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Contributions to the theory of neighborhoods and its applications

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CONTRIBUTIONS TO THE THEORY OF NEIGHBORHOODS AND ITS APPLICATIONS

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DEDICATION

To my parents

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I would like to acknowledge Sam Shore for his many contributions to my growth as a mathematician and educator. His professionalism and friendship helped tremendously in guiding this dissertation to fruition. Thanks, Sam!

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ABSTRACT

CONTRIBUTIONS TO THE THEORY OF NEIGHBORHOODS AND ITS APPLICATIONS

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Neighborhoods have played a fundamental role in general topology since the birth of the field. This work outlines the historical evolution of the notion of neighborhood and employs neighborhood assignments, weak neighborhood assignments, and a naturally induced notion of duality in a study of non-Hausdorff topological spaces. Neighborhood characterizations of various classes of spaces, among them the developable and the pseudometrizable spaces, are obtained. A generalization of topological spaces based upon a primitive notion of neighborhood is explored and examples are supplied to motivate the investigation.

INTRODUCTION

This work focuses on the mathematical notion of neighborhood.

Chapter One traces the formulation and evolution of this concept during the early part of the twentieth century and examines its connections with the historical development of topological spaces.

In Chapter Two neighborhood assignments are employed in a study of non-Hausdorff topological spaces. The important topological concepts of pseudometrizability and developability, among others, receive close scrutiny.

A generalized setting for the introduction of neighborhoods is provided in Chapter Three. Several examples are developed there to suggest the significance of non-topological neighborhood structures.

In the body of the text boldface type is used to indicate a word or phrase is being defined. Although we have suppressed the *only if*, definitions are, of course, understood to be *if and only if* statements.

We denote the set of positive integers and the set of real numbers by $\mathbf{N}$ and $\mathbf{R}$, respectively.

A sequence x will be denoted by (x_n), where x_n represents $x(n)$. If (x_n) is a sequence in a topological space (X, τ) and (x_n) converges to p, we write $x_n \xrightarrow[\tau]{} p$.

If (X, τ) is a topological space and $A \subseteq X$, we write $\overline{A}^{\tau}$ or simply $\overline{A}$ for the closure of A in (X, τ), $\mathrm{Int}_{\tau}(A)$ for the interior of A in (X, τ), and $\tau|A$ for the relative topology on A induced by τ.

The reader may refer to Munkres' *Topology, A First Course* [Mu] for standard topological terminology and notations not defined in this dissertation.

CHAPTER ONE

AN OVERVIEW OF THE EVOLUTION OF NEIGHBORHOODS IN GENERAL TOPOLOGY

Given a topological space (X, τ), an element $p \in X$, and a set $N \subseteq X$, it is common to refer to N as a **neighborhood** of p provided that there exists $G \in \tau$ such that $p \in G \subseteq N$. The collection of all neighborhoods of p in (X, τ) will be denoted by $\mathcal{N}_\tau(p)$.

This notion of neighborhood can be considered primitive and from it the theory of topological spaces can be derived. Specifically:

1.1 Theorem. If X is a nonempty set and for each $x \in X$ there is a nonempty collection $\mathcal{A}(x)$ of subsets of X satisfying

(1) $A \in \mathcal{A}(x) \;\Rightarrow\; x \in A$,

(2) $A, B \in \mathcal{A}(x) \;\Rightarrow\; A \cap B \in \mathcal{A}(x)$,

(3) $(A \in \mathcal{A}(x)$ and $A \subseteq B) \;\Rightarrow\; B \in \mathcal{A}(x)$, and

(4) $A \in \mathcal{A}(x) \;\Rightarrow\; (\exists B \in \mathcal{A}(x)$ with $B \subseteq A$ and $B \in \mathcal{A}(y) \; \forall\, y \in B)$,

then $\tau = \{S \subseteq X : \forall\, x \in S, \exists A_x \in \mathcal{A}(x), A_x \subseteq S\}$ is a topology on X and, for each $x \in X$, $\mathcal{N}_\tau(x) = \mathcal{A}(x)$.

In this chapter we sketch the evolution of the notion of neighborhood in general topology from its roles in Hilbert's axioms for an abstract plane and Veblen's definition of a linear continuum to its appearances, in more or less contemporary form, in Hausdorff's and, later, Fréchet's, axioms for a topological space. We also discuss the relationship of the neighborhood concept to other primitive notions on which mathematicians of the early part of the twentieth century attempted to base a theory of abstract spaces. Generalization and axiomatization were motivating a great deal of mathematical research during this time and it is worth remembering that many mathematics practitioners of this time were careful to distinguish between the terms *point* and *element*. *Points* were the members of well-known mathematical sets and were often capable of being interpreted geometrically, for example the points of n-dimensional Euclidean space, while *elements* were members of sets or classes which were to be understood solely through the axioms imposed on them. Thus, in paraphrasing the axioms laid out by the mathematicians cited in this chapter, we have paid more attention to this distinction than would normally be desirable (and more than will be paid to it in the following chapters of this dissertation).

We begin with David Hilbert [Hlb] who, in 1902, employs a notion of neighborhood to formulate axioms for an "abstract plane" in his book *Grundlagen der Geometrie* . Included among these axioms are the following:

(1) The plane is a set of objects called points. Every point determines certain subsets of the plane, called neighborhoods of the point. A point belongs to each of its neighborhoods.

(2) If q is any point in a neighborhood of p, then this neighborhood is also a neighborhood of q.

(3) For any two neighborhoods of a point p, there exists a neighborhood of p that is contained in their intersection.

(4) If p and q are any two points of the plane, then there exists a neighborhood of p which also contains the point q.

It needs to be pointed out that Hilbert's objective in setting down these, and several other, axioms is the exploration of the foundations of plane geometry, not the study of abstract spaces *per se*.[1] To Hilbert, the abstract plane is simply an axiomatized version of what he calls the "number plane," that is, the Cartesian plane.

[1]Thus, Hilbert uses the word *point* rather than *element*.

However, it is also apparent that Hilbert's axiomatic system documents several ideas that will later emerge as cornerstones in the formulation of an abstract notion of neighborhood and, on a broader level, the development of the theory of topological spaces. For instance, we may note that, in relation to modern terminology, Hilbert's neighborhoods of a point are open (Axiom 2) and constitute a structure closer to a neighborhood base at the point rather than an actual neighborhood system. Exactly how much influence Hilbert's conception of neighborhoods had on the development of neighborhoods in general topology is unclear because the evidence on this front is sketchy.

Although he does not use the term neighborhood, Oswald Veblen [Ve], in his 1905 definition of a linear continuum , introduces the term segment to refer to the set of all elements in a linearly ordered set that lie (strictly) between two given elements. Veblen's axioms, besides defining a linear order, define completeness in terms of Dedekind cuts and include postulates of closure, density, and uniformity. Together, these axioms allow him to conclude that every bounded set in a linear continuum has a supremum, an infimum, and at least one limiting element (for Veblen, a limiting element of a set S is an element p having the property that every segment containing p contains an element of S distinct from p), and to prove a version of the Heine-Borel Theorem.

Veblen's observations clearly convey the spirit of Cantor's point-set theory for the real line. In fact, the evolution of the notion of neighborhood cannot be separated from the attempt to generalize this theory. The formulation, in an abstract setting, of an appropriate notion of limiting element of a set had become a primary goal of mathematical analysts around the turn of the century. The idea of taking a notion of neighborhood of a point as primitive was not as popular. Since Veblen's segments may be viewed as neighborhoods of each of their elements, his work contributes not only to the explicitly defined limiting element concept, but also, quite significantly, to the development of neighborhoods. As with Hilbert, there is a sense of "openness" attaching itself to the neighborhood concept and, like Hilbert's axioms, Veblen's explorations are anchored to a specific agenda (defining a linear continuum). It is with the investigations of Maurice Fréchet and Frederic Riesz that we begin to encounter truly abstract settings for the study of point-sets.

By 1906 Maurice Fréchet [Fr1] has defined several different types of abstract spaces in an attempt to develop an axiomatic point-set theory. In each of these spaces an element p is a limit element of a set S if there is a sequence of distinct elements from S that converges to p. The set of all limit elements of a set is called the derived set of the given set. A set is then taken to be closed if it contains its derived set. The notion of convergence of a sequence varies from space to space, however.

In those spaces Fréchet referred to as being *classe (V)*, the notion of limit of a sequence is defined through the use of what Fréchet calls a *voisinage*, a kind of distance function. The *voisinage* (,) returns, for every pair of elements, a nonnegative real number to be interpreted as the distance between the elements, and is required to satisfy the following properties:

(1) $(a, b) = 0 \iff a = b$;

(2) $(a, b) = (b, a)$;

(3) there exists a nonnegative real-valued function φ defined on $\mathbf{R}$ for which $\varphi(x_n) \to 0$ whenever $x_n \to 0$, and

$$\big((a, b) < \delta,\ (b, c) < \delta\big) \implies (a, c) < \varphi(\delta).$$

A sequence (a_n) is then taken to converge to p if and only if $(a_n, p) \to 0$ as $n \to \infty$.

It should be noted that the French word *voisinage* can be translated as *neighborhood*. Thus, Fréchet, in his *classe (V)*, which we may observe is actually the class of metrizable topological spaces, introduces a notion of neighborhood. Of course, this notion of neighborhood is still quite far removed from the modern topological notion of neighborhood in the sense that, formally, a *voisinage* does not represent a set of elements, but rather a nonnegative real number. However, once our attention is fixed on a certain element p, the set $S(p, \varepsilon)$ of all elements whose distances from p are less than some specified positive number ε forms, on an intuitive level, what we may think of as a neighborhood of p, that is, a set of elements "near" the given element p. Thus, Fréchet's use of abstract distances leads naturally to a notion of sphere centered at an element and these spheres can be regarded as neighborhoods in the distance setting, although they may not represent neighborhoods in the topological setting induced by the distance.

At approximately the same time (1907), Frederic Riesz [Ri], working independently of Fréchet, puts forth axioms for what he terms a mathematical continuum. Significantly, where Fréchet makes use of distances, Riesz suppresses any notion of distance and concentrates instead on a primitive notion of derived set and a consequent notion of neighborhood of a point. Part of the motivation for Riesz's approach results from his belief that certain mathematical theories, such as Cantor's theory of ordinal numbers and order types, are not equipped with intrinsic notions of distance, and, thus, any suitably abstract conception of space should not rely on distance at the primitive level.

Riesz's axioms for a mathematical continuum employ an undefined notion of derived set built upon the principle that for any element of the continuum and any set in the continuum, either the element is isolated from the set or the element is a limit element of the set. Intuitively, Riesz is thinking of the derived set, denoted S', of a set S as being the set of all limit elements of S. Thus, an element is isolated from S if it is not a member of S'. Riesz's axioms may be stated as follows:

(1) if S is finite, then $S' = \varnothing$;

(2) if $S = T \cup U$, then $S' = T' \cup U'$;

(3) if $p \in S'$ and $q \neq p$, then there exists $T \subseteq S$ with $p \in T'$ and $q \notin T'$.

It is at this point that Riesz formally introduces a notion of neighborhood[2] of an element in a continuum X:

A set U is a neighborhood of an element p if

(1) $p \in U$, and

[2]In German the word for neighborhood is *Umbegung*. Hence, Riesz uses the letter U to represent an arbitrary neighborhood of an element.

(2) p is isolated from the complement of U (i.e. $p \notin (X - U)'$).

In this context, Riesz makes the following observations:

(1) Any finite intersection of neighborhoods of an element is a neighborhood of that element.

(2) $p \in S'$ if and only if every neighborhood of p contains infinitely many elements of S.

Both Fréchet and Riesz believe their works to be "first drafts" for a generalized version of Cantor's point-set theory. By 1910 a number of American mathematicians, among them E.H. Moore [Mo] and his student T.H. Hildebrandt [Hld], have begun to consider and build upon the research of Fréchet and Riesz. It is E.R. Hedrick [Hed], a mathematician at the University of Missouri, and Ralph Root ([Ro1], [Ro2], [Ro3]), another doctoral student of E.H. Moore at the University of Chicago, however, who offer the most significant contributions to an abstract theory of point-sets based upon a notion of neighborhood.

Following Fréchet, Hedrick (1910) considers spaces for which there is a primitive notion of sequential limiting element. The derived set of a set S is taken to be the set of all limiting elements of sequences in S. Hedrick also assumes that every infinite set has a limiting element (i.e. the space is compact) and that every derived set is closed (that is, contains all of its limiting elements). He then develops an axiom of "enclosability" that is based on the nested intervals property of the set of real numbers.

1.2 Nested Intervals Property. Suppose that, for each $n \in \mathbf{N}$, a_n and b_n are real numbers with $a_n < b_n$ and the interval $[a_{n+1}, b_{n+1}]$ is a subset of the interval $[a_n, b_n]$. If $\lim_{n \to \infty} (b_n - a_n) = 0$, then there exists a real number p such that $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{p\}$.

Hedrick's enclosability axiom is:

For each element $p \in X$ there exists a sequence $(Q_n(p))$ of closed sets such that

(1) $\bigcap_{n=1}^{\infty} Q_n(p) = \{p\}$,

(2) for each $n \in \mathbf{N}$, $Q_{n+1}(p) \subseteq Q_n(p)$, and

(3) for each $n \in \mathbf{N}$, there exists $m_n \in \mathbf{N}$ such that for any element $a \in X$, if $a \in Q_n(x)$, then $Q_n(x) \subseteq Q_{m_n}(a)$.

The closed sets whose existence is assumed in this axiom have the "flavor" of closed neighborhoods which are "shrinking down" on a particular element. Thus, the neighborhoods of an element distinguish that element from the other elements in the space. Hedrick cites Veblen's work with the linear continuum as partial motivation for his axiom.

Then, Ralph Root, in two papers published in 1914, but completed in April, 1912 [Ro2], and March, 1913 [Ro3], delineates axiom systems in which neighborhoods occupy the primitive role.[3] Root chooses a neighborhood approach based on his "thought that in most of the definitions of limit that are employed in current mathematics a notion analogous to that of 'neighborhood' or 'vicinity' of an element is fundamental."[4]

In the earlier of the two papers, Root considers a system that includes a class P of elements and a class U of what he calls ideal elements, together with a binary relation between subclasses of P and elements of $P \cup U$ that can be viewed as describing neighborhoods of elements and which results in the following set of axioms:

(1) If N is a neighborhood of $p \in P$, then $p \in N$.

(2) Every neighborhood of an ideal element contains an element of P.

(3) For any $p \in P$ there is a sequence (A_n) of neighborhoods of p such that for any neighborhood N of p there exists k such that $A_n \subseteq N$ for every $n \geq k$.

(4) For every neighborhood N of $p \in P$ there is a neighborhood M_N of p for which each element of M_N has a neighborhood that is a subset of N.

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[3] Root announced his work in an abstract [Ro1] published in the Bulletin of the American Mathematical Society in 1911.

[4] Root, *Iterated limits in general analysis*, American Journal of Mathematics 36 (1914), p. 79.

(5) Distinct elements of P have disjoint neighborhoods.

Root then defines an element to be a limit element of a set S if each neighborhood of the element includes an element of S distinct from the given element. He also defines an element to be a limit of a sequence of elements if the terms of the sequence are eventually in any neighborhood of the element. He is then able to show that x is a limit element of a set S if and only if x is a limit of some sequence in S, in other words, the "neighborhood" and the "sequential" definitions of limiting element agree, and that derived sets are closed, that is, the set of all limit elements of a set contains all of its limit elements. Further, Root shows that the derived set axioms of Riesz discussed above are satisfied.

Before completing the second paper, Root spends a year at the University of Missouri studying with Hedrick. The resulting paper is also clearly influenced by the prior work of Veblen. An undefined notion of an element being between two other elements is employed to obtain a definition of segment mirroring that used by Veblen.[5] A segment is then to be regarded as a neighborhood of each of its elements. Root then imposes the following axioms:

(1) Every element belongs to some segment (such a segment is called a neighborhood of the element).

(2) Given two neighborhoods of an element there is a neighborhood of the element that is contained in their intersection.

(3) Any two distinct elements have neighborhoods whose intersection is empty.

Finally, by providing two examples of what will soon be known as non-first countable topological spaces, Root points out that, for the spaces being considered in this paper, the neighborhood and sequential definitions of limiting element need not yield identical theories.

It is unclear how Root's investigations may have influenced Felix Hausdorff's seminal contributions to the concept of neighborhood which lead directly to the formulation

[5]A segment consists of all the elements (strictly) between two elements.

of topological space that so much mathematics of our century has relied upon [Ha]. In a series of lectures he offered in Bonn during 1912, Hausdorff describes four properties of the interiors of spheres (he refers to the interior of a sphere centered at a point p as a neighborhood of p) determined by the usual metric for the set E of points in n-dimensional Euclidean space:

(α) Each neighborhood of p contains p.

(β) For any two neighborhoods U and V of p, either $U \subseteq V$ or $V \subseteq U$.

(γ) If q is in a neighborhood U of p, then there exists a neighborhood V of q such that $V \subseteq U$.

(δ) If $p \neq q$, then there exist neighborhoods U of p and V of q such that $U \cap V = \varnothing$.

Although Hausdorff states these properties within a specific mathematical context, his subsequent remarks suggest that he intends to take them as axioms for an abstract space constructed from a primitive notion of neighborhood: "The following considerations depend only on these properties. They are valid, therefore, when E is a point set to whose points x correspond sets U_x having the four properties listed."[6]

Then, in 1914, Hausdorff's famous text, *Grundzüge der Mengenlehre*, is published. In it he considers three possible primitives for an abstract space: distance, limit element of a sequence, and neighborhood. He settles on the notion of neighborhood because he believes it to offer more generality than distances and because it is not tied to countability as are sequences. Hausdorff then introduces the notion of topological space, an abstract class in which each element is assigned a collection of subsets, called neighborhoods of the element, from the class that are subject to the following axioms:

(A) Each element of the class has at least one neighborhood and the element is contained in each of its neighborhoods.

---

[6]Taylor, *A study of Maurice Fréchet: II. Mainly about his work on general topology, 1909-1928*, Archive for History of Exact Sciences 34 (1985), p. 301 (Hausdorff as quoted by Professor Günter Bergmann of the University of Münster and translated by A.E. Taylor).

(B) If U and V are neighborhoods of p, there exists a neighborhood W of p such that $W \subseteq U \cap V$.

(C) If q is in a neighborhood U of p, then there exists a neighborhood V of q such that $V \subseteq U$.

(D) If $p \neq q$, then there exist neighborhoods U of p and V of q such that $U \cap V = \varnothing$.

We may note that axioms (A), (C), and (D) are, respectively, the properties (α), (γ), and (δ) Hausdorff considered in his Bonn lectures. It seems reasonable to view property (β) as a preliminary version of axiom (B), although, according to A.E. Taylor [Tay], a mathematician-historian who has had access to what remains of Hausdorff's unpublished notes, there does not appear to be any extant evidence suggesting how Hausdorff eventually decides to replace (β) with (B) or exactly when the replacement occurs. The obvious questions are:

(1) Was Hausdorff aware of Root's work with neighborhoods?

(2) To what degree might Hilbert's axioms have influenced the substitution of (B) for (β)?

At this time, resolution of these issues seems highly unlikely.

The neighborhoods of an element utilized in Hausdorff's axioms for a topological space are actually, in modern terminology, open neighborhoods of the element and form a neighborhood base for a (Hausdorff) topology on the given class.

1.3 Definition. Let (X, τ) be a topological space and let $p \in X$. A collection $\mathcal{B}(p) \subseteq \mathcal{N}_\tau(p)$ is a **neighborhood base** for p provided that the collection of all supersets of members of $\mathcal{B}(p)$ is $\mathcal{N}_\tau(p)$.

1.4 Theorem. If X is a nonempty set and for each $x \in X$ there is a nonempty collection $\mathcal{A}(x)$ of subsets of X satisfying

(1) $A \in \mathcal{A}(x) \Rightarrow x \in A$,

(2) $A, B \in \mathcal{A}(x) \Rightarrow (\exists C \in \mathcal{A}(x), C \subseteq A \cap B)$, and

(3) $A \in \mathcal{A}(x) \Rightarrow \left(\exists A_0 \in \mathcal{A}(x), \forall y \in A_0, \exists A_y \in \mathcal{A}(y), A_y \subseteq A\right)$,

then $\tau = \{S \subseteq X: \forall x \in S, \exists A_x \in \mathcal{A}(x), A_x \subseteq S\}$ is a topology on X and, for each $x \in X$, $\mathcal{A}(x)$ is a neighborhood base at x in (X, τ).

Of course, by omitting Hausdorff's axiom (D), we find ourselves within the realm of arbitrary topological spaces rather than just Hausdorff topological spaces.

Hausdorff's book is, at least after World War I, widely read and offers a particularly lucid and instructive account of his ideas. Meanwhile, Fréchet [Fr4], who is apparently unaware of Hausdorff's work with neighborhoods, begins (in 1917) to reformulate his various abstract spaces along the lines of a neighborhood approach.

In his 1906 thesis Fréchet had introduced a class of spaces he called *classe (L)* which are based on a primitive notion of limit element of a sequence and for which the following axioms hold:

(1) A constant sequence converges (to the obvious limit).

(2) Any subsequence of a convergent sequence converges to the same limit.

(3) Sequential limits are unique.

He now redefines *classe (V)* to refer to those spaces X of *classe (L)* having the property that to each element $x \in X$ there is assigned a sequence $(U_n(x))$ of subsets, called neighborhoods of x, of X such that $x_n \to p$ iff $\forall n, \exists k_n, \forall m \geq k_n, x_m \in U_n(p)$. It then follows that, for each $x \in X$, $\bigcap_{n=1}^{\infty} U_n(x) = \{x\}$.

However, Fréchet [Fr2] almost immediately discards this definition of *classe (V)* in favor of a more general definition that allows, as a consequence, for limit elements of sets to be defined without reference to sequences (and, thus, to do away with any direct link to countability). A space X will now be called *classe (V)* if, to each element $x \in X$, there is assigned a nonempty collection of subsets of X called neighborhoods of x. No axioms for these neighborhoods are assumed (it is not even required that an element be in each of its neighborhoods). An element $p \in X$ is then taken to be a limit element of a set $S \subseteq X$ if every neighborhood of p contains an element of S different from p. Fréchet states that this new definition of *classe (V)* is based on notes he had put together before the war, although there is no corroborating evidence of this. A fairly extensive treatment of the new *classe (V)* can be found in Sierpinski's 1934 book, *Introduction to General Topology* [Si].

In 1918 Fréchet publishes a paper [Fr3] in which he notes that the derived sets, defined in the usual manner, in a *classe (V)* satisfy the following properties:

(I) $A \subseteq B \Rightarrow A' \subseteq B'$;

(II) $p \in A' \Leftrightarrow p \in (A - \{p\})'$.

He also notes that (I) and (II) follow from the first two of Riesz's derived set axioms. Then, he observes that an arbitrary class X based upon a primitive notion of derived set that satisfies (I) and (II) generates a *classe (V)* whose induced derived sets are identical to the original primitive-based derived sets.[7]

Recall that Riesz required that an element should belong to each of its neighborhoods. Fréchet, on the other hand, believes that assuming an element is not in any of its neighborhoods provides for simpler, and more elegant, arguments. Ultimately, it makes no difference which approach is taken as the theories developed from them are parallel. With Riesz, there is a unique element that is a member of all of the neighborhoods of a given element, namely the given element; with Fréchet, the intersection of all of the neighborhoods of a given element is empty.

Fréchet next defines a *classe (H)*[8] to be a *classe (V)* satisfying:

(H1) An element belongs to each of its neighborhoods.

---

[7]Simply define N to be a neighborhood of p provided that $p \notin (X - N)'$.

[8]Fréchet mentions in his 1921 paper [Fr5], *Sur les ensembles abstraits* (Annales Ecole Norm. Sup. 38, pp. 341-388), that he has chosen this designation to honor Hedrick, whom he credited for generating the idea of this *classe*.

(H2) Given two neighborhoods of an element there is a neighborhood of the element contained in the intersection of the two given neighborhoods.

(H3) Given two distinct elements there is a neighborhood of each that does not contain the other.

(H4) Given a neighborhood N of an element there is a neighborhood of that element each of whose elements has a neighborhood contained in N.

Observe that (H1) and (H2) are the same as Hausdorff's axioms (A) and (B), respectively, while (H3) is the T_1 separation axiom which, of course, is less restrictive than Hausdorff's axiom (D).

Comparison of (H4) with Hausdorff's axiom (C) is instructive. Following the convention of Riesz, Fréchet defines an element to be an interior element of a set if the set is a neighborhood of the element. Hausdorff's axiom (C) requires that all of the elements of a neighborhood be interior elements of the neighborhood (i.e. the neighborhood must be open), while (H4) requires only that each neighborhood of an element be associated to a neighborhood of that element all of whose elements are interior elements of the original neighborhood (hence, neighborhoods need not be open). Fréchet's opinion, even when he finally does become familiar with Hausdorff's work, is that "openness" of neighborhoods may be unnecessarily restrictive and perhaps even contrary to the very nature of the neighborhood concept. In fact Fréchet's desire for generality leads him to refer to any space based upon a primitive notion of derived set satisfying only property (II) described above as a topological space.

In some respects the evolution of the neighborhood concept reaches its climax with Hausdorff's axioms. We have expounded at some length on Fréchet's contributions, some of which seem likely to have been made after the publication of *Grundzüge der Mengenlehre*, simply because it appears that he formulated his axioms concerning neighborhoods without any knowledge of Hausdorff's investigations. Root, also, must receive credit as his results generally pre-date Hausdorff's and are published in very widely circulated journals.[9]

Following the efforts of Hausdorff and Fréchet, topological spaces came to be regarded by most mathematicians as the "appropriate" setting for the study of continuity and a variety of limit processes. The field matured, a great number of problems were posed (and many of them solved), and aspects of the theory were applied to other branches of mathematics. For more than half a century the notions of topological space and neighborhood remained essentially "unrevised."

In the last twenty years, though, spawned primarily as a result of problems in applied and theoretical computer science, general topology has undergone a renaissance of sorts. Non-Hausdorff topologies have finally found meaningful applications[10] and, hence, there is a great need for additional research into and deeper understanding of the spaces they generate. Chapter Two of the present work contributes to the theory of non-Hausdorff topological spaces by focusing attention on the neighborhood assignments their topologies generate.

And, once more, mathematicians as well as computer scientists, are giving attention to foundational issues and considering structures more general than topological spaces.[11] Central to many of these generalizations is a notion of neighborhood. Chapter Three of this dissertation investigates a generalization introduced by M.B. Smyth [Sm] and known as a *neighborhood space*.

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[9]Root's initial abstract was published in the Bulletin of the American Mathematical Society (1911). The 1912 manuscript was published in the American Journal of Mathematics and the 1913 manuscript in the Transactions of the American Mathematical Society.

[10]Approximating a computer's output, for instance, requires that not all of the approximations be separated from the output in the T_2 sense.

[11]Most of these structures are, however, less general than Fréchet's *classe (V)*.

CHAPTER TWO

NEIGHBORHOODS AND NON-HAUSDORFF TOPOLOGIES

Due primarily to a variety of applications in computer science and its theoretical foundations (see, for example, [Sm] and [Ko]), non-Hausdorff topologies are playing a more prominent role in general topology than at any other time in the history of the field. That many such topologies can be generated through the use of neighborhoods is of particular interest to us. In this chapter we initiate a study of neighborhoods with the goal of developing useful non-Hausdorff generalizations of classes of topological spaces that have found significance in the Hausdorff setting. In addition we indicate how neighborhood assignments satisfying various conditions can be used to identify differences among certain fundamentally important topological properties, including pseudometrizability, developability, and pseudoquasimetrizability.

1. Neighborhoods and Weak Neighborhoods

An indexed family $\{N_\alpha(x): x\in X, \alpha\in I\}$ of subsets of X is called a **neighborhood assignment** in the topological space (X, τ) provided that

$$\text{for each } p\in X \text{ and each } \alpha\in I,\ N_\alpha(p)\in\mathcal{N}_\tau(p).$$

In what follows the word *neighborhood* will often be abbreviated to *nbhd.*

2.1.1 Theorem. For any nbhd assignment $\{N_\alpha(x): x\in X, \alpha\in I\}$ in (X, τ), the following are equivalent:

(1) for every $p\in X$, $\{N_\alpha(p): \alpha\in I\}$ is a nbhd base for p in (X, τ);

(2) $\tau = \left\{A \subseteq X : \forall p \in A, \exists \alpha_p \in I, N_{\alpha_p}(p) \subseteq A\right\}$.

Proof. Assume (1) and let $S = \left\{A \subseteq X : \forall p \in A, \exists \alpha_p \in I, N_{\alpha_p}(p) \subseteq A\right\}$. Every member of S is clearly a neighborhood of each of its points, so $S \subseteq \tau$. On the other hand, if $p \in G \in \tau$, there exists, by hypothesis, $\alpha_p \in I$ with $N_{\alpha_p}(p) \subseteq G$; hence, $\tau \subseteq S$.

Conversely, assume (2) and suppose $N \in \mathcal{N}_\tau(p)$. Then $p \in \mathrm{Int}_\tau(N) \subseteq N$ and, since $\mathrm{Int}_\tau(N) \in \tau$, there exists, by hypothesis, $\alpha_p \in I$ with $N_{\alpha_p}(p) \subseteq \mathrm{Int}_\tau(N) \subseteq N$.

Neighborhood assignments that are indexed by **N** will be particularly significant in what follows. Any family $\{S_n(x) : x \in X, n \in \mathbf{N}\}$ of subsets of X is **decreasing** [Ho2] if

for each $p \in X$, $S_{n+1}(p) \subseteq S_n(p)$ for every $n \in \mathbf{N}$.

2.1.2 Theorem. For any decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ), the following are equivalent:

(1) for every $p \in X$, $\{N_n(p) : n \in \mathbf{N}\}$ is a nbhd base for p in (X, τ);

(2) for every $p \in X$, $x_n \in N_n(p), \forall n \;\Rightarrow\; x_n \xrightarrow[\tau]{} p$;

(3) for every $p \in X$, $x_n \in N_n(p), \forall n \;\Rightarrow\; (x_n)$ clusters at p.[12]

Proof. (1)⇒(2) and (1)⇒(3): Assume (1), suppose $x_n \in N_n(p)\ \forall n$, and consider any $M \in \mathcal{N}_\tau(p)$. By hypothesis there exists $k \in \mathbf{N}$ such that $N_k(p) \subseteq M$. Then, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is decreasing, it follows that $N_n(p) \subseteq N_k(p)\ \forall n \geq k$ so that $x_n \in M\ \forall n \geq k$.

[12] Recall that a sequence (x_n) in a topological space (X, τ) *clusters at* $p \in X$ provided that, whenever $N \in \mathcal{N}_\tau(p)$ and $n \in \mathbf{N}$, there exists $m \geq n$ for which $x_m \in N$. Generally, in the neighborhood characterizations appearing in this chapter, $x_n \xrightarrow[\tau]{} p$ can be replaced with (x_n) clusters at p when the given nbhd assignment is decreasing or can itself be replaced by a decreasing nbhd assignment.

(2)$\Rightarrow$(1): Suppose that there exists $p \in X$ for which $\{N_n(p) : n \in \mathbf{N}\}$ is *not* a nbhd base for p. Then there exists $M \in N_\tau(p)$ such that $N_n(p) \not\subset M$ for any $n \in \mathbf{N}$. So, $\forall n, \exists x_n \in N_n(p)$ with $x_n \notin M$. Thus, (x_n) does *not* converge to p.

(3)$\Rightarrow$(2): Assume (3) and suppose $x_n \in N_n(p)\ \forall n$. Then (x_n) clusters at p.

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists M \in N_\tau(p)$ such that $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - M$. Now, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is decreasing, $x_{k_n} \in N_{k_n}(p) \subseteq N_n(p)$ so that, by hypothesis, (x_{k_n}) clusters at p, which is impossible since (x_{k_n}) is never in M.

2.1.3 Corollary. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is first countable;

(2) there exists a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that, for every $p \in X$, $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.

(3) there exists a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that, for every $p \in X$, $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.

Proof. (2)$\Rightarrow$(3): If the given neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is not decreasing, define $M_n(p) = \bigcap_{k=1}^{n} N_k(p)$. Clearly, $\{M_n(x) : x \in X, n \in \mathbf{N}\}$ is a decreasing nbhd assignment in (X, τ). Note that if $x_n \in N_n(p)\ \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$, then it follows that $x_n \in M_n(p)\ \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.

Given a family $\{S_\alpha(x) : x \in X, \alpha \in I\}$ of subsets of X, we define, for each $p \in X$ and each $\alpha \in I$, $S_\alpha^*(p)$ so that

$$S_\alpha^*(p) = \{x \in X : p \in S_\alpha(x)\}$$

and refer to $S_\alpha^*(p)$ as the **dual** of $S_\alpha(p)$.

2.1.4 Lemma. For any family $\{S_n(x) : x \in X, n \in \mathbf{N}\}$ of subsets of X, $\{S_n(x) : x \in X, n \in \mathbf{N}\}$ is decreasing iff $\{S_n^*(x) : x \in X, n \in \mathbf{N}\}$ is decreasing.

Also, for a family $\{S_\alpha(x) : x \in X, \alpha \in I\}$ of subsets of X, we define, for any $A \subseteq X$ and any $\alpha \in I$,

$$S_\alpha[A] = \bigcup_{x \in A} S_\alpha(x).$$

2.1.5 Remark. If $\{N_\alpha(x) : x \in X, \alpha \in I\}$ is a neighborhood assignment in (X, τ), given $p \in X$ and $\alpha \in I$, it does not necessarily follow that $N_\alpha^*(p)$ is a neighborhood of p. For instance, consider the topology $\tau = \{\mathbf{R}, \varnothing\} \cup \{(a, \infty) : a \in \mathbf{R}\}$ on $\mathbf{R}$ and, for each $p \in \mathbf{R}$ and each $n \in \mathbf{N}$, define $N_n(p) = \left(p - \frac{1}{2^n}, \infty\right)$. Clearly, $\{N_n(x) : x \in \mathbf{R}, n \in \mathbf{N}\}$ is a neighborhood assignment in $(\mathbf{R}, \tau)$. But, given $p \in \mathbf{R}$ and $n \in \mathbf{N}$, observe that $N_n^*(p) = \left(-\infty, p + \frac{1}{2^n}\right) \notin \mathcal{N}_\tau(p)$.

Corollary 2.1.3 characterizes first countability using neighborhood assignments. We now define a topological property which may be viewed as dual to first countability. A topological space (X, τ) is **semistratifiable** if there is a (decreasing) neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for every } p \in X,\ x_n \in N_n^*(p), \forall n \ \Rightarrow\ x_n \xrightarrow[\tau]{} p.$$[13]

G.D. Creede [Cr] wrote his dissertation on semistratifiability and attributes the notion to E.A. Michael. The following theorem occurs as a known formulation of semistratifiable spaces.

2.1.6 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is semistratifiable;

[13]Again, we may replace $x_n \xrightarrow[\tau]{} p$ with (x_n) clusters at p.

(2) there exists a (decreasing) nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that, for every closed set F, $F = \bigcap\{N_n[F]: n \in \mathbf{N}\}$.

Proof. Let $\{N_n(x): x \in X, n \in \mathbf{N}\}$ be a nbhd assignment in (X, τ) such that $x_n \in N_n^*(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and consider any closed set F. If $p \in F$, then, $\forall n$, $p \in N_n(p) \subseteq N_n[F]$. On the other hand, if $p \in \bigcap\{N_n[F]: n \in \mathbf{N}\}$, then $\forall n, \exists x_n \in F$ such that $p \in N_n(x_n)$. So, by hypothesis, $x_n \xrightarrow[\tau]{} p$ and, therefore, as F is closed, $p \in F$.

Conversely, assume (2), note that there is no loss of generality in assuming the nbhd assignment is decreasing, and suppose $x_n \in N_n^*(p)$ $\forall n$. If $p \in G \in \tau$, it follows that $X - G = \bigcap\{N_n[X - G]: n \in \mathbf{N}\}$ and, therefore, $p \notin N_k[X - G]$ for some $k \in \mathbf{N}$. So, as $p \in N_k(x_k)$, it follows that $x_k \notin X - G$. Then, since $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is decreasing, $\forall n \geq k$, $p \notin N_n[X - G]$ and, as $p \in N_n(x_n)$, $x_n \notin X - G$. Therefore, $\forall n \geq k$, $x_n \in G$.

In order to facilitate our study of duality, we define a topological space (X, τ) to be

(1) a γ_1**-space** if there is a nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each } p \in X,\ x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p;$$

(2) a γ_1^***-space** if there is a nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each } p \in X,\ x_n \in N_n^*(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p; \text{ and}$$

(3) a $\gamma_1 \gamma_1^*$**-space** if there is a nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for every } p \in X,\ x_n \in N_n(p) \cap N_n^*(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p.$$

Clearly, the γ_1-spaces are precisely the first countable spaces, while the γ_1^*-spaces are precisely the semistratifiable spaces.

If a topological space (X, τ) is a γ_1–space, we refer to any nbhd assignment in (X, τ) satisfying the "γ_1–space condition" as a **γ_1–nbhd assignment**. We employ similar terminology in any situation where a space is defined through the use of nbhd assignments. Thus, besides γ_1–nbhd assignments, we also have γ_1^*–nbhd assignments and $\gamma_1\gamma_1^*$–nbhd assignments. Others will be introduced in due course.

If (X, τ) is both a γ_1–space and a γ_1^*–space, we will say that (X, τ) is a γ_1–,γ_1^*–space. Similar notation is used to refer to other spaces satisfying multiple conditions involving nbhd assignments.

The diagram below summarizes how the spaces we have just introduced relate to one another:

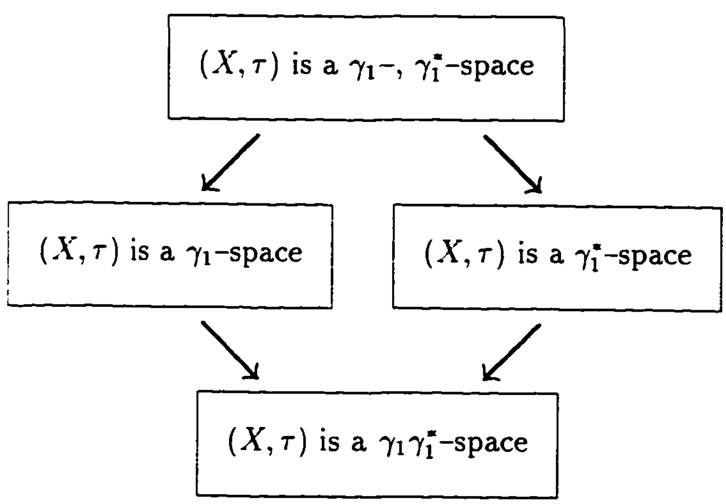

Figure 1. γ_1 and its dual γ_1^*

2.1.7 Lemma. Suppose that $\{L_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1–nbhd assignment in (X, τ) and $\{M_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1^*–nbhd assignment in (X, τ). For each $p \in X$ and each $n \in \mathbf{N}$, define

$$N_n(p) = \left(\bigcap_{k=1}^{n} L_k(p)\right) \cap \left(\bigcap_{k=1}^{n} M_k(p)\right).$$

Then $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is a decreasing nbhd assignment in (X, τ) that is both a γ_1–nbhd assignment and a γ_1^*–nbhd assignment.

The construction presented in Lemma 2.1.7 frequently provides a decreasing neighborhood assignment possessing multiple properties.

2.1.8 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is a γ_1–,γ_1^*–space;

(2) there exists a (decreasing) nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for every $p \in X$, $x_n \in N_n(p) \cup N_n^*(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.[14]

Proof. (2) $\Rightarrow$ (1): This implication is obvious.

(1) $\Rightarrow$ (2): Let $\{L_n(x): x \in X, n \in \mathbf{N}\}$ and $\{M_n(x): x \in X, n \in \mathbf{N}\}$ be γ_1– and γ_1^*–nbhd assignments, respectively, in (X, τ) and use Lemma 2.1.7 to construct a decreasing nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ that is both γ_1 and γ_1^*. Note that $N_n^*(p) = L_n^*(p) \cap M_n^*(p)$.

Now consider any $p \in X$ and suppose $x_n \in N_n(p) \cup N_n^*(p)\ \forall n$. If (x_n) does not converge to p, then there exists $U \in \mathcal{N}_\tau(p)$ such that $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - U$. Since both $\{N_n(x): x \in X, n \in \mathbf{N}\}$ and $\{N_n^*(x): x \in X, n \in \mathbf{N}\}$ are decreasing, it follows that, $\forall n$, $x_{k_n} \in N_n(p) \cup N_n^*(p)$.

Either infinitely many terms of the sequence (x_{k_n}) belong to $\bigcup\{N_n(p): n \in \mathbf{N}\}$ or infinitely many belong to $\bigcup\{N_n^*(p): n \in \mathbf{N}\}$. In the former case, $\forall n, \exists j_{k_n} \geq k_n$ with $j_{k_{n+1}} > j_{k_n}$ and $x_{j_{k_n}} \in N_{j_{k_n}}(p) \subseteq N_n(p) \subseteq L_n(p)$; so, as $\{L_n(x): x \in X, n \in \mathbf{N}\}$ is a γ_1–nbhd assignment, it follows that $x_{j_{k_n}} \xrightarrow[\tau]{} p$ so that $\left(x_{j_{k_n}}\right)$ is eventually in U, a contradiction. In the latter case, $\forall n, \exists j_{k_n} \geq k_n$ with $j_{k_{n+1}} > j_{k_n}$ and

[14]Once more we may replace $x_n \xrightarrow[\tau]{} p$ with (x_n) clusters at p.

$x_{j_{k_n}} \in N^*_{j_{k_n}}(p) \subseteq N^*_n(p) \subseteq M^*_n(p)$; so, as $\{M_n(x): x \in X, n \in \mathbb{N}\}$ is a γ_1^*-nbhd assignment, it follows that $x_{j_{k_n}} \xrightarrow[\tau]{} p$ so that $\left(x_{j_{k_n}}\right)$ is eventually in U, a contradiction.

Topological properties parallel to those of first countability and semistratifiability (among others) can be developed through the use of a more generalized neighborhood-like structure called a *weak base*. This notion points the way toward the generalization of topological spaces we will discuss in Chapter 3.

A collection $\{W_\alpha(x): x \in X, \alpha \in I\}$ of subsets of a set X is called a **weak neighborhood assignment** in a topological space (X, τ) provided that

(1) for each $p \in X$ and each $\alpha \in I$, $p \in W_\alpha(p)$, and

(2) whenever $p \in G \in \tau$, there exists $\alpha_p \in I$ with $W_{\alpha_p}(p) \subseteq G$.

The collection is called a **weak neighborhood base** or simply a **weak base** for τ if

(1) for each $p \in X$ and each $\alpha \in I$, $p \in W_\alpha(p)$,

(2) $G \in \tau \iff \forall p \in G, \exists \alpha_p \in I, W_{\alpha_p}(p) \subseteq G$, and

(3) $\forall p \in X, \forall \alpha_1, \alpha_2 \in I, \exists \alpha_3 \in I, W_{\alpha_3}(p) \subseteq W_{\alpha_1}(p) \cap W_{\alpha_2}(p)$.

Observe that we can always construct a decreasing weak base for a topology τ on X from a given weak base $\{W_n(x): x \in X, n \in \mathbb{N}\}$ for τ by defining $V_n(p) = \bigcap_{k=1}^{n} W_k(p)$. The collection $\{V_n(x): x \in X, n \in \mathbb{N}\}$ is then a decreasing weak base for τ.

2.1.9 Theorem. Let (X, τ) be a topological space and $\{S_n(x): x \in X, n \in \mathbb{N}\}$ be a decreasing family of subsets of X having the property that $p \in S_n(p)$ for each $p \in X$ and each $n \in \mathbb{N}$. The following are equivalent:

(1) $\{S_n(x): x \in X, n \in \mathbb{N}\}$ is a weak nbhd assignment in (X, τ);

(2) for each $p \in X$, $x_n \in S_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.

Proof. (1) $\Rightarrow$ (2): This implication is obvious.

(2) $\Rightarrow$ (1): Assume (2) and suppose $p \in G \in \tau$ with $S_n(p) \not\subset G, \forall n$. Then, $\forall n, \exists x_n \in S_n(p)$ such that $x_n \notin G$. By hypothesis, $x_n \xrightarrow[\tau]{} p$, a contradiction.

We now define a topological space (X, τ) to be **weakly first countable** [Ar; 1966] if there is a (decreasing) weak base $\{W_n(x): x \in X, n \in \mathbf{N}\}$ for τ. Similarly, (X, τ) is **weakly semistratifiable** if there is a (decreasing) weak base $\{W_n(x): x \in X, n \in \mathbf{N}\}$ for τ such that

$$\text{for each } p \in X,\ x_n \in W_n^*(p), \forall n \quad \Rightarrow \quad x_n \xrightarrow[\tau]{} p.$$

Weakly semistratifiable spaces can be characterized in such a way as to parallel the characterization of semistratifiable spaces given in Theorem 2.1.6. The proofs are similar as well.

2.1.10 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is weakly semistratifiable;

(2) there exists a decreasing weak base $\{W_n(x): x \in X, n \in \mathbf{N}\}$ for τ such that for every closed set F, $F = \bigcap \{W_n[F] : n \in \mathbf{N}\}$.

We conclude this section with a construction that allows us to manufacture a weak base having the potential to satisfy multiple properties.

2.1.11 Lemma. If $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a decreasing nbhd assignment in (X, τ), $\{W_n(x): x \in X, n \in \mathbf{N}\}$ is a weak base for τ, and for each $p \in X$ and each $n \in \mathbf{N}$, we define

$$V_n(p) = N_n(p) \cap W_n(p),$$

then $\{V_n(x) : x \in X, n \in \mathbf{N}\}$ is a weak base for τ.

Proof. Clearly, $p \in V_n(p)$ $\forall n$. Also, if $p \in G \in \tau$, there exists m such that $W_m(p) \subseteq G$; hence, $N_m(p) \cap W_m(p) \subseteq G$.

Claim 1. If $G \subseteq X$ has the property that $\forall p \in G, \exists n$ such that $V_n(p) \subseteq G$, then $G \in \tau$.

Pf. Consider any $p \in G$. Then $\exists m$ such that $N_m(p) \cap W_m(p) \subseteq G$ and, as $N_m(p) \in \mathcal{N}_\tau(p)$, $\exists k$ with $W_k(p) \subseteq N_m(p)$. It follows that $\exists l$ such that $W_l(p) \subseteq W_k(p) \cap W_m(p) \subseteq N_m(p) \cap W_m(p) \subseteq G$. Thus, as $\{W_n(x): x \in X, n \in \mathbf{N}\}$ is a weak base for τ, we conclude that $G \in \tau$.

Claim 2. $\forall m, n, \exists k$ such that $V_k(p) \subseteq V_m(p) \cap V_n(p)$.

Pf. Consider any $m, n \in \mathbf{N}$. There exist i, j such that $W_i(p) \subseteq N_m(p)$ and $W_j(p) \subseteq N_n(p)$. Then there exist α, β such that

$$W_\alpha(p) \subseteq W_i(p) \cap W_m(p) \subseteq N_m(p) \cap W_m(p), \text{ and}$$
$$W_\beta(p) \subseteq W_j(p) \cap W_n(p) \subseteq N_n(p) \cap W_n(p),$$

and there exists k such that $W_k(p) \subseteq W_\alpha(p) \cap W_\beta(p)$. Thus,

$$N_k(p) \cap W_k(p) \subseteq W_k(p) \subseteq V_m(p) \cap V_n(p).$$

2. An Application to the Theory of Distances

A function $d: X \times X \to [0, \infty)$ is a **distance** for the set X provided that

$$d(p, p) = 0 \text{ for each } p \in X.$$

Given a distance d for X, we define the **sphere** centered at $p \in X$ of radius $\varepsilon > 0$, denoted $S_d(p, \varepsilon)$, so that

$$S_d(p, \varepsilon) = \{x \in X : d(p, x) < \varepsilon\}.$$

We then define

$$\tau_d = \left\{A \subseteq X : \forall p \in A, \exists \varepsilon_p > 0, S_d(p, \varepsilon_p) \subseteq A\right\}.$$

Observe that τ_d is a topology on X.

A distance d for X is an **asymmetric**[15] if

$$\text{for any } p, q \in X, \ d(p, q) = 0 \ \Rightarrow \ p = q.$$

A topological space (X, τ) is **asymmetrizable** if there is an asymmetric d for X such that $\tau = \tau_d$; in this case we say that d is an asymmetric for (X, τ). An arbitrary distance for X is sometimes called a **pseudoasymmetric** for X; a topological space (X, τ) is then taken to be **pseudoasymmetrizable** if there is a distance (i.e. pseudoasymmetric) for X such that $\tau = \tau_d$.

If, for each $p \in X$, $\{S_d(p, \varepsilon) : \varepsilon > 0\}$ is a neighborhood base at p in a topological space (X, τ), it follows that $\tau = \tau_d$. In general, however, $S_d(p, \varepsilon)$ need not be a neighborhood of p in (X, τ_d). A topological space (X, τ) is said to be **(pseudo)asemimetrizable** if there is an (pseudo)asymmetric d for X such that

$$\text{for each } p \in X, \ \{S_d(p, \varepsilon) : \varepsilon > 0\} \text{ is a neighborhood base at } p \text{ in } (X, \tau);$$

in this case we say that d is an **(pseudo)asemimetric** for (X, τ).

The following lemma provides several well-known characterizations of pseudoasemimetrizable spaces.

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[15] Nedev [Ne; 1971] calls such a distance an *o-metric*.

2.2.1 Lemma. [SR] Let d be a distance for X and (X, τ) be a topological space. The following are equivalent:

(1) for each $p \in X$, $\{S_d(p, \varepsilon) : \varepsilon > 0\}$ is a neighborhood base for p in (X, τ);

(2) (X, τ) is first countable and, for each $p \in X$,

$$x_n \xrightarrow[\tau]{} p \quad \Leftrightarrow \quad d(p, x_n) \to 0;$$

(3) (X, τ) is Fréchet[16] and, for each $p \in X$,

$$x_n \xrightarrow[\tau]{} p \quad \Leftrightarrow \quad d(p, x_n) \to 0;$$

(4) for each nonempty set $A \subseteq X$, $\overline{A}^{\tau} = \{x \in X : d(x, A] = 0\}$;[17]

First countability and weak first countability can be used to distinguish pseudoasemimetrizability from the more general notion of pseudoasymmetrizability.

2.2.2 Theorem. A topological space (X, τ) is pseudoasymmetrizable iff it is weakly first countable.

Proof. If (X, τ) is pseudoasymmetrizable and d is a pseudoasymmetric for (X, τ), then $\{S_d(x, 1/2^n) : x \in X, n \in \mathbb{N}\}$ is a weak base for τ, making (X, τ) weakly first countable.

Conversely, if $\{W_n(x) : x \in X, n \in \mathbb{N}\}$ is a decreasing weak base for τ, then the distance d for X defined by

$$d(p, q) = \begin{cases} 0, \text{ if } q \in W_k(p)\ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin W_k(p)\}, \text{ otherwise}, \end{cases}$$

is a pseudoasymmetric for (X, τ), as $S_d(p, 1/2^n) = W_n(p)$.

---

[16]A topological space (X, τ) is *Fréchet* provided that, for any $p \in X$ and any $A \subseteq X$, $p \in \overline{A}^{\tau}$ iff there is a sequence in A that converges to p.

[17]If d is a distance for X, $p \in X$, and $A \subseteq X$, then we define $d(p, A] = \inf\{d(p, a) : a \in A\}$.

2.2.3 Corollary. A topological space (X, τ) is pseudoasemimetrizable iff it is first countable.

A distance d for X is **symmetric**[18] provided that

$$\text{for any } p, q \in X,\ d(p, q) = d(q, p).$$

A symmetric asymmetric is referred to as a **symmetric**. A topological space (X, τ) is said to be **symmetrizable** if there is a symmetric d for X such that $\tau = \tau_d$; we then say that d is a symmetric for (X, τ).

A topological space (X, τ) is **semimetrizable** if there is a symmetric d for (X, τ) such that, for each $p \in X$, $\{S_d(p, \varepsilon) : \varepsilon > 0\}$ is a neighborhood base for p in (X, τ); in this case we say that d is a **semimetric** for (X, τ).

Once again the prefix *pseudo* is used to allow for the possibility that $d(p, q) = 0$ even when $p \neq q$.

Given real numbers a and b, we will let $a \wedge b$ and $a \vee b$ stand for the minimum and the maximum, respectively, of a and b.

If d_1 and d_2 are distances for X, we define $(d_1 \wedge d_2), (d_1 \vee d_2) : X \times X \to [0, \infty)$ so that for any $p, q \in X$,

$$(d_1 \wedge d_2)(p, q) = d_1(p, q) \wedge d_2(p, q), \text{ and}$$
$$(d_1 \vee d_2)(p, q) = d_1(p, q) \vee d_2(p, q).$$

It follows that $(d_1 \wedge d_2)$ and $(d_1 \vee d_2)$ are distances for X.

Given distances d_1 and d_2 for X we will write $d_1 \le d_2$ provided that $d_1(p, q) \le d_2(p, q)$ for all $p, q \in X$.

2.2.4 Lemma. Let d_1 and d_2 be distances for X. Then:

(1) $d_1 \le d_2 \Rightarrow S_{d_2}(p, \varepsilon) \subseteq S_{d_1}(p, \varepsilon)$;

[18] Fréchet [Fr4] introduces this notion in 1918 as what he calls an *écart*.

(2) $S_{d_1 \wedge d_2}(p, \varepsilon) = S_{d_1}(p, \varepsilon) \cup S_{d_2}(p, \varepsilon)$;

(3) $S_{d_1 \vee d_2}(p, \varepsilon) = S_{d_1}(p, \varepsilon) \cap S_{d_2}(p, \varepsilon)$;

(4) $d_1 \le d_2 \Rightarrow \tau_{d_1} \subseteq \tau_{d_2}$.

Also, given a distance d for X, we define $d^* : X \times X \to [0, \infty)$, called the **dual** [Ko; 1993] of d, so that for any $p, q \in X$,

$$d^*(p, q) = d(q, p).$$

Note that d^* is a distance for X and that $d \wedge d^*$ and $d \vee d^*$ are symmetrics for X.

2.2.5 Lemma. Let d be a distance for X. Then:

(1) if $d^*(p, x_n) \to 0 \Rightarrow d(p, x_n) \to 0$, then $\tau_d \subseteq \tau_{d^*}$ and $\tau_d = \tau_{d \wedge d^*}$ (actually, $\tau_d \subseteq \tau_{d^*}$ iff $\tau_d = \tau_{d \wedge d^*}$);

(2) if $d(p, x_n) \to 0 \Rightarrow d^*(p, x_n) \to 0$, then $\tau_{d^*} \subseteq \tau_d$ and $\tau_d = \tau_{d \vee d^*}$ (actually, if $\tau_d = \tau_{d \vee d^*}$, then $\tau_{d^*} \subseteq \tau_d$).

Neighborhoods may be used to characterize pseudosemimetrizable spaces and weak neighborhoods to characterize pseudosymmetrizable spaces.

2.2.6 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudosemimetrizable;

(2) (X, τ) is first countable and semistratifiable;

(3) there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

(a) for each $p \in X$, $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$, and

(b) for any $p, q \in X$, $q \in N_n(p) \Rightarrow p \in N_n(q)$.

Proof. It suffices to show $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. Now $(3) \Rightarrow (2)$ is obvious and $(1) \Rightarrow (3)$ follows by taking $N_n(p) = S_d(p, 1/2^n)$.

(2)$\Rightarrow$(1): Suppose (X, τ) is first countable and semistratifiable. Then, by Lemma 2.1.7, there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) that is both a γ_1-nbhd assignment and a γ_1^*-nbhd assignment.

Define $d : X \times X \to [0, \infty)$ so that

$$d(p, q) = \begin{cases} 0, \text{ if } q \in N_k(p)\ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin N_k(p)\}, \text{ otherwise}, \end{cases}$$

and observe that $S_d(p, 1/2^n) = N_n(p)$. It then follows, since $\{N_n(p) : n \in \mathbf{N}\}$ is a nbhd base for p in (X, τ), that $\tau = \tau_d$. Thus, d is a pseudoasemimetric for (X, τ). Also, as $S_d(p, 1/2^n) = N_n(p)$, it follows that

$$x_n \xrightarrow[\tau]{} p \quad \Leftrightarrow \quad d(p, x_n) \to 0.$$

We now show that $\tau_d = \tau_{d \wedge d^*}$. It suffices, according to Lemma 2.2.5, to show that whenever $d^*(p, x_n) \to 0$, there is a subsequence (x_{k_n}) of (x_n) for which $d(p, x_{k_n}) \to 0$. So suppose $d^*(p, x_n) \to 0$. Then, $\forall n, \exists k_n$ such that $d^*(p, x_n) < \frac{1}{2^n}$ or, equivalently, $d(x_{k_n}, p) < \frac{1}{2^n}$. So, $\forall n$, $p \in S_d(x_{k_n}, 1/2^n) = N_n(x_{k_n})$. Therefore, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1^*-nbhd assignment, it follows that $x_{k_n} \xrightarrow[\tau]{} p$ so that $d(p, x_{k_n}) \to 0$.

Thus, $d \wedge d^*$ is a pseudosymmetric for (X, τ). However, since $S_{d \wedge d^*}(p, \varepsilon) = S_d(p, \varepsilon) \cup S_{d^*}(p, \varepsilon)$, it is clear that $S_{d \wedge d^*}(p, \varepsilon) \in \mathcal{N}_\tau(p)$. Hence, $(d \wedge d^*)$ is a pseudosemimetric for (X, τ).

2.2.7 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudosymmetrizable;

(2) (X, τ) is weakly semistratifiable;

(3) there is a weak base $\{W_n(x) : x \in X, n \in \mathbf{N}\}$ for τ such that

$$\text{for any } p, q \in X,\ q \in W_n(p) \quad \Rightarrow \quad p \in W_n(q).$$

Proof. It suffices to show $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. Paralleling the proof of Theorem 2.2.6, we see that $(3) \Rightarrow (2)$ is obvious and that $(1) \Rightarrow (3)$ follows by taking $W_n(p) = S_d\left(p, 1/2^n\right)$.

$(2) \Rightarrow (1)$: Suppose (X, τ) is weakly semistratifiable. Then there is a decreasing weak base $\{W_n(x) : x \in X, n \in \mathbf{N}\}$ for τ such that

$$\text{for each } p \in X,\ x_n \in W_n^*(p),\ \forall n \quad \Rightarrow \quad x_n \xrightarrow[\tau]{} p.$$

Define $A_n(p) = W_n(p) \cup W_n^*(p)$; it follows that $\{A_n(x) : x \in X, n \in \mathbf{N}\}$ is a decreasing weak base in (X, τ).

Now define $d : X \times X \to [0, \infty)$ so that

$$d(p, q) = \begin{cases} 0. \text{ if } q \in A_k(p)\ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin A_k(p)\}, \text{otherwise}. \end{cases}$$

Since $q \in A_n(p) \quad \Rightarrow \quad p \in A_n(q)$, it follows that $d(p, q) = d(q, p)\ \forall p, q \in X$.

3. The γ-spaces and their Dual, the γ^*-spaces

A topological space (X, τ) is a γ**-space** [Hol; 1972] if there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(x_n \in N_n(y_n), y_n \in N_n(p), \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

The class of γ-spaces is significant because it forms a generalization of the class of *pseudoquasimetrizable spaces* (pseudoquasimetrizable spaces are discussed in §5 of this chapter). Ralph Fox and Jacob Kofner [FK] have developed an example of a regular γ-space that is not pseudoquasimetrizable.[19]

γ-spaces can also be used in characterizing both pseudoquasimetrizable and pseudometrizable spaces.[20]

A topological space (X, τ) is

(1) a γ^***-space** if there is a (decreasing) neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(p \in N_n(y_n), y_n \in N_n(x_n), \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p, \text{ and}$$

(2) a $\gamma\gamma^*$**-space** if there is a (decreasing) neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(x_n, p \in N_n(y_n), y_n \in N_n(x_n) \cap N_n(p), \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

The γ-spaces and the γ^*-spaces are dual to each other in the sense that, by replacing each occurrence of N_n in the defining property of the γ-spaces by N_n^*, we obtain the defining property of the γ^*-spaces. The $\gamma\gamma^*$-spaces might be characterized as self-dual, since replacing each occurrence of N_n in the defining property of the $\gamma\gamma^*$-spaces by N_n^* produces the identical property.

Figure 2 summarizes how these spaces relate to one another:

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[19]This provides a counterexample to the long-standing conjecture that every γ-space is pseudoquasimetrizable.

[20]See §5 and §6 of this chapter.

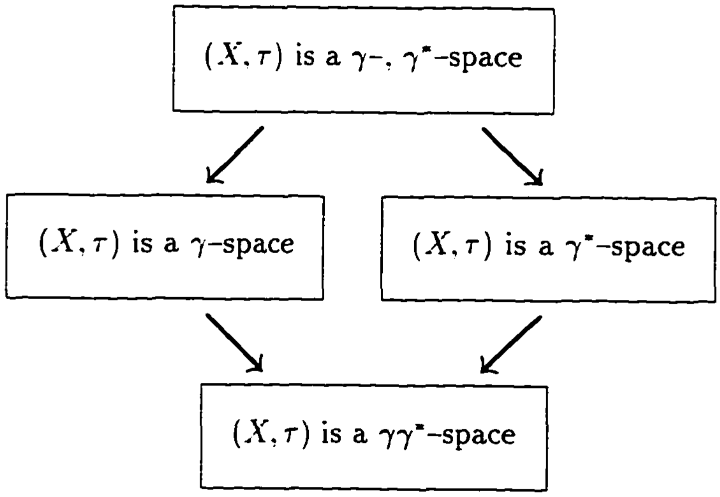

Figure 2. γ and its dual γ^*

2.3.1 Lemma. For any topological space (X, τ):

(1) (X, τ) is a γ-space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(N_n^*(x_n) \cap N_n(p) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p;$$

(2) (X, τ) is a γ^*-space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(N_n(x_n) \cap N_n^*(p) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p;$$

(3) (X, τ) is a $\gamma\gamma^*$-space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(N_n(x_n) \cap N_n^*(x_n) \cap N_n(p) \cap N_n^*(p) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p;$$

(4) (X, τ) is a γ-,γ^*-space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(\left(N_n^*(x_n) \cap N_n(p)\right) \cup \left(N_n(x_n) \cap N_n^*(p)\right) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

Given a family $\{S(x) : x \in X\}$ of subsets of X and a subset A of X, we define

$$S^2[A] = \{x \in X : \exists a \in A, \exists y \in X, x \in S(y), y \in S(a)\}.$$

2.3.2 Theorem. For any topological space (X, τ):

(1) (X, τ) is a γ-space iff there is a (decreasing) neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ in (X, τ) such that

$$(K \text{ compact}, F \text{ closed}, K \cap F \neq \varnothing) \Rightarrow N_n[K] \cap F = \varnothing \text{ for some } n;$$

(2) (X, τ) is a γ^*-space iff there is a (decreasing) neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ in (X, τ) such that

$$\text{for each closed set } F, \ F = \bigcap\{N_n^2[F] : n \in \mathbb{N}\}.$$

Proof. (1) Let $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ be a decreasing γ-nbhd assignment in (X, τ). Consider any compact K and any closed F and suppose that $\forall n, \exists x_n \in N_n[K] \cap F$ so that $\forall n, x_n \in N_n(y_n)$ for some $y_n \in K$ and $x_n \in F$. It follows that (y_n) has a cluster point $p \in K$. So, $\forall n, \exists k_n \geq n$ such that $y_{k_n} \in N_n(p)$. Then, since $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ is decreasing, $x_{k_n} \in N_n(y_{k_n})$. Thus, as $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ is a γ-nbhd assignment, $x_{k_n} \xrightarrow[\tau]{} p$. So, as (x_{k_n}) is a sequence in the closed set F, it follows that $p \in F$. Hence, $K \cap F \neq \varnothing$.

Conversely, suppose there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ in (X, τ) such that whenever K is compact, F is closed, and $K \cap F \neq \varnothing$, it follows that $N_n[K] \cap F = \varnothing$ for some n. Suppose also that (X, τ) is *not* a γ-space. Then $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ is not a γ-nbhd assignment. So there exist $p \in X$ and sequences (x_n) and (y_n) in X such that $\forall n, y_n \in N_n^*(x_n) \cap N_n(p)$, but (x_n) does not converge to p. Thus, $\exists G \in \tau$ with $p \in G$ such that (x_n) is frequently in the closed set $X - G$.

Claim 1. $p \in H \in \tau \Rightarrow \exists n, N_n(p) \subseteq H$

Pf. Suppose $p \in H \in \tau$. Now $\{p\}$ is compact, $X - H$ is closed, and $\{p\} \cap (X - H) = \varnothing$. So $N_n[\{p\}] \cap (X - H) = \varnothing$ for some n. Thus, $N_n(p) \subseteq H$.

Let $Y = \{y_n : n \in \mathbb{N}\}$ and $K = (Y \cap G) \cup \{p\}$.

Claim 2. K is compact

Pf. Consider any $H \in \tau$ with $p \in H$. By Claim 1, $\exists n$ such that $N_n(p) \subseteq H$. Since $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ is decreasing, it follows that $y_m \in N_n(p)\ \forall m \geq n$. Thus, all but finitely many members of $Y \cap G$ are in H.

Claim 3. $\forall n, N_n[K] \cap (X - G) \neq \varnothing$ (which, as K is compact, $X - G$ is closed, and $K \cap (X - G) = \varnothing$, provides a contradiction to our hypothesis)

Pf. Consider any $n \in \mathbb{N}$; then $\exists i_n \geq n$ such that $x_{i_n} \in X - G$. If $y_{i_n} \in G$, then $x_{i_n} \in N_n(y_{i_n})$ so that $x_{i_n} \in N_n(y_{i_n}) \cap (X - G) \subseteq N_n[K] \cap (X - G)$ which means $N_n[K] \cap (X - G) \neq \varnothing$. Otherwise, $y_{i_n} \in X - G$ so that $y_{i_n} \in N_n(p) \cap (X - G) \subseteq N_n[K] \cap (X - G)$ which also means $N_n[K] \cap (X - G) \neq \varnothing$.

(2) Let $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ be a γ^*–nbhd assignment in (X, τ) and consider any closed set F. If $p \in F$, then, as $p \in N_n(p)\ \forall n$, it follows that $p \in N_n^2[F]\ \forall n$. If $p \in N_n^2[F]\ \forall n$, then, $\forall n, \exists x_n \in F, \exists y_n \in X$ with $p \in N_n(y_n)$ and $y_n \in N_n(x_n)$ so that $x_n \xrightarrow[\tau]{} p$; but, as F is closed, it follows that $p \in F$.

Conversely, suppose there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ in (X, τ) such that for each closed set F, $F = \bigcap\{N_n^2[F] : n \in \mathbb{N}\}$. Consider any $p \in X$ and suppose there exist sequences (x_n) and (y_n) in X such that, $\forall n$, $y_n \in N_n(x_n) \cap N_n^*(p)$.

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists G \in \tau$ with $p \in G$ such that $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - G$. Then, since $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ is decreasing, $p \in N_n(y_{k_n})\ \forall n$ and $y_{k_n} \in N_n(x_{k_n})\ \forall n$; hence, $p \in N_n^2(x_{k_n})\ \forall n$. So, as $X - G = \bigcap\{N_n^2[X - G] : n \in \mathbb{N}\}$ and $x_{k_n} \in X - G\ \forall n$, it follows that $p \in X - G$, a contradiction.

4. The Developable Spaces and their Dual, the Nagata Spaces

A topological space (X, τ) is a **Δ-space**[21] if there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ such that

$$\text{for each } p \in X,\ x_n, p \in N_n(y_n), \forall n \ \Rightarrow\ x_n \xrightarrow[\tau]{} p.$$

A topological space (X, τ) is a **Nagata space** [Hol; 1972] or an **N-space**[22] if there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ such that

$$\text{for each } p \in X,\ N_n(p) \cap N_n(x_n) \neq \varnothing, \forall n \ \Rightarrow\ x_n \xrightarrow[\tau]{} p.$$

Note that the neighborhood assignments whose existence is provided for in the above definitions may be assumed to be decreasing, in which case it follows that, for each $p \in X$,

$$\{N_n(p) \cup N_n^*(p) : n \in \mathbf{N}\}$$

is a nbhd base for p in (X, τ).[23] Thus, by Theorem 2.2.6, both Δ-spaces and Nagata spaces are pseudosemimetrizable.

Given a collection $\mathcal{A}$ of subsets of X and $p \in X$, we define the **star** of p in $\mathcal{A}$, denoted $\text{st}(p, \mathcal{A})$, so that

$$\text{st}(p, \mathcal{A}) = \bigcup\{A \in \mathcal{A} : p \in A\}.$$

If $S \subseteq X$, we also define the **star** of S in $\mathcal{A}$, denoted $\text{st}[S, \mathcal{A})$, so that

$$\text{st}[S, \mathcal{A}) = \bigcup\{A \in \mathcal{A} : A \cap S \neq \varnothing\}.$$

Note also that $\text{st}[S, \mathcal{A}) = \bigcup\{\text{st}(p, \mathcal{A}) : p \in S\}$.

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[21] This terminology is suggested by the historical definition of a *$w\Delta$-space*.

[22] Again, the historical definition of a *wN-space* suggests this choice of terminology.

[23] If (X, τ) is a Δ-space and $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a decreasing nbhd assignment such that $x_n, p \in N_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$, to show that $\{N_n(p) \cup N_n^*(p) : n \in \mathbf{N}\}$ is a nbhd base for p, it suffices, by Theorem 2.1.2, to show that $x_n \in N_n(p) \cup N_n^*(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$. So suppose $x_n \in N_n(p) \cup N_n^*(p), \forall n$. If $x_n \in N_n(p)$, let $y_n = p$; otherwise, let $y_n = x_n$. It then follows that, $\forall n$, $x_n, p \in N_n(y_n)$ so that $x_n \xrightarrow[\tau]{} p$. If (X, τ) is a Nagata space, a similar argument can be employed.

A topological space (X, τ) is **developable** [Bi; 1951] if there is a sequence $(\mathcal{G}_n)$ of open covers of X such that for each $p \in X$, $\{\mathrm{st}(p, \mathcal{G}_n) : n \in \mathbf{N}\}$ is a nbhd base for p (such a sequence of open covers is called a **development** for (X, τ)).[24]

2.4.1 Lemma. Every developable space is semistratifiable.

Proof. Consider any developable space (X, τ) and let $(\mathcal{G}_n)$ be a development for (X, τ). It may be assumed that, $\forall n$, $\mathcal{G}_{n+1} \prec \mathcal{G}_n$[25] (if necessary, for each n replace $\mathcal{G}_n$ with $\mathcal{U}_n = \left\{\bigcap_{i=1}^{n} G_i : G_i \in \mathcal{G}_i \ \forall i \in \{1, 2, \ldots, n\}\right\}$). Then $\{\mathrm{st}(x, \mathcal{G}_n) : x \in X, n \in \mathbf{N}\}$ is a γ_1^*-nbhd assignment.

2.4.2 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is developable;

(2) (X, τ) is a Δ-space;

(3) there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for every closed set F, $F = \bigcap\{\mathrm{st}[F, \{N_n(x) : x \in X\}) : n \in \mathbf{N}\}$.

Proof. (1)$\Leftrightarrow$(2) is due to Heath [Hea; 1962].

(2)$\Rightarrow$(3): Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a Δ-nbhd assignment in (X, τ), denote $\{N_n(x) : x \in X\}$ by $\mathcal{N}_n$, and consider any closed set F. Clearly, $F \subseteq \bigcap\{\mathrm{st}[F, \mathcal{N}_n) : n \in \mathbf{N}\}$. So suppose $p \in st[F, \mathcal{N}_n)$ $\forall n$. Then $\forall n, \exists y_n \in X, \exists x_n \in F$ such that $p, x_n \in N_n(y_n)$. Since $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a Δ-nbhd assignment, it follows that $x_n \xrightarrow[\tau]{} p$ and, therefore, as F is closed, that $p \in F$.

[24]A developable T_3-space is known as a *Moore space*. Moore spaces have been the focus of much attention by topologists (the Normal Moore Space Conjecture, etc.). See [Tal] and [Fl] for a detailed discussion of Moore spaces and the issues surrounding them.

[25]We write $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ to indicate that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$.

(3)$\Rightarrow$(1): Assume (3) and denote $\{N_n(x): x \in X\}$ by $\mathcal{N}_n$. Without loss of generality, assume $\{N_n(x): x \in X, n \in \mathbf{N}\} \subseteq \tau$. It follows that $(\mathcal{N}_n)$ is a development for (X, τ).

Interest in Δ-spaces stems from the fact that the class of developable spaces generalizes the class of pseudometrizable spaces.

2.4.3 Example. *A Hausdorff developable space that is not metrizable.*

Let $A = \{(x, y): x, y \in \mathbf{R}, y > 0\}$, $B = \{(x, 0): x \in \mathbf{R}\}$, $X = A \cup B$, $\mathcal{U}$ be the usual topology on A, I be the set of interiors of disks lying in A whose boundaries are tangent to B, $T = \{D \cup \{t_D\}: D \in I\}$, where for each $D \in I$, t_D is the point of tangency of the boundary of D with B, and $\mathcal{B} = \mathcal{U} \cup T$. Then $\mathcal{B}$ is a base for a topology τ on X.

For each $p \in A$ and each $n \in \mathbf{N}$, let $U_n(p) = \{x \in A: e(p, x) < 1/2^n\}$, where $e(p, x)$ is the usual Euclidean distance between p and x.

For each $p \in B$ and each $n \in \mathbf{N}$, let $V_n(p) = D(p, n) \cup \{p\}$, where $D(p, n)$ is the interior of the disk centered at $(p, p + 1/2^n)$ having radius $1/2^n$.

Now for each $n \in \mathbf{N}$, define $\mathcal{G}_n = \{U_n(x): x \in A\} \cup \{V_n(x): x \in B\}$ and note that $(\mathcal{G}_n)$ is a development for (X, τ). Clearly, (X, τ) is Hausdorff.

Note also that (X, τ) is separable since, if $\mathbf{Q}$ is the set of rational numbers, $X \cap (\mathbf{Q} \times \mathbf{Q})$ is countable and dense. However, (X, τ) is *not* second countable since, if it were, $(B, \tau|B)$, an uncountable discrete subspace, would be as well. Hence, (X, τ) is *not* metrizable.

Observe that the developable spaces and the Nagata spaces are dual to each other in the same sense that the γ-spaces are dual to the γ^*-spaces.

Given a topological space (X, τ) and a family $\mathcal{A}$ of subsets of X, $\mathcal{A}$ is **discrete** if every point of X has a nbhd that intersects at most one member of $\mathcal{A}$, $\mathcal{A}$ is **σ-discrete** if $\mathcal{A}$ is a countable union of discrete families of subsets of X, $\mathcal{A}$ is **locally finite** if every point of X has a nbhd that intersects only finitely many members of $\mathcal{A}$, and $\mathcal{A}$ is a **net** provided that whenever $p \in G \in \tau$, there exists $A_p \in \mathcal{A}$ such that $p \in A_p \subseteq G$. Note that a net in (X, τ) is a cover of X and a discrete family in (X, τ) is locally finite.

2.4.4 Lemma. Let (X, τ) be a topological space and $\mathcal{A}$ be a locally finite family of closed sets in (X, τ). If $\mathcal{B} \subseteq \mathcal{A}$, then $\bigcup \mathcal{B}$ is closed.

Our next goal is to show that every developable space is a γ^*-space. The proof makes use of a covering property known as *subparacompactness.* A topological space (X, τ) is **subparacompact** if every open cover of X has a closed σ-discrete refining cover.

2.4.5 Lemma. Every semistratifiable space is subparacompact.

Proof. Consider any semistratifiable space (X, τ) and let $\{U_n(x) : x \in X, n \in \mathbb{N}\}$ be a γ_1^*-nbhd assignment in (X, τ) consisting of open sets. For each $G \in \tau$ and each $n \in \mathbb{N}$, define $F_n[G] = X - U_n[X - G]$ and note that these sets are closed. Now consider any open cover $\mathcal{G}$ of X and well-order $\mathcal{G}$ so that $G = \{G_\alpha : \alpha < \lambda\}$ for some λ. Define, for each $n \in \mathbb{N}$, $H_{1,n} = F_n[G_1]$ and $H_{\alpha,n} = F_n[G_\alpha] - \bigcup\{G_\beta : \beta < \alpha\}$. Then, for each $n \in \mathbb{N}$, let $\mathcal{H}_n = \{H_{\alpha,n} : \alpha < \lambda\}$. It follows that $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$ is a closed σ-discrete cover of X that refines $\mathcal{G}$.

2.4.6 Theorem. Every developable space is a γ^*-space. In fact,

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4),$$

where,

(1) (X, τ) is developable,

(2) (X, τ) has a closed σ–discrete net,

(3) there exists a γ_1^*–nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ for (X, τ) such that

$$x \in N_n(p) \Rightarrow N_n(x) \subseteq N_n(p),$$

(4) (X, τ) is a γ^*–space.

Proof. (1) $\Rightarrow$ (2): Suppose (X, τ) is developable and let $(\mathcal{G}_n)$ be a development for (X, τ). Since (X, τ) is developable, it is semistratifiable and, therefore, subparacompact. So, $\forall n, m \in \mathbf{N}$, there exists a discrete family $\mathcal{F}_{n,m}$ of closed sets such that $\mathcal{F}_{n,m} \prec \mathcal{G}_n$ and $\mathcal{F}_n = \bigcup\{\mathcal{F}_{n,m} : m \in \mathbf{N}\}$ covers X. It follows that $\mathcal{F} = \bigcup\{\mathcal{F}_{n,m} : n, m \in \mathbf{N}\}$ is a σ–discrete family of closed sets.

Now suppose $p \in G \in \tau$. As $(\mathcal{G}_n)$ is a development, there exists k such that $\text{st}(p, \mathcal{G}_k) \subseteq G$, and as $\mathcal{F}_k$ is a cover of X, there exists $F \in \mathcal{F}_k$ such that $p \in F$. Then, since $\mathcal{F}_k \prec \mathcal{G}_k$, $F \subseteq H$ for some $H \in \mathcal{G}_k$. So $p \in F \subseteq H \subseteq \text{st}(p, \mathcal{G}_k) \subseteq G$. Thus, $\mathcal{F}$ is a net.

(2) $\Rightarrow$ (3): Suppose that, $\forall n \in \mathbf{N}$, $\mathcal{F}_n$ is a discrete family of closed sets in (X, τ), and that $\mathcal{F} = \bigcup\{\mathcal{F}_n : n \in \mathbf{N}\}$ is a net. For each $n \in \mathbf{N}$, let $\mathcal{F}_n^*$ be the collection of all finite intersections of members of $\bigcup_{i=1}^{n} \mathcal{F}_i$; then let $\mathcal{F}^* = \bigcup\{\mathcal{F}_n^* : n \in \mathbf{N}\}$. Note that $\mathcal{F}^*$ is a collection of closed sets, $\mathcal{F}^*$ is a net, and, $\forall n, \mathcal{F}_n^* \subseteq \mathcal{F}_{n+1}^*$. Note also that, $\forall n$, $\mathcal{F}_n^*$ is locally finite; thus, $\forall p \in X, \forall n \in \mathbf{N}, N_n(p) = X - \bigcup\{F \in \mathcal{F}_n^* : p \notin F\} \in \tau$.

Claim 1. $x \in N_n(p) \Rightarrow N_n(x) \subseteq N_n(p)$

Pf. Suppose $x \in N_n(p)$. Consider any $y \in N_n(x)$ and any $F \in \mathcal{F}_n^*$ for which $p \notin F$. As $x \in N_n(p)$, it follows that $x \notin F$. Then, as $y \in N_n(x)$, it follows that $y \notin F$. Hence, $y \in N_n(p)$.

Claim 2. $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1^*–nbhd assignment

Pf. Assume $x_n \in N_n^*(p), \forall n$ and suppose to the contrary that (x_n) does *not* converge to p. Then, $\exists G \in \tau$ such that $p \in G$ and, $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \notin G$. Since $\mathcal{F}$ is a net, $\exists m \in \mathbf{N}, \exists F^* \in \mathcal{F}_m^*$ such that $p \in F^* \subseteq G$. Then, since $\mathcal{F}_n^* \subseteq \mathcal{F}_{n+1}^*$ $\forall n$, we observe that $F^* \in \mathcal{F}_{k_n}^*$ $\forall n$ with $k_n \geq m$. But, as $x_{k_n} \notin F^*$ $\forall n$ and $p \in F^*$, it follows that, $\forall n$ with $k_n \geq m$, $p \in \bigcup \left\{F \in \mathcal{F}_{k_n}^* : x_{k_n} \notin F\right\}$ so that, $\forall n$ with $k_n \geq m$, $p \notin N_{k_n}\left(x_{k_n}\right)$, a contradiction.

(3) $\Rightarrow$ (4): Suppose there exists a γ_1^*–nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ for (X, τ) such that $x \in N_n(p)$ $\Rightarrow$ $N_n(x) \subseteq N_n(p)$. Suppose also that, $\forall n, p \in N_n(y_n)$ and $y_n \in N_n(x_n)$.

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. It suffices, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1^*–nbhd assignment, to show that, $\forall n, p \in N_n(x_n)$. Since $y_n \in N_n(x_n), \forall n$, it follows that $N_n(y_n) \subseteq N_n(x_n), \forall n$; then, since $p \in N_n(y_n), \forall n$, it follows that $p \in N_n(x_n), \forall n$.

We now consider several classes of spaces which generalize developable spaces.

A topological space (X, τ) is a **ΔN–space** (or ***MN*–space** [HH; 1973]) if (as expected) there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(N_n(x_n) \cap N_n(p) \neq \varnothing, \forall n \text{ and } N_n^*(x_n) \cap N_n^*(p) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

A topological space (X, τ) is a **$\Delta\gamma$–space** (or **θ–space** [Hol; 1972]) if there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$N_n(p) \cap N_n^*(p) \cap N_n^*(x_n) \neq \varnothing, \forall n \Rightarrow x_n \xrightarrow[\tau]{} p.$$

A topological space (X, τ) is a **$\Delta\gamma^*$–space** if there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$N_n(x_n) \cap N_n^*(x_n) \cap N_n^*(p) \neq \varnothing, \forall n \Rightarrow x_n \xrightarrow[\tau]{} p.$$

A topological space (X, τ) is a $\Delta\gamma N$**-space** if there is a (decreasing) nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that for each $p \in X$,

$$\left(N_n(p) \cap N_n^*(p) \cap N_n^*(x_n) \neq \varnothing, \forall n \text{ and } N_n(p) \cap N_n(x_n) \neq \varnothing, \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

Hodel has shown that a topological space is developable if and only if it is a semistratifiable $\Delta\gamma$-space [Hol; 1972]. Our next theorem provides more of the interrelationships among the classes of spaces we have just defined.

2.4.7 Theorem. For any topological space (X, τ):

(1) (X, τ) is a ΔN-space iff (X, τ) is a semistratifiable $\Delta\gamma N$-space;

(2) (X, τ) is a $\Delta\gamma^*$-space iff (X, τ) is a semistratifiable $\gamma\gamma^*$-space;

(3) (X, τ) is a $(\Delta\gamma)^*$-space iff (X, τ) is a γ^*-space.

Proof. The proofs of (2) and (3) involve arguments similar to those employed in the proof of (1); hence, we provide only the proof of (1).

($\Rightarrow$) Clearly, any ΔN-nbhd assignment is both a $\Delta\gamma N$-nbhd assignment and a γ_1^*-nbhd assignment.

($\Leftarrow$) Suppose (X, τ) is a semistratifiable $\Delta\gamma N$-space. Using the construction given in Lemma 2.1.7 we can obtain a decreasing nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) that is both a γ_1^*-nbhd assignment and a $\Delta\gamma N$-nbhd assignment. Now suppose that, for each $n \in \mathbf{N}$, $\exists y_n \in N_n(x_n) \cap N_n(p)$ and $\exists z_n \in N_n^*(x_n) \cap N_n^*(p)$.

Claim 1. (x_n) clusters at p

Pf. Since $z_n \in N_n^*(p)$ and $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is a γ_1^*-nbhd assignment, it follows that $z_n \xrightarrow[\tau]{} p$. So, $\forall n, \exists k_n \geq n$ such that $z_{k_n} \in N_n(p)$. Then, as $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is decreasing, it follows that

$$\forall n, z_{k_n} \in N_{k_n}^*\left(x_{k_n}\right) \cap N_{k_n}^*(p) \subseteq N_n^*\left(x_{k_n}\right) \cap N_n^*(p);$$

therefore, $\forall n$,

$$z_{k_n} \in N_n(p) \cap N_n^*(p) \cap N_n^*\left(x_{k_n}\right).$$

Similarly, $\forall n$,

$$y_{k_n} \in N_{k_n}(p) \cap N_{k_n}\left(x_{k_n}\right) \subseteq N_n(p) \cap N_n\left(x_{k_n}\right).$$

Thus, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a $\Delta\gamma N$–nbhd assignment, it follows that $x_{k_n} \xrightarrow[\tau]{} p$.

Actually, we have now established that

(†) $\left(\forall n, N_n(a_n) \cap N_n(q) \neq \varnothing \text{ and } N_n^*(a_n) \cap N_n^*(q) \neq \varnothing\right) \Rightarrow (a_n)$ clusters at q.

Claim 2. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists M \in \mathcal{N}_\tau(p)$ such that $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - M$. Then, $\forall n$,

$$y_{k_n} \in N_{k_n}\left(x_{k_n}\right) \cap N_{k_n}(p) \subseteq N_n\left(x_{k_n}\right) \cap N_n(p)$$

and

$$z_{k_n} \in N_{k_n}^*\left(x_{k_n}\right) \cap N_{k_n}^*(p) \subseteq N_n^*\left(x_{k_n}\right) \cap N_n^*(p).$$

Thus, by (†), it follows that $\left(x_{k_n}\right)$ clusters at p, which is impossible since $\left(x_{k_n}\right)$ is never in M.

In the neighborhood characterizations of $\Delta\gamma$–spaces and $\Delta\gamma^*$–spaces given below, we make use of the following notation:

Given a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in a topological space (X, τ), we define $\tilde{N}_n(p)$ so that

$$\tilde{N}_n(p) = N_n\left[N_n(p) \cap N_n^*(p)\right].$$

Note that $\left\{\tilde{N}_n(x) : x \in X, n \in \mathbf{N}\right\}$ is also a nbhd assignment in (X, τ).

2.4.8 Theorem. For any topological space (X, τ):

(1) (X, τ) is a $\Delta\gamma$–space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each } p \in X,\ x_n \in \tilde{N}_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p;$$

(2) (X, τ) is a $\Delta\gamma^*$–space iff there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each closed set } F,\ F = \bigcap\left\{\tilde{N}_n[F] : n \in \mathbf{N}\right\}.$$

Proof. (1) Observe that

$$\begin{aligned} x_n \in \tilde{N}_n(p) &\Leftrightarrow x_n \in N_n(y_n) \text{ for some } y_n \in N_n(p) \cap N_n^*(p) \\ &\Leftrightarrow y_n \in N_n(p) \cap N_n^*(p) \cap N_n^*(x_n). \end{aligned}$$

(2) ($\Rightarrow$) Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a $\Delta\gamma^*$–nbhd assignment in (X, τ) and consider any closed set F. Clearly, $F \subseteq \bigcap\left\{\tilde{N}_n[F] : n \in \mathbf{N}\right\}$.

So suppose $p \in \tilde{N}_n[F], \forall n$. Then, $\forall n, \exists x_n \in F$ such that $p \in \tilde{N}_n(x_n)$. So, $\forall n, \exists y_n \in N_n(x_n) \cap N_n^*(x_n)$ such that $p \in N_n(y_n)$. Thus, $\forall n, y_n \in N_n(x_n) \cap N_n^*(x_n) \cap N_n^*(p)$ so that, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is $\Delta\gamma^*$, it follows that $x_n \xrightarrow[\tau]{} p$. Since F is closed and $x_n \in F, \forall n$, we may conclude that $p \in F$.

($\Leftarrow$) Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a nbhd assignment in (X, τ) such that for each closed set F, $F = \bigcap\left\{\tilde{N}_n[F] : n \in \mathbf{N}\right\}$. Note that we may assume this nbhd assignment is decreasing. It suffices, by (2) of Theorem 2.4.7, to show that $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is both γ_1^* and $\gamma\gamma^*$.

Claim 1. $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is γ_1^*

Pf. Assume $x_n \in N_n^*(p)$ $\forall n$ and suppose to the contrary that (x_n) does *not* converge to p. Then $\exists G \in \tau$ such that $p \in G$ and $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - G$. By hypothesis, $X - G = \bigcap\left\{\tilde{N}_n[X - G] : n \in \mathbf{N}\right\}$, so $\exists j$ such that $p \notin \tilde{N}_j[X - G]$. Thus, $\forall a \in X - G, \forall b \in N_j(a) \cap N_j^*(a)$, it follows that $p \notin N_j(b)$. So, $\forall a \in X - G, p \notin N_j(a)$. Choose n so that $k_n \geq j$. Then $p \notin N_j(x_{k_n})$ and, as $N_{k_n}(x_{k_n}) \subseteq N_j(x_{k_n})$, it follows that $p \notin N_{k_n}(x_{k_n})$, a contradiction.

Claim 2. $\{N_n(x): x \in X, n \in \mathbb{N}\}$ is $\gamma\gamma^*$

Pf. Assume that, $\forall n$, $y_n \in N_n(x_n) \cap N_n^*(x_n) \cap N_n(p) \cap N_n^*(p)$ and suppose to the contrary that (x_n) does *not* converge to p. Then $\exists G \in \tau$ such that $p \in G$ and $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - G$. By hypothesis, $X - G = \bigcap\left\{\tilde{N}_n[X-G]: n \in \mathbb{N}\right\}$, so $\exists j$ such that $p \notin \tilde{N}_j[X-G]$. Thus, $\forall a \in X - G, \forall b \in N_j(a) \cap N_j^*(a)$, it follows that $p \notin N_j(b)$. Choose n so that $k_n \geq j$. Then $y_{k_n} \in N_{k_n}(x_{k_n}) \cap N_{k_n}^*(x_{k_n}) \subseteq N_j(x_{k_n}) \cap N_j^*(x_{k_n})$ so that $p \notin N_{k_n}(y_{k_n})$, a contradiction.

Some of the classes of spaces we have studied are easily seen to be generalizations of Nagata spaces. These classes can also be related to several other well-known classes of spaces which we now define.

A topological space (X, τ) is

(1) **stratifiable**[26] if there is a (decreasing) neighborhood assignment $\{N_n(x): x \in X, n \in \mathbb{N}\}$ in (X, τ) such that

$$\text{for each closed set } F,\ F = \bigcap\left\{\overline{N_n[F]}: n \in \mathbb{N}\right\};$$

(2) k**-semistratifiable** [Lu] if there is a (decreasing) neighborhood assignment $\{N_n(x): x \in X, n \in \mathbb{N}\}$ in (X, τ) such that

$$(K \text{ compact},\ F \text{ closed},\ K \cap F = \varnothing) \Rightarrow K \cap N_n[F] = \varnothing \text{ for some } n; \text{and}$$

(3) N**-semistratifiable**[27] if there is a (decreasing) neighborhood assignment $\{N_n(x): x \in X, n \in \mathbb{N}\}$ in (X, τ) such that

$$\text{for each } p \in X,\ \left(y_n \in N_n(x_n), \forall n \text{ and } y_n \xrightarrow[\tau]{} p\right) \Rightarrow x_n \xrightarrow[\tau]{} p.$$

[26] Stratifiable spaces were originally known as M_3*-spaces* and were studied by Ceder [Ce] in his 1959 thesis which was directed by E.A. Michael. Borges [Bo] introduced the term stratifiable. The definition we have chosen was derived by Heath [Hea].

[27] The letter N has been chosen in honor of Nagata.

2.4.9 Theorem. For any topological space (X, τ):

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8),$$

where,

(1) (X, τ) is a Nagata space,

(2) (X, τ) is stratifiable,

(3) (X, τ) is k–semistratifiable,

(4) (X, τ) is N–semistratifiable,

(5) (X, τ) is a γ^*–space,

(6) (X, τ) is a $\Delta\gamma^*$–space,

(7) (X, τ) is a semistratifiable $\gamma\gamma^*$–space with a σ–discrete net,

(8) (X, τ) is a $\gamma\gamma^*$–space.

Proof. (1) $\Rightarrow$ (2): Let $\{N_n(x): x \in X, n \in \mathbb{N}\}$ be a decreasing Δ^*–nbhd assignment in (X, τ) and consider any closed set F. Clearly, $F \subseteq \bigcap\{\overline{N_n[F]}: n \in \mathbb{N}\}$. So suppose $p \in \overline{N_n[F]}\ \forall n$. Then, $\forall n, \exists x_n \in F$ such that $N_n(p) \cap N_n(x_n) \neq \varnothing$. Hence, as $\{N_n(x): x \in X, n \in \mathbb{N}\}$ is Δ^*, it follows that $x_n \xrightarrow[\tau]{} p$. As $x_n \in F\ \forall n$ and F is closed, we conclude that $p \in F$.

(2) $\Rightarrow$ (3): Suppose (X, τ) is *not* k–semistratifiable and consider any decreasing nbhd assignment $\{N_n(x): x \in X, n \in \mathbb{N}\}$ in (X, τ). Then there exist compact K and closed F, along with a sequence (x_n), such that $K \cap F = \varnothing$ and $x_n \in K \cap N_n[F]\ \forall n$. As (x_n) is a sequence in K, it follows that (x_n) has a cluster point $p \in K$. Thus, $p \notin F$. It suffices to show that $p \in \overline{N_j[F]}$ for each $j \in \mathbb{N}$. So consider any $j \in \mathbb{N}$ and any $M \in \mathcal{N}_\tau(p)$. As (x_n) clusters at p, $\exists i \geq j$ such that $x_i \in M$. Also, as $\{N_n(x): x \in X, n \in \mathbb{N}\}$ is decreasing, $x_i \in N_i[F] \subseteq N_j[F]$. Thus, $M \cap N_j[F] \neq \varnothing$ so that $p \in \overline{N_j[F]}$.

(3) $\Rightarrow$ (4): Let $\{N_n(x): x \in X, n \in \mathbb{N}\}$ be a decreasing k–semistratifiable nbhd assignment in (X, τ) and suppose that $y_n \in N_n(x_n)\ \forall n$ and $y_n \xrightarrow[\tau]{} p$.

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists G \in \tau$ such that $p \in G$ and $\forall n, \exists k_n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - G$. Since $y_{k_n} \xrightarrow[\tau]{} p$, $\exists j$ such that $\forall n$ with $k_n \geq j$, $y_{k_n} \in G$. Let $m = \min\{n : k_n \geq j\}$. Note that, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a decreasing k-semistratifiable neighborhood assignment, so is $\{N_n(x) : x \in X, n \geq m\}$. Now let $K = \{y_{k_n} : k_n \geq j\} \cup \{p\}$ and observe that K is compact and $K \cap (X - G) = \varnothing$. However, for any $n \geq m$, $y_{k_n} \in N_{k_n}(x_{k_n}) \subseteq N_n(x_{k_n}) \subseteq N_n[X - G]$ so that, $K \cap N_n[X - G] \neq \varnothing$, a contradiction.

(4) $\Rightarrow$ (5): Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be an N-semistratifiable nbhd assignment in (X, τ) and suppose that, $\forall n$, $p \in N_n(y_n)$ and $y_n \in N_n(x_n)$. As any N-semistratifiable nbhd assignment is γ_1^*, it follows that $y_n \xrightarrow[\tau]{} p$. Then, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is N-semistratifiable, it follows that $x_n \xrightarrow[\tau]{} p$.

(5) $\Rightarrow$ (6): This implication is obvious.

(6) $\Rightarrow$ (7): We have already shown (Theorem 2.4.7) that any $\Delta\gamma^*$-space is a semistratifiable $\gamma\gamma^*$-space. It suffices to show that a semistratifiable $\gamma\gamma^*$-space has a σ-discrete net.

Let $\{U_n(x) : x \in X, n \in \mathbf{N}\}$ be a decreasing open nbhd assignment in (X, τ) that is both γ_1^* and $\gamma\gamma^*$. Choose a well-ordering $\lhd$ of X and for each $p \in X$ let $z_n(p) = \min\{y \in X : p \in U_n(y)\}$. Now, for each $p \in X$ and all $n, k \in \mathbf{N}$, define

$$A_{n,k}(p) = X - \left(\left(\bigcup\{U_k(x) : x \notin U_n(p)\}\right) \cup \left(\bigcup\{U_n(x) : x \lhd p, x \neq p\}\right)\right),$$

and note that $\{A_{n,k}(x) : x \in X\}$ is discrete. Then, for each $p \in X$ and all $n, k, m \in \mathbf{N}$, define

$$A_{n,k,m}(p) = A_{n,k}(p) \cap U_m^*(p).$$

For all $n, k, m \in \mathbf{N}$, define

$$\mathcal{A}_{n,k,m} = \{A_{n,k,m}(x) : x \in X\}$$

and then let $\mathcal{A} = \bigcup\{\mathcal{A}_{n,k,m} : n, k, m \in \mathbf{N}\}$. It follows that $\mathcal{A}$ is a σ-discrete net.

(7) $\Rightarrow$ (8): This implication is obvious.

2.4.10 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is a Nagata space;

(2) (X, τ) is stratifiable and first countable;

(3) (X, τ) is k-semistratifiable and first countable;

(4) (X, τ) is N-semistratifiable and first countable.

Proof. Note that (1)$\Rightarrow$(2), (2)$\Rightarrow$(3), and (3)$\Rightarrow$(4) are obvious in light of the previous theorem and the observation that Nagata spaces are first countable. So suppose (X, τ) is N-semistratifiable and first countable. Then there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) that is both N-semistratifiable and γ_1. Now suppose $y_n \in N_n(p) \cap N_n(x_n)$ $\forall n$. It follows, since $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is γ_1, that $y_n \xrightarrow[\tau]{} p$. Thus, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is N-semistratifiable, it follows that $x_n \xrightarrow[\tau]{} p$.

Dennis Burke [Bu] has provided an example of a developable space that is neither a γ-space nor a Nagata space. This example is also studied in [Sa]. Since it is developable, it is a first countable γ^*-space. But as it is not a γ-space, it is not metrizable (see Theorem 2.6.2 in §6 of this chapter).

The following diagram summarizes the relationships among the spaces studied in this section; compare with the diagram in Figure 2.

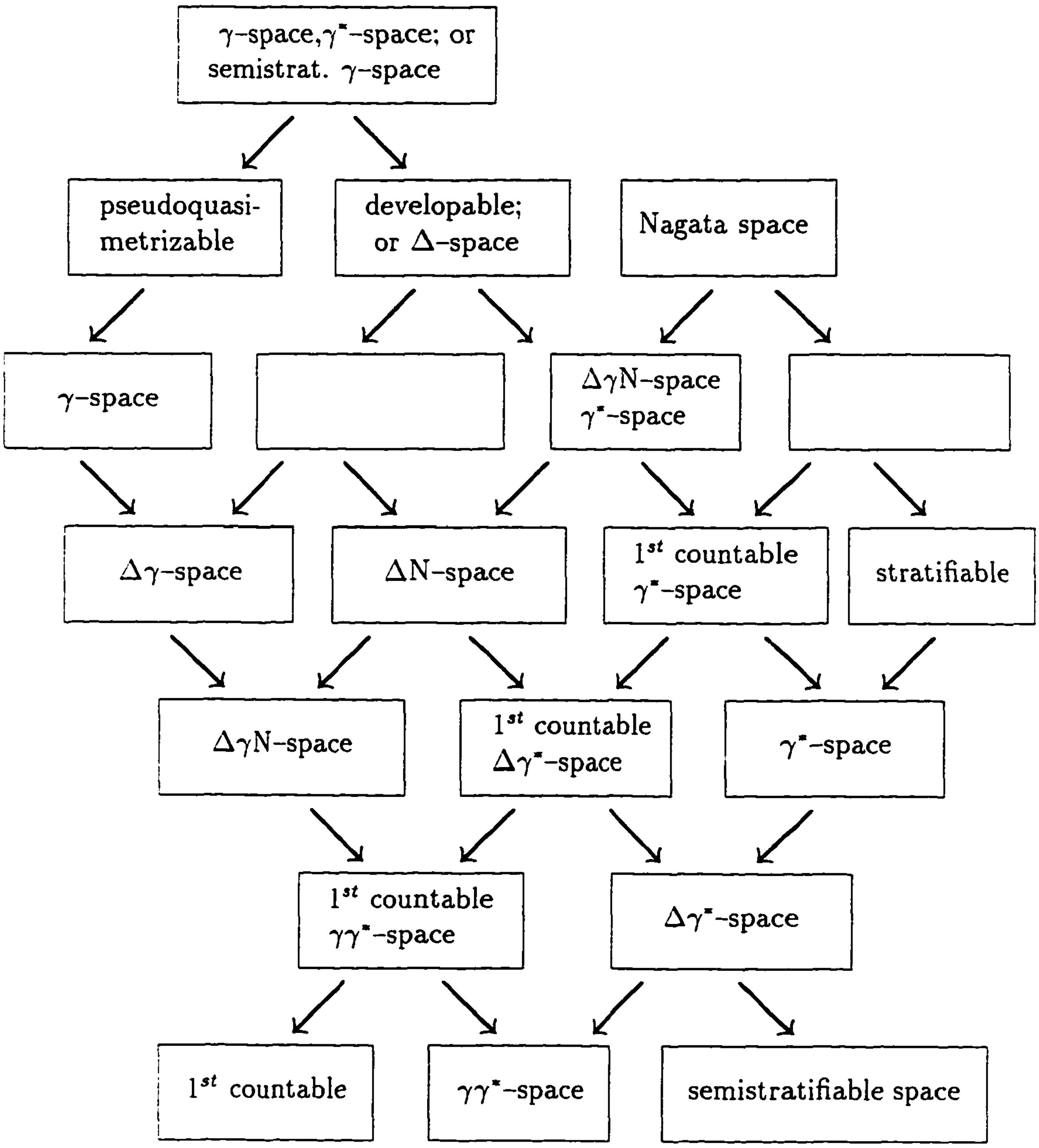

Figure 3. Developable and its dual Nagata

5. Pseudoquasimetrizability

A **pseudoquasimetric** for a set X is a distance d for X such that

$$\forall p, q, x \in X,\ d(p, q) \le d(p, x) + d(x, q).$$

In other words a pseudoquasimetric is a distance that satisfies the Triangle Inequality.

A topological space (X, τ) is **pseudoquasimetrizable** if there is a pseudoquasimetric d for X such that $\tau = \tau_d$.

Part of the significance of pseudoquasimetrics lies in the fact that any topological space can be generated from a family of pseudoquasimetrics.[28]

2.5.1 Theorem. [Gr] The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudoquasimetrizable;

(2) there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

(a) for each $p \in X$, $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$, and

(b) $q \in N_{n+1}(p) \Rightarrow N_{n+1}(q) \subseteq N_n(p)$.

Proof. (1)⇒(2): If d is a pseudoquasimetric for X such that $\tau = \tau_d$, simply let $N_n(p) = S_d\left(p, 1/2^n\right)$.

(2)⇒(1): Suppose there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) for which both $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and $q \in N_{n+1}(p) \Rightarrow N_{n+1}(q) \subseteq N_n(p)$.

Define a distance d for X so that

$$d(p, q) = \begin{cases} 0, \text{ if } q \in N_k(p)\ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin N_k(p)\}, \text{ otherwise.} \end{cases}$$

---

[28]Given a topological space (X, τ), define, for each $G \in \tau$, a pseudoquasimetric d_G for X so that $d_G(p, q) = 0$ if $p \in G$ and $q \notin G$, and $d_G(p, q) = 1$ otherwise. Then τ is the coarsest topology generated by $\{\tau_{d_G} : G \in \tau\}$.

Note that $S_d\left(p, 1/2^n\right) = N_n(p)$; hence, $\tau_d = \tau$. Note also that $d(p,p) = 0 \; \forall p \in X$; however, d may not satisfy the Triangle Inequality.

So define the distance D for X so that

$$D(p,q) = \inf\left\{\sum_{i=0}^{n} d(x_i, x_{i+1}) : n \in \mathbf{N},\ x_i \in X \ \forall i \in \{0, 1, \ldots, n+1\},\ x_0 = p, x_{n+1} = q\right\}.$$

It follows that D is a pseudoquasimetric for X and $\tau = \tau_D$.[29]

A **nonarchimedean pseudoquasimetric** for a set X is a distance d for X such that

$$\forall p, q, x \in X,\ d(p,q) \le \max\{d(p,x), d(x,q)\}.$$

A topological space (X, τ) is **nonarchimedean pseudoquasimetrizable** if there is a nonarchimedean pseudoquasimetric d for X such that $\tau = \tau_d$.

Clearly, any nonarchimedean pseudoquasimetrizable space is pseudoquasimetrizable. Kofner [Kof] provides an example of a (pseudo)quasimetrizable space that is not nonarchimedean (pseudo)quasimetrizable.

2.5.2 Theorem. [Gr] The following are equivalent for a topological space (X, τ):

(1) (X, τ) is nonarchimedean pseudoquasimetrizable;

(2) there is a (decreasing) nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

(a) for each $p \in X$, $x_n \in N_n(p)$, $\forall n \;\Rightarrow\; x_n \xrightarrow[\tau]{} p$, and

(b) $q \in N_n(p) \;\Rightarrow\; N_n(q) \subseteq N_n(p)$.

The proof of Theorem 2.5.2 is similar to that of Theorem 2.5.1.

[29] This proof makes implicit use of Frink's Lemma (see [Gr]).

Given a set X and a subset V of $X \times X$, we define

$$V^2 = \{(x, y) \in X \times X : \exists z \in X, (x, z) \in V \text{ and } (z, y) \in V\}$$

and, for each positive integer $n \geq 3$,

$$V^n = \left\{(x, y) \in X \times X : \exists z \in X, (x, z) \in V \text{ and } (z, y) \in V^{n-1}\right\},$$

and, for each $p \in X$, we write $V[p]$ for $\{x : (p, x) \in V\}$.

Hence, if $\{S_\alpha(x) : x \in X, \alpha \in I\}$ is a family of subsets of X and if, for each $\alpha \in I$, we consider S_α to be the subset of $X \times X$ having the property that $S_\alpha[x] = S_\alpha(x)$ for each $x \in X$, then, for each $n \in \mathbf{N}$, we may define $S_\alpha^n(x)$ so that $S_\alpha^n(x) = S_\alpha^n[x]$.

If (X, τ) is a topological space and $V \subseteq X \times X$, we say V is a **neighbornet** [Ju] (abbreviated *nbnet*) in (X, τ) if for each $p \in X$, $V[p] \in \mathcal{N}_\tau(p)$.

A sequence (V_n) of nbnets in (X, τ) is

(1) **decreasing** if the induced nbhd assignment $\{V_n[x] : x \in X, n \in \mathbf{N}\}$ is decreasing, and

(2) a **normal basic sequence** if $V_{n+1}^2 \subseteq V_n$ $\forall n$ and, for each $p \in X$, $\{V_n[p] : n \in \mathbf{N}\}$ is a nbhd base at p.

2.5.3 Lemma. Let (X, τ) be a topological space.

(1) [LF] (X, τ) is a γ-space iff there is a decreasing sequence (V_n) of nbnets in (X, τ) such that for each $p \in X$, $\left\{V_n^2[p] : n \in \mathbf{N}\right\}$ is a nbhd base at p.

(2) [Ju] (X, τ) is pseudoquasimetrizable iff (X, τ) has a normal basic sequence.

(3) [Fo] If (X, τ) is a developable γ-space and U is a nbnet in (X, τ), then there is a nbnet W in (X, τ) such that $W \subseteq U^2$.

Proof. (1) If $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ-nbhd assignment in (X, τ), it follows that $\left(x_n \in N_n^2(p)\ \forall n\right) \Rightarrow x_n \xrightarrow[\tau]{} p$. Define a sequence (V_n) of nbnets so that $V_n[p] = N_n(p)$ and note that $V_n^2[p] = N_n^2(p)$. Then $\left\{V_n^2[p] : n \in \mathbf{N}\right\}$ is a nbhd base at p.

Conversely, if (V_n) is a sequence of nbnets in (X, τ) such that for each $p \in X$, $\{V_n^2[p] : n \in \mathbf{N}\}$ is a nbhd base at p, it follows that $\{V_n[x] : x \in X, n \in \mathbf{N}\}$ is a γ-nbhd assignment in (X, τ).

(2) If (X, τ) is pseudoquasimetrizable, then there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) for which both $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ (so that $\{N_n(p) : n \in \mathbf{N}\}$ is a nbhd base at p) and $q \in N_{n+1}(p) \Rightarrow N_{n+1}(q) \subseteq N_n(p)$. Define a sequence (V_n) of nbnets so that $V_n[p] = N_n(p)$ and note that $V_{n+1}^2 \subseteq V_n$.

Conversely, let (V_n) be a sequence of nbnets in (X, τ) for which $V_{n+1}^2 \subseteq V_n \ \forall n$ and $\{V_n[p] : n \in \mathbf{N}\}$ is a nbhd base at p. Define $N_n(p) = V_n[p]$. As $\{V_n[p] : n \in \mathbf{N}\}$ is a nbhd base at p, it follows that $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$. Now if $q \in N_{n+1}(p)$ and $x \in N_{n+1}(q)$, then $(p, q) \in V_{n+1}$ and $(q, x) \in V_{n+1}$ so that $(p, x) \in V_{n+1}^2 \subseteq V_n$ and, therefore, $x \in N_n(p)$.

(3) The following proof is essentially that given by Fox in [Fo; 1981]. Suppose (X, τ) is a developable γ-space and U is a nbnet in (X, τ). Let $(\mathcal{G}_n)$ be a development for (X, τ) with $\mathcal{G}_{n+1} \prec \mathcal{G}_n \ \forall n$ and let (V_n) be a decreasing sequence of nbnets in (X, τ) such that, for each $p \in X$, $\{V_n^2[p] : n \in \mathbf{N}\}$ is a nbhd base at p. Observe that, for each $p \in X$, $\{V_n^5[p] : n \in \mathbf{N}\}$ is a nbhd base at p.

For each $p \in X$, let

$$j_p = \min\{n : V_n^5[p] \subseteq U[p]\} \text{ and } k_p = \min\{n \geq j_p : \mathrm{st}(p, \mathcal{G}_n) \subseteq V_{j_p}[p]\}.$$

Then, for each $A \in N_\tau(p) \cap \tau$, let

$$k_A = \min\{k_a : a \in A\}.$$

Finally, for each $p \in X$, let

$$l_p = \max\{k_A : A \in N_\tau(p) \cap \tau\}.$$

Note that $j_p \leq k_p$, $\mathrm{st}(p, \mathcal{G}_{k_p}) \subseteq V_{j_p}[p]$, and $l_p \leq k_p$ (since, for each open nbhd A of p, $k_A \leq k_p$).

Define $C_0 = \varnothing$ and, $\forall n \in \mathbf{N}$, $C_n = \{x \in X : k_x \leq n\}$, and observe that $\overline{C_n} = \{x \in X : l_x \leq n\}$. Also note that, $\forall p \in X$, $p \notin \overline{C_{l_p - 1}}$.

Now, $\forall p \in X$, let $W(p) = V_{k_p}[p] - \overline{C_{l_p - 1}}$ and note that $W(p) \in \mathcal{N}_\tau(p)$.

Claim. $\forall p \in X$, $W^4(p) \subseteq U^2[p]$

Pf. Consider any $x_4 \in W^4(p)$. Then there are $x_1, x_2, x_3 \in X$ such that $x_4 \in W(x_3)$, $x_3 \in W(x_2)$, $x_2 \in W(x_1)$, and $x_1 \in W(p)$. Note that, for $i \in \{0, 1, 2, 3, 4\}$, $x_{i+1} \notin \overline{C_{l_{x_i}} - 1}$ (where $x_0 = p$) and, therefore, $l_p \le l_{x_1} \le l_{x_2} \le l_{x_3} \le l_{x_4}$. It follows that $\exists y \in U[p] \cap \mathrm{st}(p, \mathcal{G}_{l_p})$. Then $j_y \le k_y = l_p \le l_{x_i} \le k_{x_i}$ for each $i \in \{1, 2, 3, 4\}$. So, as (V_n) is decreasing, $x_{i+1} \in W(x_i) \subseteq V_{k_{x_i}}[x_i] \subseteq V_{j_y}[x_i]$ for each $i \in \{0, 1, 2, 3, 4\}$. Hence, $x_4 \in V_{j_y}^4[p]$. But $p \in \mathrm{st}(y, \mathcal{G}_{l_p}) = \mathrm{st}(y, \mathcal{G}_{k_y}) \subseteq V_{j_y}[y]$ so that $x_4 \in V_{j_y}^5[y] \subseteq U[y]$. Thus, as $y \in U[p]$, it follows that $x_4 \in U^2[p]$.

2.5.4 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is a semistratifiable pseudoquasimetrizable space[30];

(2) (X, τ) is a developable γ-space;

(3) (X, τ) is a γ-, γ^*-space;

(4) (X, τ) is a semistratifiable γ-space.

Proof. (1)$\Rightarrow$(4): It suffices to show that any pseudoquasimetrizable space is a γ-space. So let $\{N_n(x) : x \in X, n \in \mathbb{N}\}$ be a nbhd assignment in (X, τ) for which both $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and $q \in N_{n+1}(p) \Rightarrow N_{n+1}(q) \subseteq N_n(p)$, and suppose that, for each $n \in \mathbb{N}$, $x_n \in N_n(y_n)$ and $y_n \in N_n(p)$. It follows that

$$x_{n+1} \in N_{n+1}(y_{n+1}) \subseteq N_n(p)$$

so that $x_{n+1} \xrightarrow[\tau]{} p$ and, therefore, $x_n \xrightarrow[\tau]{} p$.

(2)$\Rightarrow$(3): This implication is immediate since we have already shown (Theorem 2.4.6) that any developable space is a γ^*-space.

[30] A semistratifiable pseudoquasimetrizable space is also known as a *strongly pseudoquasimetrizable space*.

(3)$\Rightarrow$(4): Clearly, any γ^*-space is semistratifiable.

(4)$\Rightarrow$(2): It suffices to show that any semistratifiable γ-space is developable. So let $\{N_n(x): x \in X, n \in \mathbf{N}\}$ be a decreasing nbhd assignment in (X, τ) that is both γ_1^* and γ and suppose that, for each $n \in \mathbf{N}$, $x_n, p \in N_n(y_n)$. It follows that $y_n \xrightarrow[\tau]{} p$. Hence, $\forall n, \exists k_n \geq n$ such that $k_{n+1} > k_n$ and $y_{k_n} \in N_n(p)$. Then, since we also have $x_{k_n} \in N_{k_n}(y_{k_n}) \subseteq N_n(y_{k_n})$ $\forall n$ and $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is a γ-nbhd assignment, it follows that $x_{k_n} \xrightarrow[\tau]{} p$. Thus, (x_n) clusters at p.

Note that we have actually shown that

(†) $$a_n, q \in N_n(b_n)\ \forall n \Rightarrow (a_n) \text{ clusters at } p.$$

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists M \in \mathcal{N}_\tau(p)$ such that $\forall n, \exists k_n \geq n$ with $k_{n+1} > k_n$ and $x_{k_n} \in X - M$. By hypothesis, $x_{k_n}, p \in N_{k_n}(y_{k_n}) \subseteq N_n(y_{k_n})$ $\forall n$, so that, by (†), (x_{k_n}) clusters at p, which contradicts the fact that (x_{k_n}) is never in M.

(2)$\Rightarrow$(1): As we have already shown (Lemma 2.4.1) that any developable space is semistratifiable, it suffices to show that any developable γ-space is pseudoquasimetrizable. The argument presented is due to Fox [Fo; 1981]. Let (X, τ) be a developable γ-space. Then there is a decreasing sequence (V_n) of nbnets such that, for each $p \in X$, $\{V_n^2[p]: n \in \mathbf{N}\}$ is a nbhd base at p. Then, by (3) of Lemma 2.5.3, it follows that there is a sequence (W_n) of nbnets in (X, τ) such that $W_1^4 \subseteq V_1$ and, for $n \geq 2$, $W_{n+1}^4 \subseteq (W_n \cap V_{n+1})^2$. Then, $\forall n$, $(W_{n+1}^2)^2 \subseteq W_n^2$. Also, $\forall p \in X, \forall n \in \mathbf{N}$, as $W_{n+1}^2[p] \subseteq W_{n+1}^4[p] \subseteq (W_n \cap V_{n+1})^2[p] \subseteq V_{n+1}^2[p]$, it follows that $\{W_n^2[p]: n \in \mathbf{N}\}$ is a nbhd base at p. Thus, (W_n^2) is a normal basic sequence and, therefore, (X, τ) is pseudo-quasimetrizable.

Next we will examine pseudoquasimetrizability and nonarchimedean pseudo-quasimetrizability in the context of N-semistratifiable spaces (which, of course, include the Nagata spaces and the stratifiable spaces). The investigation requires some preliminary definitions.

A topological space (X, τ) is **collectionwise normal**, abbreviated *cwN*, if any discrete family of closed sets can be separated by disjoint[31] open sets and is **strongly screenable** if every open cover of X has an open σ-discrete refining cover.

A collection of subsets of X is called **point finite** if each point of X belongs to only finitely many members of the collection.

A base for a topology on a set X is **σ-discrete** if it can be written as a countable union of discrete collections of subsets of X and is **σ-point finite** if it can be written as a countable union of point finite collections of subsets of X.

2.5.5 Lemma. Let (X, τ) be a topological space.

(1) If (X, τ) is stratifiable, then (X, τ) is cwN.

(2) If (X, τ) is cwN and subparacompact, then (X, τ) is strongly screenable.

(3) [Bi] If (X, τ) is developable and strongly screenable, then τ has a σ-discrete base.

(4) [Gr] If τ has a σ-point finite base, then (X, τ) is nonarchimedean pseudo-quasimetrizable.

Proof. (1) Let $\{U_n(x) : x \in X, n \in \mathbf{N}\}$ be a stratifiable nbhd assignment in (X, τ) consisting of open sets and consider any discrete family $\mathcal{F}$ of closed sets. For each $F \in \mathcal{F}$

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[31] *Disjoint* can be replaced by *discrete* in the definition of cwN. If (X, τ) is cwN and $\mathcal{F}$ is a discrete family of closed sets, then there exists a disjoint family $H = \{H(F) : F \in \mathcal{F}\}$ of open sets such that $F \subseteq H(F)$ $\forall F \in \mathcal{F}$. As $\mathcal{F}$ is discrete, $\bigcup \mathcal{F}$ is closed. Since, clearly, any cwN space is normal and $\bigcup \mathcal{F}$ and $X - \bigcup H$ are disjoint closed sets, there exist disjoint open sets G_1 and G_2 with $\bigcup \mathcal{F} \subseteq G_1$ and $X - \bigcup H \subseteq G_2$. Now for each $F \in \mathcal{F}$, let $G(F) = H(F) \cap G_1$ and then let $\mathcal{G} = \{G(F) : F \in \mathcal{F}\}$. It follows that $\mathcal{G}$ is a discrete family of open sets that separates $\mathcal{F}$.

and each $n \in \mathbb{N}$, define $V_n(F) = \bigcup\left\{U_n\left(\tilde{F}\right) : \tilde{F} \in \mathcal{F}, \tilde{F} \neq F\right\}$ and $W_n(F) = U_n[F] - \overline{V_n(F)}$,

and then let $W(F) = \bigcup_{n=1}^{\infty} W_n(F)$. Then $\mathcal{W} = \{W(F) : F \in \mathcal{F}\}$ is a family of open sets.

Claim 1. $\forall\, F \in \mathcal{F}, F \subseteq W(F)$

Pf. Consider any $F \in \mathcal{F}$ and suppose $p \notin W(F)$. Then, $\forall n, p \notin W_n(F)$ so that $p \in \overline{V_n(F)}, \forall n$. But $\forall n$,

$$\overline{V_n(F)} = \overline{\bigcup\left\{U_n\left[\tilde{F}\right] : \tilde{F} \in \mathcal{F}, \tilde{F} \neq F\right\}} = \overline{U_n\left[\bigcup\left\{\tilde{F} \in \mathcal{F} : \tilde{F} \neq F\right\}\right]}.$$

Now, as $\mathcal{F}$ is discrete, it follows that $\bigcup\left\{\tilde{F} \in \mathcal{F} : \tilde{F} \neq F\right\}$ is closed. Then, since the nbhd assignment $\{U_n(x) : x \in X, n \in \mathbb{N}\}$ is stratifiable,

$$\bigcup\left\{\tilde{F} \in \mathcal{F} : \tilde{F} \neq F\right\} = \bigcap_{n=1}^{\infty} \overline{U_n\left[\bigcup\left\{\tilde{F} \in \mathcal{F} : \tilde{F} \neq F\right\}\right]}.$$

So $p \in \bigcup\left\{\tilde{F} \in \mathcal{F} : \tilde{F} \neq F\right\}$ which means that $p \notin F$.

Claim 2. $\mathcal{W}$ is disjoint

Pf. Suppose to the contrary that there exist distinct $F_1, F_2 \in \mathcal{F}$ and $p \in X$ such that $p \in W(F_1) \cap W(F_2)$. Then there exist n_1 and n_2 such that $p \in W_{n_1}(F_1) \cap W_{n_2}(F_2)$. Without loss of generality we may assume that $n_1 \leq n_2$. Note that $p \notin \overline{V_{n_1}(F_1)} \supseteq \overline{U_{n_1}[F_2]} \supseteq \overline{U_{n_2}[F_2]}$. But $W_{n_2}(F_2) \subseteq U_{n_2}[F_2]$ and, hence, it follows that $p \notin W_{n_2}(F_2)$, a contradiction.

(2) Suppose (X, τ) is cwN and subparacompact and consider any open cover $\mathcal{G}$ of X. Then there is a sequence $(\mathcal{F}_n)$ of discrete families of closed sets that refine $\mathcal{G}$ and for which $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ covers X. As (X, τ) is cwN, for each n, there exists a discrete family $\mathcal{H}_n = \{H_n(F) : F \in \mathcal{F}_n\}$ of open sets such that $F \subseteq H_n(F)\ \forall F \in \mathcal{F}_n$. Also, for each $n \in \mathbb{N}$ and each $F \in \mathcal{F}_n$, $\exists\, G_n(F) \in \mathcal{G}$ such that $F \subseteq G_n(F)$. Now, for each $n \in \mathbb{N}$, let

$\mathcal{U}_n = \{G_n(F) \cap H_n(F) : F \in \mathcal{F}_n\}$. It follows that, $\forall n$, $\mathcal{U}_n$ is a discrete family of open sets that refines $\mathcal{G}$ and that $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ covers X.

(3) Suppose (X, τ) is developable and strongly screenable and let $(\mathcal{G}_n)$ be a development for (X, τ). Then, for each $n \in \mathbf{N}$, there exists a sequence $(\mathcal{U}_{n,k})$ of discrete families of open sets that refine $\mathcal{G}_n$ and for which $\bigcup\{\mathcal{U}_{n,k} : k \in \mathbf{N}\}$ covers X. Note that, $\forall p \in X, \forall n, \forall k, \mathrm{st}(p, \mathcal{U}_{n,k}) \subseteq \mathrm{st}(p, \mathcal{G}_n)$. It follows that $\bigcup\{\mathcal{U}_{n,k} : k \in \mathbf{N}\}$ is a σ-discrete base for τ.

(4) Suppose that, for each $n \in \mathbf{N}$, $\mathcal{B}_n \subseteq \tau$ is point finite and $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base for τ. For each $p \in X$ and each $n \in \mathbf{N}$, let $N_n(p) = \bigcap\left\{B \in \bigcup_{i=1}^{n} \mathcal{B}_i : p \in B\right\}$, which is open since, for each n, $\mathcal{B}_n$ is point finite.

Claim 1. For each $p \in X$, $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$.

Pf. Suppose $x_n \in N_n(p), \forall n$ and consider any $M \in \mathcal{N}_\tau(p)$. Since $\mathcal{B}$ is a base for τ, there exists k such that $p \in B \subseteq M$ for some $B \in \mathcal{B}_k$. But $N_k(p) \subseteq B$ and, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is decreasing, it follows that $x_n \in B$ $\forall n \geq k$. Hence, $x_n \in M$ $\forall n \geq k$.

Claim 2. $q \in N_n(p) \Rightarrow N_n(q) \subseteq N_n(p)$

Pf. Suppose $q \in N_n(p)$ and consider any $x \in N_n(q)$. Then q belongs to every member of $\bigcup_{i=1}^{n} \mathcal{B}_i$ that p belongs to and x belongs to every member of $\bigcup_{i=1}^{n} \mathcal{B}_i$ that q belongs to. Therefore, x belongs to every member of $\bigcup_{i=1}^{n} \mathcal{B}_i$ that p belongs to. So $x \in N_n(p)$.

2.5.6 Theorem. The following are equivalent for an N-semistratifiable space (X, τ):

(1) (X, τ) is developable;

(2) (X, τ) is nonarchimedean pseudoquasimetrizable;

(3) (X, τ) is pseudoquasimetrizable;

(4) (X, τ) is a γ-space;

(5) (X, τ) is a $\Delta\gamma$-space.

Proof. The implications (2)⇒(3), (3)⇒(4), and (4)⇒(5) hold for any topological space. Both (2)⇒(3) and (4)⇒(5) are obvious.

(3)⇒(4): Let $\{N_n(x) : x \in X, n \in \mathrm{N}\}$ be a decreasing nbhd assignment in (X, τ) for which $x_n \in N_n(p), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ for each $p \in X$ and, in addition, $q \in N_{n+1}(p) \Rightarrow N_{n+1}(q) \subseteq N_n(p)$. Suppose that, $\forall n \in \mathrm{N}$, $x_n \in N_n(y_n)$ and $y_n \in N_n(p)$. Then, $\forall n$, $y_{n+2} \in N_{n+2}(p) \subseteq N_{n+1}(p)$ so that, $\forall n$, $N_{n+1}(y_{n+2}) \subseteq N_n(p)$. Also, $\forall n$, $x_{n+2} \in N_{n+2}(y_{n+2}) \subseteq N_{n+1}(y_{n+2})$. So, $\forall n$, $x_{n+2} \in N_n(p)$. Hence, $x_{n+2} \xrightarrow[\tau]{} p$ so that $x_n \xrightarrow[\tau]{} p$.

(5)⇒(1): This implication follows easily using the facts that any N-semistratifiable space is (clearly) semistratifiable and a topological space is developable iff it is a semistratifiable $\Delta\gamma$-space [Hol; 1972].

(1)⇒(2): Suppose (X, τ) is a developable space that is N-semistratifiable. As any developable space is first countable and any first countable N-semistratifiable space is a Nagata space (Theorem 2.4.10), it follows that (X, τ) is a Nagata space. Since any Nagata space is stratifiable (Theorem 2.4.9), (X, τ) is stratifiable. Since any stratifiable space is cwN , (X, τ) is cwN. Also, as any developable space is, being semistratifiable, subparacompact (Lemmas 2.4.1 and 2.4.5), and as a cwN subparacompact space is strongly screenable , it follows that (X, τ) is strongly screenable. But any developable space that is strongly screenable has a σ-discrete base ; thus, there is a σ-discrete base for τ. So, as a σ-discrete base is σ-point finite, there is a σ-point finite base for τ. Hence, by Lemma 2.5.5, (X, τ) is nonarchimedean pseudoquasimetrizable.

The diagram below summarizes the relationships among the spaces considered in this section.

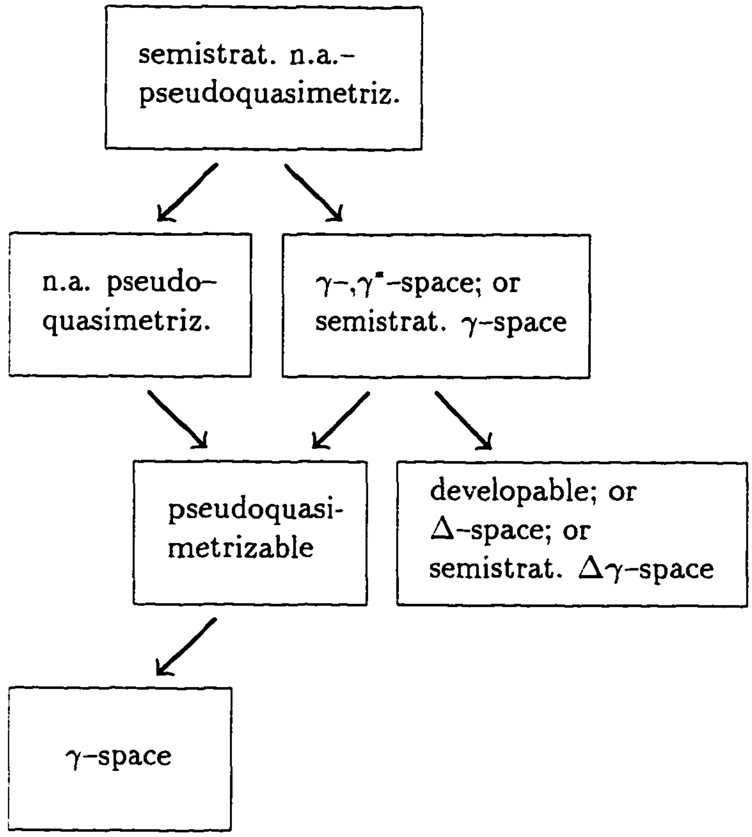

Figure 4. Pseudoquasimetrizable and nonarchimedean pseudoquasimetrizable

6. Pseudometrizability

A **pseudometric** for a set X is a symmetric pseudoquasimetric for X.

A topological space (X, τ) is **pseudometrizable** if there is a pseudometric d for X such that $\tau = \tau_d$.

The following theorem provides a weak neighborhood characterization of pseudometrizable spaces, thus generalizing a parallel theorem of Heath [Hea; 1962].

2.6.1 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudometrizable;

(2) there is a (decreasing) weak base $\{W_n(x): x \in X, n \in \mathbb{N}\}$ for τ such that

(a) for each $p \in X$, $x_n, p \in W_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$, and

(b) $q \in W_n(p) \Rightarrow p \in W_n(q)$.

Proof. (1)$\Rightarrow$(2): If d is a pseudometric for X such that $\tau = \tau_d$, let $W_n(p) = S_d(p, 1/2^n)$.

(2)$\Rightarrow$(1): Suppose $\{W_n(x): x \in X, n \in \mathbb{N}\}$ is a decreasing weak base for τ for which both $x_n, p \in W_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and $q \in W_n(p) \Rightarrow p \in W_n(q)$.

Define a distance d for X so that

$$d(p,q) = \begin{cases} 0, \text{ if } q \in W_k(p)\ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin W_k(p)\}, \text{ otherwise.} \end{cases}$$

Note that $S_d(p, 1/2^n) = W_n(p)$; hence, as $\{W_n(x): x \in X, n \in \mathbb{N}\}$ is a weak base for τ, it follows that $\tau = \tau_d$. Note, too, that d is symmetric.

It also follows that

$$\left(d(p, y_n) \to 0 \text{ and } d(y_n, x_n) \to 0\right) \Rightarrow d(p, x_n) \to 0$$

so that the spheres are actually open and

$$\forall p \in X, \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \left(d(p,x) < \delta \text{ and } d(x,q) < \delta\right) \Rightarrow d(p,q) < \varepsilon.$$

Thus, for each $p \in X$, there exists a sequence $(\delta_n(p))$ of positive real numbers and a sequence $(U_n(p))$ of open nbhds of p such that $\delta_{n+1}(p) \le \min\{\delta_n(p), 1/2^n\}$ and

$$(d(p,x) < \delta_{n+1}(p) \text{ and } d(x,q) < \delta_{n+1}(p)) \Rightarrow q \in U_n(p) \subseteq S_d(p, \delta_n(p)).$$

So, for each $p \in X$, $\{U_n(p) : n \in \mathbf{N}\}$ is a nbhd base at p and

$$(U_{n+2}(p) \cap U_{n+2}(p) \ne \varnothing \text{ and } \delta_n(q) \le \delta_n(p)) \Rightarrow U_{n+2}(p) \cup U_{n+2}(q) \subseteq U_n(p).$$

For each $n \in \mathbf{N}$, let $\mathcal{U}_n = \{U_{2n}(x) : x \in X\}$ and define a distance d' so that

$$d'(p,q) = \begin{cases} 0, \text{ if } q \in \operatorname{st}(p, \mathcal{U}_k) \ \forall k; \\ 1/2^n, \text{ where } n = \min\{k : q \notin \operatorname{st}(p, \mathcal{U}_k)\}, \text{ otherwise}, \end{cases}$$

and note that d' is symmetric and $\tau_{d'} = \tau_d = \tau$.

Now let D be the distance for X defined by

$$D(p,q) = \inf\left\{ \sum_{i=0}^{n} d'(x_i, x_{i+1}) : n \in \mathbf{N}, x_i \in X \ \forall i \in \{0, 1, \dots, n+1\}, x_0 = p, x_{n+1} = q \right\}$$

and observe that D is a pseudometric for X and $\tau_D = \tau$.

2.6.2 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudometrizable;

(2) (X, τ) is a Nagata γ-space;

(3) (X, τ) is a Nagata $\Delta\gamma$-space;

(4) (X, τ) is a Nagata developable space.

Stratifiable, or even *N-semistratifiable*, can replace *Nagata* in this theorem.

Proof. The equivalence of (2), (3), and (4) follows from Theorem 2.5.6.

(1)$\Rightarrow$(4): Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a decreasing nbhd assignment in (X, τ) for which both $x_n, p \in N_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ (so that (X, τ) is clearly developable) and $q \in N_n(p) \Rightarrow p \in N_n(q)$. If $y_n \in N_n(p) \cap N_n(x_n)$ $\forall n$, it follows that $x_n, p \in N_n(y_n)$ $\forall n$ so that $x_n \xrightarrow[\tau]{} p$; hence, (X, τ) is also a Nagata space.

(4)$\Rightarrow$(1): We show that every N-semistratifiable developable space is pseudometrizable. Let $\{N_n(x): x \in X, n \in \mathbf{N}\}$ be a decreasing nbhd assignment in (X, τ) that is both N-semistratifiable and developable. For each $p \in X$ and each $n \in \mathbf{N}$, define $M_n(p) = N_n(p) \cup N_n^*(p)$ and note that $\{M_n(x): x \in X, n \in \mathbf{N}\}$ is a nbhd assignment in (X, τ). Clearly, $q \in M_n(p) \;\Rightarrow\; p \in M_n(q)$.

Claim. $p, x_n \in M_n(y_n)\ \forall n \;\Rightarrow\; x_n \xrightarrow[\tau]{} p$

Pf. Suppose $p, x_n \in M_n(y_n)\ \forall n$.

Case (i): For infinitely many n, $p, x_n \in N_n(y_n)$.

Suppose $p, x_{k_n} \in N_{k_n}(y_{k_n})\ \forall n$. Then $x_{k_n} \xrightarrow[\tau]{} p$. Now if (x_n) does not converge to p, $\exists L \in \mathcal{N}_\tau(p)$ such that $\forall n, \exists j_n \geq n$ with $j_{n+1} > j_n$ and $x_{j_n} \in X - L$. But $p, x_{j_n} \in M_{j_n}(y_{j_n})\ \forall n$ so for infinitely many n, $p, x_{j_n} \in N_{j_n}(y_{j_n})$ or $p, x_{j_n} \in N_{j_n}^*(y_{j_n})$. From the former situation, it follows that (x_{j_n}) clusters at p, a contradiction. From the latter situation, it follows that $y_{j_n} \in N_{j_n}(p) \cap N_{j_n}(x_{j_n})$ for infinitely many n, so that (y_{j_n}) clusters at p and, therefore, (x_{j_n}) clusters at p, again a contradiction.

Case (ii): For infinitely many n, $p, x_n \in N_n^*(y_n)$.

The argument is similar to that presented in *Case (i)*.

Thus, by Theorem 2.6.1, (X, τ) is pseudometrizable.

Let $m \in \mathbf{N}$ and suppose (X, τ) is a topological space.

(1) If there is a nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

for each $p \in X$, $x_n \in N_n^m(p)\ \forall n \;\Rightarrow\; x_n \xrightarrow[\tau]{} p$,

we will say that (X, τ) is a γ_m**-space**.

(2) If there is a nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

for each $p \in X$, $p \in N_n^m(x_n)\ \forall n \;\Rightarrow\; x_n \xrightarrow[\tau]{} p$, we will say that (X, τ) is a γ_m^***-space.**

(3) If there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each } p \in X, \ \left(x_n \in N_n^m(p) \text{ and } p \in N_n^m(x_n) \ \forall n\right) \ \Rightarrow \ x_n \xrightarrow[\tau]{} p,$$

we will say that (X, τ) is an S_m**-space.**[32]

(4) If there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

$$\text{for each } A \subseteq X, \ \bar{A} \subseteq \bigcap \left\{N_n^m[A] : n \in \mathbf{N}\right\},$$

then (X, τ) is called an N_m**-space** [Ho2].[33]

The diagram below summarizes the relationships among γ_m-spaces, γ_m^*-spaces, and S_m-spaces.

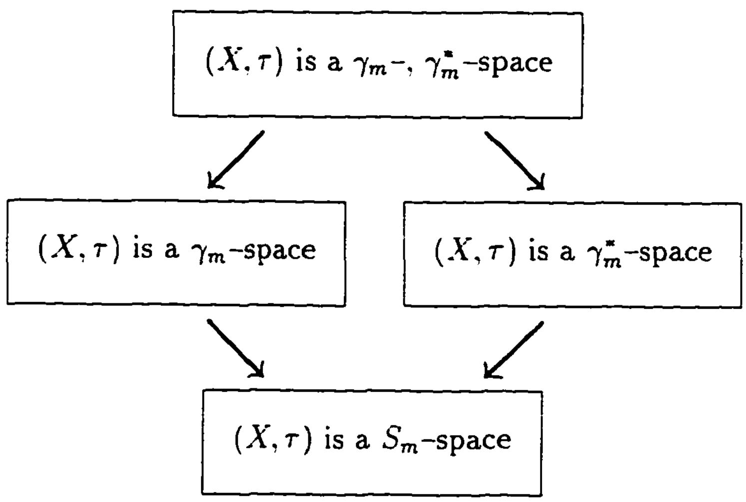

Figure 5. γ_m and its dual γ_m^*

2.6.3 Lemma. Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a nbhd assignment in (X, τ). Then for any $m \in \mathbf{N}$, $N_n^m(p) \subseteq N_n^{m+1}(p)$ for every $p \in X$ and every $n \in \mathbf{N}$. Hence:

(1) if (X, τ) is a γ_{m+1}-space, then (X, τ) is a γ_m-space;

(2) if (X, τ) is a γ_{m+1}^*-space, then (X, τ) is a γ_m^*-space;

---

[32]The letter S has been chosen because of the symmetry in the hypothesis of this condition.

[33]Hodel has chosen the letter N in recognition of Nagata.

(3) if (X, τ) is a S_{m+1}–space, then (X, τ) is a S_m–space;

(4) if (X, τ) is a N_m–space, then (X, τ) is a N_{m+1}–space.

2.6.4 Lemma. Let (X, τ) be a topological space.

(1) (X, τ) is a γ_2–space iff (X, τ) is a γ–space.

(2) (X, τ) is a γ_2^*–space iff (X, τ) is a γ^*–space.

For any $m \in \mathbf{N}$, we may assume that γ_m–, γ_m^*–, and S_m–nbhd assignments are decreasing. We may also assume that an N_1–nbhd assignment is decreasing. However, Ziqiu [Zi] shows that a decreasing N_m–nbhd assignment cannot necessarily be obtained from a given N_m–nbhd assignment when $m \geq 2$.

2.6.5 Lemma. [Ho2] The following are equivalent for a neighborhood assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ):

(1) $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is an N_m–nbhd assignment;

(2) $\forall p \in X, \forall n \in \mathbf{N}, p \in X - \overline{\{x \in X : p \notin N_n^m(x)\}}$.

Proof. (1)⇒(2): Suppose $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is an N_m–nbhd assignment and suppose to the contrary that there exist $p \in X$ and $n \in \mathbf{N}$ such that $p \in \overline{\{x \in X : p \notin N_n^m(x)\}}$. Then $p \in N_n^m\left[\{x \in X : p \notin N_n^m(x)\}\right]$. So there exists $q \in X$ such that both $p \in N_n^m(q)$ and $p \notin N_n^m(q)$, a contradiction.

(2)⇒(1): Suppose $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is *not* an N_m–nbhd assignment. Then there exist $A \subseteq X$, $p \in \overline{A}$, and $n \in \mathbf{N}$ such that $p \notin N_n^m(a)$ for any $a \in A$. Hence, $A \subseteq \{x \in X : p \notin N_n^m(x)\}$ so that $\overline{A} \subseteq \overline{\{x \in X : p \notin N_n^m(x)\}}$. It follows that $p \in \overline{\{x \in X : p \notin N_n^m(x)\}}$.

2.6.6 Theorem. [Ho2] If (X, τ) is N-semistratifiable, then (X, τ) is a γ_m^*-space for every $m \in \mathbf{N}$.

Proof. Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be an N-semistratifiable nbhd assignment in (X, τ). The proof is by induction on m. As any N-semistratifiable space is semistratifiable, it is clear that (X, τ) is a γ_1^*-space.

Now consider any $m \in \mathbf{N}$, assume (X, τ) is a γ_m^*-space, and suppose $p \in N_n^{m+1}(x_n)$ $\forall n$. Then $\forall n, \exists y_n \in N_n(x_n)$ such that $p \in N_n^m(y_n)$. So, as (X, τ) is a γ_m^*-space, $y_n \xrightarrow[\tau]{} p$. But then, as (X, τ) is N-semistratifiable, it follows that $x_n \xrightarrow[\tau]{} p$.

2.6.7 Theorem. [Ho2] The following are equivalent for a topological space (X, τ):

(1) (X, τ) is a γ-space;

(2) there is a nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) such that

for each $p \in X$, $\left(x_n \in N_n(y_n) \ \forall n \text{ and } y_n \xrightarrow[\tau]{} p\right) \Rightarrow x_n \xrightarrow[\tau]{} p$;

(3) (X, τ) is a γ_m-space for every $m \in \mathbf{N}$.

Proof. The implication (3)$\Rightarrow$(1) is obvious.

(1)$\Rightarrow$(2): Let $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ be a decreasing γ-nbhd assignment in (X, τ) and suppose $x_n \in N_n(y_n)$ $\forall n$ and $y_n \xrightarrow[\tau]{} p$. Then $\forall n, \exists k_n \geq n$ such that $y_{k_n} \in N_n(p)$ and since $x_{k_n} \in N_{k_n}(y_{k_n}) \subseteq N_n(y_{k_n})$, it follows that $x_{k_n} \xrightarrow[\tau]{} p$ so that (x_n) clusters at p.

Claim. $x_n \xrightarrow[\tau]{} p$

Pf. Otherwise, $\exists M \in \mathcal{N}_\tau(p)$ such that $\forall n, \exists j_n \geq n$ with $j_{n+1} > j_n$ and $x_{j_n} \in X - M$. Now since $y_{j_n} \xrightarrow[\tau]{} p$, it follows that $\forall n, \exists i_{j_n} \geq j_n$ such that $y_{i_{j_n}} \in N_n(p)$. Hence, as we also have $x_{i_{j_n}} \in N_{i_{j_n}}\left(y_{i_{j_n}}\right) \subseteq N_n\left(y_{i_{j_n}}\right)$, it follows that $x_{i_{j_n}} \xrightarrow[\tau]{} p$, which is impossible as $\left(x_{i_{j_n}}\right)$ is never in M.

(2)⇒(3): Assume (2). The proof is by induction on m. First, observe that $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_1–nbhd assignment.

Now consider any $m \in \mathbf{N}$, assume $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is a γ_m–nbhd assignment, and suppose $x_n \in N_n^{m+1}(p)\ \forall n$. Then $\forall n, \exists y_n \in N_n^m(p)$ such that $x_n \in N_n(y_n)$. As (X, τ) is a γ_m–space, it follows that $y_n \xrightarrow[\tau]{} p$. Thus, by our hypothesis about $\{N_n(x) : x \in X, n \in \mathbf{N}\}$, it follows that $x_n \xrightarrow[\tau]{} p$.

2.6.8 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudometrizable;

(2) (X, τ) is a γ–, N_1–space;

(3) (X, τ) is a γ^*–, N_1–space;

(4) (X, τ) is an S_m–, N_1–space for each $m \in \mathbf{N}$;

(5) (X, τ) is an S_2–, N_1–space;

(6) (X, τ) is an $\gamma\,\gamma^*$–, N_1–space;

(7) (X, τ) is an S_{m+1}–, N_m–space for each $m \in \mathbf{N}$.

Proof. (1)⇒(2) and (1)⇒(3): If (X, τ) is pseudometrizable, then there is a decreasing nbhd assignment $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ in (X, τ) for which both $x_n, p \in N_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and $q \in N_n(p) \Rightarrow p \in N_n(q)$. Note that this nbhd assignment is γ, γ^*, and N_1.

(2)⇒(4): Note that the proof of Theorem 2.6.7 shows that every γ–nbhd assignment is actually a γ_m–nbhd assignment for every $m \in \mathbf{N}$. Note also that, for every $m \in \mathbf{N}$, a γ_m–nbhd assignment is an S_m–nbhd assignment.

(1)⇒(7): As in the proofs of (1)⇒(2) and (1)⇒(3), there is a decreasing nbhd assignment $\{N_n(x): x \in X, n \in \mathbf{N}\}$ in (X, τ) for which both $x_n, p \in N_n(y_n), \forall n \Rightarrow x_n \xrightarrow[\tau]{} p$ and $q \in N_n(p) \Rightarrow p \in N_n(q)$. We have already determined that this nbhd assignment is γ and so, by our observation in the proof of (2)⇒(4), it follows that $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is γ_m for every $m \in \mathbf{N}$ and, therefore, S_m for every $m \in \mathbf{N}$. Since we have also shown that $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is N_1, and since an N_1-nbhd assignment is N_m for every $m \in \mathbf{N}$, we conclude that $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is simultaneously S_{m+1} and N_m for every $m \in \mathbf{N}$.

(4)⇒(5): This implication is obvious.

(5)⇒(6): Note that any S_2-nbhd assignment is a $\gamma\gamma^*$-nbhd assignment.

(6)⇒(1): Let $\{N_n(x): x \in X, n \in \mathbf{N}\}$ be a decreasing $\gamma\gamma^*$-, N_1-nbhd assignment in (X, τ) and define $G_n(p) = X - \overline{\{x \in X: p \notin N_n(x)\}}$.

Claim 1. $p \in G_n(p)$

Pf. Otherwise, $p \in \overline{\{x \in X: p \notin N_n(x)\}} \subseteq \bigcap_{k=1}^{\infty} N_k[\{x \in X: p \notin N_n(x)\}]$ so that $p \in N_n(x)$ for some $x \in X$ with $p \notin N_n(x)$, which clearly cannot happen.

Now note that $x \in G_n(p) \Rightarrow p \in N_n(x)$.

Define $M_n(p) = N_n(p) \cap G_n(p)$.

Claim 2. $\{M_n(x): x \in X, n \in \mathbf{N}\}$ is a Nagata nbhd assignment

Pf. Suppose $y_n \in M_n(p) \cap M_n(x_n)\ \forall n$. It follows that, $\forall n$, $y_n \in N_n(p) \cap N_n(x_n)$ and $p, x_n \in N_n(y_n)$. So, as $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is $\gamma\gamma^*$, it follows that $x_n \xrightarrow[\tau]{} p$.

Claim 3. $\{M_n(x): x \in X, n \in \mathbf{N}\}$ is a γ-nbhd assignment

Pf. Suppose $x_n \in M_n(y_n)$ and $y_n \in M_n(p)\ \forall n$. It follows that, $\forall n$, $y_n \in N_n(p) \cap N_n(x_n)$ and $p, x_n \in N_n(y_n)$. So, as $\{N_n(x): x \in X, n \in \mathbf{N}\}$ is $\gamma\gamma^*$, it follows that $x_n \xrightarrow[\tau]{} p$.

As a Nagata γ-space is pseudometrizable, it follows that (X, τ) is pseudometrizable.

(3)$\Rightarrow$(6): Note that any γ^*-nbhd assignment is a $\gamma\,\gamma^*$-nbhd assignment.

(7)$\Rightarrow$(5): This implication is obvious.

Note that the condition $q \in N_n(p) \;\Rightarrow\; p \in N_n(q)$ can replace the N_1-property in this theorem and the next.

2.6.9 Theorem. The following are equivalent for a topological space (X, τ):

(1) (X, τ) is pseudosemimetrizable;

(2) (X, τ) is a first countable N_1-space;

(3) (X, τ) is a semistratifiable N_1-space;

(4) (X, τ) is an S_1-, N_1-space.

Proof. (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3): Suppose (X, τ) is pseudosemimetrizable. Then, by Theorem 2.2.6, (X, τ) is both first countable and semistratifiable.

Claim. (X, τ) is an N_1-space

Pf. Let d be a pseudosemimetric for (X, τ). Then $\left\{S_d\left(x, 1/2^n\right) : x \in X, n \in \mathbf{N}\right\}$ is a nbhd assignment in (X, τ). Consider any $A \subseteq X$. By Lemma 2.2.1 it follows that $\bar{A} \subseteq \bigcap_{n=1}^{\infty} S_d\left[A, 1/2^n\right)$, where $S_d\left[A, 1/2^n\right) = \bigcup\left\{S_d\left(a, 1/2^n\right) : a \in A\right\}$.

(2)$\Rightarrow$(4): This implication is obvious since a γ_1-nbhd assignment is always S_1.

(3)$\Rightarrow$(4): This implication is also obvious since a γ_1^*-nbhd assignment is always S_1.

(4)$\Rightarrow$(1): Let $\left\{N_n(x) : x \in X, n \in \mathbf{N}\right\}$ be an S_1-, N_1-nbhd assignment. Then, using Lemma 2.6.5, for each $p \in X$ and each $n \in \mathbf{N}$,

$$p \in X - \overline{\{x \in X : p \notin N_n(x)\}} \subseteq X - \{x \in X : p \notin N_n(x)\} = N_n^*(p)$$

so that $N_n^*(p) \in \mathcal{N}_\tau(p)$.

Define $\tilde{N}_n(p) = N_n(p) \cap N_n^*(p)$.

Note that $q \in \tilde{N}_n(p) \Rightarrow p \in \tilde{N}_n(q)$ and, as $\{N_n(x) : x \in X, n \in \mathbf{N}\}$ is S_1,

$$x_n \in \tilde{N}_n(p) \ \forall n \Rightarrow x_n \xrightarrow[\tau]{} p,$$

so that, by Theorem 2.2.6, (X, τ) is pseudosemimetrizable.

The diagram on the next page summarizes the relationships among the spaces studied in Chapter 2 of this dissertation.

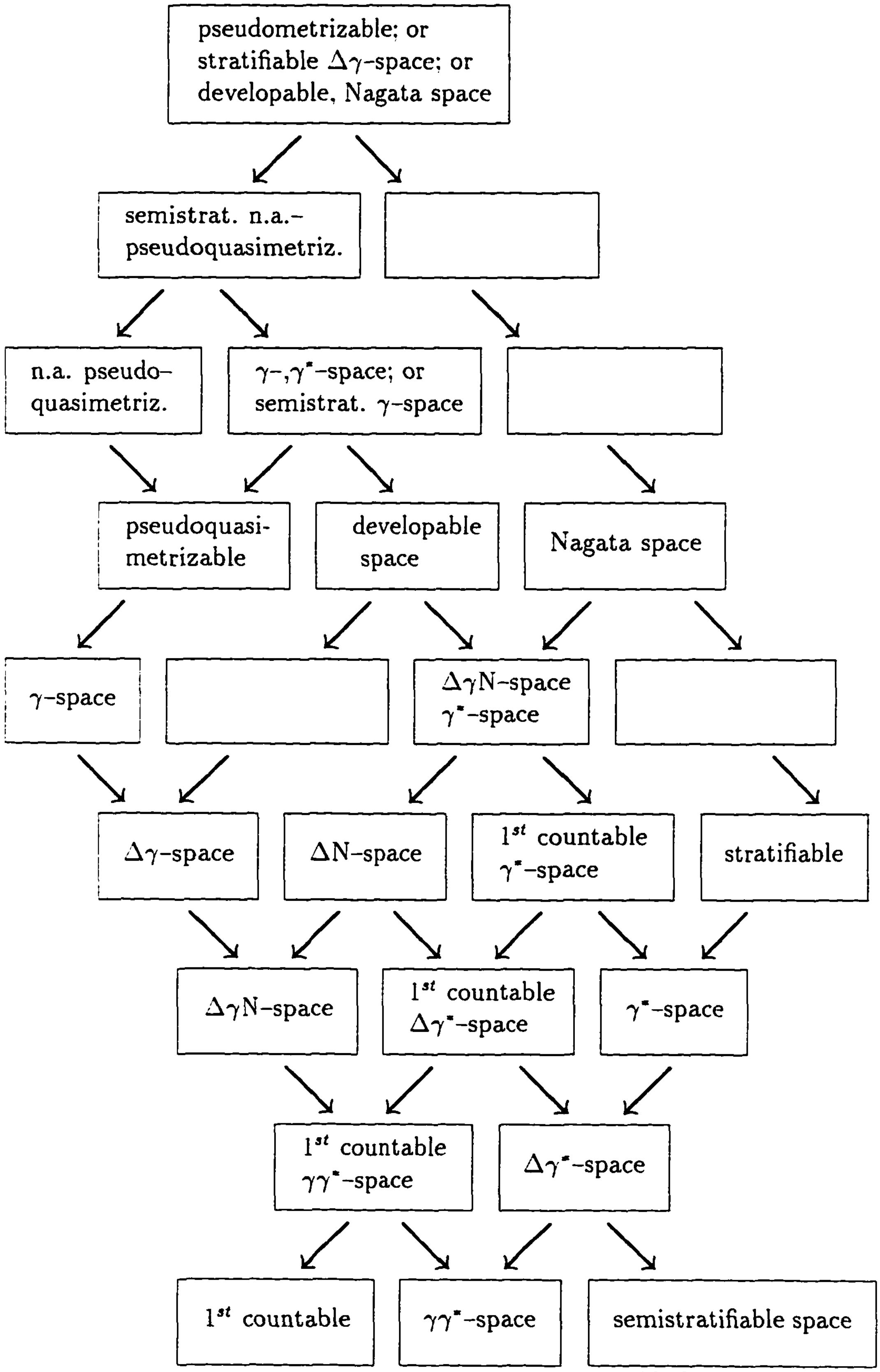

Figure 6. Generalizations of Pseudometrizable Spaces

CHAPTER THREE

NEIGHBORHOOD STRUCTURES, BINARY RELATIONS, AND WEAK DISTANCES

The aim of this chapter is to provide several examples which promote the study of structures which generalize topologies (neighborhood structures) and distances (weak distances) and to suggest some of the basic elements of the theory behind these generalizations. We shall also see that binary relations and even weak distances themselves are capable of being modeled through neighborhood structures. Significantly, each generalization supports a natural notion of continuity.

Given a set X we will let $\mathbf{P}(X)$ denote the power set of X. A **filter** in a nonempty set X is a nonempty subset of $\mathbf{P}(X)$ that is closed under finite intersections and supersets. Note that X itself is a member of any filter in X. Also, if $\varnothing$ is a member of a filter in X, then the filter is $\mathbf{P}(X)$. For any $A \in \mathbf{P}(X)$ we will let $\uparrow A$ denote the collection of all members of $\mathbf{P}(X)$ that are supersets of A.

A **neighborhood structure** [Sm] on a set X is a map $\mathcal{N}: X \to \mathbf{P}(\mathbf{P}(X))$ such that $\mathcal{N}(p)$ is a filter in X for each $p \in X$. The pair $(X, \mathcal{N})$ is then called a **neighborhood space** and the members of $\mathcal{N}(p)$ are referred to as **neighborhoods** of p.

3.1 Example. If (X, τ) is a topological space, then $\mathcal{N}_\tau$ is a nbhd structure on X and $(X, \mathcal{N}_\tau)$ is a nbhd space.

However, a nbhd structure need not be topological. In the topological setting a point must be a member of each of its nbhds, but clearly the axioms for a nbhd structure do not require that this be so.

3.2 Theorem. Let $(X, \mathcal{N})$ be a nbhd space. Even if $p \in \bigcap \mathcal{N}(p)$ for every $p \in X$, it need not be the case that $\mathcal{N} = \mathcal{N}_\tau$ for a topology on X.

Proof. Let $X = \{0, 1, 2\}$, $\mathcal{N}(0) = \uparrow\{0,1\}$, $\mathcal{N}(1) = \uparrow\{1,2\}$, and $\mathcal{N}(2) = \uparrow\{0,2\}$. Clearly, $\mathcal{N}$ is a nbhd structure on X. Now if there is a topology τ on X such that $\mathcal{N} = \mathcal{N}_\tau$, it follows that $\{0, 1\} \in \tau$ and $\{1, 2\} \in \tau$ so that $\{1\} \in \tau$ and, therefore, $\{1\} \in \mathcal{N}(1)$ (in a topological space, an open set is a nbhd of each of its points), which is not true.

The example provided in this proof demonstrates some of the versatility of nbhd structures. The only topology on $\{0, 1, 2\}$ for which $\{0, 1\}$ is a nbhd of 0, $\{1, 2\}$ is a nbhd of 1, and $\{0, 2\}$ is a nbhd of 2 is discrete. The non-topological nbhd structure in the proof can be viewed as modeling the binary relation $R = \{(0,0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 0)\}$ in the sense that $(p, q) \in R$ if and only if $q \in \bigcap \mathcal{N}(p)$. From a graph-theoretic perspective, R is a directed graph having no parallel edges; the smallest nbhd of a point (i.e. node) p is exactly the set of nodes reachable by edges emanating from p (i.e. the set of immediate successors of p).

Observe that the function $d : \{0, 1, 2\} \times \{0, 1, 2\} \to [0, \infty)$ defined by

$$d(p, q) = \begin{cases} 0, & \text{if } (p, q) \in R; \\ 1, & \text{otherwise,} \end{cases}$$

also models R. It is worth noting that d is a distance for $\{0, 1, 2\}$.

3.3 Example. For each $n \in \mathbf{N}$ we shall refer to $\{1, 2, \ldots, n\}$ as an *initial segment* of $\mathbf{N}$. Define

$$\mathcal{B}_f = \{\varnothing\} \cup \{x \mid x : A \to \{0, 1\} \text{ for some initial segment } A \text{ of } \mathbf{N}\}, \text{ and}$$

$$\mathcal{B}_i = \{x \mid x : \mathbf{N} \to \{0, 1\}\}.$$

$\mathcal{B}_f$ and $\mathcal{B}_i$ are, respectively, the sets of finite and infinite bit strings. Let $\mathcal{B} = \mathcal{B}_f \cup \mathcal{B}_i$.

Given $x, y \in \mathcal{B}$ we will say that x is a *prefix* of y, denoted $x \leq y$, provided that the domain of x is a subset of the domain of y and for each k in the domain of x, $x(k) = y(k)$. Note that $\leq$ is a partial order on $\mathcal{B}$.

If we define $\mathcal{N}_{\leq}(p) = \uparrow\{x \in \mathcal{B} : p \leq x\}$ for each $p \in \mathcal{B}$, then $\mathcal{N}_{\leq}$ is a nbhd structure on $\mathcal{B}$.

The function $d_{\leq} : \mathcal{B} \times \mathcal{B} \rightarrow [0, \infty)$ for which

$$d_{\leq}(x, y) = \begin{cases} 0, \text{ if } x \leq y; \\ 1, \text{ otherwise}, \end{cases}$$

is a distance for $\mathcal{B}$ that encapsulates the prefix ordering.

Of course there is nothing to prevent us from representing the information contained in an arbitrary binary relation R by constructing a nbhd structure $\mathcal{N}_R$ in the above fashion. Also, for any binary relation R defined on X, the function $d_R : X \times X \rightarrow [0, \infty)$ defined so that

$$d_R(p, q) = \begin{cases} 0, \text{ if } (p, q) \in R; \\ 1, \text{ otherwise}, \end{cases}$$

models R. But since it is not necessarily the case that $(p, p) \in R$, we may not have $d_R(p, p) = 0$ and, hence, d_R need not be a distance for X. For this reason we introduce the notion of a *weak distance*.

A **weak distance** for a nonempty set X is a function $d : X \times X \rightarrow [0, \infty)$. We have just shown that any binary relation on X can be represented using a weak distance.

Given a weak distance d for X we define, just as we do for distances, the **sphere** $S_d(p, \varepsilon)$ centered at $p \in X$ of radius $\varepsilon > 0$ so that

$$S_d(p, \varepsilon) = \{x \in X : d(p, x) < \varepsilon\}.$$

Weak distances generate both nbhd structures and topologies. If d is a weak distance for X, we define a nbhd structure $\mathcal{N}_d$ on X and a topology τ_d on X so that

$$\mathcal{N}_d(p) = \{A \subseteq X : \exists\, \varepsilon > 0, S_d(p, \varepsilon) \subseteq A\} \text{ for each } p \in X$$

and

$$\tau_d = \left\{A \subseteq X : \forall p \in X, \exists \varepsilon_p > 0, S_d\left(p, \varepsilon_p\right) \subseteq A\right\}.$$

As we have seen, it need *not* be the case that $S_d(p, \varepsilon) \in \mathcal{N}_{\tau_d}(p)$; however, $S_d(p, \varepsilon) \in \mathcal{N}_d(p)$ for every $p \in X$ and every $\varepsilon > 0$.

Note that if R is a binary relation on X, $\mathcal{N}_{d_R} = \mathcal{N}_R$. Hence, the nbhd structure given in the proof of Theorem 3.2 shows that, even when d is a distance (as opposed to just a weak distance), there may not be a topology τ for which $\mathcal{N}_d = \mathcal{N}_\tau$.

3.4 Theorem. If d is a weak distance for X, then, for each $p \in X$, $\mathcal{N}_{\tau_d}(p) \subseteq \mathcal{N}_d(p)$. However, even if d is a distance, it is possible that $\mathcal{N}_d \neq \mathcal{N}_{\tau_d}$.

Proof. Clearly, $\mathcal{N}_{\tau_d}(p) \subseteq \mathcal{N}_d(p)$ for each $p \in X$. If R is the binary relation described in the proof of Theorem 3.2, we have seen that $\mathcal{N}_{d_R} \neq \mathcal{N}_\tau$ for any topology on $\{0, 1, 2\}$; hence, $\mathcal{N}_{d_R} \neq \mathcal{N}_{\tau_{d_R}}$. Here is another example of a distance d for a set for which $\mathcal{N}_d \neq \mathcal{N}_{\tau_d}$:

Let $X = [0, 1]$ and $A = \left\{1/3^n : n \in \mathbf{N}\right\}$ and define a distance d so that

$$d(0, x) = d(x, 0) = 1, \text{ if } x \notin A \cup \{0\}, \text{ and}$$
$$d(x, y) = |x - y|, \text{ otherwise.}$$

Note that $S_d(0, 1) = \{0\} \cup A \in \mathcal{N}_d(0)$.

Claim. $S_d(0, 1) \notin \mathcal{N}_{\tau_d}(0)$

Pf. Otherwise, $\exists G \in \tau_d$ with $0 \in G \subseteq S_d(0, 1)$. Then, $\exists \varepsilon > 0$ such that $S_d(0, \varepsilon) \subseteq G$. But $S_d(0, \varepsilon) = \{0\} \cup \left\{1/3^n : n \in \mathbf{N}, n \geq k\right\}$ for some $k \in \mathbf{N}$. So $1/3^k \in G$. Hence, $\exists \delta > 0$ such that $S_d\left(1/3^k, \delta\right) \subseteq G$. But

$$S_d\left(1/3^k, \delta\right) = \left(1/3^k - \delta, 1/3^k + \delta\right) \cap [0, 1]$$

so that $\left(1/3^k - \delta, 1/3^k + \delta\right) \cap [0, 1] \subseteq S_d(0, \varepsilon)$, which is impossible.

3.5 Example. Let $\varepsilon > 0$ and suppose that a real number y may be used as an approximation for a real number x provided that the Euclidean distance between x and y

is less than ε. We can view ε as the smallest unit measurable by a particular measuring device. The distance d_ε for **R** defined by

$$d_\varepsilon(x, y) = \begin{cases} 0, \text{ if } |x - y| < \varepsilon; \\ 1, \text{ otherwise}, \end{cases}$$

models this situation by identifying those real numbers "close enough" to a given real number x to be regarded as approximations to x. Note that d is a pseudosymmetric for **R** that is not a pseudometric.

The previous example, although very simple, suggests the utility of distances which do not satisfy the Triangle Inequality and for which distinct points may, under certain circumstances, be identified with each other. Our next example establishes the significance of weak distances that are not distances.

3.6 Example. Let Q be the set of rational numbers in $(0, 1]$ and $P = (0,1] - Q$. Recall that each $x \in (0, 1]$ has a unique decimal representation that does not terminate; for each such x, let $x(k)$ be the decimal digit in the k-th place to the right of the decimal point in this representation of x. Define a weak distance d for $(0, 1]$ so that

$$d(x, y) = \begin{cases} 1, \text{ if } x, y \in P; \\ 0, \text{ if } x, y \in Q \text{ and } x = y; \\ 1/2^n, \text{ where } n = \min\{k \in \mathbf{N} : x(k) \neq y(k)\}, \text{ otherwise}. \end{cases}$$

The motivation behind the definition of d stems from the attempt to approximate an irrational in $(0, 1]$ using rationals in $(0, 1]$. Since it would be impractical, from a computational perspective, to employ irrationals as approximations, we have constructed d so that irrationals become "far" from one another, even from themselves. Thus, d is a weak distance that is not a distance. Intuitively, the points of $(0, 1]$ which are close to an irrational $p \in (0, 1]$, as measured by d, are exactly those rationals in $(0, 1]$ which agree with p in the first several decimal places.

A similar type of weak distance can be developed in any setting in which a set of "ideal" elements (the elements actually being computed) are understood only through their "approximations" (perhaps via a limiting process) and these approximations, distinct from the ideal elements themselves, are well-understood and capable of being stored in and retrieved from a computer's memory.

The weak distance d for (0, 1] that we have described fails to satisfy the Triangle Inequality (for example, $d\left(\sqrt{2}/2, \sqrt{2}/2\right) = 1$ and $d\left(\sqrt{2}/2, 0.7\bar{1}\right) = d\left(0.7\bar{1}, \sqrt{2}/2\right) = 1/4$ so that $d\left(\sqrt{2}/2, \sqrt{2}/2\right) > d\left(\sqrt{2}/2, 0.7\bar{1}\right) + d\left(0.7\bar{1}, \sqrt{2}/2\right)$), providing further evidence that a "reasonable" notion of distance does not necessarily require that the Triangle Inequality be satisfied.

Note also that $\sqrt{2}/2 \notin S_d\left(\sqrt{2}/2, 1\right)$ so that a sphere determined by a weak distance need not necessarily contain its center.

Next we show that there are nbhd structures which cannot be induced via weak distances.

3.7 Theorem. Given a nbhd structure $\mathcal{N}$ on X, even if $\mathcal{N} = \mathcal{N}_\tau$ for some topology on X, it need not be the case that $\mathcal{N} = \mathcal{N}_d$ for a weak distance d for X.

Proof. Let X be the collection of all real-valued functions defined on $[0,1]$. Given $f \in X$, $\varepsilon > 0$, and a finite subset F of $[0,1]$, define

$$A_{F,\varepsilon}(f) = \left\{g \in X : \forall x \in F, |f(x) - g(x)| < \varepsilon\right\}$$

and, for each $f \in X$, let

$$\mathcal{A}(f) = \left\{A_{F,\varepsilon}(f) : F \subseteq [0, 1],\ F \text{ is finite},\ \varepsilon > 0\right\}, \text{ and}$$

$$\mathcal{N}(f) = \left\{N \subseteq X : \exists A \in \mathcal{A}(f), A \subseteq N\right\}.$$

Then $\mathcal{N}$ is the nbhd structure on X induced by the topology τ of pointwise convergence.

Claim 1. For any $\mathcal{U} \subseteq \mathcal{N}(f)$ having the property that

$$N \in \mathcal{N}(f) \Rightarrow \exists U \in \mathcal{U}, U \subseteq N,$$

there exist $U_1, U_2 \in \mathcal{U}$ such that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$.

Pf. Otherwise, $\exists \mathcal{U} \subseteq \mathcal{N}(f)$ that is totally ordered by set inclusion and for which $N \in \mathcal{N}(f) \Rightarrow \exists U \in \mathcal{U}, U \subseteq N$. For each $n \in \mathbf{N}$ define $g_n \in X$ so that $g_n(x) = f(x) + 1/2^n$ and note that the sequence (g_n) converges pointwise to f. Since (X, τ) is Hausdorff, for each $n \in \mathbf{N}$ there exists $U_n \in \mathcal{U}$ such that $g_n \notin U_n$. Now, note that for any $V \in \mathcal{U}$, there exists n such that $V \not\subseteq U_n$ (otherwise, $\exists V \in \mathcal{U}$ with $V \subseteq U_n \ \forall n$ so that, $\forall n$, $g_n \notin V$, contradicting the fact that $g_n \xrightarrow[\tau]{} f$). Now suppose $f \in W \in \tau$. Then $\exists V \in \mathcal{U}$ with $f \in V \subseteq W$ and $\exists n \in \mathbf{N}$ with $V \not\subseteq U_n$. But, as $\mathcal{U}$ is totally ordered by set inclusion, it then follows that $U_n \subseteq V$ so that $U_n \subseteq W$. Therefore, $\{U_n : n \in \mathbf{N}\}$ is a countable nbhd base at f, contradicting the fact that (X, τ) is a well-known example of a topological space that is *not* first countable.

Claim 2. $\mathcal{N} \neq \mathcal{N}_d$ for any weak distance d for X

Pf. Suppose to the contrary that $\mathcal{N} = \mathcal{N}_d$ for some weak distance d for X and, for each $f \in X$, let $\mathcal{S}_d(f) = \{S_d(f, \varepsilon) : \varepsilon > 0\}$. Note that $\mathcal{S}_d(f) \subseteq \mathcal{N}(f)$. Then, given $f \in X$ and $N \in \mathcal{N}(f)$, there exists $S \in \mathcal{S}_d(f)$ such that $S \subseteq N$. Thus, by the first claim, there exist $S_1, S_2 \in \mathcal{S}_d(f)$ such that $S_1 \not\subseteq S_2$ and $S_2 \not\subseteq S_1$, contradicting the trivial observation that given two spheres centered at the same point, one is a subset of the other.

Neighborhood spaces allow, in a very natural manner, for the introduction of notions of *open set*, *interior point*, *limit point*, *closed set*, and *closure point.*

Let $(X, \mathcal{N})$ be a nbhd space, $A \subseteq X$, and $p \in X$.

(1) A is **open** in $(X, \mathcal{N})$ if it contains a nbhd of each of its points;

(2) p is an **interior point** of A if A contains a nbhd of p;

(3) p is a **limit point** of A if $(N - \{p\}) \cap A \neq \emptyset$ for each $N \in \mathcal{N}(p)$;

(4) A is **closed** in $(X, \mathcal{N})$ if it contains all of its limit points;

(5) p is a **closure point** of A if $N \cap A \neq \emptyset$ for each $N \in \mathcal{N}(p)$.

We may then define the **interior** of A, denoted $\mathrm{Int}_{\mathcal{N}}(A)$, to be the set of all interior points of A, the **derived set** of A, denoted $\mathrm{Lim}_{\mathcal{N}}(A)$, to be the set of all limit points of A, and the **closure** of A, denoted $\mathrm{Cl}_{\mathcal{N}}(A)$, to be the set of all closure points of A.

The following sequence of results outlines some of the basic theory of nbhd spaces and provides links to the theory of topological spaces. The proofs follow almost immediately from the definitions just given.

3.8 Lemma. Let $(X, \mathcal{N})$ be a nbhd space and $A \subseteq X$.

(1) $\mathrm{Int}_{\mathcal{N}}(A) = \{x \in X : A \in \mathcal{N}(x)\}$;

(2) A is open iff $A \subseteq \mathrm{Int}_{\mathcal{N}}(A)$;

(3) A is closed iff $\mathrm{Cl}_{\mathcal{N}}(A) \subseteq A$.

3.9 Theorem. [Sm] If $(X, \mathcal{N})$ is a nbhd space, the set $\tau_{\mathcal{N}}$ of all open sets in $(X, \mathcal{N})$ is a topology on X.

3.10 Corollary. Let $(X, \mathcal{N})$ be a nbhd space.

(1) $\mathcal{N}_{\tau_{\mathcal{N}}}(p) \subseteq \mathcal{N}(p)$ for each $p \in X$;

(2) $A \subseteq X$ is open in $(X, \mathcal{N}_{\tau_{\mathcal{N}}})$ iff $A \in \tau_{\mathcal{N}}$;

(3) $A \subseteq X$ is closed in $(X, \mathcal{N})$ iff A is closed in $(X, \tau_{\mathcal{N}})$.

3.11 Theorem. The following are equivalent for a nbhd space $(X, \mathcal{N})$:

(1) $\mathcal{N} = \mathcal{N}_\tau$ for some topology τ on X;

(2) $p \in \bigcap \mathcal{N}(p)$, $\forall p \in X$, and $\forall p \in X, \forall N \in \mathcal{N}(p), \exists G \in \mathcal{N}(p) \cap \tau_{\mathcal{N}}, G \subseteq N$.

Proof. (2)⇒(1): Apply Theorem 1.1.

3.12 Theorem. If τ and τ^* are topologies on X,

$$\tau = \tau^* \iff \mathcal{N}_\tau = \mathcal{N}_{\tau^*}.$$

3.13 Corollary. If $(X, \mathcal{N})$ is a nbhd space and $\mathcal{N} = \mathcal{N}_\tau$ for some topology τ on X, then $\tau_{\mathcal{N}} = \tau$.

3.14 Lemma. Let d be a weak distance for X.

(1) $\tau_{\mathcal{N}_d} = \tau_d$;

(2) for each $A \subseteq X$, $\mathrm{Cl}_{\mathcal{N}_d} = \{x \in X : d(x, A] = 0\}$.

We now introduce notions of continuity in the contexts of weak distances and nbhd structures.

If d_X and d_Y are weak distances for X and Y, respectively, a function $f : X \to Y$ is **continuous** at $p \in X$ provided that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(p, x) < \delta \implies d_Y(f(p), f(x)) < \varepsilon.$$

Note that by modeling binary relations R_X on X and R_Y on Y using the naturally associated weak distances, we can consider a function $f : X \to Y$ to be continuous at $p \in X$ exactly when $(p, x) \in R_X \implies (f(p), f(x)) \in R_Y$.

If $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ are nbhd spaces, a function $f : X \to Y$ is **continuous** at $p \in X$ provided that

for each $N \in \mathcal{N}_Y(f(p))$, there exists $M \in \mathcal{N}_X(p)$ such that $f[M] \subseteq N$.

If f is continuous at every point of X we will say f is continuous.

Our definition of continuity in the weak distance setting is consistent with that of the nbhd space setting when we consider the nbhd structures induced by weak distances defined on the domain and codomain of some function. A concrete example (the changing rate structure of a telephone system) is developed in [CKS] in which weak distances are defined on the domain and the codomain of a function in such a way that the resulting topological and potentially non-topological nbhd structures are distinct from one another and allow topological continuity to be compared with the notion of continuity derived from non-topological nbhd structures.

The topological characterizations of continuity in terms of the closure and interior operators carry over into arbitrary nbhd spaces.

3.15 Theorem. Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be nbhd spaces and $f : X \to Y$. The following are equivalent:

(1) f is continuous;

(2) for each $A \subseteq X$, $f\left[\mathrm{Cl}_{\mathcal{N}_X}(A)\right] \subseteq \mathrm{Cl}_{\mathcal{N}_Y}(f[A])$;

(3) for each $A \subseteq Y$, $f^{\leftarrow}\left[\mathrm{Int}_{\mathcal{N}_Y}(A)\right] \subseteq \mathrm{Int}_{\mathcal{N}_X}\left(f^{\leftarrow}[A]\right)$.

3.16 Lemma. Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be nbhd spaces and $f : X \to Y$ be one-to-one and continuous. Then $f\left[\mathrm{Lim}_{\mathcal{N}_X}(A)\right] \subseteq \mathrm{Lim}_{\mathcal{N}_Y}(f[A])$ for every $A \subseteq X$.

Proof. Consider any $A \subseteq X$, suppose $p \in \mathrm{Lim}_{\mathcal{N}_X}(A)$, and consider any $N \in \mathcal{N}_Y(f(p))$. Since f is continuous there exists $M \in \mathcal{N}_X(p)$ such that $f[M] \subseteq N$. Now $\exists x \in (M - \{p\}) \cap A$. Then, since f is one-to-one, $f(x) \neq f(p)$ so that $f(x) \in (N - \{f(p)\}) \cap f[A]$.

A nbhd space $(X, \mathcal{N}_X)$ is **homeomorphic** to a nbhd space $(Y, \mathcal{N}_Y)$ if there is a bijection $f : X \to Y$ such that for every $p \in X$ and every $A \subseteq X$,

$$A \in \mathcal{N}_X(p) \quad \Leftrightarrow \quad f[A] \in \mathcal{N}_Y(f(p)).$$

Such a function f is called a **homeomorphism**.

Our next result is essentially a corollary to Lemma 3.16.

3.17 Theorem. Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be nbhd spaces and $f : X \to Y$ be a homeomorphism. Then $f\left[\text{Lim}_{\mathcal{N}_X}(A)\right] = \text{Lim}_{\mathcal{N}_Y}(f[A])$ for every $A \subseteq X$.

If two topologies on a given set X induce the same derived set operator, it follows that the spaces are homeomorphic. This is not necessarily the case with arbitrary nbhd spaces, though.

3.18 Theorem. If $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ are nbhd spaces, $f : X \to Y$, and

$$f\left[\text{Lim}_{\mathcal{N}_X}(A)\right] = \text{Lim}_{\mathcal{N}_Y}(f[A]) \text{ for every } A \subseteq X,$$

it is possible that $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ are *not* homeomorphic.

Proof. Consider the nbhd structures $\mathcal{N}$ and $\mathcal{M}$ defined on $X = \{0, 1\}$ for which

$$\mathcal{N}(0) = \mathcal{N}(1) = \mathcal{M}(1) = \uparrow\{1\} \text{ and } \mathcal{M}(0) = \{X\}$$

and observe that $\text{Lim}_{\mathcal{N}}(A) = \text{Lim}_{\mathcal{M}}(A)\ \forall A \subseteq X$. But neither of the two bijections on X are homeomorphisms.

Thus, although there are many topological theorems that translate essentially word for word into results about arbitrary nbhd spaces, not all do. Theorem 3.18 is particularly compelling given that so much of the early work in general topology grew out of the study of limit elements of sets.

CHAPTER FOUR

CONCLUSION

This work has considered neighborhoods in historical, topological, and non-topological contexts. The notion of neighborhood played a crucial role in the mathematical investigations leading up to Hausdorff's formulation of topological spaces and served as the primitive concept on which his theory, and later Fréchet's, was based.

The study of topological spaces through the use of neighborhoods has the advantage of being very natural and often offers relatively simple proofs that appear to shed more light on the specific spaces being explored than other approaches. Our focus on neighborhoods has provided alternate proofs of some known results and helped us to formulate and prove a variety of new results within the setting of non-Hausdorff spaces. These spaces, once casually cast aside as pathological, are finally receiving attention due to contemporary applications in theoretical computer science.

Our use of weak neighborhoods points out that in some situations it is possible to suppress certain properties of topological neighborhoods and motivates the consideration of more general neighborhood-like structures such as Smyth's neighborhood spaces. The theory of neighborhood spaces simultaneously generalizes topologies, (weak) distances, and binary relations, and appears to be an area that demands much more exhaustive study. In particular, the formulation of appropriate notions of compactness and connectedness, along with various separation properties, especially point-separation properties less restrictive than Hausdorff's axiom, should be undertaken. Developing neighborhood space analogues to topological properties for which simple neighborhood characterizations do not (yet) exist may prove to be particularly challenging.

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