Spring 1995

Extensions of bialgebras and their cohomological description

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EXTENSIONS OF BIALGEBRAS AND THEIR COHOMOLOGICAL DESCRIPTION

BY

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B.A., University of Southern Maine. 1986

DISSERTATION

Submitted to the University of New Hampshire
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

in

Mathematics

May 1995

This dissertation has been examined and approved.

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7 April 1995
Date

Dedication

To my father

Acknowledgments

I am glad to have this chance to thank my all associates at the University of New Hampshire, they were all a joy to work with. In particular I wish to mention A. Robb Jacoby, for his inspiration. Most importantly I want to thank Donovan Van Osdol, whose unending support and suggestions were indispensable. On a more practical note I must mention TEXcad, a drawing program by Georg Horn, that enabled the typesetting of the tensor diagrams.

Table of Contents

ABSTRACT

EXTENSIONS OF BIALGEBRAS AND THEIR COHOMOLOGICAL DESCRIPTION

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University of New Hampshire, May, 1995

This paper develops the theory of crossed product Hopf algebras of pairs of arbitrary Hopf algebras. The theory generalizes crossed product algebras and abelian crossed product Hopf algebras. First, conditions are given on the structures involved that are shown to be equivalent to the existence of the crossed product. Next, a bisimplicial object is found that gives a cohomological description of the conditions. Cleft extensions of pairs of arbitrary Hopf algebras are then defined. These generalize cleft extension algebras and abelian extensions of bialgebras; while giving an internal definition of extensions. Finally, the equivalence of crossed products and extensions is proved. Throughout this paper extensive use is made of the relatively new technique of tensor diagrams, without which many of the calculations would be intractable.

Chapter 1

Preliminaries

This chapter is introductory in nature. In the first section we give an overview of the paper. The next section is a brief introduction to tensor diagrams and the third section is a collection of basic facts and notation. Finally, the last section lists some questions raised but not answered in this paper.

Throughout we will be working over a field $\mathbf{k}$. By a space we will mean a $\mathbf{k}$-vector space, by a map a $\mathbf{k}$-linear map, by an algebra a $\mathbf{k}$-algebra etc.; all tensors will be over $\mathbf{k}$. The category of sets will be denoted by Set, the category of spaces will be denoted by Vect and the natural numbers will be denoted by N. If V is a space and $n \in \mathrm{N}$ then the n-fold tensor product $V \otimes V \otimes \cdots \otimes V$ of n copies of V will be denoted V^n, where $V^1 \equiv V$ and $V^0 \equiv \mathbf{k}$.

Given two spaces, V and W, there is a natural isomorphism $T_{V,W} \in \mathsf{Vect}(V \otimes W, W \otimes V)$ given for $v \in V$ and $w \in W$ by $T_{V,W}(v \otimes w) = w \otimes v$. For each space V there are natural isomorphisms $\mathbf{k} \otimes V \overset{l_V}{\cong} V \overset{r_V}{\cong} V \otimes \mathbf{k}$, which we will usually consider as the identity on V.

1.1 An Overview

A monoidal category is a category equipped with a "product" that is associative and unitary. The category of sets is monoidal via the cartesian product with a singleton set as unit. The category of vector spaces is monoidal via the tensor product with $\mathbf{k}$ as a unit. A monoid can be defined in any monoidal category. In **Vect** the monoids are called algebras. By dualizing the definition of a monoid we have a comonoid. In **Vect** the comonoids are called a coalgebras but in **Set** (with the cartesian product) every set has a unique comonoid structure given by the diagonal map and the terminal map. A bialgebra is a space that is both an algebra and a coalgebra in a coherent way; this means that the coalgebra structure maps are algebra maps, (equivalently: vice versa!). The classical definition of group in a category requires more than a monoidal category; in particular we require a product that is actually the categorical product (giving us a diagonal map) and a unit for the product that is actually a terminal object. Since the tensor product is not a categorical product for vector spaces, this definition of a group cannot be applied to **Vect** with respect to the tensor product. However, there is a generalization here: rather than relying on a diagonal map and a terminal map to define an inverse why not employ a more general comonoid? A Hopf algebra is a bialgebra with just such an inverse (called an antipode). In this sense a Hopf algebra is a "group" in the category of vector spaces. This point of view underlies this paper; in particular we are looking at the extension theory of groups in this more general setting.

There is a general principal for making this generalization: first make the direct analogy of the group theory, next dualize the theory, finally consider the coherence between the two. The relatively new technique of tensor diagrams facilitates this generalization, first by making the direct analogy transparent and then by making the difficult formulas of the dual situation tractable. The extent to which tensor diagrams are employed in this paper is new.

This first chapter concludes with three more sections. The next section is an introduction to the use of tensor diagrams and the third section is a review of the basic definitions, facts and notation for dealing with Hopf algebras. Finally, in the last section of this chapter, we consider some open questions raised by this paper.

Chapter Two deals with crossed products. If one group acts by automorphisms on another then we can form their semi-direct product. In an analogous way if C is a bialgebra and A is an algebra and C acts on A in a particular manner then we have an algebra structure on $A \otimes C$ known as the smash product algebra. By dualizing we have a smash product coalgebra and in the presence of both structures we may have the coherence necessary to form a smash product bialgebra. The conditions for this coherence are given by Majid in [Maj90]. If we have the semidirect product of two groups and a suitable cocycle we can "twist" the multiplication on the product by using the cocycle. This is the model for the crossed product algebra on a space $A \otimes C$ (see [Swe68]). By dualizing we have the crossed product coalgebra and, in the presence of the necessary coherence, we have the crossed product bialgebra. In the abelian case, meaning that A is a commutative algebra and C is a cocommutative coalgebra, the conditions for coherence are given by Hofstetter in [Hof94]. The not-necessary-abelian case is dealt with in this chapter, where we also give an explicit formula for the antipode in the crossed product. In the first section of Chapter Two we define compatible matched (measured) pairs, which capture the basic data necessary to form crossed products. We then develop a few properties of these pairs that will be useful later. The second section develops crossed products.

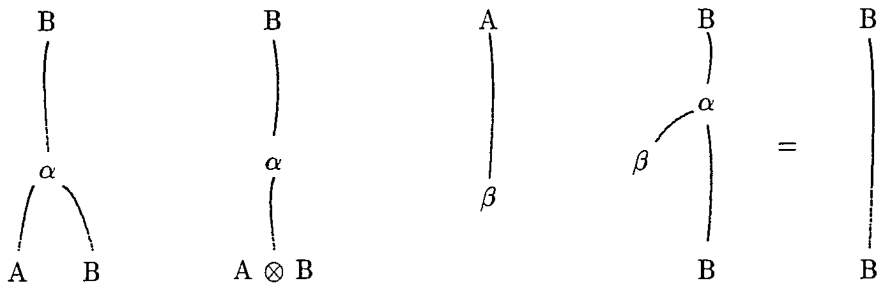
Chapter Three studies a double cosimplicial object that gives a different point of view toward the conditions necessary in forming a crossed product. The theory is restricted in that we require that the cross product be constructed on a matched pair (rather than the most general case: a measured pair, as in Chapter Two). The first section is a review of the general construction of particular cosimplicial sets associated with a triple and with a cotriple. In the next section we show that the constructions are actually cosimplicial algebras. In the general setting these algebras are not commutative. In the third section we define the middle, a substructure of the cosimplicial algebras, that gives us some control over commutativity. In the fourth section we show that the middle actually forms a sub-cosimplicial algebra. Finally, in the fifth section, we unite the middles to construct a bi-cosimplicial algebra and relate this to crossed products. The result is that we can recover the conditions necessary to form the cross product by computing the cocycles in this bi-cosimplicial algebra.

In Chapter Four we define cleft extensions of pairs of Hopf algebras. The definition of cleft extension of an algebra over a Hopf algebra was given first by Sweedler in [Swe68] as his generalization of a split extension of a group. These were commutative algebras extended over cocommutative Hopf algebras. In [DT86], [BCM86] and [BM89] extensions of arbitrary algebras over arbitrary Hopf algebras are considered. The first section of Chapter Four contains a review of cleft extensions. In the last section we define cleft coextensions, the dual notion to cleft extensions. We then consider coherence between these two structures, the result being a cleft extension Hopf algebra of a pair of Hopf algebras. This generalizes or improves several versions of extensions in the literature. Finally we show that these extensions correspond to cross products.

We begin by looking at a technique for calculating equations involving tensor products in the category of spaces.

1.2 Tensor Diagrams

Tensor diagrams were first introduced by Penrose in [Pen71] as a technique for verifying tensor calculations. The category-theoretic aspects are detailed in [JS91]. The diagrams are available in any monoidal category and provide a powerful method for calculating with morphisms. For example, in terms of tensor diagrams, a map $\alpha \in \mathsf{Vect}(A \otimes B, B)$ will be represented as in (a) below. Such diagrams are to be read from the bottom up. The edges are spaces (identity maps), labeled only on the top and bottom of the diagram. The vertices are maps, usually labeled (see the comments following definition 1.3.6 for the exceptions). Alternatively α could be represented by (b). Because of the isomorphisms l and r, $\mathbf{k}$ will be represented by no line at all; thus $\beta \in \mathsf{Vect}(\mathbf{k}, A)$ will by drawn as in (c). Composition is accomplished by joining diagrams top to bottom, and an equal sign between two diagrams means equality of the maps they represent, for example (d) states that $\alpha \circ (\beta \otimes B) = B$, (more precisely $\alpha \circ (\beta \otimes B) = l_B$).

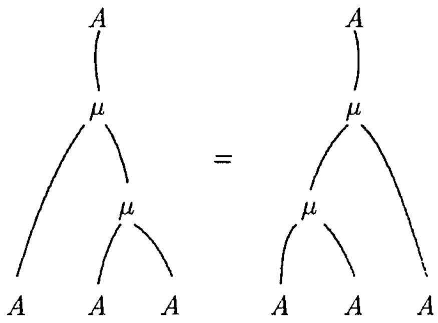


(a) (b) (c) (d)

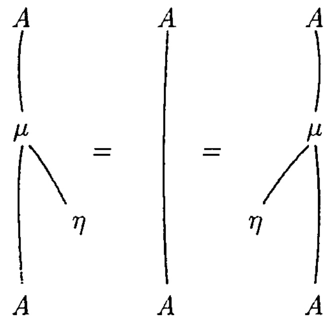
1.3 A Review of Hopf Algebras

We now look at some of the basic structures that we will be dealing with. The standard references are [Swe69] and [Abe77].

Definition 1.3.1 *An* **algebra** *is a space A together with two maps, a multiplication $\mu \in \mathsf{Vect}(A \otimes A, A)$ and a unit $\eta \in \mathsf{Vect}(\mathbf{k}, A)$, satisfying the associativity and unitary conditions: $\mu \circ (\mu \otimes A) = \mu \circ (A \otimes \mu)$, $\mu \circ (\eta \otimes A) = A$ and $\mu \circ (A \otimes \eta) = A$. In terms of tensor diagrams* *these conditions are given by (1.1) and (1.2) below.*

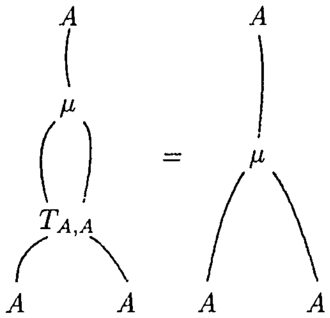


(1.1)

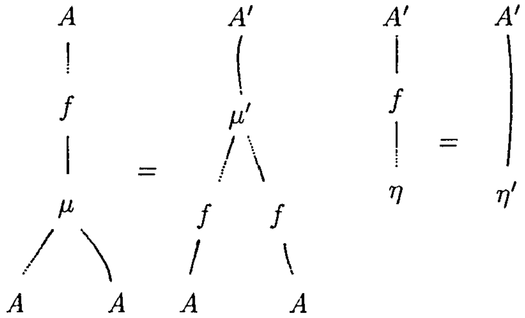


(1.2)

If in addition we have $\mu = \mu \circ T_{A,A}$ *then* A *is* **commutative**. *If* (A, μ, η) *and* (A', μ', η') *are algebras and* $f \in \mathsf{Vect}(A, A')$ *satisfies the conditions* $f \circ \mu = \mu' \circ (f \otimes f)$ *and* $f \circ \eta = \eta'$ *then* f *is an* **algebra map**. *The category of algebras will be referred to as* Alg. *The tensor diagrams depicting commutativity and the algebra map conditions are given in (1.3) and (1.4).*

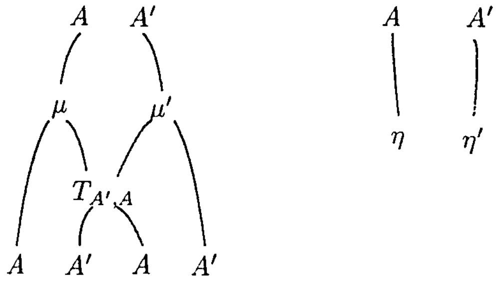


(1.3)



(1.4)

Example 1.3.2 *If* (A, μ, η) *and* (A', μ', η') *are algebras then* $A \otimes A'$ *is an algebra with multiplication given by:* $(\mu \otimes \mu') \circ (A \otimes T_{A',A} \otimes A')$ *and unit given by:* $\eta \otimes \eta'$. *In terms of diagrams the structure is given in (1.5).*

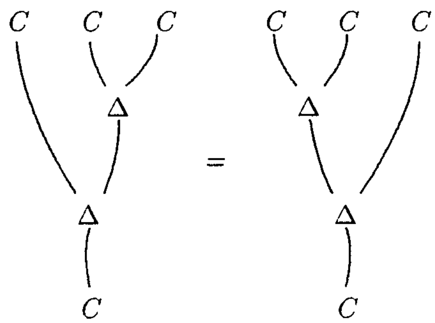


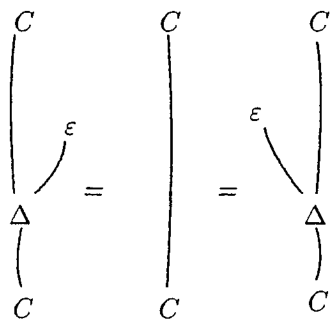
(1.5)

Definition 1.3.3 *If (A, μ, η) is an algebra then we define $\mu^{(-1)} = \eta$, $\mu^{(0)} = A$, $\mu^{(1)} = \mu$ and, for $n > 1$, $\mu^{(n)} = \mu^{(n-1)} \circ (\mu \otimes A^{n-1})$.*

By dualizing the notion of an algebra we have the definition of a coalgebra.

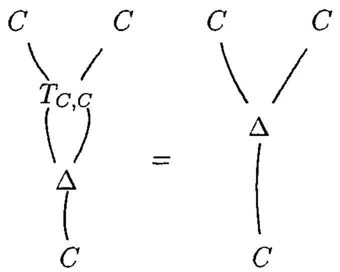
Definition 1.3.4 *A* **coalgebra** *is a space C together with two maps, a comultiplication $\Delta \in \mathsf{Vect}(C, C \otimes C)$ and a counit $\varepsilon \in \mathsf{Vect}(C, \mathrm{k})$, satisfying the coassociativity and counitary conditions: $(\Delta \otimes C) \circ \Delta = (C \otimes \Delta) \circ \Delta$, $(\varepsilon \otimes C) \circ \Delta = C$ and $(C \otimes \varepsilon) \circ \Delta = C$. In terms of diagrams these conditions are given in (1.6) and (1.7) below.*



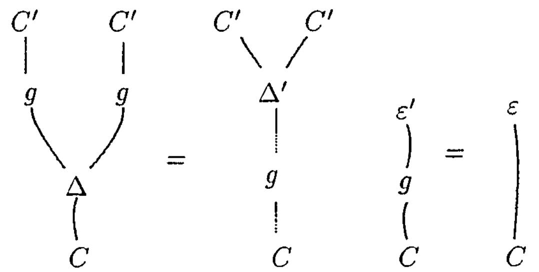


(1.6) (1.7)

If in addition we have $\Delta = T_{C,C} \circ \Delta$ then the coalgebra is **cocommutative**. *If (C, Δ, ε) and $(C', \Delta', \varepsilon')$ are coalgebras and $g \in \mathsf{Vect}(C, C')$ is a map satisfying the conditions $\Delta' \circ g = (g \otimes g) \circ \Delta$ and $\varepsilon' \circ g = \varepsilon$ then f is a* **coalgebra map**. *The category of coalgebras will be referred to as Coalg. In terms of diagrams cocommutativity and the coalgebra map* *conditions are given in (1.8) and (1.9).*

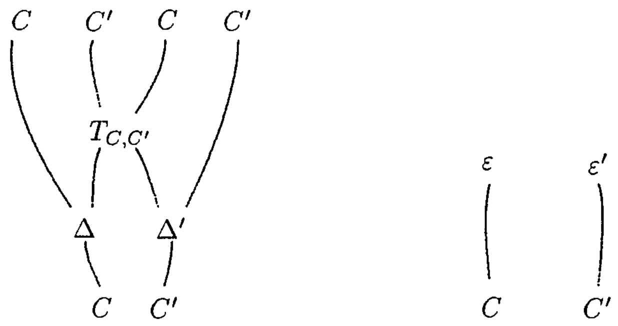


(1.8)



(1.9)

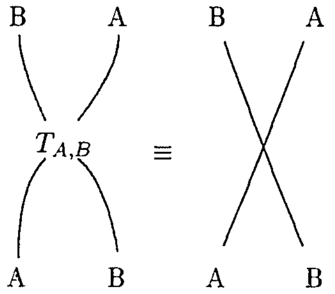
Example 1.3.5 *If (C, Δ, ε) and $(C', \Delta', \varepsilon')$ are coalgebras then $C \otimes C'$ is a coalgebra with comultiplication given by $(C \otimes T_{C,C'} \otimes C') \circ (\Delta \otimes \Delta')$ and counit given by $\varepsilon \otimes \varepsilon'$. In terms of diagrams the structure is given in (1.10).*



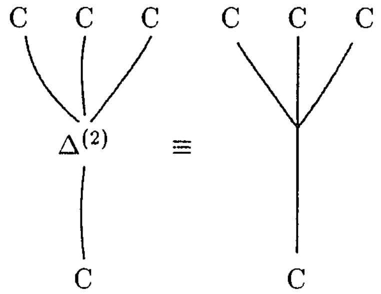
(1.10)

Definition 1.3.6 *If (C, Δ, ε) is a coalgebra then we define $\Delta^{(-1)} = \varepsilon$, $\Delta^{(0)} = C$, $\Delta^{(1)} = \Delta$ and, for $n > 1$, $\Delta^{(n)} = (\Delta \otimes C^{n-1}) \circ \Delta^{(n-1)}$.*

We now adopt a few more conventions for tensor diagrams. First of all, the twisting isomorphism T will no longer be labeled in a diagram but will be represented by a crossing of edges, as in (e) below. If C is a coalgebra with comultiplication Δ, we will no longer label the maps $\Delta^{(n)}$. Instead, these will be denoted simply as a branching edge, as $\Delta^{(2)}$ in (f). Dually we no longer label $\mu^{(n)}$ for a multiplication μ on an algebra. In addition, diagrams will be the primary method in this paper for stating and proving equalities (see theorem 2.2.7 for a proof using more conventional notation).



(e)



(f)

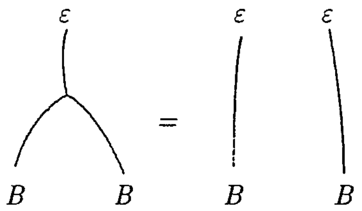
Example 1.3.7 *Suppose (C, Δ, ε) is a coalgebra and (A, μ, η) is an algebra. Then the space* Vect*(C,A) is an algebra, called the* **convolution algebra**, *when equipped with the unit element $\eta_A \circ \varepsilon_C$ and the multiplication defined for all $f, g \in$* Vect*(C, A) by 1.11.*

$$f * g = \mu \circ (f \otimes g) \circ \Delta \tag{1.11}$$

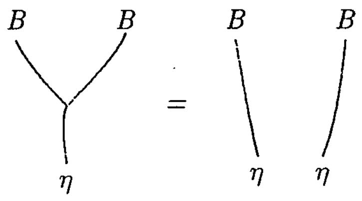
If $f \in$ Vect*(C, A) is invertible in the convolution algebra then the inverse of f will be denoted $\bar{f}$. The group of convolution invertible elements of* Vect*(C, A) will be referred to as* Reg(C, A).

It may be that a space is both an algebra and a coalgebra in a coherent way and this is the content of the next definition.

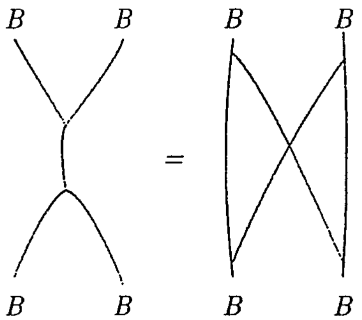
Definition 1.3.8 *Suppose (B, μ, η) is an algebra and (B, Δ, ε) is a coalgebra. If the following four equalities are satisfied then $(B, \mu, \eta, \Delta, \varepsilon)$ is called a* **bialgebra**.



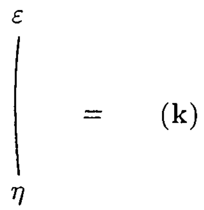
(1.12)



(1.13)

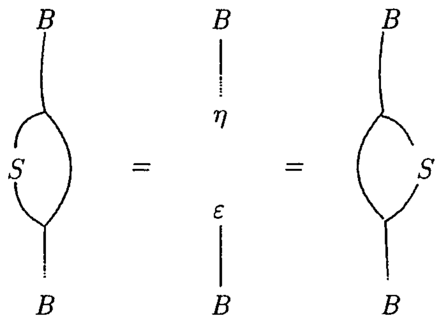


(1.14)



(1.15)

If in addition there is a map $S \in \mathsf{Vect}(B, B)$ so that S is the convolution inverse of the identity map on B, that is to say, satisfying (1.16) below, then $(B, \mu, \eta, \Delta, \varepsilon, S)$ is called a **Hopf algebra** *and S is called its* **antipode***.*



(1.16)

Notice that (1.12) and (1.14) mean that μ is a coalgebra map and that (1.13) and (1.15) mean that η is a coalgebra map. On the other hand (1.13) and (1.14) mean that Δ is an algebra map and (1.12) and (1.15) mean that ε is an algebra map.

If H is a Hopf algebra with antipode S then S is an anti-algebra map and an anti-coalgebra map; that is, S is unitary, counitary, $S \circ \mu = \mu \circ T_{H,H} \circ (S \otimes S)$ and $\Delta \circ S = (S \otimes S) \circ T_{H,H} \circ \Delta$.

Examples 1.3.9

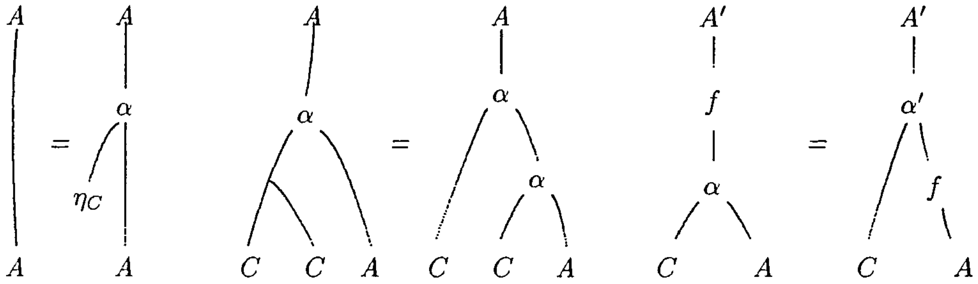
1. *If A and C are bialgebras then $A \otimes C$ is a bialgebra with the algebra structure of example 1.3.2 and the coalgebra structure of example 1.3.5. If both A and C are Hopf algebras with antipodes S_A and S_C respectively then $A \otimes C$ is a Hopf algebra with antipode $S_A \otimes S_C$.*

2. *Let G be a group and denote by* kG *the space having basis G. The usual group ring structure makes* kG *an algebra.* kG *is a cocommutative Hopf algebra by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$.*

3. *Suppose $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra and H is finite dimensional as a vector space. Let $H^* = \mathsf{Vect}(H, \mathbf{k})$, the linear dual space. Recall that there is a linear isomorphism $\Phi : H^* \otimes H^* \to (H \otimes H)^*$ given by $\Phi(f \otimes h)(a \otimes b) = f(a)h(b)$ for all $f, h \in H^*$ and $a, b \in H$, and that there is an isomorphism $\iota : \mathbf{k} \to \mathbf{k}^*$. Define $\mu^d = \Delta^* \circ \Phi$, $\eta^d = \varepsilon^* \circ \iota$, $\Delta^d = \Phi^{-1} \circ \mu^*$ and $\varepsilon^d = \iota^{-1} \circ \eta^*$. Then $(H^*, \mu^d, \eta^d, \Delta^d, \varepsilon^d, S^*)$ is a Hopf algebra. If H is cocommutative then H^* is commutative and if H is commutative then H^* is cocommutative. Note that if (C, Δ, ε) is a coalgebra then $(C^*, \Delta^d, \varepsilon^d)$ is the convolution algebra on* Vect*(C,*k*).*

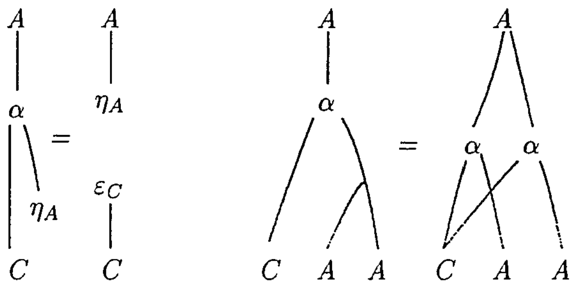
4. *Let G be a finite group. Then* kG *is a cocommutative Hopf algebra as in example 2 and* kG^* *is a commutative Hopf algebra as in example 3. Explicitly the structure is: $\mu(f \otimes h)(g) = f(g)h(g)$, $\eta(k)(g) = k$, $(\Phi \circ \Delta)(f)(g \otimes g') = f(gg')$, $\varepsilon(f) = f(e_G)$ and $S(f)(g) = f(g^{-1})$ for all $f, h \in \mathbf{k}G^*$ $g, g' \in G$ and $k \in \mathbf{k}$.*

Definition 1.3.10 *Suppose C and A are spaces and $\alpha \in$* $\mathsf{Vect}(C \otimes A, A)$. *If C is an algebra and the equalities (1.17) and (1.18) hold then (A, α) is called a* **C-module**. *If (A', α') is also a C-module and $f \in$* $\mathsf{Vect}(A, A')$ *satisfies (1.19) then f is a* **C-module morphism**. *The category of C-modules will be denoted* Cmod.



(1.17) (1.18) (1.19)

If C is a coalgebra, A is an algebra and the equalities (1.17), (1.20) and (1.21) are satisfied we say that C **measures** (A, α).

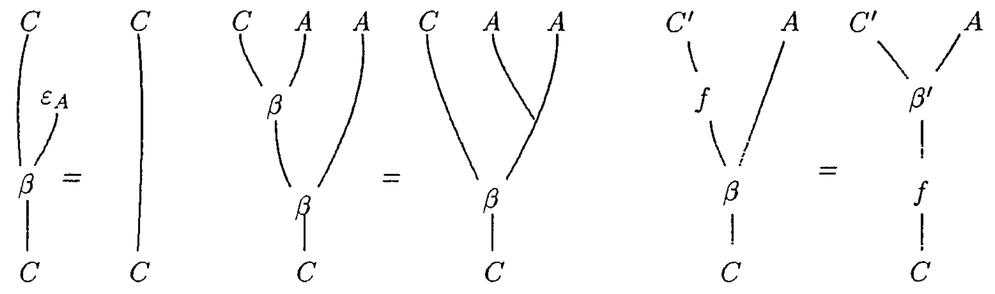


(1.20) (1.21)

When C is a bialgebra, (A, α) is a C-module, and C measures (A, α), then (A, α) is called a **C-module algebra**.

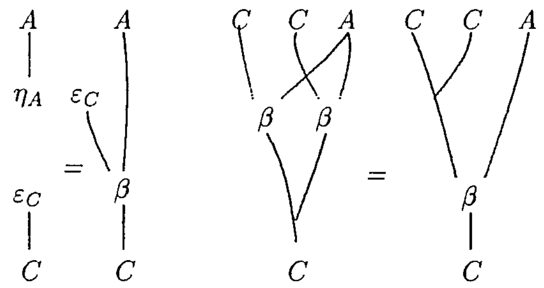
Dually Suppose C and A are spaces and $\beta \in$ $\mathsf{Vect}(C, C \otimes A)$. *If A is a coalgebra and the equalities (1.22) and (1.23) hold then (C, β) is called an* **A-comodule**. *If (C', β') is also an A-comodule and $f \in$* $\mathsf{Vect}(C, C')$ *satisfies (1.24) then f is an* **A-comodule morphism**.

The category of A-comodules will be denoted Acomod.



(1.22) (1.23) (1.24)

If A is an algebra, C is a coalgebra and the equalities (1.22), (1.25) and (1.26) are satisfied we say that A **comeasures** (C, β).



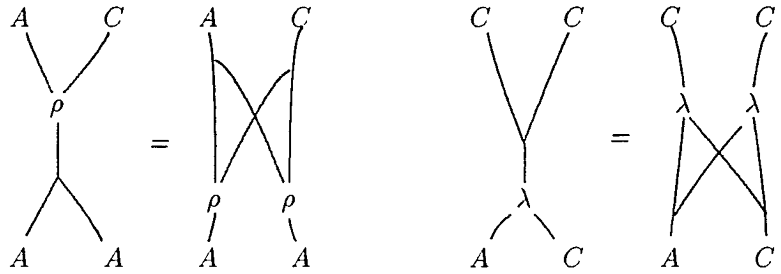
(1.25) (1.26)

If A is a bialgebra, (C, β) *is an A-comodule, and A comeasures* (C, β)*, then* (C, β) *is an* **A-comodule coalgebra**.

Note that a C-module algebra is a C-module with algebra structures that are C-module morphisms, and this leads to the next definition.

Definition 1.3.11 *Suppose C is a bialgebra and* (A, ρ) *is a C-comodule. If A is an algebra and the algebra structures are C-comodule morphisms, that is,* $\rho \circ \eta_A = \eta_A \otimes \eta_C$ *and the equality (1.27) holds then we call* (A, α) *a C-***comodule algebra**. *Dually, suppose A is a bialgebra and* (C, λ) *is an A-module. If C is a coalgebra and the coalgebra structures are*

A-module morphisms, that is, $\varepsilon_C \circ \lambda = \varepsilon_A \otimes \varepsilon_C$ *and the equality (1.28) holds then we call* (C, λ) *an A-***module coalgebra***.*



(1.27) (1.28)

1.4 Some Open Questions

We now mention a few questions raised by this paper.

1. A more general simplicial theory.

   The simplicial theory that we have developed here is based on the assumption that the pairs of Hopf algebras involved are compatible matched pairs (definition 2.1.1). Crossed products and extensions however are constructed on the more general measured pairs (definition 2.1.1). It is hoped that the cohomology can be extended to measured pairs.

2. A more complete simplicial theory.

   There is a natural definition for 2-boundaries for the bi-cosimplicial algebra given in (3.19). These should give a 2-homology and the equivalence of extensions.

3. Examples of crossed product Hopf algebras.

   Smash product Hopf algebras have been used to construct important examples of non-commutative and non-cocommutative Hopf algebras; in particular the Drinfel'd double

[Maj90]. Since crossed products are generalizations of smash products (see example (2.2.7.5)) it is hoped that these constructions will lead to some new examples.

4. A complete theory of cleft extensions.

   The cleft extensions defined here (for the first time) are the most general in the literature, and even in the abelian case they afford a more natural definition than the earlier definitions. This will be the focus of future research.

5. The category of matched (measured) pairs.

   The category of matched (measured) pairs would be the domain of of the homology functor from question 2. Also, it is interesting that in general the tensor product of two C-module algebras is not a C-module algebra but in theorem 2.1.3 it is shown that for a compatible matched pair the tensor product does carry a C-module structure. This category must be looked at in conjunction with question 2.

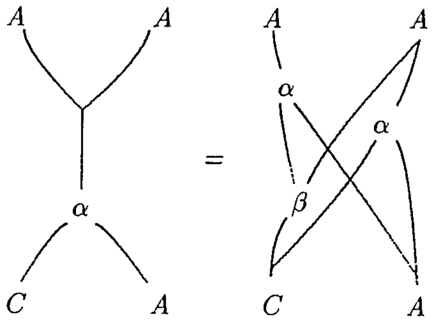
Chapter 2

Matched Pairs and Crossed Products

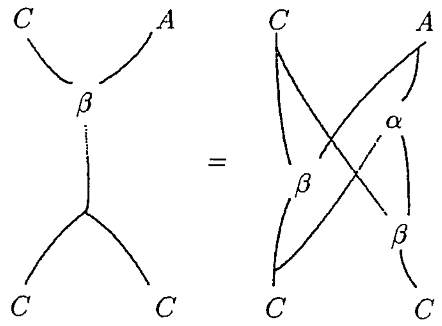
In this chapter we look at certain pairs of Hopf algebras called compatible matched pairs. Such pairs have enough structure to admit the construction of a smash product Hopf algebra on their tensor. We then turn our attention to a generalization of smash products which we will refer to as crossed products.

2.1 Compatible Matched Pairs

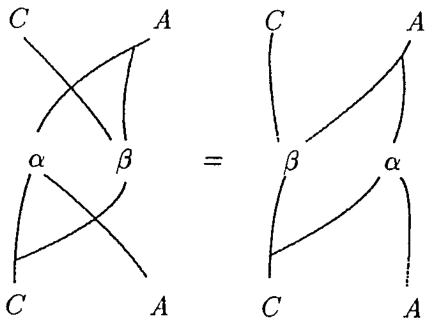
Definition 2.1.1 *Suppose C and A are bialgebras. If C measures (A, α) and A comeasures (C, β), then we call (C, A, α, β) a* **measured pair**. *If (C, A, α, β) is a measured pair and the equation (2.3) holds then (C, A, α, β) is called a* **compatible measured pair**. *If (A, α) is a C-module algebra and (C, β) is an A-comodule coalgebra then (C, A, α, β) is called a* **matched pair**. *If (C, A, α, β) is a matched pair and the equations (2.1), (2.2) and (2.3) hold then (C, A, α, β) is called a* **compatible matched pair**.



(2.1)



(2.2)

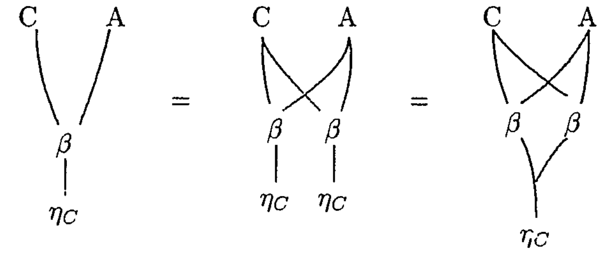


(2.3)

The abelian matched pairs of Hofstetter [Hof94] are thus generalized by this definition. An *abelian* matched pair is a matched pair (C, A, α, β) in which C is cocommutative and A is commutative. In an abelian matched pair (2.3) is always satisfied. We are thus transferring abelian conditions on A and C to a weaker commuting condition on the actions.

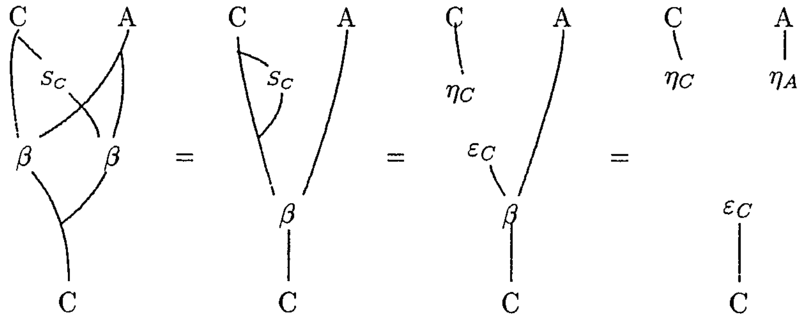
Theorem 2.1.2 *Suppose* (C, A, α, β) *is a compatible matched pair. Then* $\beta \circ \eta_C = \eta_C \otimes \eta_A$ *and dually* $\varepsilon_A \circ \alpha = \varepsilon_C \otimes \varepsilon_A$.

Proof. First we make two calculations. By preceding the equal maps of (2.2) by $\eta_C \otimes \eta_C$ and applying (1.2), (1.13), and (1.17) we get the first equality below. The second is property (1.13) of η.



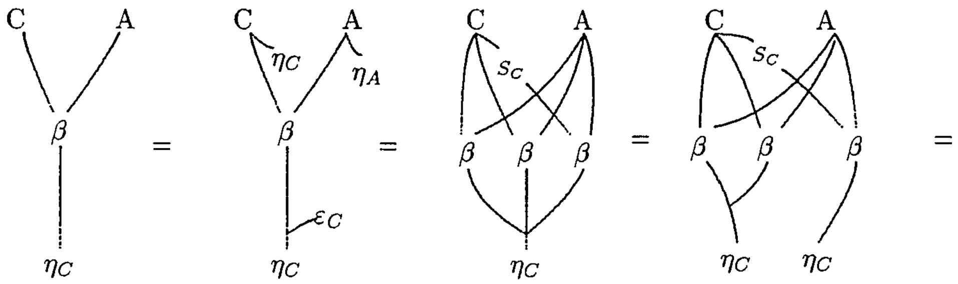
(2.4)

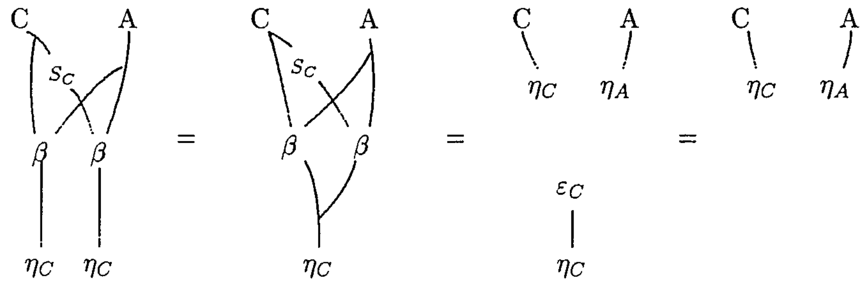
We also have the following.



(2.5)

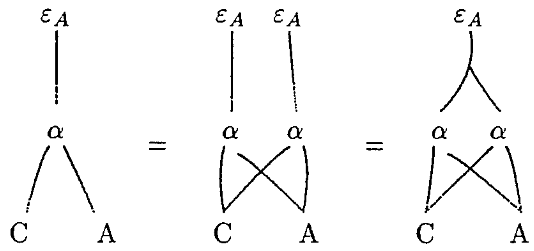
The first equality follows from (1.26), the second from the property of the antipode (1.16) and the third from (1.25). Note that (2.5) states that $(S_C \otimes A) \circ \beta$ is the right inverse of β in the convolution algebra on $\mathbf{Vect}(C, C \otimes A)$; in fact it is the inverse. Now we calculate.





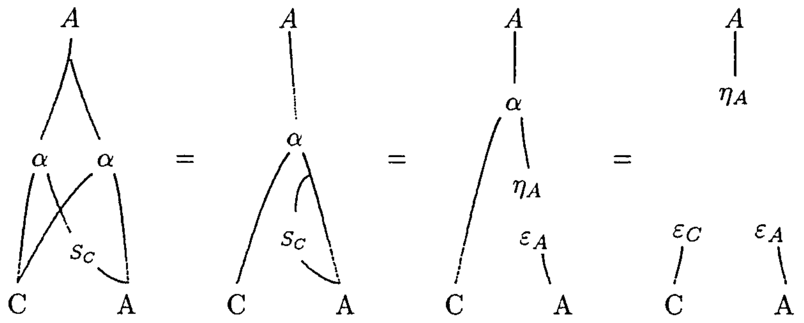
The second and sixth equalities follow from (2.5), and the fourth from (2.4). The rest are the elementary properties (1.13) and (1.15) of η and ε.

This proves the theorem as far as β is concerned. The proof of the statement concerning α is the exact dual and the proof follows for comparison. In the sequel such dual proofs will be omitted. First we make two calculations. By following the equal maps of (2.1) by $\varepsilon_A \otimes \varepsilon_A$ and applying (1.12) and (1.22) we get the first equality below. The second is the property (1.12) of ε.



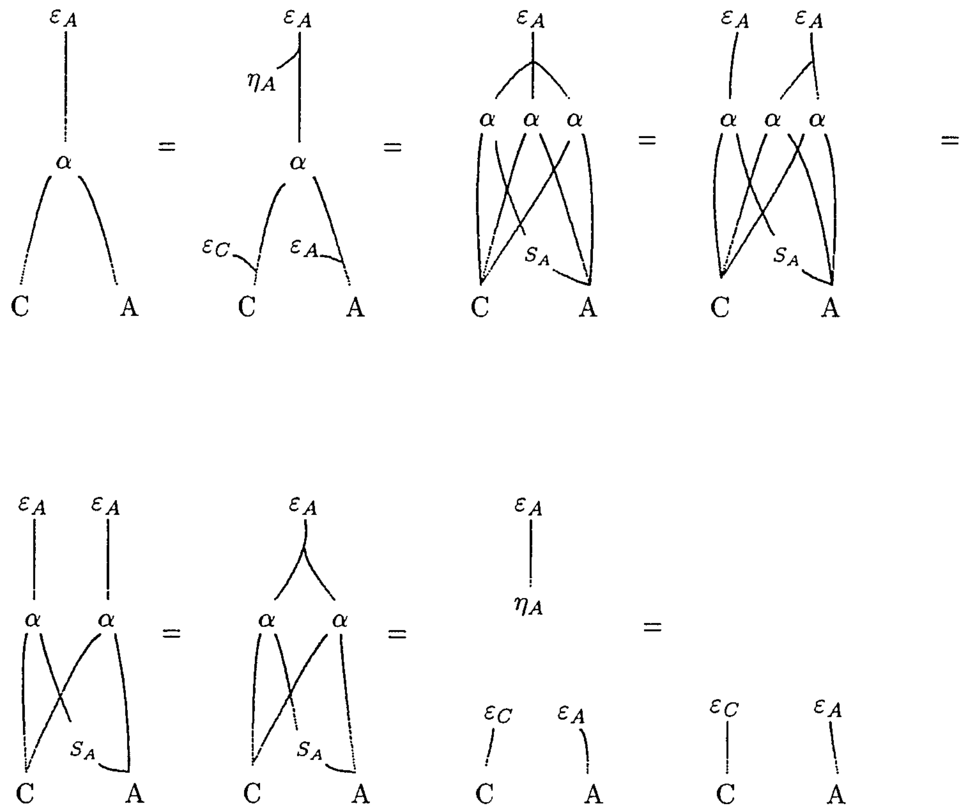
(2.4 dual)

Next we have the following.



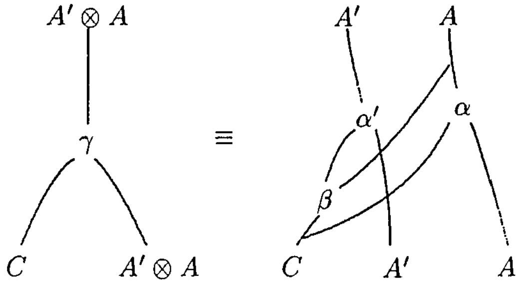
(2.5 dual)

The first equality follows from (1.21), the second from the property of the antipode (1.16) and the third from (1.20). Now we have the following calculation.



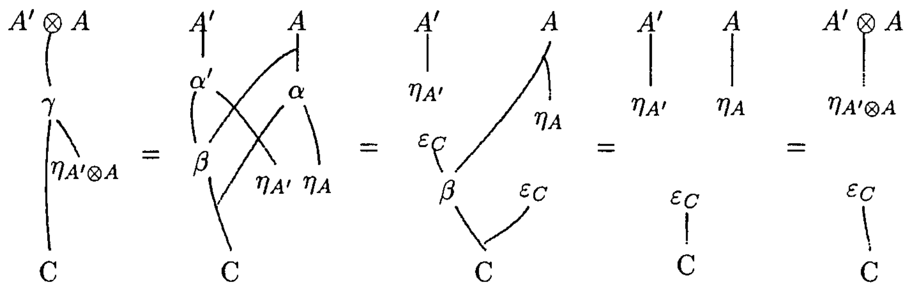
The second and sixth equalities follow from (2.5 dual), and the fourth from (2.4 dual). The rest of the equalities follow from (1.13) and (1.15) . □

Theorem 2.1.3 *Suppose (C, A, α, β) is a compatible matched pair and (A', α') is a C-module algebra. Then $(A' \otimes A, \gamma)$ is a C-module algebra with algebra structure as in example 1.3.2 and γ given by (2.6).*

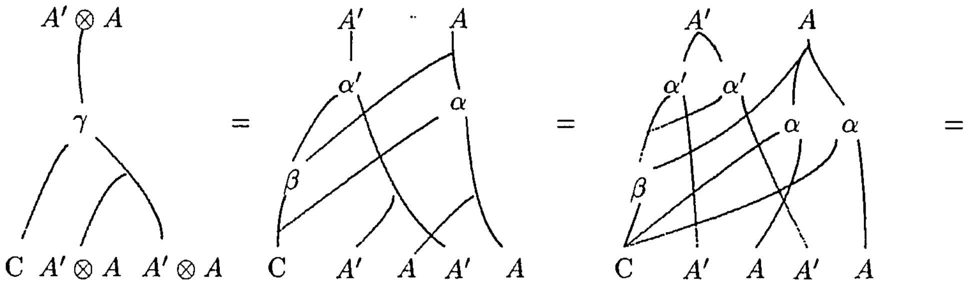


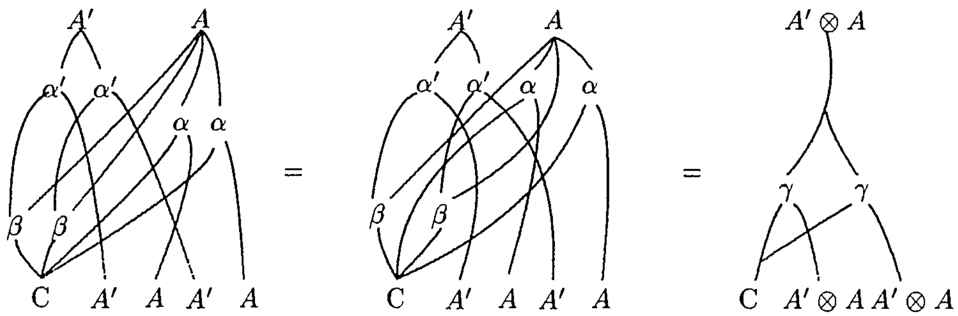
(2.6)

Proof. We must show that (1.17), (1.18), (1.20) and (1.21) are satisfied by (C, γ). First, to see that (1.20) is satisfied consider the following.



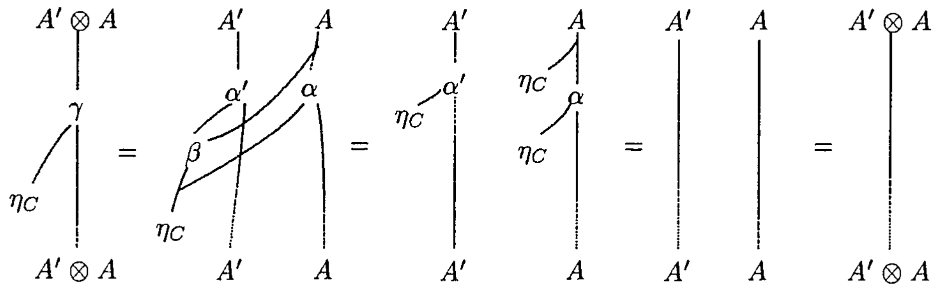
The first and last steps are the definitions. The second equality follows by applying (1.20) to α' and α. The third equality is a result of first applying (1.2) to η_A and then applying (1.25) to β. Now we see that (C, γ) satisfies (1.21).





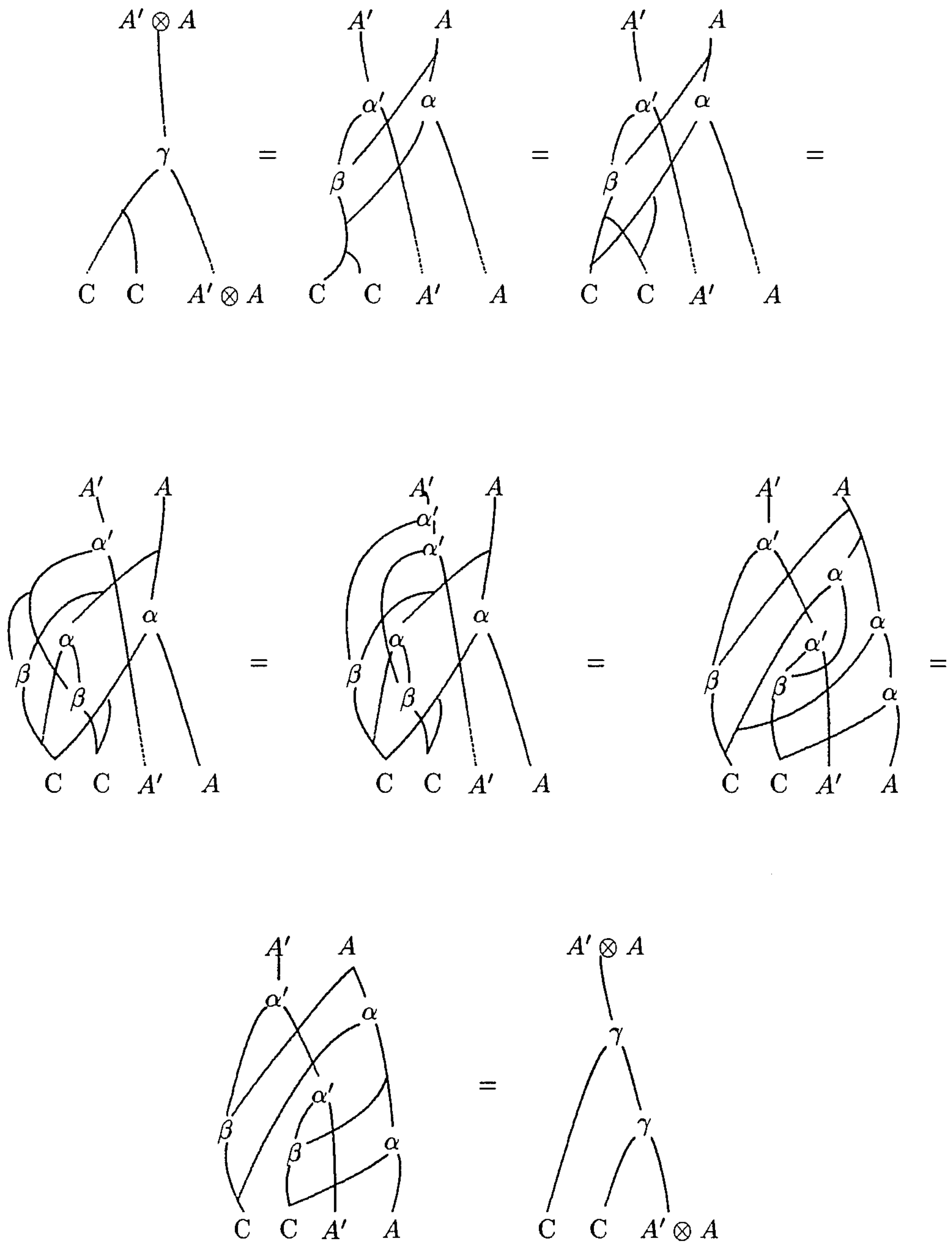
The first equality and last equalities are definitions. The second equality is the result of applying (1.21) to α' and to α. The third equality follows from (1.26) and the fourth equality from (2.3) applied to the α and β in the center of the diagram.

To see that γ satisfies (1.17) consider the following.



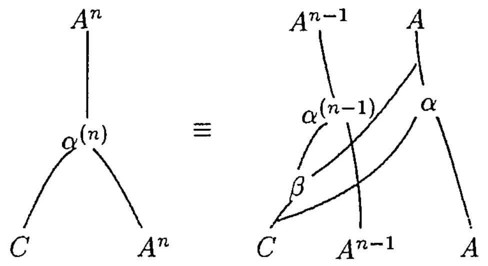
The first and last equalities are definitions. In the second equality we first used (1.13) and theorem 2.1.2, and the third equality uses (1.2) and (1.17).

Finally we show that (1.18) is satisfied.



The first and last equalities are definitions. The second equality is a result of applying (1.14) and the third equality follows from (2.2). The fourth equality is a result of (1.18) applied to α' and the fifth equality follows when (1.18) applied to α. Finally, in the sixth equality we apply (1.21) . □

Definition 2.1.4 *Suppose (C, A, α, β) is a compatible matched pair. Define $\alpha^{(0)} = \varepsilon_C$, $\alpha^{(1)} = \alpha$ and, for $n \geq 2$, $\alpha^{(n)}$ is defined by (2.7).*



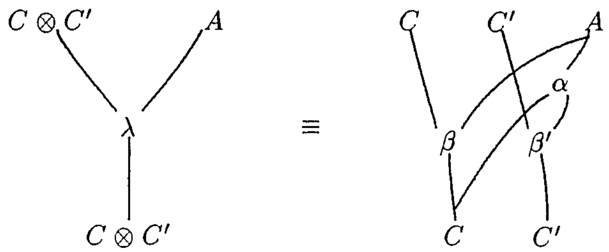
(2.7)

Corollary 2.1.5 *Suppose (C, A, α, β) is a compatible matched pair. Then for every $n \geq 0$ $(A^n, \alpha^{(n)})$ is a C-module algebra.*

Proof. The cases $n = 0$ and $n = 1$ are immediate. When $n \geq 2$ we view A^n as $(A^{n-1} \otimes A)$ and apply theorem 2.1.3 inductively. □

We also have the duals of theorem 2.1.3 and definition 2.1.4.

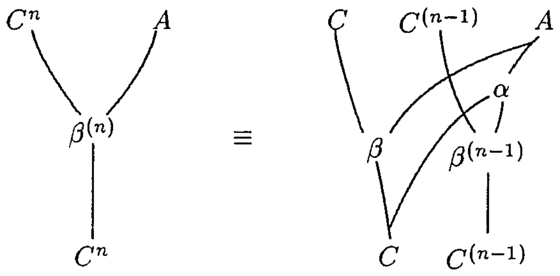
Theorem 2.1.6 *Suppose (C, A, α, β) is a compatible matched pair and (C', β') is an A-comodule coalgebra. Then $(C \otimes C', \lambda)$ is an A-comodule coalgebra with λ given by (2.8).*



(2.8)

Proof. Dualize theorem 2.1.3. □

Definition 2.1.7 *Suppose* (C, A, α, β) *is a compatible matched pair. Define* $\beta^{(0)} = \eta_A$, $\beta^{(1)} = \beta$ *and, for* $n \geq 2$, $\beta^{(n)}$ *is defined by (2.9).*



(2.9)

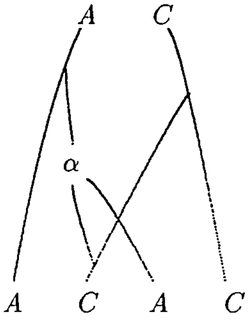
Corollary 2.1.8 *Suppose* (C, A, α, β) *is a compatible matched pair. Then for every* $n \geq 0$, $(C^n, \beta^{(n)})$ *is an A-comodule coalgebra.*

Proof. Dualize corollary 2.1.5. □

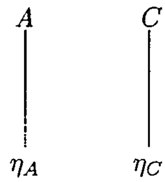
2.2 Crossed Products

In this section we will construct a new object that we will call the crossed product bialgebra. If C and A are algebras then $A \otimes C$ is an algebra, as described in (1.5). In the presence of additional structure we can impose different algebra structures on $A \otimes C$. We now consider this situation and the dual situation. The following three definitions and three theorems are minor generalizations of some previously known facts and definitions.

Definition 2.2.1 *Suppose C is a bialgebra, A is an algebra and* $\alpha \in \mathsf{Vect}(C \otimes A, A)$. *The space* $A \otimes C$ *together with the structure given by (2.10) and (2.11) below will be denoted* $A \#_\alpha C$. *If this is an algebra it will be called the* **smash product algebra**.

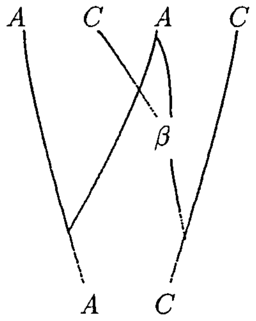


(2.10)

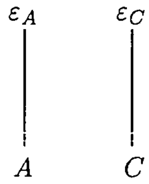


(2.11)

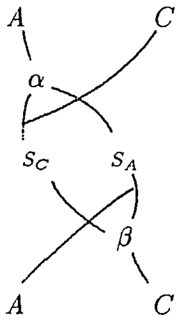
Dually, suppose A is a bialgebra, C is a coalgebra and $\beta \in \mathsf{Vect}(C, C \otimes A)$. The space $A \otimes C$ together with the structure given by (2.12) and (2.13) below will be denoted $A\#^{\beta}C$. If this is a coalgebra it will be called the **smash product coalgebra**.



(2.12)



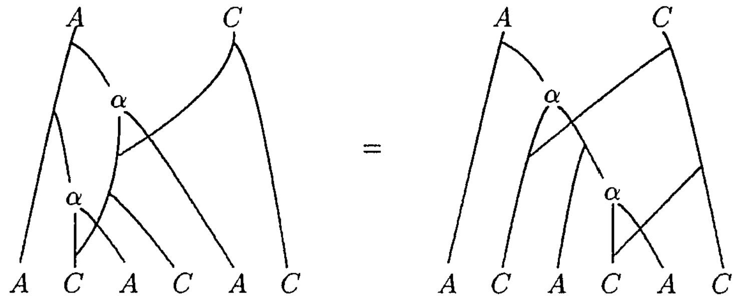
(2.13)



(2.14)

Theorem 2.2.2 *$A\#_{\alpha}C$ is a smash product algebra if and only if (A, α) is a C-module algebra. Dually, $A\#^{\beta}C$ is a smash product coalgebra if and only if (C, β) is an A-comodule coalgebra.*

Proof. If (A, α) is a C-module algebra then it is well known that $A\#_{\alpha}C$ is a smash product algebra, (see for example [Mont93]). Conversely suppose $A\#_{\alpha}C$ is a smash product algebra. The associativity condition is given in the next diagram.



(2.15)

By preceding both sides of the equality in (2.15) by $\eta \otimes C \otimes \eta \otimes C \otimes A \otimes \eta$ and following by $A \otimes \varepsilon$ we have (1.18). By preceding both sides of the equality in (2.15) by $\eta \otimes C \otimes A \otimes \eta \otimes A \otimes \eta$ and following by $A \otimes \varepsilon$ we have (1.21). The unitary conditions are similar, as is the dualization. □

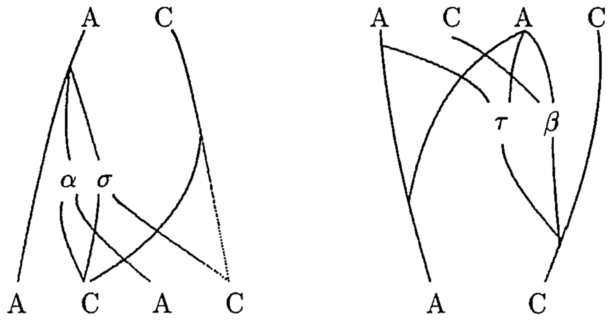
It may happen that both structures, $A\#_\alpha C$ and $A\#^\beta C$, are given and we wish to know if they are compatible, in the sense of theorem 1.3.8, resulting in a bialgebra structure on the space $A \otimes C$.

Definition 2.2.3 *Suppose A and C are bialgebras $\alpha \in \mathsf{Vect}(C \otimes A, A)$ and $\beta \in \mathsf{Vect}(C, C \otimes A)$. The space $A \otimes C$, together with the structures given by (2.10)—(2.13) will be denoted $A\#_\alpha^\beta C$, if this is a bialgebra it will be called the* **smash product bialgebra**.

Theorem 2.2.4 *$A\#_\alpha^\beta C$ is a smash product bialgebra if and only if (C, A, α, β) is a compatible matched pair. If A and C are Hopf algebras then $A\#_\alpha^\beta C$ is a Hopf algebra with antipode given by (2.14).*

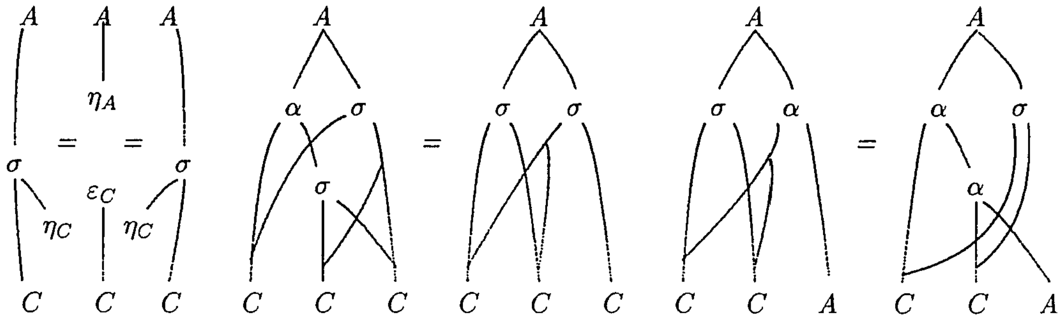
Proof. Suppose (C, A, α, β) is a compatible matched pair. If we invoke theorem 2.1.2 we have exactly the data necessary to to cite theorem 3.3 of [Maj90] and $A\#_\alpha^\beta C$ is a smash product bialgebra. Conversely if $A\#_\alpha^\beta C$ is a smash product bialgebra then theorem 3.3 of [Maj90] tells us that (2.1)—(2.3) hold. This, and the results of the previous theorem complete the proof. □

We now want to allow for a further "twist" in these structures on $A \otimes C$. Suppose $\sigma \in \mathsf{Vect}(C \otimes C, A)$ and that we further alter the algebra structure on $A \otimes C$ as in (2.16) below. Under what conditions do we get an algebra? This is answered in the following definition and theorem.



(2.16) (2.17)

Definition 2.2.5 *Suppose A is an algebra, C is a bialgebra, $\alpha \in \mathsf{Vect}(C \otimes A, A)$ and $\sigma \in \mathsf{Vect}(C \otimes C, A)$. The space $A \otimes C$ together with the structure given by (2.16) and (2.11) will be denoted $A\#_{\alpha,\sigma}C$. If this is an algebra it will be called the* **crossed product algebra**.



(2.18) (2.19) (2.20)

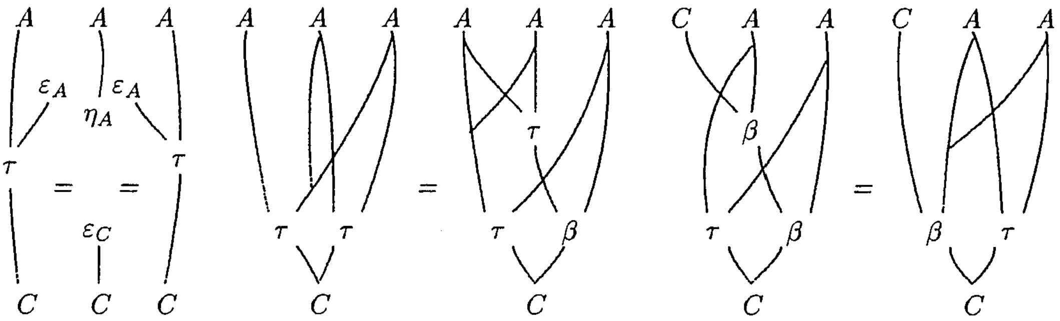
Theorem 2.2.6 *Suppose α satisfies (1.20), then $A\#_{\alpha,\sigma}C$ is a crossed product algebra if and only if C measures (A, α) and the conditions (2.18)—(2.20) are satisfied.*

Proof. For the most part this is well known, (see for example lemma 7.1.2, [Mont93]). We only need to show that C measures (A, α) and the proof, in its dual form, is given in the next theorem. □

We now dualize definition 2.2.5 and theorem 2.2.6.

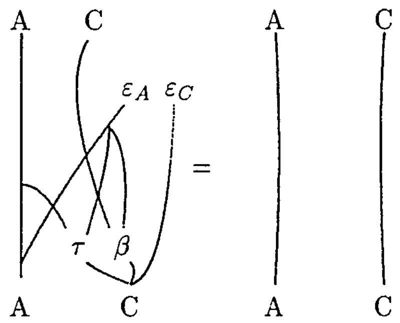
Definition 2.2.7 *Suppose C is a coalgebra, A is a bialgebra, $\beta \in \mathsf{Vect}(C, C \otimes A)$ and $\tau \in \mathsf{Vect}(C, A \otimes A)$. The space $A \otimes C$ together with the structure given by (2.17) and (2.13) will be denoted $A\#^{\beta,\tau}C$. If this is a coalgebra it will be called the* **crossed product coalgebra**.

Theorem 2.2.8 *Suppose β satisfies (1.25), then $A\#^{\beta,\tau}C$ is a crossed product coalgebra if and only if A comeasures (C, β) and the conditions (2.21)—(2.23) are satisfied.*

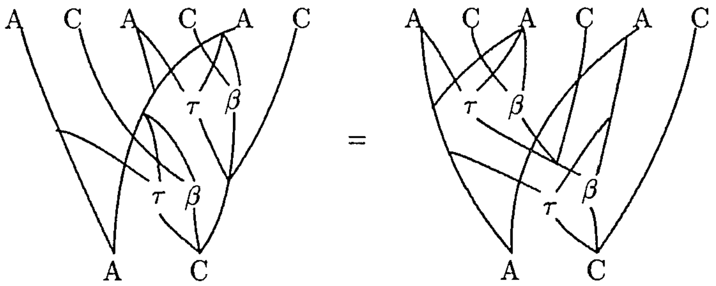


(2.21) (2.22) (2.23)

Proof. First suppose $A\#^{\beta,\tau}C$ is a coalgebra. The right counitary axiom for $A\#^{\beta,\tau}C$ gives us the following equality.

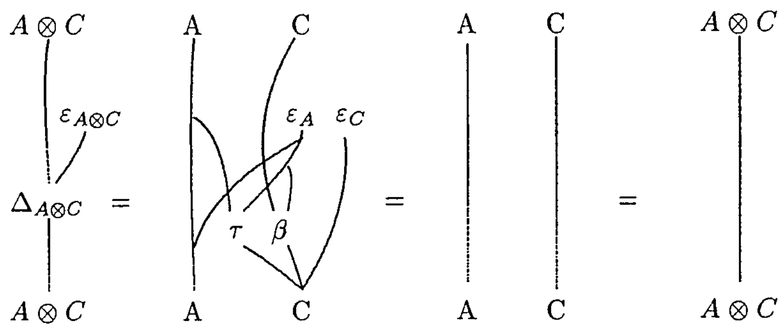


Preceding these two equal maps by $\eta_A \otimes C$ and following them by $A \otimes \varepsilon_C$ we have the left hand side of (2.21). The right hand side is similar, as is (1.22). The coassociativity of $A\#^{\beta,\tau}C$ is shown in the next diagram.

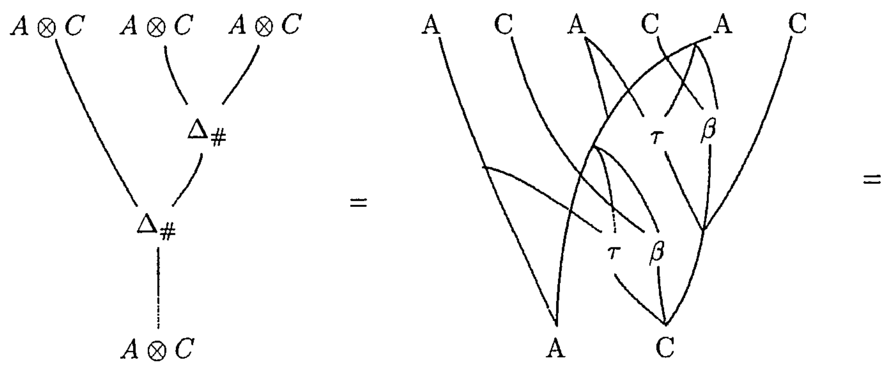


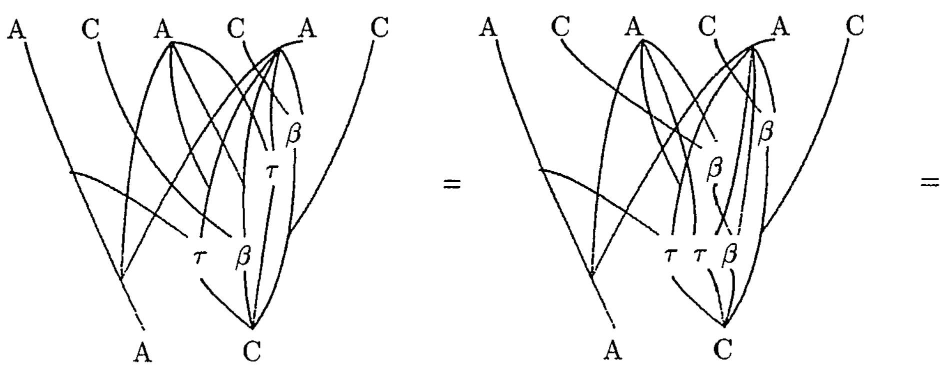
Preceding both sides of this equality by $\eta_A \otimes C$ and following by $(A \otimes \varepsilon_C)^3$ yields (2.22). Preceding both sides of this equality by $\eta_A \otimes C$ and following by $\varepsilon_A \otimes C \otimes (A \otimes \varepsilon_C)^2$ yields (2.23). Preceding both sides of this equality by $\eta_A \otimes C$ and following by $\varepsilon_A \otimes C \otimes \varepsilon_A \otimes C \otimes A \otimes \varepsilon_C$ yields (1.26).

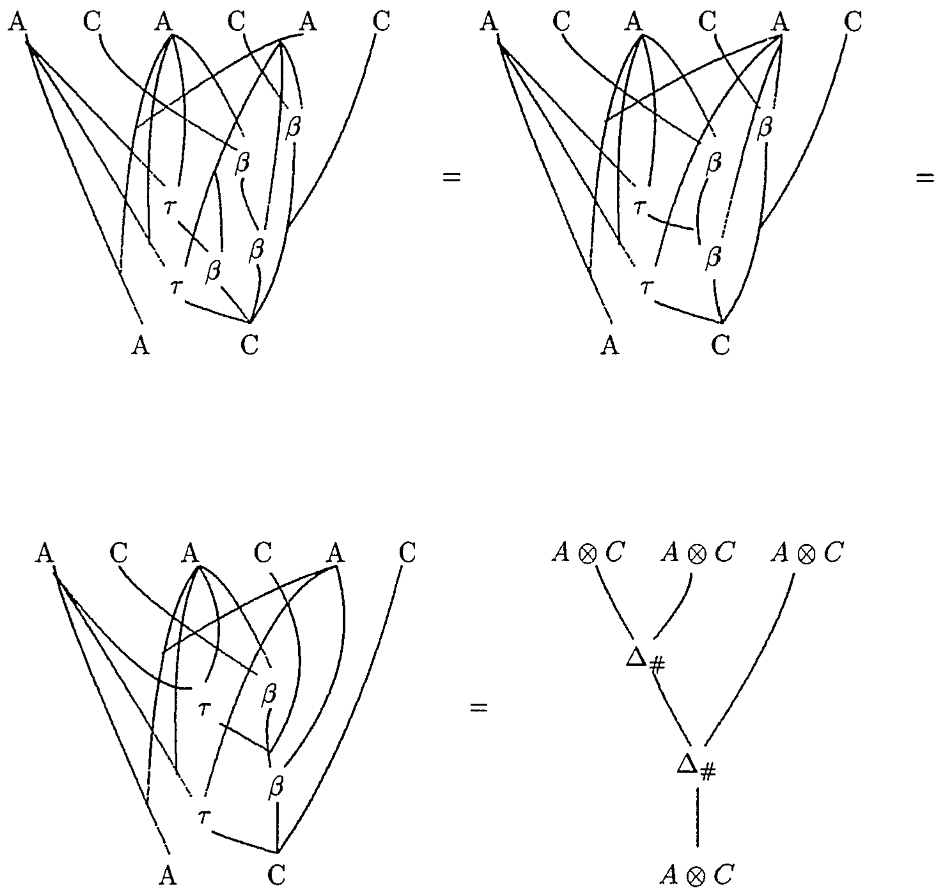
Now for the converse, suppose A comeasures (C, β) and that conditions (2.21)—(2.23) are satisfied. The counitary property (1.7) follows easily from the counitary properties of ε_A and ε_C as seen in the next calculation.



The first and last equalities are definitions. The second equality follows from (1.7), (1.12), (1.22) and (2.21). Now we prove coassociativity.







The first and last equalities are definitions. The second equality is (1.14) applied to $\Delta_A \circ \mu_A^2$. The third equality is (2.23) applied to β and τ, and the fourth equality is (2.22). The fifth and sixth equalities are (1.26) applied first to the two β's toward the bottom of the diagram and then to the two β's toward the lower right. This shows coassociativity and the theorem has been proved.

Now, for the interested reader, the above proof of coassociativity will be restated in the more traditional notation. Let (C, Δ, ε) be a coalgebra. The Sweedler notation for Δ, [Swe69], is given as follows: for $c \in C$,

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}.$$

The axioms of a coalgebra (1.6) and (1.7) are:

$$\sum_{(c)} c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = \sum_{(c)} c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$$

and

$$\sum_{(c)} \varepsilon(c_{(1)})c_{(2)} = c = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}).$$

Now suppose (C, β) is an A-comodule coalgebra. For $c \in C$ we will write $\beta(c) = \sum_{[c]} c_{[1]} \otimes c_{[2]}$. In these terms the conditions (1.25)—(1.23) are, for $c \in C$.

$$\eta(\varepsilon(c)) = \sum_{[c]} \varepsilon(c_{[1]})c_{[2]}$$

$$\sum_{(c),[c_{(1)}],[c_{(2)}]} c_{(1)[1]} \otimes c_{(2)[1]} \otimes c_{(1)[2]}c_{(2)[2]} = \sum_{[c],(c_{[1]})} c_{[1](1)} \otimes c_{[1](2)} \otimes c_{[2]}.$$

$$\sum_{[c]} c_{[1]}\varepsilon(c_{[2]}) = c$$

$$\sum_{[c],[c_{[1]}]} c_{[1][1]} \otimes c_{[1][2]} \otimes c_{[2]} = \sum_{[c],(c_{[2]})} c_{[1]} \otimes c_{[2](1)} \otimes c_{[2](2)}.$$

The coalgebra structure on $A\#^{\beta}C$ given by (2.12) and (2.13) is:

$$\Delta_{\#}(a \otimes c) = \sum_{(a),(c),[c_{(1)}]} a_{(1)} \otimes c_{(1)[1]} \otimes a_{(2)}c_{(1)[2]} \otimes c_{(2)}, \qquad \varepsilon_{\#} = \varepsilon_A \otimes \varepsilon_C.$$

For $c \in C$ we will write

$$\tau(c) = \sum_{\{c\}} c_{\{1\}} \otimes c_{\{2\}}$$

In these terms the conditions (2.21)—(2.23) are: for $c \in C$:

$$\sum_{\{c\}} \varepsilon(c_{\{1\}})c_{\{2\}} = \sum_{\{c\}} c_{\{1\}}\varepsilon(c_{\{2\}}) = \varepsilon(c)1_A$$

$$\sum_{\substack{(c),\{c_{(1)}\},\\ \{c_{(2)}\},(c_{(1)\{2\}})}} c_{(1)\{1\}} \otimes c_{(1)\{2\}(1)}c_{(2)\{1\}} \otimes c_{(1)\{2\}(2)}c_{(2)\{2\}} =$$

$$\sum_{\substack{(c),\{c_{(1)}\},[c_{(2)}],\\ (c_{(1)\{2\}}),\{c_{(2)[1]}\}}} c_{(1)\{1\}(1)}c_{(2)[1]\{1\}} \otimes c_{(1)\{1\}(2)}c_{(2)[1]\{2\}} \otimes c_{(1)\{2\}}c_{(2)[2]}$$

$$\sum_{\substack{(c),\{c_{(1)}\},\\ [c_{(2)}],[c_{(2)[1]}]}} c_{(2)[1][1]} \otimes c_{(1)\{1\}}c_{(2)[1][2]} \otimes c_{(1)\{2\}}c_{(2)[2]} =$$

$$\sum_{\substack{(c),[c_{(1)}],\\ \{c_{(2)}\},(c_{(1)[2]})}} c_{(1)[1]} \otimes c_{(1)[2](1)}c_{(2)\{1\}} \otimes c_{(1)[2](2)}c_{(2)\{2\}}$$

The comultiplication (2.17) is now given by:

$$\Delta(a \otimes c) = \sum_{\substack{(a),(c),\\ \{c_{(1)}\},[c_{(2)}]}} a_{(1)}c_{(1)\{1\}} \otimes c_{(2)[1]} \otimes a_{(2)}c_{(1)\{2\}}c_{(2)[2]} \otimes c_{(3)}$$

Now we are ready to show that Δ is coassociative using the traditional notation; the steps are exactly the same as in the proof of coassociativity using diagrams.

$$((A \otimes C) \otimes \Delta) \circ \Delta(a \otimes c)$$

$$\equiv \sum_{\substack{(a),(c),\{c_{(1)}\},[c_{(2)}],\\ (a_{(2)}c_{(1)\{2\}}c_{(2)[2]}),\{c_{(3)}\},[c_{(4)}]}} a_{(1)}c_{(1)\{1\}} \otimes c_{(2)[1]} \otimes (a_{(2)}c_{(1)\{2\}}c_{(2)[2]})_{(1)}c_{(3)\{1\}} \otimes$$

$$c_{(4)[1]} \otimes (a_{(2)}c_{(1)\{2\}}c_{(2)[2]})_{(2)}c_{(3)\{2\}}c_{(4)[2]} \otimes c_{(5)}$$

$$= \sum_{\substack{(a),(c),\{c_{(1)}\},(c_{(1)\{2\}}),\\ [c_{(2)}],(c_{(2)[2]}),\{c_{(3)}\},[c_{(4)}]}} a_{(1)}c_{(1)\{1\}} \otimes c_{(2)[1]} \otimes a_{(2)}c_{(1)\{2\}(1)}c_{(2)[2](1)}c_{(3)\{1\}} \otimes$$

$$c_{(4)[1]} \otimes a_{(3)}c_{(1)\{2\}(2)}c_{(2)[2](2)}c_{(3)\{2\}}c_{(4)[2]} \otimes c_{(5)}$$

$$= \sum_{\substack{(a),(c),\{c_{(1)}\},(c_{(1)\{2\}}),\\ \{c_{(2)}\},[c_{(3)}],[c_{(3)[1]}],[c_{(4)}]}} a_{(1)}c_{(1)\{1\}} \otimes c_{(3)[1][1]} \otimes a_{(2)}c_{(1)\{2\}(1)}c_{(2)\{1\}}c_{(3)[1][2]} \otimes$$

$$c_{(4)[1]} \otimes a_{(3)}c_{(1)\{2\}(2)}c_{(2)\{2\}}c_{(3)[2]}c_{(4)[2]} \otimes c_{(5)}$$

$$= \sum_{\substack{(a),(c),\{c_{(1)}\},[c_{(2)}],[c_{(3)}],\\ [c_{(4)}],(c_{(1)\{1\}}),\{c_{(2)[1]}\},[c_{(3)[1]}]}} a_{(1)}c_{(1)\{1\}(1)}c_{(2)[1]\{1\}} \otimes c_{(3)[1][1]} \otimes a_{(2)}c_{(1)\{1\}(2)}c_{(2)[1]\{2\}}c_{(3)[1][2]} \otimes$$

$$c_{(4)[1]} \otimes a_{(3)}c_{(1)\{2\}}c_{(2)[2]}c_{(3)[2]}c_{(4)[2]} \otimes c_{(5)}$$

$$= \sum_{\substack{(a),(c),\{c_{(1)}\},[c_{(2)}],[c_{(3)}],(c_{(1)\{1\}}),\\ (c_{(2)[1]}),\{c_{(2)[1](1)}\}[c_{(2)[1](2)}]}} a_{(1)}c_{(1)\{1\}(1)}c_{(2)[1](1)\{1\}} \otimes c_{(2)[1](2)[1]} \otimes a_{(2)}c_{(1)\{1\}(2)}c_{(2)[1](1)\{2\}}c_{(2)[1](2)[2]} \otimes$$

$$c_{(3)[1]} \otimes a_{(3)}c_{(1)\{2\}}c_{(2)[2]}c_{(3)[2]} \otimes c_{(5)}$$

$$= \sum_{\substack{(a),(c),\{c_{(1)}\},[c_{(2)}],(c_{(1)\{1\}}),(c_{(2)[1]}),\\ (c_{(2)[1](1)})[c_{(2)[1](1)(1)}]\{c_{(2)[1](1)(1)}\}}} a_{(1)}c_{(1)\{1\}(1)}c_{(2)[1](1)(1)\{1\}} \otimes c_{(2)[1](1)(2)[1]} \otimes$$

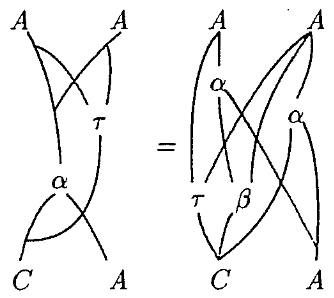
$$a_{(2)}c_{(1)\{1\}(2)}c_{(2)[1](1)(1)\{2\}}c_{(2)[1](1)(2)[2]} \otimes c_{(2)[1](2)} \otimes a_{(3)}c_{(1)\{2\}}c_{(2)[2]} \otimes c_{(5)}$$

$$\equiv (\Delta \otimes (A \otimes C)) \circ \Delta(a \otimes c). \quad \square$$

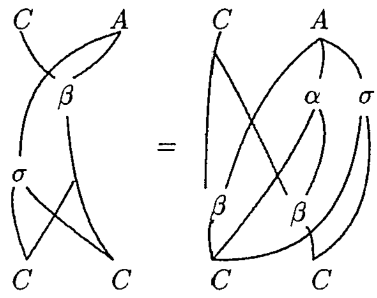
Suppose it happens that we have both $A\#_{\alpha,\sigma}C$ and $A\#^{\beta,\tau}C$; the next definition and theorem tell us when they are compatible, resulting in a bialgebra.

Definition 2.2.9 *Suppose A and C are bialgebras, $\alpha \in \mathsf{Vect}(C \otimes A, A)$, $\sigma \in \mathsf{Vect}(C \otimes C, A)$, $\beta \in \mathsf{Vect}(C, C \otimes A)$, and $\tau \in \mathsf{Vect}(C, A \otimes A)$. The space $A \otimes C$, together with the structures given by (2.16), (2.17), (2.11) and (2.13) will be denoted $A\#^{\beta,\tau}_{\alpha,\sigma}C$. If this is a bialgebra it will be called the* **crossed product bialgebra**.

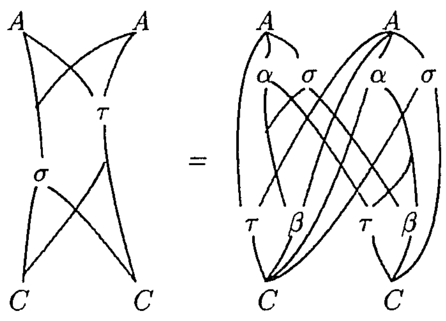
Theorem 2.2.10 *Suppose α satisfies (1.20), β satisfies (1.25), $A\#_{\alpha,\sigma}C$ is a crossed product algebra, and $A\#^{\beta,\tau}C$ is a crossed product coalgebra. Then $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a crossed product bialgebra if and only if (C, A, α, β) is a compatible measured pair and (2.24)—(2.28) below are satisfied.*



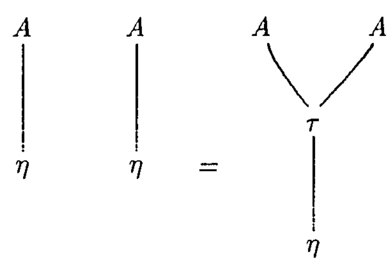
(2.24)



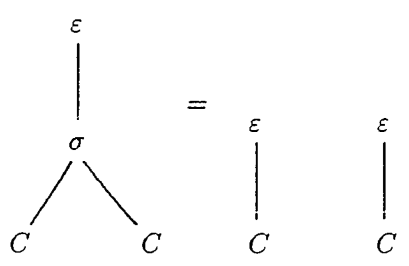
(2.25)



(2.26)

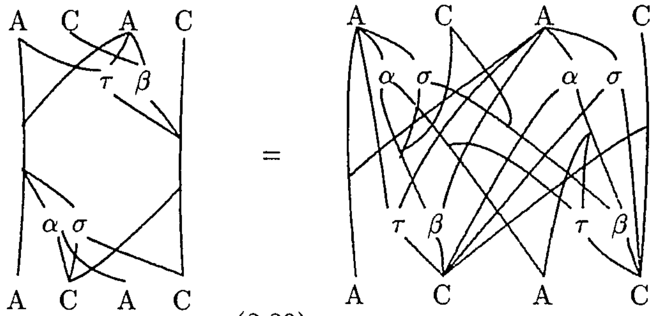


(2.27)



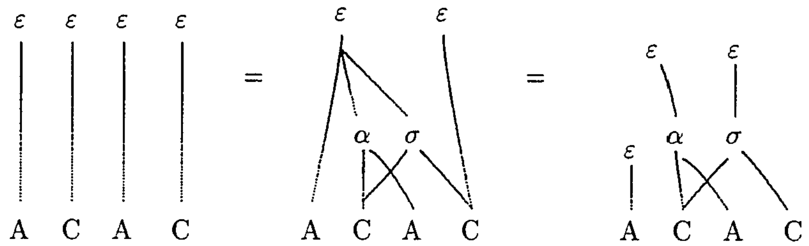
(2.28)

Proof. First we prove the necessity of the conditions. The assumption that the multiplication and comultiplication are compatible as in (1.14) is depicted in the following picture.



(2.29)

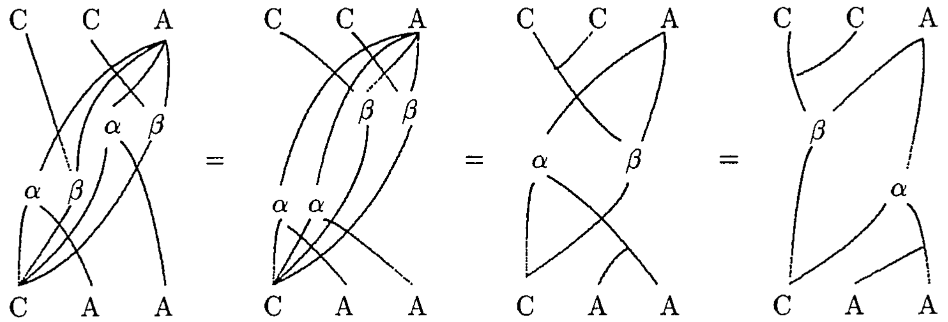
By preceding the equal maps of (2.29) by $\eta_A \otimes C \otimes A \otimes \eta_C$ and following them by $A \otimes \varepsilon_C \otimes A \otimes \varepsilon_C$ we have (2.24). If we precede (2.29) by $\eta_A \otimes C \otimes \eta_A \otimes C$ and follow it by $\varepsilon_A \otimes C \otimes A \otimes \varepsilon_C$ we have (2.25) and if we precede (2.29) by $\eta_A \otimes C \otimes \eta_A \otimes C$ and follow it by $A \otimes \varepsilon_C \otimes A \otimes \varepsilon_C$ we have (2.26). To see that we have a compatible measured pair (that is (2.3) is satisfied) precede (2.29) by $\eta_A \otimes C \otimes A \otimes \eta_C$ and follow it by $\varepsilon_A \otimes C \otimes A \otimes \varepsilon_C$. To see that condition (2.28) is necessary we make the following calculation.



(2.30)

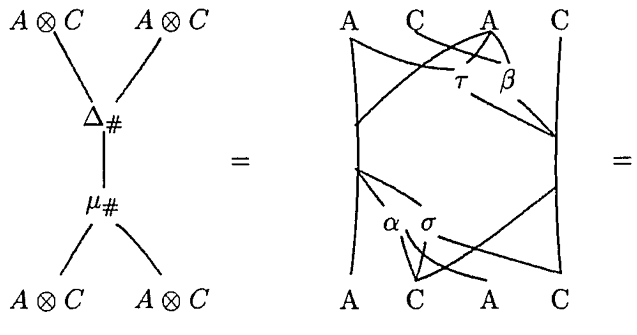
The first equality is the assumption that (1.12) holds for the crossed product and the second equality follows from the properties of ε. By preceding the equal maps on the two ends of (2.30) by $\eta_A \otimes C \otimes \eta_A \otimes C$ and using the fact that (1.20) holds we have (2.28). The necessity of (2.27) has a dual proof.

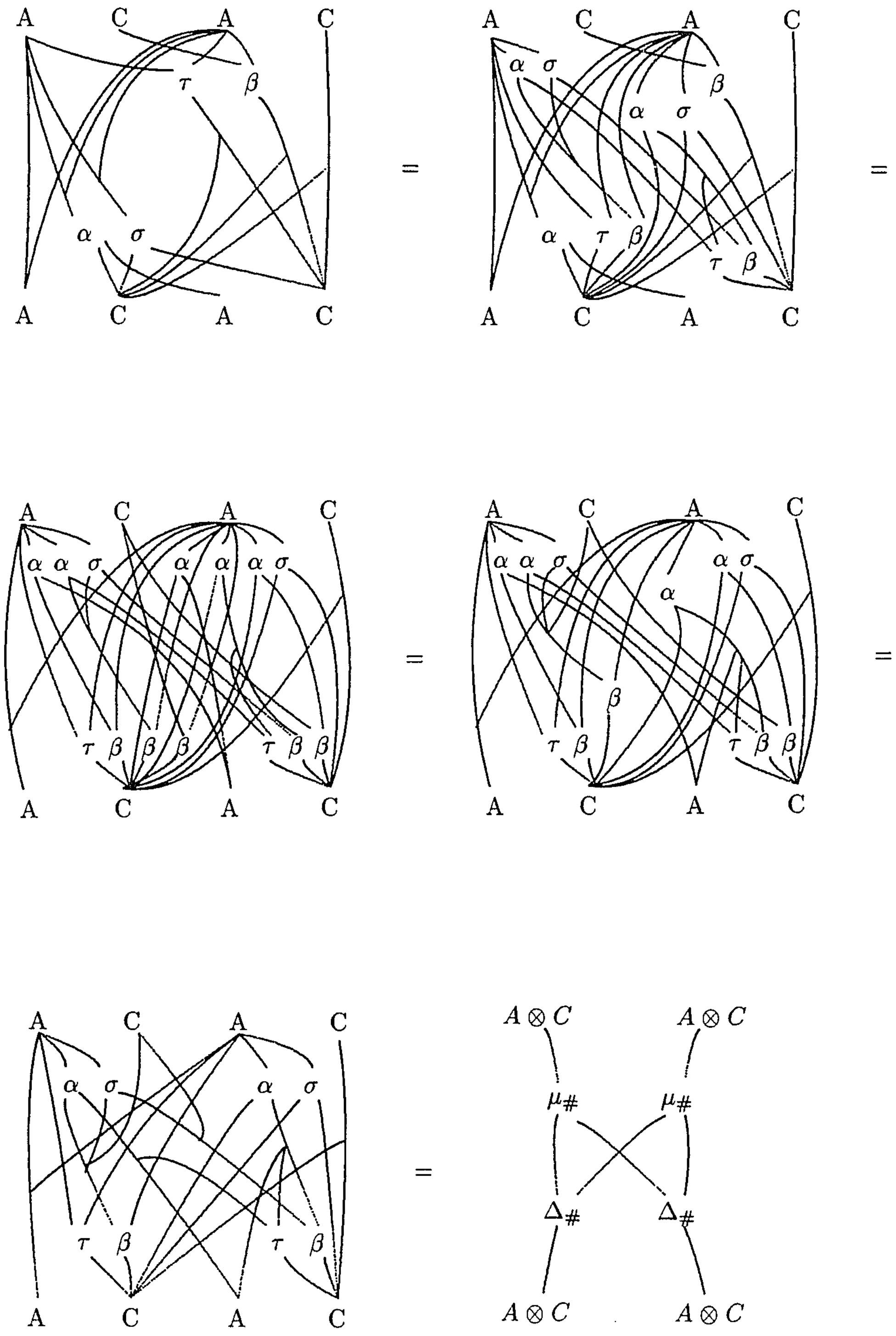
Conversely, suppose (C, A, α, β) is a compatible measured pair and the conditions (2.24)—(2.28) hold. The compatibility axioms (1.12), (1.13) and (1.15) follow easily from theorem 2.1.2. The next calculation will help us later.



(2.31)

The first and third equalities are had by applying (2.3). The second is a result of applying (1.21) and (1.26). Now we calculate the compatibility (1.14) of the two structures.

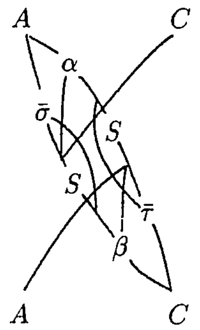



A C A C
α σ τ β
A C A C
=
A C A C
A C A C
=
A ⊗ C
μ#
Δ#

The first and last equalities are definitions. The second equality results from applications of (1.14) and the third is a result of applying (2.26). The fourth equality is an application of both (2.24) and (2.25). The fifth equality is the above calculation, (2.31), and the sixth equality is a result of (1.21) and (1.26), each applied twice. The other axioms of a bialgebra, (1.12), (1.13) and (1.15), follow easily from (2.27), (2.28) and the definitions of ε and η in the crossed product. □

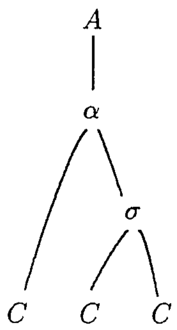
The next theorem shows that the crossed product bialgebra of two Hopf algebras is a Hopf algebra.

Theorem 2.2.11 *Suppose A and C are Hopf algebras, $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a crossed product bialgebra, and τ and σ are convolution invertible. Then $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a Hopf algebra with antipode given by (2.32).*

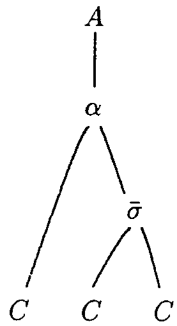


(2.32)

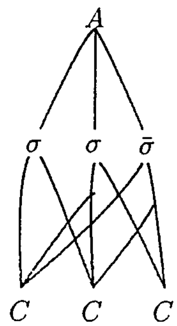
Proof. First we make an observation. Consider the four maps in the next diagram.



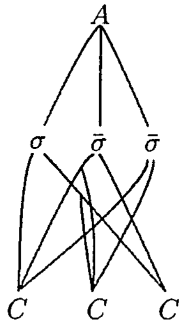
(2.33)



(2.34)

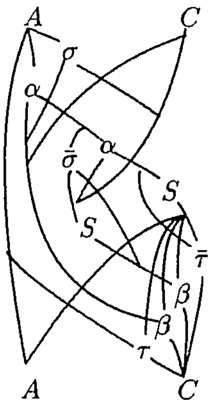


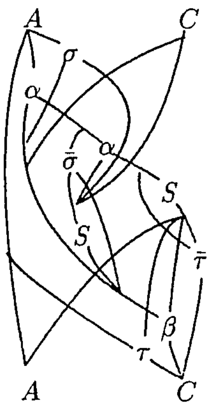
(2.35)

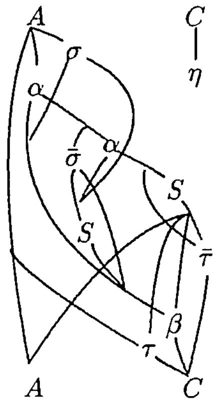


(2.36)

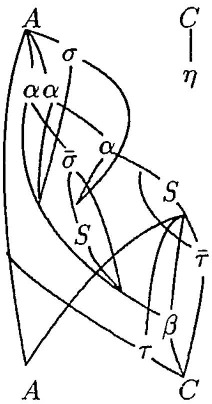
Using the fact that $\bar{\sigma}$ is the convolution inverse of σ and (2.19) we see that the map (2.33) is equal to (2.35). Moreover easy calculations show that (2.33) is the convolution inverse of (2.34) and that (2.35) is the convolution inverse of (2.36). We conclude that (2.34) is equal to (2.36). Now we calculate.

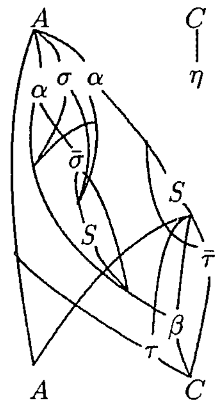
$id * S =$ 

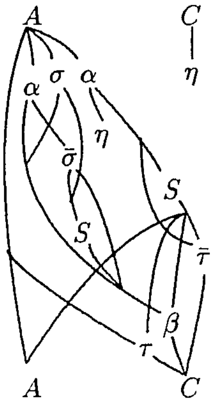
$=$ 

$=$ 

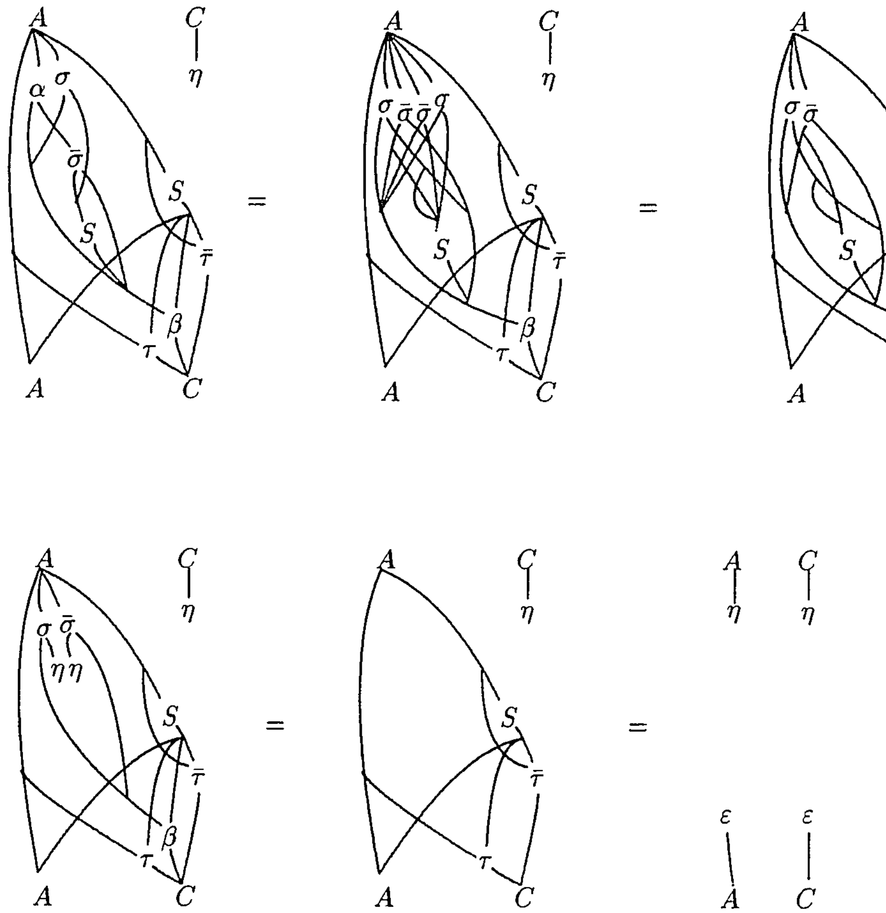
$=$



$=$ 

$=$ 

$=$



$$= \eta_{A\otimes C} \circ \varepsilon_{A\otimes C}$$

The first and last equalities are definitions. The second equality is a result of (1.26) and the third, sixth and tenth equalities are a result of (1.16) and the fact that S is an antialgebra map. The fourth equality follows from (1.21) and the fifth is a result of (2.20). The seventh equality follows from (1.17) and the eighth is the above equality of (2.34) and (2.36). The ninth equality is a result of the fact that σ and $\bar{\sigma}$ are inverses. The eleventh equality uses (2.18) and (1.25) and the twelfth equality follows from (1.16) and (2.21). Thus S is a right inverse for the identity, the proof that S is a left inverse is similar. □

Examples 2.2.12

1. *[Mont93] Suppose* A *is an algebra,* C *is a bialgebra and* $\alpha \in \mathsf{Vect}(C \otimes A, A)$*. Define* $\sigma^{\mathrm{t}} \in \mathsf{Vect}(C \otimes C, A)$ *by* $\sigma^{\mathrm{t}} = \eta_A \circ (\varepsilon_C \otimes \varepsilon_C)$*. Then* $A\#_{\alpha,\sigma^{\mathrm{t}}}C$ *is a crossed product algebra if and only if* (A, α) *is a* C*-module algebra. This follows from the fact that in this case (2.19) reduces to (1.18). Moreover it is easy to see that* $A\#_{\alpha,\sigma^{\mathrm{t}}}C = A\#_{\alpha}C$*, the smash product algebra.*

2. *Dually, suppose* A *is a bialgebra,* C *is a coalgebra and* $\beta \in \mathsf{Vect}(C, C \otimes A)$*. Define* $\tau^{\mathrm{t}} \in \mathsf{Vect}(C, A \otimes A)$ *by* $\tau^{\mathrm{t}} = (\eta_A \otimes \eta_A) \circ \varepsilon_C$*. Then* $A\#^{\beta,\tau^{\mathrm{t}}}C$ *is a crossed product coalgebra if and only if* (C, β) *is a* A*-comodule coalgebra. This follows from the fact that in this case (2.23) reduces to (1.23). Moreover it is easy to see that* $A\#^{\beta,\tau^{\mathrm{t}}}C = A\#^{\beta}C$*, the smash product coalgebra.*

3. *[Mont93] Suppose* A *is an algebra,* C *is a bialgebra and* $\sigma \in \mathsf{Vect}(C \otimes C, A)$*. Define* $\alpha^{\mathrm{t}} \in \mathsf{Vect}(C \otimes A, A)$ *by* $\alpha^{\mathrm{t}} = (\varepsilon_C \otimes A)$*. Then* (A, α^{t}) *is a* C*-module algebra and in the case that* $A\#_{\alpha^{\mathrm{t}},\sigma}C$ *is an algebra it is known as the twisted algebra.*

4. *Dually, suppose* A *is a bialgebra,* C *is a coalgebra and* $\tau \in \mathsf{Vect}(C, A \otimes A)$*. Define* $\beta^{\mathrm{t}} \in \mathsf{Vect}(C, C \otimes A)$ *by* $\beta^{\mathrm{t}} = (C \otimes \eta_A)$*. Then* (C, β^{t}) *is an* A*-comodule coalgebra and in the case that* $A\#^{\beta^{\mathrm{t}},\tau}C$ *is a coalgebra we will call it the twisted coalgebra.*

5. *Suppose* A *and* C *are bialgebras,* $\alpha \in \mathsf{Vect}(C \otimes A, A)$*, and* $\beta \in \mathsf{Vect}(C, C \otimes A)$*. Then* $A\#^{\beta,\tau^{\mathrm{t}}}_{\alpha,\sigma^{\mathrm{t}}}C$ *is a crossed product bialgebra if and only if* (A, C, α, β) *is a compatible matched pair. This follows from the fact that in this case (2.24) reduces to (2.1) and (2.25) reduces to (2.2). Moreover it is easy to see that* $A\#^{\beta,\tau^{\mathrm{t}}}_{\alpha,\sigma^{\mathrm{t}}}C = A\#^{\beta}_{\alpha}C$*, the smash product bialgebra. If* A *and* C *are Hopf algebras then, since* τ^{t} *and* σ^{t} *are invertible,* $A\#^{\beta,\tau^{\mathrm{t}}}_{\alpha,\sigma^{\mathrm{t}}}C$ *is a Hopf algebra and the antipode given in (2.32) reduces to the antipode given in (2.14).*

6. *Suppose A and C are bialgebras, $\sigma \in \mathsf{Vect}(C \otimes C, A)$, and $\tau \in \mathsf{Vect}(C, A \otimes A)$. Then $(A, C, \alpha^{\mathrm{t}}, \beta^{\mathrm{t}})$ is a compatible matched pair and in the case that $A\#^{\beta^{\mathrm{t}},\tau}_{\alpha^{\mathrm{t}},\sigma}C$ is a bialgebra we will call it the twisted bialgebra.*

7. *Suppose A and C are bialgebras. Then $A\#^{\beta^{\mathrm{t}},\tau^{\mathrm{t}}}_{\alpha^{\mathrm{t}},\sigma^{\mathrm{t}}}C = A \otimes C$, the tensor product bialgebra of example 1.3.9.1.*

8. *Suppose A and C are Hopf algebras then $A\#^{\beta^{\mathrm{t}},\tau^{\mathrm{t}}}_{\alpha,\sigma^{\mathrm{t}}}C$ is a Hopf algebra if and only if (A, α) is a C-module algebra and a C-module coalgebra. This follows from the fact that in this case (2.24) reduces to (1.28). This situation is studied in [Mol75].*

9. *Suppose A and C are Hopf algebras then $A\#^{\beta^{\mathrm{t}},\tau^{\mathrm{t}}}_{\alpha,\sigma}C$ is a Hopf algebra if and only if it is an algebra, (A, α) is a C-module coalgebra, σ commutes with μ_C in the convolution algebra, and σ is a coalgebra map. The last three conditions follow from (2.24), (2.25), (2.26) and (2.28).*

10. *Suppose G is a group and N is a normal subgroup of G. Define $Ad \in \mathsf{Vect}(\mathbf{k}G \otimes \mathbf{k}N, \mathbf{k}N)$ by $Ad(g \otimes n) = n^g$ $(= gng^{-1})$ for all $g \in G$ and $n \in N$. Then $(\mathbf{k}N, Ad)$ is a $\mathbf{k}G$-module algebra and a $\mathbf{k}G$-module coalgebra. Thus $\mathbf{k}N\#^{\beta^{\mathrm{t}},\tau^{\mathrm{t}}}_{Ad,\sigma^{\mathrm{t}}}\mathbf{k}G$ is a Hopf algebra as in example 8. When G is finite and $N = G$ it is shown in [Maj90] that this is equivalent to the "quantum double" of $\mathbf{k}G$; this is an important example of a non-cocommutative Hopf algebra.*

11. *Suppose G is a finite group and A is an abelian normal subgroup of G. Suppose $s \in \mathsf{Set}(G \times G, A)$ is a "factor set" for conjugation, that is: $s(h,k)^g s(g,hk) = s(g,h)s(gh,k)$ for all $g,h,k \in G$. Let $\sigma \in \mathsf{Vect}(\mathbf{k}G \otimes \mathbf{k}G, \mathbf{k}A)$ be the linear extension of s. Then $\mathbf{k}A\#^{\beta^{\mathrm{t}},\tau^{\mathrm{t}}}_{Ad,\sigma}\mathbf{k}G$ is a Hopf algebra. The condition that A is abelian can be weakened to the condition that α and σ satisfy (2.20).*

Chapter 3

The Middle

In this chapter we define the middle, a bi-cosimplicial algebra associated with a given compatible matched pair. By imitating the constructions of homology on a complex we recapture the conditions required to form the crossed product bialgebra of theorem 2.2.10. The first section is a review of the construction of two cosimplicial spaces on $\mathsf{Vect}(C^n, A^m)$. In the second section we show that these cosimplicial spaces are actually cosimplicial algebras. The middle, a subalgebra of $\mathsf{Vect}(C^n, A^m)$, is defined in the third section and in the fourth section we show that the middle is actually a cosimplicial algebra. Finally, in the last section of this chapter we describe the connection between the middle and crossed product bialgebras.

3.1 Review

In this section we review the construction of two cosimplicial spaces on $\mathsf{Vect}(C^n, A^m)$ that are the basis for our further constructions.

Suppose (C, μ, η) is an algebra; then C is a C-module by μ and $\mathbf{k}$ is a C-module by η. If V is any space then we have a functor $F : \mathsf{Vect} \longrightarrow \mathsf{Cmod}$ given by $F(V) = (C \otimes V, \mu \otimes V)$. The functor F is a left adjoint of the underlying functor U given by $U(M, \alpha) = M$. This can be seen by the natural isomorphisms:

$$\mathsf{Cmod}(C \otimes V, M) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{Vect}(V, UM)$$

(3.1)

where, for $g \in \mathsf{Cmod}(C \otimes V, M)$ and $f \in \mathsf{Vect}(V, UM)$, $\Phi(g) = Ug \circ (\eta_C \otimes V)$ and $\Psi(f) = \alpha \circ (C \otimes f)$.

Thus we have a cotriple on **Cmod**, (FU, ϵ, δ) where $\epsilon_{(M,\phi)} = \phi$ and $\delta_{(M,\phi)} = C \otimes \eta_C \otimes M$. This gives rise to a simplicial object in **Cmod** as described in [Mac71], explicitly:

$$C \otimes M \;\; \underset{s_0^1}{\overset{d_i^1}{\leftleftarrows\!\!\rightarrow}} \;\; C \otimes C \otimes M \;\; \underset{s_j^2}{\overset{d_i^2}{\leftleftarrows\!\!\leftarrow\!\!\rightrightarrows}} \;\; C \otimes C \otimes C \otimes M \qquad \cdots$$

(3.2)

where

$$d_i^n = C^i \otimes \mu_C \otimes C^{n-1-i} \otimes M \quad \text{if} \quad 0 \leq i \leq n-1$$

$$d_n^n = C^n \otimes \phi \quad \text{and}$$

$$s_j^n = C^{j+1} \otimes \eta_C \otimes C^{n-j-1} \otimes M \;\; \text{for} \;\; 0 \leq j \leq n-1.$$

Suppose (A, α) is a C-module. If the contravariant functor $\mathsf{Cmod}(\_\,, \mathrm{A})$ is applied to (3.2) the result is the cosimplicial space at the top of (3.3). By passing through the natural isomorphisms of the adjoint pair (F, U) we have the cosimplicial space at the bottom of (3.3).

$$\begin{array}{ccccccc}
\mathsf{Cmod}(C\otimes M,A) & \overset{d_i^1*}{\underset{s_0^1*}{\rightrightarrows \atop \leftarrow}} & \mathsf{Cmod}(C^2\otimes M,A) & \overset{d_i^2*}{\underset{s_j^2*}{\rightrightarrows\!\rightarrow \atop \leftleftarrows}} & \mathsf{Cmod}(C^3\otimes M,A) & & \cdots \\
\Psi \uparrow \downarrow \Phi & & \Psi \uparrow \downarrow \Phi & & \Psi \uparrow \downarrow \Phi & & \cdots \\
\mathsf{Vect}(M,UA) & \overset{u_i^1}{\underset{v_0^1}{\rightrightarrows \atop \leftarrow}} & \mathsf{Vect}(C\otimes M,UA) & \overset{u_i^2}{\underset{v_j^2}{\rightrightarrows\!\rightarrow \atop \leftleftarrows}} & \mathsf{Vect}(C^2\otimes M,UA) & & \cdots
\end{array}$$

(3.3)

Explicitly, for each $n \in \mathrm{N}$, $n > 0$ the cofaces and codegeneracies are given, for $g \in \mathsf{Vect}(C^{n-1} \otimes M, UA)$ and $f \in \mathsf{Vect}(C^n \otimes M, UA)$ by:

$$u_0^n(g) = \alpha \circ (C \otimes g),$$

$$u_n^n = g \circ (C^{n-1} \otimes \phi),$$

$$u_i^n = g \circ (C^{i-1} \otimes \mu_C \otimes C^{n-1-i} \otimes M) \quad \text{for} \quad 0 < i < n \quad \text{and}$$

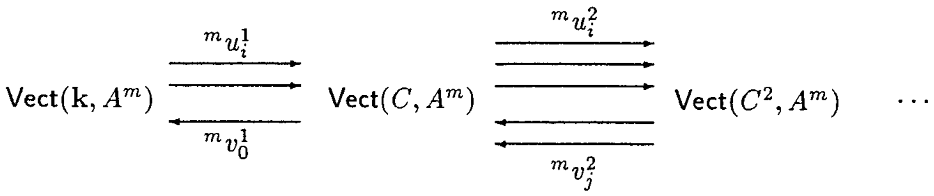
$$v_j^n = f \circ (C^j \otimes \eta_C \otimes C^{n-1-j} \otimes M) \quad \text{for} \quad 0 \leq j \leq n.$$

Now, by taking $M = \mathbf{k}$ in the cosimplicial space at the bottom of (3.3), we have (3.4).

$$\begin{array}{ccccccc}
\mathsf{Vect}(\mathbf{k},A) & \overset{u_i^1}{\underset{v_0^1}{\rightrightarrows \atop \leftarrow}} & \mathsf{Vect}(C,A) & \overset{u_i^2}{\underset{v_j^2}{\rightrightarrows\!\rightarrow \atop \leftleftarrows}} & \mathsf{Vect}(C^2,A) & & \cdots
\end{array}$$

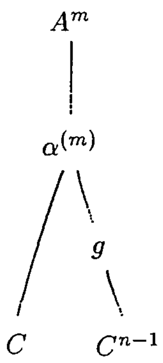
(3.4)

If (C, A, α, β) is a compatible matched pair then for each $m \in \mathrm{N}$, A^m is a C-module as in (2.1.4). Hence we can perform the above construction on A^m and we have, for each $m \in \mathrm{N}$, the cosimplicial space given in (3.5) below.

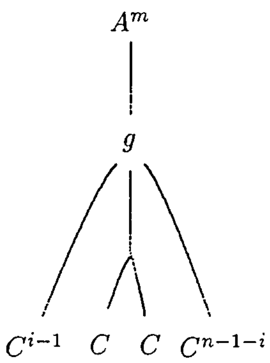


(3.5)

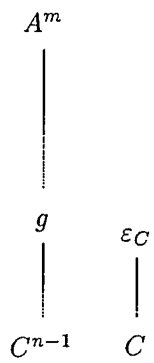
We now describe the cofaces, $^m u_i^n$, and the codegeneracies, $^m v_j^n$, of (3.5) in terms of tensor diagrams. Suppose $m, n \in \mathrm{N}$, $n > 0$ and $g \in \mathsf{Vect}(C^{n-1}, A^m)$ then $^m u_0^n(g)$ is given by (3.6) below and $^m u_n^n(g)$ is given by (3.8), (recalling that $\mathbf{k}$ is a C-module by ε_C). If $n > 1$ and $0 < i < n$ then $^m u_i^n(g)$ is given by (3.7). If $h \in \mathsf{Vect}(C^n, A^m)$ then for $0 \leq j < n$, $^m v_j^n(g)$ is given by (3.9).



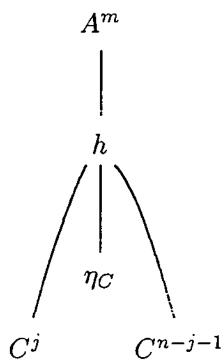
(3.6)



(3.7)

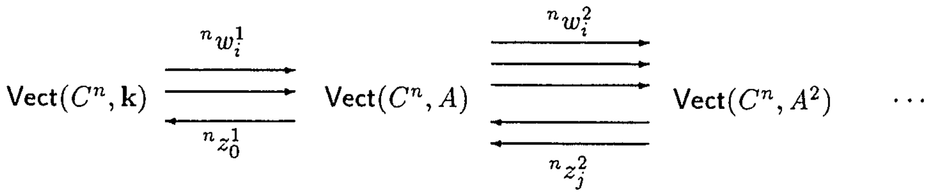


(3.8)



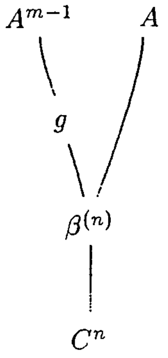
(3.9)

The above discussion can be dualized, the result being that given a compatible matched pair (C, A, α, β) and $n \in \mathrm{N}$ we have a cosimplicial space given by (3.10).

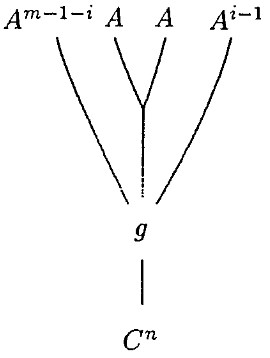


(3.10)

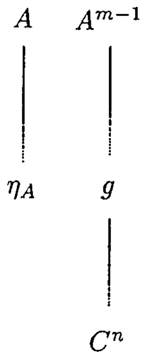
The cofaces ${}^n w_i^m$ and the codegeneracies ${}^n z_i^m$ of (3.10) are now given in terms of tensor diagrams. Suppose $n, m \in \mathrm{N}$, $m > 0$ and $g \in \mathsf{Vect}(C^n, A^{m-1})$. Then ${}^n w_0^m(g)$ is given by (3.11) below and ${}^n w_m^m(g)$ is given by (3.13). If $m > 1$ and $0 < i < m$ then ${}^n w_i^m(g)$ is as in (3.12). If $h \in \mathsf{Vect}(C^n, A^m)$ then, for $0 \leq j \leq m-1$, ${}^n z_j^m(h)$ is given by (3.14).



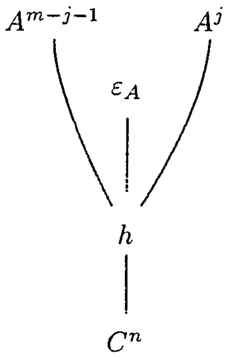
(3.11)



(3.12)



(3.13)



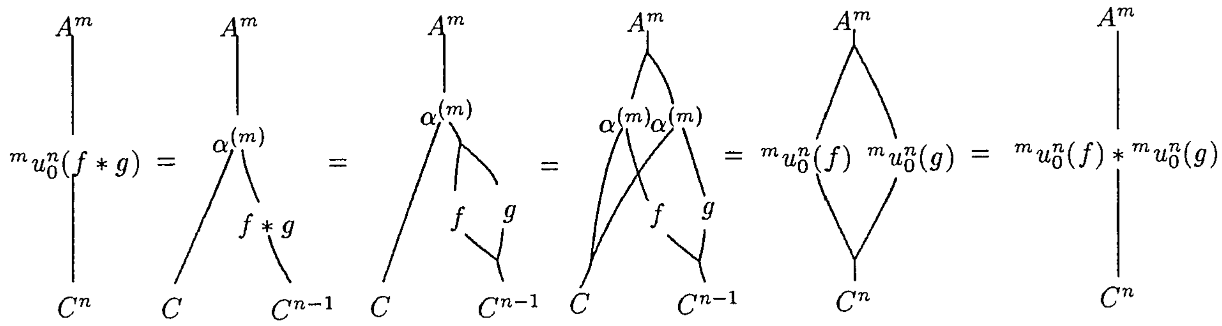
(3.14)

3.2 $\mathsf{Vect}(C^*, A^m)$ as a Cosimplicial Algebra

In this section we show that the cofaces and codegeneracies of (3.5) and (3.10) are algebra maps. Thus these cosimplicial spaces are in fact cosimplicial algebras.

Lemma 3.2.1 *Suppose (C, A, α, β) is a compatible matched pair and $n, m \in \mathrm{N}$. Then ${}^m u_0^n$ of (3.5) is an algebra map.*

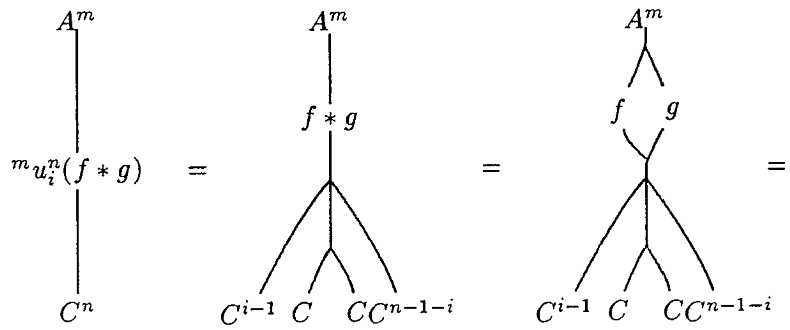
Proof. Suppose $f, g \in \mathsf{Vect}(C^{n-1}, A^m)$. We need to show that ${}^m u_0^n(f * g) = {}^m u_0^n(f) * {}^m u_0^n(g)$, so we calculate.

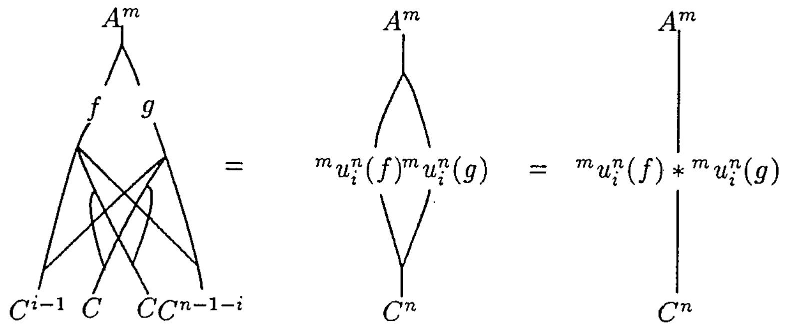


The first and fourth equalities follow from (3.6). The second and last equalities follow from the definition of $*$; (1.11). The third equality follows from (1.21). The fact that ${}^m u_0^n$ preserves the unit follows easily from (1.20) and (1.25). □

Lemma 3.2.2 *Suppose (C, A, α, β) is a compatible matched pair, $n, m, i \in \mathbb{N}, n > 1$ and $0 < i < n$. Then ${}^m u_i^n$ of (3.5) is an algebra map.*

Proof. Suppose $f, g \in \mathsf{Vect}(C^{n-1}, A^m)$. We need to show that ${}^m u_i^n(f * g) = {}^m u_i^n(f) * {}^m u_i^n(g)$, so we calculate.

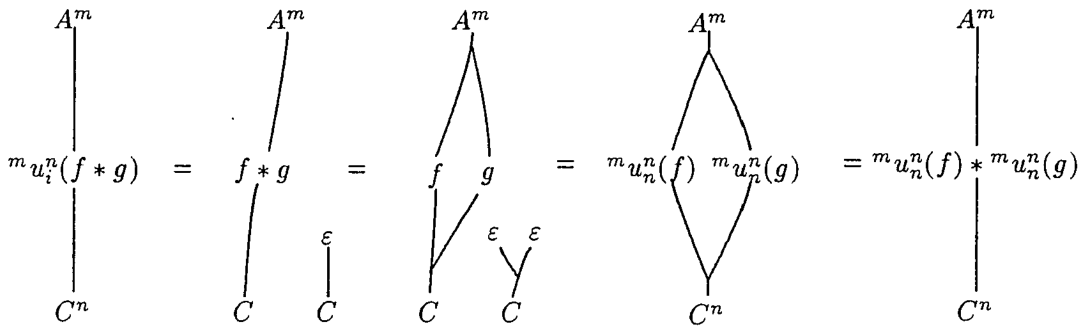




The first and fourth equalities follow from (3.7). The second and last equalities follow from (1.11). The third equality follows from (1.14). The fact that $^m u_i^n$ preserves the unit follows easily from (1.12). □

Lemma 3.2.3 *Suppose* (C, A, α, β) *is a compatible matched pair,* $n, m \in \mathbf{N}$, *and* $n > 0$. *Then* $^m u_n^n$ *of (3.5) is an algebra map.*

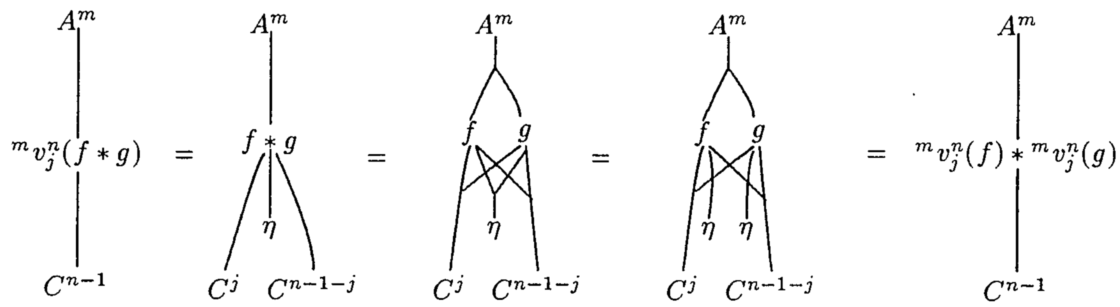
Proof. Suppose $f, g \in \mathsf{Vect}(C^{n-1}, A^m)$. We need to show that $^m u_n^n(f * g) = {}^m u_n^n(f) * {}^m u_n^n(g)$, so we calculate.



The first and third equalities follow from (3.8). The second equality follows from the definition of $*$;that is (1.11), and from (1.7). The last equality follows from the definition of $*$. The fact that $^m u_n^n$ preserves the unit follows immediately. □

Lemma 3.2.4 *Suppose (C, A, α, β) is a compatible matched pair, $n, m, j \in \mathbb{N}$, and $0 \le j < n$. Then ${}^m v_j^n$ of (3.5) is an algebra map.*

Proof. Suppose $f, g \in \mathsf{Vect}(C^n, A^m)$. We want to show that ${}^m v_j^n(f * g) = {}^m v_j^n(f) * {}^m v_j^n(g)$, so we calculate.



The first equality follows from (3.9) and the second equality follows from (1.11). The third equality follows from (1.13) and the last equality follows from (1.11) and (3.9). The fact that ${}^m v_j^n$ preserves the unit follows from (1.15). □

Theorem 3.2.5 *The cosimplicial space of (3.5) is a cosimplicial algebra.*

Proof. This is the content of the last four lemmas. □

Similarly we have the dual of the last theorem.

Theorem 3.2.6 *The cosimplicial space of (3.10) is a cosimplicial algebra.*

Proof. Dualize the last four lemmas. □

3.3 The Middle

In this section we construct a subalgebra of $\mathsf{Vect}(C^n, A^m)$ which we will call the middle. In the general case the algebra $\mathsf{Vect}(C^n, A^m)$ may not be commutative but the middle will give us enough commutativity to perform the constructions we need.

Definition 3.3.1 *Suppose C is an algebra and (A,α) is a C-module. Define*

$$\alpha^{<0>} = A, \quad \alpha^{<1>} = \alpha \quad \textit{and for} \quad n > 1 \quad , \quad \alpha^{<n>} = \alpha \circ (C \otimes \alpha^{<n-1>}).$$

Dually, for a coalgebra A and an A-comodule (C,β) we define

$$\beta^{<0>} = C, \quad \beta^{<1>} = \beta \quad \textit{and for} \quad n > 1, \quad \beta^{<n>} = (\beta^{<n-1>} \otimes A) \circ \beta.$$

Note that $\alpha^{<n>}$ should not be confused with $\alpha^{(n)}$ of definition 2.1.4, and observe that by (1.18), $\alpha^{<n>} = \alpha \circ (\mu^{(n-1)} \otimes A)$ and that by (1.23), $\beta^{<n>} = (C \otimes \Delta^{(n-1)}) \circ \beta$ for $n \geq 1$.

Definition 3.3.2 *Suppose (A,α) is a C-module, (C,β) is an A-comodule and $m,n \in \mathbf{N}$. Define $P^{n,m} \in \mathsf{Vect}(C^n \otimes A, A^m)$ and $Q^{n,m} \in \mathsf{Vect}(C^n, C \otimes A^m)$ by:*

$$P^{n,m} = \Delta_A^{(m-1)} \circ \alpha^{<n>} \quad \textit{and} \quad Q^{n,m} = \beta^{<m>} \circ \mu_C^{(n-1)}.$$

Now suppose $f \in \mathsf{Vect}(C^n, A^m)$. Define $f^{(n)} \in \mathsf{Vect}(C^n \otimes A, A^m)$ and ${}^{(m)}f \in \mathsf{Vect}(C^n, C \otimes A^m)$ by:

$$f^{(n)} = f \circ (C^n \otimes \varepsilon_A) \quad \textit{and} \quad {}^{(m)}f = (\eta_C \otimes A^m) \circ f.$$

Notice that if $f \in \mathsf{Vect}(C^n, A^m)$ then $f^{(n)}$ and $P^{n,m}$ are both in $\mathsf{Vect}(C^n \otimes A, A^m)$ and ${}^{(m)}f$ and $Q^{n,m}$ are both in $\mathsf{Vect}(C^n, C \otimes A^m)$.

Definition 3.3.3 *Suppose (A,α) is a C-module, (C,β) is an A-comodule and $m,n \in \mathbf{N}$. We now define two subsets of the convolution algebra $\mathsf{Vect}(C^n, A^m)$.*

$$\mathsf{Vect}^*(C^n, A^m) = \{f \in \mathsf{Vect}(C^n, A^m) \mid P^{n,m} * f^{(n)} = f^{(n)} * P^{n,m}\} \quad \textit{and}$$

$$^*\mathsf{Vect}(C^n, A^m) = \{f \in \mathsf{Vect}(C^n, A^m) \mid Q^{n,m} * {}^{(m)}f = {}^{(m)}f * Q^{n,m}\}.$$

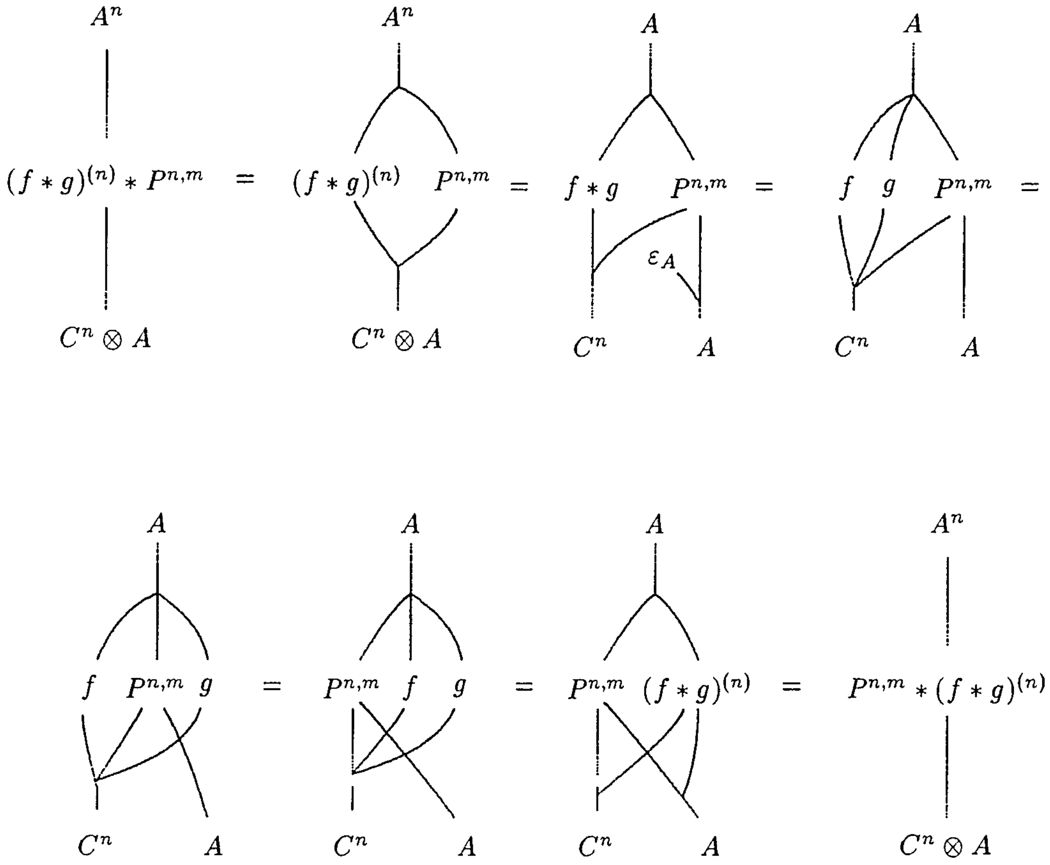
Notice that if C is cocommutative and A is commutative in definition 3.3.3 then $\mathsf{Vect}^*(C^n, A^m) = {}^*\mathsf{Vect}(C^n, A^m) = \mathsf{Vect}(C^n, A^m)$. That is, we are generalizing the abelian case.

Theorem 3.3.4 *Suppose (A, α) is a C-module, (C, β) is an A-comodule and $m, n \in \mathbb{N}$. Then $\mathsf{Vect}^*(C^n, A^m)$ and ${}^*\mathsf{Vect}(C^n, A^m)$ are subalgebras of $\mathsf{Vect}(C^n, A^m)$.*

Proof. First, the fact that $\mathsf{Vect}^*(C^n, A^m)$ and ${}^*\mathsf{Vect}(C^n, A^m)$ are subspaces follows from elementary properties of the spaces involved. For instance if $f, g \in \mathsf{Vect}^*(C^n, A^m)$ then $f + g \in \mathsf{Vect}^*(C^n, A^m)$ by the following calculation.

$$\begin{aligned}
(f+g)^{(n)} * P^{(n,m)} &= (f+g) \circ (C^n \otimes \varepsilon_A) * P^{(n,m)} \\
&= ((f \circ (C^n \otimes \varepsilon_A)) + (g \circ (C^n \otimes \varepsilon_A))) * P^{(n,m)} \\
&= ((f \circ (C^n \otimes \varepsilon_A)) * P^{(n,m)}) + ((g \circ (C^n \otimes \varepsilon_A)) * P^{(n,m)}) \\
&= (P^{(n,m)} * (f \circ (C^n \otimes \varepsilon_A))) + (P^{(n,m)} * (g \circ (C^n \otimes \varepsilon_A))) \\
&= P^{(n,m)} * (f \circ (C^n \otimes \varepsilon_A) + (g \circ (C^n \otimes \varepsilon_A)) \\
&= P^{(n,m)} * ((f+g)^{(n)})
\end{aligned}$$

Calculating with tensor diagrams makes it easy to see that $\mathsf{Vect}^*(C^n, A^m)$ and ${}^*\mathsf{Vect}(C^n, A^m)$ are subalgebras. For instance, if $f, g \in \mathsf{Vect}^*(C^n, A^m)$ then $f * g \in \mathsf{Vect}^*(C^n, A^m)$ by the following calculation.



The fourth equality is the fact that $g \in \mathsf{Vect}^*(C^n, A^m)$ and the fifth equality is the fact that $f \in \mathsf{Vect}^*(C^n, A^m)$. The other equalities are definitions. □

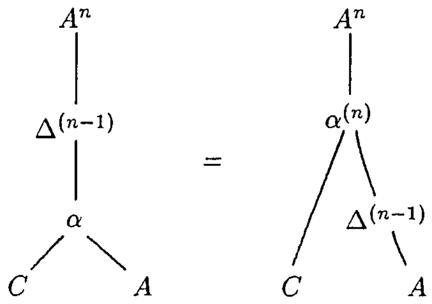
Definition 3.3.5 *Suppose* (A, α) *is a* C*-module,* (C, β) *is an* A*-comodule, and* $m, n \in \mathbf{N}$*. The* **middle** *of* $\mathsf{Vect}(C^n, A^m)$*, denoted by* $\mathrm{Mid}(C^n, A^m)$*, is defined as:*

$$\mathrm{Mid}(C^n, A^m) = \mathsf{Vect}^*(C^n, A^m) \cap {}^*\mathsf{Vect}(C^n, A^m).$$

3.4 $\mathrm{Mid}(C^*, A^m)$ as a Cosimplicial Algebra

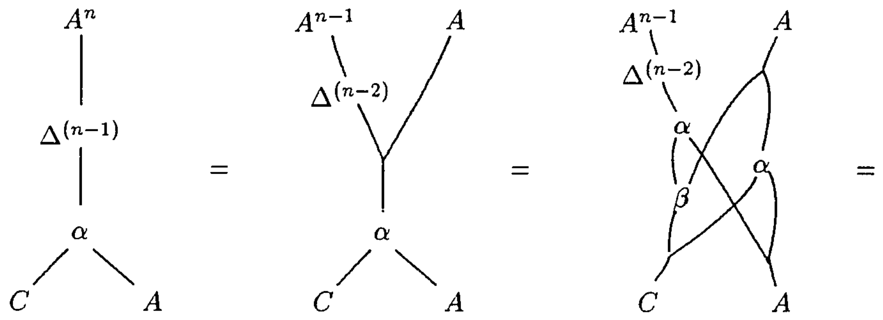
In this section we show that the cofaces and codegeneracies in (3.5) and (3.10) restrict to the middles. As a result the appropriate middles form cosimplicial algebras. We begin with a few lemmas.

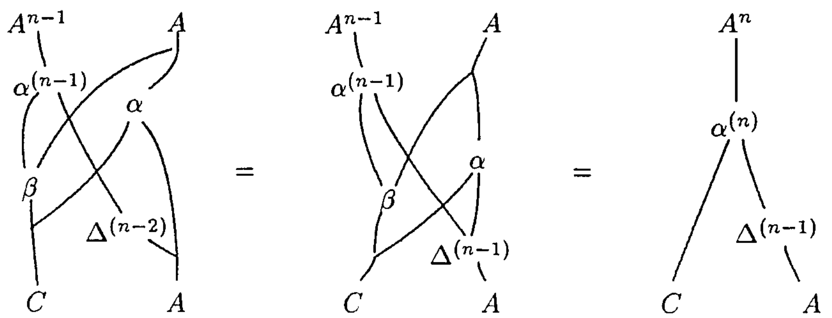
Lemma 3.4.1 *Suppose (C, A, α, β) is a compatible matched pair. Then for every $n \in \mathbb{N}$, the following equality holds.*



(3.15)

Proof. The proof is inductive, the case $n = 0$ is theorem 2.1.2 and the case $n = 1$ is trivial. If $n > 1$ then we have the following.

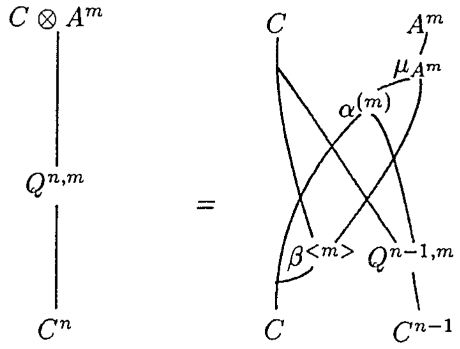




The first equality follows from (1.6) and the second equality is (2.1). The third equality is the inductive hypothesis. The fourth equality follows from (1.6) and the last equality is definition (2.7). □

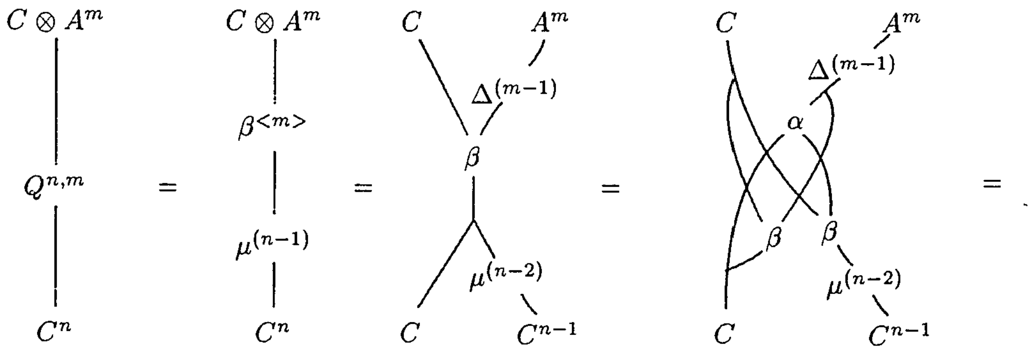
The next lemma gives us $Q^{n,m}$ in an inductive form.

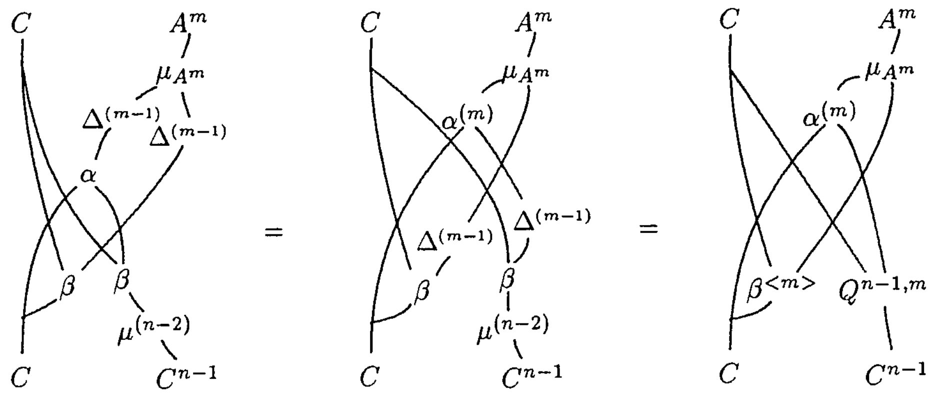
Lemma 3.4.2 *Suppose* (C, A, α, β) *is a compatible matched pair,* $m, n \in \mathbf{N}$ *and* $n > 0$. *Then the following equality holds.*



(3.16)

Proof.

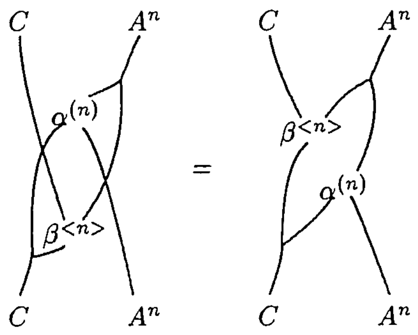




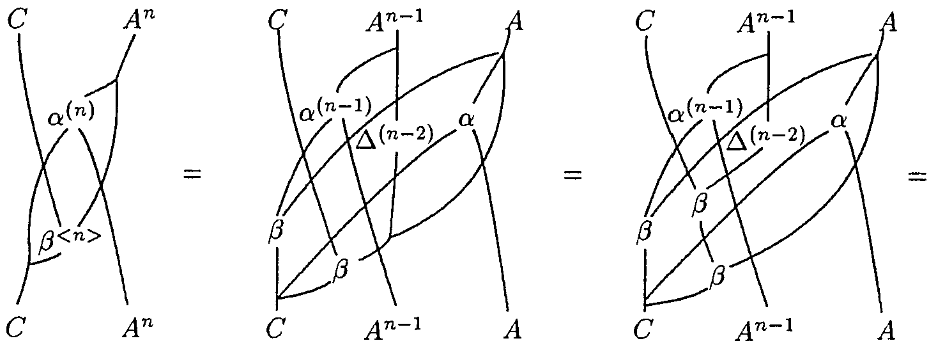
The first equality is definition 3.3.2 and the second is definition 3.3.1 and (1.1). The third equality follows from (2.2) and (2.3). The fourth equality is a multiple application of (1.14) and the fifth is lemma 3.4.1. The last equality follows from the definitions 3.3.1 and 3.3.2. □

The next lemma generalizes (2.3).

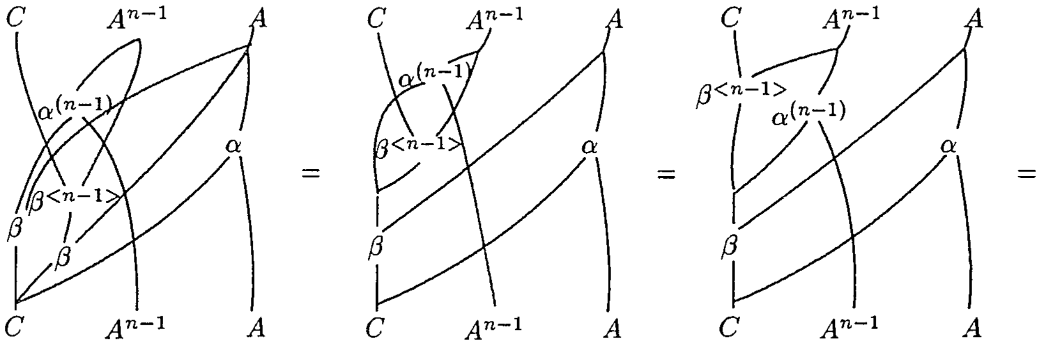
Lemma 3.4.3 *Suppose* (C, A, α, β) *is a compatible matched pair. Then for every* $n \in \mathrm{N}$ *we have the following equality.*



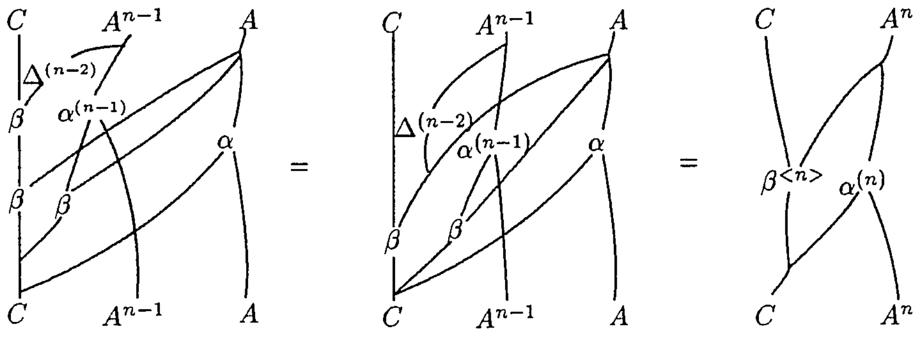
Proof. The proof is inductive. For $n = 0$ the claim is trivial and for $n = 1$ the claim is (2.3). If $n > 1$ then we have,



C
A^n
$\alpha^{(n)}$
$\beta^{<n>}$
C
A^n
C
A^{n-1}
A
$\alpha^{(n-1)}$
$\Delta^{(n-2)}$
α
β
β
C
A^{n-1}
A
C
A^{n-1}
A
$\alpha^{(n-1)}$
$\Delta^{(n-2)}$
α
β
β
β
C
A^{n-1}
A



C
A^{n-1}
A
$\alpha^{(n-1)}$
α
$\beta^{<n-1>}$
β
β
C
A^{n-1}
A
C
A^{n-1}
A
$\alpha^{(n-1)}$
$\beta^{<n-1>}$
α
β
C
A^{n-1}
A
C
A^{n-1}
A
$\beta^{<n-1>}$
$\alpha^{(n-1)}$
α
β
C
A^{n-1}
A

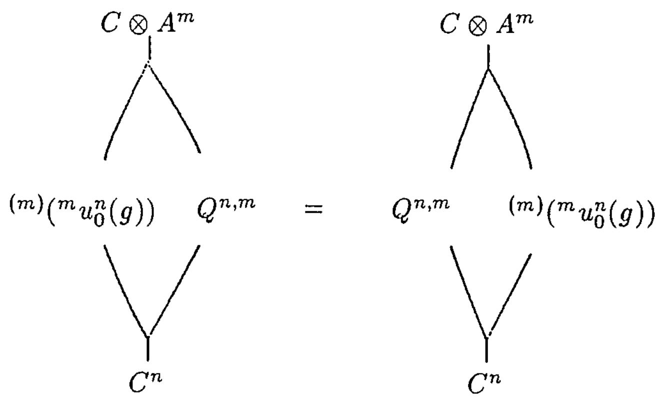


C
A^{n-1}
A
$\Delta^{(n-2)}$
β
$\alpha^{(n-1)}$
α
β
β
C
A^{n-1}
A
C
A^{n-1}
A
$\Delta^{(n-2)}$
$\alpha^{(n-1)}$
α
β
β
C
A^{n-1}
A
C
A^n
$\beta^{<n>}$
$\alpha^{(n)}$
C
A^n

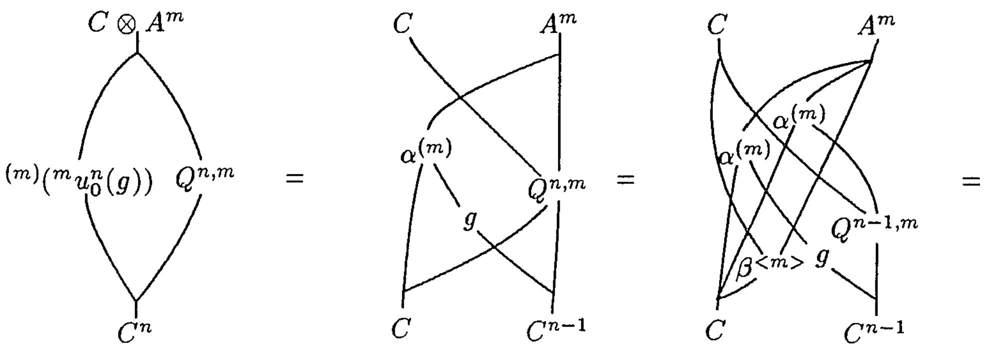
The first and last equalities follow from the definitions 2.1.4 and 3.3.1 as well as an application of (1.6). The second and seventh equalities follow from (1.23) and the fourth and sixth follow from (1.26). The third equality is (2.3) and the fifth is the inductive hypothesis. □

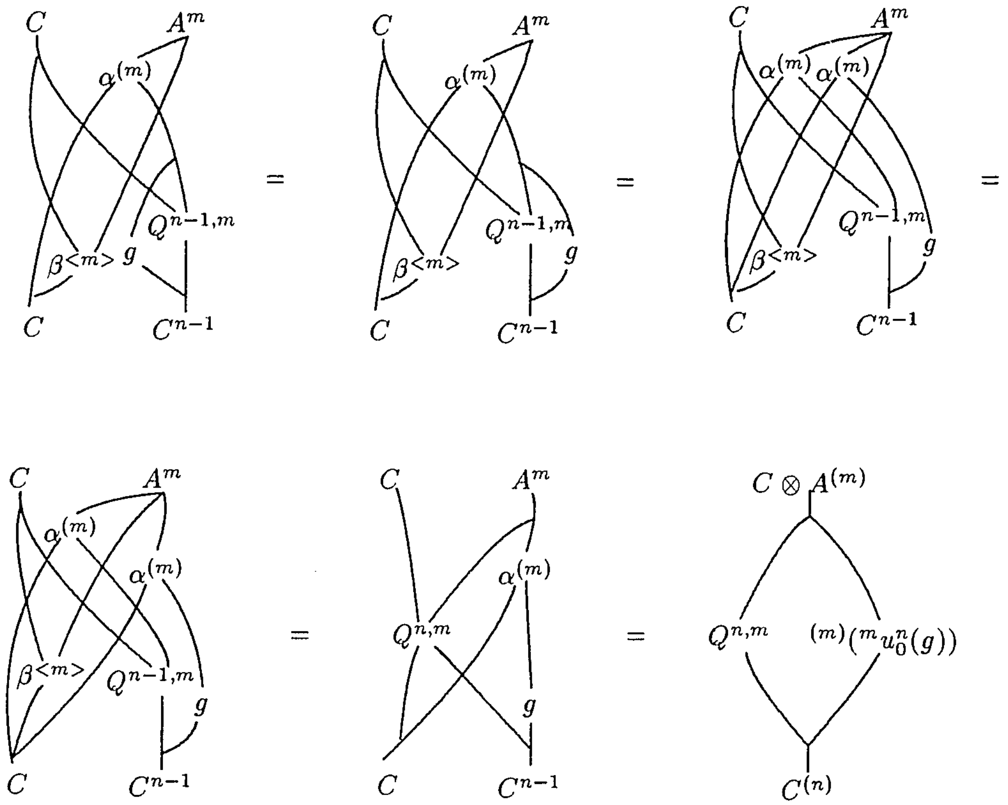
Theorem 3.4.4 *Suppose (C, A, α, β) is a compatible matched pair, $m, n \in \mathbb{N}$, $n > 0$ and $g \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$. Then ${}^m u_0^n(g) \in {}^*\mathsf{Vect}(C^n, A^m)$.*

Proof. We must show that the following equality holds,



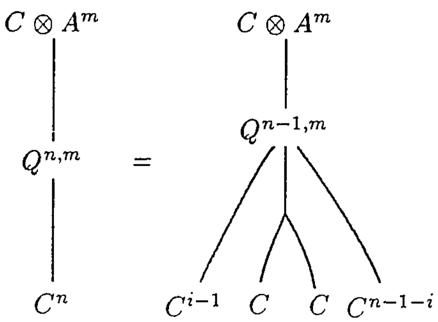
so we calculate.



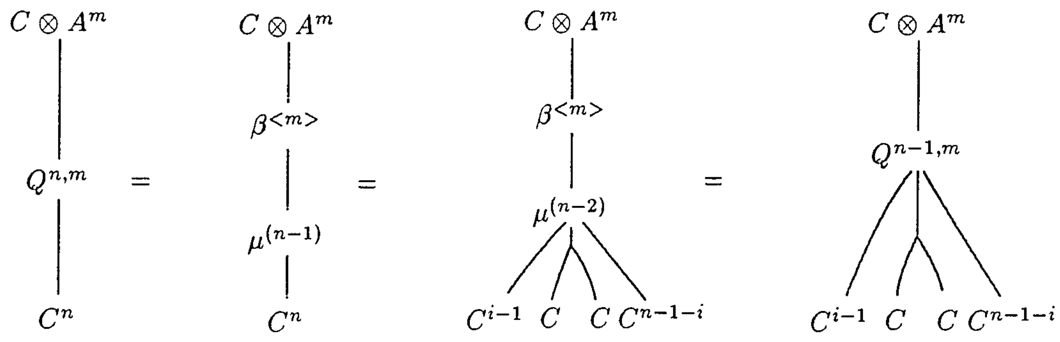


The first and last equalities follow from definition 3.3.2. The second and seventh equalities follow from lemma 3.4.2. The third and fifth equalities follow from (1.21). The fourth equality follows from the fact that $g \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$ and the sixth equality follows from lemma 3.4.3. $\square$

Lemma 3.4.5 *Suppose* (C, A, α, β) *is a compatible matched pair,* $m, n \in \mathrm{N}$, $n > 1$ *and* $0 < i < n$. *Then the following equality holds.*



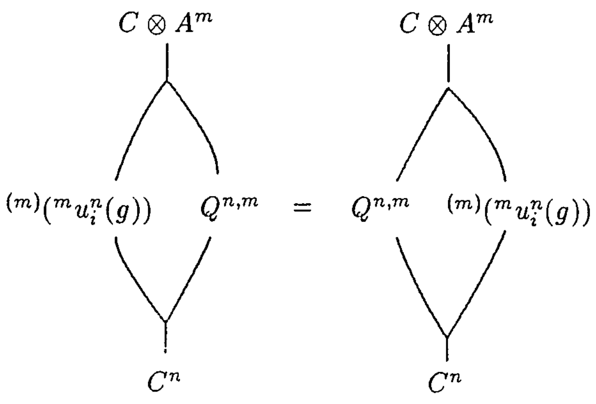
Proof.



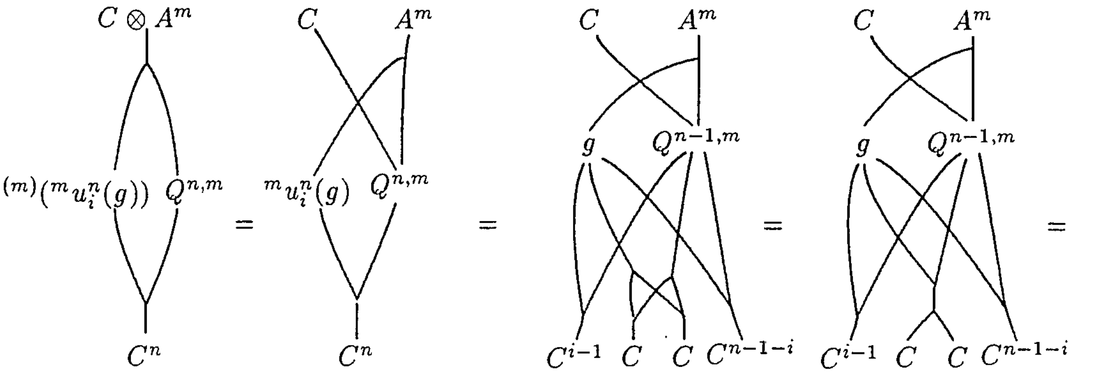
The first and last equalities follow from definition 3.3.2. The second equality follows from associative property of μ . $\square$

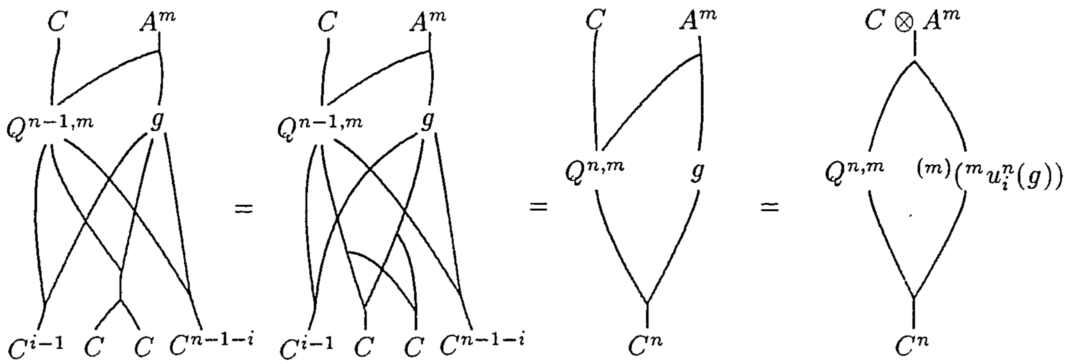
Theorem 3.4.6 *Suppose* (C, A, α, β) *is a compatible matched pair,* $m, n \in \mathrm{N}$, $n > 1$, $0 < i < n$ *and* $g \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$. *Then* ${}^m u_i^n(g) \in {}^*\mathsf{Vect}(C^n, A^m)$.

Proof. We must show that the following equality holds,



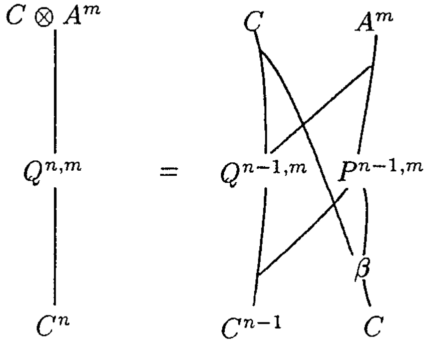
so we calculate.



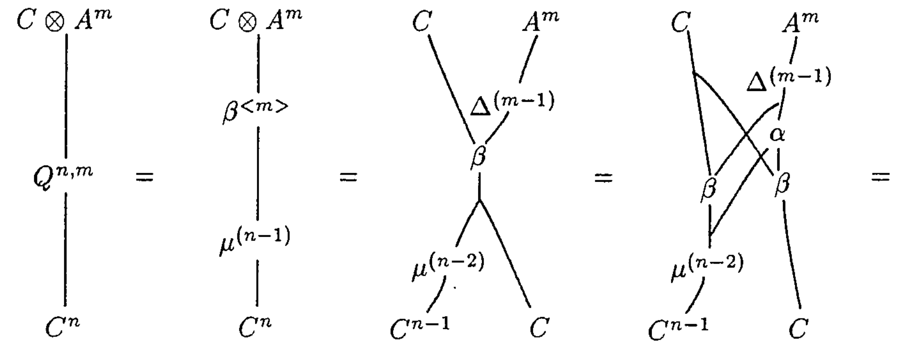


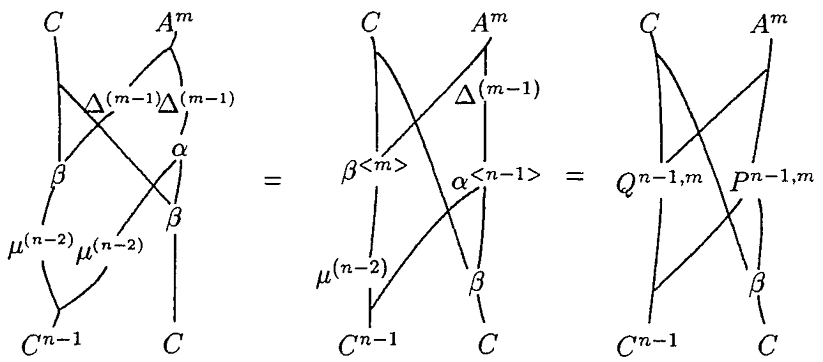
The first and seventh equalities follow from definition 3.3.2. The second and sixth equalities follow from lemma 3.4.5. The third and fifth equalities follow from (1.14) and the fourth is the fact that $g \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$. $\square$

Lemma 3.4.7 *Suppose (C, A, α, β) is a compatible matched pair, $m, n \in \mathbf{N}$, and $n > 0$. Then the following equality holds.*



Proof.

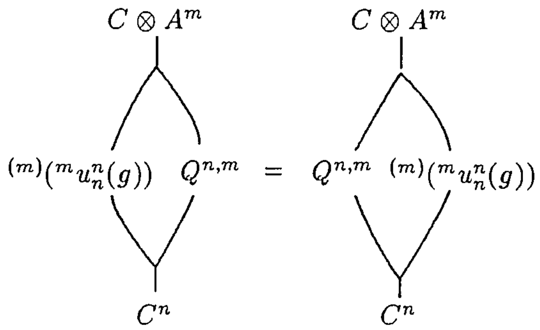




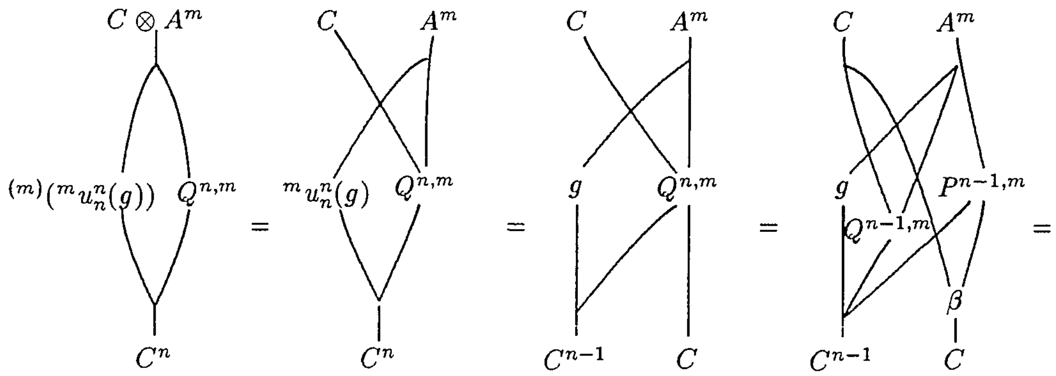
The first and last equalities follow from definition 3.3.2 and the second equality follows from definition 3.3.1 and the associative property of μ. The third equality follows from (2.1) and the fourth is a result of multiple applications of (1.14). The fifth equality follows from definition 3.3.1. □

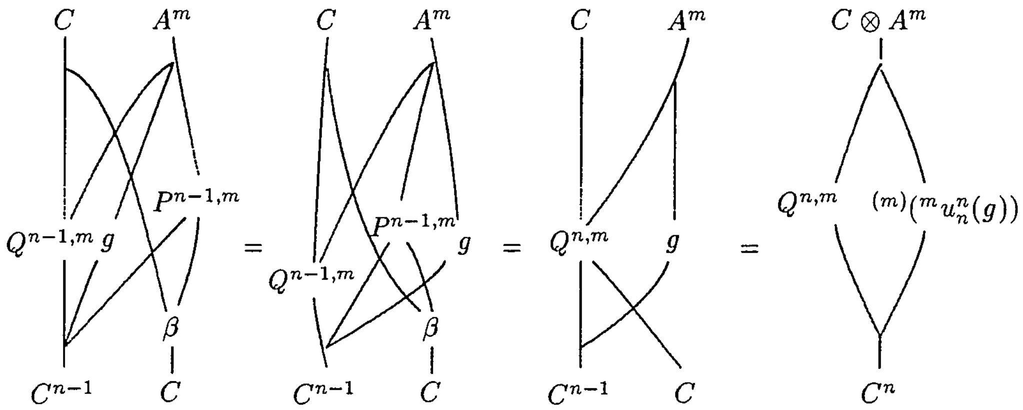
Theorem 3.4.8 *Suppose* (C, A, α, β) *is a compatible matched pair,* $m, n \in \mathrm{N}$, $n > 0$ *and* $g \in \mathrm{Mid}(C^{n-1}, A^m)$. *Then* ${}^m u_n^n(g) \in {}^*\mathsf{Vect}(C^n, A^m)$.

Proof. We must show that the following equality holds,



so we calculate.

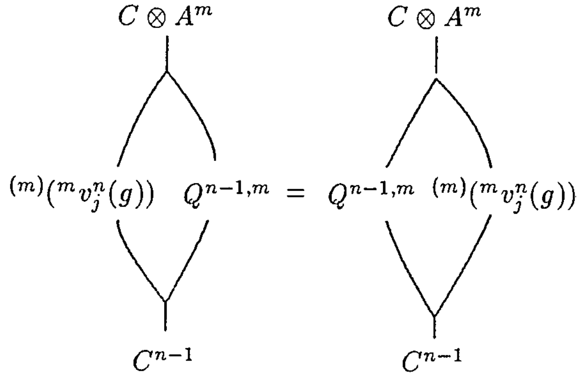




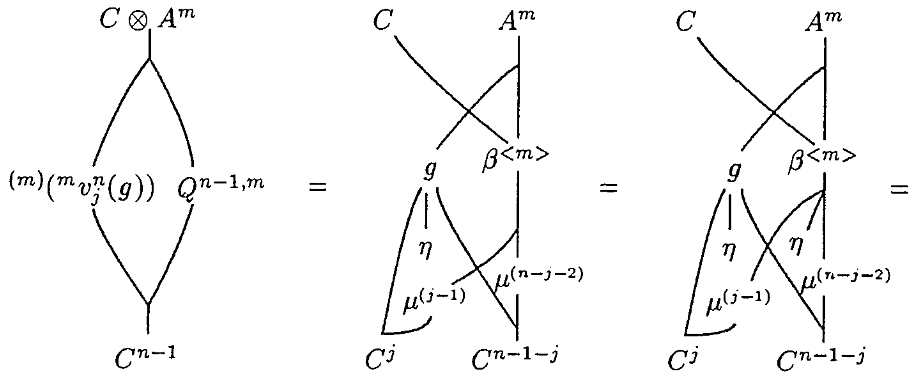
The first equality follows from definition 3.3.2 and the second follows (3.8). The third and sixth equalities follow from lemma 3.4.7. The fourth and fifth equalities result from the facts that $g \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$ and $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$ respectively. The last equality is a result of definition 3.3.2 and (3.8). $\square$

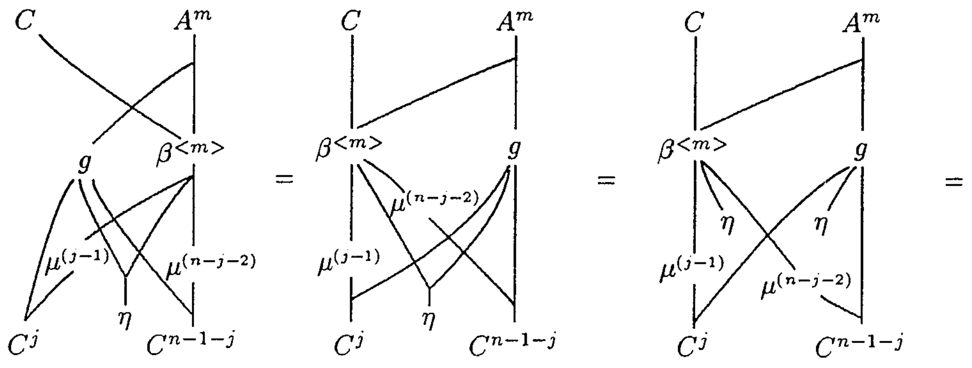
Theorem 3.4.9 *Suppose (C, A, α, β) is a compatible matched pair, $m, n, j \in \mathbf{N}$, $n \geq 1$, $0 \leq j \leq n-1$, and $g \in {}^*\mathsf{Vect}(C^n, A^m)$. Then ${}^m v_j^n(g) \in {}^*\mathsf{Vect}(C^{n-1}, A^m)$.*

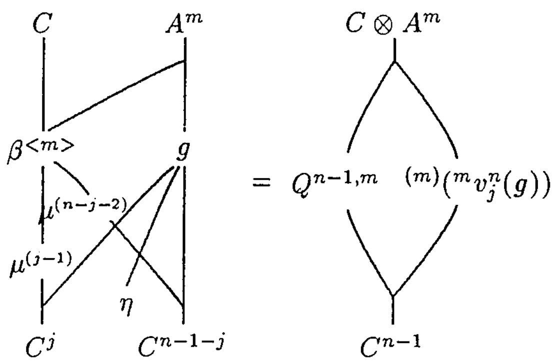
Proof. We must show that the following equality holds;



so we calculate.



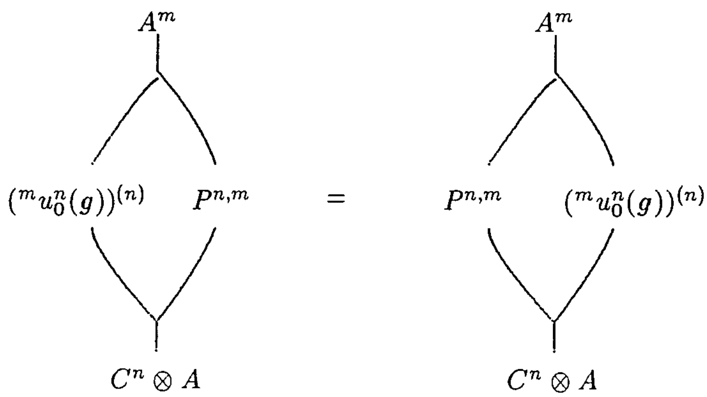




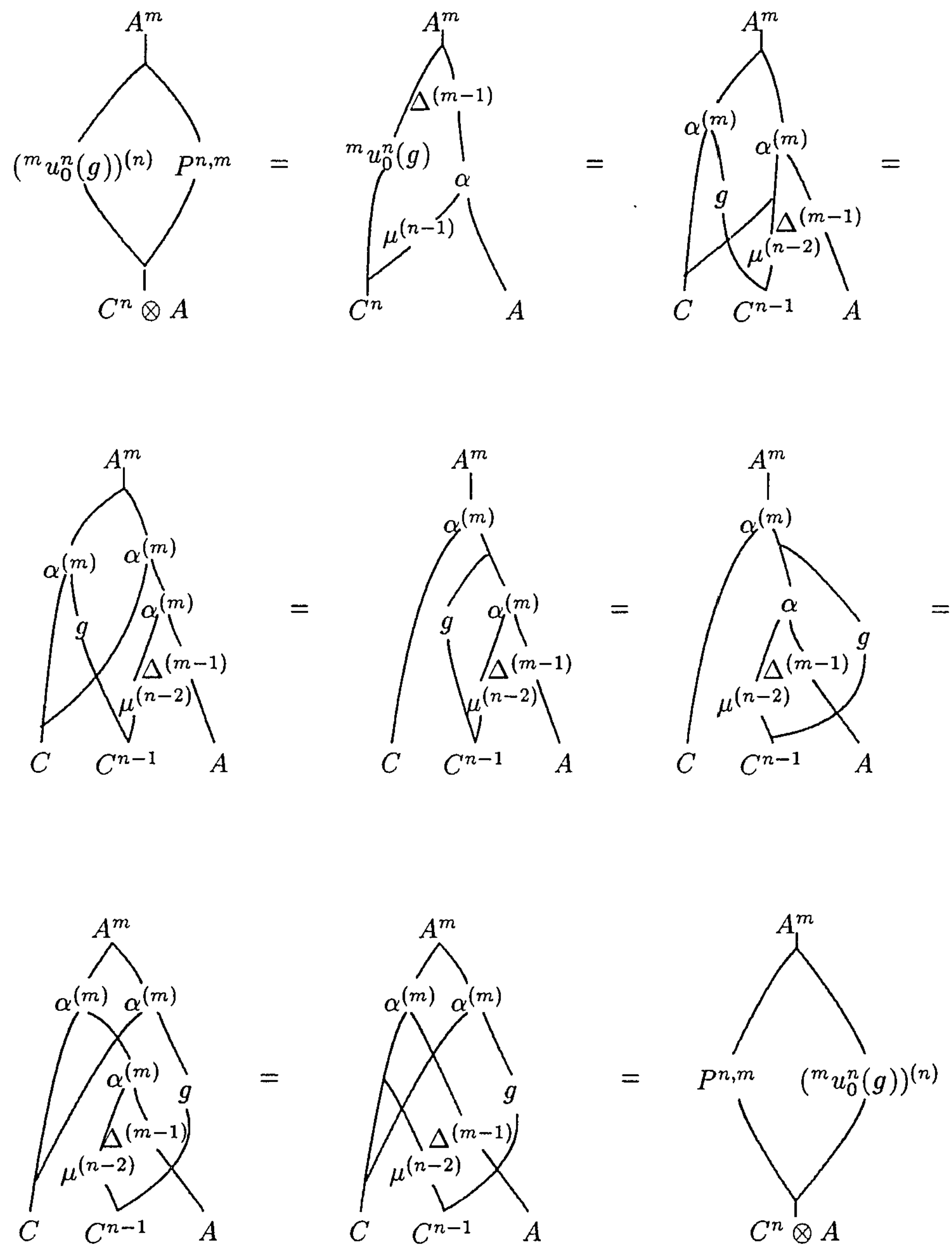
The first and last equalities follow from the definition 3.3.2 and (3.9). The second and sixth equalities follow from the fact that η is a unit; that is (1.2). The third and fifth equalities follow from (1.13) and the fourth is the fact that $g \in {}^*\mathsf{Vect}(C^n, A^m)$. □

Theorem 3.4.10 *Suppose (C, A, α, β) is a compatible matched pair, $m, n \in \mathbb{N}$, $n \geq 1$, and $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. Then ${}^m u_0^n(g) \in \mathsf{Vect}^*(C^n, A^m)$.*

Proof. We must show that the following equality holds;

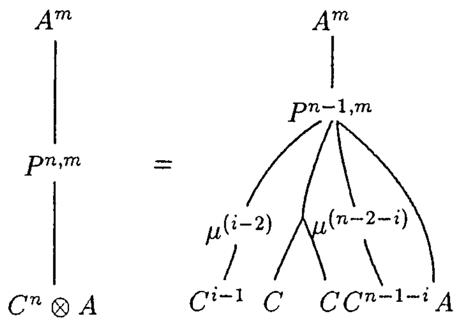


so we calculate.

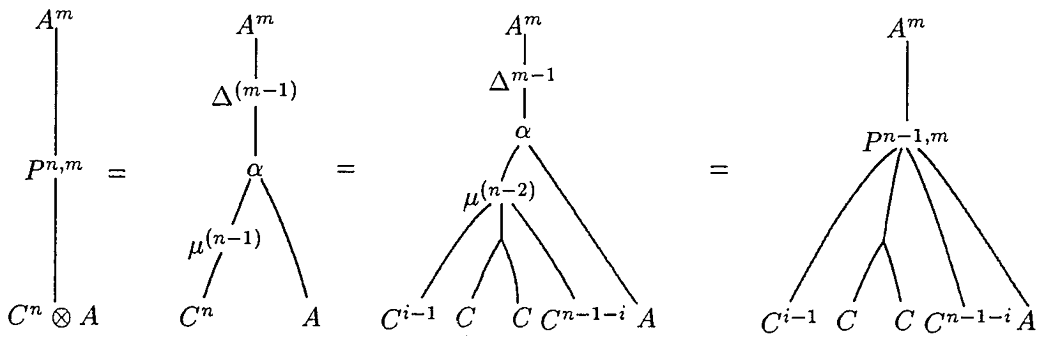


The first equality follows from the definition 3.3.2 and the second follows from (3.6) and (3.15). The third and seventh equalities follow from (1.18) and the fourth and sixth equalities follow from (1.21). The fifth equality is the fact that $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$ and the last equality is the result of definition 3.3.2 and (3.6). □

Lemma 3.4.11 *Suppose (C, A, α, β) is a compatible matched pair, $n, m, i \in \mathbf{N}$, $n > 1$, and $0 < i < n$. Then we have the following equality.*



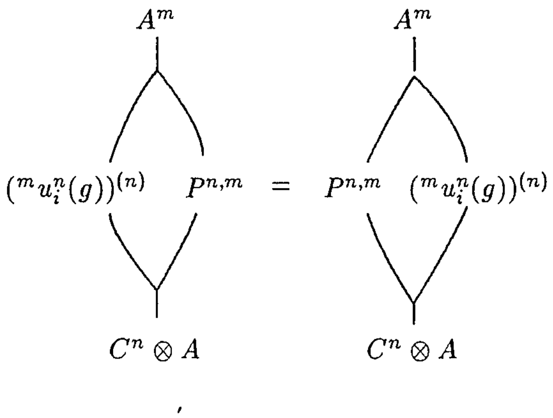
Proof.



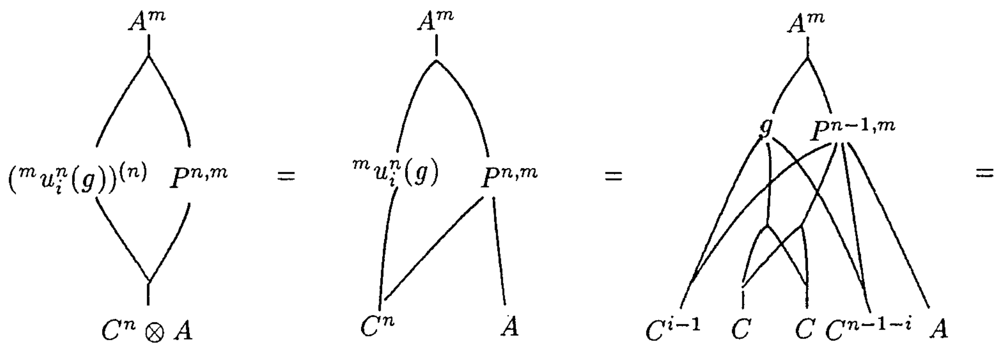
The first and last equalities follow from definition 3.3.2 and the second equality follows from (1.1). □

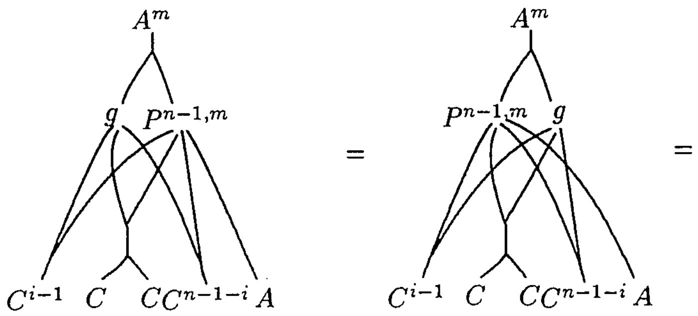
Theorem 3.4.12 *Suppose (C, A, α, β) is a compatible matched pair, $m, n, i \in \mathbf{N}$, $n > 1$, $0 < i < n$ and $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. Then ${}^m u_i^n(g) \in \mathsf{Vect}^*(C^n, A^m)$.*

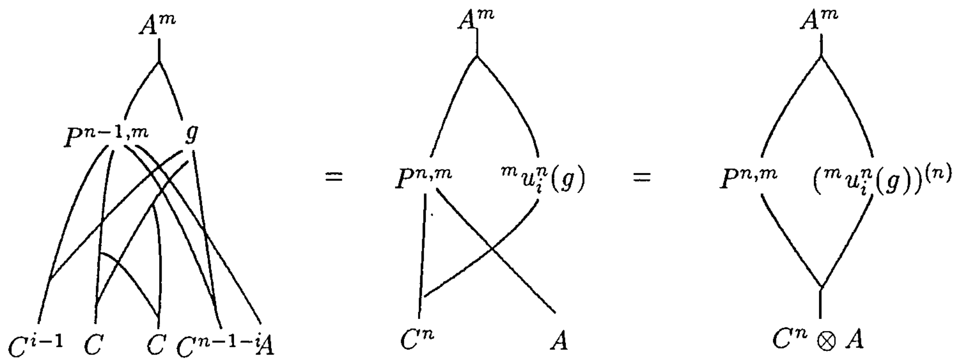
Proof. We must show that the following equality holds:



so we calculate.



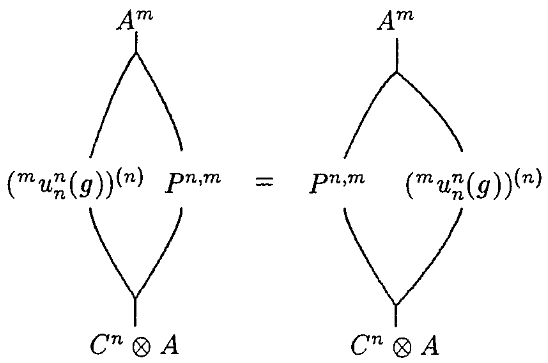




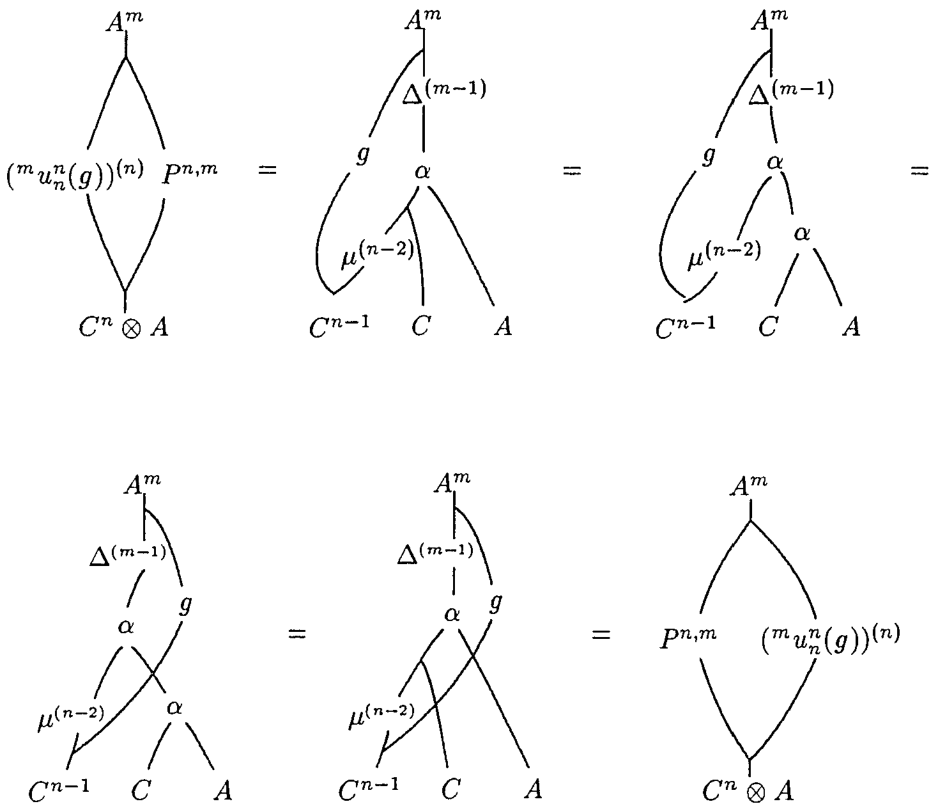
The first and last equalities follow from definition 3.3.2. The second and sixth equalities follow from the definition of ${}^m u_i^n$ and lemma 3.4.11. The third and fifth equalities follow from (1.14) and the fourth equality is the fact that $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. $\square$

Theorem 3.4.13 *Suppose (C, A, α, β) is a compatible matched pair, $m, n \in \mathbb{N}$, $n > 0$, and $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. Then ${}^m u_n^n(g) \in \mathsf{Vect}^*(C^n, A^m)$.*

Proof. We must show that the following equality holds:



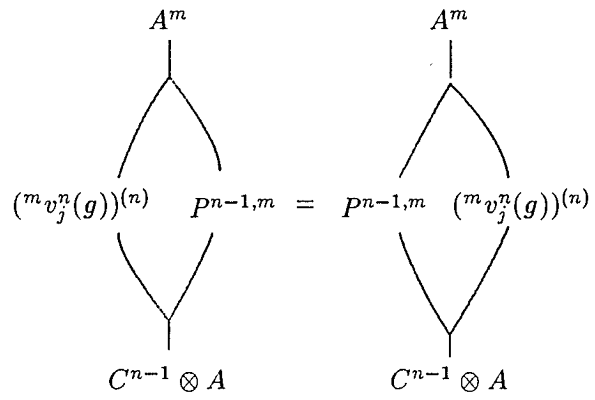
so we calculate.



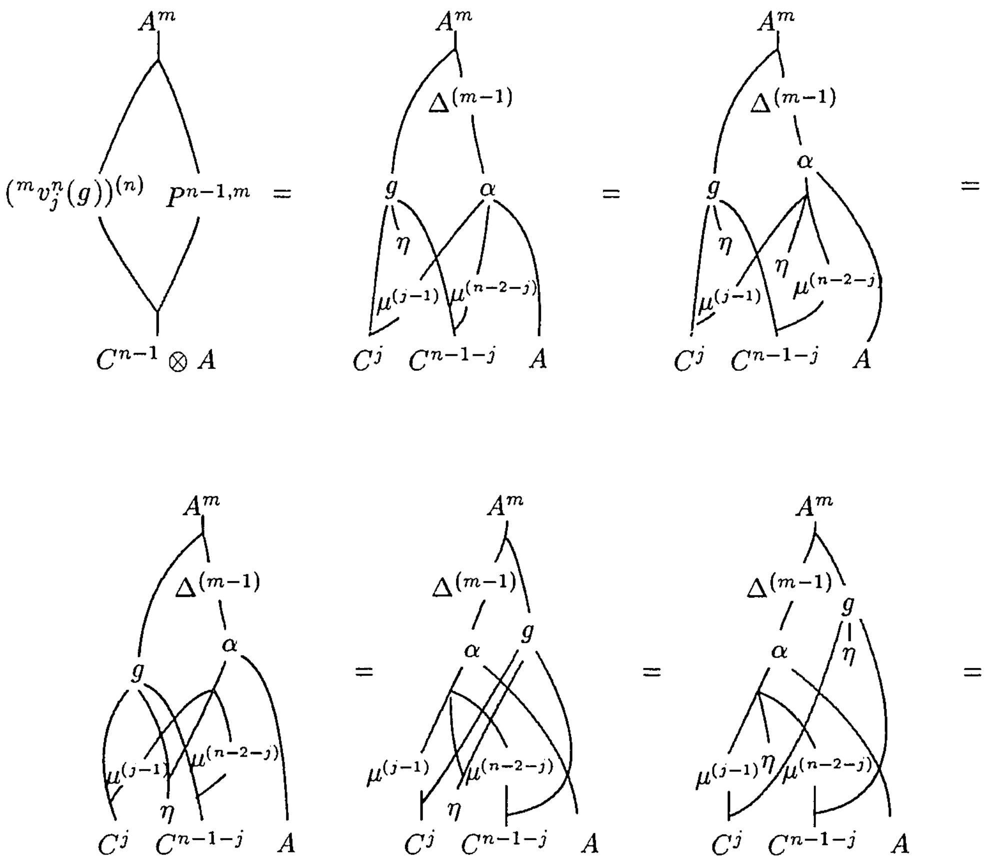
The first and last equalities follow from definition 3.3.2 and (3.8). The second and fourth equalities follow from (1.18) and the third equality follows from the assumption that $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. $\square$

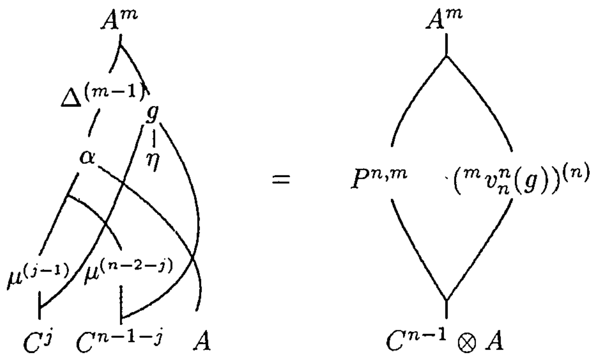
Theorem 3.4.14 *Suppose* (C, A, α, β) *is a compatible matched pair,* $m, n, j \in \mathbf{N}$, $n > 0$, $0 \leq j < n$, *and* $g \in \mathsf{Vect}^*(C^n, A^m)$. *Then* ${}^m v_j^n(g) \in \mathsf{Vect}^*(C^{n-1}, A^m)$.

Proof. We must show that the following equality holds:



so we calculate.

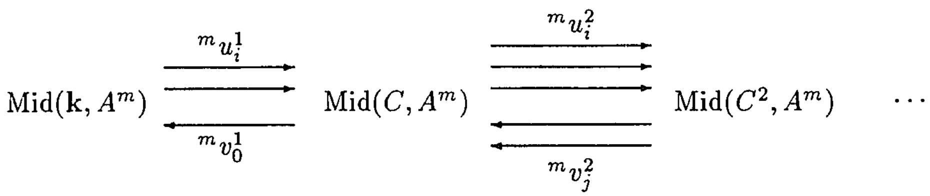




The first and last equalities follow from definition 3.3.2 and (3.9). The second and sixth equalities follow from (1.2) and the third and fifth equalities follow from (1.13). The fourth equality follows from the assumption that $g \in \mathsf{Vect}^*(C^n, A^m)$. The third and fifth equalities follow from (1.14) and the fourth equality is the fact that $g \in \mathsf{Vect}^*(C^{n-1}, A^m)$. □

The next theorem summarizes the results of this section.

Theorem 3.4.15 *Suppose (C, A, α, β) is a compatible matched pair and $m \in \mathrm{N}$, then (3.17) is a sub-cosimplicial algebra of (3.5).*

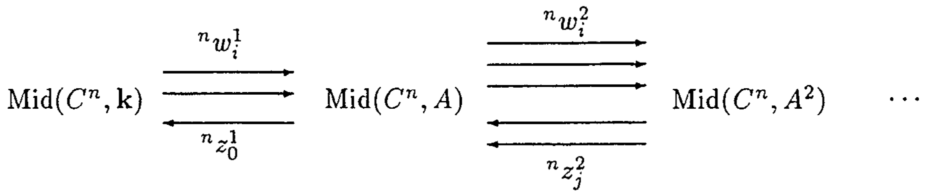


(3.17)

Proof. This is the content of the seven previous theorems of this section. □

By dualizing we have the next theorem.

Theorem 3.4.16 *Suppose (C, A, α, β) is a compatible matched pair and $n \in \mathrm{N}$, then (3.18) is a sub-cosimplicial algebra of (3.10).*



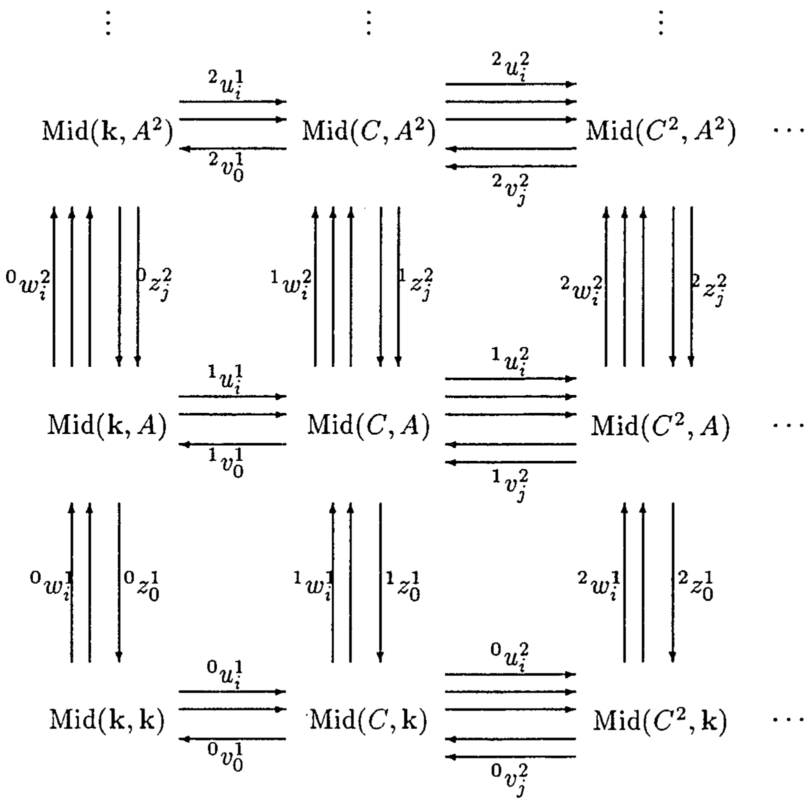
(3.18)

Proof. Dualize the arguments of this section. □

3.5 The Middle and Crossed Products

In this section we draw the connection between the cosimplicial algebras (3.17) and (3.18) and the crossed product bialgebra of theorem and definition 2.2.9.

Given the cosimplicial algebras (3.17) and (3.18) we construct the following bisimplicial algebra.



(3.19)

The rows are given by (3.17) and the columns are given by (3.18). The diagram commutes serially, that is, for $m, n, i, j \in \mathrm{N}$, $n > 0$, $m > 0$, $0 \leq i < n$ and $0 \leq j < m$, we have:

$$ {}^{n}w_j^m \circ {}^{m-1}u_i^n = {}^{m}u_i^n \circ {}^{n-1}w_j^m \quad \text{and} $$

$$ {}^{n-1}z_j^m \circ {}^{m}v_i^n = {}^{m-1}v_i^n \circ {}^{n}z_j^m . $$

The next definition imitates the normalization of a complex, see for instance [Mac63].

Definition 3.5.1 *Suppose (C, A, α, β) is a compatible matched pair, $m, n, j \in \mathrm{N}$, $n > 0$, $0 \leq j < n$, and ${}^{m}v_j^n$ is as in (3.17). By $\mathrm{Ker}({}^{m}v_j^n)$ we mean the kernel of ${}^{m}v_j^n$ as a morphism of monoids, that is, $\mathrm{Ker}({}^{m}v_j^n) = \{f \in \mathrm{Mid}(C^n, A^m) \mid {}^{m}v_j^n(f) = \eta \circ \varepsilon\}$. Define:*

$$ K^{0,m} = \mathrm{Mid}(\mathbf{k}, A^m) \quad \textit{and} $$

$$ K^{n,m} = \bigcap_{j=0}^{n-1} \mathrm{Ker}\,{}^{m}v_j^n \quad \textit{for} \;\; n > 0. $$

Similarly for $m, n, j \in \mathrm{N}$, $m > 0$, $0 \leq j < m$ and ${}^{n}z_j^m$ is as in (3.18), define:

$$ L^{n,0} = \mathrm{Mid}(C^n, \mathbf{k}) \quad \textit{and} $$

$$ L^{n,m} = \bigcap_{j=0}^{m-1} \mathrm{Ker}\,{}^{n}z_j^m \quad \textit{for} \;\; m > 0. $$

For $m, n \in \mathrm{N}$ define:

$$ N^{n,m} = K^{n,m} \cap L^{n,m}. $$

The next definition imitates the formation of the cocycles of a complex.

Definition 3.5.2 *Suppose (C, A, α, β) is a compatible matched pair and $n, m \in \mathrm{N}$. For*

$n > 0$ and all m define:

$$Z_h^{n,m} = \{f \in N^{n,m} \mid \ast_i {}^m u_i^n(f) = \ast_j {}^m u_j^n(f)\}.$$

Here the convolution product on the left is taken in increasing order over $0 \leq i \leq n$ where i is even. The convolution product on the right is taken in decreasing order over $0 \leq j \leq n$ where j is odd.

For $m > 0$ and all n define:

$$Z_v^{n,m} = \{f \in N^{n,m} \mid \ast_i {}^n w_i^m(f) = \ast_j {}^n w_i^m(f)\}.$$

Here the convolution product on the left is taken in decreasing order over $0 \leq i \leq m$ where i is even. The convolution product on the right is taken in increasing order over $0 \leq j \leq m$ where j is odd.

Definition 3.5.3 *Suppose (C, A, α, β) is a compatible matched pair; define*

$$T^2(C, A, \alpha, \beta) =$$

$$\{(\sigma, \tau) \mid \sigma \in \mathrm{Mid}(C^2, A), \tau \in \mathrm{Mid}(C, A^2), {}^2w_1^2(\sigma) \ast {}^2u_1^2(\tau) = {}^2u_2^2(\tau) \ast {}^2u_0^2(\tau) \ast {}^2w_0^2(\sigma) \ast {}^2w_2^2(\sigma)\}$$

The next definition imitates the formation of the cocycles of a bicomplex, see for instance [Mac63].

Definition 3.5.4 *Suppose (C, A, α, β) is a compatible matched pair, define*

$$Z^2(C, A, \alpha, \beta) = \{(\sigma, \tau) \in T^2(C, A, \alpha, \beta) \mid \sigma \in Z_h^{2,1}, \tau \in Z_v^{1,2}\}$$

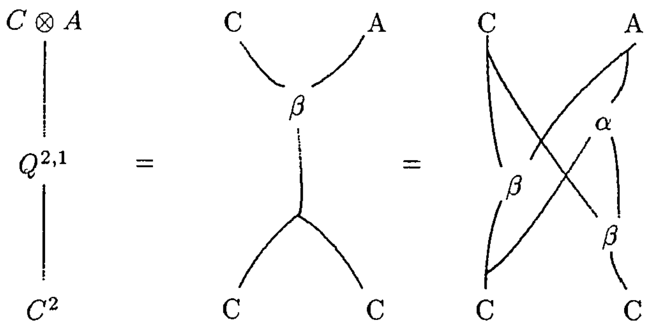
We now show the relationship between Z^2 and the crossed product. The following lemmas culminate in theorem 3.5.17.

Lemma 3.5.5 *Suppose* (C, A, α, β) *is a compatible matched pair and* $\sigma \in \mathsf{Vect}(C^2, A)$. *Then* $\sigma \in \mathsf{Vect}^*(C^2, A)$ *if and only if (2.20) holds.*

Proof. Simply observe that $P^{2,1} = \alpha \circ (C \otimes \alpha) = \alpha \circ (\mu \otimes A)$ where the first equality is definition 3.3.2 and the second follows from (1.18). The lemma now follows easily. □

Lemma 3.5.6 *Suppose* (C, A, α, β) *is a compatible matched pair and* $\sigma \in \mathsf{Vect}(C^2, A)$. *Then* $\sigma \in {}^*\mathsf{Vect}(C^2, A)$ *if and only if (2.25) holds.*

Proof. Just observe that the following equalities hold.



The first equality is definition 3.3.2 and the second is (2.2). The lemma now follows easily. □

We also have the duals of the last two lemmas.

Lemma 3.5.7 *Suppose* (C, A, α, β) *is a compatible matched pair and* $\tau \in \mathsf{Vect}(C, A^2)$. *Then* $\tau \in {}^*\mathsf{Vect}(C, A^2)$ *if and only if (2.23) holds.*

Lemma 3.5.8 *Suppose* (C, A, α, β) *is a compatible matched pair and* $\tau \in \mathsf{Vect}(C, A^2)$. *Then* $\tau \in \mathsf{Vect}^*(C, A^2)$ *if and only if (2.24) holds.*

Lemma 3.5.9 *Suppose (C, A, α, β) is a compatible matched pair and $\sigma \in \mathsf{Vect}(C^2, A)$. Then $\sigma \in K^{2,1}$ if and only if the two equalities of (2.18) hold.*

Proof. This follows immediately from the definitions. □

Lemma 3.5.10 *Suppose (C, A, α, β) is a compatible matched pair and $\sigma \in \mathsf{Vect}(C^2, A)$. Then $\sigma \in L^{2,1}$ if and only if (2.28) holds.*

Proof. This follows immediately from the definitions. □

We also have the duals of the last two lemmas.

Lemma 3.5.11 *Suppose (C, A, α, β) is a compatible matched pair and $\tau \in \mathsf{Vect}(C, A^2)$. Then $\tau \in L^{1,2}$ if and only if the two equalities of (2.21) hold.*

Lemma 3.5.12 *Suppose (C, A, α, β) is a compatible matched pair and $\tau \in \mathsf{Vect}(C, A^2)$. Then $\tau \in K^{1,2}$ if and only if (2.27) holds.*

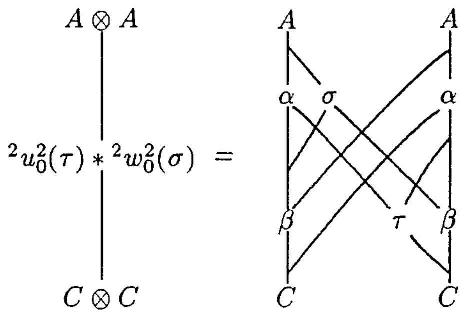
Lemma 3.5.13 *Suppose (C, A, α, β) is a compatible matched pair and $\sigma \in \mathrm{Mid}(C^2, A)$. Then $\sigma \in Z_h^{2,1}$ if and only if the equality (2.19) holds.*

Proof. The condition is ${}^1u_0^2(\sigma) *^1 u_2^2(\sigma) =^1 u_3^2(\sigma) *^1 u_1^2(\sigma)$, which is exactly (2.19). □

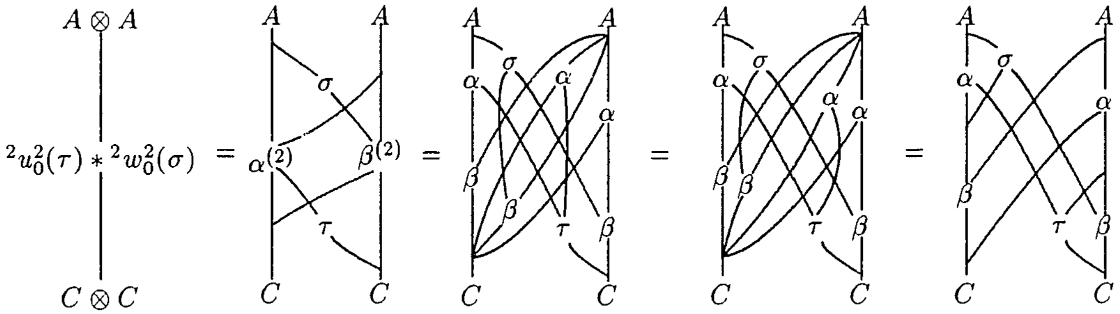
Lemma 3.5.14 *Suppose (C, A, α, β) is a compatible matched pair and $\tau \in \mathrm{Mid}(C, A^2)$. Then $\tau \in Z_v^{1,2}$ if and only if the equality (2.22) holds.*

Proof. The condition is ${}^1w_2^2(\tau) *^1 w_0^2(\tau) =^1 w_1^2(\tau) *^1 w_3^2(\tau)$, which is exactly (2.22). □

Lemma 3.5.15 *Suppose (C, A, α, β) is a compatible matched pair, $\sigma \in \mathrm{Mid}(C^2, A)$ and $\tau \in \mathrm{Mid}(C, A^2)$. Then the following equality holds:*



Proof. We calculate:



The first equality follows from the definition of $*$; that is (1.11) and from (3.6) and (3.11). The second equality follows from the definition of $\alpha^{(2)}$ and $\beta^{(2)}$; namely (2.7) and (2.9). The third equality results from (2.3) and the last equation follows from (1.21) and (1.26). □

Lemma 3.5.16 *Suppose* (C, A, α, β) *is a compatible matched pair,* $\sigma \in \mathrm{Mid}(C^2, A)$ *and* $\tau \in \mathrm{Mid}(C, A^2)$. *Then* $(\sigma, \tau) \in T^2(C, A, \alpha, \beta)$ *if and only if the equality (2.26) holds.*

Proof. The left hand side of (2.26) is easily seen to be ${}^2w_1^2(\sigma) * {}^2u_1^2(\tau)$. With the aid of the last lemma it is also clear that the right hand side of (2.26) is ${}^2u_2^2(\tau) * {}^2u_0^2(\tau) * {}^2w_0^2(\sigma) * {}^2w_2^2(\sigma)$. □

Theorem 3.5.17 *Suppose* (C, A, α, β) *is a compatible matched pair,* $\sigma \in \mathsf{Vect}(C^2, A)$ *and* $\tau \in \mathsf{Vect}(C, A^2)$. *Then* $A\#_{\alpha,\sigma}^{\beta,\tau}C$ *is a bialgebra if and only if* $(\sigma, \tau) \in Z^2(C, A, \alpha, \beta)$.

Proof. Suppose (C, A, α, β) is a compatible matched pair, $\sigma \in \mathsf{Vect}(C^2, A)$ and $\tau \in \mathsf{Vect}(C, A^2)$. By theorems 2.2.5, 2.2.7 and 2.2.9 $A\#_{\alpha,\sigma}^{\beta,\tau}C$ is a bialgebra if and only if the eleven equalities (2.18)—(2.28) hold. By analyzing definition 3.5.4 we see that $(\sigma, \tau) \in Z^2(C, A, \alpha, \beta)$ if and only if the following eleven conditions hold: $\sigma \in \mathsf{Vect}^*(C^2, A)$, $\sigma \in {}^*\mathsf{Vect}(C^2, A)$, $\tau \in \mathsf{Vect}^*(C, A^2)$, $\tau \in {}^*\mathsf{Vect}(C, A^2)$, $\sigma \in K^{2,1}$, $\tau \in L^{1,2}$, $\sigma \in L^{2,1}$, $\tau \in K^{1,2}$, $\sigma \in Z_h^{2,1}$, $\tau \in Z_v^{1,2}$ and $(\sigma, \tau) \in T^2(C, A, \alpha, \beta)$. The previous lemmas in this section match up these eleven equalities and conditions. □

Chapter 4

Cleft Extensions and Crossed Products

In this chapter we characterize crossed products as special kinds of extensions of pairs of Hopf algebras. In the first section we review the definition of a cleft extension of an algebra by a Hopf algebra. In the second section we dualize this notion, defining the cleft extension of a coalgebra by a Hopf algebra. We then juxtapose the two structures, the result being a cleft extension of a pair of Hopf algebras. Finally we compare these extensions with crossed products.

4.1 A Review of Cleft Extensions of Algebras

Cleft extensions of algebras were first considered in [Swe68] in the case of a commutative algebra extended by a cocommutative Hopf algebra. In [DT86], [BCM86] and [BM89] they are studied in the general setting; we also refer to the summary given in [Mont93].

Definition 4.1.1 [Mont93] *Suppose A is a space, B is an algebra, and C is a Hopf algebra. If (B,ρ) is a C-comodule algebra and $A = \{b \in B \mid \rho(b) = b \otimes 1\}$ then we call (A,B,ρ) a* C-**extension**. *If (A,B,ρ) is a C-extension and there exists $t \in \mathrm{Reg}(C,B)$ so that t is a C-comodule map and $t(1) = 1$, then (A,B,ρ,t) is called a* C-**cleft extension**.

Notice in the above definition that A is the equalizer of the algebra maps ρ and $B \otimes \eta_C$ in the category of vector spaces, and since the category of algebras is tripleable over vector spaces [Mac63], A is actually an equalizer in the category of algebras.

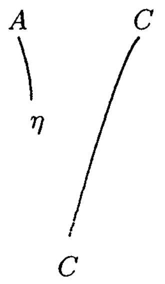
Definition 4.1.2 [Swe68] *Suppose* (A, B, ρ) *and* (A, B', ρ') *are* C*-extensions. A map* $T \in \mathsf{Alg}(B, B')$ *is called a* **morphism of** C**-extensions** *if the diagram (4.1) commutes, where* ι *and* ι' *are inclusions.*

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & B & \xrightarrow{\rho} & B \otimes C \\ \Vert & & \downarrow T & & \downarrow T \otimes C \\ A & \xrightarrow{\iota'} & B' & \xrightarrow{\rho'} & B' \otimes C \end{array}$$

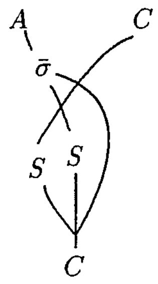
(4.1)

The next two theorems show that cleft extensions correspond to crossed products.

Theorem 4.1.3 [BM89] *Suppose* C *is a Hopf algebra,* $A\#_{\alpha,\sigma}C$ *is a crossed product algebra,* $\sigma \in \mathrm{Reg}(C \otimes C, A)$*, and* t *is given by (4.2). Then* $t \in \mathrm{Reg}(C, A \otimes C)$ *with convolution inverse* $\bar{t}$ *given by (4.3); moreover* $(A, A\#_{\alpha,\sigma}C, A \otimes \Delta_C, t)$ *is a* C*-cleft extension.*

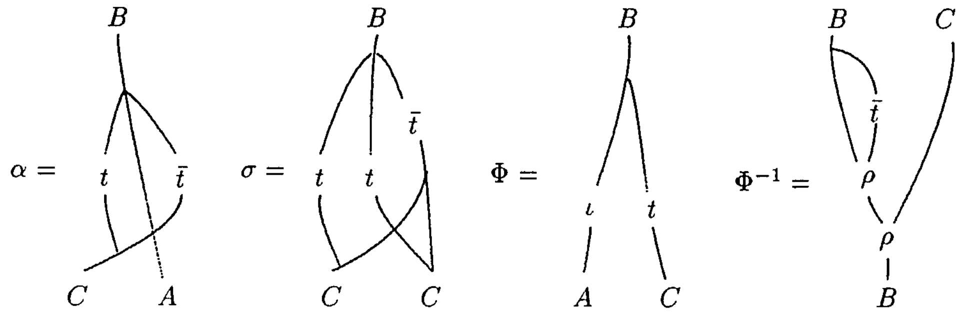


(4.2)



(4.3)

Theorem 4.1.4 [DT86] *Suppose* C *is a Hopf algebra,* (A, B, ρ, t) *is a* C*-cleft extension, and* α*,* σ*,* Φ *and* Φ^{-1} *are the maps given in (4.4). Then* $\alpha \in \mathsf{Vect}(C \otimes A, A)$*,* $\sigma \in \mathsf{Vect}(C \otimes C, A)$*,* *and* $\Phi^{-1} \in \mathsf{Vect}(B, A \otimes C)$. *Moreover* $A\#_{\alpha,\sigma}C$ *is a crossed product algebra and* Φ *is an isomorphism of* C*-extensions.*



(4.4)

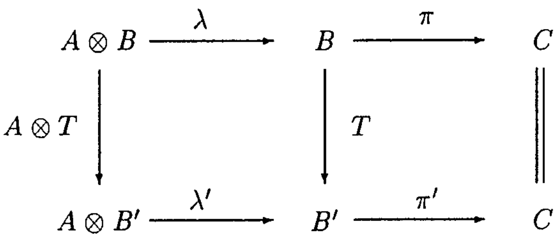
4.2 Cleft Extensions of Pairs of Hopf Algebras

We now dualize the notion of cleft product algebras.

Definition 4.2.1 *Suppose* C *is a space,* B *is a coalgebra, and* A *is a Hopf algebra. If* (B, λ) *is an* A*-module coalgebra and* $C \cong B/(\lambda - (\varepsilon_A \otimes B))[A \otimes B]$, *then we call* (B, C, λ) *an* A-**coextension**. *If* (B, C, λ) *is an* A*-coextension and there exists* $r \in \mathrm{Reg}(B, A)$ *so that* r *is an* A*-module map and* $\varepsilon_A \circ r = \varepsilon_B$, *then* (B, C, λ, r) *is called an* A-**cleft coextension.**

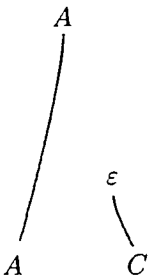
Notice in the above definition that C is a coequalizer of the coalgebra maps λ and $\varepsilon_A \otimes B$ in vector spaces, and since the category of coalgebras is cotripleable over vector spaces [VanO72], C is actually a coequalizer in the category of coalgebras.

Definition 4.2.2 *Suppose* (B, C, λ) *and* (B', C, λ') *are* A*-coextensions and* $T \in \mathsf{Coalg}(B, B')$. *Then* T *is called a* **morphism of** A**-coextensions** *if the diagram (4.5) commutes, where* π *and* π' *are the projections.*

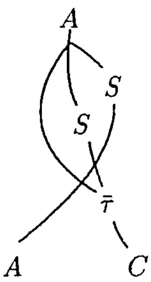


(4.5)

Theorem 4.2.3 *Suppose A is a Hopf algebra, $A \#^{\beta,\tau} C$ is a crossed product coalgebra, $\tau \in$ $\mathrm{Reg}(C, A \otimes A)$, and r is given by (4.6). Then $r \in \mathrm{Reg}(A \otimes C, A)$ with convolution inverse $\bar{r}$ given by (4.7); moreover $(C, A\#^{\beta,\tau}C, \mu_A \otimes C, r)$ is an A-cleft coextension.*

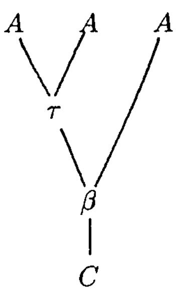


(4.6)

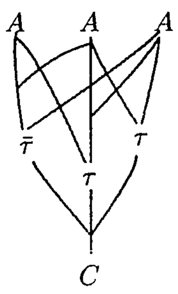


(4.7)

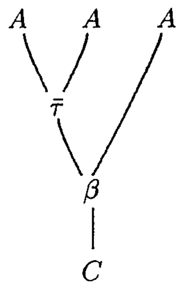
Proof. In order to show that $\bar{r}$ is a left inverse of r we first make an observation. Consider the four maps in the next diagram.



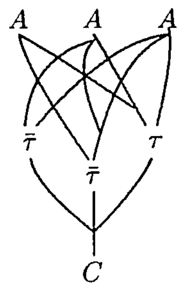
(4.8)



(4.9)



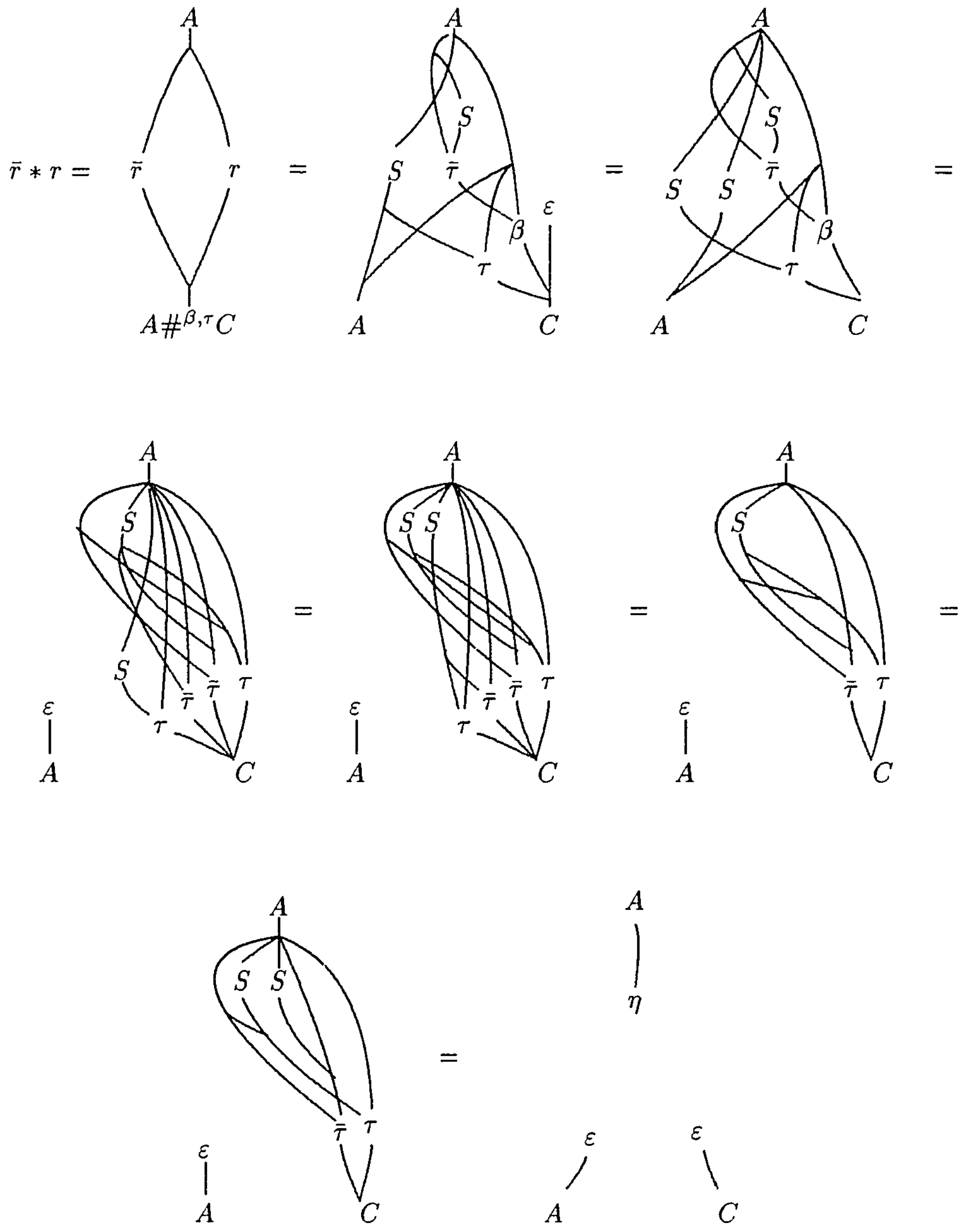
(4.10)



(4.11)

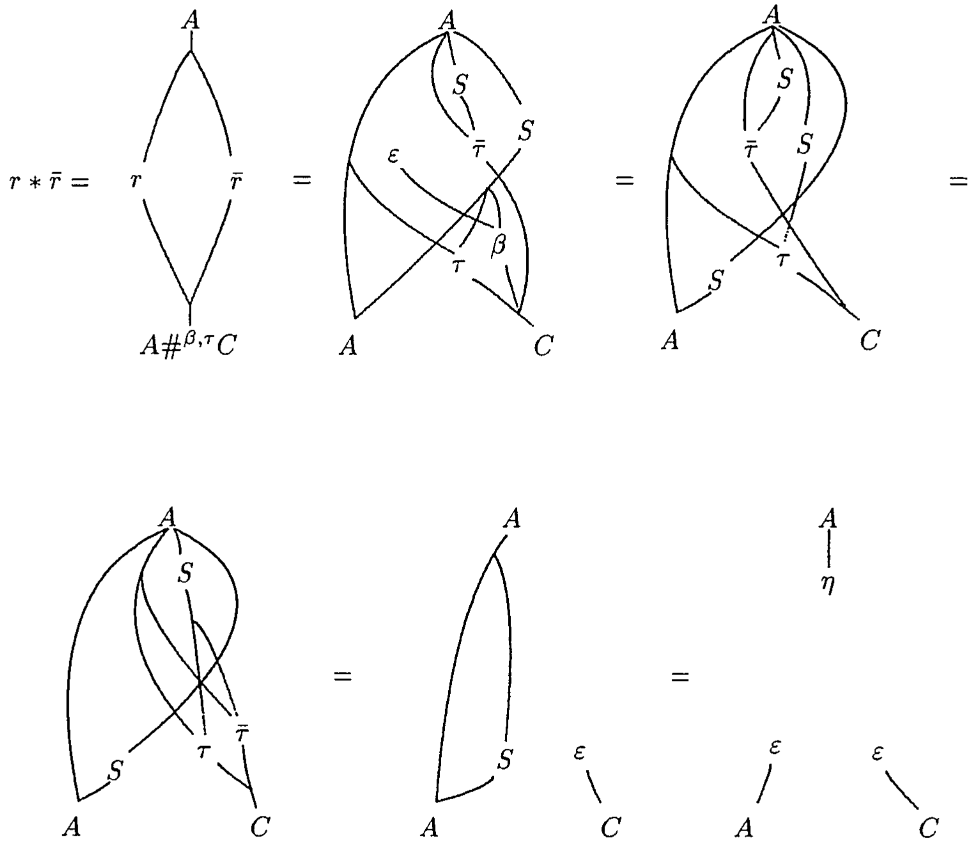
Using the fact that $\bar{\tau}$ is the convolution inverse of τ and (2.22), we see that the map (4.8) is equal to the map (4.9). Moreover an easy calculation shows that the map (4.8) is the convolution inverse of (4.10) and the map (4.9) is the convolution inverse of (4.11). We conclude that (4.10) is equal to (4.11).

Now we calculate:



The first and second equalities are definitions. The third, fifth, sixth, and seventh equalities use the fact that S is an anti-algebra map. The fourth equality follows from (1.16) and the above equality of (4.10) and (4.11). The sixth equality follows since τ and $\bar{\tau}$ are inverses, and the last equality follows from (1.16) and (2.21).

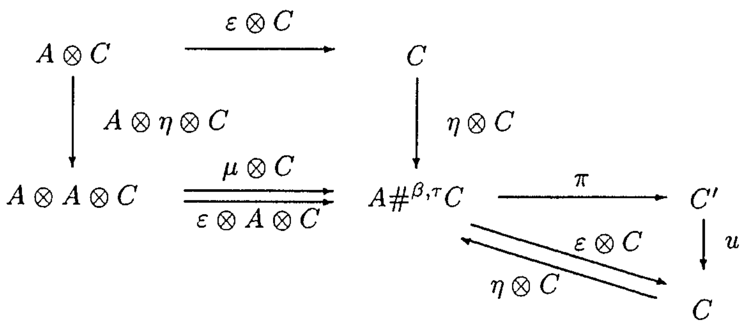
Now we want to show that $\bar{\tau}$ is a right inverse of τ, so we calculate:



The first and second equalities are definitions. The third equality is a result of the fact that S is an anti-algebra map and (1.25) and the fourth equality is another application of the fact that S is an anti-algebra map. The fifth equality is the fact that τ and $\bar{\tau}$ are inverses and the last equality follows from (1.16).

It is easy to see that $(A\#^{\beta,\tau}C, \mu \otimes C)$ is an A-module coalgebra.

Let $C' = A\#^{\beta,\tau}C/((\mu \otimes C) - (\varepsilon \otimes A \otimes C))[A \otimes A \otimes C]$. In order to complete the proof we must show that C is isomorphic as a coalgebra to C'. To this end consider the following diagram.



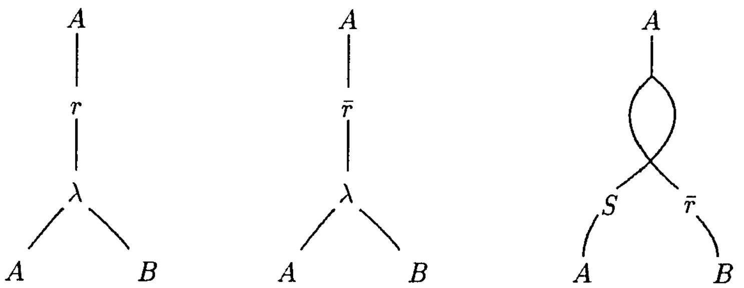
(4.12)

By (1.12) $\varepsilon \otimes C$ coequalizes the pair in (4.12) and thus we have the coalgebra map u making the triangle commute. An easy calculation shows that $u \circ (\pi \circ (\eta \otimes C)) = C$. To see that $(\pi \circ (\eta \otimes C)) \circ u = C'$ we precede both by the surjection π and calculate:

$$\begin{aligned}
\pi \circ (\eta \otimes C) \circ u \circ \pi &= \pi \circ (\eta \otimes C) \circ (\varepsilon \otimes C) \\
&= \pi \circ (\varepsilon \otimes A \otimes C) \circ (A \otimes \eta \otimes C) \\
&= \pi \circ (\mu \otimes C) \circ (A \otimes \eta \otimes C) \\
&= \pi \circ (A \otimes C) \\
&= \pi
\end{aligned}$$

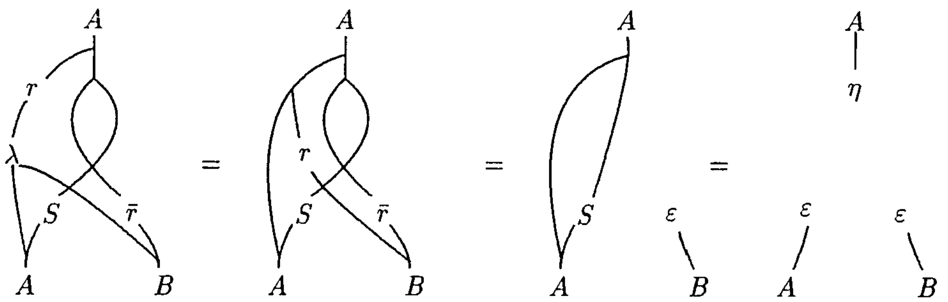
Thus u is an isomorphism. □

Lemma 4.2.4 *Suppose (B, C, λ, r) is an A-cleft coextension, then (4.14) and (4.15) below are equal.*



(4.13) (4.14) (4.15)

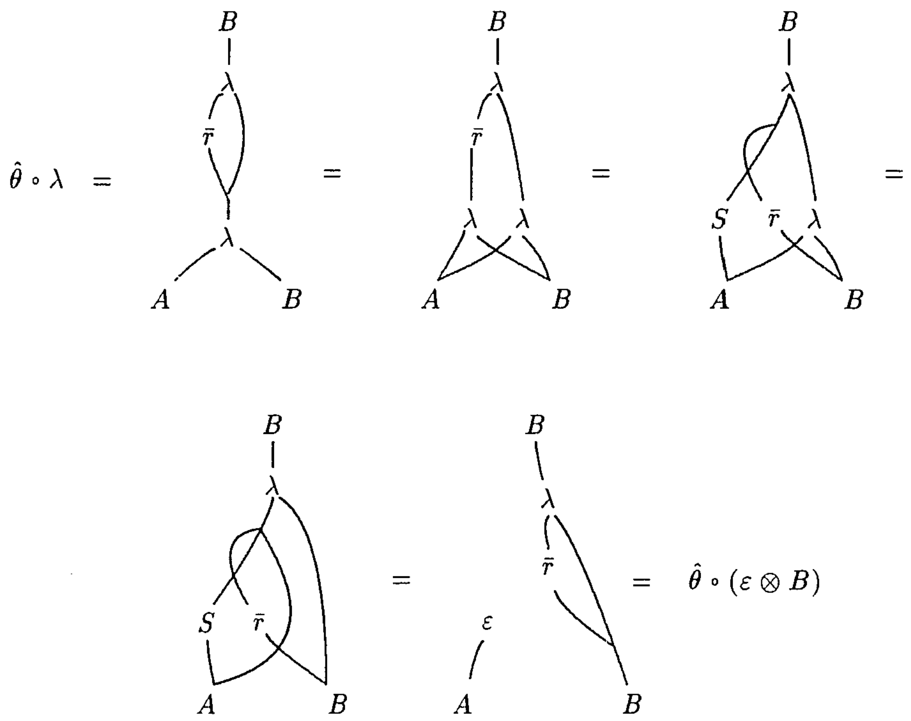
Proof. Since λ is a coalgebra map, it follows that (4.14) is the convolution inverse of (4.13). The proof concludes by showing that (4.15) is a right inverse of (4.13); thus by uniqueness of inverses (4.14) and (4.15) are equal. We calculate the convolution product of (4.13) and (4.15):



The first equality is a result of the fact that r is an A-module map. The second equality follows from the fact that r and $\bar{r}$ are inverses and the last equality is (1.16). □

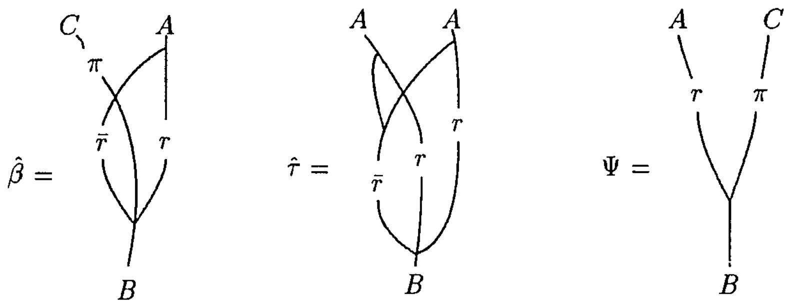
Lemma 4.2.5 *Suppose* (B, C, λ, r) *is an A-cleft coextension. If* $\hat{\theta} \in \mathsf{Vect}(B, B)$ *is defined by* $\hat{\theta} = \lambda \circ \bar{r} \otimes B \circ \Delta$*, then* $\hat{\theta}$ *coequalizes* λ *and* $\varepsilon \otimes B$*. Thus we have a map* $\theta \in \mathsf{Vect}(C, B)$ *so that* $\theta \circ \pi = \hat{\theta}$.

Proof. We need to show that $\hat{\theta} \circ \lambda = \hat{\theta} \circ (\varepsilon \otimes B)$, so we calculate:



The first and last equalities are definitions. The second equality follows from the fact that λ is a coalgebra map, and the third equality results from lemma 4.2.4. The fourth equality follows from the fact that λ is a module structure; that is (1.18), and the fifth equality is (1.16). $\square$

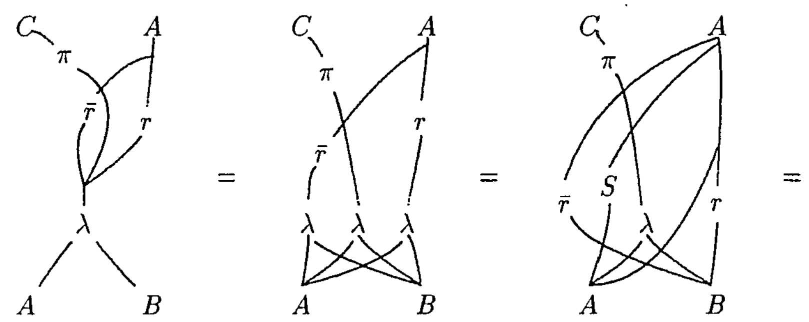
Theorem 4.2.6 *Suppose* (B, C, λ, r) *is an A-cleft coextension and* $\hat{\beta}$, $\hat{\tau}$ *and* Ψ *are the maps given by (4.16). Then we have* $\beta \in \mathsf{Vect}(C, C \otimes A)$ *and* $\tau \in \mathrm{Reg}(C, A \otimes A)$ *so that* $A\#^{\beta,\tau}C$ *is a crossed product coalgebra and* $\Psi : B \to A\#^{\beta,\tau}C$ *is an isomorphism of A-coextensions.*

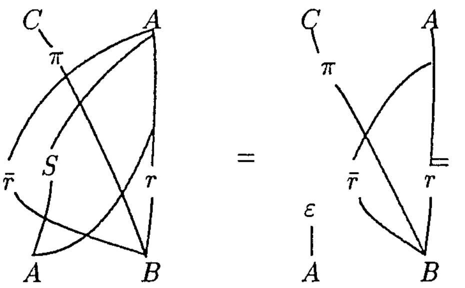


(4.16)

Proof. First we show that $\hat{\beta} \circ \lambda = \hat{\beta} \circ (\varepsilon \otimes B)$, and thus we have $\beta \in \mathsf{Vect}(C, C \otimes A)$ so that $\beta \circ \pi = \hat{\beta}$.

$\hat{\beta} \circ \lambda \; =$

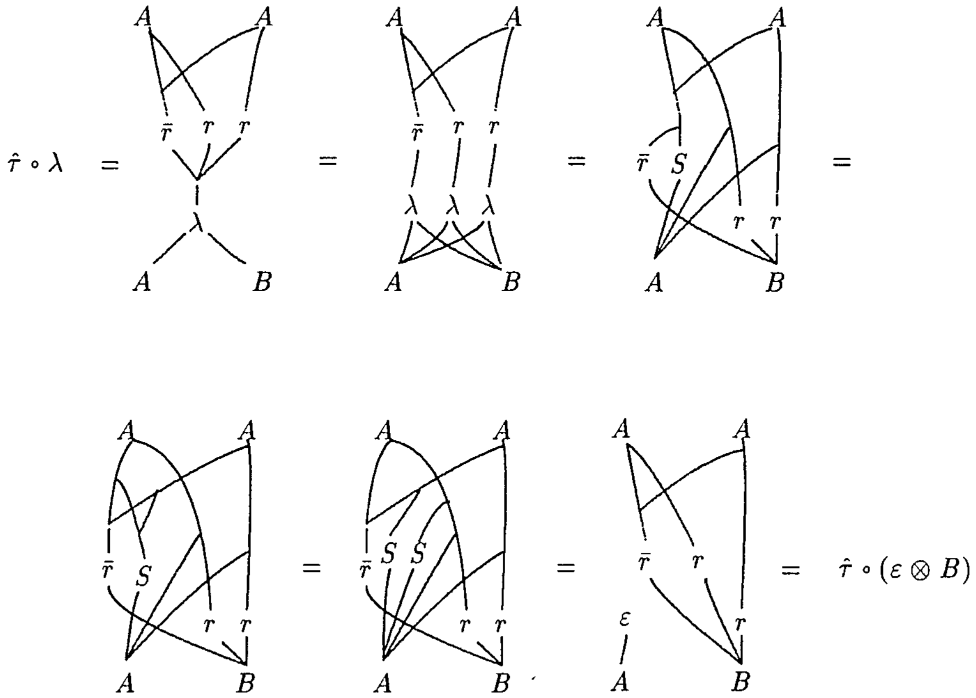




$= \hat{\beta} \circ (\varepsilon \otimes B)$

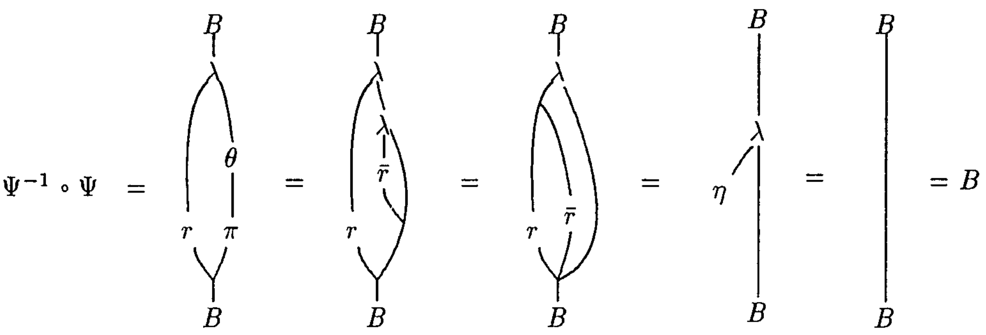
The first and last equalities are definitions and the second equality follows from the fact that λ is a coalgebra map. The third equality follows from lemma 4.2.4 and the fact that r is a module map. The fourth equality follows from the fact that π coequalizes λ and $\varepsilon \otimes B$, and the fifth equality is a result of (1.16).

Now we show that $\hat{\tau} \circ \lambda = \hat{\tau} \circ (\varepsilon \otimes B)$, and thus we have $\tau \in \mathsf{Vect}(C, A \otimes A)$ so that $\tau \circ \pi = \bar{\tau}$.



The first and last equalities are definitions. The second equality follows since λ is a coalgebra map and the third equality is a result of lemma 4.2.4 and the fact that r is a module map. The fourth equality is a result of(1.14), and the fifth equality is a result of the fact that S is an anti-algebra map. The sixth equality follows from two applications of (1.16). We now have $A\#^{\beta,\tau}C$, at this point a space with a potential comultiplication and counit.

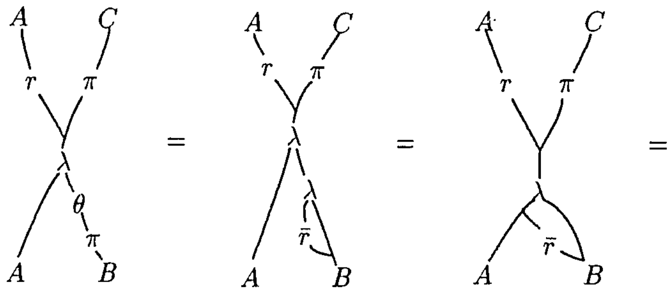
Define $\Psi^{-1} = \lambda \circ (A \otimes \theta)$ where θ is as given in lemma 4.2.5; we now show that this is indeed the inverse of Ψ.

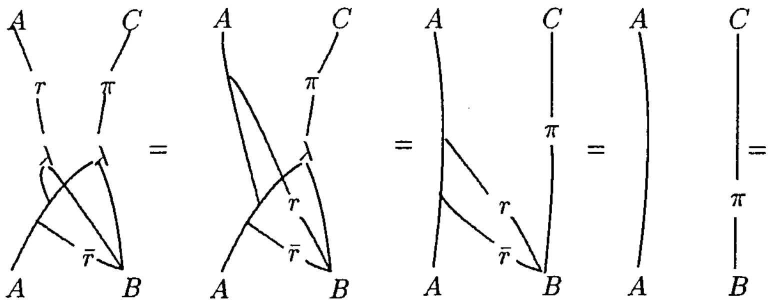


The first and last equalities are definitions. The second equality is lemma 4.2.5 and the third equality follows from (1.18). The fourth equality is the fact that r and $\bar{r}$ are inverses and the fifth equality is (1.17).

Since $A \otimes \pi$ is a surjection, the next calculation shows that $\Psi \circ \Psi^{-1} = A \otimes C$.

$\Psi \circ \Psi^{-1} \circ (A \otimes \pi) =$

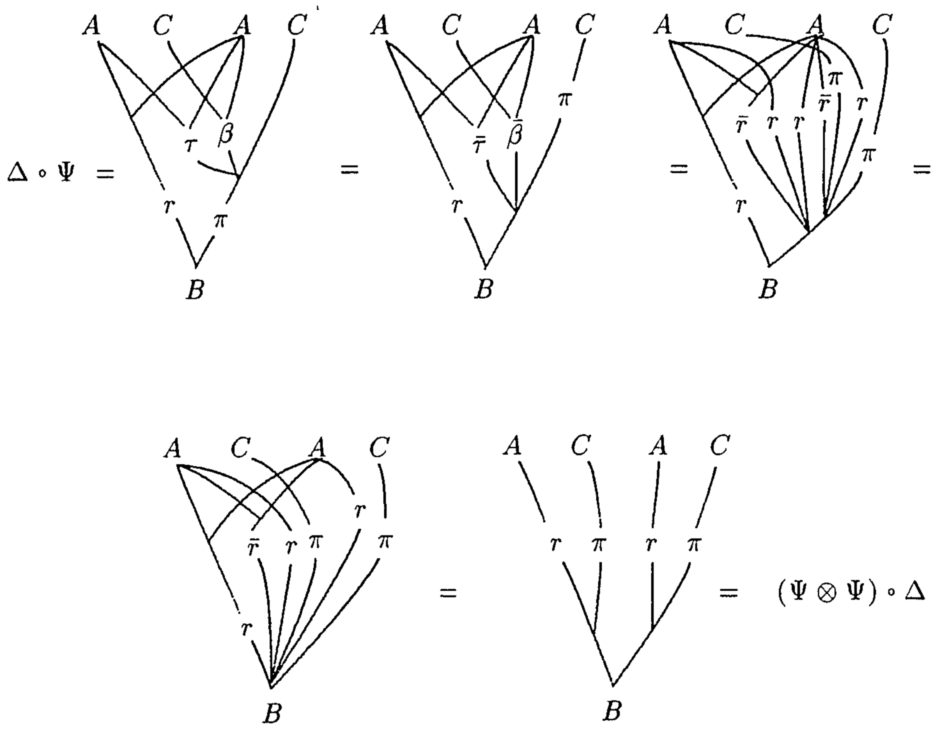




$= (A \otimes C) \circ (A \otimes \pi)$

The first and last equalities are definitions. The second equality follows from lemma 4.2.5 and the third equality follows from (1.18). The fourth equality follows from the fact that λ is a coalgebra map and the fifth is a result of the fact that r is a module map. The sixth equality results from the equality $\pi \circ \lambda = \pi \circ (\varepsilon \otimes B)$. The seventh equality is simply the fact that r and $\bar{r}$ are inverses.

Next we want to show that Ψ respects the comultiplication B and the (not-necessarily-coassociative) comultiplication of $A\#^{\beta,\tau}C$, which is the next calculation.

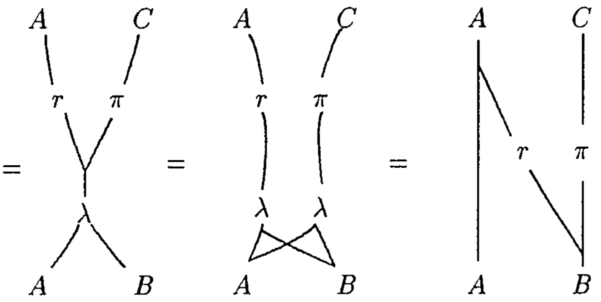


The first and last equalities are definitions. The second equality is the fact that π is a coalgebra map and the definition of β and τ, and the third equality results from definition of $\hat{\beta}$ and $\hat{\tau}$. The rest of the equalities use the fact that r and $\bar{r}$ are inverses. It is easy to see that Ψ respects the counits.

Now, since B is a (coassociative) coalgebra so is $A\#^{\beta,\tau}C$, that is $A\#^{\beta,\tau}C$ is a crossed product coalgebra.

Finally, to show that Ψ is a morphism of A-coextensions we must show that the diagram (4.5) commutes. The right hand side of (4.5) is trivial; the left hand side is the next calculation.

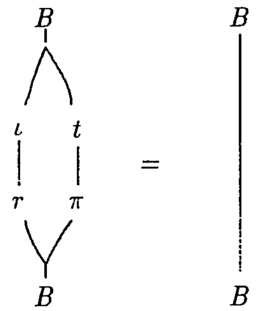
$\Psi \circ \lambda =$



$= (\mu \otimes C) \circ (A \otimes \Psi)$

The first and last equalities are definitions and the second equality follows from the fact that λ is a coalgebra map. The third equality uses the fact that r is an algebra map and that π coequalizes λ and $\varepsilon \otimes B$. □

Definition 4.2.7 *Suppose A and C are Hopf algebras, B is a bialgebra, (B, C, λ, r) is an A-cleft coextension and (A, B, ρ, t) is a C-cleft extension. If the equality (4.17) holds then $(A, B, C, \rho, \lambda, t, r)$ is called an (A, C)-***cleft extension***.*

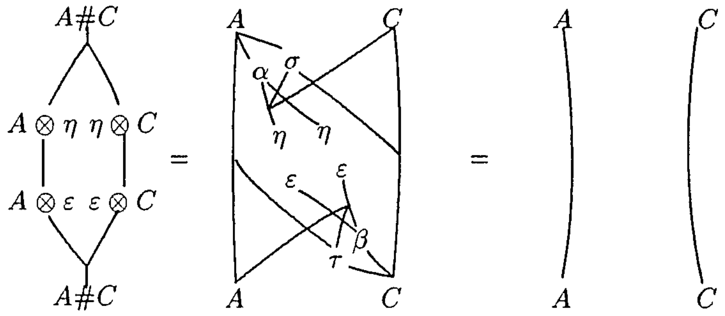


(4.17)

A map between (A, C)-cleft extensions that is both a morphism of C-extensions and a morphism of A-coextensions is a **morphism of (A,C)-cleft extensions***.*

Theorem 4.2.8 *Suppose $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a crossed product bialgebra, A and C are Hopf algebras and σ and τ are invertible. Then $(A, A\#^{\beta,\tau}_{\alpha,\sigma}C, C, A\otimes\Delta, \mu\otimes C, \eta\otimes C, A\otimes\varepsilon)$ is an (A, C)-cleft extension.*

Proof. We need only verify that (4.17) holds in this situation; this is the next calculation.



This follows from the properties of the unit and counit; (1.20), (1.25), (2.18) and (2.21). □

Theorem 4.2.9 *Suppose $(A, B, C, \rho, \lambda, t, r)$ is an (A, C)-cleft extension and α, σ, β, τ and Φ are as given in theorem 4.1.4 and theorem 4.2.6. Then $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a crossed product bialgebra and Φ is an isomorphism of (A, C)-extensions.*

Proof. We know that Φ is an isomorphism of C-extensions and Ψ from (4.16) is an isomorphism of A-coextensions. It is also apparent that (4.17) means that $\Phi \circ \Psi = id_{A\otimes C}$, Thus $\Phi^{-1} = \Psi$ and Φ is an isomorphism of (A, C) extensions. Moreover, since Φ is a bialgebra map and B is a bialgebra, $A\#^{\beta,\tau}_{\alpha,\sigma}C$ is a crossed product bialgebra. □

These extensions generalize the extensions of [By93] and [Hof94].

Bibliography

[Abe77] E. Abe. *Hopf Algebras.* Cambridge University Press, Cambridge, 1977.

[BCM86] R.J. Blattner, M. Cohen and S. Montgomery. Crossed products and inner actions of Hopf algebras. *Trans. AMS* **298**, (1986) 671—711.

[BM89] R.J. Blattner, and S. Montgomery. Crossed products and Galois Extentions of Hopf algebras. *Pacific J. Math* **137**, (1989) 37—54.

[By93] N.P. Byott. Cleft Extensions of Hopf Algebras. *Journal of Algebra* **157**, (1993) 405—429.

[DT86] Y. Doi and M. Takeuchi. Cleft comodule algebras for a bialgebra. *Comm. Alg.* **14**, (1986) 801—818.

[Hof94] I. Hofstetter. Extensions of Hopf Algebras and Their Cohomological Description. *Journal of Algebra* **164**, (1994) 264—298.

[JS91] A. Joyal and R. Street. The Geometry of Tensor Calculus, I. *Advances in Mathematics* **88**, (1991) 55—112.

[Mac63] S. Mac Lane. *Homology.* Springer-Verlag, Berlin, 1963.

[Mac71] S. Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag, New York, 1971.

[Maj90] S. Majid. Physics for Algebraists: Non-commutative and Non-cocommutative Hopf Algebras by a Bicrossproduct Construction. *Journal of Algebra* **130** (1990) 17—64.

[Mol75] R. Molnar. Semi-Direct Products of Hopf Algebras. *Journal of Algebra* **47** (1975) 29—51.

[Mont93] S. Montgomery. *Hopf Algebras and Their Actions on Rings* CBMS Regional Conference Series in Mathematics **82**. AMS , Providence, RI, 1993.

[Pen71] R. Penrose. Applications of Negative Dimensional Tensors. In *Combinatorial Mathematics and its Applications*, (D.J.A. Welsh, Ed.). 221—224 Academic Press, New York, 1971.

[Swe68] M. Sweedler. Cohomology of algebras over Hopf algebras. *Trans. AMS* **127**, (1968) 205—239.

[Swe69] M. Sweedler. *Hopf Algebras.* W.A.Benjamin, New York, 1969.

[VanO72] D. H. Van Osdol. Coalgebras, Sheaves, and Cohomology. *Proc. AMS* **33**, #2, (1972) 257—263.