})$. Similarly, we will say that $(X, \mathcal{T})$ is *locally semimetrizable* iff there is a local semimetric for $(X, \mathcal{T})$; again by *local semimetric* we mean a locally symmetric asemimetric.

Given any distance function d for X, it is possible to define on the same set another distance d^* for X by:

$$d^*(p,q) = d(q,p) \text{ for every } p,q \in X.$$

Using Kopperman's terminology [Kop$_2$, 1995], we refer to d^* as the *dual* of d. It follows immediately that $(d \vee d^*)$ and $(d \wedge d^*)$ are symmetric distances for X. Applying the results of the previous section, we get the following:

Lemma 3.12 *If d is a distance then:*

(1) *If d is locally symmetric, then $\mathcal{T}_{d^*} \subseteq \mathcal{T}_d$ and $\mathcal{T}_d = \mathcal{T}_{d \vee d^*}$.*

In fact, if $\mathcal{T}_d = \mathcal{T}_{d \vee d^*}$, then $\mathcal{T}_{d^*} \subseteq \mathcal{T}_d$.

(2) *If $d^*(p, x_n) \to 0 \Rightarrow d(p, x_n) \to 0$, then $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$ and $\mathcal{T}_d = \mathcal{T}_{d \wedge d^*}$.*

Actually, $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$ iff $\mathcal{T}_d = \mathcal{T}_{d \wedge d^*}$.

Lemma 3.13 *$(X, \mathcal{T})$ is symmetrizable iff there is a distance d for $(X, \mathcal{T})$ such that*

$$\textit{for any } p \in X, \textit{ if } d^*(p, x_n) \to 0 \textit{ then } x_n \to p \textit{ in } (X, \mathcal{T}).$$

Proof: Assume that d is a distance function with the given property. Then it follows from Lemma 3.6 that $\mathcal{T} = \mathcal{T}_d \subseteq \mathcal{T}_{d^*}$. Now as a result of Lemma 3.12, $(d \wedge d^*)$ is a symmetric for $(X, \mathcal{T})$. The converse is obvious.

Corollary 3.14 *There is a symmetric distance for* $(X, \mathcal{T})$ *iff there is a distance d for* $(X, \mathcal{T})$ *such that for any* $p \in X$ *if* $d^*(p, x_n) \to 0$ *then* $d(p, x_n) \to 0$.

Recall, that first countable spaces are pseudo-semimetrizable; therefore, we have:

$(X, \mathcal{T})$ is pseudo-semimetrizable iff $(X, \mathcal{T})$ is first countable and asymmetrizable.

We have from Lemmas 3.5 and 3.4.

Corollary 3.15 *If* $(X, \mathcal{T})$ *has unique limits, then*

$(X, \mathcal{T})$ *is semimetrizable iff* $(X, \mathcal{T})$ *is first countable and symmetrizable.*

Semimetrizable spaces were studied extensively in the 1960's and 1970's. Still, the question of whether Corollary 3.15 holds in general remains an open question. We note the following generalization of a theorem of Heath [He_1, 1962].

Theorem 3.16 $(X, \mathcal{T})$ *is pseudo-semimetrizable iff there is* $\{U_n(x) | x \in X, n \in \mathbf{N}\} \subseteq \mathcal{T}$ *such that for every p,*

(1) *if* $x_n \in U_n(p)$ *for every n, then* $x_n \to p$ *and*

(2) *if* $p \in U_n(x_n)$ *for every n, then* $x_n \to p$.

Proof: If d is a pseudo-semimetric for $(X, \mathcal{T})$ and $U_n(p)$ is the interior of $S_d(p, 1/2^n)$, then $\{U_n(x) | x \in X, n \in \mathbf{N}\}$ has the needed properties.

Conversely, we may assume $\{U_n(x) | x \in X, n \in \mathbf{N}\}$ has the additional property that

$U_{n+1}(p) \subseteq U_n(p)$. Then the distance d such that

$$d(p,q) = \begin{cases} 1/2^n, & \text{if } n = \min\{k | q \notin U_k(p)\} \\ 0, & \text{if } q \in U_n(p) \text{ for every } k \end{cases};$$

is a pseudo-asemimetric as in Lemma 3.2. Moreover, $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$ so that $(d \wedge d^*)$ is a pseudosemimetric for $(X, \mathcal{T})$.

Following Sabella [Sa$_1$, 1973], we say that $\{U_n(x) | x \in X, n \in \mathbf{N}\}$ is an *open neighborhood assignment* in $(X, \mathcal{T})$ iff

$$p \in U_n(p) \in \mathcal{T} \text{ for every } p \in X \text{ and every } n \in \mathbf{N}.$$

A space $(X, \mathcal{T})$ is *semistratifiable* iff there is $\{U_n(x) | x \in X, n \in \mathbf{N}\}$, a open neighborhood assignment in $(X, \mathcal{T})$, such that

$$\text{for every } p, \text{ if } p \in U_n(x_n) \text{ for every } n, \text{ then } x_n \to p.$$

Thus, Theorem 3.16 can be restated as:

Corollary 3.17 *$(X, \mathcal{T})$ is pseudo-semimetrizable iff $(X, \mathcal{T})$ is first countable and semistratifiable.*

Open questions:

1. If $\mathcal{T}_{d^*} \subseteq \mathcal{T}_d$, is $\mathcal{T}_d = \mathcal{T}_{d \vee d^*}$? See Lemma 3.12.

2. Is every first countable symmetrizable space semimetrizable? See Corollary 3.15.

Chapter 4

Local properties of the triangle for a distance

4.1 Introduction

A distance function d for X satisfies *the triangle inequality* iff for any p and q in X,

$$d(p,q) \leq d(p,x) + d(x,q) \text{ for every } x \text{ in } X;$$

d is a *quasimetric* for X iff d is an asymmetric that satisfies the triangle inequality. Although initially considered by Fréchet, the geometric nature of the triangle inequality prompted other mathematicians to consider properties that might prove more appealing in abstract spaces. Niemytzki in [Ni_1, 1927], while extending the results of Pitcher and Chittenden [PC, 1918], introduced a property he called *the local axiom of the triangle*:

for every $\varepsilon > 0$ and every point p

$$\text{there is } \delta > 0 \text{ such that, if } d(p,x) < \delta \text{ and } d(x,y) < \delta, \text{ then } d(p,y) < \varepsilon.$$

Working in the context of symmetric distance functions, Pitcher and Chittenden [PC, 1918] had introduced an equivalent condition which they called *coherent* and defined as

$$\text{for any } p \in X, \text{ when } d(p, y_n) \to 0 \text{ and } d(y_n, x_n) \to 0, \text{ then } d(p, x_n) \to 0.$$

If we focus our attention on the coherent distance functions in the absence of symmetry, it becomes important to distinguish between the different types of convergence as $d(p, x_n) \to 0$ and $d(x_n, p) \to 0$ might now be different. In fact, we can generate four different non-symmetric conditions corresponding to the coherent distances of Pitcher and Chittenden.

If d is a distance for X, then:

d is a *γ-distance*[1], if for any $p \in X$,

$$\text{when } d(p, y_n) \to 0 \text{ and } d(y_n, x_n) \to 0, \text{ then } d(p, x_n) \to 0;$$

d is a *γ^*-distance*, if for any $p \in X$,

$$\text{when } d(y_n, p) \to 0 \text{ and } d(x_n, y_n) \to 0, \text{ then } d(p, x_n) \to 0;$$

d is a *Π_1-distance* [Ne], if for any $p \in X$,

$$\text{when } d(p, y_n) \to 0 \text{ and } d(x_n, y_n) \to 0, \text{ then } d(p, x_n) \to 0;$$

---

[1]This property was introduced by Kofner[Ko_2] using the terminology *γ-metric*.

d is a Π_2-***distance*** [Ne], if for any $p \in X$,

$$\text{when } d(y_n, p) \to 0 \text{ and } d(y_n, x_n) \to 0, \text{ then } d(p, x_n) \to 0.$$

Note that for each of these distances there is an equivalent "***epsilon-delta***" definition. For example, d is a γ^*-distance, if for every $\varepsilon > 0$ and every point p,

$$\text{there is } \delta > 0 \text{ such that, if } d(y, p) < \delta \text{ and } d(x, y) < \delta, \text{ then } d(p, x) < \varepsilon.$$

In this chapter, we will study the topological spaces generated by γ-distance functions, γ^*-distance functions and Π_1-distance functions. We will turn our attention to the Π_2-distance functions in the next chapter with our investigation of developable spaces.

4.2 γ-distance functions

We begin our study by observing that γ-distances are precisely those distances that satisfy the local axiom of the triangle introduced by Niemytzki so that every pseudo-quasimetric is a γ-distance.

It is a standard exercise to prove that the spheres $S_d(p, \varepsilon)$ generated by a pseudo-quasimetric are open in $(X, \mathcal{T}_d)$. This may fail for a γ-symmetric and thus also for γ-distances. However, if d is a γ-distance, for any $S_d(p, \varepsilon)$ there is $\delta > 0$ such that for any $x \in S_d(p, \delta)$, there exists $\alpha > 0$ with $S_d(x, \alpha) \subseteq S_d(p, \varepsilon)$. Hence, $S_d(p, \varepsilon)$ is a neighborhood of p in $(X, \mathcal{T}_d)$, and so $\{S_d(p, \varepsilon) | \varepsilon > 0\}$ is a neighborhood base for p in $(X, \mathcal{T}_d)$. Thus, by applying Lemma 3.3 we observe the following:

Lemma 4.1 *If d is a γ-distance, then*

(1) *$(X, \mathcal{T}_d)$ is first countable and*

(2) *d is a distance for $(X, \mathcal{T})$ iff d is a pseudo-asemimetric for $(X, \mathcal{T})$.*

Corollary 4.2 *For a space $(X, \mathcal{T})$,*

(1) *$(X, \mathcal{T})$ is γ-asymmetrizable iff $(X, \mathcal{T})$ is γ-asemimetrizable;*

(2) *$(X, \mathcal{T})$ is γ-symmetrizable iff $(X, \mathcal{T})$ is γ-semimetrizable.*

Although γ-distances generate first countable topologies in which the topological convergence coincides with the distance convergence, the spaces generated this way need not have unique limits. As the next example shows this holds even for non-archimedean quasimetrics. Recall that d is a *non-archimedean quasimetric* if and only if

$$d(p, q) \leq \max\{d(p, x), d(x, q)\} \text{ for every } x \in X.$$

Example 4.3 *A non-archimedean quasimetrizable space need not have unique limits.*

Let $X = (0,1)$, and let d be the asymmetric for X such that:

$$d(p,q) = \begin{cases} q, & \text{if } p > q; \\ 1, & \text{if } p < q. \end{cases}$$

Then, d is a non-archimedean quasimetric for $(X, \mathcal{T}_d)$ that does not have unique limits. The sequence $\langle 1/2^n \rangle$ in X converges to every point p in X.

Lemma 4.4 *If d is a locally symmetric γ-distance for $(X, \mathcal{T})$, then $(d \vee d^*)$ is a symmetric γ-distance for $(X, \mathcal{T})$.*[2]

Proof: Since d is a locally symmetric distance for $(X, \mathcal{T})$, then by Lemma 3.12 $(d \vee d^*)$ is a symmetric for $(X, \mathcal{T})$. We must now establish that $(d \vee d^*)$ is also a γ-distance.

Suppose that $(d \vee d^*)(p, y_n) \to 0$ and $(d \vee d^*)(y_n, x_n) \to 0$, then $d(p, y_n) \to 0$ and $d(y_n, x_n) \to 0$ and thus $d(p, x_n) \to 0$. Since d is locally symmetric $d^*(p, x_n) \to 0$ and thus $(d \vee d^*)(p, x_n) \to 0$.

A well known theorem due to Arhangel'skiĭ [Ar] states that:

> a Hausdorff space $(X, \mathcal{T})$ is metrizable iff there is a symmetric d for $(X, \mathcal{T})$ such that $d[F_1, F_2] > 0$ for any disjoint pair of closed sets, one of which is compact,

where $d[F_1, F_2]$, the distance between the sets F_1 and F_2, is defined by,

$$d[F_1, F_2] = \inf\{d(p,q) \mid p \in F_1,\ q \in F_2\}.$$

---

[2]This is, in fact, a pseudometrizability result. See Theorem 6.3.

We will obtain a similar characterization for γ-spaces, which, interestingly, involves the dual distance.

Another metrization theorem in this context is the "uniform separation" theorem of Kenton [Ke] and Harley–Faulkner [HF]:

> a Hausdorff space $(X, \mathcal{T})$ is metrizable iff there is a symmetric d for $(X, \mathcal{T})$ such that, when p is not in a closed set F in $(X, \mathcal{T})$, then there is $\epsilon > 0$ such that $S_d(p, \varepsilon) \cap S_d[F, \varepsilon) \neq \emptyset$,

where

$$S_d[F, \varepsilon) = \bigcup \{S_d(x, \varepsilon) | x \in F\}.$$

Again we find analogous separations for each of the distances with a "local property of the triangle." In these cases we want to introduce distances that we call *weak*. For example, we say that a distance d is a *weak γ-distance* for $(X, \mathcal{T})$ if, for any $p \in X$,

$$\text{when } d(p, y_n) \to 0 \text{ and } d(y_n, x_n) \to 0, \text{ then } x_n \to p \text{ in } (X, \mathcal{T}).$$

Lemma 4.5 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is a weak γ-distance for $(X, \mathcal{T})$;*

(2) *when F is a closed set in $(X, \mathcal{T})$ and p is not in F,*

$$\textit{there is an } \varepsilon > 0 \textit{ such that } S_d(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) = \emptyset.$$

Proof: Suppose that d is a weak γ-distance for $(X, \mathcal{T})$ and that for every $\varepsilon > 0$, $S_d(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) \neq \emptyset$ for some closed set F in $(X, \mathcal{T})$. Then for every $n \in \mathbf{N}$, there is a $y_n \in$

$S_d(p, 1/2^n) \cap S_{d^*}(x_n, 1/2^n)$ for some $x_n \in F$. Since d is a weak γ-distance for $(X, \mathcal{T})$ and F is closed, $p \in F$.

Conversely, assume that $d(p, y_n) \rightarrow 0$ and that $d(y_n, x_n) \rightarrow 0$. Suppose $p \in G$ and G is open. Then $S_d(p, \varepsilon) \cap S_{d^*}[X \setminus G, \varepsilon) = \emptyset$ for some $\varepsilon > 0$. Since $d(p, y_n) \rightarrow 0$, there exist $k \in \mathbf{N}$ such that for $n \geq k$, $y_n \in S_d(p, \varepsilon)$ and $d(y_n, x_n) < \varepsilon$, and thus $x_n \in G$.

Lemma 4.6 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *if $d(y_n, x_n) \rightarrow 0$ and $y_n \rightarrow p$, then $x_n \rightarrow p$;*

(2) *for a sequentially compact set K and a closed set F, if $K \cap F = \emptyset$ then $d[K, F] > 0$.*

Proof: Assume that if $y_n \rightarrow p$ and $d(y_n, x_n) \rightarrow 0$ then $x_n \rightarrow p$. Suppose that F is closed and K is sequentially compact with $d[K, F] = 0$. Then there is a sequence $\langle x_n \rangle$ in F and a sequence $\langle y_n \rangle$ in K with $d(y_n, x_n) \rightarrow 0$ and thus there is a subsequence $\langle y_{k_n} \rangle$ of $\langle y_n \rangle$ and $p \in K$ with $y_{k_n} \rightarrow p$. Since $d(y_{k_n}, x_{k_n}) \rightarrow 0$, $p \in F$ and thus F intersects K.

Conversely, assume (2) and that $y_n \rightarrow p$ and $d(y_n, x_n) \rightarrow 0$. Let G be an open set containing p. Then $K = \{p\} \cup \{y_n | y_n \in G\}$ is sequentially compact. Hence, $d[K, X \setminus G] > 0$ and, since $d(y_n, x_n) \rightarrow 0$, x_n is eventually in G.

Lemma 4.7 *If d is a distance for $(X, \mathcal{T})$, such that $d[K, F] > 0$ when F is a closed set and K is a disjoint sequentially compact set, then d a weak γ-distance for $(X, \mathcal{T})$.*

Proof: The implication follows from the conditions (1) of Lemma 4.5 and 4.6.

We can now establish our main result.

Theorem 4.8 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is a γ-distance for $(X, \mathcal{T})$;*

(2) (a) $x_n \to p$ *in* $(X, \mathcal{T})$ *implies that* $d(p, x_n) \to 0$ *for any* $p \in X$, *and*

(b) $d[K, F] > 0$, *for any closed set* F *and disjoint sequentially compact set* K;

(3) (a) $x_n \to p$ *in* $(X, \mathcal{T})$ *implies that* $d(p, x_n) \to 0$ *for any* $p \in X$, *and*

(b) *if* $p \notin F$, *a closed set in* $(X, \mathcal{T})$, *then* $S_d(p, \epsilon) \cap S_{d^*}[F, \epsilon) = \emptyset$ *for some* $\epsilon > 0$.

Proof: Each of the conditions (1), (2) and (3) implies that $d(p, x_n) \to 0 \Leftrightarrow x_n \to p$. From this the equivalence follows from Lemmas 4.5 and 4.6.

Since spaces generated by γ-distances are first countable, we also get the following theorem.

Theorem 4.9 *If* d *is a distance for* $(X, \mathcal{T})$, *then the following are equivalent:*

(1) d *is a* γ*-distance for* $(X, \mathcal{T})$;

(2) $x_n \to p$ *in* $(X, \mathcal{T})$ *implies that* $d(p, x_n) \to 0$ *for any* $p \in X$, *and* $d[K, F] > 0$ *for any closed set* F *and disjoint compact set* K *in* $(X, \mathcal{T})$;

Proof: (1⇒2) Since d is a γ-distance for $(X, \mathcal{T})$, it follows that $d(p, x_n) \to 0$ iff $x_n \to p$ in $(X, \mathcal{T})$, and that $(X, \mathcal{T})$ is first countable. Thus, being compact, K is sequentially compact.

Conversely, assume (2) and suppose that $p \in X$. Assume further that F is a closed set and that, for every $n \in \mathbb{N}$, there is $x_n \in F$ and $y_n \in S_d(p, 1/2^n) \cap S_{d^*}(x_n, 1/2^n)$. Then $y_n \to p$ and thus $K = \{p\} \cup \{y_n | y_n \in X \setminus F\}$ is compact. Since $d^*(x_n, y_n) \to 0$, $d[K, F] = 0$, and thus $p \in F$. By applying the preceding theorem we obtain the desired conclusion.

Example 4.10 *A distance* d *may be a weak* γ*-distance for* $(X, \mathcal{T})$ *without being a* γ*-distance for* $(X, \mathcal{T})$.

Let $X = \{0\} \cup A \cup B$, where $A = \{1/3^n | n \in \mathbf{N}\}$ and $B = \{2/3^n | n \in \mathbf{N}\}$. Let d be the distance for X such that:

$$d(p,q) = \begin{cases} 1, & \text{if } p = 0 \text{ and } q \in B, \text{ or } p \in B \text{ and } q = 0, \\ & \text{or } p, q \in B, \text{ or } p \in B \text{ and } q \in A; \\ p, & \text{if } p \in A \text{ and } q = 0; \\ q, & \text{if } p = 0 \text{ and } q \in A, \text{ or } p \in A \text{ and } q \in B; \\ |p-q|, & \text{if } p, q \in A. \end{cases}$$

A direct computation shows that d is a weak γ-distance for $(X, \mathcal{T}_d)$. But, d is not a γ-distance, since

$$d(0, 1/3^n) \to 0 \text{ and } d(1/3^n, 2/3^n) \to 0, \text{ while } d(0, 2/3^n) = 1 \text{ for every } n \in \mathbf{N}.$$

Spaces defined by distance functions are often related to spaces defined by sequences of open covers; see, for example: [He$_1$, 1962], [Ho$_1$, 1972], [Gr, 1984]. We now turn our attention to such spaces and how they relate to the spaces defined by γ-distances.

Generalizing Hodel's definition for Hausdorff spaces [Ho$_1$], we define a space $(X, \mathcal{T})$ to be a *γ-space* iff there is an open neighborhood assignment $\{U_n(x) | x \in X, n \in \mathbf{N}\}$ in $(X, \mathcal{T})$ such that

$$\text{for every } p, \text{ if } x_n \in U_n(y_n) \text{ and } y_n \in U_n(p) \text{ for every } n, \text{ then } x_n \to p.$$

Such an open neighborhood assignment will be called a *γ-neighborhood assignment for* $(X, \mathcal{T})$. This next theorem explains our choice of terminology for γ-distances.

Theorem 4.11 *$(X, \mathcal{T})$ is a γ-space iff there is a γ-distance for $(X, \mathcal{T})$.*

Proof: Suppose that $\{U_n(x)|x \in X, n \in \mathbf{N}\}$ is γ-neighborhood assignment for $(X, \mathcal{T})$. Let

$$V_n(p) = \bigcap_{k=1}^{n} U_k(p);$$

then $\{V_n(x)|x \in X, n \in \mathbf{N}\}$ is also a γ-neighborhood assignment for $(X, \mathcal{T})$ and for every $p \in X$, $\{V_n(p)|n \in \mathbf{N}\}$ is an open neighborhood base for p in $(X, \mathcal{T})$. Define

$$d(p,q) = \begin{cases} 0 & \text{if } q \in V_n(p) \text{ for every } n; \\ 1/2^n & \text{where } n = \min\{k|q \notin V_k(p)\}\ . \end{cases}$$

It follows from our construction that $S_d(p, 1/2^n) = V_n(p)$, and thus d is a pseudo-asemimetric for $(X, \mathcal{T})$.

It is left to show that d is a γ-distance. Suppose that $d(p, y_n) \to 0$ and $d(y_n, x_n) \to 0$, but that $d(p, x_n) \not\to 0$. Without loss of generality, we can assume that for every $n \in \mathbf{N}$, $d(p, x_n) > 1/2^i$ for some i; otherwise, consider a subsequence with this property.

Since $d(y_n, x_n) \to 0$, for every $n \in \mathbf{N}$ there exists $k_n > n$ such that $d(y_{k_n}, x_{k_n}) < 1/2^n$; hence $d(p, y_{k_n}) \to 0$, $d(y_{k_n}, x_{k_n}) < 1/2^n$, while $d(p, x_{k_n}) > 1/2^i$.

Since $d(p, y_{k_n}) \to 0$, for every $n \in \mathbf{N}$ there exists $j_{k_n} > k_n$ such that $d(p, y_{j_{k_n}}) < 1/2^n$, and also $d(y_{j_{k_n}}, x_{j_{k_n}}) < 1/2^n$. Therefore $y_{j_{k_n}} \in V_n(p)$ and $x_{j_{k_n}} \in V_n(y_{j_{k_n}})$, which implies that $x_{j_{k_n}} \to p$. This is a contradiction, since for all n, $x_{j_{k_n}} \notin V_i(p)$.

Conversely, suppose that d is a γ-distance function for $(X, \mathcal{T})$, then by Lemma 4.1, $\{S_d(p, 1/2^n)|n \in \mathbf{N}\}$ is neighborhood base for p in $(X, \mathcal{T})$. If for each $n \in \mathbf{N}$ and each $p \in X$

$$U_n(p) = \text{int}_{\mathcal{T}}(S_d(p, 1/2^n)),$$

then $\{U_n(p)|p \in X \text{ and } n \in \mathbf{N}\}$ is an γ-neighborhood assignment for $(X, \mathcal{T})$.

We summarize the implications among the spaces that we have discussed with the following diagram. These properties of distance will form a basis for our subsequent studies of other distance functions.[3]

$$\left.\begin{array}{lcl} d \text{ is a } \gamma\text{-distance (with} & & d(y_n, x_n)\to 0, \\ & ;\ \text{or}, & \\ x_n\to p \text{ iff } d(p, x_n)\to 0) & & y_n\to p \end{array}\right\} \Rightarrow d(p, x_n)\to 0$$

$$\downarrow$$

$$\left.\begin{array}{l} d(y_n, x_n)\to 0, \\ y_n\to p \end{array}\right\} \Rightarrow x_n\to p$$

$$\downarrow$$

$$d \text{ is a weak } \gamma\text{-distance}$$

Corollary 4.12 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is a γ-distance;*

(2) *when $d(y_n, x_n) \to 0$ and $y_n \to p$, then $d(p, x_n) \to 0$.*

Proof: Either condition implies that the topological convergence and the distance convergence coincide. From this the equivalence of the conditions follows.

Open questions:

1. Do either of the converses hold? In particular, if there is a weak γ-distance for $(X, \mathcal{T})$, is there always a γ-distance for $(X, \mathcal{T})$?

[3] All implications follow from the convergence property: $d(p, x_n) \to 0 \Rightarrow x_n \to p$ in $(X, \mathcal{T})$. The equivalence in the first box is shown in Corollary 4.12.

4.3 γ^*-distance functions

We now turn our attention to γ^*-distances. Contrary to the situation with γ-distances, where the set of spheres centered at a point form a neighborhood base for that point in $(X, \mathcal{T}_d)$, and thus $x_n \to p$ iff $d(p, x_n) \to 0$, the situation is far more complicated for spaces determined by a γ^*-distance. In particular, $x_n \to p$ in $(X, \mathcal{T}_d)$ may not always imply that $d(p, x_n) \to 0$.[4] Hence, when studying topological spaces generated by γ^*-distances, we must consider a number of possibilities summarized in the following diagram. Each implication can be established by noting that $d(p, x_n) \to 0 \Rightarrow x_n \to p$ in $(X, \mathcal{T}_d)$.

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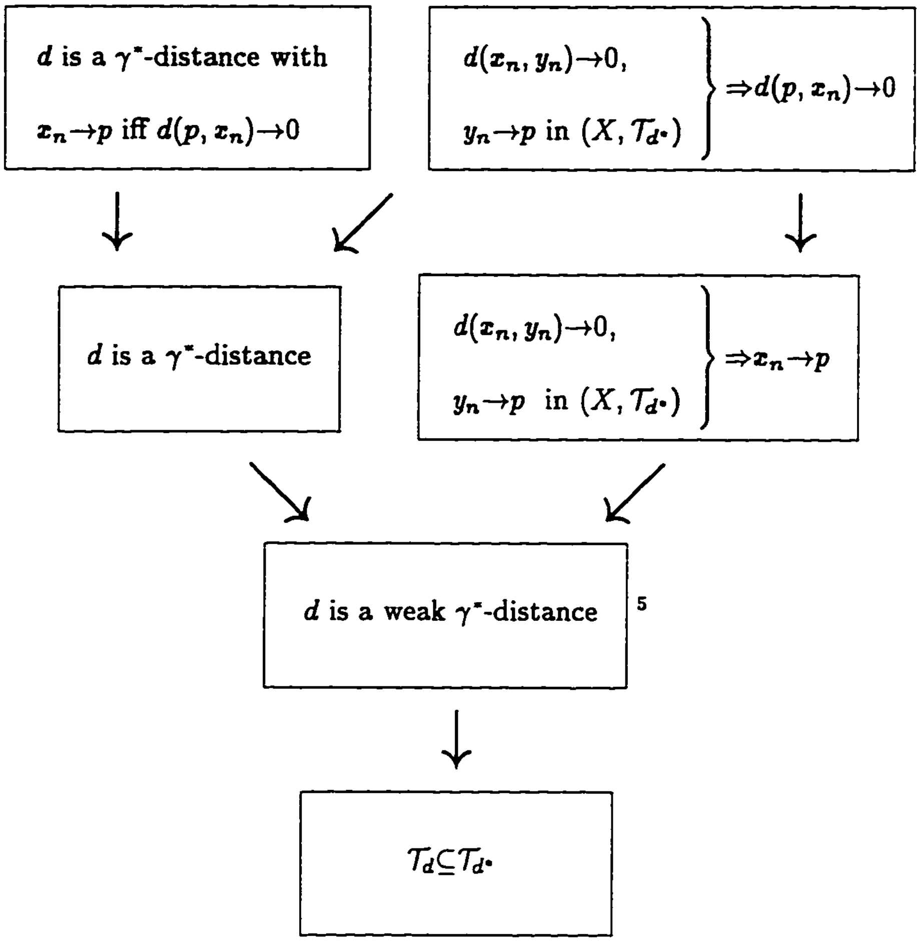
[4]See Example 4.15.



The following results are immediate.

Lemma 4.13 *If d is a γ^*-distance for $(X, \mathcal{T})$, then $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$ and thus $\mathcal{T}_d = \mathcal{T}_{d \wedge d^*}$.*

Proof: See Lemma 3.12.

Lemma 4.14 *If d is a locally symmetric, γ^*-distance for $(X, \mathcal{T})$, then $(d \vee d^*)$ is a symmetric γ-distance for $(X, \mathcal{T})$.*

---

[5] We say that a distance d for $(X, \mathcal{T})$ is a *weak γ^*-distance* for $(X, \mathcal{T})$ if,

for any $p \in X$, when $d(x_n, y_n) \to 0$ and $d(y_n, p) \to 0$, then $x_n \to p$ in $(X, \mathcal{T})$.

Proof: See Lemma 4.4 and note that $(d \vee d^*)$ is a symmetric γ-distance, if d is a γ^*-distance.

We also observe that, if d is a symmetric γ^*-distance for $(X, \mathcal{T})$, then as a result of symmetry it follows from Corollary 4.2 that d is a semimetric for $(X, \mathcal{T})$. Thus, $(X, \mathcal{T})$ is first countable, and $x_n \to p$ in $(X, \mathcal{T})$ iff $d(p, x_n) \to 0$. The following example shows that this need not be the case for non-symmetric γ^*-distances.

Example 4.15 *A Hausdorff space $(X, \mathcal{T})$ can be γ^*-asymmetrizable without being γ^*-asemimetrizable.*

Let $X = [0, 1]$ and $A = \{1/3^n | n \in \mathbf{N}\} \cup \{0\}$. Define a distance d for X as follows:

$$d(x, y) = \begin{cases} 1, & \text{if } y = 0; \text{ or } x = 0 \text{ and } y \notin A; \text{ or } x \notin A \text{ and } y \in A; \\ |x - y|, & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} = \mathcal{T}_d$, and note that d is a γ^*-distance.

Since $d \geq e$, where e is the Euclidean metric on X, it follows from Lemma 3.12 that the usual topology on X is contained in $\mathcal{T}_d$ so that $(X, \mathcal{T}_d)$ is Hausdorff. Therefore, d is an asymmetric with unique limits and (from Lemma 3.5) $x_n \to p$ in $(X, \mathcal{T}_d)$ iff $d(p, x_n) \to 0$.

However, any sphere $S_d(0, \epsilon)$ centered at 0 has empty interior and thus is not a neighborhood of 0. In fact, $(X, \mathcal{T})$ is not first countable at 0, and thus *there can be no asemimetric of any kind for* $(X, \mathcal{T})$. Therefore $(X, \mathcal{T})$ is not metrizable. However, since $\mathcal{T}_e = \mathcal{T}_d$ on $(0, 1]$, it follows that $(X, \mathcal{T})$ is Hausdorff, regular and Lindelöf, and thus paracompact.

This example can be attributed to Arhangel'skiĭ, as it appeared in a different form in [Ar, p.127], but we should also note that the essence of this example was introduced by Fréchet in 1918 [Fr_5, p.55–56].

Pursuing our study of γ^*-distances, we now turn our attention to the separation of closed sets and sequentially compact sets, modeling our investigation after the results of the previous section. The situation for spaces generated by γ^*-distance is more complicated than that for spaces generated by γ-distances. It now becomes important to differentiate between a γ^*-distance and a γ^*-asymmetric.

We begin our investigation by studying the situation for spaces generated by weak γ^*-distances.

Lemma 4.16 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d a weak γ^*-distance for $(X, \mathcal{T})$;*

(2) *for $p \notin F$, a closed set in $(X, \mathcal{T})$, there is an $\epsilon > 0$ with $S_{d^*}(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$.*

Proof: See the proof of Lemma 4.1, as the argument is similar.

Theorem 4.17 *If d is a distance for $(X, \mathcal{T})$ such that $x_n \to p$ iff $d(p, x_n) \to 0$, then the following are equivalent:*

(1) *d is a γ^*-distance;*

(2) *for $p \notin F$, F a closed set, there exists $\epsilon > 0$ such that $S_{d^*}(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$.*

If we compare these results to the results obtained for γ-distance, we observe that we do not obtain an equivalence between a γ^*-distance and a distance that separates closed and disjoint sequentially compact or compact sets. The problems lies in the fact that for γ^*-distances, we must also consider the difference between the topological convergence in $\mathcal{T}_{d^*}$ and the convergence with respect to d^*. These problems are resolved for spaces generated by γ^*-asymmetrics, since these spaces must be T_1-spaces.

Lemma 4.18 *If d is an asymmetric for X and K is a countably compact set in $(X, \mathcal{T}_d)$, then any sequence $\langle x_n \rangle$ in K has a subsequence $\langle x_{k_n} \rangle$ such that $d(p, x_{k_n}) \to 0$ for some $p \in X$.*

Proof: Suppose that d is an asymmetric for X and that K is countably compact in $(X, \mathcal{T}_d)$. Let $\langle x_n \rangle$ be a sequence in K, and suppose that for no subsequence $\langle x_{k_n} \rangle$ of $\langle x_n \rangle$, is there a $p \in X$ such that $d(p, x_{k_n}) \to 0$. Then for every $p \in X$, there exists ϵ_p and $M_p \in \mathbf{N}$ such that for every $n \geq M_p$, $x_n \notin S_d(p, \epsilon_p)$.

Define $F_j = \{x_n | n \geq j\}$. We claim that every F_j is closed.

> Suppose $x \notin F_j$, define $\alpha = \min(\{\epsilon_x\} \cup \{d(x, x_i) | i \leq \max\{j, M_x\}, x_i \neq x\})$. This is possible since $\mathcal{T}_d$ is a T_1 topology. Note that $S_d(x, \alpha) \subseteq (X \setminus F_j)$, and so F_j is closed.

Hence, $\{(X \setminus F_j) | j \in \mathbf{N}\}$ is an countable open cover of K with no finite subcover, which contradicts that K is countably compact.

Our next lemma will play the role of Lemma 4.7 in this context.

Lemma 4.19 *If d is an asymmetric for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *if $d(x_n, y_n) \to 0$ and $y_n \to p$ in $(X, \mathcal{T}_{d^*})$ then $x_n \to p$;*

(2) *for F a closed set in $(X, \mathcal{T})$ and K a closed sequentially compact set in $(X, \mathcal{T}_{d^*})$, if $F \cap K = \emptyset$, then $d[F, K] > 0$.*

Proof: Assume that $y_n \to p$ in $(X, \mathcal{T}_{d^*})$ and $d(y_n, x_n) \to 0$ implies that $x_n \to p$, suppose that F is closed in $(X, \mathcal{T})$ and K is closed and sequentially compact in $(X, \mathcal{T}_{d^*})$. Assume that $d[F, K] = 0$, then there is a sequence $\langle x_n \rangle$ in F and a sequence $\langle y_n \rangle$ in K with $d(x_n, y_n) \to 0$

and thus there is a subsequence $\langle y_{k_n}\rangle$ of $\langle y_n\rangle$ and $p \in K$ with $y_{k_n} \to p$ in $(X, \mathcal{T}_{d^*})$. Since $d(y_{k_n}, x_{k_n}) \to 0$, $p \in F$ and thus F intersects K.

Conversely, assume that $d[F, K] > 0$ for F a closed in set in $(X, \mathcal{T})$ and K a closed, sequentially compact set in $(X, \mathcal{T}_{d^*})$, whenever $F \cap K = \emptyset$.

We observe that $\mathcal{T} \subseteq \mathcal{T}_{d^*}$. Let $G \in \mathcal{T}$, and suppose that $p \notin G$. Since d^* is also an asymmetric, is follows that $\{p\}$ is a closed sequentially compact set in $(X, \mathcal{T}_{d^*})$, and thus $d[X \setminus G, \{p\}] > 0$. Therefore, $S_{d^*}(p, \epsilon) \subseteq G$ for some $\epsilon > 0$, proving that $G \in \mathcal{T}_{d^*}$.

Suppose that $d(x_n, y_n) \to 0$, and $y_n \to p$ in $(X, \mathcal{T}_{d^*})$. Let $p \in G$, for an open set G, define $K = \{p\} \cup \{y_n | y_n \in G\}$. It follows that K is closed and sequentially compact in $(X, \mathcal{T}_{d^*})$, and thus $d[X \setminus G, K] = \epsilon$, for some positive ϵ, and since $d(x_n, y_n) \to 0$, x_n is eventually in G.

This lemma, along with Theorem 4.18, yields the following theorem.

Theorem 4.20 *If d is an asymmetric for $(X, \mathcal{T})$ such that $d(p, x_n) \to 0 \Leftrightarrow x_n \to p$, then the following are equivalent:*

(1) *d is a γ^*-asymmetric for $(X, \mathcal{T})$;*

(2) *for F, a closed set in $(X, \mathcal{T})$, and, K, a closed sequentially compact set in $(X, \mathcal{T}_{d^*})$, if $F \cap K = \emptyset$ then $d[F, K] > 0$;*

(3) *if $p \notin F$, F a closed set in $(X, \mathcal{T})$, then $S_{d^*}(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$ for some $\epsilon > 0$.*

Proof: (1$\Rightarrow$2) Suppose that d is a γ^*-asymmetric for $(X, \mathcal{T})$, and that F is a closed set in $(X, \mathcal{T})$, and K a closed sequentially compact set in $(X, \mathcal{T}_{d^*})$. Furthermore, assume that $d[F, K] = 0$, then there is a sequence $\langle x_n\rangle$ in F and a sequence $\langle y_n\rangle$ in K such that $d(x_n, y_n) \to 0$. Since d^* is an asymmetric for $\mathcal{T}_{d^*}$, and K is closed, there is a subsequence $\langle y_{k_n}\rangle$ of $\langle y_n\rangle$, and $p \in K$ such that $d^*(p, y_{k_n}) \to 0$. Hence, it follows that $p \in F \cap K$.

(2⇒3) is proved using an argument similar to that used in Lemma 4.7.

(3⇒1) follows from Theorem 4.17.

We conclude our study of γ^*-distances, as we did for γ-distances, by introducing a related neighborhood characterization.

A space $(X,\mathcal{T})$ is a γ^*-*space* iff there is $\{U_n(p)|p \in X, n \in \mathbf{N}\}$, an open neighborhood assignment in $(X,\mathcal{T})$, such that for every p:

$$\text{if } p \in U_n(y_n) \text{ and } y_n \in U_n(x_n) \text{ for every } n, \text{ then } x_n \to p.$$

We, then, say that $\{U_n(p)|p \in X, n \in \mathbf{N}\}$ is a γ^*-*neighborhood assignment for* $(X,\mathcal{T})$ and observe that $\{U_n(p)|n \in \mathbf{N}\}$ need not be a local base for p in $(X,\mathcal{T})$.

Theorem 4.21 *For any first countable space* $(X,\mathcal{T})$

$$(X,\mathcal{T}) \textit{ is a } \gamma^*\textit{-space iff } (X,\mathcal{T}) \textit{ is pseudo } \gamma^*\textit{-asemimetrizable.}$$

Proof: Suppose that $\{U_n(p)|p \in X, n \in \mathbf{N}\}$ is a γ^*-neighborhood assignment for $(X,\mathcal{T})$. Since $(X,\mathcal{T})$ is first countable, there is a neighborhood assignment $\{V_n(p)|p \in X, n \in \mathbf{N}\}$ in $(X,\mathcal{T})$, such that: if $x_n \in V_n(p)$ for every n, then $x_n \to p$. For every $p \in X$ and every $n \in \mathbf{N}$, let $G_n(p) = U_n(p) \cap V_n(p)$, then $\{G_n(p)|p \in X, n \in \mathbf{N}\}$ is a γ^*-neighborhood assignment for $(X,\mathcal{T})$ and if $x_n \in G_n(p)$ for every n, then $x_n \to p$. We may also assume that $G_{n+1}(p) \subseteq G_n(p)$. Now, let d be the distance for X such that:

$$d(p,q) = \begin{cases} 0 & \text{if } q \in G_n(p) \text{ for every } n; \\ 1/2^n & \text{where } n = \min\{k|q \notin G_k(p)\}, \end{cases}$$

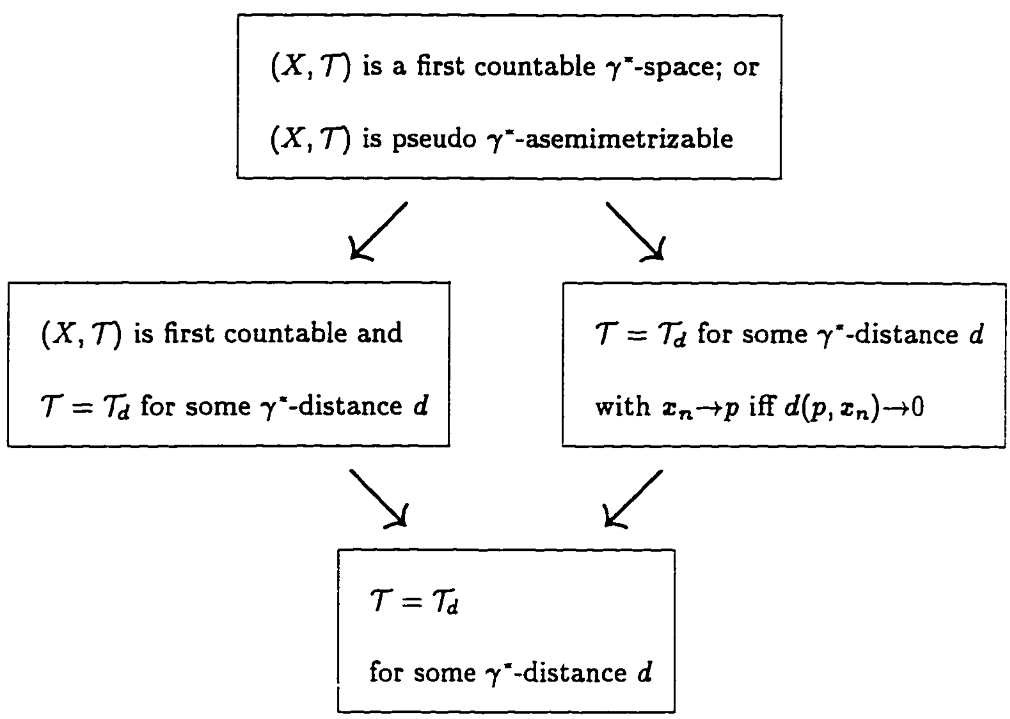
then, as in Lemma 3.2 d is a pseudo-asemimetric for $(X, \mathcal{T})$. That d is a γ^* distance follows from the fact that $\{G_n(p) | p \in X, n \in \mathbf{N}\}$ is a γ^*-neighborhood assignment for $(X, \mathcal{T})$.

Conversely, if d is a pseudo γ^*-asemimetric for $(X, \mathcal{T})$ then,

$$\{int_{\mathcal{T}}(S_d(p, 1/2^n)) | p \in X, n \in \mathbf{N}\},$$

is a γ^*-neighborhood assignment for $(X, \mathcal{T})$.

As we have seen in Example 4.15, a topological space determined by a γ^*-distance may be such that the topological convergence is the same as the distance convergence, while the space is not first countable. Hence, when spaces are determined by γ^*-distances, it becomes important to distinguish between first countability and the equivalence of the convergence of the sequences. The situation is summarized by the following:



Directly from Lemma 3.5 and Lemma 3.3 we obtain our next result, which addresses the issue of the asemimetrizability of asymmetric spaces.

Corollary 4.22 *If $(X, \mathcal{T})$ is a first countable space with unique limits, then*

there is a γ^*-distance for $(X, \mathcal{T})$ iff $(X, \mathcal{T})$ is Hausdorff and γ^*-asemimetrizable.

Open questions:

1. If $(X, \mathcal{T})$ is a *first countable space* which is determined by a γ^*-distance, is there a *pseudo γ^*-asemimetric* for $(X, \mathcal{T})$?

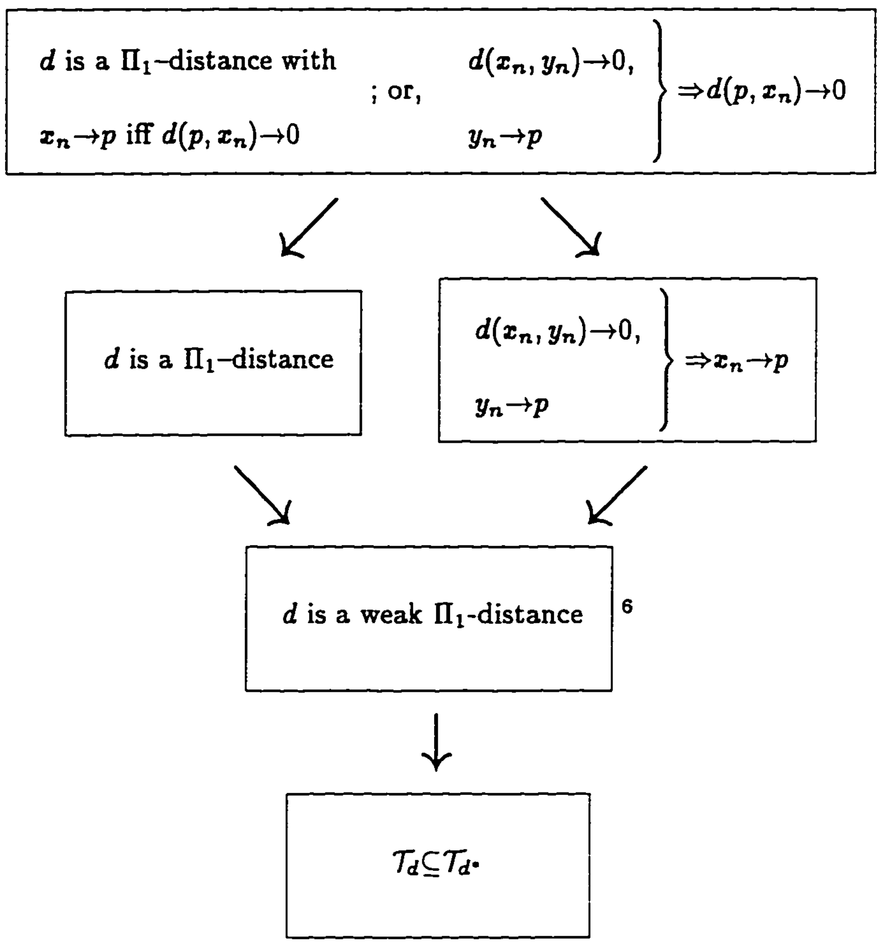
2. If $(X, \mathcal{T})$ is a a γ^*-space, is there a γ^*-distance for $(X, \mathcal{T})$?

3. If d is a γ^*-distance for $(X, \mathcal{T})$, is there a γ^*-distance d_1, such that $x_n \to p$ in $(X, \mathcal{T})$ iff $d_1(p, x_n) \to 0$?

4.4 Π_1-distance functions

Clearly, a symmetric distance d is a Π_1-distance iff it is a γ-distance.

We begin with the following overview. Note that, except for the equivalence in the first box and the last implication, all implications are immediate.



Lemma 4.23 *If d is distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is a Π_1-distance such that $x_n \to p \Leftrightarrow d(p, x_n) \to 0$;*

(2) *d is a distance such that for any $p \in X$, if $d(x_n, y_n) \to 0$ and $y_n \to p$ then $d(p, x_n) \to 0$.*

---

[6] A distance d for $(X, \mathcal{T})$ is a *weak* Π_1*-distance* for $(X, \mathcal{T})$ if,

for any $p \in X$, when $d(p, y_n) \to 0$ and $d(x_n, y_n) \to 0$, then $x_n \to p$ in $(X, \mathcal{T})$;

Proof: Either condition implies that $x_n \to p \Leftrightarrow d(p, x_n) \to 0$ and from this the equivalence follows.

Furthermore, if d is a distance for $(X, \mathcal{T})$ such that $x_n \to p$ when $d(p, y_n) \to 0$ and $d(x_n, y_n) \to 0$, then $d(x_n, p) \to 0$ implies that $x_n \to p$. Hence, it follows from Lemma 3.13 that there is a symmetric distance for $(X, \mathcal{T})$.

For a Π_1-distance d we also observe that $(d(x_n, p) \to 0 \Rightarrow d(p, x_n) \to 0)$, and thus we obtain the following:

Lemma 4.24 *For any* Π_1*-distance d*

(1) *d is a* γ^**-distance and, thus,*

(2) $(d \wedge d^*)$ *is a symmetric distance for* $(X, \mathcal{T}_d)$.

Proof: See Lemma 3.12.

Example 4.25 *A* γ^**-asymmetrizable space that is not* Π_1*-asymmetrizable.*

The distance given in Example 4.10 is γ^*-distance for a T_1-space $(X, \mathcal{T})$. However, $(X, \mathcal{T})$ is not Π_1-asymmetrizable, since it fails to have unique limits (see Theorem 4.27). Therefore, *no distance for* $(X, \mathcal{T})$ *can be a* Π_1*-distance.*

Corollary 4.26 *If d is a locally symmetric,* Π_1*-distance for* $(X, \mathcal{T})$*, then* $(d \vee d^*)$ *is a symmetric* Π_1*-distance, and thus a* γ*-distance, for* $(X, \mathcal{T})$.

The next theorem emphasizes the importance of point separation when studying spaces determined by Π_1-distances.

Theorem 4.27 *If d is a* Π_1*-distance, then the following are equivalent:*

(1) $(X, \mathcal{T}_d)$ *is a* T_0*-space;*

(2) d *is separating;*

(3) d *has unique limits;*

(4) d *is an asymmetric.*

Proof: (1⇒2) and (4⇒1) follow directly from Lemma 3.1.

(2⇒3): Suppose that d is a separating Π_1-distance and that $d(p, x_n) \to 0$ and $d(q, x_n) \to 0$, then $d(p,q) = d(q,p) = 0$ and thus $p = q$.

(3⇒4): Suppose that d is a distance with unique limits and that $d(p,q) = 0$. Let $\langle x_n \rangle$ be the constant sequence with $x_n = q$, then $d(p,q) \to 0$ and $d(q,q) \to 0$ and thus $p = q$.

In light of this theorem and Lemma 3.5 we obtain the next corollary.

Corollary 4.28 *If* d *is a* Π_1*-distance for a* T_0*-space* $(X, \mathcal{T})$ *then*

$$d(p, x_n) \to 0 \;\Leftrightarrow\; x_n \to p.$$

We also note the next fact which is immediate from Theorem 4.27 and Lemma 3.3.

Corollary 4.29 $(X, \mathcal{T})$ *is* Π_1*-asemimetrizable iff* $(X, \mathcal{T})$ *is* Π_1*-asymmetrizable and first countable.*

Example 4.30 *A* Π_1*-asymmetrizable space need not be first countable even though it is Hausdorff.*

The distance given in Example 4.15 is a Π_1-asymmetric for a Hausdorff paracompact space that is not first countable.

Turning our attention to the separation of closed and sequentially compact sets, we obtain some interesting characterizations of spaces determined by Π_1-distance, and observe that the results are different than the ones obtained for γ-distances or γ^*-distance, with T_0-spaces playing an important role.

We begin with a few preliminary lemmas.

Lemma 4.31 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is weak Π_1-distance;*

(2) *for $p \notin F$, a closed set in $(X, \mathcal{T})$, there is an $\epsilon > 0$ with $S_d(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$.*

Proof: Suppose that d is a weak Π_1-distance for $(X, \mathcal{T})$, and assume that $p \in X$, F is a closed set. Suppose that for every $\epsilon > 0$, $S_d(p, \epsilon) \cap S_d[F, \epsilon) \neq \emptyset$, then for ¿every $n \in \mathbf{N}$, there is a $y_n \in S_d(p, 1/2^n) \cap S_d(x_n, 1/2^n)$ for some $x_n \in F$. Since d is a weak Π_1-distance for $(X, \mathcal{T})$ and F is closed, $p \in F$.

Conversely, assume that $d(p, y_n) \to 0$ and that $d(x_n, y_n) \to 0$. Suppose $p \in G$ and G is open, then $S_d(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$ for some $\epsilon > 0$. Since $y_n \to p$, there exist $k \in \mathbf{N}$ such that for $n \geq k$, $y_n \in S_d(p, \epsilon) \cap G$ and $d(x_n, y_n) < \epsilon$, establishing that $x_n \to p$.

Lemma 4.32 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *if $d(x_n, y_n) \to 0$ and $y_n \to p$ then $x_n \to p$;*

(2) *if K is a sequentially compact set and F is a closed set, with $F \cap K = \emptyset$, then $d[F, K] > 0$.*

Proof: Assume that $y_n \to p$ and $d(x_n, y_n) \to 0$ implies that $x_n \to p$, suppose that F is closed and K is sequentially compact. Assume that $d[F, K] = 0$, then there is a sequence $\langle x_n \rangle$

in F and a sequence $\langle y_n \rangle$ in K with $d(x_n, y_n) \to 0$ and thus there is a subsequence $\langle y_{k_n} \rangle$ of $\langle y_n \rangle$ and $p \in K$ with $y_{k_n} \to p$. Since $d(x_{k_n}, y_{k_n}) \to 0$, $p \in F$ and thus F intersects K.

Conversely, assume that $y_n \to p$ and $d(x_n, y_n) \to 0$. Let G be an open set containing p, then $K = \{p\} \cup \{x_n | x_n \in G\}$ is sequentially compact. Hence, $d[X \setminus G, K] > 0$ and since $d(x_n, y_n) \to 0$, x_n is eventually in G.

The next theorem establishes our "separation theorem."

Theorem 4.33 *If d is a distance function for $(X, \mathcal{T})$ such that the topological convergence is the same as the distance convergence, then the following are equivalent:*

(1) *d is a Π_1-distance for $(X, \mathcal{T})$;*

(2) *for F a closed set and K a sequentially compact set, if $F \cap K = \emptyset$, then $d[F, K] > 0$;*

(3) *for F a closed set and $p \notin F$, there is an $\epsilon > 0$ with $S_d(p, \epsilon) \cap S_d[F, \epsilon) = \emptyset$.*

Proof: It is enough to observe that if d is a distance function for $(X, \mathcal{T})$ such that the topological convergence and the distance convergence coincide then the following are equivalent:

(1) d is a Π_1-distance for $(X, \mathcal{T})$;

(2) if $d(x_n, y_n) \to 0$ and $y_n \to p$ then $x_n \to p$;

(3) d is a weak Π_1-distance for $(X, \mathcal{T})$.

We have the following corollaries as a result of Theorem 4.27.

Corollary 4.34 *If d is a distance for a T_0-space $(X, \mathcal{T})$, the following are equivalent:*

(1) *d is a Π_1-distance;*

(2) *for F a closed set and K a sequentially compact set, if $F \cap K = \emptyset$, then $d[F, K] > 0$;*

(3) *for F a closed set and $p \notin F$, there is an $\epsilon > 0$ with $S_d(p,\epsilon) \cap S_d[F,\epsilon) = \emptyset$.*

Proof: We only have to show that (3) implies (1). Suppose that d is a weak Π_1-distance for $(X,\mathcal{T})$, since $(X,\mathcal{T})$ is a T_0-space, d has unique limits. A distance for $(X,\mathcal{T})$ with unique limits, is an asymmetric for $(X,\mathcal{T})$ with $d(p,x_n)\to 0$ iff $x_n \to p$.

Corollary 4.35 *$(X,\mathcal{T})$ is Π_1-asymmetrizable iff there is a weak Π_1-asymmetric for $(X,\mathcal{T})$.*

The class of space generated by Π_1-distances is related to the class of Nagata spaces which were introduced by Hodel [Ho_1] in 1972. A space $(X,\mathcal{T})$ is a *Nagata space* iff there is an open neighborhood assignment, $\{U_n(p)|p \in X, n \in \mathbf{N}\}$, for $(X,\mathcal{T})$ such that:

$$\text{for every } p, \text{ if } U_n(p) \cap U_n(x_n) \neq \emptyset \text{ for every } n, \text{ then } x_n \to p.$$

We then say that $\{U_n(p)|p \in X, n \in \mathbf{N}\}$ is a *Nagata* neighborhood assignment for $(X,\mathcal{T})$. We also observe that $\{U_n(p)|n \in \mathbf{N}\}$ is an open neighborhood base for p in $(X,\mathcal{T})$, and thus $(X,\mathcal{T})$ is first countable. This implies that any distance characterization of a Nagata space must involve at least a pseudo-asemimetric, as stated in Lemma 3.2.

Theorem 4.36 *A space $(X,\mathcal{T})$ is a Nagata space iff $(X,\mathcal{T})$ is pseudo Π_1-asemimetrizable.*

Proof: The proof uses a similar strategy as the proof of Theorem 4.21.

As a result of Theorem 4.27, and the previous corollary, we observe that:

Corollary 4.37 *$(X,\mathcal{T})$ is Π_1-asemimetrizable iff $(X,\mathcal{T})$ is a Hausdorff Nagata space.*

The results of this section reiterate the importance of distinguishing between first countable spaces and spaces where the topological convergence agrees with the distance convergence when studying spaces determined by distance functions. The following diagrams summarize the situation.

$(X,\mathcal{T})$ is a Nagata space; or

$(X,\mathcal{T})$ is pseudo Π_1-asemimetrizable

↙ ↘

$(X,\mathcal{T})$ is first countable and

$\mathcal{T}=\mathcal{T}_d$ for some Π_1-distance d

$\mathcal{T}=\mathcal{T}_d$ for some Π_1-distance d

with $x_n \to p$ iff $d(p,x_n)\to 0$

↘ ↙

$\mathcal{T}=\mathcal{T}_d$

for some Π_1-distance d

On the other hand, if $(X,\mathcal{T})$ is a $\mathcal{T}_0$-space, we have:

$(X,\mathcal{T})$ is Π_1-asemimetrizable; or

$(X,\mathcal{T})$ is a Hausdorff Nagata space; or

$(X,\mathcal{T})$ is first countable, Π_1-asymmetrizable

↓

$(X,\mathcal{T})$ is Π_1-asymmetrizable

Open questions:

1. In first countable spaces does having a Π_1-distance imply that there is a pseudo Π_1-*asemimetric* for $(X,\mathcal{T})$?

2. If there is a Π_1-distance for $(X,\mathcal{T})$, is there another Π_1-distance for $(X,\mathcal{T})$ such that the distance convergence and the topological convergence coincide for the new distance?

Chapter 5

A contribution to the theory of developable spaces

5.1 Introduction

The notion of ***developable*** topological spaces was introduced by R.H. Bing in [Bi_1]. He defines $(X, \mathcal{T})$ to be developable iff there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that:

$$\{st(p, \mathcal{G}_n) | n \in \mathbf{N}\} \text{ is a local base for } p \text{ in } (X, \mathcal{T}),$$

where $st(p, \mathcal{G}_n) = \bigcup \{G \in \mathcal{G}_n | p \in G\}$.

Developable spaces have attracted the interest of general topologists since their introduction. However, these spaces do occur implicitly as spaces with *refining sequences* in Alexandroff and Urysohn's *Une condition nécessaire pour qu'une classe* $(\mathcal{L})$ *soit une classe* $(\mathcal{D})$ [AU]. The importance, as well as the evolution of developable spaces is studied in greater detail by Shore [Sh_2] in *From Developments to Developable Spaces: The evolution of a topological idea.*

A distance d for X is *developable* [Ke, 1971] iff for any $p \in X$,

$$\text{when } d(p, x_n) \to 0 \text{ and } d(p, y_n) \to 0 \text{ then } d(x_n, y_n) \to 0.$$

This property first appeared in the work of Pitcher and Chittenden [PC] in 1918 who referred to it as property (3)[1]. It was later studied by Alexandroff and Niemytzki [AN], who called this property *a Cauchy condition.*

The importance of developable distance function is made clear, by the following lemma, which also explains the terminology chosen by Alexandroff and Niemytzki.

Lemma 5.1 *For any distance d the following are equivalent:*

(1) *d is developable;*

(2) *every d-convergent sequence is d-Cauchy;*

(3) *for each p in X there is a sphere, centered at p, of arbitrarily small diameter.*

The connection between developable spaces and developable distances is illustrated by the following result noted by Brown [Br].

Theorem 5.2 *A Hausdorff space, $(X, \mathcal{T})$ is developable iff there is a developable semimetric for $(X, \mathcal{T})$.*

This chapter will give a concise overview of developable distance functions and then focus on an unexpected alternative distance function characterization for developable spaces.

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[1]See page 21 of this dissertation.

5.2 Developable distance functions

We begin our overview of developable distance functions by observing that if d is a developable distance, then d is locally symmetric, which along with Lemma 3.12 yield the next result.

Lemma 5.3 *If d is a developable distance function for $(X, \mathcal{T})$, then $(d \vee d^*)$ is also a developable distance for $(X, \mathcal{T})$.*

We are now ready to make the connection between developable distances and Π_2-distances.

Theorem 5.4 *For any space $(X, \mathcal{T})$, the following are equivalent:*

(1) *there is a developable distance for $(X, \mathcal{T})$;*

(2) *there is a developable, symmetric distance for $(X, \mathcal{T})$;*

(3) *there is Π_2-distance for $(X, \mathcal{T})$.*

Proof: We observe immediately from the previous lemma that conditions (1) and (2) are equivalent.

Assume that d is a symmetric developable distance for $(X, \mathcal{T})$. Then for each $p \in X$, there is a decreasing sequence $\langle \delta_n(p) \rangle$ of radii, such that $\delta_n(p) < 1/2^n$ for each n, and if $x, y \in S_d(p, \delta_n(p))$, then $d(x, y) < 1/2^n$. Define a distance function d_1 by

$$d_1(p, q) = \begin{cases} 0, & \text{if } q \in S_d(p, \delta_n(p)) \text{ for every } n; \\ 1/2^n, & \text{where } n = \min\{k | q \notin S_d(p, \delta_k(p))\}. \end{cases}$$

Since $S_{d_1}(p, 1/2^n) = S_d(p, \delta_n(p))$, $\mathcal{T}_{d_1} = \mathcal{T}_d$ and $d_1(p, x_n) \to 0$ iff $d(p, x_n) \to 0$.

It is left to show that d_1 is a Π_2-distance. Suppose that $d_1(y_n, p) \to 0$ and $d_1(y_n, x_n) \to 0$. Then for every $k \in \mathbf{N}$, there is exists $m \in \mathbf{N}$ such that for every $n > m$,

$$d_1(y_n, p) < 1/2^k, \text{ and } d_1(y_n, x_n) < 1/2^k.$$

Hence, $p, x_n \in S_{d_1}\left(y_n, 1/2^k\right) = S_d(y_n, \delta_k(y_n))$, and thus $d(p, x_n) < 1/2^k$. Now, since $d(p, x_n) \to 0$, then $d_1(p, x_n) \to 0$ as desired.

Conversely, suppose that d is a Π_2-distance for $(X, \mathcal{T})$. We will construct a developable, symmetric distance for $(X, \mathcal{T})$.

For every $n \in \mathbf{N}$ and every $x \in X$, let $\mathcal{G}_n = \{S_d(x, 1/2^n) | x \in X\}$, and $st(x, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n | x \in G\}$. Define the distance function d_1 for X by:

$$d_1(p, q) = \begin{cases} 0, & \text{if } q \in st(p, \mathcal{G}_n) \text{ for every } n; \\ 1/2^n, & \text{where } n = \min\{k | q \notin st(p, \mathcal{G}_k\}. \end{cases}$$

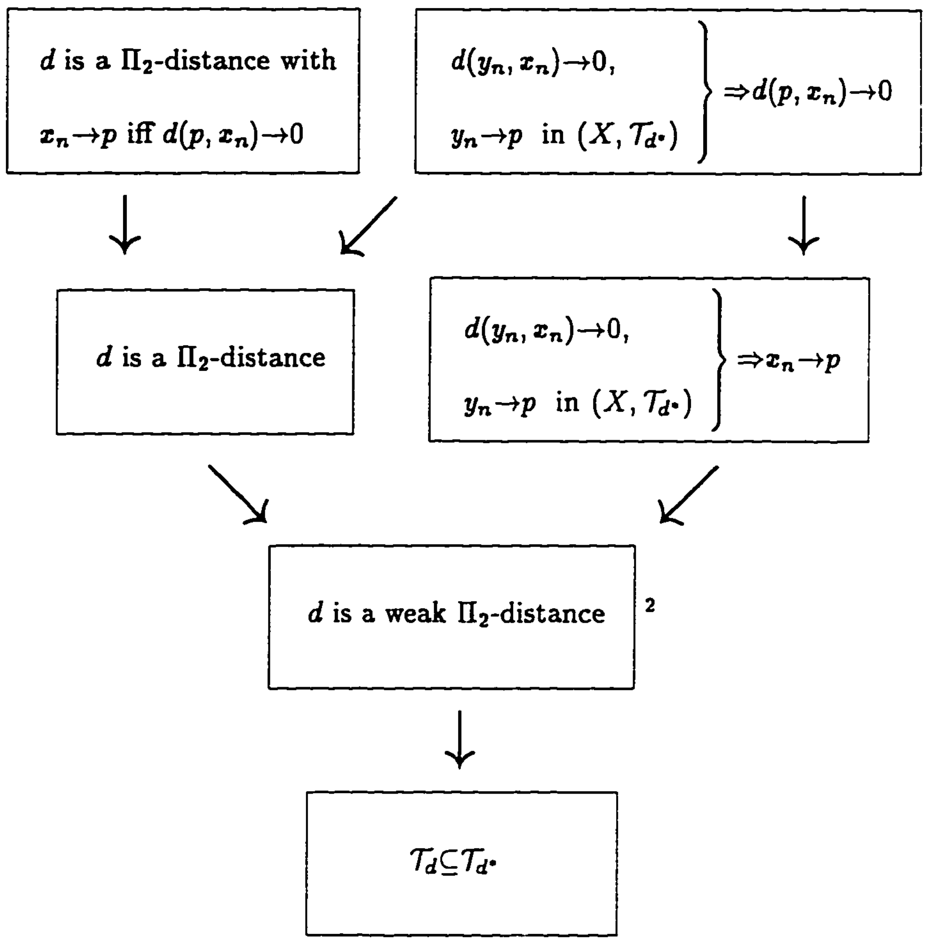
Since $p \in st(x, \mathcal{G}_n)$ iff $x \in st(p, \mathcal{G}_n)$, it follows that d_1 is a symmetric distance function. Also, observe that $S_d(p, 1/2^n) \subseteq S_{d_1}(p, 1/2^n)$ and thus $d(p, x_n) \to 0$ implies that $d_1(p, x_n) \to 0$. Furthermore, assume that $d_1(p, x_n) \to 0$. Then for every $k \in \mathbf{N}$, there is exists $m \in \mathbf{N}$ such that for every $n > m$, $d_1(p, x_n) < 1/2^k$, and so $x_n \in st(p, \mathcal{G}_k)$, and thus, there is $y_n \in X$ such that $x_n, p \in S_d\left(y_n, 1/2^k\right)$. It follows that, since $d(y_n, p) \to 0$ and $d(y_n, x_n) \to 0$, $d(p, x_n) \to 0$, establishing that $d(p, x_n) \to 0$ iff $d_1(p, x_n) \to 0$ and thus that $\mathcal{T}_d = \mathcal{T}_{d_1}$.

We must now show that d_1 is a developable distance. Suppose that $d_1(p, x_n) \to 0$ and $d_1(p, y_n) \to 0$, then $d(p, x_n) \to 0$ and $d(p, y_n) \to 0$. Hence, for every $k \in \mathbf{N}$, there is exists $m \in \mathbf{N}$ such that for every $n > m$, $d(p, x_n) + d(p, y_n) < 1/2^k$. Thus, $y_n \in st(x_n, \mathcal{G}_k)$ for every $n > m$, and therefore, $d_1(x_n, y_n) \to 0$.

Having established the connection between developable distance functions and Π_2-distances, we now turn our attention to the study of Π_2-distances.

5.3 Π_2-distance functions

The importance of developable spaces in general topology and the relationship of spaces determined by Π_2-distances with developable spaces motivates our further study of Π_2-distances. First, we observe the following overview.



As with the other distances studied in this work, we note the following corollaries, the proofs of which are similar to results presented previously.

Corollary 5.5 *If d is a Π_2-distance for $(X, \mathcal{T})$, then $(d \wedge d^*)$ is a symmetric distance for $(X, \mathcal{T})$.*

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[2]We say that a distance d for $(X, \mathcal{T})$ is a *weak Π_2-distance* for $(X, \mathcal{T})$ if,

for any $p \in X$, when $d(y_n, x_n) \to 0$ and $d(y_n, p) \to 0$, then $x_n \to p$ in $(X, \mathcal{T})$.

Proof: Note that for a Π_2-distance d, if $d(x_n, p) \to 0$, then $d(p, x_n) \to 0$; and use Corollary 3.14.

Corollary 5.6 *If d is a locally symmetric Π_2-distance for $(X, \mathcal{T})$, then $(d \vee d^*)$ is a symmetric γ-distance for $(X, \mathcal{T})$.*

Proof: See Lemma 4.4 and note that a symmetric Π_2-distance is a γ-distance.

Turning our attention to possible characterizations of spaces defined by Π_2-distances by means of a distance separation between closed and sequentially compact sets, we find ourselves in a situation similar to the one encountered for γ^*-distances. We must consider simultaneously the difference between the topological convergence and the distance convergence in both $(X, \mathcal{T}_d)$ and $(X, \mathcal{T}_{d^*})$ (see initial overview), and thus will consider asymmetrics as well as distances. We also find that for spaces determined by Π_2-distances, we can consider countably compact sets instead of sequentially compact sets.

Lemma 5.7 *If d is a distance for $(X, \mathcal{T})$, then the following are equivalent:*

(1) *d is a weak Π_2-distance for $(X, \mathcal{T})$;*

(2) *for $p \notin F$, a closed set in $(X, \mathcal{T})$, there is an $\varepsilon > 0$ with $S_{d^*}(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) = \emptyset$.*

Proof: See Lemma 4.1.

As was the case for γ^*-distances, we get the following immediate result when considering a Π_2-distance for $(X, \mathcal{T})$ where $d(p, x_n) \to 0$ iff $x_n \to p$.

Theorem 5.8 *If d is a distance for $(X, \mathcal{T})$ such that $d(p, x_n) \to 0$ iff $x_n \to p$, then the following are equivalent:*

(1) *d is a Π_2-distance;*

(2) *for* $p \notin F$*, a closed set, there exists* $\varepsilon > 0$ *such that* $S_{d^*}(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) = \emptyset$.

If, on the other hand, we assume that d is an asymmetric, we obtain different results. We begin by observing this next result.

Lemma 5.9 *If* d *is an asymmetric for* $(X, \mathcal{T})$*, then the following are equivalent:*

(1) *if* $d(y_n, x_n) \to 0$*, and* $y_n \to p$ *in* $(X, \mathcal{T}_{d^*})$*, then* $x_n \to p$ *in* $(X, \mathcal{T})$*;*

(2) d *is a weak* Π_2*-asymmetric for* $(X, \mathcal{T})$*;*

(3) *if* p *is not in a closed set* F*, there is an* $\varepsilon > 0$ *such that* $S_{d^*}(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) = \emptyset$.

Proof: Note that ($1 \Rightarrow 2$) always holds, while (2) being equivalent to (3) follows from Lemma 5.7. We observe, that if d satisfies condition (3), then since d^* is an asymmetric for $(X, \mathcal{T}_{d^*})$, for two distinct points p and q in X, there is an $\varepsilon > 0$ such that $S_{d^*}(p, \varepsilon) \cap S_{d^*}[\{q\}, \varepsilon) = \emptyset$. Applying Lemma 3.4, d^* is an asymmetric with unique limits. It follows then, from Lemma 3.5, that $x_n \to p$ in $(X, \mathcal{T}_{d^*})$ iff $d^*(p, x_n) \to 0$.

($2 \Rightarrow 1$) If d is a weak Π_2-asymmetric for $(X, \mathcal{T})$, then $x_n \to p$ in $(X, \mathcal{T}_{d^*})$ iff $d^*(p, x_n) \to 0$, and therefore d satisfies condition (1).

We also note that if d is a weak Π_2-asymmetric for $(X, \mathcal{T})$, then, by Lemma 3.4, if K is a countably compact set in $(X, \mathcal{T}_{d^*})$, it is a closed set in $(X, \mathcal{T}_{d^*})$. Our next lemma is a direct consequence of Lemma 4.18, and the preceding remark.

Lemma 5.10 *If* d *is a weak* Π_2*-asymmetric for* $(X, \mathcal{T})$ *then, for* F *a closed set and* K *a countably compact set in* $(X, \mathcal{T}_{d^*})$*, if* $K \cap F = \emptyset, d[K, F] > 0$*. The converse holds, if we further assume that* $\mathcal{T} \subseteq \mathcal{T}_{d^*}$*.*

Proof: See Lemma 4.19.

The next theorem establishes our main separation result.

Theorem 5.11 *If d is an asymmetric for $(X, \mathcal{T})$ such that $d(p, x_n) \to 0$ iff $x_n \to p$, then* (1) *is equivalent to* (2), *and* (2) *implies* (3);

(1) *d is a Π_2-distance for $(X, \mathcal{T})$;*

(2) *for $p \notin F$, F a closed set, $S_{d^*}(p, \varepsilon) \cap S_{d^*}[F, \varepsilon) = \emptyset$, for some $\varepsilon > 0$;*

(3) *for F, a closed set, and K, a countably compact set in $(X, \mathcal{T}_{d^*})$, if $K \cap F = \emptyset$ then $d[K, F] > 0$.*

If we also assume that $\mathcal{T} \subseteq \mathcal{T}_{d^*}$, then (1), (2) *and* (3) *are all equivalent.*

We do not know that if d is an asymmetric that satisfies condition (3) of Theorem 5.11, it is always the case that $\mathcal{T} \subseteq \mathcal{T}_{d^*}$, which reinforces the role played by that condition in this type of study.

Having established the separation properties for space determined by Π_2-distances, we now focus on the relationship between the distance characterization and the neighborhood characterizations of developable spaces in order to establish results similar to the ones obtained for other distances. The results presented will involve both Π_2-distances as well as the developable distances introduced earlier in this chapter.

We begin with an open neighborhood assignment characterization for developable spaces due to Heath [He_1].

Lemma 5.12 *A space $(X, \mathcal{T})$ is developable iff there is an open neighborhood assignment $\{U_n(p) | p \in X, n \in \mathbb{N}\}$ for $(X, \mathcal{T})$ such that:*

$$\text{if for each } n,\ x_n, p \in U_n(y_n) \text{ for some } y_n,\ \text{then } x_n \to p.$$

As before, we may assume that $U_{n+1}(p) \subseteq U_n(p)$ for every n, and we also note that for developable spaces we may assume that $\{U_n(p) | n \in \mathbf{N}\}$ is a local base for p in $(X, \mathcal{T})$. Having the above characterization for developable spaces we obtain the following theorem which uses a similar proof as in Theorems 4.11 and 5.4.

Theorem 5.13 *For any space $(X, \mathcal{T})$, the following are equivalent:*

(1) *$(X, \mathcal{T})$ is developably pseudo-asemimetrizable;*

(2) *$(X, \mathcal{T})$ is developably pseudo-semimetrizable;*

(3) *$(X, \mathcal{T})$ is pseudo Π_2-asemimetrizable;*

(4) *$(X, \mathcal{T})$ is developable.*

Proof: (1⇒4) Assume d is a developable pseudo-asemimetric for $(X, \mathcal{T})$. Then, as in Theorem 5.4, it is possible for each $p \in X$ to get a decreasing sequence $\langle \delta_n(p) \rangle$ such that the d-diameter of $S_d(p, \delta_n(p))$ is less than $1/2^n$. We now observe that $\{\mathrm{int}_{\mathcal{T}}(S_d(p, \delta_n(p))) | p \in X\}$ is an open developable neighborhood assignment for $(X, \mathcal{T})$.

(4⇒3) See Theorem 4.11, (3⇒2) and (2⇒1) follow from Theorem 5.4.

Corollary 5.14 *For any T_1-space $(X, \mathcal{T})$, the following are equivalent:*

(1) *$(X, \mathcal{T})$ is developably asemimetrizable;*

(2) *$(X, \mathcal{T})$ is developably semimetrizable;*

(3) *$(X, \mathcal{T})$ is Π_2-asemimetrizable;*

(4) *$(X, \mathcal{T})$ is developable.*

A consequence of Lemma 3.5 is that a *weak* Π_2-distance for a topological space $(X, \mathcal{T})$ with unique limits is a Π_2-distance for that space. This fact, along with Theorem 5.4 and 5.13, combine to give the next corollary.

Corollary 5.15 *For any space $(X, \mathcal{T})$ with unique limits,*

(1) *$(X, \mathcal{T})$ is developably semimetrizable iff it is weakly Π_2-asymmetrizable and first countable;*

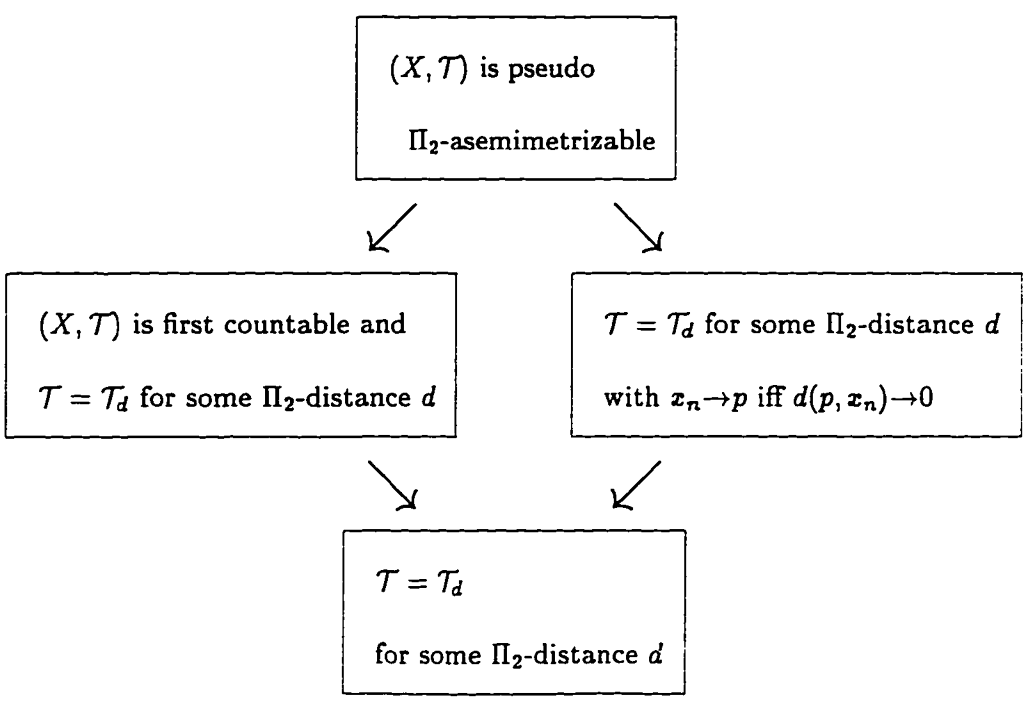
(2) *$(X, \mathcal{T})$ developably symmetrizable iff it is weakly Π_2-asymmetrizable.*

The next example shows the importance of having unique limits.

Example 5.16 *A weak Π_2-distance for $(X, \mathcal{T})$ need not be a Π_2-distance.*

Consider the space in Example 4.10. Then d is weak Π_2-distance for $(X, \mathcal{T})$. However, it is not a Π_2-distance for $(X, \mathcal{T})$, since $d(1/3^n, 0) \to 0$ and $d(1/3^n, 2/3^n) \to 0$ but $d(0, 2/3^n) = 1$ for every $n \in \mathbf{N}$.

As in the previous sections we note the following results.



The interesting connection, established by Corollary 5.14 between developable T_1-spaces and developably asemimetrizable spaces, points toward some important generalizations in the class of developable asymmetrizable spaces or even spaces determined by developable distances. The next examples will indeed shed some light on the generalizations possible, when doing away with first countability even in the presence of a strong separation axiom, or when doing away with Hausdorffness while retaining first countability; and show some surprising loss of structure in those spaces. This serves to establish developable asymmetrizable spaces as interesting generalizations of developable spaces. We begin by investigating the case for non-Hausdorff spaces.

Example 5.17 *A developable semimetric need not have unique limits, and thus is not metrizable.*

Let $X = (0, 1)$, consider the symmetric distance d for X with

$$d(p, q) = \min\{p, q\}.$$

Then, d is developable since the diameter of any sphere centered at p of radius $\varepsilon < p$ is less than ε. We also observe that $S_d(p, \varepsilon) \in \mathcal{T}_d$, and therefore, d is a *semimetric* for $(X, \mathcal{T}_d)$. Furthermore, note that $\langle 1/2^n \rangle$ converges to any point $p \in X$.

We also observe that $A = \{1/3^n | \, n \in \mathrm{N}\}$ with its relative topology (the finite complement topology) is a *compact*, semimetrizable T_1-space which is not metrizable (since it does not have unique limits).

This example reinforces the importance that Hausdorffness plays in the classical metrization theorems of Alexandroff and Niemytzki [AN]:

a compact, semimetrizable Hausdorff space is metrizable,

or in the Bing's result [Bi_1]:

a paracompact, developably semimetrizable Hausdorff space is metrizable.

Our next example shows that first countability, also plays an important role in Bing's result.

Example 5.18 *A paracompact, developably symmetrizable Hausdorff space that is not first countable.*

Let $X = [0,1]$ and $A = \{1/3^n |\, n \in \mathbb{N}\} \cup \{0\}$. Consider the symmetric d for X such that:

$$d(p,q) = \begin{cases} 1, & \text{if } p = 0 \text{ and } q \notin A; \\ |p-q|, & \text{otherwise.} \end{cases}$$

Then, d is a developable symmetric (with unique limits) for $(X, \mathcal{T}_d)$, but note that $\mathcal{T}_d$ is the same topology as the one presented in Example 4.15.

As a final remark, we note that although the symmetry does not play an important role for developable distances, it is an important differentiating factor for Π_2-distances. We have noted before that a space is developably *asemimetrizable* iff it is developably *semimetrizable.* The situation for Π_2-distances is different, in fact we have proved that a space is Π_2-*asemimetrizable* iff it is T_1 and developable, while we will show in the next chapter that a Π_2-*semimetrizable* space is in fact a metrizable.[3]

Open questions:

[3]See Example 5.17 for a T_1, developable space which is not metrizable

1. In first countable spaces does having a Π_2-distance imply that there is a pseudo Π_2-asemimetric for the space?

2. If there is a Π_2-distance for $(X, \mathcal{T})$, is there another Π_2-distance for $(X, \mathcal{T})$ such that the new distance convergence coincides with the topological convergence?

5.4 Other related spaces

In this section, we investigate the connections that exist between developable spaces and other types of topological spaces. For the most part the results are known, but the intention is to further our investigation of developable spaces.

We begin with a generalization of developable spaces.

In 1972, Hodel [Ho_1] introduced θ-spaces as a generalization of both developable spaces and γ-spaces. A topological space $(X, \mathcal{T})$ is a *θ-space* iff there is an open neighborhood assignment $\{U_n(x) | x \in X, n \in \mathbf{N}\}$ in $(X, \mathcal{T})$ such that

$$\text{for every } p, \text{ if } x_n, p \in U_n(y_n) \text{ and } y_n \in U_n(p) \text{ for every } n, \text{ then } x_n \to p.$$

Such an open neighborhood assignment will be called a *θ-neighborhood assignment* for $(X, \mathcal{T})$. We also note that θ-spaces are first countable.

Following Hodel's definition, if d is a distance for X, then d is a *θ-distance*, if for any $p \in X$,

$$\text{when } d(p, y_n) \to 0 \text{ and } d(y_n, p) \to 0 \text{ and } d(y_n, x_n) \to 0, \text{ then } d(p, x_n) \to 0.$$

We observe the following lemma.

Lemma 5.19 *If d is a γ-distance or a Π_2-distance for X, then d is a θ-distance.*

Proof: This follows from the definitions of γ-distance and Π_2-distance.

As is the case for other neighborhood characterizations we have the following theorem.

Theorem 5.20 *A space $(X,\mathcal{T})$ is a θ-space iff $(X,\mathcal{T})$ is pseudo θ-asemimetrizable.*

Proof: The proof uses a similar strategy as the proof of Theorem 4.21

In his 1972 paper, Hodel [Ho_1] also showed that:

a T_1-space is developable iff it is a semistratifiable θ-space.

We generalize this result.

Theorem 5.21 *$(X,\mathcal{T})$ is developable iff $(X,\mathcal{T})$ is a semistratifiable θ-space.*

Proof: Suppose that $(X,\mathcal{T})$ is developable then $(X,\mathcal{T})$ is semistratifiable and pseudo Π_2-asemimetrizable. Therefore $(X,\mathcal{T})$ is θ-asemimetrizable and thus a θ-space.

Conversely, suppose that $(X,\mathcal{T})$ is a semistratifiable θ-space. Then there is an open neighborhood assignment that is simultaneously a θ-neighborhood assignment and satisfies the semistratifiable condition. Using a construction similar to the one used in Lemma 3.2, there is a pseudo θ-asemimetric d for $(X,\mathcal{T})$ such that:

$$d(x_n,p)\to 0 \Rightarrow x_n\to p.$$

Then, we note that d is a pseudo Π_2-asemimetric for $(X,\mathcal{T})$.

Since any γ-space is a θ-space we have the following corollary.

Corollary 5.22 *If $(X,\mathcal{T})$ is a semistratifiable γ-space, then $(X,\mathcal{T})$ is developable.*

In 1981, Fox [Fo] proved that

a developable γ-space is quasimetrizable.

His work was done in the context of T_1-spaces. His result carries over for non T_1-spaces if quasimetrizable is replaced by pseudo-quasimetrizable.

We establish the following.

Theorem 5.23 *$(X, \mathcal{T})$ is pseudo-semimetrizable and pseudo-quasimetrizable iff $(X, \mathcal{T})$ is a semistratifiable γ-space.*

Proof: Suppose that $(X, \mathcal{T})$ is pseudo-semimetrizable and pseudo-quasimetrizable. Then it follows from Lemma 3.17 that $(X, \mathcal{T})$ is semistratifiable, and every pseudo-quasimetrizable space is a γ-space.

Conversely, suppose that $(X, \mathcal{T})$ is a semistratifiable γ-space. Then $(X, \mathcal{T})$ is first countable and semistratifiable and thus from Lemma 3.17 $(X, \mathcal{T})$ is pseudo-semimetrizable. Also, from Lemma 5.22 $(X, \mathcal{T})$ is developable and thus $(X, \mathcal{T})$ is pseudo-quasimetrizable.

This concludes our investigation of developable spaces.

Chapter 6

Contribution to the theory of metrization

Metrization has been and remains an important area of investigation in general topology, with its history dating back to Fréchet's initial contribution. In the ninety years since the introduction of metrics as a mathematical concept, a great number of mathematicians have contributed solutions to the so-called Metrization Problem, that is, the problem of determining when the mathematical structure being investigated can be constructed from a metric. Before the creation of topological spaces, the problem took the form of determining, when a more general distance was introduced, that the same structure could be generated using a metric. However, as seen in the classic papers of Bing [Bi_1], Nagata [Na_1] and Smirnov [Sm], the more desirable solutions in the context of topological spaces focus on topologically inherent properties of the space and forgo any mention of distance [1].

In this chapter, we will present a number of metrization, or pseudometrization, theorems involving the distances studied in this work.

We begin by extending the classical 1927 result of Niemytzki [Ni_1] to non-Hausdorff spaces and give an outline of the proof.

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[1]This is not to say that distances disappeared entirely; see Frink [Fri] for example.

Recall, Niemytzki's metrization theorem in our terminology:

Theorem 6.1 *There is a metric for* $(X, \mathcal{T})$ *iff there is a* γ*-symmetric for* $(X, \mathcal{T})$.

Theorem 6.2 $(X, \mathcal{T})$ *is pseudometrizable iff there is a symmetric* γ*-distance for* $(X, \mathcal{T})$.

Proof: Suppose that d is a symmetric γ-distance for $(X, \mathcal{T})$. It follows from Lemma 4.1, that d is a symmetric pseudo-asemimetric for $(X, \mathcal{T})$, and thus for every $n \in \mathbf{N}$, and for every $p \in X$, $\mathrm{int}_{\mathcal{T}}(S_d(p, 1/2^n))$ is an open neighborhood of p. We also note that for every $p \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$\text{if } d(p, x) < \delta \text{ and } d(x, q) < \delta, \text{ then } d(p, q) < \varepsilon.$$

Using those two facts, we obtain for each $p \in X$, a decreasing sequence of positive radii $\langle \delta_n(p) \rangle$ and a decreasing sequence $\langle U_n(p) \rangle$ of open neighborhoods of p such that $\delta_{n+1}(p) \leq \min\{\delta_n(p), 1/2^n\}$ and,

$$\text{if } d(p, x) < \delta_{n+1}(p) \text{ and } d(x, q) < \delta_{n+1}(p), \text{ then } q \in U_n(p) \subseteq S_d(p, \delta_n(p)).$$

We conclude that $\{U_n(p) | n \in \mathbf{N}\}$ is a local base for p in $(X, \mathcal{T})$ and that

$$\text{if } U_{n+2}(p) \cap U_{n+2}(q) = \emptyset \text{ and } \delta_n(q) \leq \delta_n(p), \text{ then } U_{n+2}(p) \cup U_{n+2}(q) \subseteq U_n(p).$$

Let $\mathcal{U}_n = \{U_{2n}(x) | x \in X\}$, and let d_1 be the distance for $(X, \mathcal{T})$ given by:

$$d_1(p, q) = \begin{cases} 0, & \text{if } q \in st(p, \mathcal{U}_n) \text{ for every } n; \\ 1/2^n, & \text{otherwise, where } n = \min\{k | q \notin st(p, \mathcal{U}_n)\}. \end{cases}$$

It follows from the construction that d_1 is a symmetric distance for $(X, \mathcal{T})$ and that

$$\text{if } d_1(p,x) < \varepsilon \text{ and } d_1(x,q) < \varepsilon, \text{ then } d_1(p,q) < 2\varepsilon.$$

As in Frink's construction [Fri, 1937], we define a pseudometric ρ for $(X, \mathcal{T})$ by:

$$\rho(p,q) = \inf\left\{ \sum_{i=1}^{n-1} d_1(x_i, x_{i+1}) \mid n \in \mathbf{N} \text{ with } x_1 = p, x_n = q \text{ and } x_i \in X \text{ for every } i \right\}.$$

We generalize this theorem, requiring only that the γ-distance be *locally symmetric.*

Theorem 6.3 *$(X, \mathcal{T})$ is pseudometrizable iff there is a locally symmetric γ-distance for $(X, \mathcal{T})$.*

Proof: If d is a locally symmetric γ-distance for $(X, \mathcal{T})$, then by Lemma 4.4 $(d \vee d^*)$ is a symmetric γ-distance for $(X, \mathcal{T})$, and thus, there is a pseudometric for $(X, \mathcal{T})$ from Theorem 6.2.

This theorem appeared in Collins and Roscoe [CR_2, 1984]. Their proof, however, involved a complicated neighborhood construction. Our investigation also gives rise to the next corollary which introduces some additional equivalences for pseudometric spaces.

Corollary 6.4 *For any space $(X, \mathcal{T})$, the following are equivalent:*

(1) *$(X, \mathcal{T})$ is pseudometrizable;*

(2) *there is a locally symmetric, pseudoquasimetric for $(X, \mathcal{T})$;*

(3) *there is a locally symmetric, Π_1-distance for $(X, \mathcal{T})$;*

(4) *there is a locally symmetric, Π_2-distance for $(X, \mathcal{T})$;*

(5) *there is a locally symmetric, γ^*-distance for $(X, \mathcal{T})$.*

Proof: It follows from Lemmas 3.12, 4.14, 4.26,and 5.6 that in each instance there is a symmetric γ-distance, and thus a pseudometric, for $(X, \mathcal{T})$ according to Theorem 6.2.

Theorem 6.5 *For any T_0-space $(X, \mathcal{T})$, the following are equivalent:*

(1) *$(X, \mathcal{T})$ is metrizable;*

(2) *there is a local symmetric d for $(X, \mathcal{T})$ such that: $d[F, K] > 0$, for any closed set F and any disjoint compact set K;*

(3) *there is a local symmetric d for $(X, \mathcal{T})$ such that: for $p \notin F$, F a closed set, there is an $\varepsilon > 0$ such that $S_d(p, \varepsilon) \cap S_d[F, \varepsilon) = \emptyset$.*

Proof: (1$\Rightarrow$2) follows easily.

(2$\Rightarrow$3) Let d be a locally symmetric distance for $(X, \mathcal{T})$ and suppose that, if F is a closed set and K is a disjoint compact set, then $d[F, K] > 0$. Using a similar argument as the one used in the proof of Lemma 4.32, we can show that, if $y_n \to p$ and $d(x_n, y_n) \to 0$, then $x_n \to p$, which, in turn, implies that d is a weak Π_1-distance for $(X, \mathcal{T})$. Therefore, we get the desired result from Lemma 4.31.

(3$\Rightarrow$1) Since $(X, \mathcal{T})$ is a T_0-space, it follows from Corollary 4.34 that d is a Π_1-asymmetric[2] for $(X, \mathcal{T})$. And, since d is locally symmetric, it follows from Corollary 6.4 that $(X, \mathcal{T})$ is pseudometrizable. Since the space is T_1, it is metrizable.

Corollary 6.6 *For any topological space $(X, \mathcal{T})$ the following are equivalent:*

[2] A Π_1-distance for a T_0-space is an asymmetric. See Theorem 4.27

(1) $(X, \mathcal{T})$ *is metrizable;*

(2) *there is a symmetric* d *for* $(X, \mathcal{T})$ *such that* $d[F, K] > 0$, *for any closed set* F *and disjoint compact set* K*;*

(3) *there is a symmetric* d *for* $(X, \mathcal{T})$ *such that for any closed set* F *and* $p \notin F$, *there is an* $\varepsilon > 0$ *such that* $S_d(p, \varepsilon) \cap S_d[F, \varepsilon) = \emptyset$.

This corollary has an interesting history. The equivalence of (1) and (2) was proved originally for Hausdorff spaces by Arhangel'skiĭ [Ar] in 1966. Kenton [Ke, 1971] showed that Hausdorff could be omitted and also proved the equivalence of (2) and (3). Martin [Ma, 1972], independently of Kenton, also showed that Hausdorffness could be removed. Finally, Harley and Faulkner [HF, 1975], also independently of Kenton, proved that (2) is equivalent to (3). We remark that their results, however, implicitly assume a Hausdorff space.

We have shown that Example 5.17 is semimetrizable and γ-asymmetrizable[3], but that there is no *locally symmetric* γ*-distance* for the space. This means that Theorem 6.3 is not a *factorization*[4] of pseudometrizability. Hence, pseudometrizability is assured only when the distance function is simultaneously locally symmetric and a γ-distance.

On the other hand, we have seen that being γ-asymmetrizable and Π_1-asymmetrizable are both necessary, and *independent*, conditions for metrization. Example 4.30 is a Hausdorff Π_1-asymmetrizable space which is not γ-asymmetrizable, since it is not first countable. Example 5.17 is a γ-asymmetrizable T_1-space that fails to be Π_1-asymetrizable, since it lacks unique limits.

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[3]See Example 4.3.

[4]Pseudometrizability factors into two (necessary) conditions when these conditions are *independent* (that is, neither implies the other) and taken together they implies pseudometrizability.

However, Hodel [Ho_1] has shown the following:

a Hausdorff space $(X, \mathcal{T})$ is metrizable iff $(X, \mathcal{T})$ is a Nagata γ-space.

Our next result introduces an analogous factorization of metrization using distances with a proof that follows easily from our initial study of distances.

Theorem 6.7 *$(X, \mathcal{T})$ is metrizable iff $(X, \mathcal{T})$ is Π_1-asymmetrizable and γ-asymmetrizable.*

Proof: Suppose that d_1 is a Π_1-asymmetric for $(X, \mathcal{T})$ and that d_2 is a γ-asymmetric for $(X, \mathcal{T})$. Then, $(X, \mathcal{T})$ has unique limits (see Lemma 4.27 and Lemma 3.4) and is first countable (see Lemma 4.1). This implies that d_1 and d_2 are both asemimetrics for $(X, \mathcal{T})$ so that $x_n \to p$ iff both $d_1(p, x_n) \to 0$ and $d_2(p, x_n) \to 0$.

Let $\rho = d_1 \vee d_2$. Then ρ is simultaneously a γ-distance and a Π_1-distance for $(X, \mathcal{T})$. Therefore, if F is a closed set and K is a compact set in $(X, \mathcal{T})$, then $\rho[K, F] > 0$ and $\rho[F, K] > 0$ according to Theorems 4.8 and 4.33. Applying Lemma 4.24, we conclude that $d = \rho \wedge \rho^*$ is a symmetric distance for $(X, \mathcal{T})$. Finally, since d satisfies the second condition of Corollary 6.6, $(X, \mathcal{T})$ is metrizable.

We now extend Hodel's result to non-Hausdorff spaces. In this setting we lose the property that compact sets are closed. This suggests that different strategies will need to be applied. Our proof is based on a distance function construction that we have exploited in this work.

Theorem 6.8 *$(X, \mathcal{T})$ is pseudometrizable iff $(X, \mathcal{T})$ is a Nagata, γ-space.*

Proof: Suppose that $(X, \mathcal{T})$ is a Nagata, γ-space. It follows from Theorem 4.36 that there is a pseudo Π_1-asemimetric for $(X, \mathcal{T})$, and from Theorem 4.11 that there is γ-distance for

$(X, \mathcal{T})$. Let d_1 be a pseudo Π_1-asemimetric for $(X, \mathcal{T})$, and let d_2 be a γ-distance for $(X, \mathcal{T})$. We note then that,

$$\begin{aligned} x_n \to p &\Leftrightarrow d_1(p, x_n) \to 0 \\ &\Leftrightarrow d_2(p, x_n) \to 0 \\ &\Leftrightarrow d_1(p, x_n) \to 0 \text{ and } d_2(p, x_n) \to 0 \\ &\Leftrightarrow (d_1 \vee d_2)(p, x_n) \to 0. \end{aligned}$$

Let $\rho = (d_1 \vee d_2)$. Then ρ is a pseudo Π_1-asemimetric and a pseudo γ-asymmetric. Since ρ is a Π_1-distance,

$$\rho(x_n, p) \to 0 \Rightarrow \rho(p, x_n) \to 0.$$

Define $d = (\rho \wedge \rho^*)$ then

$$\begin{aligned} d(p, x_n) \to 0 &\Leftrightarrow \rho(p, x_n) \to 0 \\ &\Leftrightarrow x_n \to p. \end{aligned}$$

Hence, d is a pseudo-semimetric for $(X, \mathcal{T})$. Since d is symmetric Π_1-distance, d a γ-distance, and therefore $(X, \mathcal{T})$ is pseudometrizable according to our Theorem 6.2.

In his 1972 paper, Hodel [Ho_1] also proved that:

a T_0-space is metrizable iff it is a developable Nagata space.

We now generalize this result.

Theorem 6.9 *A T_0-space is metrizable iff it is a first countable space that is both weakly Π_2-asymmetrizable and weakly Π_1-asymmetrizable space.*

Proof: Let $(X, \mathcal{T})$ be a first countable, weakly Π_1-asymmetrizable, weakly Π_2-asymmetrizable, T_0-space. It follows from Corollary 4.35 and Corollary 4.29 that $(X, \mathcal{T})$ is Π_1-asemimerizable and thus by Theorem 4.36 that $(X, \mathcal{T})$ is a Nagata space. Since $(X, \mathcal{T})$ is Π_1-asemimetrizable, it has unique limits, and hence, it follows from Corollary 5.15 that $(X, \mathcal{T})$ is developably semimetrizable. Therefore, $(X, \mathcal{T})$ is a developable space (see Theorem 5.13). Now, apply Hodel's result.

First countability plays a crucial part in this theorem.

Example 6.10 *A* Π_1*-asymmetrizable,* Π_2*-asymmetrizable Hausdorff space need not be metrizable.*

The distance given for Example 4.15 is a Π_1-asymmetric as well as a Π_2-asymmetric for $(X, \mathcal{T}_d)$. As we have already noted, this space is not metrizable.

We conclude this chapter by showing that for weak Π_2-asymmetric, local symmetry is enough to obtain metrizability.

Theorem 6.11 $(X, \mathcal{T})$ *is metrizable iff there is a locally symmetric, weak* Π_2*-asymmetric for* $(X, \mathcal{T})$.

Proof: Let d be a locally symmetric, weak Π_2-asymmetric for $(X, \mathcal{T})$. It follows then that $(d \wedge d^*)$ is a symmetric for $(X, \mathcal{T})$. We will establish that $(d \wedge d^*)$ is a Π_1-symmetric and thus a γ-symmetric for $(X, \mathcal{T})$ so that metrizability will follow from Theorem 6.7.

We begin by observing that $(d \wedge d^*)$ has unique limits for, if $(d \wedge d^*)(p, x_n) \to 0$ and $(d \wedge d^*)(q, x_n) \to 0$, then $d(x_n, p) \to 0$ and $d(x_n, q) \to 0$ and thus $p = q$ by the T_1-propety of $(X, \mathcal{T})$.. Finally, if $(d \wedge d^*)(p, y_n) \to 0$ and $(d \wedge d^*)(x_n, y_n) \to 0$, then $d(y_n, p) \to 0$ and

$d(y_n, x_n) \to 0$ so that $x_n \to p$. However, since $(d \wedge d^*)$ has unique limits, it follows that $(d \wedge d^*)(p, x_n) \to 0$ (see Lemma 3.5), and thus, $(d \wedge d^*)$ is a Π_1-symmetric for $(X, \mathcal{T})$.

Corollary 6.12 *$(X, \mathcal{T})$ is metrizable iff there is a weak Π_2-symmetric for $(X, \mathcal{T})$.*

Chapter 7

Conclusion

As non-Hausdorff spaces become more important in topology, there is a need to consider new notions in topology to supplement the usual structures. This work has been a step in this direction with the introduction of *asymmetrizable* spaces as a generalization of symmetrizable spaces. This generalization took place in a non-Hausdorff setting, and uses distance functions that lack the axiom of symmetry, which according to Reilly [Rei] have the following advantages:

> most of the distance functions we meet in everyday life late in the twentieth century seem to be inherently non-symmetrical. Examples are the "shortest-time-taken" distance and the "minimum-energy-consumed" distance, and these have relevance when consideration is taken of such things as topography, prevailing winds, river and ocean currents, and barrier to travel such as one-way streets systems. If mathematical models should reflect reality, then the metric model of distance is too restrictive.[1]

The first part of this work is a historical overview of the evolution topology. In keeping with the theme of non-Hausdorffness, we have focused mainly on the contribution of

[1] Reilly, *On non-Hausdorff spaces*, p.332.

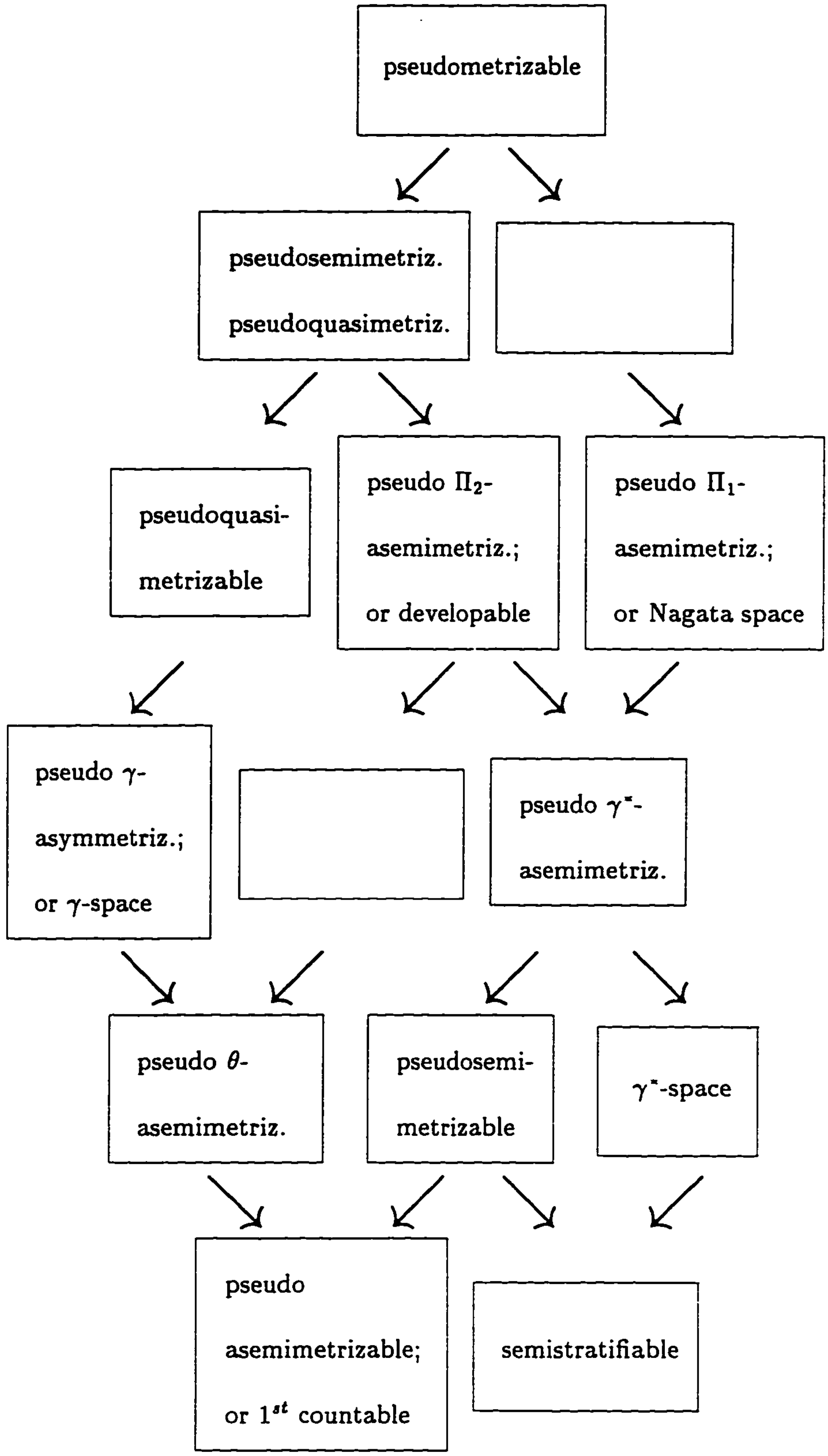
Maurice Fréchet and the mathematicians that followed his approach. This does not downplay the importance of Felix Hausdorff, whose contribution to topology cannot possibly be *understated*. It does however shed some light on the early years of topology from a different point of view. It is in the work of mathematicians who sought to generalize Fréchet's contribution by means of distances that we find the most useful information for this study.

In Chapter 4 we provided a number of interesting distinguishing characteristics for spaces determined by our asymmetrics and in turn provided a generalization for a number of spaces studied in the area of generalized metric space.

In Chapter 5 we established developably asymmetrizable spaces as a generalization of developable spaces.

In Chapter 6 we obtain a number of pseudometrization and metrization results using techniques developed in the previous chapters. In particular, we introduce a generalization of the metrization theorem of Roscoe and Collins.

The following diagram summarizes the relationships among the topological spaces considered in this work:



In retrospect, we have established that asymmetrizable spaces provide, in a non-Hausdorff setting, significant generalizations for spaces that have traditionally been important in the

Hausdorff context. We also note that the arguments used in the proofs are for the most part straightforward, which comes as a surprise considering the class of spaces involved in this study.

Perhaps more significantly, we have initiated a study of what we have called ***weak distances***, which we have noted as having strong ties with the topological spaces they generate; see Lemmas 4.5, 4.16, 4.31 and 5.7.

In addition, the classes of spaces generated by our weak distances always properly generalize their counterparts, which are necessarily classes of first countable spaces. The role that these distances can play has not been fully explored, and the properties of these spaces need to be more clearly understood. This warrants further studies.

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