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Decomposable functions and universal C^*-algebras

Kaonga, Llolsten, Ph.D.

University of New Hampshire, 1994

DECOMPOSABLE FUNCTIONS AND UNIVERSAL
C^*-ALGEBRAS

BY

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DISSERTATION

Submitted to the University of New Hampshire
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in

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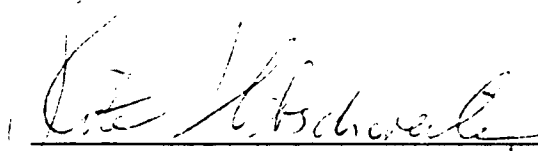May 1994

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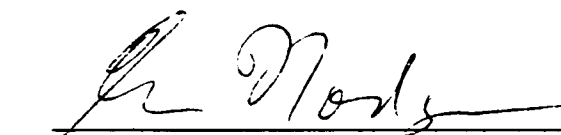
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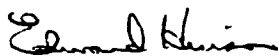
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04/27/94
Date

Dedication

To my Mother, VLK

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I wish to thank my many professors from whom I have learned a lot during my stay at the University of New Hampshire. I would like to thank Professor Nordgren and Rita Hibschweiler for being such great teachers. I particularly wish to thank my adviser, Don Hadwin, without whom this dissertation would not have become a reality. He is both a great teacher and a friend. Thanks also go to my colleagues who have helped me in one way or another. Last but not least, I wish to thank my family for their support.

Foreword

Suppose U_n^{nc} is the unital C^*−algebra generated by the elements u_{ij}, $1 \leq i,j \leq n$ subject to the condition that the matrix (u_{ij}) be unitary. McClanahan has shown that this C^*−algebra has no non-trivial projections.

We prove a more general result for which the above is a special case. Our result applies to a wide class of C^*−algebras.

In Chapter 1, we discuss some of the background material we shall need in our discussions later. We define decomposable functions and give a brief outline of some of their properties. We define decomposable functions of several variables in the second section of this chapter. Completions of the space of decomposable functions are given in Chapter 2 and we show that some of these completions are in fact equivalent.

We use decomposable functions to compute the K−groups of some C^*−algebras generated by elements satisfying certain relations in the first section of Chapter 3. In Section 3.2 we prove some results involving spaces with non-trivial projections.

Contents

ABSTRACT

DECOMPOSABLE FUNCTIONS AND UNIVERSAL C^*-ALGEBRAS

by

Llolsten Kaonga
University of New Hampshire, May, 1994

This paper deals with universal C^*-algebras generated by matricial relations on the generators, for example, the universal C^*-algebra with generators a_{ij}, $1 \leq i,j \leq n$, subject to the condition that the matrix (a_{ij}) be normal and have spectrum in a designated compact subset $\mathcal{K}$ of the complex plane.

The main thrust of the paper is to compute the K-groups of some of these C^*-algebras and to determine when they contain non-trivial projections. In the above example, we show that the K-groups of the algebra coincide with the topological K-groups of the set $\mathcal{K}$. We show, in general, that if the algebra has a multiplicative linear functional, then the K-theory is independent of n, when the matricial constraints are fixed.

It is also shown that if the constraints are fixed and $\mathcal{A}_n$ is the algebra with n^2 generators, then the tensor product of $\mathcal{A}_n$ with the algebra M_n of complex $n \times n$ matrices is isomorphic to the free product of $\mathcal{A}_1$ with M_n.

Also in the example above, the algebra contains no non-trivial projections when n is not less than the number of connected components of $\mathcal{K}$. These results have also been extended to include the case in which the constraints are in several variables.

Chapter 1

Preliminaries

Throughout our discussion, $\mathcal{H}$ denotes a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$. The set of all real numbers will be denoted by $\mathbf{R}$ and the positive real numbers will be denoted by $\mathbf{R}^+$. We shall denote the set of all complex numbers by $\mathbb{C}$. The notation $\mathcal{M} \leq \mathcal{H}$ means $\mathcal{M}$ is a subspace of $\mathcal{H}$. Also $A \subset B$ will mean A is a subset of B. The notation $a \simeq b$ will mean a is unitarily equivalent to b. We shall call a complex polynomial $p(x_1, x_2, \ldots, x_n)$ in the non-commutative variables $x_1, x_2, \ldots, x_n$ a *non-commutative polynomial*.

1.1 Decomposable Functions of one variable

Decomposable functions were created by D. Hadwin and introduced in [BFH]. They were studied in [H1], [H2], [H3] and [H4]. In this section, we present the basic theory of these functions.

Definition 1.1.1 *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. A decomposable function φ on $\cup\{\mathcal{B}(\mathcal{M}) : \mathcal{M} \leq \mathcal{H}\}$ is a function such that*

i) $\varphi(\mathcal{B}(\mathcal{M})) \subset \mathcal{B}(\mathcal{M}) \quad \forall \mathcal{M} \leq \mathcal{H}$;

ii) if $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is a reducing subspace of T, then $\mathcal{M}$ reduces $\varphi(T)$ and

$$\varphi(T|\mathcal{M}) = \varphi(T)|\mathcal{M};$$

iii) if $\mathcal{M},\mathcal{N} \leq \mathcal{H}$, $S \in \mathcal{B}(\mathcal{M})$ and $U : \mathcal{N} \to \mathcal{M}$ is unitary, then

$$\varphi(U^*SU) = U^*\varphi(S)U.$$

A decomposable function is *continuous* if $\varphi|\mathcal{B}(\mathcal{M})$ is continuous for all $\mathcal{M} \leq \mathcal{H}$.

Definition 1.1.2 *A decomposable function φ is null-bounded in case it is continuous and*

$$\sup\{\|T\| : \varphi(T) = 0 \quad and \quad T \in \mathcal{B}(\ell^2)\} < \infty.$$

The collection of all decomposable functions is a *-algebra that is closed under composition and under limits that converge point-wise in the weak operator topology [H1, BFH]. It was shown in [H1] that a decomposable function φ on $\mathcal{H}$ can be unambiguously defined at any operator T on any Hilbert space $\mathcal{H}_1$. We outline how this is done. First, suppose $\mathcal{H}_1$ is

separable and infinite dimensional. Then choose a unitary operator U such that $U : \mathcal{H}_1 \to \mathcal{H}$ is one-to-one and onto. Since $T \in \mathcal{B}(\mathcal{H}_1)$, it follows that $UTU^* \in \mathcal{B}(\mathcal{H})$. Define

$$\varphi(T) = U^*\varphi(UTU^*)U.$$

This defines a decomposable function on $\mathcal{H}_1$ and in fact the definition is independent of U. Indeed if $V : \mathcal{H} \to \mathcal{H}_1$ is unitary, then

$$
\begin{aligned}
V^*\varphi[(UV^*)^*TUV^*]V &= \varphi(V^*VU^*TUV^*V) \quad \text{by iii) of Definition 1.1.1} \\
&= \varphi(U^*TU).
\end{aligned}
$$

If $\dim\mathcal{H}_1 < \infty$, consider $T^{(\infty)} \in \mathcal{B}(\mathcal{H}_1^{(\infty)})$ and define $\varphi(T)$ by

$$\varphi(T)^{(\infty)} = \varphi(T^{(\infty)}).$$

To handle the case in which $\mathcal{H}_1$ is not separable, we need the following two lemmas.

Lemma 1.1.3 *Let $E \subset \mathcal{H}$ and suppose $T(E) \subset E$. Then $T(\overline{E}) \subset \overline{E}$ and $T(\text{span} E) \subset \text{span} E$.*

Proof Suppose $h \in \overline{E}$. Then there exists a net h_n in E converging to h. But $T \in \mathcal{B}(\mathcal{H})$ implies that $Th_n \to Th$. Since $Th_n \in E$, we have $\lim_n Th_n \in \overline{E}$. Therefore

$$T(\overline{E}) \subset \overline{E}.$$

If $e \in \text{span} E$, then

$$e = \sum_{e_i \in E} a_i e_i \quad i \in I, I \text{ a finite set}.$$

Since $Te_i \in E$, we have $Te = \sum a_i Te_i \in \text{span} E$.

This proves the lemma.

□

Lemma 1.1.4 *If $\mathcal{H}_1$ is a non-separable Hilbert space and $\mathcal{S}$ is a norm-separable subset of $\mathcal{B}(\mathcal{H}_1)$, then there is an orthogonal family $\{\mathcal{H}_{1_i} : i \in I\}$ of subspaces of $\mathcal{H}_1$ such that*

1) $\mathcal{H}_{1_i}$ reduces $\mathcal{S}$ $\quad \forall i \in I$;

2) $\sum^{\oplus} \mathcal{H}_{1_i} = \mathcal{H}_1$;

3) $\mathcal{H}_{1_i}$ is separable.

Proof Let $T \in \mathcal{S}$. Let $\mathcal{G}$ be a collection of subspaces of $\mathcal{H}_1$ such that if $\mathcal{M}, \mathcal{N} \in \mathcal{G}$, $\mathcal{M} \neq \mathcal{N}$, then $\mathcal{M} \perp \mathcal{N}$ and $\mathcal{M} \in \mathcal{G}$ implies that $\mathcal{M}$ is separable and reduces T. Let $\mathcal{Z}$ be the set of all such $\mathcal{G}$'s.

Since $\emptyset \in \mathcal{Z}$, $\mathcal{Z}$ is not empty. Order $\mathcal{Z}$ by set inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{Z}$ and let $\hat{\mathcal{K}}$ be the union of all elements in $\mathcal{C}$. If $\mathcal{M} \in \hat{\mathcal{K}}$, then there exists $\mathcal{K}_0 \in \mathcal{C}$ such that $\mathcal{M} \in \mathcal{K}_0$. Let $\mathcal{N} \in \hat{\mathcal{K}}$, $\mathcal{N} \neq \mathcal{M}$. Then there exists $\mathcal{K}_1 \in \mathcal{C}$ such that $\mathcal{N} \in \mathcal{K}_1$. Since $\mathcal{C}$ is a chain, $\mathcal{M}, \mathcal{N} \in \max(\mathcal{K}_0, \mathcal{K}_1)$. Thus $\hat{\mathcal{K}} \in \mathcal{Z}$ and clearly $\hat{\mathcal{K}}$ is an upper bound for $\mathcal{Z}$. So by Zorn's lemma, $\mathcal{Z}$ has a maximal element $\mathcal{K}$, say.

We claim that

$$\overline{\text{span}\mathcal{K}} = \mathcal{H}_1.$$

Suppose the claim is not true. Then the span of the orthogonal space, $\text{span}\mathcal{K}^{\perp}$, of $\mathcal{K}$ is nontrivial. Let $e \in (\text{span}\mathcal{K})^{\perp}$, $\|e\| = 1$. Let

$$\mathcal{M}_e = \overline{\{p(T, T^*)e : p(x, y) \text{ non-commutative polynomial}\}}.$$

Then $\mathcal{M}_e \leq \mathcal{H}_1$. If $h \in \mathcal{M}_e$, then

$$h = \lim_n p_n(T, T^*)e.$$

$$Th = \lim_n Tp_n(T, T^*)e = \lim_m p_m(T, T^*).$$

So $Th \in \mathcal{M}_e$. Similarly, $T^*h \in \mathcal{M}_e$.

Since both $Tp(T, T^*)$ and $T^*p(T, T^*)$ are polynomials in T and T^*, applying the preceding lemma to $\mathcal{M}_e$ we conclude that $\mathcal{M}_e$ is a reducing subspace for T.

Also the set of polynomials $p(T, T^*)$ with complex-rational coefficients, that is, with coefficients from $\mathbf{Q} + i\mathbf{Q}$ is dense in $\mathcal{M}_e$. Thus $\mathcal{M}_e$ is separable. Since $\mathcal{M}_e \subset (\mathrm{span}\mathcal{K})^\perp$, it is orthogonal to $\mathrm{span}\mathcal{K}$. So $\mathrm{span}\mathcal{K} \cup \{\mathcal{M}_e\}$ gives us an element that is larger than the maximal element in $\mathcal{Z}$, a contradiction. Therefore

$$\overline{\mathrm{span}\mathcal{K}} = \mathcal{H}_1.$$

This means that we can write

$$\mathcal{H}_1 = \sum_{\mathcal{M} \in \mathcal{K}}^{\oplus} \mathcal{M}.$$

a direct sum of separable Hilbert spaces. Thus $T = \sum_{\mathcal{M} \in \mathcal{K}}^{\oplus} T_{\mathcal{M}}$ and we can thus uniquely extend the definition of φ to any Hilbert space $\mathcal{H}_1$.

$\square$

The following results are proved in [H1, H2], and will play an important role in our discussion.

Theorem 1.1.5 *A function φ is decomposable on $\mathcal{H}$ if and only if there exists a net $< p_\lambda(x, y) >$ of non-commutative polynomials such that for all $T \in \mathcal{B}(\mathcal{H})$, $p_\lambda(T, T^*) \to \varphi(T)$*

in the strong operator topology.

Theorem 1.1.6 *If $T \in \mathcal{B}(\mathcal{H})$, then $W^*(T) = \{\varphi(T) : \varphi$ is a decomposable function on $\mathcal{H}\}$.*

Proposition 1.1.7 *Let φ be a decomposable function on $\mathcal{H}$. Then the following are equivalent.*

(i) φ is continuous;

(ii) $\varphi(T) \in C^*(T)$ for every $T \in \mathcal{B}(\mathcal{H})$;

(iii) $\varphi(\pi(T)) = \pi(\varphi(T))$ for every $T \in \mathcal{B}(\mathcal{H})$ and every representation π of $C^*(T, \varphi(T))$;

(iv) there is a sequence $< p_n(x, y) >$ of non-commutative polynomials such that $\|p_n(T, T^*) - \varphi(T)\| \to 0$ uniformly (in T) on bounded subsets of $\mathcal{B}(\mathcal{H})$.

The next result is a continuous analog of Theorem 1.1.6 (see [H1]).

Theorem 1.1.8 *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$C^*(T) = \{\varphi(T) : \varphi \quad \text{is a continuous decomposable function on } H\}.$$

Thus if φ is continuous, then $\varphi(a)$ makes sense on an arbitrary C^*−algebra, $C^*(a)$.

We apply these results to prove the following

Theorem 1.1.9 *Let $A = \sum_{i \in I}^{\oplus} A_i$, $T = \sum_{i \in I}^{\oplus} T_i$ where I is an indexing set. Then $A \in C^*(T)$ if and only if for all countable $I_0 \subset I$, $\sum_{i \in I_0}^{\oplus} A_i \in C^*(\sum_{i \in I}^{\oplus} T_i)$.*

Proof First suppose $A \in C^*(T)$. By [H1, Proposition 1.5],

$$C^*(T) = \overline{\{p(T, T^*) : p(x, y) \text{ is a non-commutative polynomial}\}}.$$

Thus since $A \in C^*(T)$, we can find a net $< p_n(T, T^*) >$ of polynomials in $C^*(T)$ converging to $\sum_{i \in I}^{\oplus} A_i$. Now

$$p_n(T, T^*) = \sum_{i \in I}^{\oplus} p_n(T_i, T_i^*).$$

Taking $p_n(T,T^*)$ to be zero except possibly for countably many i, that is taking $i \in I_0$ or what is the same thing, repeating the sequence of p_n's, we get the right hand side.

Now assume the right hand side. $T = \sum_{i \in I}^{\oplus} T_i$ and we may take each T_i to be defined on a separable space. Suppose $A \notin C^*(T)$. Then there exists $\epsilon > 0$ such that

$$
\begin{aligned}
\epsilon \quad &< \quad d(A, C^*(T)) \\
&= \inf_{p(x,y)} \|A - p(T,T^*)\| \\
&= \inf_n \sup_i \|A_i - p_n(T_i, T_i^*)\|
\end{aligned}
$$

where $\{p_n(x,y) : n > 0\}$ is the set of non-commutative polynomials with complex-rational coefficients.

This means that for each non-commutative polynomial $p_n(x,y)$ over the complex-rationals, we can find an index i_n such that

$$
\epsilon < \|A_{i_n} - p_n(T_{i_n}, T_{i_n}^*)\|.
$$

Let $I_0 = \{i_n\}$. Then I_0 is countable and $\sum_{i \in I_0}^{\oplus} A_i \notin C^*(\sum_{i \in I_0}^{\oplus} T_i)$, a contradiction.

$\square$

Corollary 1.1.10 *If* $\dim \mathcal{H} = \aleph_0$ *and* $\varphi(T) \in C^*(T)$ *for all* $T \in B(\mathcal{H})$, *then* $\varphi(S) \in C^*(S)$ *for all* S *on every Hilbert space.*

Proof First, write $S = \sum_{i \in I}^{\oplus} S_i$ where S_i is defined on a separable subspace for all $i \in I$. From the theorem,

$$
\varphi(S) \in C^*(S) \Leftrightarrow \varphi(\sum_{i \in I_0}^{\oplus} S_i) \in C^*(\sum_{i \in I_0}^{\oplus} S_i)
$$

for all countable $I_0 \subset I$. But the right hand side follows by hypothesis since S_i is defined

on a separable space and $\sum_{i \in I_0}^{\oplus} S_i$ is thus defined on a separable space since I_0 is countable.

Note that if $\mathcal{K}$ is another Hilbert space with $\dim \mathcal{K} = \aleph_0$, then there exists a unitary

operator $U : \mathcal{K} \to \mathcal{H}$. If $S \in \mathcal{B}(\mathcal{K})$, then $USU^* \in \mathcal{H}$. Let $\pi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ be defined by

$\pi(S) = USU^*$. Then π is a *-isomorphism and

$$
\begin{aligned}
\pi(\varphi(S)) &= U\varphi(S)U^* \\
&= \varphi(USU^*) \\
&\in C^*(USU^*).
\end{aligned}
$$

But

$$
\begin{aligned}
C^*(USU^*) &= C^*(\pi(S)) \\
&= \pi(C^*(S)).
\end{aligned}
$$

So $\varphi(S) \in C^*(S)$.

If $\dim \mathcal{K} = n < \infty$ and $S \in \mathcal{B}(\mathcal{K})$, write

$$
S = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus S_1 \oplus \cdots \oplus S_n \oplus \cdots
$$

and apply what we have just shown for $\dim \mathcal{K} = \aleph_0$.

$\square$

In [H2], Hadwin proves the following theorem:

Theorem 1.1.11 (Asymptotic Double Commutant) *Let* $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{S}$ *countable or*

norm separable. Let $T \in \mathcal{B}(\mathcal{H})$ *and* $\dim \mathcal{H} = \aleph_0$. *Then the following are equivalent.*

i) $T \in C^*(\mathcal{S})$;

ii) $\|U_n T - T U_n\| \to 0$ *whenever* $< U_n >$ *is a sequence of unitary operators such that for all* $S \in \mathcal{S}$, $\|U_n S - S U_n\| \to 0$;

iii) $\|P_n T - T P_n\| \to 0$ *whenever* $< P_n >$ *is a sequence of projections such that for all* $S \in \mathcal{S}$, $\|P_n S - S P_n\| \to 0$;

iv) $\|T A_n - A_n T\| \to 0$ *whenever* $< A_n >$ *is a bounded sequence in* $\mathcal{B}(\mathcal{H})$ *such that for all* $S \in \mathcal{S} \cup \mathcal{S}^*$, $\|S A_n - A_n S\| \to 0$.

We conclude this section with the following theorems which we state without proof. First, some definitions (see [**BFH, H1**]).

Definition 1.1.12 *Let* $\mathcal{S} \subset \cup\{\mathcal{B}(\mathcal{M}) : \mathcal{M} \leq \mathcal{H}\}$. *Then* $\mathcal{S}$ *is a* part class *in case it is closed under unitary equivalence and every operator* $T \in \mathcal{B}(\mathcal{H})$ *can be uniquely decomposed into the direct sum of an operator in* $\mathcal{S}$ *(the* $\mathcal{S}$-part*) and an operator with no suboperator in* $\mathcal{S}$ *(the non-$\mathcal{S}$-part).*

Definition 1.1.13 *Two operators* S *and* T *are said to be* approximately equivalent, *denoted* $S \sim_a T$ *if there is a sequence* $< U_n >$ *of unitary operators such that*

$$\|U_n^* S U_n - T\| \to 0.$$

Theorem 1.1.14 [**H1**, Theorem 5.1] *Suppose* $\mathcal{S} \subset \cup\{\mathcal{B}(\mathcal{M}) : \mathcal{M} \leq \mathcal{H}\}$, $\mathcal{S}$ *is a part class, and* $\mathcal{S} \cap \mathcal{B}(\mathcal{H})$ *is norm bounded. Then the following are equivalent:*

i) *there is a family* $\mathcal{F}$ *of continuous decomposable functions such that* $\mathcal{S} = \{S : \varphi(S) = 0 \quad \varphi \in \mathcal{F}\}$;

ii) *there is a continuous decomposable function* ψ *such that* $\mathcal{S} = \{S : \psi(S) = 0\}$;

iii) $\mathcal{S} \cap \mathcal{B}(\mathcal{H})$ *is norm closed;*

iv) $S \cap B(\mathcal{H})$ *is* *-strongly closed;*

v) there is a $T \in B(\mathcal{H})$ *such that* $S = \sum_{\max}(T)$, *where* $\sum_{\max}(T)$ *is the maximal reducing operator spectrum of* T *and is given by*

$$\sum_{\max} (T) = \cup\{\sum(\pi(T)) : \pi : C^*(T) \to B(\mathcal{H}) \text{ is a faithful representation}\}.$$

The following versions of the above result will be very important in our discussion.

Theorem 1.1.15 *Let* S *be a part class and suppose that* $S \cup B(\mathcal{H})$ *is norm bounded. Then the following are equivalent.*

(i) S *is closed under norm limits;*

(ii) S *is closed under* $\sim_a$;

(iii) S *is closed under* *−strong operator topology limits;*

(iv) there exists a continuous decomposable function φ *such that if* $T \in S$, *then* $\varphi(T) = 0$.

Theorem 1.1.16 (C^*−**property**) *Let* $\mathcal{P}$ *be a property of elements of unital* C^*−*algebras that is closed under direct sums. Let* $m \in \mathbb{R}$, *and suppose that* $T \in \mathcal{P}$ *implies that* $\|T\| \leq m$. *Suppose that for each* $a \in \mathcal{P}$ *and each unital representation* π *of* $C^*(a)$, *we have* $\pi(a) \in \mathcal{P}$. *Then there exists a decomposable function* φ *such that* $a \in \mathcal{P}$ *if and only if* $\varphi(a) = 0$.

Thus if $\mathcal{A}$ is the universal C^*−algebra generated by a, subject to a family of relations, then there exists a continuous decomposable function φ such that the family of relations is equivalent to $\varphi(a) = 0$.

1.2 Decomposable Functions of Several Variables

We now extend the definition of decomposable functions to several variables [H4]. Let $m \in \mathbf{Z}^+$. Define $\mathcal{B}(\mathcal{H})^m$ by

$$\mathcal{B}(\mathcal{H})^m = \{(A_1, A_2, \ldots, A_m) : A_1, A_2, \ldots, A_m \in \mathcal{B}(\mathcal{H})\}.$$

Definition 1.2.1 *Let $\mathcal{B}(\mathcal{M})^m$ be a subalgebra of $\mathcal{B}(\mathcal{H})^m$, where $\mathcal{M}$ is an arbitrary subspace of $\mathcal{H}$. A decomposable function of m variables is a function*

$$\varphi : \cup_{\mathcal{M} \leq \mathcal{H}} \mathcal{B}(\mathcal{M})^m \to \cup_{\mathcal{M} \leq \mathcal{H}} \mathcal{B}(\mathcal{M})$$

such that

1) $\varphi(\mathcal{B}(\mathcal{M})^m) \subset \mathcal{B}(\mathcal{M})^m$;

2) if $A_1, A_2, \ldots, A_m \in \mathcal{B}(\mathcal{H})$, $\mathcal{M}$ a subspace of $\mathcal{H}$ and $\mathcal{M}$ reduces $A_1, A_2, \ldots, A_m$, then $\mathcal{M}$ reduces $\varphi(A_1, A_2, \ldots, A_m)$ and

$$\varphi(A_1, A_2, \ldots, A_m)|\mathcal{M} = \varphi(A_1|\mathcal{M}, A_2|\mathcal{M}, \ldots, A_m|\mathcal{M});$$

3) if $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{H}$ and $U : \mathcal{M} \to \mathcal{N}$ is unitary and $A_1, A_2, \ldots, A_m \in \mathcal{B}(\mathcal{N})$, then

$$\varphi(U^* A_1 U, U^* A_2 U, \ldots, U^* A_m U) = U^* \varphi(A_1, A_2, \ldots, A_m) U.$$

The function φ is (norm) continuous if $\varphi|\mathcal{B}(\mathcal{H})^m$ is continuous (with the product topology on $\mathcal{B}(\mathcal{H})^m$). Also, we observe that all the results stated for decomposable functions of one

variable translate into ones for decomposable functions of several variables with only minor changes. The following is an easy example of a decomposable function of m variables. Let $p(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$ be a non-commutative polynomial of $2m$ variables and define φ by

$$\varphi(A_1, A_2, \ldots, A_m) = p(A_1, A_1^*, \ldots, A_m, A_m^*).$$

Then φ is a decomposable function.

The following result will be useful in the next chapter when we discuss completions of spaces of decomposable functions.

Let

$$\mathcal{D} = \{\varphi : \varphi \text{ is a continuous decomposable function of } m \text{ variables}\}.$$

Let $T = (T_1, T_2, \ldots, T_m) \in \mathcal{B}(\mathcal{H})^m$. Let $\mathcal{S}_n$ be the set of all operators $T \in \mathcal{B}(\mathcal{H})^m$ such that $\|T\| \leq n$. Define $\| \ \|_n : \mathcal{D} \to [0, \infty)$ by

$$\| \ \|_n(\varphi) = \sup_{T \in \mathcal{S}_n} \|\varphi(T)\|.$$

Then

$$
\begin{aligned}
\|\varphi\|_n &= \sup_{T \in \mathcal{S}_n} \|\varphi(T)\| \\
&= \| \oplus_{T \in \mathcal{S}_n} \varphi(T)\| \\
&= \|\varphi(\oplus_{T \in \mathcal{S}_n})\| \\
&< \infty.
\end{aligned}
$$

It is also clear that $\| \ \|_n$ is a seminorm on $\mathcal{D}$.

Define a metric on $\mathcal{D}$ as follows

$$d(\varphi_1, \varphi_2) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|\varphi_1 - \varphi_2\|_j}{1 + \|\varphi_1 - \varphi_2\|_j}.$$

Theorem 1.2.2 *The set $\mathcal{D}$ together with the above metric is a complete separable metric space.*

Proof We observe that

$$
\begin{aligned}
\|\varphi^*\varphi\|_n &= \sup_{T \in \mathcal{S}_n} \|\varphi^*(T)\varphi(T)\| \\
&= \sup_{T \in \mathcal{S}_n} \|\varphi(T)^*\varphi(T)\| \\
&= \sup_{T \in \mathcal{S}_n} \|\varphi(T)\|^2 \\
&= \|\varphi\|_n^2.
\end{aligned}
$$

Let $A = \oplus_{T \in \mathcal{S}_n} T$, where $T \in \mathcal{B}(\mathcal{H})^m$. Since φ is continuous, there exists a non-commutative polynomial $p_n(x_1, y_1, \ldots, x_m, y_m)$ such that

$$\|\varphi(A) - p_n(A, A^*)\| < \frac{1}{n}.$$

Thus

$$\|(\oplus_{T \in \mathcal{S}_n} \varphi(T)) - p_n(T, T^*)\| < \frac{1}{n}$$

or equivalently,

$$\|\varphi - p\|_n < \frac{1}{n}.$$

Thus polynomials are dense in $\mathcal{D}$ and taking those polynomials with coefficients in $\mathbf{Q} + i\mathbf{Q}$, we conclude that $\mathcal{D}$ is separable.

We will demonstrate the completeness of $\mathcal{D}$ in the next chapter.

$\square$

Chapter 2

Completions of the space of

non-commutative polynomials

Let $\mathcal{P}$ be the set of all non-commutative polynomials $p(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$. We shall define several topologies on $\mathcal{P}$ and show that the completion of $\mathcal{P}$ with respect to some of these topologies give the same set.

Completions of the space of non-commutative polynomials give us a natural way to represent decomposable functions and in fact enables one to naturally extend the definition of a continuous decomposable function to arbitrary C^*-algebras.

2.1 Locally convex topological vector spaces

In this section, we give a brief review of some topological results that we shall need later in our discussion. The reader is referred to [KN] for a more complete discussion of the background. We include proofs of some of the theorems here for completeness' sake.

Definition 2.1.1 *Let (X, τ) be a locally convex topological vector space. Then a net $< x_\lambda >$ is Cauchy if and only if $x_\lambda - x_\mu \to 0$ where $\lambda, \mu \in \mathcal{D}$ for some directed set $\mathcal{D}$ in which $(\lambda_1, \mu_1) \le (\lambda_2, \mu_2) \Leftrightarrow \lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$.*

Theorem 2.1.2 [KN] *Every locally convex topological vector space has a completion which is also a locally convex topological space.*

Proof Let X be a locally convex topological vector space. Then the topology on X is given by seminorms, say $\{\rho_i : i \in I\}$. Let $E_i = \{x \in X : \rho_i(x) = 0\}$. For $x + E_i \in X/E_i$, let $\overline{\rho_i}(x + E_i) = \rho_i(x)$. Then $(X/E_i, \overline{\rho_i})$ is a normed space. Now $(\overline{X/E_i}, \overline{\rho_i})$ is a completion of $(X/E_i, \overline{\rho_i})$. Put $Y = \Pi_{i \in I}(\overline{X/E_i}, \overline{\rho_i})$. Then Y is complete. Indeed suppose $< y_\alpha >$ is a Cauchy net in Y. Then $< y_\alpha^i >$ is Cauchy in $(\overline{X/E_i}, \overline{\rho_i})$ for all $i \in I$. Since $(\overline{X/E_i}, \overline{\rho_i})$ is complete, $< y_\alpha^i >$ converges in $(\overline{X/E_i}, \overline{\rho_i})$ for each $i \in I$. But this happens if and only if $< y_\alpha >$ converges in Y. Therefore Y is complete.

Let $\pi : X \to Y$ be given by $\pi(x)(i) = x + E_i$. If $x_\alpha \to x$, then

$$
\begin{aligned}
\pi(x_\alpha)(i) &= x_\alpha + E_i \\
&\to x + E_i \\
&= \pi(x)(i).
\end{aligned}
$$

Thus π is continuous. Also π is 1-1 since

$$\pi(x)(i) = E_i \quad \Leftrightarrow \quad x \in E_i \quad \text{for all } i \in I$$

$$\Leftrightarrow \quad \rho_i(x) = 0 \quad \text{for all } i \in I$$

$$\Leftrightarrow \quad x = 0.$$

Finally,

$$\pi(x_\alpha) \to \pi(x) \in \pi(X) \quad \Leftrightarrow \quad \pi(x_\alpha)(i) \to \pi(x)(i) \quad \forall i \in I$$

$$\Leftrightarrow \quad x_\alpha + E_i \to x + E_i \quad \forall i \in I$$

$$\Leftrightarrow \quad \overline{\rho_i}((x_\alpha - x + E_i)) \to 0 \quad \forall i \in I$$

$$\Leftrightarrow \quad \rho_i(x_\alpha - x) \to 0 \quad \forall i \in I$$

$$\Leftrightarrow \quad x_\alpha \to x.$$

Thus π^{-1} is continuous and so π is a homeomorphism from X onto $\pi(X)$.

Put $\tilde{X} = \overline{\pi(X)}$.

$\square$

The next two theorems are modifications of an exercise in [KN]. We assume that X and Y are Hausdorff.

Theorem 2.1.3 *If $T : X \to Y$ is a continuous linear map, then there exists a unique map $\tilde{T} : \tilde{X} \to \tilde{Y}$, with $\tilde{T}$ continuous and linear such that $\tilde{T}|X = T$.*

Proof Let $x \in \tilde{X}$. Then there exists a net $< x_\alpha > \subset X$ such that $x_\alpha \to x$. Since T is continuous linear and so uniformly continuous, $< Tx_\alpha >$ is Cauchy in $\tilde{Y}$ and thus converges there. Let $\tilde{T}x = \lim_\alpha Tx_\alpha$. Note that the limit is unique since Y is Hausdorff.

$\tilde{T}$ is well-defined. To see this, let $< x_\alpha >$ and $< y_\beta >$ be Cauchy nets in X both converging to $x \in \tilde{X}$. Let $\xi = \lim_\alpha T x_\alpha$ and $\zeta = \lim_\beta T y_\beta$. Then

$$\begin{aligned} \xi - \zeta &= \lim_\alpha T x_\alpha - \lim_\beta T y_\beta \\ &= \lim_{\alpha,\beta}(T x_\alpha - T y_\beta) \\ &= \lim_{\alpha,\beta} T(x_\alpha - y_\beta) \\ &= 0. \end{aligned}$$

Note that addition of nets in the preceding statements is as given in the proof of the preceding theorem. Next, we show that $\tilde{T}$ is uniformly continuous. Let W be a neighborhood of 0 in $\tilde{Y}$ and let V be a neighborhood of 0 in $\tilde{Y}$ with $V = -V$ and $V + V \subset W$. Let U be a neighborhood of 0 in $\tilde{X}$ such that if $x \in X$ and $x \in U$, then $Tx \in V$.

Such a neighborhood exists since if $U_0 \subset X$ then $U_0 = U \cap \tilde{X}$ for some open $U \subset \tilde{X}$. Choose V a neighborhood of 0 in $\tilde{Y}$ with $\overline{V} \subset W$ and such that $x \in U_0$ implies that $Tx \in V$. Now if $x_0 \in U_0$, then we can find a net x_λ in U_0 such that $x_\lambda \to x_0$. But this implies that $\tilde{T}x = \lim_\lambda T x_\lambda \in \overline{V} \subset W$. So $\tilde{T}(\overline{U}) \subset W$. On the other hand if $x \in U$ then there exists a net $x_\alpha \in X$ such that $x_\alpha \to x$. So eventually, $x_\alpha \in U \Rightarrow x_\alpha \in U_0$. Thus there exists a subnet $x_{\alpha_k} \in U_0$ converging to x. But this means that $x \in \overline{U_0}$.

Let $u \in \tilde{X}$ with $u \in U$ and let $< u_\alpha >$ be a net in X converging to u. Then for a sufficiently "large" α, $u_\alpha \in U$ and so $T u_\alpha \in V$. Since $T u_\alpha \to \tilde{T} u$, for a sufficiently "large" α,

$$\begin{aligned} Tu &\in Tu_\alpha + V \\ &\subset V + W \\ &\subset W. \end{aligned}$$

Thus T is uniformly continuous.

Clearly $\tilde{T}$ is unique for if S also extends T and if $x_\alpha \in X$ converges to $x \in \tilde{X}$, then by continuity,

$$
\begin{aligned}
Sx &= \lim_\alpha Sx_\alpha \\
&= \lim_\alpha Tx_\alpha \\
&= \tilde{T}x.
\end{aligned}
$$

$\square$

Theorem 2.1.4 *The map $\tilde{T}$ is 1-1 if and only if T is 1-1 and if $x_\alpha \to x_0$ whenever $< x_\alpha >$ is Cauchy in X with $Tx_\alpha \to Tx_0$.*

Proof ($\Rightarrow$) Without loss of generality, we may assume $Tx_0 = 0$. Now since $< x_\alpha >$ is Cauchy in X, $x_\alpha \to x$ for some $x \in \tilde{X}$. Continuity of T implies that

$$
Tx_\alpha = \tilde{T}x_\alpha \to \tilde{T}x.
$$

But we also already know that $Tx_\alpha \to 0$. Since $\tilde{X}$ is Hausdorff, $\tilde{T}x = 0$. Since $\tilde{T}$ is 1-1, $x = 0$. Clearly T is 1-1, being a restriction of a 1-1 map.

($\Leftarrow$) Suppose the converse is true and let $\tilde{T}x = 0$. Since $x \in \tilde{X}$, there exists a net $< x_\alpha >$ in X converging to x. Continuity of T implies

$$
\tilde{T}x_\alpha = Tx_\alpha \to 0.
$$

Clearly $< x_\alpha >$ is Cauchy in X and so by hypothesis, $x_\alpha \to 0$. Since $\tilde{X}$ is Hausdorff, $x = 0$.

Thus $\tilde{T}$ is 1-1. $\square$

2.2 Completions of the space $\mathcal{P}$

In this section, we plan to extend the idea of polynomials to a more general class of functions. This will enable us to get analogs of Theorems 1.1.6 and 1.1.8. We show that certain completions of the family of non-commutative polynomials are in fact equivalent. We shall restrict our discussion to decomposable functions of one variable but the results we present here apply to decomposable functions of several variables.

Let

$$\mathcal{P} = \{p(x,y) \; : \; p \;\; \text{non-commutative polynomial}\}.$$

This is an algebra with involution. To get involution, because we know that for operators A and B, $(AB)^* = B^* A^*$, and $(\lambda A)^* = \overline{\lambda} A^*$, we want $x^* = y$ and $y^* = x$. Thus for example if $p(x,y) = 2ix^2 y^3$, then

$$(2ix^2 y^3)^* = -2ix^3 y^2.$$

That is, interchange x and y, reverse the order of the factors and take the conjugate of the complex coefficients.

We now define several seminorms on the family $\mathcal{P}$. Let $T \in \mathcal{B}(\mathcal{H})$. Define seminorms on $\mathcal{P}$ as follows.

0) Put $\|p\|_T = \|p(T, T^*)\|$.

1) For $n \geq 1$, let

$$\|p\|_n = \sup\{\|p\|_T : \|T\| \leq n, \;\; T \in \mathcal{B}(\mathcal{H}), \;\; \mathcal{H} \;\; \text{a Hilbert space}\}.$$

Let $T \in \mathcal{B}(\mathcal{H})$, $f \in \mathcal{H}$ and define

2) $\|p\|_{T,T^*,f} = \|p\|_{T,f} + \|p\|_{T^*,f}$.

3) $\|p\|_{T,f} = \|p(T,T^*)f\|$.

Let $\mathcal{P}_1$ be the family of polynomials $\mathcal{P}$ with the topology obtained from the family of seminorms given in (1); $\mathcal{P}_2$ be $\mathcal{P}$ with topology obtained from the family of seminorns in (2); $\mathcal{P}_3$ be $\mathcal{P}$ with the topology given by the family of seminorns in (3). Each of these $\mathcal{P}_j$'s, $1 \le j \le 3$ is a locally convex topological vector space. By Theorem 2.1.2, each $\mathcal{P}_j$ has a completion $\tilde{\mathcal{P}}_j$ which is also a locally convex topological vector space. The elements in $\tilde{\mathcal{P}}_j$ are equivalence classes of Cauchy nets in $\mathcal{P}_j$.

Note that in the above discussion, we have not said anything about the family (0) seminorms. The following result explains why we do not need to define a topology for that family of seminorns.

Theorem 2.2.1 *The spaces $\mathcal{P}_0$ and $\mathcal{P}_1$ are equivalent.*

Proof Let $\epsilon > 0$ and suppose $p_0 \in \mathcal{P}$. Put

$$V_1 = \{p \in \mathcal{P} : \|p - p_0\|_T < \epsilon\}.$$

$$V_2 = \{p \in \mathcal{P} : \|p - p_0\|_n < \epsilon\}.$$

If $p \in V_2$, then

$$\|p - p_0\|_n = \sup_{\|T\| \le n} \|(p - p_0)(T,T^*)\|$$
$$< \epsilon.$$

This implies that $\|(p - p_0)(T,T^*)\| < \epsilon$ for all T with $\|T\| \le n$. So $p \in V_1$ and we have $V_2 \subset V_1$.

Conversely, consider the subset $\mathcal{B}$ of all operators $T \in \mathcal{B}(\ell^2)$ such that $\|T\| \le n$. We know

that all norms are equivalent on $\mathcal{B}$. Take a $*$−strong dense subset of $\mathcal{B}$, which we shall denote by $\mathcal{B}_d$. If $T \in \mathcal{B}(\ell^2)$, then we may write T as a direct sum of operators. If S is any operator with $\|S\| \leq n$, there exists an operator $S' \in \mathcal{B}(\ell^2)$ with $\|S\| = \|S'\|$ (see proof of Theorem 1.2.2). We have

$$
\begin{aligned}
\| \ \|_n &= \sup_{\|T\| \leq n} \|p(T, T^*)\| \\
&= \sup_{T \in \mathcal{B}(\ell^2)} \|p(T, T^*)\| \\
&= \|p(T, T^*)\| \qquad \text{write } T = \sum^{\oplus}_{T_j \in \mathcal{B}} T_j \\
&= \|p\|_T.
\end{aligned}
$$

Thus $\mathcal{P}_0$ and $\mathcal{P}_1$ are equivalent.

$\square$

Thus we shall restrict our discussion only to the spaces $\mathcal{P}_j$, $1 \leq j \leq 3$.

We observe that the map that sends $\mathcal{P}_{j-1} \to \mathcal{P}_j$, $2 \leq i \leq 3$, is continuous. So by Theorem 2.1.4, it is one-to-one. Thus we have

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3.$$

Thus we get the following picture.

$$
\begin{array}{ccccc}
\mathcal{P}_1 & \xrightarrow{i_1} & \mathcal{P}_2 & \xrightarrow{i_2} & \mathcal{P}_3 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\mathcal{P}}_1 & \xrightarrow{i_1} & \tilde{\mathcal{P}}_2 & \xrightarrow{i_2} & \tilde{\mathcal{P}}_3
\end{array}
$$

We now extend the definition of a decomposable function as follows:

Definition 2.2.2 *Let $\mathcal{L}(\mathcal{M})$ be the set of all linear transformations on $\mathcal{M}$, where $\mathcal{M} \leq \mathcal{H}$. The map $\varphi : \cup_{\mathcal{M} \leq \mathcal{H}} \{\mathcal{B}(\mathcal{M})\} \to \cup \mathcal{L}(\mathcal{M})$ is a* quasi-decomposable *function in case*

a) $\varphi(\mathcal{B}(\mathcal{M})) \subset \mathcal{L}(\mathcal{M})$ whenever $\mathcal{M} \leq \mathcal{H}$

b) if $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{M} \leq \mathcal{H}$ and $\mathcal{M}$ reduces T, then $\mathcal{M}$ reduces $\varphi(T)$ and $\varphi(T|\mathcal{M}) = \varphi(T)|\mathcal{M}$.

c) if $\mathcal{M}, \mathcal{N} \leq \mathcal{H}$ and $S \in \mathcal{B}(\mathcal{M})$ and $U : \mathcal{N} \to \mathcal{M}$ is unitary, then $\varphi(U^*SU) = U^*\varphi(S)U$.

Proposition 2.2.3 *If $\dim\mathcal{H} = \aleph_0$ and φ is a quasi-decomposable function on $\mathcal{H}$, then φ is a decomposable function on $\mathcal{H}$.*

Proof It is sufficient to show that $\varphi(\mathcal{B}(\mathcal{H})) \subset \mathcal{B}(\mathcal{H})$. Suppose not. Then there exists a unit vector $x_n \in \mathcal{H}$ such that

$$\|\varphi(T)x_n\| \geq 2^n.$$

Let $x = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$ and put $T^{(\infty)} = \sum^{\oplus} T$. Then

$$\begin{aligned}
\infty &> \|\varphi(T^{(\infty)}x)\|^2 \\
&= \sum_{n=1}^{\infty} \|\varphi(T)\frac{x_n}{n}\| \\
&\geq \sum_{n=1}^{\infty} \frac{2^n}{n} \\
&= \infty \quad \text{a contradiction.}
\end{aligned}$$

Thus $\varphi(T)$ is bounded.

$\square$

We shall also need the following lemma [KR].

Lemma 2.2.4 *Suppose $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is linear in the first coordinate, conjugate linear in the second and for all (x,y), there exists $M > 0$ such that $|\beta(x,y)| \leq M\|x\|\|y\|$. Then there exists a unique $A \in \mathcal{B}(\mathcal{H})$ such that*

i) for all $x, y \in \mathcal{H}$, $\beta(x, y) = < Ax, y >$,

ii) $\|A\| \leq M$.

Here is a $*-$strong version of [H1, Proposition 1.3].

Proposition 2.2.5 *If φ is a decomposable function on $\mathcal{H}$, then there is a net $< p_n(x, y) >$ of non-commutative polynomials such that $p_n(T, T^*) \rightarrow \varphi(T)$ in the $*-$strong operator topology for every $T \in \mathcal{B}(\mathcal{H})$.*

Proof Let φ be a decomposable function. Let $\epsilon > 0$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\} \subset \mathcal{B}(\mathcal{H})$. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$. Put $S = \sum_{1 \leq i \leq k}^{\oplus} S_i$, $f = \sum_{1 \leq i \leq k}^{\oplus} f_i$. Then since $\varphi(S) \in W^*(S)$, there exists a non-commutative polynomial $p(x, y)$ such that

$$\|p(S, S^*)f - \varphi(S)f\| \leq \epsilon.$$

Now because $\mathcal{S}$ is a bounded set of operators, the norm, weak and $*-$strong closures of the set of such non-commutative polynomials are all equivalent. Thus we can find a non-commutative polynomial $p(x, y)$ with

$$\|p(S, S^*)f - \varphi(S)f\| \leq \epsilon,$$

and

$$\|p(S, S^*)f - \varphi(S)^*f\| \leq \epsilon.$$

Therefore for all $h \in \mathcal{F}$, $1 \leq j \leq k$,

$$\|p(S_j, S_j^*)h - \varphi(S_j)h\| \leq \epsilon,$$

and

$$\|p(S_j, S_j^*)h - \varphi(S_j)^*h\| \le \epsilon.$$

Put $n = (S, \mathcal{F}, \epsilon)$, $p_n(x, y) = p(x, y)$ and define

$$(S_1, \mathcal{F}_1, \epsilon_1) \le (S_2, \mathcal{F}_2, \epsilon_2)$$

$$\Leftrightarrow$$

$$S_1 \subset S_2, \quad \mathcal{F}_1 \subset \mathcal{F}_2 \ \& \ \epsilon_1 \ge \epsilon_2.$$

Then $< p_n(x, y) >$ is a net and $p_n(T, T^*) \to \varphi(T)$ *−strongly for each $T \in \mathcal{B}(\mathcal{H})$.

□

Suppose that $\varphi \in \tilde{\mathcal{P}}_3$. Then we can associate φ with an equivalence class of Cauchy nets in $\mathcal{P}_3$, say $[< p_\lambda(x, y) >]$ with $< p_\lambda(x, y) >$ Cauchy in $\mathcal{P}_3$. This means that for all $T \in \mathcal{B}(\mathcal{H})$, for all $f \in \mathcal{H}$, $< p_\lambda(T, T^*)f >$ is norm Cauchy in $\mathcal{H}$. Since $\mathcal{H}$ is complete, the net coverges to some element $\tilde{\varphi}(T)f \in \mathcal{H}$.

We are now ready to show the equivalence of the completions $\tilde{\mathcal{P}}_2$ and $\tilde{\mathcal{P}}_3$.

Theorem 2.2.6

$$\tilde{\mathcal{P}}_2 = \tilde{\mathcal{P}}_3.$$

Proof First, let $\xi \in \tilde{\mathcal{P}}_2$. Then $\xi = [< p_n(x, y) >]$. For all e and all indexing sets I, $< \{p_n(x, y)\} >$ is norm Cauchy in $\mathcal{P}_2$. Define

$$\tilde{\xi}(T)e = \lim_n p_n(T, T^*)e$$

where $T \in \mathcal{B}(\ell^2)$, $e \in \ell^2$. Then $\tilde{\xi}$ is well-defined. Indeed suppose $p_n(x,y)$, $q_m(x,y)$ are Cauchy nets in the same equivalence class with

$$\xi = [< p_n(x,y) >] \text{ and } \xi' = [< q_m(x,y) >].$$

Let $T \in \mathcal{B}(\ell^2)$, $e \in \ell^2$. Then

$$\tilde{\xi}(T)e = \lim_n p_n(T,T^*)e.$$

$$\tilde{\xi}'(T)e = \lim_m q_m(T,T^*)e.$$

Now

$$
\begin{aligned}
\tilde{\xi}(T)e - \tilde{\xi}'(T)e &= \lim_n p_n(T,T^*)e - \lim_m q_m(T,T^*)e \\
&= \lim_{(n,m)} (p_n(T,T^*)e - q_m(T,T^*)e) \\
&= 0.
\end{aligned}
$$

We note that $\tilde{\xi} : \tilde{\mathcal{P}}_2 \rightarrow \mathcal{F}(\mathcal{B}(\ell^2), \mathcal{B}(\ell^2))$, that is, $\tilde{\xi}$ maps $\tilde{\mathcal{P}}_2$ into functions on $\mathcal{B}(\ell^2)$.

Claim

$\xi = \alpha \Leftrightarrow \tilde{\xi} = \tilde{\alpha}$.

Proof of claim

($\Rightarrow$) Suppose $\xi = [< p_n(x,y) >]$ and $\alpha = [< q_m(x,y) >]$. Then $\alpha = \xi$ implies

$$
\begin{aligned}
\tilde{\alpha}(T)e &= \lim_n p_n(T,T^*)e \\
&= \lim_m q_m(T,T^*)e \\
&= \tilde{\xi}(T)e.
\end{aligned}
$$

($\Leftarrow$) Conversely, if $\tilde{\alpha} = \tilde{\xi}$, then

$$\lim_m q_m(T,T^*)e = \lim_n p_n(T,T^*)e \Leftrightarrow \lim_{(n,m)} (q_m(T,T^*)e - p_n(T,T^*)e) = 0.$$

So $< q_m(x,y) - p_n(x,y) > \in [0]$ which proves the claim.

If φ is a decomposable function, then given $\epsilon > 0$ there exists a non-commutative polynomial $p(x,y)$ such that

$$\|\varphi(T)e - p(T,T^*)\| < \epsilon \quad \text{and} \quad \|\varphi(T)^*e - p^*(T,T^*)e\| < \epsilon.$$

Thus for any decomposable function φ, there exists $\xi \in \tilde{\mathcal{P}}_2$ such that

$$\tilde{\xi} = \varphi|\mathcal{B}(\ell^2).$$

Now let $\alpha = [< p_\lambda(x,y) >]$ and let $\tilde{\alpha} : \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2)$ be given by

$$\tilde{\alpha}(T)e = \text{norm} \lim_\lambda p_\lambda(T,T^*)e.$$

Let $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ arbitrary. Let $\tilde{\tilde{\alpha}}(T)f = \lim_\lambda p_\lambda(T,T^*)f$ with $\tilde{\tilde{\alpha}}|\mathcal{B}(\ell^2) = \tilde{\alpha}$. Then $\tilde{\tilde{\alpha}}$ is a decomposable function. To show this, first we note that if $T \in \mathcal{B}(\mathcal{H})$, then so is $\tilde{\tilde{\alpha}}(T)$. Suppose not. Then for each n, we can find $x_n \in \mathcal{H}$, $\|x_n\| < \frac{1}{n}$ such that $\|\tilde{\tilde{\alpha}}(T)x_n\| \geq 2^n$. Let $x = (x_1, x_2, \ldots)$ and let $T^{(\infty)} = \sum^\oplus T$. Then

$$\begin{aligned} \tilde{\tilde{\alpha}}(T^{(\infty)})x &= (y_1, y_2, \ldots) \\ &= \lim_\lambda p_\lambda(T^{(\infty)}, T^{(\infty)*})x \end{aligned}$$

$$= \lim_\lambda (p_\lambda(T,T^*)x_1, p_\lambda(T,T^*)x_2, \ldots).$$

So

$$
\begin{aligned}
y_n &= \lim_\lambda p_\lambda(T,T^*)x_n \\
&= \tilde{\tilde{\alpha}}(T)x_n.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\infty &> \|\tilde{\tilde{\alpha}}(T^{(\infty)})x\| \\
&= \sqrt{\sum_n \|\tilde{\tilde{\alpha}}(T)x_n\|^2} \\
&\geq \sqrt{\sum_n 2^{2n}} \\
&= \infty \qquad \text{a contradiction.}
\end{aligned}
$$

Thus $\tilde{\tilde{\alpha}}(\mathcal{B}(\mathcal{M})) \subset \mathcal{B}(\mathcal{M})$.

If $\mathcal{M}$ reduces T and $e \in \mathcal{H}$, then

$$\tilde{\tilde{\alpha}}(T)(\mathcal{M}) = \lim_\lambda p_\lambda(T,T^*)(\mathcal{M}) \subset \mathcal{M}$$

and

$$
\begin{aligned}
\tilde{\tilde{\alpha}}(T|\mathcal{M})e &= \lim_\lambda p_\lambda(T|\mathcal{M}, T^*|\mathcal{M})e \\
&= \lim_\lambda p_\lambda(T,T^*)|\mathcal{M}e \\
&= \tilde{\tilde{\alpha}}(T)|\mathcal{M}e.
\end{aligned}
$$

If $\mathcal{M}, \mathcal{N} \leq \mathcal{H}$, $S \in \mathcal{B}(\mathcal{M})$ and $U : \mathcal{N} \to \mathcal{M}$ is unitary, then (for $e \in \mathcal{N}$)

$$
\begin{aligned}
\tilde{\alpha}(U^*SU)e &= \lim_\lambda p_\lambda(U^*SU, U^*S^*U)e \\
&= \lim_\lambda U^* p_\lambda(S, S^*)Ue \\
&= U^*\tilde{\alpha}(S)Ue.
\end{aligned}
$$

So $\tilde{\alpha}(U^*SU) = U^*\tilde{\alpha}(S)U$. Thus we have shown that $\tilde{\alpha}$ is a decomposable function.

To complete the proof of the equivalence of the two spaces, we show that

$$\tilde{\tilde{\alpha}} = \tilde{\tilde{\beta}} \Leftrightarrow \tilde{\alpha} = \tilde{\beta} \Leftrightarrow \alpha = \beta.$$

We have already proven the second $\Leftrightarrow$. For the first part,

($\Rightarrow$) Suppose $\tilde{\tilde{\alpha}} = \tilde{\tilde{\beta}}$. Then since $\tilde{\tilde{\alpha}}|\mathcal{B}(\ell^2) = \tilde{\alpha}$ and $\tilde{\tilde{\beta}}|\mathcal{B}(\ell^2) = \tilde{\beta}$, we have $\tilde{\alpha} = \tilde{\beta}$.

($\Leftarrow$) Now suppose $\tilde{\alpha} = \tilde{\beta}$. Then on $\mathcal{B}(\ell^2)$,

$$\tilde{\tilde{\alpha}} = \tilde{\tilde{\beta}}.$$

If $\mathcal{H}$ is an arbitrary Hilbert space, write

$$\mathcal{H} = \sum^{\oplus} \mathcal{H}_j$$

where $\mathcal{H}_j$ is separable and $\dim\mathcal{H}_j = \aleph_0$. Then $\mathcal{H}_j \simeq \ell^2$ for all j and so

$$\tilde{\tilde{\alpha}}|\mathcal{B}(\mathcal{H}_j) = \tilde{\tilde{\beta}}|\mathcal{B}(\mathcal{H}_j).$$

But since $\tilde{\tilde{\alpha}}$ and $\tilde{\tilde{\beta}}$ are decomposable functions and since any $T \in \mathcal{B}(\mathcal{H})$ may be written as

$$T = \sum^{\oplus} T_j \quad T_j \in \mathcal{B}(\mathcal{H}_j),$$

$\tilde{\tilde{\alpha}}$ and $\tilde{\tilde{\beta}}$ must agree on $\mathcal{H}$. Applying [H1, Proposition 1.3] again but with $\tilde{\mathcal{P}}_3$ in place of $\tilde{\mathcal{P}}_2$, we get a one-to-one correspondence of $\tilde{\mathcal{P}}_3$ with $\mathcal{D}$. So $\tilde{\mathcal{P}}_3$ and $\tilde{\mathcal{P}}_2$ are equivalent.

$\square$

Remark Let $\varphi \in \tilde{\mathcal{P}}_3$. Then it follows from the above proof that for all $T \in \mathcal{B}(\mathcal{H})$, and all $f \in \mathcal{H}$,

Theorem 2.2.7 *The map $\tilde{\varphi}(T) : \mathcal{H} \to \mathcal{H}$ given by*

$$\tilde{\varphi}(T)f = \lim_{\lambda} p_{\lambda}(T, T^*)f$$

is linear and bounded.

Theorem 2.2.8 *Let $\mathcal{D}_c$ be the set of all continuous decomposable functions. Endow $\mathcal{D}_c$ with the metric discussed in Section 1.2. Then*

$$\tilde{\mathcal{P}}_1 = \mathcal{D}_c.$$

Proof First we consider the case when $T \in \mathcal{B}(\ell^2)$. Let $\xi \in \tilde{\mathcal{P}}_1$. Then $\xi = [< p_{\lambda}(x, y) >]$. For all $e \in \ell^2$ and all indexing sets I, $< \{p_{\lambda}(T, T^*)\} >$ is norm Cauchy. Put

$$\tilde{\xi}(T)e = \lim_{\lambda} p_{\lambda}(T, T^*)e.$$

Then $\tilde{\xi}$ is well-defined and $\xi = \zeta$ if and only if $\tilde{\xi} = \tilde{\zeta}$. If $\varphi \in \mathcal{D}_c$, then for any $\epsilon > 0$ and

any operator T in a bounded subset of $\mathcal{B}(\ell^2)$, there exists a non-commutative polynomial $p(x, y)$ such that

$$\|p(T, T^*) - \varphi(T)\| \to 0 \quad \text{uniformly.}$$

Thus for any continuous decomposable function φ, we can find $\xi \in \tilde{\mathcal{P}}_1$ such that

$$\tilde{\xi} = \varphi | \mathcal{B}(\ell^2).$$

If $\alpha = [< p_\lambda(x, y) >]$, let $\tilde{\alpha} : \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2)$ be given by

$$\tilde{\alpha}(T) = \text{norm} \lim_\lambda p_\lambda(T, T^*).$$

For an arbitrary Hilbert space $\mathcal{H}$, define

$$\tilde{\tilde{\alpha}}(T) = \lim_\lambda p_\lambda(T, T^*)$$

with

$$\tilde{\tilde{\alpha}} | \mathcal{B}(\ell^2) = \tilde{\alpha}.$$

Then $\tilde{\tilde{\alpha}}$ is a continuous decomposable function. We also note that if $\alpha = \beta$, where α is as above and $\tilde{\beta}(T) = \text{norm-}\lim_\mu q_\mu(T, T^*)$, then

$$
\begin{aligned}
\tilde{\alpha}(T) &= \lim_\lambda p_\lambda(T, T^*) \\
&= \lim_\mu q_\mu(T, T^*) \\
&= \tilde{\beta}.
\end{aligned}
$$

Conversely, we note that $\tilde{\alpha} = \tilde{\beta}$ implies that $p_\lambda(x,y) - q_\mu(x,y)$ belongs to the equivalence class $[0]$.

Also, since $\tilde{\alpha}|\mathcal{B}(\ell^2) = \tilde{\alpha}$ and $\tilde{\beta}|\mathcal{B}(\ell^2) = \tilde{\beta}$, equality of $\tilde{\alpha}$ and $\tilde{\beta}$ implies equality of $\tilde{\alpha}$ and $\tilde{\beta}$. Conversely, if $\tilde{\alpha} = \tilde{\beta}$, then on $\mathcal{B}(\ell^2)$, $\tilde{\alpha} = \tilde{\beta}$. For an arbitrary Hilbert space, we apply Lemma 1.1.4 to write $\mathcal{H}$ as a direct sum of orthogonal separable subspaces $< \mathcal{H}_j >$ and we get that $\tilde{\alpha} = \tilde{\beta}$ on $\mathcal{B}(\mathcal{H}_j)$ for all j. Since $\tilde{\alpha}$ and $\tilde{\beta}$ are (continuous) decomposable functions and since any T may be written as $T = \sum_j^\oplus T_j$, where $T_j \in \mathcal{B}(\mathcal{H}_j)$, it follows that $\tilde{\alpha}$ and $\tilde{\beta}$ must agree on $\mathcal{H}$. Thus $\tilde{\mathcal{P}}_1$ and $\mathcal{D}_c$ are equivalent and the theorem is proved.

$\square$

What we have shown in the preceding results is that the diagram below commutes.

$$
\begin{array}{ccccc}
\mathcal{P}_1 & \xrightarrow{i_1} & \mathcal{P}_2 & \xrightarrow{i_2} & \mathcal{P}_3 \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{\mathcal{P}}_1 & \xrightarrow{i_1} & \tilde{\mathcal{P}}_2 & \xrightarrow{i_2} & \tilde{\mathcal{P}}_3 \\
\downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 \\
\mathcal{F}(\mathcal{B}(\ell^2)) & \xrightarrow{id} & \mathcal{F}(\mathcal{B}(\ell^2)) & \xrightarrow{id} & \mathcal{F}(\mathcal{B}(\ell^2))
\end{array}
$$

We observe also that the maps h_j, $1 \leq j \leq 3$ are all one-to-one and so i_1 and i_2 are also one-to-one. Our results show that h_2 and h_3 have the same range. It follows from this that the map i_2 is onto. Also note that the inclusion $\mathrm{ran}h_1 \subset \mathrm{ran}h_2$ is strict.

Chapter 3

Finitely generated free

C^*-algebras

In this chapter we will study free C^*-algebras generated by elements satisfying certain matricial relations. We shall compute the $K-$groups of some of these C^*-algebras and determine the conditions under which they have no non-trivial projections.

Throughout our discussion, Rep$(\mathcal{A},\mathcal{B})$ will represent the collection of all representations from $\mathcal{A}$ into $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. The collection of all representations from the C^*-algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ will be denoted by Rep$(\mathcal{A},\mathcal{H})$.

3.1 K−theory consequences

Let $\mathcal{A}$ be a C^*−algebra generated by $a_1, a_2, \ldots, a_m$ subject to relations that are closed under representations, direct sums, norm limits and summands and such that if $A_1, A_2, \ldots, A_m$ is a collection of operators on a Hilbert space satisfying the same relations, then there exists a unital representation π such that $\pi(a_j) = A_j$, $1 \leq j \leq m$.

Definition 3.1.1 *The C^*−algebra $\mathcal{A}$ satisfying the defining condition above is called a* universal C^*−algebra.

Let φ be a null-bounded decomposable function (see Definition 1.1.2) of m variables. Define $\mathcal{F}_{n,\varphi}$ to be the universal C^*−algebra generated by $\{a_{ijk}\}$, $1 \leq i, j \leq n$, $1 \leq k \leq m$ subject to the condition that $\varphi((a_{ij1}), (a_{ij2}), \ldots, (a_{ijm})) = 0$. The notation $\mathcal{F}_\varphi$ will denote the universal C^*−algebra generated by the elements $a_1, a_2, \ldots, a_m$ subject to the condition that $\varphi(a_1, a_2, \ldots, a_m) = 0$.

We note here that $\mathcal{F}_{n,\varphi}$ makes sense only for null-bounded decomposable functions. To see this, suppose φ is a continuous decomposable function and suppose that $\{T_n\}$ is a set of operators such that $\|T_n\| \geq n$ for each n and $\varphi(T_n) = 0$. Let $\mathcal{F}_{n,\varphi} = C^*(a)$ where for each n, there exists a representation π_n such that $\pi_n(a) = T_n$. Then $n \leq \|T_n\| \leq \|a\|$ which is clearly a contradiction.

From [H1, Corollary 3.2], we get

Theorem 3.1.2 *A collection of free relations on a set $a_1, a_2, \ldots, a_m$ of generators can be represented as a single equation $\varphi(a_1, a_2, \ldots, a_m) = 0$, where φ is a null-bounded decomposable function of m variables.*

Thus the universal C^*−algebra generated by $a_1, a_2, \ldots, a_m$ subject to the given relations is isomorphic to $\mathcal{F}_\varphi$.

Here are some examples of universal C^*-algebras.

Example 3.1.3 *Consider the C^*-algebra generated by $a_1, a_2, \ldots, a_m$ where*

$$\sum_{i=1}^{m}[(a_i a_i^* - 1)^2 + (a_i^* a_i - 1)^2] = 0.$$

This is $C^*(\mathbf{F}_m)$, the C^*-algebra generated by the free group on m generators.

If we put $\varphi(a_1, a_2, \ldots, a_m) = \sum_{i=1}^{m}[(a_i a_i^* - 1)^2 + (a_i^* a_i - 1)^2]$, then φ is a null-bounded decomposable function of m variables and annihilates every element satisfying the defining condition of the above C^*-algebra. Thus this C^*-algebra is isomorphic to $\mathcal{F}_\varphi$.

Example 3.1.4 (The Cuntz algebra) *Let*

$$\mathcal{O}_m = C^*(\{v_1, v_2, \ldots, v_m : v_i^* v_i = 1, \ \sum_{i=1}^{m} v_i v_i^* = 1\}).$$

Here, we define a null-bounded decomposable function φ by

$$\varphi(v_1, v_2, \ldots, v_m) = (\sum_{i=1}^{m}|1 - v_i^* v_i|) + |\sum_{i=1}^{m} v_i v_i^* - 1|.$$

If $v_1, v_2, \ldots, v_m$ satisfy the conditions defining the Cuntz algebra, then $\varphi(v_1, v_2, \ldots, v_m) = 0$. Thus in this case $\mathcal{O}_m$ is isomorphic to $\mathcal{F}_\varphi$.

Example 3.1.5 *Let θ be an irrational number and suppose that u and v are unitary elements of some C^*-algebra such that $uv = e^{2\pi i\theta}vu$. Then we get the irrational rotation C^*-algebra*

$$\mathcal{A}_\theta = C^*(u, v).$$

We observe that in this case, $\mathcal{A}_\theta$ is isomorphic to $\mathcal{F}_\varphi$ where $\varphi(a_1, a_2) = |a_1 a_2 - e^{2\pi i\theta} a_2 a_1|$.

Example 3.1.6 *Let K be a compact subset of $\mathbb{C}^m$. Let*

$$\mathcal{F}_{m,K} = C^*(A_1, A_2, \ldots, A_m)$$

where A_i, $1 \le i \le m$, are commuting normal operators with joint spectrum in the set K. Then $\mathcal{F}_{m,K}$ is a universal C^*-algebra.

We note here that $\mathcal{F}_{1,K} = C(K)$, the continuous functions on K. If we take K to be the unit circle, we get the following example of a universal C^*-algebra studied in [Mc1].

Example 3.1.7 *Let U_n^{nc} be the C^*-algebra generated by $\{a_{ij}\}$, $1 \le i, j \le n$, subject to the condition that the matrix (a_{ij}) be unitary.*

Before we give our last example, we need the following definition.

Definition 3.1.8 *Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then T is called C^*-universal if for any contraction S, there is a $*$-homomorphism $\pi : C^*(T) \to C^*(S)$ such that $\pi(T) = S$.*

Example 3.1.9 *Consider the universal C^*-algebra generated by the elements $\{a_{ij}\}$, $1 \le i, j \le n$ subject to the condition that $\|(a_{ij})\| \le 1$. Define φ by*

$$\varphi(a) = 1 - a^* a - |1 - a^* a|.$$

Then φ is a continuous decomposable function and $\varphi((a_{ij})) = 0$. Thus $C^*(a_{ij}) \simeq \mathcal{F}_{n,\varphi}$.

We shall now give the definition of the concept of free products of C^*-algebras (see [Mc1]).

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and suppose $\mathcal{C}$ is a subalgebra of both $\mathcal{A}$ and $\mathcal{B}$. For each n, $1 \le n \le \aleph_0$, let $\mathcal{H}_n$ be a Hilbert space with $\dim \mathcal{H}_n = n$. Let

$$\Im = \{\mathcal{H}_n : \dim \mathcal{H}_n = n, \ 1 \le n \le \aleph_0\}.$$

Definition 3.1.10 *We define the* free product *of the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ over the subalgebra C, denoted by $\mathcal{A} *_C \mathcal{B}$ by*

$$\mathcal{A} *_C \mathcal{B} = C^*(\tilde{\pi}(\mathcal{A}) \cup \tilde{\rho}(\mathcal{B}))$$

where $\tilde{\pi}$ is the direct sum of all $\pi \in Rep(\mathcal{A}, \mathcal{H})$, for all $\mathcal{H} \in \Im$. Similarly $\tilde{\rho}$ is the direct sum of all $\rho \in Rep(\mathcal{B}, \mathcal{H})$.

From this definition, we observe that the defining property for $\mathcal{A} *_C \mathcal{B}$ is that if $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$ and $\rho \in \mathrm{Rep}(\mathcal{B}, \mathcal{H})$ such that $\pi|C = \rho|C$, then there exists $\sigma \in \mathrm{Rep}(\mathcal{A} *_C \mathcal{B}, \mathcal{H})$ such that $\sigma|\mathcal{A} = \pi$ and $\sigma|\mathcal{B} = \rho$.

In [Mc1, Theorem 2.3], McClanahan shows that if a unital C^*-algebra $\mathcal{A}$ has a unital *-homomorphism into the set of complex numbers, then the K-groups of $M_n *_{\mathbb{C}} \mathcal{A}$ and $\mathcal{A}$ are isomorphic. We use this result to compute the K-groups of $\mathcal{F}_{n,\varphi}$.

The next result is a version of Paschke's result given in [Mc1].

Theorem 3.1.11 *There exists a *-isomorphism $\sigma : \mathcal{F}_{n,\varphi} \otimes M_n \to \mathcal{F}_\varphi * M_n$ so that $\sigma|M_n$ is the natural inclusion of M_n into $\mathcal{F}_\varphi * M_n$.*

Proof Let u be a generator for $\mathcal{F}_\varphi$ and put

$$u_{ij} = \sum_{k=1}^{n} e_{ki} u e_{jk}$$

where e_{rs} is a matrix unit of M_n. Put

$$v = (u_{ij}).$$

Then

$$v \;=\; (e_{ij}) \, \mathrm{diag}(u)(e_{ij}) \qquad 1 \le i,j \le n$$

$$=\; W^* \, \mathrm{diag}(u) W$$

where $W = (e_{ij})$. We observe that

$$W = W^* = W^{-1}.$$

Thus $C^*((u_{ij})) \simeq C^*(u)$.

Also

$$u_{ij} e_{rs} \;=\; \sum_{k=1}^{n} (e_{ki} u e_{jk}) e_{rs}$$

$$=\; e_{ri} u e_{js}$$

and

$$e_{rs} u_{ij} \;=\; e_{rs}(\sum_{k=1}^{n} e_{ki} u e_{jk})$$

$$=\; e_{ri} u e_{js}.$$

Therefore $u_{ij} \in M_n^c$, the commutant of M_n. So there exists a *-homomorphism

$$\rho : \mathcal{F}_{n,\varphi} \to C^*(\{u_{ij} : 1 \le i,j \le n\})$$

such that $\rho(1) = 1$, and $\rho(v_{ij}) = u_{ij}$. We also know that there exists a *-homomorphism (the inclusion map)

$$\imath : M_n \rightarrow \mathcal{F}_\varphi * M_n$$

such that

$$\imath(e_{ij}) = e_{ij}.$$

Therefore it follows from the definition of tensor products that there exists a unique *-homomorphism

$$\pi : \mathcal{F}_{n,\varphi} \otimes M_n \rightarrow \mathcal{F}_\varphi * M_n$$

such that

$$\pi(a \otimes b) = \rho(a)\imath(b).$$

We show that π is both one-to-one and onto.

Since

$$
\begin{aligned}
\mathrm{diag}(u) &= \sum_{i,j=1}^{n} e_{ii}ue_{jj} \\
&= \sum_{i,j=1}^{n} e_{ir}e_{ri}ue_{js}e_{sj} \\
&= \sum_{i,j=1}^{n} e_{ir}e_{rs}u_{ij}e_{sj},
\end{aligned}
$$

we have that

$$
\begin{aligned}
\mathrm{ran}\pi &= C^*(\mathrm{ran}\rho \cup \mathrm{ran}\imath) \\
&= C^*(\{u_{ij} : \quad 1 \leq i,j \leq n\} \cup \{e_{rs} : \quad 1 \leq r,s \leq n\}).
\end{aligned}
$$

Thus π is onto.

To see that π is one-to-one, first observe that $(v_{ij}) \in \mathcal{F}_{n,\varphi} \otimes M_n$ and

$$\varphi((v_{ij})) = 0.$$

So there exist *-homomorphisms ρ_1 and ρ_2,

$$\rho_1 : \mathcal{F}_\varphi \to \mathcal{F}_\varphi \otimes M_n$$

$$\rho_2 : M_n \to \mathcal{F}_{n,\varphi} \otimes M_n$$

with

$$\rho_1(1) = 1, \quad \rho_1(u) = (v_{ij}).$$

$$\rho_2(1) = 1, \quad \rho_2(A) = 1 \otimes A.$$

We also have a *-homomorphism

$$\bar{\rho} : \mathcal{F}_\varphi * M_n \to \mathcal{F}_{n,\varphi} \otimes M_n$$

such that

$$\bar{\rho}|_{\mathcal{F}_\varphi} = \rho_1 \quad \text{and} \quad \bar{\rho}|_{M_n} = \rho_2.$$

Now

$$\bar{\rho} \circ \pi : \mathcal{F}_{n,\varphi} \otimes M_n \to \mathcal{F}_{n,\varphi} \otimes M_n$$

and

$$\bar{\rho} \circ \pi(v_{ij} \otimes 1) = v_{ij} \otimes 1.$$

$$\tilde{\rho} \circ \pi(1 \otimes A) = 1 \otimes A.$$

And the proof is complete.

□

We note that the relative commutant of M_n in $\mathcal{F}_{n,\varphi} \otimes M_n$ is isomorphic to the relative commutant of $\pi(\mathcal{F}_{n,\varphi} \otimes M_n)$ in $\mathcal{F}_\varphi * M_n$. But the commutant of $\mathcal{F}_{n,\varphi} \otimes M_n$ is $\mathcal{F}_{n,\varphi} \otimes 1$ which is isomorphic to $\mathcal{F}_{n,\varphi}$. Thus we have

Corollary 3.1.12 $\mathcal{F}_{n,\varphi}$ *is isomorphic to the relative commutant of* $\mathcal{F}_\varphi * M_n$.

□

The following theorem is due to McClanahan [**Mc1**, Theorem 2.3].

Theorem 3.1.13 *Suppose* $\mathcal{A}$ *is a unital* $C^*-algebra$ *and suppose*

$$\alpha : \mathcal{A} \to \mathbb{C}$$

*is a unital *-homomorphism. Then*

$$K_j(\mathcal{A} * M_n) \cong K_j(\mathcal{A}) \quad for \quad j \geq 1, \quad j = 0, 1.$$

□

Applying the above theorem, we get the following corollaries to Theorem 3.1.11.

Corollary 3.1.14 *Let φ be a null-bounded decomposable function and suppose $\mathcal{F}_\varphi$ admits a multiplicative linear functional. Then*

$$K_j(\mathcal{F}_{n,\varphi}) = K_j(\mathcal{F}_\varphi) \quad j = 0, 1.$$

□

Corollary 3.1.15 *Consider the C^*-algebra $\mathcal{F}_{n,\varphi}$ given in Example 3.1.9 which is generated by the elements $\{a_{ij}\}$ subject to the condition that $\|(a_{ij})\| \leq 1$. Then*

$$K_j(\mathcal{F}_{n,\varphi}) = K_j(C^*(T))$$

where T is a C^*-universal operator.

□

Since

$$K_j(C^*(T)) = K_j(\mathbb{C}), \qquad j = 0, 1, \quad (\text{see [FH, Theorem 3.3]}),$$

we get $K_0(\mathcal{F}_{n,\varphi}) = \mathbb{Z}$ and $K_1(\mathcal{F}_{n,\varphi}) = 0$.

Corollary 3.1.16 *Let K be a compact subset of the complex plane. Then*

$$K_j(\mathcal{F}_{n,K}) = K_j(\mathcal{F}_{1,K}) = K_j(C(K)), \qquad j = 0, 1.$$

□

The next result is due to McClanahan [Mc1, Corollary 2.4]. We note that in our notation, $U_n^{nc} = \mathcal{F}_{n,\mathbb{T}}$, where $\mathbb{T}$ is the unit circle (see Example 3.1.7).

Corollary 3.1.17

$$K_j(U_n^{nc}) = K_j(C(\mathbf{T})) = \mathbf{Z}$$

□

Consider the C^*−algebra $C^*((a_{ij}))$, $1 \le i, j \le n$ subject to the condition that

$(a_{ij}^*)(a_{ij}) = I$. If we define φ by

$$\varphi(a) = |1 - a^*a|,$$

then this defines a null-bounded decomposable function and we have $C^*((a_{ij})) \simeq \mathcal{F}_{n,\varphi}$.

When $n = 1$, we get $\mathcal{F}_\varphi \simeq \mathcal{T}$, the *Toeplitz* algebra. The algebra $\mathcal{T}$ is generated by the

unilateral shift [**NW**]. Hence we have

Corollary 3.1.18

$$K_0(\mathcal{F}_{n,\varphi}) = \mathbf{Z}$$

and

$$K_1(\mathcal{F}_{n,\varphi}) = 0.$$

□

Note: Suppose $\mathcal{A}$ is a C^*−algebra and suppose $a \in \mathcal{A}$. Consider the class of all operators

T such that there exists a representation $\pi : C^*(a) \to C^*(T)$, $\pi(1) = 1$ and $\pi(T) = a$. This

class is defined by a null-bounded decomposable function φ so that $\varphi(T) = 0$ if and only

if T is in the class. Then $C^*(a) \simeq \mathcal{F}_\varphi$. Let $\mathcal{F}_{n,\varphi}$ be the universal C^*−algebra generated

by $\{a_{ij}\}$, $1 \le i, j \le n$ subject to the condition that $\varphi((a_{ij})) = 0$. Then there exists a

representation $\pi : C^*(a) \to C^*(\{a_{ij}\})$, with $\pi(1) = 1$ and $\pi(a) = (a_{ij})$.

Thus for example if a is normal, $C^*(a) \simeq C(\sigma(a)) \simeq \mathcal{F}_{\sigma(a)}$. Now $C^*(a)$ admits a multiplica-

tive linear functional, so $K_j(C^*(a)) = K_j(C^*(\{a_{ij}\}))$.

We can get similar results for other classes of operators such as hyponormal, hermitian, and

subnormal operators.

3.2 Projectionless Spaces

In [Mc1], McClanahan studies the unital C^*−algebra generated by the elements u_{ij} , $1 \le i,j \le n$ satisfying the relations which make (u_{ij}) a unitary matrix. It is shown that this C^*−algebra has no non-trivial projections. Also, Froelich and Salas show in [FH, Theorem 1.3] that if T is a C^*−universal operator then $C^*(T)$ has no non-trivial projections. In this section, we use decomposable functions to prove a more general result for which the above are special cases.

Definition 3.2.1 *Let φ be a null-bounded decomposable function of m variables. Define*

$$\mathcal{G}_{n,\varphi} = \{T = (T_1, T_2, \ldots, T_m) \in M_n(\mathcal{B}(\ell)^2))^m \ : \ \varphi(T) = 0\}.$$

Let

$$\mathcal{G}_{n,\varphi,e} = \{T \in \mathcal{G}_{n,\varphi} \ : \ C^*(\{T_1, T_2, \ldots, T_m\}) \cap M_n(\mathcal{K}(\ell^2)) = \{0\}\}.$$

We are now ready to prove the main result in this section.

Theorem 3.2.2 *Suppose φ is a null-bounded decomposable function of m variables and every connected component of $\mathcal{G}_{n,\varphi,e}$ contains an element $((b_{ij1}), (b_{ij2}), \ldots, (b_{ijm}))$ such that $C^*(\{b_{ijk} \ : \ 1 \le i,j \le n, \ 1 \le k \le m\})$ contains no non-trivial projections. Then $\mathcal{F}_{n,\varphi}$ contains no non-trivial projections.*

Proof Since $\mathcal{F}_{n,\varphi}$ is a separable C^*−algebra, there is a faithful unital representation $\pi : \mathcal{F}_{n,\varphi} \to \mathcal{B}(\ell^2)$ such that π is unitarily equivalent to $\pi \oplus \pi \oplus \cdots$. Let $\{a_{ijk} \ : \ 1 \le i,j \le n, \ 1 \le k \le m\}$ be the generators of $\mathcal{F}_{n,\varphi}$ with

$\varphi((a_{ij1}),(a_{ij2}),\ldots,(a_{ijm})) = 0$. Let

$$T = ((\pi(a_{ij1})),(\pi(a_{ij2})),\ldots,(\pi(a_{ijm}))).$$

Since $\pi \simeq \pi \oplus \pi \oplus \cdots$, clearly $T \in \mathcal{G}_{n,\varphi,e}$. Let $\mathcal{E}$ be the connected component of $\mathcal{G}_{n,\varphi,e}$

containing T and let $S = ((b_{ij1}),(b_{ij2}),\ldots,(b_{ijm})) \in \mathcal{E}$ be an element such that

$$\mathcal{A} = C^*(\{b_{ijk} \ : \ 1 \le i,j \le n, \ 1 \le k \le m\})$$

contains no non-trivial projections. Let $\mathcal{C}$ denote the C^*-algebra of bounded norm contin-

uous functions $g : \mathcal{E} \to \mathcal{B}(\ell^2)$ such that $g(S) \in \mathcal{A}$. For $1 \le i,j \le n$, $1 \le k \le m$, define

$\alpha_{ijk} : \mathcal{E} \to \mathcal{B}(\ell^2)$ such that for each $A \in \mathcal{E}$,

$$A = ((\alpha_{ij1}(A)),(\alpha_{ij2}(A)),\ldots,(\alpha_{ijm}(A))).$$

Then $\alpha_{ijk} \in \mathcal{C}$ for $1 \le i,j \le n, 1 \le k \le m$ and

$$\varphi(((\alpha_{ij1}),(\alpha_{ij2}),\ldots,(\alpha_{ijm}))) = 0.$$

Thus there exists a unital representation $\rho : \mathcal{F}_{n,\varphi} \to \mathcal{C}$ such that

$$\rho(a_{ijk}) = \alpha_{ijk} \quad \text{for} \quad 1 \le i,j \le n, \ 1 \le k \le m.$$

Since evaluation at T with ρ is π, that is, $\rho(a_{ijk})(T) = \pi(a_{ijk})$, ρ must be faithful.

The proof is completed by showing that $\mathcal{C}$ has no non-trivial projections. To see this, suppose

P is a projection in $\mathcal{C}$. Then $P(S)$ is a projection in $\mathcal{A}$. Thus $P(S) = 0$ or $P(S) = I$. By

considering $I - P$ if necessary, we may assume that $P(S) = 0$. Let $h : \mathcal{E} \to \{0, 1\}$ be defined by

$$g(A) = \|P(A)\|.$$

Then h is a continuous function and so $h(\mathcal{E})$ is a connected subset of $\{0, 1\}$. Since $h(S) = 0$, h must be identically 0. Thus $P = 0$.

$\square$

Remark: The preceding theorem remains true if $\mathcal{G}_{n,\varphi,e}$ is replaced by $\mathcal{G}_{n,\varphi}$. To see this, let $m = 1$ and note that the component $\mathcal{E}_0$ of $\mathcal{G}_{n,\varphi,e}$ containing an element T contains the closure of the unitary orbit of T. It follows from Voiculescu's theorem [H3] that the closure of the unitary orbit of T contains an operator unitarily equivalent to $T^{(\infty)} = T \oplus T \oplus \cdots$. The map $A \mapsto A^{(\infty)}$ defines a map from $\mathcal{G}_{n,\varphi}$ to $\mathcal{G}_{n,\varphi,e}$ that maps the connected component $\mathcal{E}_1$ containing T in $\mathcal{G}_{n,\varphi}$ to $\mathcal{E}_1^{(\infty)}$. Thus $\mathcal{E}_1^{(\infty)} \subset \mathcal{E}_0$. If $\mathcal{E}_1$ contains an element (b_{ij}) such that $C^*(\{b_{ij} \ : \ 1 \le i, j \le n\})$ contains no non-trivial projections, then $(b_{ij}^{(\infty)}) \in \mathcal{E}_0$ and $C^*(\{b_{ij}^{(\infty)} \ : \ 1 \le i, j \le n\})$ also contains no non-trivial projections.

Our first application of Theorem 3.2.2 is the result of Froelich and Salas [FH, Theorem 1.3]. In [FH], a C^*−universal operator T is defined by the single relation $\|T\| \le 1$. Thus $C^*(T) = \mathcal{F}_\varphi$, where

$$\varphi(a) = 1 - a^* a - |1 - a^* a|.$$

Note that $\|a\| \le 1$ if and only if $\varphi(a) = 0$. In this case

$$\mathcal{F}_{n,\varphi} = \{T \in M_n(\mathcal{B}(\ell^2)) \ : \ \|T\| \le 1\}.$$

Clearly, $\mathcal{G}_{n,\varphi}$ is connected and contains 0, whose matrix elements belong to the projectionless

C^*-algebra $\mathbb{C}I$.

Corollary 3.2.3 *If $\varphi(a) = 1 - a^*a - |1 - a^*a|$, then $\mathcal{F}_{n,\varphi}$ contains no non-trivial projections.*

$\square$

A similar argument applies to the C^*-algebra generated by $a_1, a_2, \ldots, a_m$ subject to the condition that $\|a_k\| \leq 1$ for $1 \leq k \leq m$.

Corollary 3.2.4 *If $\varphi(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m}(|1 - a_k^*a_k| - (1 - a_k^*a_k))$, then $\mathcal{F}_{n,\varphi}$ contains no non-trivial projections for $n = 1, 2, \ldots$.*

$\square$

In [Ch], M-D Choi proved that the C^*-algebra of the free group on m generators contains no non-trivial projections. This C^*-algebra is $\mathcal{F}_\varphi$ with

$$\varphi(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m} |1 - a_k^*a_k| + |1 - a_ka_k^*|.$$

In this case,

$$\mathcal{G}_{n,\varphi} = \{(T_1, T_2, \ldots, T_m) \in M_n(\mathcal{B}(\ell^2))^m \; : \; T_1, 1_2, \ldots, T_m \text{ unitary}\}.$$

Since the set of unitary operators is connected, $\mathcal{G}_{n,\varphi}$ is connected and contains $(I, I, \ldots, I)$.

Corollary 3.2.5 *If*

$$\varphi(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m} |1 - a_k^* a_k| + |1 - a_k a_k^*|,$$

then $\mathcal{F}_{n,\varphi}$ contains no non-trivial projections for $n = 1, 2, \ldots$.

□

We now turn our attention to a generalization of McClanahan's result on U_n^{nc} mentioned in the introduction to this section.

Suppose K is a subset of $\mathbb{C}^m$. We know from [H1] that there exists a continuous decomposable function φ of m variables such that $\varphi(a_1, a_2, \ldots, a_m) = 0$ if and only if $a_1, a_2, \ldots, a_m$ are commuting normal operators with joint spectrum contained in K. For this φ, we denote $\mathcal{F}_{n,\varphi}$ by $\mathcal{N}_{n,K}$. Note that if $n = 1$ and K is the unit circle, then $\mathcal{N}_{n,K}$ is the C^*–algebra U_n^{nc} in [Mc1]. For notational convenience, we restrict ourselves to the case in which $m = 1$. However, our results are true for arbitrary $1 \le m < \infty$.

Lemma 3.2.6 *Suppose K has p connected components, $1 \le p \le n < \infty$. Then every connected component of $\mathcal{G}_{n,\varphi}$ contains a matrix $diag(\lambda_1 I, \lambda_2 I, \ldots, \lambda_n I)$.*

In the statement above, $diag(a_1, a_2, \ldots, a_n)$ stands for the diagonal matrix

$$\begin{pmatrix} \lambda_1 I & & & \\ & \lambda_2 I & & \\ & & \ddots & \\ & & & \lambda_3 I \end{pmatrix}$$

whose only non-zero entries appear along the main diagonal.

Proof Suppose $T \in \mathcal{G}_{n,\varphi,\epsilon}$ and $\mathcal{E}$ is the connected component of $\mathcal{G}_{n,\varphi,\epsilon}$ containing T. Since the set of unitary operators is connected, $\mathcal{E}$ is closed under approximate equivalence. But $C^*(T) \cap \mathcal{K}(\ell^2) = 0$ implies that T is approximately equivalent to a diagonal operator D such that

$$D = D \oplus D \oplus \cdots = D^{(\infty)}.$$

Let $K_1, K_2, \ldots, K_p$ denote the connected components of K and let $\alpha_i \in K_i$, $1 \leq i \leq p$. Then since $p \leq n$, D is unitarily equivalent to

$$\begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix}$$

with $\sigma(D_i) \subset K_{q_i}$, D_i diagonal and

$$D_i \simeq D_i^{(\infty)} \quad \text{for} \quad 1 \leq i \leq n.$$

Note that for each integer $r \geq 1$, and each K_i, the set of diagonal operators of the form $\mathrm{diag}(\lambda_1^{(\infty)}, \lambda_2^{(\infty)}, \ldots, \lambda_r^{(\infty)})$ with $\lambda_1, \lambda_2, \ldots, \lambda_r \in K_i$ is homeomorphic to the cartesian product of r copies of K_i and is therefore connected. Since the set of unitary operators is connected, the set $\mathcal{D}_{i,r}$ of all diagonal operators A such that $A \simeq A^{(\infty)}$ and has finite spectrum contained in K_i and $\mathrm{card}\,\sigma(A) \leq r$ is connected and contains $\alpha_i I$. Thus the norm closure of the union of the $\mathcal{D}_{i,r}$'s which is the set of all the diagonal operators A such that $A = A^{(\infty)}$ and $\sigma(A) \subset K_i$ is connected and contains $\alpha_i I$.

Thus the connected component $\mathcal{E}$ which contains D must contain $\mathrm{diag}(\alpha_{q_1} I, \alpha_{q_2} I, \ldots, \alpha_{q_n} I)$.

$\square$

Theorem 3.2.7 *If K is a nonempty compact subset of $\mathbb{C}^m$ with p connected components, and if $p \leq n < \infty$, then $\mathcal{N}_{n,K}$ contains no non-trivial projections.*

$\square$

Corollary 3.2.8 *If K is connected, then $\mathcal{N}_{n,K}$ contains no non-trivial projections for all $n \geq 1$.*

$\square$

Example 3.2.9 *Suppose K has two components. Then $\mathcal{N}_{n,K} = C(K)$ contains a non-trivial projection.*

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