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Approximate equivalence invon Neumann algebras

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APPROXIMATE EQUIVALENCE IN VON NEUMANN ALGEBRAS

BY

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DISSERTATION

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This dissertation has been examined and approved.

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Dedication

To my husband and my family, for their love and encouragement throughout this endeavour.

Acknowledgments

Many people have helped make this paper possible. I would like to thank all of my professors at the University of New Hampshire for their academic contributions and my colleagues for their friendship and personal help. In particular I wish to thank Eric Nordgren and Rita Hibschweiler for their help in writing this thesis. Most importantly I must thank my advisor, Donald W. Hadwin for his patience and many helpful suggestions.

Foreword

Suppose $\mathcal{A}$ is a unital C^*-algebra, $B(H)$ is the set of all operators on a Hilbert space H and $\pi, \rho : \mathcal{A} \longmapsto B(H)$ are unital $*$-homomorphisms. We say π and ρ are approximately equivalent, denoted by $\pi \sim_a \rho$, if there is a net $\{u_n\}$ of unitary operators in $B(H)$ such that

$$\|u_n^* \pi(a) u_n - \rho(a)\| \longrightarrow 0 \text{ for every } a \text{ in } \mathcal{A}.$$

In [VOI 1], D. Voiculescu proved a very deep theorem that characterizes approximate equivalence for representations when $\mathcal{A}$ and H are both separable. Later D. Hadwin ([HAD 2]) showed that Voiculescu's characterization could be formulated in terms of the "rank" function; more precisely,

$\pi \sim_a \rho$ if and only if rank $\pi(a) =$ rank $\rho(a)$ for every a in $\mathcal{A}$.

D. Hadwin ([HAD 2]) also proved that the "rank" characterization holds when $\mathcal{A}$ or H is nonseparable.

We will look at a "localized" version of Voiculescu's theorem where we replace $B(H)$ with a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H. If $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital $*$-homomorphisms, we say that π is approximately equivalent to ρ in $\mathcal{R}$, denoted by $\pi \sim_a \rho\ (\mathcal{R})$, if there is a net $\{u_n\}$ of unitary operators in $\mathcal{R}$ such that

$$\|u_n^* \pi(a) u_n - \rho(a)\| \longrightarrow 0 \text{ for every } a \text{ in } \mathcal{A}.$$

The role of "rank" will be played by our newly-defined function "$\mathcal{R}$−rank". If $T \in B(H)$, then rank T is the Hilbert-space dimension of the closure of the range of T. Hence the rank of T is a function of the projection onto the closure of the range of T. In $B(H)$ two projections P, Q have the same rank if and only if there is a partial isometry V in $B(H)$ such that $P = V^*V$ and $Q = VV^*$.

In other words two projections in $B(H)$ have tha same rank if and only if they are Murray-von Neumann equivalent. This equivalence for projections in a von Neumann algebra is one of the fundamental concepts used in the classification and structure theory for von Neumann algebras.

We define the "$\mathcal{R}$−rank" of an operator T in the von Neumann $\mathcal{R}$ to be the Murray-von Neumann equivalence class in $\mathcal{R}$ of the projection onto the closure of the range of T.

The main focus of this thesis is trying to determine if the following version of Voiculescu's theorem is true:

Problem: $\pi \sim_a \rho\ (\mathcal{R}) \Longleftrightarrow \mathcal{R}-\text{rank } \pi(a) = \mathcal{R}-\text{rank } \rho(a)$ for every a in $\mathcal{A}$.

This paper is organized as follows.

Chapter 1 introduces the sufficient and necessary condition for two normal operators A and B in any von Neumann algebra $\mathcal{R}$, that acts on a separable Hilbert space, to be approximately equivalent with unitaries in the given von Neumann algebra $\mathcal{R}$, that is $\mathcal{R}-\text{rank } f(A) = \mathcal{R}-\text{rank } f(B)$ for every continuous function f. In the first section, we give the definition of "$\mathcal{R}$−rank" function, then we summarize the definitions and propositions in the literature, that will be used in our paper. Section §1.2 proves that the condition is sufficient. In the third section we present some results of direct integrals, which are related to our work. Next we investigate the properties of $\mathcal{R}$−rank function. We prove that the set of operators T in $\mathcal{R}$, with property $\mathcal{R}-\text{rank } T \preceq \mathcal{R}-\text{rank } A$ for a fixed operator A in $\mathcal{R}$, is closed under $*$-strong sequential limits. First we prove the result for factor von

Neumann algebras of type I_n, type I_∞, type II_1, type II_∞ and type III. Then we extend the result to any von Neumann algebra acting on a separable Hilbert space. Finally in this chapter we finish the proof of the necessity of the condition for approximately equivalent normal operators in any von Neumann algebra acting on a separable Hilbert space.

In Chapter 2, we classify approximately equivalent unital representations π and ρ, from a certain class of C^*-algebras to all von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space, by the "$\mathcal{R}-$rank" function. The conclusion is that π and ρ are approximately equivelent with unitaries in $\mathcal{R}$ if and only if $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$. In the first section we prove the necessary condition for the general case: if π and ρ are unital representations from any C^*-algebra into any von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space, that are approximately equivalent, then $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$. In Section §2.2, we study a class of C^*-algebras, we denote it by Q. A C^*-algebra $\mathcal{A}$ is in Q provided for every von Neumann algebra $\mathcal{S}$, for all unital representations π and ρ from $\mathcal{A}$ into $\mathcal{S}$, if $\mathcal{S}-\text{rank} \circ \pi = \mathcal{S}-\text{rank} \circ \rho$, then π and ρ are approximately equivalent in $\mathcal{S}$. We prove that if every von Neumann algebra $\mathcal{S}$ is acting on a separable Hilbert space, then $C(X)$ is contained in Q and that if $\mathcal{A}$ is in Q, then $M_n(\mathcal{A})$ is also contained in Q for every $n \geq 1$. We also prove that Q is closed under direct sum, direct limit and quotient map from a C^*-algebra onto the quotient C^*-algebra. A more interesting result is that if a C^*-algebra $\mathcal{A}$ is in Q, π and ρ are unital representations from $\mathcal{A}$ into a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space, such that for each a in $\mathcal{A}$ there are sequences $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ in $\mathcal{R}$ all depending on a such that $A_n\pi(a)B_n$ convergent to $\rho(a)$ and $C_n\rho(a)D_n$ convergent to $\pi(a)$ $*$-strongly, then π and ρ are approximately equivalent in $\mathcal{R}$.

Contents

ABSTRACT

APPROXIMATE EQUIVALENCE IN VON NEUMANN ALGEBRAS

by

Hui-Ru Ding
University of New Hampshire, December, 1993

In this paper we investigate approximate equivalence in von Neumann algebras. We find a necessary and sufficient condition for two normal operators to be approximately equivalent in any von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H with unitaries in $\mathcal{R}$. For the approximate equivalence of two unital representations from a given C^*- algebra to any von Neumann algebra acting on a separable Hilbert space, we find the necessary condition for the general case. Finally we investigate an interesting class of C^*-algebras, closed under direct sum, direct limit and quotient map, which contains $C(X)$ and $M_n(\mathcal{A})$, where $\mathcal{A}$ is in Q.

Chapter 1

Approximately Equivalent Normal Operators in von Neumann Algebras

Motivated by D. Voiculescu and D. Hadwin's works about the approximately unitary equivalence of any two normal operators in an operator algebra $B(H)$, where H is a separable Hilbert space, we use the "$\mathcal{R}$−rank" function to classify approximately equivalent normal operators in a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space.

The main result in this chapter is : For any two normal operators A and B in a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H, A and B are approximately equivalent with unitaries in $\mathcal{R}$ if and only if $\mathcal{R}$−rank $f(A) = \mathcal{R}$−rank $f(B)$ for every continuous function f.

Throughout this thesis $\mathcal{R}$ is a von Neumann algebra, I is the identity operator in the corresponding algebra and $\sigma(A)$ is the spectrum of operator A. The range and kernal of an arbitrary function F are denoted by ran F and ker F respectively. Let $\mathbf{C}$ be the set of complex numbers and $\mathbf{R}$ be the set of real numbers. By continuous function, we mean a complex-valued continuous function on the spectrum of the corresponding operator.

1.1 Preliminaries

Definition 1.1.1 *[KAP 1] Two projections E and F are said to be Murray-von Neumann equivalent in $\mathcal{R}$ (written $E \sim F$ $(\mathcal{R})$), when $V^*V = E$ and $VV^* = F$ for some partial isometry V in $\mathcal{R}$. A projection E is weaker than a projection F in $\mathcal{R}$ (written $E \prec F$ $(\mathcal{R})$), when E is equivalent to a subprojection of F. When $E \sim F$ $(\mathcal{R})$ or $E \prec F$ $(\mathcal{R})$, we write $E \preceq F$ $(\mathcal{R})$.*

Definition 1.1.2 *Two operators A and B in $\mathcal{R}$ are said to be approximately equivalent in $\mathcal{R}$ (written $A \sim_a B$ $(\mathcal{R})$) if there is a sequence $\{U_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{R}$ such that*

$$\|U_n A U_n^* - B\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Definition 1.1.3 *For an operator A in $\mathcal{R}$, $\mathcal{R}-$rank A is the Murray-von Neumann equivalence class of the projection $P_{\overline{\text{ran } A}}$ onto the closure of the range of A. We say $\mathcal{R}-$rank $A \preceq \mathcal{R}-$rank B if and only if $P_{\overline{\text{ran } A}} \preceq P_{\overline{\text{ran } B}}$ $(\mathcal{R})$.*

Example 1.1.4 *The following examples give equivalent conditions for equality of "$\mathcal{R}-$rank" function in some von Neumann algebras.*

1. *If $\mathcal{R} = B(H)$, and A and B are in $\mathcal{R}$, then*

$$\mathcal{R}-\text{rank } A = \mathcal{R}-\text{rank } B \Longleftrightarrow \dim\,(\overline{\text{ran } A}) = \dim\,(\overline{\text{ran } B}).$$

2. *If $\mathcal{R}$ is a type II_1 factor von Neumann algebra, τ is the central value trace on $\mathcal{R}$, then*

$$\mathcal{R}-\text{rank } A = \mathcal{R}-\text{rank } B \Longleftrightarrow \tau(P_{\overline{\text{ran } A}}) = \tau(P_{\overline{\text{ran } B}}).$$

The following definitions and propositions will be used throughout this thesis.

Definition 1.1.5 *[ARV 2] A polish space is a topological space which is homeomorphic to a separable metric space.*

Example 1.1.6 *The following are examples of some polish spaces.*

1. *Let* **N** *be the set of positive integers endowed with the discrete topology. Then* **N** *is a polish space.*
2. *A countable direct product of polish spaces is a polish space.*
3. *A closed subspace of a polish space is a polish space.*

Definition 1.1.7 *[ARV 2] A subset of a polish space P is called analytic if it has the form $f(Q)$, where Q is a polish space and f is a continuous map of Q into P.*

Definition 1.1.8 *[ARV 2] Let X be a separable metric space. A subset E of X is absolutely measurable if for every σ-finite Borel measure μ on X, E is μ-measurable. (i.e. $E = A \cup B$, $\mu(A) = 0, B$ is a Borel set).*

Definition 1.1.9 *[ARV 2] Let X and Y be topological spaces, $f : X \longmapsto Y$ a Borel function. A Borel cross section for f is a Borel function $g : Y \longmapsto X$ such that $f \circ g = id_Y$, where id_Y is the identity map on Y.*

Definition 1.1.10 *[KAP 1] A projection E in a von Neumann algebra $\mathcal{R}$ is said to be an abelian projection in $\mathcal{R}$ if $E\mathcal{R}E$ is abelian.*

Definition 1.1.11 *[KR 1] The central carrier of an operator A in a von Neumann algebra $\mathcal{R}$ is the projection $I - P$, where P is the union of all central projections P_α in $\mathcal{R}$ such that $P_\alpha A = 0$.*

Definition 1.1.12 *[KR 2] A projection E in a von Neumann algebra $\mathcal{R}$ is said to be infinite (relative to $\mathcal{R}$) when $E \sim E_0$ ($\mathcal{R}$) and $E_0 < E$ for some projection E_0 in $\mathcal{R}$. Otherwise, E is said to be finite (relative to $\mathcal{R}$). If E is infinite and PE is either 0 or infinite, for each central projection P, then E is said to be properly infinite.*

Definition 1.1.13 *[MN 1] A von Neumann algebra $\mathcal{R}$ is said to be a factor if the center of $\mathcal{R}$ consists of scalar multiples of I.*

Definition 1.1.14 *[KR 2] A von Neumann algebra $\mathcal{R}$ is said to be of type I if it has an abelian projection with central carrier the identity - of type I_n if the identity is the sum of n equivalent abelian projections. If $\mathcal{R}$ has no non-zero abelian projections but has a finite projection with central carrier the identity, then $\mathcal{R}$ is said to be of type II - of type II_1 if the identity is finite - of type II_∞ if the identity is properly infinite. If $\mathcal{R}$ has no non-zero finite projections, the $\mathcal{R}$ is said to be of type III.*

Definition 1.1.15 *[KR 2] Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$ and unitary group $\mathcal{U}$. By a center-valued trace on $\mathcal{R}$ we mean a linear mapping $\tau : \mathcal{R} \longmapsto \mathcal{C}$ such that:*

1. $\tau(AB) = \tau(BA)$ $(A, B \in \mathcal{R})$,
2. $\tau(C) = C$ $(C \in \mathcal{C})$,
3. $\tau(A) > 0$ $(A \in \mathcal{R}, A > 0)$.

Definition 1.1.16 *[KR 2] A weight on a von Neumann algebra $\mathcal{R}$ is a mapping ρ from $\mathcal{R}^+$ (the positive operators in $\mathcal{R}$) into the interval $[0, \infty]$ such that:*

1. $\rho(A + B) = \rho(A) + \rho(B)$ $(A, B \in \mathcal{R}^+)$,
2. $\rho(aA) = a\rho(A)$ $(A \in \mathcal{R}^+, a \geq 0)$.

A weight ρ is a tracial weight if, in addition

3. $\rho(AA^*) = \rho(A^*A)$.

   A weight ρ is normal when there is a family $\{\rho_a : a \in \Omega\}$ of positive normal functionals ρ_a on $\mathcal{R}$ such that

4. $\rho(A) = \sum_{a\in\Omega} \rho_a(A)$, *for each* $A \in \mathcal{R}^+$.

A weight ρ is semifinite when the linear span of $\mathcal{F}_\rho = \{A \in \mathcal{U}^+ : \rho(A) < \infty\}$ is weak-operator dense in $\mathcal{R}$, where $\mathcal{U}^+$ is the set of positive unitary operators in $\mathcal{R}$.

A weight ρ is faithful if $\rho(A) > 0$, whenever $A \in \mathcal{R}$ and $A > 0$.

Definition 1.1.17 *[KR 2] Let Ω be a σ-compact, locally compact (Borel measure) space. Let μ be the completion of a Borel measure on Ω. Suppose $\{H_p\}$ is a family of separable Hilbert spaces indexed by the points p of Ω. We say that a separable Hilbert space H is the direct integral of $\{H_p\}$ over (Ω, μ) (written as $H = \int_\Omega^\oplus H_p d\mu(p)$) when, to each x in H, there corresponds a function $p \longmapsto x(p)$ on Ω such that $x(p) \in H_p$ for each p and*

1. $p \longmapsto < x(p), y(p) >$ *is μ-integrable and* $< x, y > = \int_\Omega < x(p), y(p) > d\mu(p)$, *where* $x, y \in H$, $< \ , \ >$ *is the inner product in the corresponding Hilbert space.*

2. *If $u_p \in H_p$ for all p in Ω and $p \longmapsto < u_p, y(p) >$ is integrable for each $y \in H$, then there is a $u \in H$ such that $u(p) = u_p$ for almost every p.*

   We say that $\int_\Omega^\oplus H_p d\mu(p)$ and $p \longmapsto x(p)$ are the (direct integral) decompositions of H and x respectively.

Example 1.1.18 *A direct sum of Hilbert spaces is the case of a direct integral decomposition over a discrete measure space.*

Definition 1.1.19 *[KR 2] Suppose that H is the direct integral of $\{H_p\}$ over (Ω, μ), then an operator $T \in B(H)$ is said to be decomposable when there is a function $p \longmapsto T(p)$ on Ω such that $T(p) \in B(H_p)$ and for each $x \in H$, $T(p)(x(p)) = (T(x))(p)$ for almost every p.*

Definition 1.1.20 *[KR 2] Suppose that H is the direct integral of Hilbert spaces $\{H_p\}$ over (Ω, μ). A representation φ of a C^*-algebra $\mathcal{A}$ on H is said to be decomposable over (Ω, μ) when there is a representation φ_p of $\mathcal{A}$ on H_p such that $\varphi(A)$ is decomposable for each $A \in \mathcal{A}$ and $\varphi(A)(p) = \varphi_p(A)$ almost everywhere. The mapping $p \longmapsto \varphi_p$ is said to be a decomposition of φ.*

Definition 1.1.21 *[KR 2] Let H be the direct integral of Hilbert spaces $\{H_p\}$ over (Ω, μ). A von Neumann algebra $\mathcal{R}$ on H is said to be decomposable with decomposition $p \longmapsto \mathcal{R}_p$ when $\mathcal{R}$ contains a norm-separable strong-operator-dense C^*-algebra $\mathcal{A}$ for which the identity representation i is decomposable and such that $i_p(\mathcal{A})$ is strong-operator dense in $\mathcal{R}_p$ almost everywhere. In this case we write $\mathcal{R} = \int_\Omega^\oplus \mathcal{R}_p d\mu(p)$.*

Proposition 1.1.22 *[KAP 1] Every von Neumann algebra is uniquely a direct sum of algebras of type I, II_1, II_∞ and III.*

Proposition 1.1.23 *[KAP 2] A type I von Neumann algebra $\mathcal{R}$ can be decomposed uniquely into a direct sum of type I_n von Neumann algebras $\mathcal{R}_n$ $(n \in K)$, where K is a family of mutually distinct cardinal numbers.*

Proposition 1.1.24 *[KR 2] If $\mathcal{R}$ is a type I_n factor, where n is finite, then $\mathcal{R}$ is $*$-isomorphic to $B(H)$, where H has dimension n.*

Proposition 1.1.25 *[KR 2] If $\mathcal{R}$ is a finite von Neumann algebra with center $\mathcal{C}$, then there is a unique positive linear mapping τ from $\mathcal{R}$ into $\mathcal{C}$ such that*

1. $\tau(AB) = \tau(BA)$ $(A, B \in \mathcal{R})$,
2. $\tau(C) = C$ $(C \in \mathcal{C})$.

Moreover, if $A \in \mathcal{R}$ and $C \in \mathcal{C}$, then

3. $\tau(A) > 0$ *if* $A > 0$,

4. $\tau(CA) = C\tau(A)$ $(C \in \mathcal{C}, A \in \mathcal{R})$,

5. $\|\tau(A)\| \leq \|A\|$, *and*

6. *The mapping* τ *is ultraweakly continuous.*

Proposition 1.1.26 *[KR 2] If* $\mathcal{R}$ *is a factor of type* I_∞ *or* II_∞*, then there is a faithful, normal, semi-finite, tracial weight* ρ *on* $\mathcal{R}$.

Proposition 1.1.27 *[DIX 5] Every von Neumann algebra is expressed as a direct integral of factors. If* $\mathcal{R}$ *is a von Neumann algebra of type* I_n, II_1, II_∞, *or* III *acting on a separable Hilbert space* H, *then the components* $\mathcal{R}_p$ *of* $\mathcal{R}$ *in its direct integral decomposition relative to its center are, almost everywhere, factors of type* I_n, II_1, II_∞ *or* III *respectively.*

Proposition 1.1.28 *[SUND 1] Suppose* $\mathcal{R}$ *is a factor. If* E *and* F *are projections in* $\mathcal{R}$, *then* $E \preceq F$ $(\mathcal{R})$ *or* $F \preceq E$ $(\mathcal{R})$.

Proposition 1.1.29 *[SUND 1] Suppose* $\mathcal{R}$ *is a factor and* E *and* F *are infinite projections in* $\mathcal{R}$. *Then* $E \sim F$ $(\mathcal{R})$.

Proposition 1.1.30 *[ARV 2] A continuous image of an analytic set is analytic.*

Proposition 1.1.31 *[ARV 2] Let* A *be an analytic set in a polish space* P. *Then* A *is* μ*-measurable for every finite Borel measure* μ *on* P, *i.e.* A *is absolutely measurable.*

Proposition 1.1.32 *[ARV 2] Suppose* X *is analytic and* Y *is a countably separated Borel space. Let* f *be a Borel map of* X *onto* Y. *Then* f *has an absolutely measurable cross section.*

Corollary 1.1.33 *Suppose* X *and* Y *are analytic spaces and* f *is a Borel map of* X *onto* Y. *Then* f *has an absolutely measurable cross section.*

Proposition 1.1.34 *[DUG 1] Suppose Y is a Hausdorff, normal space and E and F be disjoint closed subsets in Y. Then there is a continuous function $f : Y \longmapsto \mathbb{R}$ such that $f|_E = 0, f|_F = 1$ and $0 \leq f \leq 1$. The function f is called a Uryshon function for E and F.*

Moreover a necessary and sufficient condition for the existence of a Uryshon function satisfying $E = f^{-1}(0)$ is that E is a G_δ set.

Proposition 1.1.35 *[KR 2] Suppose H is the direct integral of Hilbert spaces $\{H_\omega\}$ over (Ω, μ). If $\mathcal{R}$ is a decomposable von Neumann algebra on H and E is a projection in $\mathcal{R}$, then the following assertions hold almost everywhere:*

1. *E_ω is a projection in $\mathcal{R}_\omega$.*
2. *If $E \sim F$ $(\mathcal{R})$, then $E_\omega \sim F_\omega$ $(\mathcal{R}_\omega)$.*
3. *If E is abelian in $\mathcal{R}$, then E_ω is abelian in $\mathcal{R}_\omega$.*

Proposition 1.1.36 *[DIX 5] Let $T_n = \int_\Omega^\oplus T_n(p)d\mu(p)$ $(n = 1, 2, \cdots)$ and $T = \int_\Omega^\oplus T(p)d\mu(p)$ be decomposable operators.*

1. *If $T_n \xrightarrow{\text{SOT}} T$, there exists a subsequence $\{T_{n_k}\}$ such that $T_{n_k}(p) \xrightarrow{\text{SOT}} T(p)$ almost everywhere.*
2. *If $T_n(p) \xrightarrow{\text{SOT}} T(p)$ almost everywhere, and if $\sup_n \|T_n\| < \infty$, then $T_n \xrightarrow{\text{SOT}} T$.*

1.2 Sufficient Condition

In this section we prove:

Theorem 1.2.1 *Let $\mathcal{R}$ be a von Neumann algebra acting on a separable Hilbert space H, and let A and B be two normal operators in $\mathcal{R}$ such that $\mathcal{R}-$rank $f(A) = \mathcal{R}-$rank $f(B)$ for all continuous function f. Then there is a sequence $\{U_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{R}$ such that*

$$\|U_n A U_n^* - B\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Throughout this section H is a separable Hilbert space unless specifically noted.

Lemma 1.2.2 *Suppose $\{P_k\}_{k=1}^{n}$ and $\{Q_k\}_{k=1}^{n}$ are two sets of orthogonal projections in $\mathcal{R}$ both with sum I, where $1 \le n \le \aleph_0$. Furthermore suppose $P_k \sim Q_k$ $(\mathcal{R})$ for $1 \le k \le n$. Then there is a unitary U in $\mathcal{R}$ such that $UP_kU^* = Q_k$ for $1 \le k \le n$.*

Proof: Since $P_k \sim Q_k$ $(\mathcal{R})$ for $1 \le k \le n$, by Definition 1.1.1 there are partial isometries V_k in $\mathcal{R}$ such that $V_k^*V_k = P_k, V_kV_k^* = Q_k$ for $1 \le k \le n$. Define $U = \sum_{k=1}^{n}{}^{\oplus} V_kP_k$. It follows that U is a unitary in $\mathcal{R}$, since

$$U^*U = \sum_{k=1}^{n}{}^{\oplus} P_k^*V_k^*V_kP_k = \sum_{k=1}^{n}{}^{\oplus} P_k = I,$$

$$UU^* = \sum_{k=1}^{n}{}^{\oplus} V_kP_kP_k^*V_k^* = \sum_{k=1}^{n}{}^{\oplus} Q_k = I,$$

and for $1 \le k \le n$,

$$UP_kU^* = V_kP_kV_k^* = Q_k.$$

□

Lemma 1.2.3 *Suppose A is a normal operator in $\mathcal{R}$ and f is a continuous function. Then*

$$\begin{aligned} P_{\overline{\text{ran } A}} &= \chi_{(\mathbb{C}\setminus\{0\})\cap\sigma(A)}(A) \\ P_{\overline{\text{ran } f(A)}} &= \chi_{f^{-1}(\mathbb{C}\setminus\{0\})\cap\sigma(A)}(A). \end{aligned}$$

Proof: Since A is normal, $AA^* = A^*A$. Note

$$\overline{\text{ran } (A^*A)} = \overline{\text{ran } A^*} = (\ker A)^{\perp},$$

$$\overline{\text{ran } (AA^*)} = \overline{\text{ran } A}.$$

It follows that

$$\overline{\text{ran } A} = (\ker A)^{\perp}.$$

Now we show that

$$P_{\overline{\text{ran } A}} = \chi_{(\mathbb{C}\setminus\{0\})\cap\sigma(A)}(A).$$

This is equivalent to showing that

$$P_{\overline{\text{ran } A}^{\perp}} = \chi_{\{0\}\cap\sigma(A)}(A),$$

i.e.

$$P_{\ker A} = \chi_{\{0\}\cap\sigma(A)}(A). \tag{1.1}$$

Equation (1.1) is true since ker A is the set of eigenvectors of A corresponding to the eigenvalue 0, and $\chi_{\{0\}\cap\sigma(A)}(A)$ is the projection onto ker A. We have proved that

$$P_{\overline{\text{ran } A}} = \chi_{(\mathbf{C}\setminus\{0\})\cap\sigma(A)}(A).$$

Therefore, for any continuous function f,

$$\begin{aligned} P_{\overline{\text{ran } f(A)}} &= \chi_{(\mathbf{C}\setminus\{0\})\cap\sigma(f(A))}(f(A)) \\ &= (\chi_{(\mathbf{C}\setminus\{0\})\cap f(\sigma(A))} \circ f)(A) \\ &= \chi_{f^{-1}(\mathbf{C}\setminus\{0\})\cap\sigma(A)}(A). \end{aligned}$$

□

Lemma 1.2.4 *Suppose A and B are two normal operators in $\mathcal{R}$. Suppose that for all continuous function f, $\mathcal{R}-$rank $f(A) = \mathcal{R}-$rank $f(B)$. Then $\sigma(A) = \sigma(B)$.*

Proof: We show $\sigma(A) \subseteq \sigma(B)$ via contradiction.

Suppose $a \in \sigma(A)$ and $a \notin \sigma(B)$.

Since $\sigma(A)$ and $\sigma(B)$ are compact subsets of $\mathbf{C}$, and $a \in \sigma(A)$ and $a \notin \sigma(B)$, therefore there is an open rectangle $E = (c_1, d_1) \times (c_2, d_2)$ containing a such that $E \cap \sigma(B) = \phi$. Note that $\mathbf{C} \setminus E$ is a G_δ set. By Proposition 1.1.34, there is a continuous function f such that $f(a) = 1$ and $f^{-1}(0) = \mathbf{C} \setminus E$. Hence $f(B) = 0$ and $\|f(A)\| = \sup_{x\in\sigma(A)}|f(x)| \neq 0$, i.e., $f(A) \neq 0$. It follows that

$$P_{\overline{\text{ran } f(A)}} \neq 0 \text{ and } P_{\overline{\text{ran } f(B)}} = 0.$$

But by the hypothesis, $\mathcal{R}$−rank $f(A) = \mathcal{R}$−rank $f(B)$, thus

$$P_{\overline{\text{ran } f(A)}} \sim P_{\overline{\text{ran } f(B)}} \ (\mathcal{R}),$$

i.e. $P_{\overline{\text{ran } f(A)}} \neq 0 \iff P_{\overline{\text{ran } f(B)}} \neq 0$. This is a contradiction since $P_{\overline{\text{ran } f(A)}} \neq 0$ and $P_{\overline{\text{ran } f(B)}} = 0$.

We have proved that $\sigma(A) \subseteq \sigma(B)$.

Similarly we can show that $\sigma(B) \subseteq \sigma(A)$. Hence $\sigma(A) = \sigma(B)$. □

Lemma 1.2.5 *Let A and B be as in the preceding Lemma. Suppose a, b, c and d are real numbers such that $a < b$, $c < d$ and $E = (a, b) \times (c, d)$. Then $\chi_E(A) \sim \chi_E(B)$ $(\mathcal{R})$.*

Proof: Choose $\epsilon > 0$ such that $a+\epsilon < b-\epsilon$ and $c+\epsilon < d-\epsilon$. Let $F = [a+\epsilon, b-\epsilon] \times [c+\epsilon, d-\epsilon]$. Since F and $\mathbf{C} \setminus E$ are disjoint closed subsets of a metrizable space $\mathbf{C}$, and $\mathbf{C} \setminus E$ is a G_δ set, there is a continuous function f such that $f|_F = 1$, $f^{-1}(0) = \mathbf{C} \setminus E$ and $0 \leq f \leq 1$ by Proposition 1.1.34. Applying Lemma 1.2.3 gives

$$\begin{aligned} P_{\overline{\text{ran } f(A)}} &= \chi_{f^{-1}(\mathbf{C} \setminus \{0\}) \cap \sigma(A)}(A) \\ &= \chi_{E \cap \sigma(A)}(A), \\ P_{\overline{\text{ran } f(B)}} &= \chi_{f^{-1}(\mathbf{C} \setminus \{0\}) \cap \sigma(B)}(B) \\ &= \chi_{E \cap \sigma(B)}(B). \end{aligned}$$

By the hypothesis, $\mathcal{R}$−rank $f(A) = \mathcal{R}$−rank $f(B)$. Therefore $P_{\overline{\text{ran } f(A)}} \sim P_{\overline{\text{ran } f(B)}}$ $(\mathcal{R})$, i.e. $\chi_{E \cap \sigma(A)}(A) \sim \chi_{E \cap \sigma(B)}(B)$ $(\mathcal{R})$. By Lemma 1.2.4, $\sigma(A) = \sigma(B)$, and it follows that

$$\chi_E(A) \sim \chi_E(B) \ (\mathcal{R}).$$

□

Lemma 1.2.6 *Let A, B and E be as in Lemma 1.2.5. Suppose $F = (a,b] \times (c,d]$, and*

$$\chi_{\{b\}\times(c,d)}(A) = \chi_{\{b\}\times(c,d)}(B) = \chi_{(a,b)\times\{d\}}(A) = \chi_{(a,b)\times\{d\}}(B) = 0,$$

$$\chi_{\{b\}\times\{d\}}(A) = \chi_{\{b\}\times\{d\}}(B) = 0.$$

Then $\chi_F(A) \sim \chi_F(B)$ $(\mathcal{R})$.

Proof: Note that

$$\begin{aligned}
\chi_F(A) &= \chi_E(A) \oplus \chi_{\{b\}\times(c,d)}(A) \oplus \chi_{(a,b)\times\{d\}}(A) \oplus \chi_{\{b\}\times\{d\}}(A) \\
&= \chi_E(A), \textit{and} \\
\chi_F(B) &= \chi_E(B) \oplus \chi_{\{b\}\times(c,d)}(B) \oplus \chi_{(a,b)\times\{d\}}(B) \oplus \chi_{\{b\}\times\{d\}}(B) \\
&= \chi_E(B).
\end{aligned}$$

Lemma 1.2.5 implies that $\chi_F(A) \sim \chi_F(B)$ $(\mathcal{R})$. □

Lemma 1.2.7 *Suppose $\mathcal{R}$ is a von Neumann algebra acting on H and A and B are normal operators in $\mathcal{R}$.*

Let

$$E_1 = \{a \in \mathbf{R} : \chi_{\{a+ti\}}(A) \neq 0 \textit{ and } \chi_{\{a+ti\}}(B) \neq 0, \quad -\infty < t < \infty\}$$

and

$$E_2 = \{a \in \mathbf{R} : \chi_{\{t+ai\}}(A) \neq 0 \textit{ and } \chi_{\{t+ai\}}(B) \neq 0, \quad -\infty < t < \infty\},$$

where $i^2 = -1$. Then E_j is at most countable for $1 \leq j \leq 2$.

Proof: Since $\{\chi_{\{a+ti\}}(A)\}_{a\in\mathbf{R}}$ is a family of orthogonal projections in $B(H)$ and H is separable, the set $\{a \in \mathbf{R} : \chi_{\{a+ti\}}(A) \neq 0, -\infty < t < \infty\}$ is at most countable. This is also true for operator B. So E_1 is at most countable. Similarly, E_2 is countable. □

Proposition 1.2.8 *Suppose A and B are normal operators in a von Neumann algebra $\mathcal{R}$ acting on H such that $\mathcal{R}-$rank $f(A) = \mathcal{R}-$rank $f(B)$ for all continuous function f. Then for every $\epsilon > 0$, there is a unitary U_ϵ in $\mathcal{R}$ such that $\|U_\epsilon A U_\epsilon^* - B\| < \epsilon$.*

Proof: By Lemma 1.2.4, $\sigma(A) = \sigma(B)$. Given $\epsilon > 0$, there is a partition $\{F_{i,j}\}$ of $\sigma(A)(= \sigma(B))$ such that for $1 \le i \le n$ and $1 \le j \le m$,

1. $F_{i,j} = (a_i, a_{i+1}] \times (b_j, b_{j+1}]$,
2. $\text{diam}(F_{i,j}) < \frac{\epsilon}{2}$.

By Lemma 1.2.7, we can choose a partition $\{F_{i,j}\}$ such that for $1 \le i \le n$ and $1 \le j \le m$,

$$\chi_{\{a_{i+1}\}\times[b_j,b_{j+1}]}(A) = \chi_{\{a_{i+1}\}\times[b_j,b_{j+1}]}(B) = 0,$$

and

$$\chi_{[a_i,a_{i+1}]\times\{b_{j+1}\}}(A) = \chi_{[a_i,a_{i+1}]\times\{b_{j+1}\}}(B) = 0.$$

So for $1 \le i \le n$ and $1 \le j \le m$,

$$\chi_{\{a_{i+1}\}\times(b_j,b_{j+1})}(A) = \chi_{\{a_{i+1}\}\times(b_j,b_{j+1})}(B) = 0,$$

$$\chi_{(a_i,a_{i+1})\times\{b_{j+1}\}}(A) = \chi_{(a_i,a_{i+1})\times\{b_{j+1}\}}(B) = 0,$$

$$\chi_{\{a_{i+1}\}\times\{b_{j+1}\}}(A) = \chi_{\{a_{i+1}\}\times\{b_{j+1}\}}(B) = 0.$$

By Lemma 1.2.6, $\chi_{F_{i,j}}(A) \sim \chi_{F_{i,j}}(B)$ $(\mathcal{R})$ for $1 \le i \le n$ and $1 \le j \le m$. Note that $\{\chi_{F_{i,j}}(A)\}$ and $\{\chi_{F_{i,j}}(B)\}$ are two sets of orthogonal projections in $\mathcal{R}$ with sum I respectively. By Lemma 1.2.2, there is a unitary $U_\epsilon \in \mathcal{R}$ such that $U_\epsilon \chi_{F_{i,j}}(A)U_\epsilon^* = \chi_{F_{i,j}}(B)$ for $1 \le i \le n$ and $1 \le j \le m$.

Choose $\alpha_{i,j} \in F_{i,j}$ for $1 \le i \le n$ and $1 \le j \le m$, so

$$\|z - \sum_{1\le i\le n,\ 1\le j\le m} \alpha_{i,j}\chi_{F_{i,j}}(z)\|_\infty < \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned}\|A - \sum_{1\le i\le n,\ 1\le j\le m} \alpha_{i,j}\chi_{F_{i,j}}(A)\| &< \frac{\epsilon}{2}, \text{ and}\\ \|B - \sum_{1\le i\le n,\ 1\le j\le m} \alpha_{i,j}\chi_{F_{i,j}}(B)\| &< \frac{\epsilon}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\|U_\epsilon A U_\epsilon^* - B\| &\le \|U_\epsilon A U_\epsilon^* - U_\epsilon(\sum_{1\le i\le n,\ 1\le j\le m} \alpha_{i,j}\chi_{F_{i,j}}(A))U_\epsilon^*\| + \|\sum_{1\le i\le n,\ 1\le j\le m} \alpha_{i,j}\chi_{F_{i,j}}(B) - B\|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon.\end{aligned}$$

□

Now we prove Theorem 1.2.1.

Proof: For every positive integer n, let $\epsilon_n = \dfrac{1}{n}$. Applying Proposition 1.2.8 to see that there is a unitary U_n in $\mathcal{R}$ such that $\|U_n A U_n^* - B\| < \frac{1}{n}$ for $n \ge 1$. Hence there is a sequence $\{U_n\}_{n=1}^\infty$ of untaries in $\mathcal{R}$ such that $\|U_n A U_n^* - B\| \longrightarrow 0$ as $n \longrightarrow \infty$. □

Theorem 1.2.9 *Suppose $\mathcal{R}$ is a type III factor and S and T are normal in $\mathcal{R}$. Then*

$$S \sim_a T\ (\mathcal{R}) \Longleftrightarrow \sigma(S) = \sigma(T).$$

Proof: ($\Longrightarrow$) Suppose $S \sim_a T$ ($\mathcal{R}$).

There is a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{R}$ such that

$$\|u_n S u_n^* - T\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore for every continuous function f,

$$\|u_n f(S) u_n^* - f(T)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $f(S) \neq 0 \Longleftrightarrow f(T) \neq 0$, i.e. $P_{\overline{\text{ran } f(S)}} \neq 0 \Longleftrightarrow P_{\overline{\text{ran } f(T)}} \neq 0$.

Since $\mathcal{R}$ is a type III factor, it follows that for every continuous function f,

$$P_{\overline{\text{ran } f(S)}} \sim P_{\overline{\text{ran } f(T)}} \ (\mathcal{R}),$$

i.e. $\mathcal{R}$−rank $f(S) = \mathcal{R}$−rank $f(T)$ for all continuous function f. Applying Lemma 1.2.4 gives that $\sigma(S) = \sigma(T)$.

($\Longleftarrow$) Suppose $\sigma(S) = \sigma(T)$.

Since

$$\|f(S)\| = \sup_{t \in \sigma(S)} |f(t)| = \sup_{t \in \sigma(T)} |f(t)| = \|f(T)\|.$$

Therefore for every continuous function f, $f(S) \neq 0 \Longleftrightarrow f(T) \neq 0$.

Hence

$$P_{\overline{\text{ran } f(S)}} \neq 0 \Longleftrightarrow P_{\overline{\text{ran } f(T)}} \neq 0.$$

Since $\mathcal{R}$ is a type III factor, $P_{\overline{\text{ran } f(S)}} \sim P_{\overline{\text{ran } f(T)}}$ ($\mathcal{R}$) for every continuous function f. Applying Theorem 1.2.1 to see $S \sim_a T$ ($\mathcal{R}$). □

1.3 Direct Integrals

In this section we will prove some results about direct integrals.

Throughout this section, $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space H. For each $\omega \in \Omega$, let $\mathcal{R}_\omega$ be the von Neumann algebra acting on the separable Hilbert space K. Let $\mathcal{R} = \int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega) \subseteq L^\infty(\mu, B(K))$.

Definition 1.3.1 *Two operators A and B in $\mathcal{R}$ are said to be unitarily equivalent in $\mathcal{R}$, if there is a unitary U in $\mathcal{R}$ such that $UAU^* = B$. We denote this by $A \simeq B$ ($\mathcal{R}$).*

Proposition 1.3.2 *Suppose A and B are in $\mathcal{R} \subseteq L^\infty(\mu, B(K))$. Suppose $A = \int_\Omega^\oplus A_\omega d\mu(\omega)$ and $B = \int_\Omega^\oplus B_\omega d\mu(\omega)$. Then*

$$A_\omega \simeq B_\omega \ (\mathcal{R}_\omega) \text{ almost every } \omega \in \Omega \iff A \simeq B \ (\mathcal{R}).$$

Proof: ($\Leftarrow$) Suppose $A \simeq B$ ($\mathcal{R}$).

By Definition 1.3.1, there is a unitary $U \in \mathcal{R}$ such that $UAU^* = B$. Since we can decompose U into the direct integral of unitaries in $\mathcal{R}_\omega$, write $U = \int_\Omega^\oplus U_\omega d\mu(\omega)$, where U_ω is a unitary in $\mathcal{R}_\omega \subseteq B(K)$ almost everywhere. For almost all $\omega \in \Omega$, U_ω is a unitary. Therefore we may assume U_ω is a unitary in $\mathcal{R}_\omega$ for every $\omega \in \Omega$.

It follows from

$$\begin{aligned} B &= UAU^* \\ &= \int_\Omega^\oplus U_\omega A_\omega U_\omega^* d\mu(\omega) \\ &= \int_\Omega^\oplus B_\omega d\mu(\omega), \end{aligned}$$

$U_\omega A_\omega U_\omega^* = B_\omega$ almost everywhere. Thus for almost every $\omega \in \Omega$,

$$A_\omega \simeq B_\omega \ (\mathcal{R}_\omega).$$

($\Rightarrow$) Suppose $A_\omega \simeq B_\omega$ $(\mathcal{R}_\omega)$ almost everywhere.

Without loss of generality, we may assume that $\|A\| \le 1$ and $\|B\| \le 1$. (If not replace A and B by $A/\max(\|A\|, \|B\|)$ and $B/\max(\|A\|, \|B\|)$, respectively)

For almost every $\omega \in \Omega$, there is a unitary U_ω in $\mathcal{R}_\omega$ such that $U_\omega A_\omega U_\omega^* = B_\omega$. Neglecting a set of measure 0, we assume for every $\omega \in \Omega$, there is a unitary $U_\omega \in \mathcal{R}_\omega$ such that $U_\omega A_\omega U_\omega^* = B_\omega$.

Let $\mathcal{U} = \{U \in B(K) : U \text{ is a unitary}\}$ with the $*$-strong operator topology (write $*$-SOT). Let $\mathcal{V} = \{T \in B(K) : \|T\| \le 1\}$ with the $*$-strong operator topology. Since K is separable, Ball$B(K)$ is $*$-SOT separable and metrizable. Since $\mathcal{U}$ and $\mathcal{V}$ are $*$-SOT closed in Ball$B(K)$, by Definition 1.1.5 and Example 1.1.6, $\mathcal{U}$ and $\mathcal{V}$ are polish spaces. Therefore $\mathcal{U} \times \mathcal{V} \times \mathcal{V}$ with the product topology is a polish space.

Let

$$X = \{(U, A, B) \in \mathcal{U} \times \mathcal{V} \times \mathcal{V} : UAU^* = B\}.$$

We show that X is a polish space, for which it suffices to show X is a closed subset of $\mathcal{U} \times \mathcal{V} \times \mathcal{V}$.

Suppose $(U_n, A_n, B_n) \in X$ for $n \ge 1$, and $(U_n, A_n, B_n) \longrightarrow (U, A, B)$ as $n \longrightarrow \infty$, i.e.

$$U_n \overset{*\text{-SOT}}{\longrightarrow} U,$$

$$A_n \overset{*\text{-SOT}}{\longrightarrow} A \text{ and}$$

$$B_n \overset{*\text{-SOT}}{\longrightarrow} B \text{ as } n \longrightarrow \infty.$$

Therefore U, A, and B are in $B(K)$, $\|A\| \leq 1$, $\|B\| \leq 1$ and U is a unitary in $B(K)$.

Since

$$\begin{aligned} U_n A_n U_n^* &\xrightarrow{*\text{-SOT}} UAU^* \text{ as } n \longrightarrow \infty \text{ and} \\ U_n A_n U_n^* &= B_n \\ &\xrightarrow{*\text{-SOT}} B \text{ as } n \longrightarrow \infty. \end{aligned}$$

It follows that $UAU^* = B$. We have proved that X is closed, and hence X is a polish space.

Define

$$\pi : X \longmapsto \mathcal{V} \times \mathcal{V} \text{ by } \pi(U, A, B) = (A, B).$$

π is continuous since π is a coordinate projection. Thus $\pi(X)$ is an analytic subset of $\mathcal{V} \times \mathcal{V}$ by Definition 1.1.7. Since $\pi : X \longmapsto \pi(X)$ is an onto Borel function, it follows from Corollary 1.1.33 that π has an absolutely measurable cross section $\alpha : \pi(X) \longmapsto X$ such that $\pi \circ \alpha = \mathrm{id}_{\pi(X)}$.

Note $\mathcal{V}$ is a polish space and hence the Borel structure of $\mathcal{V} \times \mathcal{V}$ equals the product Borel structure. Define $\beta : \Omega \longmapsto \mathcal{V} \times \mathcal{V}$ by $\beta(\omega) = (A_\omega, B_\omega)$.

Since

$$\begin{aligned} A &= \int_\Omega^\oplus A_\omega d\mu(\omega) \text{ and} \\ B &= \int_\Omega^\oplus B_\omega d\mu(\omega), \end{aligned}$$

the maps $\omega \longmapsto A_\omega$ and $\omega \longmapsto B_\omega$ are μ-measurable functions. It follows that β is μ-measurable.

Note $(U_\omega, A_\omega, B_\omega) \in X$ for every ω in Ω,

$$\alpha \circ \beta(\omega) = \alpha(A_\omega, B_\omega) = (U_\omega, A_\omega, B_\omega), \text{ and}$$

$$U_\omega = \pi_1 \circ \alpha \circ \beta(\omega),$$

where π_1 is the first coordinate projection of X. Therefore

$$\pi_1 \circ \alpha \circ \beta : \Omega \longmapsto \mathcal{U} \text{ defined by } \pi_1 \circ \alpha \circ \beta(\omega) = U_\omega$$

is a μ-measurable function, since π_1, α and β are μ- measurable. We have shown that the mapping $\omega \longmapsto U_\omega$ is μ-measurable. □

Define $U = \int_\Omega^\oplus U_\omega d\mu(\omega)$. So U is a unitary in $\mathcal{R}$ and

$$\begin{aligned} UAU^* &= \int_\Omega^\oplus U_\omega A_\omega U_\omega^* d\mu(\omega) \\ &= \int_\Omega^\oplus B_\omega d\mu(\omega) \\ &= B, \end{aligned}$$

i.e. $A \simeq B$ $(\mathcal{R})$. □

Proposition 1.3.3 *Suppose P and Q are projections in $\mathcal{R}$. Suppose $P = \int_\Omega^\oplus P_\omega d\mu(\omega)$ and $Q = \int_\Omega^\oplus Q_\omega d\mu(\omega)$ in $L^\infty(\mu, B(K))$. Then*

$$P \sim Q\ (\mathcal{R}) \Longleftrightarrow P_\omega \sim Q_\omega\ (\mathcal{R}_\omega) \text{ almost everywhere.}$$

Proof: Note that P_ω and Q_ω are projections in $\mathcal{R}_\omega \subseteq B(K)$ almost everywhere. Without loss of generality, we may assume P_ω and Q_ω are projections in $\mathcal{R}_\omega$ for each $\omega \in \Omega$.

($\Rightarrow$) Applying Proposition 1.1.35 gives that $P_\omega \sim Q_\omega$ $(\mathcal{R}_\omega)$ almost everywhere.

($\Leftarrow$) Suppose $P_\omega \sim Q_\omega$ $(\mathcal{R}_\omega)$ almost everywhere.

There are partial isometries V_ω in $\mathcal{R}_\omega$ such that $V_\omega^* V_\omega = P_\omega$ and $V_\omega V_\omega^* = Q_\omega$ almost everywhere. We may assume that for every $\omega \in \Omega$ there is a partial isometry $V_\omega \in \mathcal{R}_\omega$ such that

$$V_\omega^* V_\omega = P_\omega \text{ and } V_\omega V_\omega^* = Q_\omega.$$

Let $\mathcal{U} = \{V \in B(K) : V \text{ is a partial isometry}\}$ with the $*$-strong operator topology. Let $\mathcal{V} = \{T \in B(K) : T \text{ is a projection}\}$ with the $*$-strong operator topology. Since K is separable, Ball$B(K)$ is $*$-strong separable and metrizable. Since $\mathcal{U}$ and $\mathcal{V}$ are $*$-SOT closed subsets of Ball$B(K)$, hence $\mathcal{U}$ and $\mathcal{V}$ are polish spaces. It follows that $\mathcal{U} \times \mathcal{V} \times \mathcal{V}$ is a polish space, which is endowed with the product topology.

Let $X = \{(V, P, Q) \in \mathcal{U} \times \mathcal{V} \times \mathcal{V} : V^*V = P \text{ and } VV^* = Q\}$. Now we prove that X is a $*$-SOT closed subset of $\mathcal{U} \times \mathcal{V} \times \mathcal{V}$. It will follow that X is a polish space.

Suppose $(V_n, P_n, Q_n) \in X$ for every positive integer n, and $(V_n, P_n, Q_n) \longrightarrow (V, P, Q)$ as $n \longrightarrow \infty$, i.e.

$$\begin{aligned} V_n &\xrightarrow{*-SOT} V, \\ P_n &\xrightarrow{*-SOT} P, \text{ and} \\ Q_n &\xrightarrow{*-SOT} Q \text{ as } n \longrightarrow \infty. \end{aligned}$$

Hence P and Q are projections in $B(K)$, and

$$\begin{aligned} V_n^* V_n &\xrightarrow{*-SOT} V^*V \text{ and} \\ V_n^* V_n &= P_n \end{aligned}$$

$$\overset{*\text{-SOT}}{\longrightarrow} \quad P \text{ as } n \longrightarrow \infty.$$

This follows that $V^*V = P$. Using a similar argument, we can show that $VV^* = Q$. We have proved that $(V, P, Q) \in X$. Hence X is a $*$-SOT closed subset of $\mathcal{U} \times \mathcal{V} \times \mathcal{V}$.

Define $\pi : X \longmapsto \mathcal{V} \times \mathcal{V}$ by $\pi(V, P, Q) = (P, Q)$. The map π is continuous since π is the coordinate projection. Applying Definition 1.1.7 to see $\pi(X)$ is an analytic subset of $\mathcal{V} \times \mathcal{V}$. Because $\pi : X \longmapsto \pi(X)$ is an onto Borel function, applying Corollary 1.1.33 we see that π has an absolutely measurable cross section $\alpha : \pi(X) \longmapsto X$ such that $\pi \circ \alpha = \mathrm{id}_{\pi(X)}$. Define $\beta : \Omega \longmapsto \mathcal{V} \times \mathcal{V}$ by $\beta(\omega) = (P_\omega, Q_\omega)$. By the hypothesis

$$\begin{aligned} P &= \int_\Omega^\oplus P_\omega d\mu(\omega) \text{ and} \\ Q &= \int_\Omega^\oplus Q_\omega d\mu(\omega), \end{aligned}$$

hence $\omega \longmapsto P_\omega$ and $\omega \longmapsto Q_\omega$ are μ- measurable functions. Since $\mathcal{V}$ is a polish space, it follows that the Borel structure of $\mathcal{V} \times \mathcal{V}$ equals the product Borel structure, and therefore β is a μ-measurable function.

Note $(V_\omega, P_\omega, Q_\omega) \in X$ for every ω in Ω,

$$\alpha \circ \beta(\omega) = \alpha(P_\omega, Q_\omega) = (V_\omega, P_\omega, Q_\omega), \text{ and}$$

$$\pi_1 \circ \alpha \circ \beta(\omega) = V_\omega,$$

where π_1 is the first coordinate projection of X. Therefore $\pi_1 \circ \alpha \circ \beta : X \longmapsto \mathcal{U}$, defined by $\pi_1 \circ \alpha \circ \beta(\omega) = V_\omega$, is a μ-measurable function, since π_1, α and β are μ- measurable functions. We have defined a μ-measurable mapping $\omega \longmapsto V_\omega$.

Define $V = \int_{\Omega}^{\oplus} V_\omega d\mu(\omega)$. Since

$$\begin{aligned} V^*V &= \int_{\Omega}^{\oplus} V_\omega^* V_\omega d\mu(\omega) \\ &= \int_{\Omega}^{\oplus} P_\omega d\mu(\omega) \\ &= P, \text{ and} \\ VV^* &= \int_{\Omega}^{\oplus} V_\omega V_\omega^* d\mu(\omega) \\ &= \int_{\Omega}^{\oplus} Q_\omega d\mu(\omega) \\ &= Q, \end{aligned}$$

V is a partial isometry in $\mathcal{R}$ and $P \sim Q$ $(\mathcal{R})$. □

Proposition 1.3.4 *Suppose $\mathcal{R} = \int_{\Omega}^{\oplus} \mathcal{R}_\omega d\mu(\omega)$. Suppose A and B are normal operators in $\mathcal{R}$, $A = \int_{\Omega}^{\oplus} A_\omega d\mu(\omega)$ and $B = \int_{\Omega}^{\oplus} B_\omega d\mu(\omega)$. Without loss of generality, we may assume A_ω and B_ω are normal operators in $\mathcal{R}_\omega$ for every $\omega \in \Omega$. Moreover suppose $A \sim_a B$ $(\mathcal{R})$. Then $A_\omega \sim_a B_\omega$ $(\mathcal{R}_\omega)$ almost everywhere.*

Proof: Since $A \sim_a B$ $(\mathcal{R})$, there is a sequence $\{U_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{R}$ such that

$$\|U_n A U_n^* - B\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Let $U_n = \int_{\Omega}^{\oplus} U_\omega^n d\mu(\omega)$. Then for every $n \geq 1$, U_ω^n is a unitary in $\mathcal{R}_\omega$ almost everywhere.

Let $\Omega_n = \{\omega \in \Omega : U_\omega^n \text{ is a unitary in } \mathcal{R}_\omega\}$ for $n \geq 1$. Note that $\mu(\Omega \setminus \Omega_n) = 0$. Let $\Omega_o = \cap_{n=1}^{\infty} \Omega_n$. For every $\omega \in \Omega_o$, $\{U_\omega^n\}_{n=1}^{\infty}$ is a sequence of unitaries in $\mathcal{R}_\omega$, and

$$\mu(\Omega \setminus \Omega_o) = \mu(\cup_{n=1}^{\infty}(\Omega \setminus \Omega_n)) = 0.$$

Note that

$$\begin{aligned}
\left\| \int_{\Omega}^{\oplus} U_{\omega}^{n} A_{\omega} U_{\omega}^{n*} d\mu(\omega) - \int_{\Omega}^{\oplus} B_{\omega} d\mu(\omega) \right\| &= \text{ess sup}_{\omega \in \Omega} \| U_{\omega}^{n} A_{\omega} U_{\omega}^{n*} - B_{\omega} \| \\
&= \| U_n A U_n^{*} - B \| \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

It follows that for almost every $\omega \in \Omega_o$,

$$\| U_{\omega}^{n} A_{\omega} U_{\omega}^{n*} - B_{\omega} \| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

i.e. for almost every $\omega \in \Omega_o$, $A_{\omega} \sim_a B_{\omega}$ $(\mathcal{R}_{\omega})$. Hence for almost every $\omega \in \Omega$, $A_{\omega} \sim_a B_{\omega}$ $(\mathcal{R}_{\omega})$.

□

1.4 $\mathcal{R}$-rank Function

In this section, we will investigate some properties of the $\mathcal{R}$-rank function, where the $\mathcal{R}$-rank function is as in Section 1.1.

Throughout this chapter, $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space H.

Lemma 1.4.1 *For every operator T in $\mathcal{R}$, $\mathcal{R}-rank\ T = \mathcal{R}-rank\ TT^*$.*

Proof: For every operator T, $\overline{\text{ran } T} = \overline{\text{ran } TT^*}$. It follows that $P_{\overline{\text{ran } T}} = P_{\overline{\text{ran } TT^*}}$, i.e. $\mathcal{R}-\text{rank } T = \mathcal{R}-\text{rank } TT^*$. □

Lemma 1.4.2 *For all operators A and B in $\mathcal{R}$,*

$$\mathcal{R}-rank\ AB \preceq \mathcal{R}-rank\ A \text{ and } \mathcal{R}-rank\ AB \preceq \mathcal{R}-rank\ B.$$

Proof: Note that for all operators A and B in $\mathcal{R}$, ran $AB = AB(H) \subseteq A(H) =$ ran A, and hence $\overline{\text{ran } AB} \subseteq \overline{\text{ran } A}$. Thus $P_{\overline{\text{ran } AB}} \leq P_{\overline{\text{ran } A}}$, i.e. $\mathcal{R}-\text{rank } AB \preceq \mathcal{R}-\text{rank } A$.

Note that $(\ker AB)^{\perp} = \overline{\text{ran } (AB)^*} = \overline{\text{ran } B^*A^*} \subseteq \overline{\text{ran } B^*} = (\ker B)^{\perp}$. It follows that

$$P_{(\ker AB)^{\perp}} \leq P_{(\ker B)^{\perp}}. \tag{1.2}$$

Applying the Polar decomposition, we see that $P_{(\ker B)^{\perp}} \sim P_{\overline{\text{ran } B}}\ (\mathcal{R})$ and $P_{(\ker AB)^{\perp}} \sim P_{\overline{\text{ran } AB}}\ (\mathcal{R})$. Hence $P_{\overline{\text{ran } AB}} \preceq P_{\overline{\text{ran } B}}\ (\mathcal{R})$ by (1.2), i.e.

$$\mathcal{R}-\text{rank } AB \preceq \mathcal{R}-\text{rank } B.$$

□

Lemma 1.4.3 *If U is a unitary in $\mathcal{R}$ and $S \in \mathcal{R}$, then $\mathcal{R}-rank\ USU^* = \mathcal{R}-rank\ S$.*

Proof: Since U is a unitary in $\mathcal{R}$, we have

$$\begin{aligned} U^*(\text{ran } USU^*) &= U^*(USU^*(H)) \\ &= SU^*(H) \\ &= S(H) \\ &= \text{ran } S. \end{aligned}$$

It follows that $U^*(\overline{\text{ran } USU^*}) = \overline{\text{ran } S}$, i.e. the unitary U^* in $\mathcal{R}$ is such that

$$U^* : \overline{\text{ran } USU^*} \longmapsto \overline{\text{ran } S}.$$

Let $V = U^* P_{\overline{\text{ran } USU^*}}$. V is a partial isometry in $\mathcal{R}$, and

$$V^*V = P_{\overline{\text{ran } USU^*}} \text{ and } VV^* = U^* P_{\overline{\text{ran } USU^*}} U = P_{\overline{\text{ran } S}}.$$

Therefore $P_{\overline{\text{ran } USU^*}} \sim P_{\overline{\text{ran } S}}$ $(\mathcal{R})$, i.e. $\mathcal{R}$−rank $USU^* = \mathcal{R}$−rank S. □

Lemma 1.4.4 *Suppose $S \in \mathcal{R}$ and $0 \leq S \leq I$. Then $P_{\overline{\text{ran } S}} \geq S$.*

Proof: Since $(\overline{\text{ran } S})^{\perp} = \ker S^* = \ker S$, for all $x \in (\overline{\text{ran } S})^{\perp}$,

$$\langle (P_{\overline{\text{ran } S}} - S)x, x \rangle = \langle 0, x \rangle = 0.$$

Since $\|S\| \leq 1$, for all $x \in \overline{\text{ran } S}$,

$$\begin{aligned} \langle (P_{\overline{\text{ran } S}} - S)x, x \rangle &= \langle x - Sx, x \rangle \\ &= \langle x, x \rangle - \langle Sx, x \rangle \end{aligned}$$

$$\geq \|x\|^2 - \|S\|\|x\|^2$$

$$\geq 0.$$

Therefore $P_{\overline{\text{ran } S}} - S \geq 0$. □

Lemma 1.4.5 *Suppose S is a normal operator in $\mathcal{R}$ and f is a continuous function with $0 \leq f \leq 1$ and $f(0) = 0$. Then $P_{\overline{\text{ran } f(S)}} \leq P_{\overline{\text{ran } S}}$.*

Proof: Note that $f \leq \chi_{\mathbf{C}\setminus\{0\}}$ for every continuous function f with $0 \leq f \leq 1$ and $f(0) = 0$, and hence

$$f(S) \leq \chi_{\mathbf{C}\setminus\{0\}}(S).$$

It follows that ran $f(S) \subseteq$ ran $\chi_{\mathbf{C}\setminus\{0\}} = \overline{\text{ran } S}$, which implies the result. □

Lemma 1.4.6 *Suppose S is a normal operator in $\mathcal{R}$. Then*

$$P_{\overline{\text{ran } S}} = sup\{g(S) : 0 \leq g \leq 1,\ g(0) = 0 \text{ and } g \text{ is continuous}\}.$$

Proof: Applying the preceding Lemma to see that $g(S) \leq P_{\overline{\text{ran } S}}$ for every continuous function g with $0 \leq g \leq 1$ and $g(0) = 0$.

Note that there is an increasing sequence $\{g_n\}$ of continuous functions convergent to $\chi_{(\mathbf{C}\setminus\{0\})\cap\sigma(S)}$. For instance we can choose g_n to be

$$g_n(z) = \begin{cases} 0 & \text{if } z = 0 \\ 1 & \text{if } |z| \geq \frac{1}{n} \\ \text{linear} & \text{if } 0 < |z| < \frac{1}{n}. \end{cases}$$

So

$$g_n(S) \xrightarrow{\text{WOT}} \chi_{(\mathbb{C}\setminus\{0\})\cap\sigma(S)}(S) = P_{\overline{\operatorname{ran} S}} \text{ as } n \longrightarrow \infty,$$

i.e. $P_{\overline{\operatorname{ran} S}} \leq \sup\{g(S) : 0 \leq g \leq 1,\ g(0) = 0 \text{ and } g \text{ is continuous}\}$.

This proves that

$$P_{\overline{\operatorname{ran} S}} = \sup\{g(S) : 0 \leq g \leq 1,\ g(0) = 0 \text{ and } g \text{ is continuous}\}.$$

□

Lemma 1.4.7 *Suppose τ is the unique positive center-valued trace on the factor von Neumann algebra $\mathcal{R}$ of type I_n with n finite or type II_1 and E and F are projections in $\mathcal{R}$. Then*

$$E \sim F\ (\mathcal{R}) \iff \tau(E) = \tau(F) \text{ and}$$

$$E \prec F \iff \tau(E) < \tau(F).$$

Proof: ($\Rightarrow$)Suppose $E \sim F$ $(\mathcal{R})$.

By Definition 1.1.1, there is a partial isometry V in $\mathcal{R}$ such that $V^*V = E$ and $VV^* = F$. Therefore

$$\tau(E) = \tau(V^*V) = \tau(VV^*) = \tau(F).$$

($\Leftarrow$) Suppose $\tau(E) = \tau(F)$.

Proposition 1.1.28 implies that either $E \preceq F$ $(\mathcal{R})$ or $F \preceq E$ $(\mathcal{R})$. Without loss of generality, we asume $E \preceq F$ $(\mathcal{R})$. We will prove $E \sim F$ $(\mathcal{R})$ via contradiction.

Assume $E \prec F$ $(\mathcal{R})$. By Definition 1.1.1, there is a projection F_o in $\mathcal{R}$ such that

$$E \sim F_o < F\ (\mathcal{R}).$$

Since τ is the center-valued trace, $\tau(F_o) < \tau(F)$, it follows that $\tau(E) = \tau(F_o) < \tau(F)$, a contradiction. Therefore $E \sim F$ $(\mathcal{R})$.

Similarly we can show $E \prec F \Longleftrightarrow \tau(E) < \tau(F)$. □

Lemma 1.4.8 *Suppose ρ is the faithful, normal, semifinite tracial weight on the factor von Neumann algebra $\mathcal{R}$ of type I_∞, or type II_∞. Then*

$$E \sim F \quad \Longleftrightarrow \quad \rho(E) = \rho(F) \text{ and}$$
$$E \prec F \quad \Longleftrightarrow \quad \rho(E) < \rho(F).$$

Proof: Use a similar argument to that in the preceding Lemma. □

Suppose A is in $\mathcal{R}$. We define $\mathcal{E} = \{T \in \mathcal{R}\colon \mathcal{R}\text{–rank } T \preceq \mathcal{R}\text{–rank } A\}$.

Now we prove $\mathcal{E}$ is closed under $*$-strong sequential limits.

Theorem 1.4.9 *If $\mathcal{R}$ is acting on a separable Hilbert space, then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

First we prove Theorem 1.4.9 for factor von Neumann algebras acting on any Hilbert space.

Proposition 1.4.10 *If $\mathcal{R}$ is a factor von Neumann algebra of type I_n (where n is finite) or type II_1, then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

Proof: Since $\mathcal{R}$ is a factor von Neumann algebra of type I_n (with n finite) or type II_1, $\mathcal{R}$ is a finite von Neumann algebra. Proposition 1.1.25 implies that there is a unique central value trace τ and that τ is weak operator topology continuous.

Suppose $\{T_m\}_{m=1}^{\infty} \subseteq \mathcal{E}$ and $T_m \overset{*-\mathrm{SOT}}{\longrightarrow} T$ as $m \longrightarrow \infty$. Hence $T_mT_m^* \overset{*-\mathrm{SOT}}{\longrightarrow} TT^*$ as $m \longrightarrow \infty$ and $\{\|T_mT_m^*\|\}_{m=1}^{\infty}$ is bounded. Let $\sup_{m\geq 1}\|T_mT_m^*\| = M$ and let $\overline{D(0,M)}$ be the closed disk centered at the orign with radius M. Then for every continuous function $f : \overline{D(0,M)} \longmapsto \mathbf{C}$,

$$f(T_mT_m^*) \overset{*-\mathrm{SOT}}{\longrightarrow} f(TT^*) \text{ as } m \longrightarrow \infty. \tag{1.3}$$

Applying Lemma 1.4.1 and Lemma 1.4.6, we see that for every $m \geq 1$,

$$\begin{aligned} P_{\overline{\mathrm{ran}\ T_m}} &= P_{\overline{\mathrm{ran}\ T_mT_m^*}} \\ &= \sup \{\ g(T_mT_m^*) : 0 \leq g \leq 1, g(0) = 0 \text{ and } g \text{ is continuous } \}. \end{aligned}$$

Since $\{T_m\}_{m=1}^{\infty} \subseteq \mathcal{E}$, $\mathcal{R}$−rank $T_mT_m^* = \mathcal{R}$−rank $T_m \preceq \mathcal{R}$−rank A for every $m \geq 1$. Thus $\tau(P_{\overline{\mathrm{ran}\ T_mT_m^*}}) \leq \tau(P_{\overline{\mathrm{ran}\ A}})$ for every $m \geq 1$. Therefore for every continuous function g with $0 \leq g \leq 1$ and $g(0) = 0$,

$$\tau(g(T_mT_m^*)) \leq \tau(P_{\overline{\mathrm{ran}\ T_mT_m^*}}) \leq \tau(P_{\overline{\mathrm{ran}\ A}}) \text{ for every } m \geq 1. \tag{1.4}$$

Since for every continuous function g, $\tau(g(T_mT_m^*)) \longrightarrow \tau(g(TT^*))$ as $m \longrightarrow \infty$, therefore by (1.4), for every continuous function g with $0 \leq g \leq 1$ and $g(0) = 0$,

$$\tau(g(TT^*)) \leq \tau(P_{\overline{\mathrm{ran}\ A}}).$$

Note that

$$\tau(P_{\overline{\mathrm{ran}\ TT^*}}) = \sup \{\tau(g(TT^*)) : 0 \leq g \leq 1, g(0) = 0 \text{ and } g \text{ is continuous } \}.$$

Thus

$$\tau(P_{\overline{\text{ran } TT^*}}) \leq \tau(P_{\overline{\text{ran } A}}).$$

It follows that $P_{\overline{\text{ran } TT^*}} \preceq P_{\overline{\text{ran } A}}$ $(\mathcal{R})$, i.e. $\mathcal{R}-$rank $TT^* \preceq \mathcal{R}-$rank A.

By Lemma 1.4.1, $\mathcal{R}-$rank $T = \mathcal{R}-$rank $TT^* \preceq \mathcal{R}-$rank A. We have proved that $T \in \mathcal{E}$. This shows that $\mathcal{E}$ is closed under $*$-strong sequential limits. □

Proposition 1.4.11 *If $\mathcal{R}$ is a factor von Neumann algebra of type I_∞ or II_∞, then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

Proof: Since $\mathcal{R}$ is a factor von Neumann algebra of type I_∞ or II_∞, Proposition 1.1.26 implies that there is a faithful, normal, semifinite, tracial weight ρ on $\mathcal{R}$ such that $\rho = \sum_{\alpha\in\Omega}\rho_\alpha$, where ρ_α is a positive normal functional. Hence ρ_α is weak operator topology continuous.

Suppose $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{E}$ and $T_n \xrightarrow{*-\text{SOT}} T$ as $n \longrightarrow \infty$.

Hence $T_nT_n^* \xrightarrow{*-\text{SOT}} TT^*$ as $n \longrightarrow \infty$ and $\{\|T_nT_n^*\|\}_{n=1}^{\infty}$ is bounded. Let $\sup_{n\geq 1}\|T_nT_n^*\| = M$. Let $\overline{D(0,M)}$ be the closed disk centered at the origin with radius M. For every continuous function $f : \overline{D(0,M)} \longmapsto \mathbf{C}$, $f(T_nT_n^*) \xrightarrow{*-\text{SOT}} f(TT^*)$ as $n \longrightarrow \infty$.

By Lemma 1.4.6 and 1.4.8, for every continuous function f with $0 \leq f \leq 1$ and $f(0) = 0$,

$$\rho(P_{\overline{\text{ran } T_nT_n^*}}) \geq \rho(f(T_nT_n^*)).$$

Since $\mathcal{R}-$rank $T_n = \mathcal{R}-$rank $T_nT_n^* \preceq \mathcal{R}-$rank A, and

$$\rho(f(T_nT_n^*)) = \sum_{\alpha\in\Omega}\rho_\alpha(f(T_nT_n^*)),$$

it follows that for every finite subset Ω_o of Ω and for every $n \geq 1$,

$$\begin{aligned} \rho(P_{\overline{\text{ran } A}}) &\geq \rho(P_{\overline{\text{ran } T_nT_n^*}}) \\ &\geq \rho(f(T_nT_n^*)) \\ &\geq \sum_{k\in\Omega_o} \rho_k(f(T_nT_n^*)). \end{aligned}$$

Since for every finite subset Ω_o of Ω,

$$\sum_{k\in\Omega_o} \rho_k(f(T_nT_n^*)) \longrightarrow \sum_{k\in\Omega_o} \rho_k(f(TT^*)) \text{ as } n \longrightarrow \infty,$$

it follows that for every finite subset Ω_o of Ω.

$$\rho(P_{\overline{\text{ran } A}}) \geq \sum_{k\in\Omega_o} \rho_k(f(TT^*)).$$

Therefore

$$\rho(P_{\overline{\text{ran } A}}) \geq \sup \Big\{ \sum_{k\in\Omega_o} \rho_k(f(TT^*)) : \Omega_o \text{ is finite } \Big\}.$$

Hence $\rho(P_{\overline{\text{ran } A}}) \geq \sum_{\alpha\in\Omega} \rho_\alpha(f(TT^*)) = \rho(f(TT^*))$ for every continuous function f with $0 \leq f \leq 1$ and $f(0) = 0$. By Lemma 1.4.1 and 1.4.6,

$$P_{\overline{\text{ran } T}} = P_{\overline{\text{ran } TT^*}} = \sup \{ f(TT^*) : 0 \leq f \leq 1, f(0) = 0 \text{ and } f \text{ is continuous } \}.$$

Thus $\rho(P_{\overline{\text{ran } A}}) \geq \rho(P_{\overline{\text{ran } T}})$, i.e. $\mathcal{R}-$rank $T \preceq \mathcal{R}-$rank A. This proves that $T \in \mathcal{E}$, and therefore $\mathcal{E}$ is closed under $*$-strong sequential limits. □

Proposition 1.4.12 *If $\mathcal{R}$ is a factor von Neumann algebra of type III, then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

Proof: Suppose $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{E}$ and $T_n \xrightarrow{*-SOT}$ as $n \longrightarrow \infty$.

In the case $A = 0$, note that $\mathcal{R}$−rank $T_n \preceq \mathcal{R}$−rank A for all $n \geq 1$, therefore $T_n = 0$ for all $n \geq 1$. Hence $T = 0$. It follows that $\mathcal{R}$−rank $T = \mathcal{R}$−rank A.

If $A \neq 0$, note that any two infinite projections in a factor von Neumann algebra are Murray-von Neumann equivalent by Proposition 1.1.29. Therefore

$$\mathcal{R}\text{-rank } T \begin{cases} = & \mathcal{R}\text{-rank } A \quad \text{if } T \neq 0 \\ \prec & \mathcal{R}\text{-rank } A \quad \text{if } T = 0 \end{cases}$$

We have proved $T \in \mathcal{E}$. Therefore $\mathcal{E}$ is closed under ∗- strong sequential limits. □

Next we prove Theorem 1.4.9 for type I_n (n is finite), I_∞, II_1, II_∞ or III von Neumann algebras acting on a separable Hilbert space.

Lemma 1.4.13 *Suppose $H = \int_\Omega^\oplus H_\omega d\mu(\omega) \subseteq L^2(\mu, K)$, where K is a separable Hilbert space and $\mathcal{R} = \int_\Omega^\oplus \mathcal{R}_\omega d\mu(\omega) \subseteq L^\infty(\mu, B(K))$. Suppose $A = \int_\Omega^\oplus A(\omega) d\mu(\omega)$ and $T = \int_\Omega^\oplus T(\omega) d\mu(\omega)$ in $\mathcal{R}$. Then*

$\mathcal{R}$−rank $T \preceq \mathcal{R}$−rank $A \iff \mathcal{R}_\omega$−rank $T(\omega) \preceq \mathcal{R}_\omega$−rank $A(\omega)$ almost everywhere.

Proof: ($\Longrightarrow$) Suppose $\mathcal{R}$−rank $T \preceq \mathcal{R}$ −rank A.

There is a projection P in $\mathcal{R}$ such that

$$P_{\overline{\text{ran } T}} \sim P \leq P_{\overline{\text{ran } A}} (\mathcal{R}). \tag{1.5}$$

Let $P = \int_\Omega^\oplus P(\omega) d\mu(\omega)$. $P(\omega)$ is a projection in $\mathcal{R}_\omega$ and $P(\omega) \leq P_{\overline{\text{ran } A(\omega)}}$ almost everywhere. Without loss of generality, we assume that $P(\omega)$ is a projection in $\mathcal{R}_\omega$ and that $P(\omega) \leq P_{\overline{\text{ran } A(\omega)}}$ for every ω in Ω. By Proposition 1.1.35 and (1.5), $P_{\overline{\text{ran } T(\omega)}} \sim P(\omega)$ $(\mathcal{R}_\omega)$ almost everywhere. We assume this is true for every ω in Ω. Therefore for every ω in Ω,

$$P_{\overline{\text{ran } T(\omega)}} \sim P(\omega)\ (\mathcal{R}_\omega) \leq P_{\overline{\text{ran } A(\omega)}}.$$

This proves that $\mathcal{R}_\omega$−rank $T(\omega) \preceq \mathcal{R}_\omega$−rank $A(\omega)$ almost everywhere.

($\Longleftarrow$) Suppose $\mathcal{R}_\omega$−rank $T(\omega) \preceq \mathcal{R}_\omega$−rank $A(\omega)$ almost everywhere.

For almost every ω in Ω, there is a projection $P(\omega)$ in $\mathcal{R}_\omega$ such that

$$P_{\overline{\text{ran } T(\omega)}} \sim P(\omega)\ (\mathcal{R}_\omega) \leq P_{\overline{\text{ran } A(\omega)}}. \tag{1.6}$$

Without loss of generality, we assume this is valid for every ω in Ω. Therefore by (1.6), and by similar argument to that in Proposition 1.3.3, there is a projection $P = \int_\Omega^\oplus P(\omega) d\mu(\omega)$ in $\mathcal{R}$ such that

$$P_{\overline{\text{ran } T}} \sim P\ (\mathcal{R}) \leq P_{\overline{\text{ran } A}} = \int_\Omega^\oplus P_{\overline{\text{ran } A(\omega)}}\, d\mu(\omega), \tag{1.7}$$

i.e. $\mathcal{R}$−rank $T \preceq \mathcal{R}$−rank A. □

Lemma 1.4.14 *Suppose $\mathcal{R}$ and $\overline{\mathcal{R}}$ are von Neumann algebras on Hilbert spaces H and K respectively. Suppose $u : H \longmapsto K$ is a unitary such that $u\mathcal{R}u^* = \overline{\mathcal{R}}$. Suppose S and T are normal operators in $\mathcal{R}$. Then*

$$\mathcal{R}\text{−rank } S \preceq \mathcal{R}\text{−rank } T \Longleftrightarrow \overline{\mathcal{R}}\text{−rank } uSu^* \preceq \overline{\mathcal{R}}\text{−rank } uTu^*.$$

Proof: ($\Longrightarrow$) Suppose $\mathcal{R}$−rank $S \preceq \mathcal{R}$ −rank T.

There is a partial isometry V in $\mathcal{R}$ and a closed subspace M of H such that

$$V : \overline{\text{ran } S} \longmapsto M \subseteq \overline{\text{ran } T} \text{ is an isometry.}$$

Therefore

$$uVu^* : \overline{\text{ran } uSu^*} \longmapsto uMu^* \subseteq \overline{\text{ran } uTu^*} \text{ is an isometry,}$$

i.e. $P_{\overline{\text{ran } uSu^*}} \sim P_{uMu^*} \ (\overline{\mathcal{R}}) \leq P_{\overline{\text{ran } uTu^*}}$.

Hence $\overline{\mathcal{R}}$–rank $uSu^* = \overline{\mathcal{R}}$–rank $P_{\overline{uMu^*}} \preceq \overline{\mathcal{R}}$–rank uTu^*.

($\Longleftarrow$) Suppose $\overline{\mathcal{R}}$–rank $uSu^* \preceq \overline{\mathcal{R}}$–rank uTu^*.

There is a partial isometry W in $\mathcal{R}$ and a closed subspace M of K such that

$$uWu^* : \overline{\text{ran } uSu^*} \longmapsto M \subseteq \overline{\text{ran } uTu^*} \text{ is an isometry .}$$

Hence

$$uWu^* : u\ \overline{\text{ran } S}\ u^* \longmapsto M \subseteq u\ \overline{\text{ran } T}\ u^* \text{ is an isometry.}$$

It follows that

$$W : \overline{\text{ran } S} \longmapsto u^*Mu \subseteq \overline{\text{ran } T} \text{ is an isometry ,}$$

i.e. $\mathcal{R}$–rank $S = \mathcal{R}$–rank $P_{u^*Mu} \preceq \mathcal{R}$–rank T. □

Proposition 1.4.15 *Let H be a separable Hilbert space. Suppose $\mathcal{R}$ is a type I_n von Neumann algebra acting on H, where n is finite. Then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

Proof: Suppose $\{T_m\}_{m=1}^{\infty} \subseteq \mathcal{E}$ and $T_m \xrightarrow{*-\text{SOT}} T$ as $m \longrightarrow \infty$. Suppose $\mathcal{R}$ is a type I_n von Neumann algebra acting on a separable Hilbert space H.

Let $\mathcal{C}$ be the center of $\mathcal{R}$. There is a (locally compact, complete separable metric) measure space (X,μ) such that H is (unitarily equivalent to) the direct integral of Hilbert spaces $\{H_p\}$ over (X,μ), and $\mathcal{R}$ is (unitarily equivalent to) the direct integral of type I_n factors almost everywhere relative to $\mathcal{C}$. ([DIX 5])

Note that there is a separable Hilbert space K and a family $\{v_p\}_{p\in X}$ of unitary transformations such that v_p maps H_p into K, $p \longmapsto v_p x(p)$ is measurable for each x in $\int_X^\oplus H_p d\mu(p)$, and $p \longmapsto v_p A_p v_p^*$ is measurable for each A in $\mathcal{R}$ ([DIX 5]). Thus $\int_X^\oplus v_p H_p d\mu(p) = L^2(\mu, K)$.

Hence there is a unitary $u : H \longmapsto \int_X^\oplus v_p H_p d\mu(p) \subseteq L^2(\mu, K)$ such that

$$u\mathcal{R}u^* = \int_X^\oplus \mathcal{R}_p d\mu(p) \subseteq L^\infty(\mu, B(K)),$$

where $\{\mathcal{R}_p\}_{p\in X}$ is a family of type I_n factors on the separable Hilbert space K almost everywhere. Since $T_m \xrightarrow{*-SOT} T$ as $m \longrightarrow \infty$, it follows that $uT_m u^* \xrightarrow{*-SOT} uTu^*$ as $m \longrightarrow \infty$.

Let

$$\begin{aligned} uT_m u^* &= \int_X^\oplus T_m(p) d\mu(p), \\ uTu^* &= \int_X^\oplus T(p) d\mu(p) \text{ and} \\ uAu^* &= \int_X^\oplus A(p) d\mu(p). \end{aligned}$$

Note that

$$uT_m u^* \xrightarrow{SOT} uTu^* \text{ as } m \longrightarrow \infty.$$

Proposition 1.1.36 implies that there is a subsequence $\{T_{m_k}\}$ such that for almost every p in X,

$$T_{m_k}(p) \xrightarrow{SOT} T(p) \text{ as } k \longrightarrow \infty.$$

Note that

$$(uT_{m_k}u^*)^* \xrightarrow{\text{SOT}} (uTu^*)^* \text{ as } k \longrightarrow \infty.$$

By Proposition 1.1.36 again, there is a subsequence $\{T_{m_{k_j}}\}$ such that for almost every p in X,

$$T_{m_{k_j}}(p)^* \xrightarrow{\text{SOT}} T(p)^* \text{ as } j \longrightarrow \infty.$$

Therefore there is a subsequence $\{T_{m_{k_j}}\}$ such that for almost every p in X,

$$T_{m_{k_j}}(p) \xrightarrow{*-\text{SOT}} T(p) \text{ as } j \longrightarrow \infty. \tag{1.8}$$

Without loss of generality, we assume $\mathcal{R}_p$ is a type I_n factor and (1.8) is true for every p in X. Since $\{T_{m_{k_j}}\} \subseteq \mathcal{E}$, $\mathcal{R}-$rank $T_{m_{k_j}} \preceq \mathcal{R}-$rank A for every $j \geq 1$. By Lemma 1.4.14, for every $j \geq 1$,

$$u\mathcal{R}u^*-\text{rank } uT_{m_{k_j}}u^* \preceq u\mathcal{R}u^*-\text{rank } uAu^*.$$

By Lemma 1.4.13, for every $j \geq 1$ and for almost every p in X,

$$\mathcal{R}_p-\text{rank } T_{m_{k_j}}(p) \preceq \mathcal{R}_p-\text{rank } A(p).$$

Proposition 1.4.10 and (1.8) imply that for almost every p in X,

$$\mathcal{R}_p-\text{rank } T(p) \preceq \mathcal{R}_p-\text{rank } A(p).$$

By Lemma 1.4.13, $u\mathcal{R}u^*-$rank $uTu^* \preceq u\mathcal{R}u^*-$rank uAu^*. Lemma 1.4.14 implies that

$$\mathcal{R}-\text{rank } T \preceq \mathcal{R}-\text{rank } A.$$

We have proved $T \in \mathcal{E}$. Hence $\mathcal{E}$ is closed under $*$-strong sequential limits. □

Proposition 1.4.16 *Suppose $\mathcal{R}$ is a type I_∞ (or II_1, II_∞, III) von Neumann algebra acting on a separable Hilbert space. Then $\mathcal{E}$ is closed under $*$-strong sequential limits.*

Proof: Use an analogous proof to that of the preceding Proposition. □

Now we prove some results about direct sums.

Lemma 1.4.17 *Suppose $\mathcal{R} = \sum_{\alpha\in\Omega}^{\oplus} \mathcal{R}_\alpha$. Suppose $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{R}$ and $T_n \overset{*-\mathrm{SOT}}{\longrightarrow} T$. Suppose $T_n = \sum_{\alpha\in\Omega}^{\oplus} T_n(\alpha)$ for every $n \geq 1$ and $T = \sum_{\alpha\in\Omega}^{\oplus} T(\alpha)$. Then for every α in Ω,*

$$T_n(\alpha) \overset{*-\mathrm{SOT}}{\longrightarrow} T(\alpha) \text{ as } n \longrightarrow \infty.$$

Proof: Let $H = \sum_{\alpha\in\Omega}^{\oplus} H_\alpha$, where $\mathcal{R}_\alpha \subseteq B(H_\alpha)$.

For a fixed $\alpha_o \in \Omega$ and for every $x \in H_{\alpha_o}$, let $y = \sum_{\alpha\in\Omega}^{\oplus} y(\alpha)$, where

$$y(\alpha) = \begin{cases} x & \text{if } \alpha = \alpha_o \\ 0 & \text{if } \alpha \neq \alpha_o. \end{cases}$$

Since $T_n \overset{\mathrm{SOT}}{\longrightarrow} T$ as $n \longrightarrow \infty$,

$$\|(T_n(\alpha_o) - T(\alpha_o))x\| = \|(T_n - T)y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This proves that $T_n(\alpha_o) \overset{\mathrm{SOT}}{\longrightarrow} T(\alpha_o)$ as $n \longrightarrow \infty$. Therefore $T_n(\alpha) \overset{\mathrm{SOT}}{\longrightarrow} T(\alpha)$ as $n \longrightarrow \infty$ for every α in Ω.

Similarly we can prove that for every α in Ω, $T_n(\alpha)^* \overset{\mathrm{SOT}}{\longrightarrow} T(\alpha)^*$ as $n \longrightarrow \infty$.

Hence for every α in Ω, $T_n(\alpha) \overset{*-\mathrm{SOT}}{\longrightarrow} T(\alpha)$ as $n \longrightarrow \infty$. □

Lemma 1.4.18 *Suppose $\mathcal{R}$ and $\mathcal{R}_n$ are von Neumann algebras such that $\mathcal{R} = \sum_{n\in K}^{\oplus} \mathcal{R}_n$,* *where K is an index set. Let E and F be projections in $\mathcal{R}$, and E_n and F_n be projections in $\mathcal{R}_n$ for $n \in K$, such that $E = \sum_{n\in K}^{\oplus} E_n$ and $F = \sum_{n\in K}^{\oplus} F_n$. Then*

$$E_n \sim F_n \ (\mathcal{R}_n) \text{ for every } n \in K \Longleftrightarrow E \sim F \ (\mathcal{R}).$$

Proof: ($\Rightarrow$) Suppose $E_n \sim F_n$ $(\mathcal{R}_n)$ for every $n \in K$.

By Definition 1.1.1, there are partial isometries $V_n \in \mathcal{R}_n$ such that $V_n^* V_n = E_n$ and $V_n V_n^* = F_n$ for every $n \in K$. Define $V = \sum_{n\in K}^{\oplus} V_n$. Then V is a partial isometry in $\mathcal{R}$.

Since

$$\begin{aligned} V^*V &= (\sum_{n\in K}^{\oplus} V_n^*)(\sum_{n\in K}^{\oplus} V_n) \\ &= \sum_{n\in K}^{\oplus} V_n^* V_n \\ &= \sum_{n\in K}^{\oplus} E_n \\ &= E, \text{ and} \\ VV^* &= (\sum_{n\in K}^{\oplus} V_n)(\sum_{n\in K}^{\oplus} V_n^*) \\ &= \sum_{n\in K}^{\oplus} V_n V_n^* \\ &= \sum_{n\in K}^{\oplus} F_n \\ &= F, \end{aligned}$$

it follows that $E \sim F$ $(\mathcal{R})$.

($\Leftarrow$) Suppose $E \sim F$ $(\mathcal{R})$.

By Definition 1.1.1, there is a partial isometry V in $\mathcal{R}$ such that $V^*V = E$ and $VV^* = F$. Decompose V into the direct sum of partial isometries in $\mathcal{R}_n$ $(n \in K)$, say $V = \sum_{n\in K}^{\oplus} V_n$, where V_n is a partial isometry in $\mathcal{R}_n$ for $n \in K$. Since

$$\begin{aligned} V^*V &= (\sum_{n\in K}^{\oplus} V_n^*)(\sum_{n\in K}^{\oplus} V_n) \\ &= \sum_{n\in K}^{\oplus} V_n^* V_n \\ &= E \\ &= \sum_{n\in K}^{\oplus} E_n, \end{aligned}$$

it follows that $V_n^* V_n = E_n$ for $n \in K$. Similarly we can show that $V_n V_n^* = F_n$ for $n \in K$. By Definition 1.1.1 again, $E_n \sim F_n$ $(\mathcal{R}_n)$ for $n \in K$. □

Lemma 1.4.19 *Suppose $\mathcal{R}$ and $\{\mathcal{R}_\alpha\}_{\alpha\in\Omega}$ are as in Lemma 1.4.17. Suppose A and T are in $\mathcal{R}$ such that $T = \sum_{\alpha\in\Omega}^{\oplus} T_\alpha$ and $A = \sum_{\alpha\in\Omega}^{\oplus} A_\alpha$. Then*

$$\mathcal{R}\text{-}rank\ T \preceq \mathcal{R}\text{-}rank\ A \Longleftrightarrow \mathcal{R}_\alpha\text{-}rank\ T_\alpha \preceq \mathcal{R}_\alpha\text{-}rank\ A_\alpha \text{ for every } \alpha \in \Omega.$$

Proof: $(\Longrightarrow)$ Suppose $\mathcal{R}$−rank $T \preceq \mathcal{R}$−rank A.

There is a projection P in $\mathcal{R}$ such that

$$P_{\overline{\text{ran } T}} \sim P\ (\mathcal{R}) \le P_{\overline{\text{ran } A}}. \tag{1.9}$$

Let $P = \sum_{\alpha\in\Omega}^{\oplus} P_\alpha$. For every α in Ω, P_α is a projection in $\mathcal{R}_\alpha$ and

$$P_\alpha \le P_{\overline{\text{ran } A_\alpha}}. \tag{1.10}$$

By Lemma 1.4.18 and (1.9), for every α in Ω,

$$P_{\overline{\mathrm{ran}\ T_\alpha}} \sim P_\alpha\ (\mathcal{R}_\alpha). \tag{1.11}$$

By (1.10) and (1.11), $\mathcal{R}_\alpha$–rank $T_\alpha \preceq \mathcal{R}_\alpha$–rank A_α for every α in Ω.

($\Longleftarrow$) Suppose $\mathcal{R}_\alpha$–rank $T_\alpha \preceq \mathcal{R}_\alpha$–rank A_α for every α in Ω.

For every α in Ω, there is a projection P_α in $\mathcal{R}_\alpha$ such that

$$P_{\overline{\mathrm{ran}\ T_\alpha}} \sim P_\alpha\ (\mathcal{R}_\alpha) \leq P_{\overline{\mathrm{ran}\ A_\alpha}}. \tag{1.12}$$

Let $P = \sum_{\alpha\in\Omega}^{\oplus} P_\alpha$.

By (1.12), $P \leq P_{\overline{\mathrm{ran}\ A}}$. By Lemma 1.4.18 and (1.12), $P_{\overline{\mathrm{ran}\ T}} \sim P\ (\mathcal{R})$. Hence $P_{\overline{\mathrm{ran}\ T}} \preceq P_{\overline{\mathrm{ran}\ A}}\ (\mathcal{R})$, i.e. $\mathcal{R}$–rank $T \preceq \mathcal{R}$–rank A. □

Finally we prove Theorem 1.4.9.

Proof: By Proposition 1.1.22, $\mathcal{R}$ is the direct sum of type I, type II_1, type II_∞ and type III von Neumann algebras. Write $\mathcal{R} = \mathcal{R}_I \oplus \mathcal{R}_{II_1} \oplus \mathcal{R}_{II_\infty} \oplus \mathcal{R}_{III}$.

By Propostion 1.1.23, $\mathcal{R}_I$ is the direct sum of type I_n von Neumann algebras, write $\mathcal{R}_I = \sum_{n\in K}^{\oplus} \mathcal{R}_{I_n}$, where K is a family of mutually distinct cardinal numbers.

Suppose $\{T_m\}_{m=1}^{\infty} \subseteq \mathcal{E}$ and $T_m \xrightarrow{*-\mathrm{SOT}} T$ as $m \longrightarrow \infty$. Hence $\mathcal{R}$–rank $T_m \preceq \mathcal{R}$–rank A for every $m \geq 1$. Write

$$\begin{aligned}
T_m &= \sum_{n\in K}^{\oplus} T_m^{I_n} \oplus T_m^{II_1} \oplus T_m^{II_\infty} \oplus T_m^{III} \text{ for every } m \geq 1, \\
A &= \sum_{n\in K}^{\oplus} A^{I_n} \oplus A^{II_1} \oplus A^{II_\infty} \oplus A^{III}, \\
T &= \sum_{n\in K}^{\oplus} T^{I_n} \oplus T^{II_1} \oplus T^{II_\infty} \oplus T^{III}.
\end{aligned}$$

By Lemma 1.4.19,

$$\mathcal{R}_{I_n}\text{-rank } T_m^{I_n} \preceq \mathcal{R}_{I_n}\text{-rank } A^{I_n} \text{ for every } n \in K, \tag{1.13}$$

$$\mathcal{R}_{II_1}\text{-rank } T_m^{II_1} \preceq \mathcal{R}_{II_1}\text{-rank } A^{II_1}, \tag{1.14}$$

$$\mathcal{R}_{II_\infty}\text{-rank } T_m^{II_\infty} \preceq \mathcal{R}_{II_\infty}\text{-rank } A^{II_\infty}, \tag{1.15}$$

$$\mathcal{R}_{III}\text{-rank } T_m^{III} \preceq \mathcal{R}_{III}\text{-rank } A^{III}. \tag{1.16}$$

Since $T_m \overset{*-\text{SOT}}{\longrightarrow} T$ as $m \longrightarrow \infty$, Lemma 1.4.17 implies that

$$T_m^{I_n} \overset{*-\text{SOT}}{\longrightarrow} T^{I_n} \text{ for every } n \in K, \tag{1.17}$$

$$T_m^{II_1} \overset{*-\text{SOT}}{\longrightarrow} T^{II_1}, \tag{1.18}$$

$$T_m^{II_\infty} \overset{*-\text{SOT}}{\longrightarrow} T^{II_\infty}, \tag{1.19}$$

$$T_m^{III} \overset{*-\text{SOT}}{\longrightarrow} T^{III} \text{ as } m \longrightarrow \infty. \tag{1.20}$$

Hence by Propositions 1.4.15, Proposition 1.4.16 and (1.13) - (1.20),

$$\mathcal{R}_{I_n}\text{-rank } T^{I_n} \preceq \mathcal{R}_{I_n}\text{-rank } A^{I_n} \text{ for every } n \in K,$$

$$\mathcal{R}_{II_1}\text{-rank } T^{II_1} \preceq \mathcal{R}_{II_1}\text{-rank } A^{II_1},$$

$$\mathcal{R}_{II_\infty}\text{-rank } T^{II_\infty} \preceq \mathcal{R}_{II_\infty}\text{-rank } A^{II_\infty},$$

$$\mathcal{R}_{III}\text{-rank } T^{III} \preceq \mathcal{R}_{III}\text{-rank } A^{III}.$$

Therefore an application of Lemma 1.4.19 shows that $\mathcal{R}$−rank $T \preceq \mathcal{R}$−rank A, i.e. $\mathcal{E}$ is closed under $*$-strong sequential limits. □

Actually, we have proved that the $\mathcal{R}$−rank function is sequentially lower-semicontinuous in the $*$-strong operator topology in the following sense.

Definition 1.4.20 *Suppose X is a topological space and $(Y, \leq)$ is a partial ordered set. We say that $\varphi : X \longmapsto (Y, \leq)$ is sequentially lower-semicontinuous if for every element α in Y, the inverse image of $\{y \in Y : y \leq \alpha\}$ under φ is sequentially closed in X.*

Lemma 1.4.21 *Let $Y = \{\mathcal{R}\text{-rank}\, T : T \in \mathcal{R}\}$. Then "$\preceq$" is a partial order in Y.*

Proof: It's obvious since Murray-von Neumann equivalence is an equivalence relation. □

Theorem 1.4.22 *Let $X = \mathcal{R}$ with $*$-strong operator topology, where $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space. Let $Y = \{\mathcal{R}\text{-rank}\, T : T \in \mathcal{R}\}$ with partial order "$\preceq$". Then $\mathcal{R}\text{-rank} : \mathcal{R} \longmapsto Y$ is sequentially lower-semicontinuous.*

Proof: Suppose $A \in \mathcal{R}$ and $\alpha = \mathcal{R}\text{-rank}\, A$ in Y. Suppose $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, where $\mathcal{F}$ is the inverse image of $\{y \in Y : y \preceq \alpha\}$ under $\mathcal{R}$-rank function, and $T_n \xrightarrow{*-SOT} T$ as $n \longrightarrow \infty$.

Since $\{T_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, therefore $\mathcal{R}\text{-rank}\, T_n \preceq \alpha = \mathcal{R}\text{-rank}\, A$ for $n \geq 1$. Since $T_n \xrightarrow{*-SOT} T$ as $n \longrightarrow \infty$, Theorem 1.4.9 implies that $\mathcal{R}\text{-rank}\, T \preceq \mathcal{R}\text{-rank}\, A$, i.e. $T \in \mathcal{F}$. We have proved that $\mathcal{F}$ is closed in X under $\mathcal{R}$-rank function. Hence $\mathcal{R}$-rank function is $*$-strong sequentially lower-semicontinuous. □

1.5 Necessary Condition

In this last section, we prove a necessary condition for two normal operators in a von Neumann algebra acting on a separable Hilbert space to be approximately equivalent in the algebra.

Theorem 1.5.1 *Suppose A and B are normal operators in a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H. If $A \sim_a B$ $(\mathcal{R})$, then $\mathcal{R}$−rank $f(A) = \mathcal{R}$−rank $f(B)$ for all continuous function f.*

Proof: Since $A \sim_a B$ $(\mathcal{R})$, there is a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{R}$ such that

$$\|u_n A u_n^* - B\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence for every continuous function f,

$$\|u_n f(A) u_n^* - f(B)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore

$$u_n f(A) u_n^* \xrightarrow{*-\mathrm{SOT}} f(B) \text{ as } n \longrightarrow \infty.$$

Note that

$$\mathcal{R}\text{−rank } u_n f(A) u_n^* = \mathcal{R}\text{−rank } f(A) \text{ for every } n \geq 1.$$

Applying Theorem 1.4.9 gives that

$$\mathcal{R}\text{−rank } f(B) \preceq \mathcal{R}\text{−rank } f(A). \tag{1.21}$$

Similarly, since

$$u_n^* f(B) u_n \xrightarrow{*-\text{SOT}} f(A) \text{ as } n \longrightarrow \infty,$$

it follows that

$$\mathcal{R}\text{-rank } f(B) \preceq \mathcal{R}\text{-rank } f(A). \tag{1.22}$$

By (1.21) and (1.22), for all continuous function f,

$$\mathcal{R}\text{-rank } f(A) = \mathcal{R}\text{-rank } f(B).$$

□

Chapter 2

Approximately Equivalent Representations in von Neumann Algebras

In this chapter, we classify two unital representations π and ρ from a C^*-algebra $\mathcal{A}$ to a von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space H by the $\mathcal{R}-$rank function, where the $\mathcal{R}-$rank function is as before.

We start by giving some definitions.

Definition 2.0.1 *Suppose $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations. If for every element $a \in \mathcal{A}$, $\mathcal{R}-rank\ \pi(a) = \mathcal{R}-rank\ \rho(a)$, then we say $\mathcal{R}-rank \circ \pi = \mathcal{R}-rank \circ \rho$.*

Definition 2.0.2 *We say that two representations $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are approximately equivalent in $\mathcal{R}$ (written $\pi \sim_a \rho\ (\mathcal{R})$) if there is a net $\{U_\alpha\}_\alpha$ of unitaries in $\mathcal{R}$ such that*

$$\|U_\alpha \pi(a) U_\alpha^* - \rho(a)\| \longrightarrow 0 \text{ for every } a \in \mathcal{A}.$$

Throughout this chapter $\mathcal{A}$ is a C^*-algebra, $C(X)$ is the set of complex-valued continuous functions defined on the compact Hausdorff space X and $\mathrm{Bor}(X)$ is the set of complex-valued bounded Borel functions defined on X. The set of $n \times n$ matrices with entries in $\mathcal{A}$ is denoted by $M_n(\mathcal{A})$.

2.1 Necessary Condition

Throughout this section $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space.

Theorem 2.1.1 *Suppose $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations. If $\pi \sim_a \rho\ (\mathcal{R})$, then $\mathcal{R}$−rank $\circ\ \pi = \mathcal{R}$−rank $\circ\ \rho$.*

Proof: Since $\pi \sim_a \rho\ (\mathcal{R})$, there is a net $\{u_\alpha\}_\alpha$ of unitaries in $\mathcal{R}$ such that for every a in $\mathcal{A}$,

$$\|u_\alpha \pi(a) u_\alpha^* - \rho(a)\| \longrightarrow 0.$$

Thus for a fixed a in $\mathcal{A}$, there is a sequence $\{u_n\}_{n=1}^\infty \subseteq \{u_\alpha\}_\alpha$ such that

$$\|u_n \pi(a) u_n^* - \rho(a)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore

$$\|u_n \pi(a)\pi(a)^* u_n^* - \rho(a)\rho(a)^*\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $\pi(a)\pi(a)^*$ and $\rho(a)\rho(a)^*$ are normal in $\mathcal{R}$, an application of Theorem 1.5.1 shows that $\mathcal{R}$−rank $\pi(a)\pi(a)^* = \mathcal{R}$−rank $\rho(a)\rho(a)^*$. By Lemma 1.4.1,

$$\begin{aligned} \mathcal{R}\text{−rank } \pi(a) &= \mathcal{R}\text{−rank } \pi(a)\pi(a)^*, \\ \mathcal{R}\text{−rank } \rho(a) &= \mathcal{R}\text{−rank } \rho(a)\rho(a)^*. \end{aligned}$$

Hence $\mathcal{R}$−rank $\pi(a) = \mathcal{R}$−rank $\rho(a)$ for every a in $\mathcal{A}$. Thus $\mathcal{R}$−rank $\circ\ \pi = \mathcal{R}$−rank $\circ\ \rho$.

□

Theorem 2.1.2 *Suppose $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations. Suppose that for each a in $\mathcal{A}$ there are sequences $\{A_n\}_{n=1}^\infty$, $\{B_n\}_{n=1}^\infty$, $\{C_n\}_{n=1}^\infty$ and $\{D_n\}_{n=1}^\infty$ in $\mathcal{R}$ all depending on a*

such that $A_n\pi(a)B_n \xrightarrow{*-SOT} \rho(a)$ *and* $C_n\rho(a)D_n \xrightarrow{*-SOT} \pi(a)$ *as* $n \longrightarrow \infty$. *Then* $\mathcal{R}$*−rank* $\circ\pi$ = $\mathcal{R}$*−rank* $\circ\rho$.

Proof: Lemma 1.4.2 implies that for every $n \geq 1$,

$$\mathcal{R}\text{−rank } A_n\pi(a)B_n \preceq \mathcal{R}\text{−rank } A_n\pi(a) \preceq \mathcal{R}\text{−rank } \pi(a), \tag{2.1}$$

$$\mathcal{R}\text{−rank } C_n\rho(a)D_n \preceq \mathcal{R}\text{−rank } C_n\rho(a) \preceq \mathcal{R}\text{−rank } \rho(a). \tag{2.2}$$

Since $A_n\pi(a)B_n \xrightarrow{*-SOT} \rho(a)$ as $n \longrightarrow \infty$, Theorem 1.4.9 and (2.1) imply that

$$\mathcal{R}\text{−rank } \rho(a) \preceq \mathcal{R}\text{−rank } \pi(a). \tag{2.3}$$

Since $C_n\rho(a)D_n \xrightarrow{*-SOT} \pi(a)$ as $n \longrightarrow \infty$, Theorem 1.4.9 and (2.2) imply that

$$\mathcal{R}\text{−rank } \pi(a) \preceq \mathcal{R}\text{−rank } \rho(a). \tag{2.4}$$

By (2.3) and (2.4), $\mathcal{R}$−rank $\pi(a) = \mathcal{R}$−rank $\rho(a)$ for every a in $\mathcal{A}$, i.e.

$$\mathcal{R}\text{−rank} \circ \pi = \mathcal{R}\text{−rank} \circ \rho.$$

□

2.2 Sufficient Condition

In this section, we study a class Q of well-behaved C^*-algebras. A C^*-algebra $\mathcal{A}$ is in Q provided for every von Neumann algebra $\mathcal{S}$ and for all unital representations π and ρ from $\mathcal{A}$ into $\mathcal{S}$, if $\mathcal{S}-$rank $\circ\pi = \mathcal{S}-$rank $\circ\rho$, then $\pi \sim_a \rho\ (\mathcal{S})$.

First we prove that Q contains $C(X)$.

Theorem 2.2.1 *If every von Neumann algebra $\mathcal{S}$ is acting on a separable Hilbert space, then $C(X)$ is contained in Q.*

Lemma 2.2.2 *[MUR 1] Suppose $\pi, \rho : C(X) \longmapsto \mathcal{R}$ are unital representations. Then there are unital representations $\tilde{\pi}, \tilde{\rho} : Bor(X) \longmapsto \mathcal{R}$ such that $\tilde{\pi}|_{C(X)} = \pi$ and $\tilde{\rho}|_{C(X)} = \rho$.*

Lemma 2.2.3 *Suppose $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ is a subset of $C(X)$. Let $C^*(f_1, f_2, \cdots, f_n)$ be the C^*-algebra generated by $\mathcal{F}$. Then $C^*(f_1, f_2, \cdots, f_n)$ is $*$- isomorphic to $C(Y)$, where Y is a closed bounded subset of $\mathbf{C}^{(n)} = \mathbf{R}^{(2n)}$.*

Proof: Let $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$ be the maximal ideal space of $C^*(f_1, f_2, \cdots, f_n)$, i.e. $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n)) = \{\alpha |\ \alpha : C^*(f_1, f_2, \cdots, f_n) \longmapsto \mathbf{C}$ is a $*$-homomorphism, $\alpha(1) = 1\}$. Since $C^*(f_1, f_2, \cdots, f_n)$ is a commutative C^*-algebra, it is isometric, $*$-isomorphic to $C(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n)))$, the set of continuous functions defined on $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$. Define

$$\Phi : \mathcal{M}(C^*(f_1, f_2, \cdots, f_n)) \longmapsto \mathbf{C}^{(n)} \text{ by } \Phi(\alpha) = (\alpha(f_1), \alpha(f_2), \cdots, \alpha(f_n)).$$

Since $\alpha \in \mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$, it follows that $\alpha \in \mathcal{M}(C^*(f_i))$ for $1 \leq i \leq n$, and therefore $\alpha(f_i) \subseteq \sigma(f_i)$, since

$$\sigma(f_i) = \{\alpha(f_i) : \alpha \in \mathcal{M}(C^*(f_i))\}.$$

We have proved

$$\Phi(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))) \subseteq \prod_{1\le i\le n} \sigma(f_i).$$

Let $Y = \Phi(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n)))$.

Now we prove that Φ is a homeomorphism.

Since a one-one, continuous map from a compact space onto a Hausdorff space is a homeomorphism ([WILD 1]), $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$ is compact and Y is Hausdorff, it is sufficient to show that Φ is one-one and continuous. This is proved next.

Suppose $\Phi(\alpha) = \Phi(\beta)$, $\alpha, \beta \in \mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$, i.e.

$$(\alpha(f_1), \alpha(f_2), \cdots, \alpha(f_n)) = (\beta(f_1), \beta(f_2), \cdots, \beta(f_n)).$$

Therefore $\alpha(f_i) = \beta(f_i)$ for $1 \le i \le n$, and it follows that

$$\alpha(f) = \beta(f) \text{ for every } f \in C^*(f_1, f_2, \cdots, f_n),$$

i.e. $\alpha = \beta$. We have proved that Φ is one-one.

Suppose $\alpha_m \longrightarrow \alpha$ as $m \longrightarrow \infty$ in $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$ (with the weak*-topology). Hence $\alpha_m(f) \longrightarrow \alpha(f)$ as $m \longrightarrow \infty$ for every $f \in C^*(f_1, f_2, \cdots, f_n)$. Therefore $\alpha_m(f_i) \longrightarrow \alpha(f_i)$ as $m \longrightarrow \infty$, for $1 \le i \le n$. So $\Phi(\alpha_m) \longrightarrow \Phi(\alpha)$ in $\mathbf{C}^{(n)}$ as $m \longrightarrow \infty$. This proves that Φ is continuous.

Hence Φ is a homeomorphism.

Suppose Y is compact. Since $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$ is compact and Hausdorff, Y is compact and Hausdorff and $\Phi : \mathcal{M}(C^*(f_1, f_2, \cdots, f_n)) \longmapsto Y$ is a homeomorphism, it follows that $C(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n)))$ is $*$-isomorphic to $C(Y)$ ([KR 1]). Since $C^*(f_1, f_2, \cdots, f_n)$

is isometric, $*$-isomorphic to $C(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n)))$, it follows that $C^*(f_1, f_2, \cdots, f_n)$ is $*$-isomorphic to $C(Y)$.

It remains to show that Y is compact.

Note that Φ is continuous and $\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))$ is compact. Hence

$$Y = \Phi(\mathcal{M}(C^*(f_1, f_2, \cdots, f_n))) \text{ is compact.}$$

Note that $\prod_{1 \leq i \leq n} \sigma(f_i)$ is Hausdorff and Y is a compact subset of $\prod_{1 \leq i \leq n} \sigma(f_i)$, Y is closed.

We have completed the proof. □

Lemma 2.2.4 *Suppose $\pi, \rho : C(Y) \longmapsto \mathcal{R}$ are unital representations, where Y is a compact subset of $\mathbf{C}^{(n)} = \mathbf{R}^{(2n)}$ and $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space H. $\tilde{\pi}, \tilde{\rho}$ are extensions of π, ρ to $Bor(Y)$ respectively. Suppose $\mathcal{R}-rank \circ \pi = \mathcal{R}-rank \circ \rho$. Then $\tilde{\pi}(\chi_E) \sim \tilde{\rho}(\chi_E)$ $(\mathcal{R})$, where $E = \prod_{1 \leq i \leq 2n} (a_i, b_i)$, a_i, b_i are real numbers.*

Proof: For $E = \prod_{1 \leq i \leq 2n} (a_i, b_i)$, there is a $\epsilon > 0$ such that $a_i + \epsilon < b_i - \epsilon$ for $1 \leq i \leq 2n$.

Let $F = \prod_{1 \leq i \leq 2n} [a_i + \epsilon, b_i - \epsilon]$. Then F is closed in $\mathbf{R}^{(2n)}$, and $F \cap (Y \setminus E) = \phi$. Since Y is a compact, Hausdorff space, Urysohn's lemma implies that there is a continuous function f such that $f|_F = 1$, $f|_{Y \setminus E} = 0$ and $0 \leq f \leq 1$. Since $Y \setminus E$ is a G_δ set, Proposition 1.1.34 implies that we can choose f such that f is continuous, $f|_F = 1$, $0 \leq f \leq 1$ and $f^{-1}(0) = Y \setminus E$. Lemma 1.2.3 implies that for every continuous function f,

$$\begin{aligned} P_{\overline{\text{ran } \pi(f)}} &= P_{\overline{\text{ran } \tilde{\pi}(f)}} \\ &= \chi_{\mathbf{C} \setminus \{0\}}(\tilde{\pi}(f)) \\ &= \tilde{\pi}(\chi_{\mathbf{C} \setminus \{0\}} \circ f) \end{aligned}$$

$$
\begin{aligned}
&= \tilde{\pi}(\chi_E), \\
P_{\overline{\text{ran } \rho(f)}} &= P_{\overline{\text{ran } \tilde{\rho}(f)}} \\
&= \chi_{\mathbf{C}\setminus\{0\}}(\tilde{\rho}(f)) \\
&= \tilde{\rho}(\chi_{\mathbf{C}\setminus\{0\}} \circ f) \\
&= \tilde{\rho}(\chi_E).
\end{aligned}
$$

Since $\mathcal{R}$−rank $o\pi$ = $\mathcal{R}$−rank $o\rho$ by the hypothesis, $P_{\overline{\text{ran } \pi(f)}} \sim P_{\overline{\text{ran } \rho(f)}}$ $(\mathcal{R})$.

This establishes that $\tilde{\pi}(\chi_E) \sim \tilde{\rho}(\chi_E)$ $(\mathcal{R})$. □

Lemma 2.2.5 *Let $\pi, \rho, E, C(Y)$ and $\mathcal{R}$ be as in the preceding Lemma. Let $F = \prod_{1 \le i \le 2n} (a_i, b_i]$, $F_k = \prod_{1 \le i \le k-1} (a_i, b_i) \times \{b_k\} \times \prod_{k+1 \le i \le 2n} (a_i, b_i)$ for $1 \le k \le 2n$ and $F' = \prod_{1 \le i \le 2n} \{b_i\}$. Suppose $\tilde{\pi}(\chi_{F_k}) = \tilde{\rho}(\chi_{F_k}) = 0$ for $1 \le k \le 2n$ and $\tilde{\pi}(\chi_{F'}) = \tilde{\rho}(\chi_{F'}) = 0$. Then $\tilde{\pi}(\chi_F) \sim \tilde{\rho}(\chi_F)$ $(\mathcal{R})$.*

Proof: Let $E = \prod_{1 \le i \le 2n} (a_i, b_i)$. Note that $F = E \cup \bigcup_{k=1}^{2n} F_k \cup F'$. Since $\{E, F_k, F'\}$ are disjoint subsets of Y,

$$\chi_F = \chi_E + \sum_{k=1}^{2n} \chi_{F_k} + \chi_{F'}.$$

Therefore $\tilde{\pi}(\chi_F) = \tilde{\pi}(\chi_E)$ and $\tilde{\rho}(\chi_F) = \tilde{\rho}(\chi_E)$ by the hypothesis. By the preceding Lemma, we see that $\tilde{\pi}(\chi_E) \sim \tilde{\rho}(\chi_E)$ $(\mathcal{R})$. Hence $\tilde{\pi}(\chi_F) \sim \tilde{\rho}(\chi_F)$ $(\mathcal{R})$. □

Proposition 2.2.6 *Let $\pi, \rho, C(Y)$ and $\mathcal{R}$ be as in Lemma 2.2.4. Suppose $\mathcal{F} = \{f_1, f_2, \cdots f_n\}$, $f_i \in C(Y)$ for $1 \le i \le n$. Then for every given $\epsilon > 0$, there is a unitary $U_\epsilon \in \mathcal{R}$ such that $\|U_\epsilon \pi(f_i) U_\epsilon^* - \rho(f_i)\| < \epsilon$ for $1 \le i \le n$. .*

Proof: For $1 \le k \le 2n$, let

$$\mathcal{S}_k = \{b \in \mathbf{R} : \tilde{\pi}(\chi_{F_b}) \ne 0, \tilde{\rho}(\chi_{F_b}) \ne 0, \text{ where } F_b = \prod_{\substack{1 \le i \le 2n \\ i \ne k}} \mathbf{R} \times \{b\}\}.$$

Let

$$\mathcal{T} = \{(b_1, b_2, \cdots, b_{2n}) \in \mathbf{R}^{(2n)} : \tilde{\pi}(\chi_F) \neq 0, \tilde{\rho}(\chi_F) \neq 0, \text{ where } F = \prod_{1 \leq i \leq 2n} \{b_i\}\}.$$

Since H is separable, $\{\tilde{\pi}(\chi_{F_b})\}_{b \in \mathbf{R}}$ and $\{\tilde{\rho}(\chi_{F_b})\}_{b \in \mathbf{R}}$ are two sets of orthogonal projections in $\mathcal{R}$ respectively, and hence $\mathrm{card}(\mathcal{S}_k) \leq \aleph_0$ for $1 \leq k \leq 2n$. Similarly $\mathrm{card}(\mathcal{T}) \leq \aleph_0$. Therefore by Lemma 2.2.5, for a given $\epsilon > 0$, there is a partition $\{F_l\}_{l=1}^N$ of Y such that

1. $F_l = \prod_{1 \leq i \leq 2n} (a_l^i, a_{l+1}^i]$,
2. $\tilde{\pi}(\chi_{F_l}) \sim \tilde{\rho}(\chi_{F_l})(\mathcal{R})$ for $1 \leq l \leq N$,
3. $\|f_i - \sum_{l=1}^N \alpha_l \chi_{F_l}\|_\infty < \epsilon/2$ for $1 \leq i \leq n$ and $\alpha_l \in \mathbf{C}$ for $1 \leq l \leq N$.

Since $\tilde{\pi}$ and $\tilde{\rho}$ are unital representations, for $1 \leq i \leq n$,

$$\begin{aligned} \|\tilde{\pi}(f_i) - \sum_{l=1}^N \alpha_l \tilde{\pi}(\chi_{F_l})\| &< \epsilon/2 \text{ and} \\ \|\tilde{\rho}(f_i) - \sum_{l=1}^N \alpha_l \tilde{\rho}(\chi_{F_l})\| &< \epsilon/2. \end{aligned}$$

Note that $\{\tilde{\pi}(\chi_{F_l})\}_{l=1}^N$ and $\{\tilde{\rho}(\chi_{F_l})\}_{l=1}^N$ are two sets of orthogonal projections in $\mathcal{R}$ with sum I respectively, and $\tilde{\pi}(\chi_{F_l}) \sim \tilde{\rho}(\chi_{F_l})(\mathcal{R})$ for $1 \leq l \leq N$. Lemma 1.2.2 implies that there is a unitary $U_\epsilon \in \mathcal{R}$ such that for $1 \leq l \leq N$,

$$U_\epsilon \tilde{\pi}(\chi_{F_l}) U_\epsilon^* = \tilde{\rho}(\chi_{F_l}).$$

Hence for $1 \leq i \leq n$,

$$\|U_\epsilon \tilde{\pi}(f_i) U_\epsilon^* - \tilde{\rho}(f_i)\| \leq \|U_\epsilon \tilde{\pi}(f_i) U_\epsilon^* - U_\epsilon(\sum_{l=1}^N \alpha_l \tilde{\pi}(\chi_{F_l}))U_\epsilon^*\| + \|\sum_{l=1}^N \alpha_l \tilde{\rho}(\chi_{F_l}) - \tilde{\rho}(f_i)\|$$

$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon.$$

□

Now we prove Theorem 2.2.1.

Proof: Suppose $\pi, \rho : C(X) \longmapsto \mathcal{R}$ are unital representations, where $\mathcal{R}$ is a von Neumann algebra acting on a separable Hilbert space. Suppose $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$.

By an application of Lemma 2.2.2, there are unital $*$-homomorphisms $\tilde{\pi}, \tilde{\rho} : Bor(X) \longmapsto \mathcal{R}$ such that $\tilde{\pi}|_{C(X)} = \pi$, $\tilde{\rho}|_{C(X)} = \rho$.

First we show that for every finite subset $\mathcal{F} = \{f_1, f_2, \cdots, f_n\}$ of $C(X)$ and for every $\epsilon > 0$, there is a unitary $U_\epsilon \in \mathcal{R}$ such that $\|U_\epsilon \pi(f_i) U_\epsilon^* - \rho(f_i)\| < \epsilon$ for $1 \leq i \leq n$.

Lemma 2.2.3 implies that $C^*(f_1, f_2, \cdots, f_n)$ is $*$-isomorphic to $C(Y)$, the set of continuous functions defined on Y, where Y is a closed, bounded subset of $\mathbf{C}^{(n)} = \mathbf{R}^{(2n)}$. Suppose $\Phi : C^*(f_1, f_2, \cdots, f_n) \longmapsto C(Y)$ is the $*$-isomorphism such that $\Phi(f_i) = g_i$ for $1 \leq i \leq n$. Therefore $\pi \circ \Phi^{-1}$ and $\rho \circ \Phi^{-1} : C(Y) \longmapsto \mathcal{R}$ are unital $*$-homomorphisms and

$$\mathcal{R}-\text{rank} \circ \pi \circ \Phi^{-1} = \mathcal{R}-\text{rank} \circ \rho \circ \Phi^{-1}.$$

According to Proposition 2.2.6, there is a unitary $U_\epsilon \in \mathcal{R}$ such that for $1 \leq i \leq n$,

$$\|U_\epsilon \pi \circ \Phi^{-1}(g_i) U_\epsilon^* - \rho \circ \Phi^{-1}(g_i)\| < \epsilon,$$

i.e. for $1 \leq i \leq n$,

$$\|U_\epsilon \pi(f_i) U_\epsilon^* - \rho(f_i)\| < \epsilon.$$

Let $\mathcal{S} = \{(\mathcal{F}, \epsilon) : \mathcal{F}$ is a finite subset of $C(X)$ and $\epsilon > 0\}$, ordered by

$$(\mathcal{F}_1, \epsilon_1) \geq (\mathcal{F}_2, \epsilon_2) \Longleftrightarrow \mathcal{F}_2 \subseteq \mathcal{F}_1, \epsilon_1 \leq \epsilon_2.$$

Then $\mathcal{S}$ is a directed set. By the above argument, for every $(\mathcal{F}, 1/\#\mathcal{F}) \in \mathcal{S}$, where $\#\mathcal{F}$ is the cardinality of $\mathcal{F}$, there is a unitary $U_{\mathcal{F}} \in \mathcal{R}$ such that for all $f \in \mathcal{F}$,

$$\|U_{\mathcal{F}}\pi(f)U_{\mathcal{F}}^* - \rho(f)\| < \frac{1}{\#\mathcal{F}}.$$

It follows that there is a net $\{U_{\mathcal{F}}\}$ of unitaries of $\mathcal{R}$ such that for every $f \in C(X)$,

$$\|U_{\mathcal{F}}\pi(f)U_{\mathcal{F}}^* - \rho(f)\| \longrightarrow 0,$$

i.e. $\pi \sim_a \rho$ $(\mathcal{R})$. □

Next we prove that if $\mathcal{A}$ is in Q, then $M_n(\mathcal{A})$ is in Q, where $M_n(\mathcal{A})$ is the set of $n \times n$ matrices with entries in $\mathcal{A}$.

Let I be the identity in the corresponding algebras. Let $E(I)$ be the $n \times n$ matrix that each entry on the first diagonal above the main diagonal is I and all other entries are 0. For each A in $\mathcal{A}$ and for $1 \leq i, j \leq n$, let $E_{i,j}(A)$ be the $n \times n$ matrix with a A in the (i, j) position and 0's elsewhere.

Theorem 2.2.7 *If $\mathcal{A}$ is in Q, then $M_n(\mathcal{A})$ is in Q for $n \geq 1$.*

Lemma 2.2.8 *$M_n(\mathcal{A})$ is the C^*-algebra generated by* $E = \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & \ddots & I \\ & & & & 0 \end{pmatrix}_{n \times n}$ *and* $E_{1,1}(A) =$

$$\begin{pmatrix} A & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}_{n\times n}, \text{ where } A \in \mathcal{A}.$$

Proof: Note that $E_{1,2}(I) = (E(I)E(I)^* - E(I)^*E(I))E(I)$ and $E_{1,1}(I) = E_{1,2}(I)E_{1,2}(I)^*$. Therefore $E_{1,2}(I)$ and $E_{1,1}(I)$ are generated by $E(I)$. Note that $E_{1,j+1}(I) = E_{1,j}(I)E(I)$ for $1 \le j \le n-1$. Hence for $1 \le j \le n$, $E_{1,j}(I)$ and $E_{j,1}(I) = E_{1,j}(I)^*$ are generated by $E(I)$.

Inductively $E_{i,j}(I)$ is generated by $E(I)$ for $1 \le i,j \le n$. Thus

$$E_{i,j}(A) = E_{1,i}(I)E_{1,1}(A)E_{1,j}(I)$$

is generated by $E_{1,1}(A)$ and $E(I)$ for every $A \in \mathcal{A}$. Therefore

$$F = (A_{i,j})_{n\times n} = \sum_{i,j=1}^{n} E_{i,j}(A_{i,j}) \in M_n(\mathcal{A})$$

is generated by $E(I)$ and $E_{1,1}(A)$ for every $F \in M_n(\mathcal{A})$. □

Lemma 2.2.9 *Suppose $\{H_k\}_{k=1}^n$ is a set of Hilbert spaces and $H = \sum_{k=1}^{n}{}^{\oplus} H_k$. Suppose $A = (A_{i,j})_{n\times n} \in B(H)$, where $A_{i,j} \in B(H_j, H_i)$ for $1 \le i,j \le n$. Then $\|A_{i,j}\| \le \|A\|$ for $1 \le i,j \le n$.*

Proof: For $1 \le i,j \le n$ and for every unit vector x in H_j, let

$$y = \underbrace{0 \oplus 0 \oplus \cdots \oplus 0}_{j-1} \oplus x \oplus \underbrace{0 \oplus \cdots \oplus 0}_{n-j}.$$

y is a unit vector in H. It follows that for $1 \leq i, j \leq n$,

$$\|Ay\| = (\sum_{l=1}^{n} \|A_{l,j}x\|^2)^{\frac{1}{2}} \geq \|A_{i,j}x\|.$$

Therefore for $1 \leq i, j \leq n$,

$$\|A\| = \sup_{\{y \in H, \|y\|=1\}} \|Ay\| \geq \sup_{\{x \in H_j, \|x\|=1\}} \|A_{i,j}x\| = \|A_{i,j}\|.$$

□

Now we prove Theorem 2.2.7.

Proof: Let $\mathcal{A}$ be a C^*-algebra in Q. Suppose that $\pi, \rho : M_n(\mathcal{A}) \longmapsto \mathcal{R}$ are unital representations, and that $\mathcal{R}-$rank $\circ\ \pi = \mathcal{R}-$rank $\circ\ \rho$.

Let

$$P_i = \pi(E_{i,i}(I)) \text{ and } Q_i = \rho(E_{i,i}(I)) \text{ for } 1 \leq i \leq n.$$

Then $\{P_i\}_{i=1}^n$ and $\{Q_i\}_{i=1}^n$ are two sets of orthogonal projections in $\mathcal{R}$ with sum I respectively, since $\{E_{i,i}(I)\}_{i=1}^n$ is a set of orthogonal projections in $M_n(\mathcal{A})$ with sum I, and π and ρ are unital representations. Also $\mathcal{R}-$rank $P_i = \mathcal{R}-$rank Q_i, i.e. $P_i \sim Q_i$ $(\mathcal{R})$ for $1 \leq i \leq n$, since $\mathcal{R}-$rank $\circ\ \pi = \mathcal{R}-$rank $\circ\ \rho$. By Lemma 1.2.2, there is a unitary u in $\mathcal{R}$ such that $u\pi(E_{i,i}(I))u^* = \rho(E_{i,i}(I))$ for $1 \leq i \leq n$. Without loss of generality, we may assume

$$P_i = \pi(E_{i,i}(I)) = \rho(E_{i,i}(I)) = Q_i \text{ for } 1 \leq i \leq n.$$

For otherwise, we replace π by $u\pi()u^*$ and using Lemma 1.4.3, we obtain

$$\mathcal{R}-\text{rank } \circ\ u\pi()u^* = \mathcal{R}-\text{rank } \circ\ \pi = \mathcal{R}-\text{rank } \circ\ \rho.$$

For $1 \le i \le n-1$,

$$\begin{aligned}
\pi(E_{i,i+1}(I))\pi(E_{i,i+1}(I))^* &= P_i, \\
\pi(E_{i,i+1}(I))^*\pi(E_{i,i+1}(I)) &= P_{i+1}, \\
\rho(E_{i,i+1}(I))\rho(E_{i,i+1}(I))^* &= P_i, \\
\rho(E_{i,i+1}(I))^*\rho(E_{i,i+1}(I)) &= P_{i+1}.
\end{aligned}$$

Note that dim ran P_i = dim ran P_{i+1} and that $\pi(E_{i,i+1}(I))$ and $\rho(E_{i,i+1}(I))$: ran $P_{i+1} \longmapsto$ ran P_i are isometries for $1 \le i \le n$. Let H_i = ran P_i for $1 \le i \le n$. Therefore $H = \sum_{i=1}^{n} {}^{\oplus} H_i$. There exist isometries A_i and B_i in $B(H_{i+1}, H_i) \cap \mathcal{R}$ for $1 \le i \le n-1$, such that

$$\pi(E(I)) = \begin{pmatrix} 0 & A_1 & & & \\ & 0 & A_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & A_{n-1} \\ & & & & 0 \end{pmatrix}_{n \times n} \in \mathcal{R}, \text{ and} \tag{2.5}$$

$$\rho(E(I)) = \begin{pmatrix} 0 & B_1 & & & \\ & 0 & B_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & B_{n-1} \\ & & & & 0 \end{pmatrix}_{n \times n} \in \mathcal{R}. \tag{2.6}$$

Therefore

$$\pi(E_{1,1}(I)) = E_{1,1}(I) = \rho(E_{1,1}(I)). \tag{2.7}$$

Let $\phi : \mathcal{A} \longmapsto M_n(\mathcal{A})$ be defined by $\phi(A) = E_{1,1}(A)$ for every $A \in \mathcal{A}$. Then ϕ is a one-one, $*$-homomorphism. Therefore $\pi \circ \phi, \rho \circ \phi : \mathcal{A} \longmapsto \mathcal{R}$ are unital $*$-homomorphisms (restrict to the image of $\pi \circ \phi$ and $\rho \circ \phi$ respectively), and since $\mathcal{R}$−rank $\circ\, \pi = \mathcal{R}$−rank $\circ\, \rho$, $\mathcal{R}$−rank $\circ\, (\pi \circ \phi) = \mathcal{R}$−rank $\circ\, (\rho \circ \phi)$. Since $\mathcal{A} \in Q$, there is a net $\{u_\alpha\}_\alpha$ of unitaries in $\mathcal{R}$

such that

$$\|u_\alpha \pi \circ \phi(A) u_\alpha^* - \rho \circ \phi(A)\| \longrightarrow 0 \text{ for every } A \in \mathcal{A}, \tag{2.8}$$

i.e.

$$\|u_\alpha \pi(E_{1,1}(A)) u_\alpha^* - \rho(E_{1,1}(A))\| \longrightarrow 0 \text{ for every } A \in \mathcal{A}. \tag{2.9}$$

We can write $u_\alpha = (u_{i,j}^\alpha)_{n \times n}$, where $u_{i,j}^\alpha \in B(H_j, H_i)$ for $1 \leq i, j \leq n$.

By (2.7) and (2.9),

$$\|u_\alpha E_{1,1}(I) u_\alpha^* - E_{1,1}(I)\| \longrightarrow 0. \tag{2.10}$$

By an application of Lemma 2.2.9 and (2.10),

$$\|u_{1,1}^\alpha u_{1,1}^{\alpha\,*} - I\| \longrightarrow 0, \text{ and} \tag{2.11}$$

$$\|u_{1,1}^{\alpha\,*} u_{1,1}^\alpha - I\| \longrightarrow 0. \tag{2.12}$$

Hence for sufficiently large α, $u_{1,1}^\alpha$ is invertible, and $Z_{1,1}^\alpha = (u_{1,1}^\alpha u_{1,1}^{\alpha\,*})^{-\frac{1}{2}} u_{1,1}^\alpha$ is a unitary in $B(H_1) \cap \mathcal{R}$.

Define

$$U^\alpha = \begin{pmatrix} Z_{1,1}^\alpha & & & & \\ & X_2^\alpha & & & \\ & & X_3^\alpha & & \\ & & & \ddots & \\ & & & & X_n^\alpha \end{pmatrix}, \tag{2.13}$$

where $X_2^\alpha = B_1^* Z_{1,1}^\alpha A_1$ is a unitary in $B(H_2) \cap \mathcal{R}$, and $X_i^\alpha = B_{i-1}^* X_{i-1}^\alpha A_{i-1}$ is a unitary in $B(H_i) \cap \mathcal{R}$ for $3 \leq i \leq n$. U^α is a unitary in $\mathcal{R}$.

Since $M_n(\mathcal{A})$ is generated by $E(I)$ and $\{E_{1,1}(A) : A \in \mathcal{A}\}$, to show $\pi \sim_a \rho$ $(\mathcal{R})$ it is sufficient to show that

$$\|U^\alpha \pi(E(I))U^{\alpha *} - \rho(E(I))\| \longrightarrow 0 \text{ and} \tag{2.14}$$

$$\|U^\alpha \pi(E_{1,1}(A))U^{\alpha *} - \rho(E_{1,1}(A))\| \longrightarrow 0 \text{ for every } A \in \mathcal{A}. \tag{2.15}$$

By (2.5), (2.6) and (2.13),

$$U^\alpha \pi(E(I))U^{\alpha *} - \rho(E(I)) = 0.$$

This proves (2.14). It remains to show (2.15).

Since for $2 \leq i \leq n$ and for every $A \in \mathcal{A}$,

$$\begin{aligned} P_i \pi(E_{1,1}(A)) &= 0 \text{ and} \\ \pi(E_{1,1}(A))P_i &= 0, \end{aligned}$$

we can write $\pi(E_{1,1}(A)) = E_{1,1}(C)$ for some $C \in B(H_1)$. Similarly, we can write $\rho(E_{1,1}(A)) = E_{1,1}(D)$ for some $D \in B(H_1)$. By Lemma 2.2.9 and (2.9),

$$\|u^\alpha_{1,1} C u^{\alpha}_{1,1}{}^* - D\| \longrightarrow 0. \tag{2.16}$$

Note that

$$\|U^\alpha \pi(E_{1,1}(A))U^{\alpha *} - \rho(E_{1,1}(A))\| = \|Z^\alpha_{1,1} C Z^{\alpha}_{1,1}{}^* - D\|.$$

It remains to show

$$\|Z^\alpha_{1,1} C Z^{\alpha}_{1,1}{}^* - D\| \longrightarrow 0. \tag{2.17}$$

By (2.11),

$$\|(u^{\alpha}_{1,1}u^{\alpha}_{1,1}{}^{*})^{-\frac{1}{2}} - I\| \longrightarrow 0.$$

Therefore

$$\|Z^{\alpha}_{1,1} - u^{\alpha}_{1,1}\| \leq \|(u^{\alpha}_{1,1}u^{\alpha}_{1,1}{}^{*})^{-\frac{1}{2}} - I\|\|u^{\alpha}_{1,1}\| \longrightarrow 0. \tag{2.18}$$

By (2.16) and (2.18), it follows that

$$\begin{aligned} &\|Z^{\alpha}_{1,1}CZ^{\alpha}_{1,1}{}^{*} - D\| = \\ &\quad \|Z^{\alpha}_{1,1}CZ^{\alpha}_{1,1}{}^{*} - u^{\alpha}_{1,1}CZ^{\alpha}_{1,1}{}^{*} + u^{\alpha}_{1,1}CZ^{\alpha}_{1,1}{}^{*} - u^{\alpha}_{1,1}Cu^{\alpha}_{1,1}{}^{*} + u^{\alpha}_{1,1}Cu^{\alpha}_{1,1}{}^{*} - D\| \\ &\leq \|Z^{\alpha}_{1,1} - u^{\alpha}_{1,1}\|\|C\|\|Z^{\alpha}_{1,1}{}^{*}\| + \|u^{\alpha}_{1,1}\|\|C\|\|Z^{\alpha}_{1,1}{}^{*} - u^{\alpha}_{1,1}{}^{*}\| + \|u^{\alpha}_{1,1}Cu^{\alpha}_{1,1}{}^{*} - D\|, \end{aligned}$$

and this last quantity tends to 0, hence (2.17) is established

We have proved that $\pi \sim_a \rho\ (\mathcal{R})$. □

Then we will prove that Q is closed under direct sum, direct limit and quotient map.

First we prove that Q is closed under direct sum.

Theorem 2.2.10 *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are in Q. Then $\mathcal{A}_1 \oplus \mathcal{A}_2$ is in Q.*

Proof: Suppose $\pi, \rho : \mathcal{A}_1 \oplus \mathcal{A}_2 \longmapsto \mathcal{R}$ are unital representations. Suppose $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$.

We can write $\pi = \pi_1 \oplus \pi_2$, $\rho = \rho_1 \oplus \rho_2$ and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\pi_i, \rho_i : \mathcal{A}_i \longmapsto \mathcal{R}_i$ are unital representations for $1 \leq i \leq 2$. Since $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$, it follows that

$$\mathcal{R}_1-\text{rank} \circ \pi_1 = \mathcal{R}_1-\text{rank} \circ \rho_1, \text{ and } \mathcal{R}_2-\text{rank} \circ \pi_2 = \mathcal{R}_2-\text{rank} \circ \rho_2.$$

Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are in Q. Hence $\pi_i \sim_a \rho_i\ (\mathcal{R}_i)$ for $1 \leq i \leq 2$.

For every $\epsilon > 0$ and for every finite subset $F \subseteq \mathcal{A}_1 \oplus \mathcal{A}_2$, suppose that

$$F = \{a_1 \oplus b_1, a_2 \oplus b_2, \cdots, a_n \oplus b_n\}.$$

Since $\{a_1, a_2, \cdots, a_n\} \subseteq \mathcal{A}_1$ and $\{b_1, b_2, \cdots, b_n\} \subseteq \mathcal{A}_2$, there are unitaries $U_F^{(1)}$ and $U_F^{(2)}$ in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively such that for $1 \leq k \leq n$,

$$\begin{aligned} \|U_F^{(1)}\pi_1(a_k)U_F^{(1)^*} - \rho_1(a_k)\| &< \epsilon/2, \\ \|U_F^{(2)}\pi_2(b_k)U_F^{(1)^*} - \rho_2(b_k)\| &< \epsilon/2. \end{aligned}$$

Define $U_F = U_F^{(1)} \oplus U_F^{(2)}$. Then U_F is a unitary in $\mathcal{R}$ such that for $1 \leq k \leq n$,

$$\begin{aligned} & \|U_F\pi(a_k \oplus b_k)U_F^* - \rho(a_k \oplus b_k)\| \\ = \ & \|U_F(\pi_1(a_k) \oplus \pi_2(b_k))U_F^* - \rho_1(a_k) \oplus \rho_2(b_k)\| \\ = \ & \sup \{ \ \|U_F^{(1)}\pi_1(a_k)U_F^{(1)^*} - \rho_1(a_k)\|, \|U_F^{(2)}\pi_2(b_k)U_F^{(2)^*} - \rho_2(b_k)\| \ \} \\ < \ & \epsilon. \end{aligned}$$

Let $\mathcal{S} = \{(F, \epsilon) : F$ is a finite subset of $\mathcal{A}_1 \oplus \mathcal{A}_2$ and $\epsilon > 0\}$, ordered by

$$(F_1, \epsilon_1) \leq (F_2, \epsilon_2) \Longleftrightarrow F_1 \subseteq F_2 \text{ and } \epsilon_1 \geq \epsilon_2.$$

$\mathcal{S}$ is a directed set. By the above argument, for every $\beta = (F, 1/\#F)$ in $\mathcal{S}$, there is a unitary U_β in $\mathcal{R}$ such that for every a in F

$$\|U_\beta\pi(a)U_\beta^* - \rho(a)\| < 1/\#F.$$

Hence $\pi \sim_a \rho\ (\mathcal{R})$, i.e. $\mathcal{A}_1 \oplus \mathcal{A}_2$ is in Q. □

Next we will prove that Q is closed under direct limit.

Theorem 2.2.11 *Suppose $\{\mathcal{A}_\lambda : \lambda \in \Omega\}$ is an increasing net of C^*-algebras in Q. Then the direct limit $\mathcal{A} = \varinjlim \mathcal{A}_\lambda$ is in Q.*

Proof: Suppose $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations. Suppose $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$.

Let $\pi_\lambda = \pi|_{\mathcal{A}_\lambda}$ and $\rho_\lambda = \rho|_{\mathcal{A}_\lambda}$ for every λ. Then $\pi_\lambda, \rho_\lambda : \mathcal{A}_\lambda \longmapsto \mathcal{R}$ are unital representations for every λ. Also $\mathcal{R}-\text{rank} \circ \pi_\lambda = \mathcal{R}-\text{rank} \circ \rho_\lambda$ for every λ in Ω, since $\mathcal{R}-\text{rank} \circ \pi = \mathcal{R}-\text{rank} \circ \rho$.

For every λ in Ω, since $\mathcal{A}_\lambda \in Q$, there exists a net $\{u_\alpha^\lambda\}_\alpha$ of unitaries in $\mathcal{R}$ such that

$$\|u_\alpha^\lambda \pi_\lambda(a) u_\alpha^{\lambda *} - \rho_\lambda(a)\| \longrightarrow 0 \text{ for every } a \in \mathcal{A}_\lambda.$$

Now we show that for every $\epsilon > 0$, for every finite subset $\mathcal{F}$ of $\cup_{\lambda\in\Omega}\mathcal{A}_\lambda$, there is a unitary u in $\mathcal{R}$ such that

$$\|u\pi(a)u^* - \rho(a)\| < \epsilon \text{ for all } a \in \mathcal{F}.$$

Since $\{\mathcal{A}_\lambda\}$ is an increasing net of C^*-algebras and $\mathcal{F}$ is a finite subset of $\cup_{\lambda\in\Omega}\mathcal{A}_\lambda$, there is a β in Ω such that $\mathcal{F} \subseteq \mathcal{A}_\beta$. Thus

$$\begin{aligned} \|u_\alpha^\beta \pi(a) u_\alpha^{\beta *} - \rho(a)\| &= \|u_\alpha^\beta \pi_\beta(a) u_\alpha^{\beta *} - \rho_\beta(a)\| \\ &\longrightarrow 0 \text{ for all } a \in \mathcal{F}. \end{aligned}$$

It follows that there is a unitary $u \in \{u_\alpha^\beta\}_\alpha$ such that

$$\|u\pi(a)u^* - \rho(a)\| < \epsilon \text{ for all } a \in \mathcal{F}.$$

Let

$$\mathcal{T} = \{(\mathcal{F}, \epsilon) : \mathcal{F} \text{ is a finite subset of } \cup_{\lambda \in \Omega} \mathcal{A}_\lambda \text{ and } \epsilon > 0\},$$

ordered by $(\mathcal{F}_1, \epsilon_1) \leq (\mathcal{F}_2, \epsilon_2) \Longleftrightarrow \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\epsilon_1 \geq \epsilon_2$. Then $\mathcal{T}$ is a directed set. By the above argument, for every $\gamma = (\mathcal{F}, \frac{1}{\#\mathcal{F}}) \in \mathcal{T}$, there is a unitary $u_\gamma \in \mathcal{R}$ such that

$$\|u_\gamma \pi(a) u_\gamma^* - \rho(a)\| < \frac{1}{\#\mathcal{F}} \text{ for all } a \in \mathcal{F}.$$

Thus there exists a net $\{u_\gamma\}_\gamma$ of unitaries in $\mathcal{R}$ such that

$$\|u_\gamma \pi(a) u_\gamma^* - \rho(a)\| \longrightarrow 0 \text{ for all } a \in \cup_{\lambda \in \Omega} \mathcal{A}_\lambda.$$

Since $\mathcal{S} = \{a \in \mathcal{A} : \|u_\gamma \pi(a) u_\gamma^* - \rho(a)\| \longrightarrow 0\}$ is a norm-closed linear space containing $\cup_{\lambda \in \Omega} \mathcal{A}_\lambda$, it contains $\mathcal{A} = \overline{\cup_{\lambda \in \Omega} \mathcal{A}_\lambda}^{\text{Norm}}$, i.e. $\pi \sim_a \rho \ (\mathcal{R})$. □

Now we prove that Q is closed under quotient map.

Theorem 2.2.12 *Suppose that $\mathcal{A}$ is in Q and that $\mathcal{J}$ is a closed ideal in $\mathcal{A}$. Then $\mathcal{A}/\mathcal{J}$ is in Q.*

Proof: Suppose $\pi, \rho : \mathcal{A}/\mathcal{J} \longmapsto \mathcal{R}$ are unital representations such that $\mathcal{R}$−rank $\circ\, \pi =$ $\mathcal{R}$−rank $\circ\, \rho$.

Suppose $\eta : \mathcal{A} \longmapsto \mathcal{A}/\mathcal{J}$ is the canonical map. Therefore $\pi \circ \eta$ and $\rho \circ \eta : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations and $\mathcal{R}$−rank $\circ\, (\pi \circ \eta) = \mathcal{R}$−rank $\circ\, (\rho \circ \eta)$. Since $\mathcal{A}$ is in Q,

$\pi \circ \eta \sim_a \rho \circ \eta$ $(\mathcal{R})$. It follows that $\pi \sim_a \rho$ $(\mathcal{R})$, i.e. $\mathcal{A}/\mathcal{J}$ is in Q. □

The following results are somewhat more interesting.

Theorem 2.2.13 *Suppose $\mathcal{R}$ is a factor von Neumann algebra of type III and $\mathcal{A}$ is a C^*-algebra in Q. Suppose $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations. Then*

$$\pi \sim_a \rho \ (\mathcal{R}) \Longleftrightarrow \mathit{ker}\ \pi = \mathit{ker}\ \rho.$$

Proof: ($\Longrightarrow$) Suppose $\pi \sim_a \rho$ $(\mathcal{R})$.

There is a net $\{u_\alpha\}_\alpha$ of unitaries in $\mathcal{R}$ such that

$$\|u_\alpha \pi(a) u_\alpha^* - \rho(a)\| \longrightarrow 0 \text{ for every } a \in \mathcal{A}.$$

Hence $\pi(a) = 0 \Longleftrightarrow \rho(a) = 0$, i.e. ker π = ker ρ.

($\Longleftarrow$) Suppose ker π = ker ρ.

For every a in $\mathcal{A}$, $\pi(a) \neq 0 \Longleftrightarrow \rho(a) \neq 0$. Hence

$$P_{\overline{\text{ran } \pi(a)}} \neq 0 \Longleftrightarrow P_{\overline{\text{ran } \rho(a)}} \neq 0.$$

Therefore $P_{\overline{\text{ran } \pi(a)}} \sim P_{\overline{\text{ran } \rho(a)}}$ $(\mathcal{R})$ for every a in $\mathcal{A}$, since $\mathcal{R}$ is a type III factor, i.e.

$$\mathcal{R}\text{-rank} \circ \pi = \mathcal{R}\text{-rank} \circ \rho.$$

Thus $\pi \sim_a \rho$ $(\mathcal{R})$, since $\mathcal{A}$ is in Q. □

Theorem 2.2.14 *Suppose $\mathcal{A}$ is in Q and $\pi, \rho : \mathcal{A} \longmapsto \mathcal{R}$ are unital representations, where $\mathcal{R}$ is acting on a separable Hilbert space. Furthermore suppose for every a in $\mathcal{A}$, there are* *sequences* $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$ *and* $\{D_n\}_{n=1}^{\infty}$ *in* $\mathcal{R}$ *all depending on a such that*

$$A_n\pi(a)B_n \xrightarrow{*-\mathrm{SOT}} \rho(a) \text{ and } C_n\rho(a)D_n \xrightarrow{*-\mathrm{SOT}} \pi(a) \text{ as } n \longrightarrow \infty.$$

Then $\pi \sim_a \rho\ (\mathcal{R})$.

Proof: By Theorem 2.1.2, $\mathcal{R}$−rank $\circ\ \pi = \mathcal{R}$−rank $\circ\ \rho$. Therefore $\pi \sim_a \rho\ (\mathcal{R})$, since $\mathcal{A}$ is in Q. □

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