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# Finite groups as a generalization of vector spaces through the use of splitting systems

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# Finite Groups as a Generalization of Vector Spaces Through the Use of Splitting Systems

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# Dedication

To my wife Cindy and son Timothy, for their love and support throughout this endeavor.

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# Contents

## ABSTRACT

**Finite Groups as a Generalization of Vector Spaces Through the Use of Splitting Systems**

by

Joseph Kirtland
University of New Hampshire, September, 1992

The structure of a finite group is investigated through a geometry induced by the splitting systems of the group. The method is based on the one used to induce a geometry on a finite dimensional vector space over a finite field and as a result, concepts related to the special and projective linear group are extended to arbitrary groups. One major by-product is the classification of solvable multiprimitive groups of arbitrary derived length. This leads to a necessary and sufficient condition for a solvable nC-group to be multiprimitive.

# Introduction

This investigation concerns finite groups. A subgroup A of a group G is **complemented** in G if there is a subgroup B of G, such that G = AB and A ∩ B = { 1 }. If A is normal in G, G is said to **split** over A, denoted by G = [A]B. A group G that splits over each normal subgroup is called an nC-group. A group is **inseparable** if it has order 1 or if it does not split over any proper, non-trivial normal subgroup. All other groups are **separable.** If a group G is separable, it can be expressed in the form $G = S_1S_2\dots S_n$ such that for each $i$, $1 \leq i \leq n$, $S_i$ is an inseparable subgroup of G and for $2 \leq i \leq n$, $S_1\dots S_i = [S_1\dots S_{i-1}]S_i$. The ordered set $\{ S_1, S_2, \dots, S_n \}$ is called a **splitting system** for G of **length n**. Note that $S_1\dots S_{i-1}$ is normal in $S_1\dots S_i$ and not necessarily in G. An inseparable group has only one splitting system { G }. Splitting systems were first introduced by Bechtell in [6]. Inseparable groups have been studied by Bechtell in [4], [5], and [7] and by Scarselli in [24], [25], and [26].

A separable group has two or more splitting systems. For example, consider the group $S_3 = \langle a, b \mid a^3 = b^2 = 1, a^b = a^2\rangle$. It has three splitting systems $\Sigma_1 = \{\langle a\rangle, \langle b\rangle\}$, $\Sigma_2 = \{\langle a\rangle, \langle ab\rangle\}$, and $\Sigma_3 = \{\langle a\rangle, \langle a^2b\rangle\}$. Each splitting system for $S_3$ is of length 2. However, not every group has all its splitting systems of the same length. The dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3\rangle$ is a counter example. It has splitting systems $\Sigma_1 = \{\langle a\rangle, \langle b\rangle\}$ and $\Sigma_2 = \{\langle a^2\rangle, \langle b\rangle, \langle ab\rangle\}$, where $\Sigma_1$ is of length 2 and $\Sigma_2$ is of length 3.

Two splitting systems $\Sigma = \{ S_1, S_2, \dots, S_n \}$ and $\Sigma' = \{ T_1, T_2, \dots, T_m \}$ for a group G are **equivalent** if $n = m$ and there is an $n$-tuple $\hat{g} = (g_1, g_2, \dots, g_n) \in G \times \dots \times G$

such that $\Sigma^{\hat{g}} = \{ S_1^{g_1}, S_2^{g_2}, \ldots, S_n^{g_n} \} = \{ T_1, T_2, \ldots, T_n \} = \Sigma'$. They are **congugate** if there is an element $g \in$ G such that $\Sigma^g = \{ S_1^g, S_2^g, \ldots, S_n^g \} = \{ T_1, T_2, \ldots, T_n \} = \Sigma'$. H. Bechtell, in [6], studied finite solvable nC-groups through splitting systems. In particular, he classified all finite solvable nC-groups which have all their splitting systems congugate. The next avenue of study is to investigate those groups which have all their splitting systems equivalent. These groups are studied and it is proven that a solvable nC-group with all its splitting systems equivalent is a multiprimitive group. Multiprimitive groups were first introduced by T. Hawkes in [15]. A solvable multiprimitive group is a group which has a unique chief series and all its chief factors complemented.

Not every multiprimitive group has all its splitting systems equivalent. This leads to an investigation of the structure of multiprimitive groups through the use of splitting systems. The study of multiprimitive groups, done only within the framework of splitting systems, does not yield many results. Consequently, splitting systems are used to induce a geometry on the group. Within the framework of this induced geometry, the structure of multiprimitive groups is studied. This method can be generalized to arbitrary finite groups and permits an investigation, through an induced geometry, into the structure of finite groups.

The use of splitting systems to induce a geometry on a finite group is based on the geometry associated with finite dimensional vector spaces over the field of characteristic $p$. Suzuki, in 3.3 of [29], presents a study of the geometry of linear groups. A brief overview of this work is given.

Let V be a finite dimensional vector space over the field of characteristic p. The set of 1-dimensional subspaces of V form **points** in a set $\mathcal{P}$ associated with V. A subset $\alpha$ of $\mathcal{P}$ is called a **subspace** of $\mathcal{P}$ if there is a subspace U of V such that $\alpha$ coincides with the set of 1-dimensional subspaces of U. A **frame** for $\mathcal{P}$ is a set of points $\Sigma = \{P_0, P_1, \ldots, P_d\}$

from $\mathcal{P}$ such that the points in $\Sigma$ generate V and the dimension of V is $d+1$. A nested sequence of non-trivial subspaces A: $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_d$ of $\mathcal{P}$ is called a **flag.** The number of subspaces in a flag is called the **rank** of the flag. Given two flags A: $\alpha_1 \supset \ldots \supset \alpha_d$ and B: $\beta_1 \supset \ldots \supset \beta_e$, A is **contained** in B, denoted A $\subseteq$ B, if $d \leq e$ and for each $i$, $1 \leq i \leq d$, $\alpha_i = \beta_j$ for some $j$, $1 \leq j \leq e$. A partial ordering, via containment, can be defined on the collection of all flags associated with $\mathcal{P}$. The collection of all flags, together with this partial ordering, is called the **flag space** $\Delta(\mathcal{P})$ of G.

A flag in $\Delta(\mathcal{P})$ is called **maximal** if it is not properly contained in any other flag in $\Delta(\mathcal{P})$. All the maximal flags in $\Delta(\mathcal{P})$ are of the same rank. In addition, given two flags A and B in $\Delta(\mathcal{P})$, **A $\cap$ B** is defined to be the flag C which is maximal with respect to containment in A and B. Two flags A and B in $\Delta(\mathcal{P})$ are **connected** if there is an ordered set $\{ C_1, C_2, \ldots, C_n \}$ of maximal flags in $\Delta(\mathcal{P})$ of rank d such that A is contained in $C_1$, B is contained in $C_n$, and for each $i$, $1 \leq i \leq n-1$, $C_i \cap C_{i+1}$ is of rank d-1. $\Delta(\mathcal{P})$ is connected if each pair of flags in $\Delta(\mathcal{P})$ are connected. One geometric aspect of $\Delta(\mathcal{P})$ is that it is connected.

Associated with $\Delta(\mathcal{P})$ is a collection of collineations. A **collineation** $\sigma$ is a permutation of the subspaces in $\mathcal{P}$ such that A $\subset$ B if and only if $A^\sigma \subset B^\sigma$. Hence each collineation induces a bijective map of $\Delta(\mathcal{P})$. Furthermore, each element $\phi$ in GL$(n,p)$ induces a collineation. The collection of all collineations induced by GL$(n,p)$ is denoted by PGL$(n,p)$.

A finite dimensional vector space V over the field of characteristic $p$ is an elementary abelian $p$-group. If G is an elementary abelian $p$-group, then each frame $\Sigma = \{P_0, P_1, \ldots, P_d\}$ for G is also a splitting system for G. This follows, since for each $i$, $0 \leq i \leq d$, $P_i$ is inseparable and the subgroup U generated by the points $P_0, P_1, \ldots, P_i$ is the semi-direct product of the subgroup W generated by the points $P_0, P_1, \ldots, P_{i-1}$ and the subgroup $P_i$.

Thus there is a natural connection between splitting systems and frames. Furthermore, the subgroups in a splitting system $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for an arbitrary G form a basis for G in the sense that for $i, j$, $1 \leq i, j \leq n$, $i \neq j$, $S_i \cap S_j = \{ 1 \}$ and each element $g \in$ G can be uniquely written $g = s_1 s_2 \ldots s_n$ where $s_i \in S_i$. In this manner, each subgroup $S_i$ in $\Sigma$ is a generalization of a 1-dimensional subspace of a vector space V. When G is an elementary abelian $p$-group, the subgroups in all the splitting systems for G correspond to the 1-dimensional subspaces of G as a vector space over $Z_p$. Consequently, a generalization of the construction of $\Delta(\mathcal{P})$ for finite groups is made through the use of splitting systems.

To begin the construction, each splitting system $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G is called a **frame** for G. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a nested sequence of subgroups in G. A is called a **flag** for G if there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G and $1 \leq i_1 < i_2 < \ldots < i_d \leq n-1$ such that $\alpha_1 = S_1 \ldots S_{i_1}, \alpha_2 = S_1 \ldots S_{i_2}, \ldots$, and $\alpha_d = S_1 \ldots S_{i_d}$. The number of subgroups $d$ in A is called the **rank** of A. This definition presents a flag for a group G to be a sequence of subgroups of G. A flag defined for a finite dimensional vector space V over the field of characteristic p is a sequence of subspaces of $\mathcal{P}$, where each subspace $\alpha$ of $\mathcal{P}$ coincides with the set of 1-dimensional subspaces of a subspace U of V. The definition of a subspace of $\mathcal{P}$ creates a one-to-one correspondence between the subspaces of $\mathcal{P}$ and the subspaces of V. This correspondence and the fact that elementary abelian p-groups split over each subgroup indicate, that without loss of generality, the definition presented here of a flag for a group G reduces to the definition of a flag for a finite dimensional vector space over the field of characteristic p when G is an elementary abelian p-group. Also note that for a flag, as defined here, subgroups are ordered with respect to containment beginning with the subgroup of least order. There is no loss of generality and this is done as the definition of a flag is based on a splitting system for G.

The collection of all flags together with a partial ordering defined on this collection forms the **projective space** $\Delta(\mathcal{P})$. Once $\Delta(\mathcal{P})$ has been defined for an arbitrary finite group, **maximal flags** and **A $\wedge$ B** can be defined, where A $\wedge$ B is the flag C which is maximal with respect to containment in A and B. The concept of maximal flags leads to the definition of a **point** in $\Delta(\mathcal{P})$. The creation of $\Delta(\mathcal{P})$ for an arbitrary finite group enables an investigation of the structure of G with respect to this induced geometry.

The concept of a collineation can also be generalized. A permutation $\sigma$ of the flags in $\Delta(\mathcal{P})$ is called a **collineation** of $\Delta(\mathcal{P})$ if for each flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, $A^\sigma = \alpha_1' \subset \alpha_2' \subset \ldots \subset \alpha_d'$, where $\alpha_i \cong \alpha_i'$ for each $i$, $1 \leq i \leq d$; and for each pair of flags A and B in $\Delta(\mathcal{P})$, $(A \wedge B)^\sigma = A^\sigma \wedge B^\sigma$. If G is an elementary abelian $p$-group, the two definitions are equivalent. The collection of all collineations of $\Delta(\mathcal{P})$ is denoted by **Col($\Delta(\mathcal{P})$)**. Each automorphism in Aut(G) induces a collineation. This is not to say that all the elements of Col($\Delta(\mathcal{P})$) are induced by the elements of Aut(G).

The construction of $\Delta(\mathcal{P})$ and Col($\Delta(\mathcal{P})$) associated with a finite group G induce a geometry on G. Once this construction is complete, the geometric properties of $\Delta(\mathcal{P})$ and Col($\Delta(\mathcal{P})$) are studied. Questions, such as when an arbitrary group G has a connected flag space and when Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags and points in $\Delta(\mathcal{P})$ are investigated. It is proven that abelian groups and solvable nC-groups have connected flag spaces. In addition, it is proven that Col($\Delta(\mathcal{P})$) acts point and maximal flag transitively on $\Delta(\mathcal{P})$ of an abelian group G if and only if G is homocyclic.

Another aspect in the study of finite dimensional vector spaces over a finite field is the study of hyperplanes and transvections. A **hyperplane** H for a finite dimensional vector space V over a finite field is a subspace of codimension 1. An invertible linear transformation $\tau$ of V is called a **transvection** of V, if for some hyperplane H of V, $h^\tau = h$ for all $h \in$ H and $v^\tau v^{-1} \in$ H for all elements $v \in$ V. For an elementary abelian $p$-group G, the collection of transvections generates SL($n, p$) and GL($n, p$)/SL($n, p$) $\cong Z_p^*$, the multiplicative group of the units in $Z_p$.

Splitting systems are used to generalize the concepts of a hyperplane and transvection. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a splitting system for a separable group G. Since each subgroup in $\Sigma$ is a generalization of a 1-dimensional subspace, a subgroup H of G is defined to be a **hyperplane** in G if there is a splitting system $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G such that H $= S_1 S_2 \ldots S_{n-1}$. Through splitting systems, all the hyperplanes in G can be identified. If G is an elementary abelian $p$-group, this definition is equivalent to the standard one. An automorphism $\sigma$ in Aut(G) is a **transvection** associated with the hyperplane H if $h^\sigma = h$ for all $h \in$ H and $g^\sigma g^{-1} \in$ H for all $g \in$ G. A similar generalization of hyperplanes and transvections was done by Liebert in [20] for abelian p-groups. The subgroup of Aut(G) generated by all the transvections in Aut(G) is denoted by $\mathcal{T}$**(G)**. For an elementary abelian $p$-group, $\mathcal{T}$(G) = SL($n, p$). Consequently, $\mathcal{T}$(G) is a generalization of the Special Linear Group. Many results concerning GL($n, p$) and SL($n, p$) are proven as corollaries to results obtained here.

Once an investigation into the induced geometry of a group is complete, multiprimitive groups are studied through their induced geometry. An answer is given to the question as to when a multiprimitive group has all its splitting systems equivalent. A method is established to classify all solvable multiprimitive groups and a necessary and sufficient condition is given as to when a solvable nC-group is multiprimitive.

This dissertation is divided into 8 chapters. The first chapter introduces primitive and multiprimitive groups. The concept of a splitting system is introduced and their fundamental properties are established. **SE-groups** are defined to be those groups which have all their splitting systems equivalent. A subclass of these groups are **SC-groups**, which have all their splitting systems conjugate. Fundamental results concerning SE-groups are proven in this chapter. In particular, it is proven that a finite solvable nC-group with all its splitting systems equivalent is multiprimitive. An example is given to show the converse is not true.

Chapter 2 focuses on the development of the flag space $\Delta(\mathcal{P})$ for an arbitrary group G. The concepts of a frame and flag are introduced and their fundamental properties are established. After $\Delta(\mathcal{P})$ is defined, points are defined and $\mathcal{P}$ is defined to be the collection of all points in $\Delta(\mathcal{P})$. The construction of Col($\Delta(\mathcal{P})$), the collection of all collineations of $\Delta(\mathcal{P})$, is developed. In addition, it is proven that Col($\Delta(\mathcal{P})$) is a group and that each automorphism of G induces an element in Col($\Delta(\mathcal{P})$). As a result, there is a homomorphism $\kappa$: Aut(G) $\rightarrow$ Col($\Delta(\mathcal{P})$). **Normal** and **complemented** flags are defined in this chapter. Lastly, conditions are studied as to when a flag space $\Delta(\mathcal{P})$ is connected. There are two main results in this chapter. The first result is that all abelian groups have a connected flag space. It is a corollary to this result that the flag space of an elementary abelian $p$-group is connected. The other main result is that if two flags A and B contain a normal, complemented flag C: $\gamma_1 \subset \gamma_2 \subset \ldots \subset \gamma_d$, where the flag space associated with $\gamma_i/\gamma_{i-1}$ is connected for each $i$, $1 \leq i \leq d+1$ with $\alpha_0 = \{ 1 \}$ and $\alpha_{d+1} =$ G, then A and B are connected. This theorem is used to show that the flag space of a solvable nC-group is connected.

Chapter 3 discusses flag spaces where Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags and points in $\Delta(\mathcal{P})$. It is proven that an abelian group G has Col($\Delta(\mathcal{P})$) acting transitively on the maximal flags in $\Delta(\mathcal{P})$ if and only if G is homocyclic. It is a corollary to this result that when G is an elementary abelian $p$-group, Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$. Conditions are also given as to when Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in the flag space $\Delta(\mathcal{P})$ associated with a solvable, non-$p$-group of derived length 2.

In Chapter 4, the induced geometric structure of a group is used to investigate the algebraic structure of the group. One aspect of the flag space of an elementary abelian $p$-group is that each maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$ has the property that $\alpha_i/\alpha_{i-1} \cong Z_p$. This results in each pair of maximal flags in $\Delta(\mathcal{P})$ being equivalent. In this chapter, groups are examined in which each pair of maximal flags A and B in $\Delta(\mathcal{P})$ are equivalent. It is proven that solvable nC-groups, SE-groups, groups in which each element is of prime order, and certain metacyclic groups satisfy this condition. An **f-primary group of type** S is defined to be a group G such that each maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$ has the property that for each $i$, $1 \leq i \leq d+1$, $\alpha_i/\alpha_{i-1} \cong$ S, where S is an inseparable group with $\alpha_0 = \{ 1 \}$ and $\alpha_{d+1} =$ G. Groups which are f-primary groups of type S are studied in this chapter. In addition, those groups are identified whose induced geometric structure most closely matches the induced geometric structure of elementary abelian $p$-groups. This investigation results in a necessary and sufficient condition for a group G to be complemented.

In Chapter 5, the properties of Col($\Delta(\mathcal{P})$) are investigated. In Chapter 2, the existence of a homomorphism $\kappa$ from Aut(G) to Col($\Delta(\mathcal{P})$) was established. The first question pertains to when this map is onto. It is proven that even for elementary abelian $p$-groups G with $p \geq 5$, this is not valid. For an elementary abelian p-group of rank $n$, PGL$(n,p) \cong$ GL$(n,p)$/Z(GL$(n,p)$), where PGL$(n,p)$ is the image of GL$(n,p)$ in Col($\Delta(\mathcal{P})$). Therefore, ker($\kappa$) = Z(GL$(n,p)$). This leads to the study of those groups which satisfiy the condition ker($\kappa$) = Z(Aut(G)). This is proven to be valid for abelian groups. In addition, each automorphism in Z(GL($n, p$)) fixes each 1-dimensional subspace in G. **St(Aut(G))** is defined to be the collection of those automorphisms in Aut(G), which fix each element of each frame for G. It is proven that Stab(G) $\subseteq$ Z(Aut(G)) when G is a solvable nC-group.

In Chapter 6, hyperplanes and transvections are defined through the use of splitting systems. Fundamental results concerning hyperplanes and transvections are proven here. Many theorems concerning GL($n, p$) and SL($n, p$) are proven as corollaries to results obtained in this chapter.

The next chapter, Chapter 7, is restricted to multiprimitive groups. In Chapter 1 it was proven that a solvable nC-group with all its splitting systems equivalent is multiprimitive. A counter example was given to show that the converse is not always valid. In this chapter an answer is given as to when a multprimitive group G has all its splitting systems equivalent. The geometric results used to answer this question lead to a method of classifying all solvable multiprimitive groups. **Invariant** flags are defined in this chapter and are those flags A in $\Delta(\mathcal{P})$ such that $A^\sigma = A$ for each $\sigma \in \mathrm{Col}(\Delta(\mathcal{P}))$. Some basic results concerning invariant flags are established. Through the use of invariant flags , a necessary and sufficient condition is given as to when a solvable nC-group is multiprimitive.

The last chapter states some questions motivated by this work and presents avenues to be explored.

# Chapter 1

# Multiprimitive Groups, Splitting Systems, and Their Connection

The notation used is standard and all groups mentioned are assumed to be finite.

## 1.1 Multiprimitive Groups

Given a group G, a subgroup A of G is said to be **complemented** in G if there is a subgroup B of G such that $G = AB$ and $A \cap B = \{1\}$. If A is normal in G, G is said to **split** over A and is denoted by $G = [A]B$. Of special interest are **nC-groups**, which are those groups which split over every normal subgroup. Solvable nC-groups were first studied by C. Christensen in [9]. H. Bechtell shows in [3] that the class of solvable nC-groups constitutes a formation.

A group G is said to **primitive** if it has a faithful, primitive, permutation representation. Equivalent to this condition is that G has a maximal subgroup M such that $\text{core}(M) = \{1\}$. Hence, if a solvable group G is primitive, it has a unique minimal normal subgroup N which is the Fitting subgroup of G. Furthermore, since $N \cap M = \{ 1 \}$, N is complemented in G by M and $G = [N]M$. Primitive groups play an important role in the study of solvable and non-solvable groups.

A group G is said to be **multiprimitive** if for each normal subgroup N in G, G/N is primitive. Multiprimitive groups were first introduced and studied by T. Hawkes in [15]. He defines a group G to be multiprimitive if each epimorphic image of G is primitive. Hawkes also developes a method, involving the twisted wreath product, to construct a multiprimitive group given two other multiprimitive groups. Hawkes shows in [15] that an arbitrary finite solvable group H can be embedded in a solvable multiprimitive group G. Thus solvable multiprimitive groups are an important class of groups which deserve investigation. The symmetric group $S_4$ and the alternating group $A_4$ are examples of multiprimitive groups.

From now on, when referring to a multiprimitive group, it shall be assumed to be solvable. Hawkes was able to prove the following theorem.

**Theorem 1.1** *(2.1 of [15]) Each of the following is necessary and sufficient for a non-trivial solvable group G to be multiprimitive:*

*a) G has a unique chief series and all its chief factors are complemented;*

*b) The lower nilpotent series of G is a chief series;*

*c) The upper nilpotent series of G is a chief series;*

*d) G is extreme(definition 3 of [8]) and G has prefrattini subgroups of order 1.*

By Theorem 1.1 (a), the class of multiprimitive groups is a special class of solvable nC-groups. Let $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_r \lhd N_{r+1} = G$ be the unique chief series for a multiprimitive group G. Let $\mid N_i/N_{i-1} \mid = p_i^{t_i}$ for $1 \leq i \leq r+1$ and primes $p_i$. Note that $t_{r+1} = 1$ or else $N_r$ is not the unique maximal normal subgroup of G. In addition, suppose for some $j$, $1 \leq j \leq r-1$, that $\mid N_j/N_{j-1} \mid = p_j$. Since Aut($N_j/N_{j-1}$) is abelian and G/$N_{j-1}$ is multiprimitive, G/$N_{j-1}$ is a supersolvable group of derived length at most 2. Furthermore, this series is the derived series and the Fitting series for G. Therefore, for each $i$, $1 \leq i \leq r$, $p_i \neq p_{i+1}$ or else the Fitting subgroup F(G/$N_{i-1}$) $\neq N_i/N_{i-1}$, a contradiction.

For each $i$, $1 \leq i \leq r$, $N_i$ is complemented in G. Hence $G = [N_i]B_i$ and $N_{i+1} \cap G = N_{i+1} \cap N_i B_i = N_i(N_{i+1} \cap B_i)$. Therefore, $N_i$ is complemented in $N_{i+1}$ by $C_{i+1} = N_{i+1} \cap B_i$, which is a finite dimensional vector space over the field of characteristic $p$. Thus G can be written as $G = C_1 C_2 \ldots C_{r+1}$, where for each $j$, $1 \leq j \leq r+1$, $C_j$ is a finite dimensional vector space over a field of characteristic $p$ and $C_1 \ldots C_{j+1} = [C_1 \ldots C_j]C_{j+1}$.

In a communication, T. Hawkes indicated that an extensive study of multiprimitive groups had not yet been attempted. One of the goals of this work is to make an investigation into the structure of multiprimitive groups and to classify, to the extent possible, these groups.

At this point, the concept of a splitting system is introduced and how the study of splitting systems leads to multiprimitive groups.

## 1.2 Splitting Sytems and Their Fundamental Properties

**Definition 1.2** *A group is* **inseparable** *if it has order 1 or if it does not split over any proper, non-trivial normal subgroup. All other groups are* **separable**.

Inseparable groups have been studied by H. Bechtell in [4], [5], and [7] and by Scarselli in [24], [25], and [26].

If a group G is separable, it can be expressed in the form $G=S_1 \ldots S_n$ where for which each $i$, $1 \leq i \leq n$, $S_i$ is a non-trivial inseparable group and for $2 \leq i \leq n$, $S_1 \ldots S_i = [S_1 \ldots S_{i-1}]S_i$. Each element $g \in G$ can be uniquely written as $g = s_1 s_2 \ldots s_n$ where $s_i \in S_i$ for $1 \leq i \leq n$. Suppose there is another representation $g = t_1 t_2 \ldots t_n$ where $t_i \in S_i$ for $1 \leq i \leq n$. If $n = 1$, then $s_1 = g = t_1$. Proceed by induction on $n$. Since $s_1 s_2 \ldots s_n = t_1 t_2 \ldots t_n$, $(t_1 t_2 \ldots t_{n-1})^{-1} s_1 s_2 \ldots s_{n-1} = t_n s_n^{-1}$. Given that $S_1 S_2 \ldots S_{n-1} \cap S_n = \{1\}$, $s_n = t_n$ and $s_1 s_2 \ldots s_{n-1} = t_1 t_2 \ldots t_{n-1}$. By the induction hypothesis, $s_1 = t_1$, $s_2 = t_2$, ...

. $s_{n-1} = t_{n-1}$. Therefore, the representation $g = s_1 \dots s_n$ is unique.

**Definition 1.3** *An ordered set* $\{S_1, \dots, S_n\}$ *of non-trivial inseparable subgroups of* $G$ *is called a* **splitting system** *for* $G$ *of* **length n** *if* $G = S_1 \dots S_n$ *and* $S_1 \dots S_i = [S_1 \dots S_{i-1}]S_i$ *for each* $i$, $2 \le i \le n$.

If G is inseparable, then it has only one splitting system { G }. Splitting systems were first introduced by H. Bechtell in [6].

**Lemma 1.4** *Let* $\Sigma = \{ S_1, S_2, \dots, S_n \}$ *be a splitting system for a group* $G$ *and* $\sigma$ *be an automorphism of* $G$. *Then* $\Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \dots, S_n^\sigma \}$ *is also a splitting system for* $G$.

Proof. Since G = $S_1 \dots S_n$, G = $G^\sigma = (S_1 \dots S_n)^\sigma = S_1^\sigma \dots S_n^\sigma$. For each $i$, $1 \le i \le n-1$, $S_1^\sigma \dots S_i^\sigma$ is normal in $S_1^\sigma \dots S_{i+1}^\sigma$ since $S_1 \dots S_i \lhd S_1 \dots S_{i+1}$. For $\Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \dots S_n^\sigma \}$ to be a splitting system for G, it must shown that $S_1^\sigma \dots S_i^\sigma \cap S_{i+1}^\sigma = 1$. Suppose for some $i$, $1 \le i \le n-1$, there is a x $\neq$ 1, such that $x \in S_1^\sigma \dots S_i^\sigma \cap S_{i+1}^\sigma$. Then x = $\sigma(s_1 \dots s_i) = \sigma(s_{i+1})$ for $s_j \in S_j$, $1 \le j \le i+1$, and $\sigma^{-1}$(x) $\in S_1 \dots S_i \cap S_{i+1}$. This is a contradiction and $\{ S_1^\sigma, \dots, S_n^\sigma \}$ is a splitting system for G □

**Corollary 1.5** *If* $\Sigma = \{S_1, \dots, S_n\}$ *is a splitting system for a group* $G$, *then* $\Sigma^g = \{S_1^g, \dots, S_n^g\}$ *is also a splitting system for* $G$ *for any* $g \in G$.

Proof. Since conjugation by any element of G induces an automorphism on G, Corollary 1.5 follows directly from Lemma 1.4 □

**Theorem 1.6** *A group with precisely one splitting system is inseparable.*

Proof. Suppose a group G has only one splitting system $\Sigma = \{ S_1, S_2, \dots, S_n \}$ and that $n \ge 2$. By Corollary 1.5, every conjugate of $\Sigma$ is also a splitting system for G. So for all $g \in G$, $\Sigma^g = \Sigma$. Therefore $S_i^g = S_i$ and $S_i$ is normal in G for each $i$, $1 \leq i \leq n$. More specifically, $S_1S_2 = S_1 \times S_2$ and $S_1S_2 = [S_1]S_2 = [S_2]S_1$ implying that both $\{S_1, S_2, \ldots, S_n\}$ and $\{S_2, S_1, \ldots, S_n\}$ are splitting systems for G. This is a contradiction. Thus $n = 1$ and G is inseparable □

**Lemma 1.7** *Let* $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ *be splitting system for a group G. If for some i,* $2 \leq i \leq n-1$, $S_1 \ldots S_{i-1} \lhd S_1 \ldots S_{i+1}$ *and* $S_i$ *and* $S_{i+1}$ *commute elementwise, then* $\Sigma' = \{S_1, \ldots, S_{i-1}, S_{i+1}, S_i, \ldots, S_n\}$ *is also a splitting system for G.*

Proof. Let $N = S_1 \ldots S_{i-1}$. Since $N \lhd S_1 \ldots S_{i+1}$, $NS_{i+1}$ is a subgroup of G. For $\Sigma'$ to be a splitting system for G, $NS_{i+1}$ must be normal in $NS_iS_{i+1}$ and $NS_iS_{i+1}$ must split over it. Let $s_i \in S_i$. Since $N \lhd S_1 \ldots S_{i+1}$, $N^{s_i} = N$ and since $[S_i, S_{i+1}] = 1$, $S_{i+1}^{s_i} = S_{i+1}$. Therefore, $(NS_{i+1})^{s_i} = N^{s_i}S_{i+1}^{s_i} = NS_{i+1}$. Suppose $NS_{i+1} \cap S_i \neq 1$. Then there exist non-trivial elements $ns_{i+1} \in NS_{i+1}$ and $s_i \in S_i$ such that $ns_{i+1} = s_i$. Hence $n = s_is_{i+1}^{-1}$, which implies $N \cap S_iS_{i+1} \neq \{ 1 \}$. This is a contradiction. Thus $NS_iS_{i+1}$ splits over $NS_{i+1}$.

Since $N \lhd NS_{i+1}$ and $N \cap S_{i+1} = \{ 1 \}$, it follows that $NS_{i+1}$ splits over N. Thus $\Sigma' = \{S_1, \ldots, S_{i-1}, S_{i+1}, S_i, \ldots, S_n\}$ is also a splitting system for G □

**Lemma 1.8** *Let G be a group and N a proper normal subgroup in G such that G = [N]B. Then there is a splitting system* $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ *for G such that for some i, 1* $\leq i \leq n-1$, $N = S_1 \ldots S_i$ *and* $B = S_{i+1} \ldots S_n$.

Proof. Let $\Sigma_1 = \{T_1, \ldots, T_m \}$ and $\Sigma_2 = \{R_1, \ldots, R_t \}$ be splitting systems for N and B respectively. The theorem is proven by establishing that $\Sigma_3 = \{ T_1, \ldots, T_m, R_1, \ldots, R_t \}$ is a splitting system for G.

Since $N = T_1 \ldots T_m \lhd G$, $T_1 \ldots T_m R_1 \ldots R_i$ is a subgroup of G for $1 \leq i \leq t$. Furthermore, $R_1 \ldots R_i \lhd R_1 \ldots R_{i+1}$ and $T_1 \ldots T_m R_1 \ldots R_i$ is normal in $T_1 \ldots T_m R_1 \ldots R_{i+1}$.

Suppose that $T_1 \dots T_m R_1 \dots R_i \cap R_{i+1} \neq \{ 1 \}$. Then there is a non-trivial element $nr_1 \dots r_i \in T_1 \dots T_m R_1 \dots R_i$, where $n \in N$ and $r_j \in R_j$ for $1 \leq j \leq i$, and a non-trivial element $r_{i+1} \in R_{i+1}$ such that $nr_1 \dots r_i = r_{i+1}$. This implies that $n = r_{i+1} r_i^{-1} \dots r_1^{-1}$ and that $N \cap B \neq \{ 1 \}$. This is a contradiction. Therefore $T_1 \dots T_m R_1 \dots R_{i+1}$ splits over $T_1 \dots T_m R_1 \dots R_i$ and $\Sigma_3$ is a splitting system for G □

**Lemma 1.9** *Let $G$ be a group and $N$ be a proper normal subgroup of $G$, such that $G$ splits over $N$. If $\Sigma = \{ T_1, \dots, T_n \}$ is a splitting system for $N$, then $\Sigma$ can be extended to a splitting system $\Sigma' = \{ T_1, \dots, T_n, T_{n+1}, \dots, T_m \}$ for $G$.*

Proof. By Lemma 1.8, there is a splitting system $\Sigma_G = \{ S_1, \dots, S_r \}$ for G, such that for some $i$, $1 \leq i \leq r-1$, N $= S_1 \dots S_i$. Since N $= T_1 \dots T_n$, $\Sigma' = \{T_1, \dots, T_n, S_{i+1}, \dots S_r\}$ is a splitting system for G □

**Lemma 1.10** *If $\Sigma = \{ S_1, S_2, \dots, S_n \}$ is a splitting system for $G$ and for some $i$, 1 $\leq i \leq n-1$, $N = S_1 \dots S_i \lhd G$, then $\Sigma' = \{S_{i+1}N/N, \dots, S_nN/N \}$ is a splitting system for $G/N$.*

Proof. For each $j$, $i+1 \leq j \leq n$, $S_jN/N \cong S_j/N \cap S_j \cong S_j$ and each element in $\Sigma'$ is inseparable. Since $S_{i+1}N/N \dots S_jN/N = S_{i+1} \dots S_jN/N$ and $NS_{i+1} \dots S_j = S_1 \dots S_j \lhd S_1 \dots S_{j+1}$, $S_{i+1}N/N \dots S_jN/N \lhd S_{i+1}N/N \dots S_{j+1}N/N$. Suppose $S_{i+1}N/N \dots S_jN/N \cap S_{j+1}N/N \neq \{ 1_{G/N} \}$. Then there is a non-trivial element $s_{i+1} \dots s_j$N $\in S_{i+1}N/N \dots S_jN/N$ and a non-trivial element $s_{j+1}$N $\in S_{j+1}N/N$ such that $s_{i+1} \dots s_j$N $=$ $s_{j+1}$N. Therefore, $s_{j+1}^{-1} s_{i+1} \dots s_j$N $=$ N and there are elements $n_1$ and $n_2$ in N such that $s_{j+1}^{-1} s_{i+1} \dots s_j n_1 = n_2$. This implies that $s_{i+1} \dots s_j n_1 n_2^{-1} = s_{j+1}$ and that $S_1 \dots S_j \cap S_{j+1} \neq \{ 1 \}$. This is a contradiction. Hence $\Sigma'$ is a splitting system for G/N □

**Lemma 1.11** *Let* $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ *be a splitting system for* $G$ *and* $N$ *a proper normal subgroup in* $G$, *such that* $G = [N]B$. *Suppose for some* $i$, $1 \leq i \leq n-1$, *that* $N = S_1 \ldots S_i$. *If* $\Sigma' = \{ N_1/N, \ldots, N_r/N \}$ *is a splitting system for* $G/N$, *then* $\{S_1, \ldots, S_i, N_1 \cap B, \ldots, N_r \cap B\}$ *is a splitting system for* $G$.

Proof. Let $1 \leq j \leq r$. Since $N_j = N_j \cap NB = N(N_j \cap B)$ and $N \neq N_j$, $N_j \cap B \neq 1$. Furthermore, $N_j/N = N_j \cap NB/N = N(N_j \cap B)/N \cong N_j \cap B/N \cap N_j \cap B \cong N_j \cap B$. Thus $N_j \cap B$ is inseparable.

For $j = 1$, $S_1 \ldots S_i(N_1 \cap B) =$ N$(N_1 \cap B) =$ NB $\cap N_1 = N_1$. Proceed by induction on $j$. Hence, $S_1 \ldots S_i(N_1 \cap B) \ldots (N_j \cap B) = N_1 \ldots N_{j-1}(N_j \cap B)$. Since $N \subset N_i$ for $1 \leq i \leq j-1$, and $N \triangleleft$ G, $N_1 \ldots N_{j-1} = N_1 \ldots N_{j-1}N$. Therefore, $S_1 \ldots S_i(N_1 \cap B) \ldots (N_j \cap B) = N_1 \ldots N_{j-1}(N_j \cap B) = N_1 \ldots N_{j-1}N(N_j \cap B) = N_1 \ldots N_{j-1}(N_j \cap NB) = N_1 \ldots N_{j-1}N_j$.

Since $N_1 \ldots N_j \triangleleft N_1 \ldots N_{j+1}$, $S_1 \ldots S_i(N_1 \cap B) \ldots (N_j \cap B)$ is normal in $S_1 \ldots S_i(N_1 \cap B) \ldots (N_{j+1} \cap B)$. Furthermore, $S_1 \ldots S_i(N_1 \cap B) \ldots (N_j \cap B) \bigcap (N_{j+1} \cap B) = N_1 \ldots N_j \bigcap (N_{j+1} \cap B) = (N_1 \ldots N_j \cap N_{j+1}) \cap B = N \cap B = \{ 1 \}$. Therefore, for each $j$, $1 \leq j \leq r-1$, $S_1 \ldots S_i(N_1 \cap B) \ldots (N_{j+1} \cap B)$ splits over $S_1 \ldots S_i(N_1 \cap B) \ldots (N_j \cap B)$ and $\{S_1, \ldots, S_i, N_1 \cap B, \ldots, N_r \cap B \}$ is a splitting system for G □

**Lemma 1.12** *Let* $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ *be a splitting system for* $G$ *and* $N$ *a normal subgroup in* $G$ *such that for some* $l$, $1 \leq l \leq n-1$, $N = S_1 \ldots S_l$. *If* $\{A_1N/N, \ldots, A_mN/N\}$ *is a splitting system for* $G/N$ *such that* $A_i \cap N = 1$ *for* $1 \leq i \leq m$, *then* $\{S_1, \ldots, S_l, A_1, \ldots, A_m\}$ *is a splitting system for* $G$.

Proof. For $1 \leq i \leq m$, $A_iN/N \cong A_i/A_i \cap N \cong A_i$. Thus $A_i$ is inseparable. Since $A_1N/N \ldots A_iN/N = A_1 \ldots A_iN/N$ and $A_1N/N \ldots A_iN/N \triangleleft A_1N/N \ldots A_{i+1}N/N$, $NA_1 \ldots A_i$ is normal in $NA_1 \ldots A_{i+1}$. Suppose $NA_1 \ldots A_i \cap A_{i+1} \neq \{ 1 \}$. Then there are elements $n \in N$ and $a_j \in A_j$ for $1 \leq j \leq i+1$, such that at least one is non-trivial and $na_1 \ldots a_i = a_{i+1}$. Thus $a_1 \ldots a_i N = a_{i+1} N$ and $A_1 \ldots A_i N/N \cap A_{i+1} N/N \neq \{ 1_{G/N} \}$. This is a contradiction and $\{S_1, \ldots S_l, A_1, \ldots, A_m\}$ is a splitting system for G □

## 1.3 SE-Groups

**Definition 1.13** *Two splitting systems* $\Sigma = \{S_1, S_2, \ldots, S_n\}$ *and* $\Sigma' = \{T_1, T_2, \ldots, T_m\}$ *are* **equivalent** *if* $n = m$ *and if there exists an n-tuple* $\hat{g} = (g_1, g_2, \ldots, g_n) \in G \times \ldots \times G$ *such that* $S_i^{g_i} = T_i$ *for* $i = 1, \ldots, n$. *This is denoted by* $\Sigma^{\hat{g}} = \Sigma'$. *If* $n = m$ *and there is an element* $g \in G$ *such that* $S_i^g = T_i$ *for* $i = 1, \ldots, n$, *then the two splitting systems are* **conjugate** *and it is denoted by* $\Sigma^g = \Sigma'$.

For any group G, the concept of "equivalence" defines an equivalence relation on the collection of splitting systems for G. Clearly any splitting system is equivalent to itself. Suppose that $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ are splitting systems for G which are equivalent. Then there is a $\hat{g} = (g_1, g_2, \ldots, g_n)$ such that $\Sigma^{\hat{g}} = \Sigma'$. Therefore, for each $i$, $1 \leq i \leq n$, $S_i^{g_i} = T_i$ and $S_i = T_i^{g_i^{-1}}$. Thus $\Sigma'^{\hat{g}^{-1}} = \Sigma$, where $\hat{g}^{-1} = (g_1^{-1}, \ldots, g_n^{-1})$, and $\Sigma'$ is equivalent to $\Sigma$. If $\Sigma$ is equivalent to $\Sigma'$ and $\Sigma'$ is equivalent to $\Sigma''$, then there are $\hat{g}$ and $\hat{h}$ in $G \times \ldots \times G$ such that $\Sigma^{\hat{g}} = \Sigma'$ and $\Sigma'^{\hat{h}} = \Sigma''$. Hence, $\Sigma^{\hat{g}\hat{h}} = \Sigma''$ and $\Sigma$ is equivalent to $\Sigma''$.

Similarly, the relation defined on two splitting systems by conjugation is also an equivalence relation.

H. Bechtell, in [6], studied groups which have all their splitting systems conjugate. He classified those solvable nC-groups which have all their splitting systems conjugate. The next step is to study those groups which have all their splitting systems equivalent and to classify solvable nC-groups which have all their splitting systems equivalent.

**Definition 1.14** *A group $G$ is called an* **SE-group** *if it is inseparable or if the splitting systems of $G$ are pairwise equivalent. A group is called an* **SC-group** *if the splitting systems are pairwise conjugate.*

The class of SC-groups is contained in the class of SE-groups. A subgroup of an SE-group need not be an SE-group. $A_4$ is an SE-group, but $K_4 < A_4$ is not. Epimorphic images of SE-groups also need not be SE-groups. Consider the quaternion group $Q_8$ and $\Phi(Q_8)$. $Q_8$ is inseparable and the quotient group $Q_8/\Phi(Q_8) \cong K_4$, which is not an SE-group.

**Theorem 1.15** *Let $G$ be an SC-group, $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ a splitting system for $G$ and $A_o$ the collection of automorphisms on $G$ which leave $\Sigma$ invariant. Then $A_o$ is a subgroup of $Aut(G)$ such that $Aut(G) = A_o Inn(G)$.*

Proof. Let $\sigma \in$ Aut(G). By Lemma 1.4, $\Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \ldots, S_n^\sigma \}$ is also a splitting system for G. Since G is an SC-group, there is an element $g \in$ G such that $\Sigma^g = \{ S_1^g, S_2^g, \ldots, S_n^g \}$ $= \Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \ldots, S_n^\sigma \}$. Then $\Sigma^{\sigma g^{-1}} = \Sigma$ and $\sigma i_{g^{-1}} = \alpha \in A_o$ where $i_{g^{-1}}$ is conjugation by $g^{-1}$. Then $\sigma = \alpha i_g$ and Aut(G) $= A_o$Inn(G) □

**Examples.**

1. $S_3 = \langle a, b \mid a^3 = b^2 = 1, bab = a^2 \rangle$ has three splitting systems: $\Sigma_1 = \{\langle a \rangle, \langle b \rangle\}$, $\Sigma_2 = \{\langle a \rangle, \langle ab \rangle\}$, and $\Sigma_3 = \{\langle a \rangle, \langle a^2 b \rangle\}$. Since $b^a = ab$ and $(ab)^a = a^2 b$, $S_3$ is an SC-group.

2. SE-groups of length 2 are always SC-groups since the first term in each splitting system is normal in G.

3. Consider $A_4 = \langle a, b, c \mid a^2 = b^2 = c^3 = 1, ab = ba, c^{-1}ac = ab, c^{-1}bc = a \rangle = [\langle a \rangle \times \langle b \rangle]\, \langle c \rangle$. M $= \langle a, b \rangle$ has six splitting systems $\{ \langle a \rangle, \langle b \rangle \}$, $\{ \langle b \rangle, \langle a \rangle \}$, $\{\langle a \rangle, \langle ab \rangle\}$, $\{\langle ab \rangle, \langle a \rangle\}$, $\{\langle ab \rangle, \langle b \rangle\}$, $\{ \langle b \rangle, \langle ab \rangle \}$, where $a^c = ab$, $(ab)^c = b$ and $b^c = a$. The

third element of each splitting systems is one of the following four Sylow 3-subgroups of G: $\langle c \rangle$ , $\langle abc \rangle$ , $\langle bc \rangle$ and $\langle ac \rangle$ where $c^a = abc$ , $(abc)^b = bc$ and $(bc)^a = ac$. G is an SE-group and there are 24 splitting systems for G. Let $\mathcal{Z}$ denote the collection of all the splitting systems for $A_4$. $\mathcal{Z}$ can be partitioned, via conjugation, into two equivalence classes whose representatives are $\Sigma_1 = \{ \langle a \rangle , \langle b \rangle , \langle c \rangle \}$ and $\Sigma_2 = \{\langle ab \rangle, \langle b \rangle, \langle abc \rangle\}$.

4. Another example of an SE-group is $S_4$, which has 144 splitting systems. $S_4 = \langle x, y, a, b \mid x^2 = y^2 = a^3 = b^2 = 1$ and $x^y = x, x^a = y, x^b = x, y^a = xy, y^b = xy, a^b = xya^2 \rangle$ and $S_4 = [[ \langle x \rangle \times \langle y \rangle ] \langle a \rangle ] \langle b \rangle = [[K_4] \langle a \rangle ] \langle b \rangle = [A_4] \langle b \rangle$ . $\Sigma = \{\langle x \rangle, \langle y \rangle , \langle a \rangle , \langle b \rangle \}$ is a splitting system for $S_4$ and the other splitting systems for $S_4$ can be produced via the following generating relations. The generators of $K_4$ are $x^a = y, y^a = xy, (xy)^a = x$. The generators of $A_4$ are those of $K_4$ along with $a^x = ya, (ya)^{xy} = xya, (xya)^x = xa, (xa)^{xy} = a$. For $S_4$, there are those of $A_4$, along with $b^y = xb$, $(xb)^{ya} = xab$, $(xab)^a = xa^2b, (xa^2b)^{xa^2} = xyab, (xyab)^a = ya^2b$, and $(ya^2b)^{xa} = b$.

5. Let $G = [Q_8]S_3 = \langle a, b, x, y \mid a^4 = b^4 = x^3 = y^2 = 1, a^2 = b^2, x^y = x^2, b^a = b^3, a^y = b, b^y = a, a^x = b, b^x = ab \rangle$. Suppose G = [N]H. Since $\langle x, y \rangle$ acts faithfully on $\langle a, b \rangle$, N $\neq \langle x \rangle$. Therefore 2 divides the order of N. If $y \in N$, then $\mathrm{Cl}(y) = \langle y^k \mid k \in G \rangle \leq N$. As a result, $\langle a, b \rangle < N$. Suppose $y \notin N$. Since 2 divides the order of N, there is a $q \in \langle a, b \rangle$ such that $q \in N$. Therefore, $\langle a, b \rangle \leq N$ unless $q = a^2$ and $\langle a^2 \rangle$ is the Sylow 2-subgroup of N. If N = $\langle a^2 \rangle$, this is a contradiction since $\Phi(\langle a, b \rangle) = \langle a^2 \rangle$. Thus the order of N is 6 and $x \in$ N. Thus $x^a = bx$ and $x^b = a^3bx$ are in N. Therefore $b$ and $a^3b$ are in N and $\langle a, b \rangle \leq$ N. Hence if G splits over a normal subgroup N, $\langle a, b \rangle \leq$ N. Given that $\langle a, b \rangle \lhd$ G and the only normal subgroup of $G/\langle a, b \rangle$ is $\langle a, b, x \rangle / \langle a, b \rangle$, $\langle a, b \rangle$ and $\langle a, b, x \rangle$ are the only normal subgroups in G over which G splits. Since $\langle a, b \rangle$ is

inseparable and normal in G, it will always be the first element in any splitting system for G. $G/\langle a, b\rangle \cong S_3$, which is an SE-group, and G is an SE-group.

Some fundamental results concerning SE-groups are now presented.

**Lemma 1.16** *The direct product of a finite number of groups is never an SE-group.*

Proof. Let $G = H_1 \times \ldots \times H_t$ where $t$ is finite. Suppose $t \geq 2$. Therefore G splits over $H_1$. By Lemma 1.8, there is a splitting system $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G such that for some $i$, $1 \leq i \leq n$, $H_1 = S_1 \ldots S_i$. G also splits over $H_2$. By Lemma 1.8, there is a splitting system $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ for G, such that for some $j$, $1 \leq j \leq m$, $H_2 = T_1 \ldots T_j$. Since $H_1$ and $H_2$ are both normal in G and $H_1 \cap H_2 = \{ 1 \}$, there is no element $g \in G$, such that $S_1^g = T_1$. Thus G is not an SE-group □

**Theorem 1.17** *An SE-group possesses a normal subgroup M which is unique with respect to G=[M]B where B is inseparable. Futhermore, all complements of M are conjugate and M is characteristic in G.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a splitting system for G and consider $M = S_1 \ldots S_{n-1}$. By definition, $M \triangleleft G$. Let $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ be another splitting system for G. Since G is an SE-group, $n = m$ and there is an element $\hat{g} = (g_1, \ldots, g_n) \in G \times \ldots \times G$ where $\Sigma^{\hat{g}} = \Sigma'$. Given that $M \triangleleft G$, $S_i^{g_i} = T_i \leq M$ for each $i$, $1 \leq i \leq n-1$, and $T_1 \ldots T_{n-1} \leq M$. Since $| S_i | = | T_i |$ for $1 \leq i \leq n-1$, $| M | = | S_1 \ldots S_{n-1} | = | T_1 \ldots T_{n-1} |$ and $T_1 \ldots T_{n-1} = M$. Therefore, M is unique with respect to G = [M]B, where B is inseparable. Furthermore, $S_n^{g_n} = T_n$, so all the complements of M are conjugate. Finally, since M is unique with respect to this property, it is characteristic in G □

**Corollary 1.18** *A nilpotent SE-group G is inseparable.*

Proof. If two or more primes divide the order of G, then G is the direct product of its Sylow-p subgroups. By Lemma 1.16, G is not an SE-group.

Suppose that G is a separable p-group. Let M be a normal subgroup of G such that G = [M]B and B is inseparable. There exists a subgroup N normal in G such that N is contained in M and $| M/N |$ = p. Then G/N = [M/N]BN/N and since $| M/N |$ = p, G/N = M/N $\otimes$ BN/N. Let M/N = $\langle mN \mid m^p N \subseteq N \rangle$ and $bN \in$ BN/N, such that $| bN | = p$ and $\langle bN \rangle \lhd$ BN/N. Form the subgroup generated by the element $mb$N whose order is p and is complemented by BN/N. Let K/N = $\langle mbN \rangle$ which is normal in G/N. Since G/N = [K/N]BN/N, G = KB. Furthermore, N = K $\cap$ BN = N(K $\cap$ B) and K $\cap$ B $\subseteq$ N $\cap$ B = {1}. Thus G=[K]B where B is inseparable and K $\neq M$, which contradicts Theorem 1.16 □

**Theorem 1.19** *Let G be an SE-group and $\mathcal{C}$ be the collection of normal subgroups over which G splits. Then there exists an ordering of this collection such that $G = N_0 \supset N_1 \supset \ldots \supset N_m = 1$, where $\mathcal{C} = \{ N_0, N_1, \ldots, N_m \}$.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a splitting system for G and suppose that G=[N]B. The theorem is proven by showing that N = $S_1 \ldots S_i$ for some $i$, $1 \leq i \leq n-1$.

If B is inseparable, N = $S_1 \ldots S_{n-1}$ by Theorem 1.17. Suppose B is separable and let $\Sigma''$ = $\{ R_1, \ldots, R_t \}$ be a splitting system for B. Let $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ be a splitting system for N. By Lemma 1.8, $\Sigma''' = \{ T_1, \ldots, T_m, R_1, \ldots, R_t \}$ is a splitting system for G. G is an SE-group and there is an n-tuple $\hat{g} = (g_1, \ldots, g_n) \in$ G × ... × G such that for $1 \leq j \leq m$, $T_j^{g_j} = S_j$ and for $1 \leq l \leq t$, $R_l^{g_{m+l}} = S_{m+l}$. Since N is normal in G, $T_j^{g_j} \leq N$. Therefore, $T_1^{g_1} \ldots T_m^{g_m} = S_1 \ldots S_m \leq N$. Since $| T_1 \ldots T_m | = | S_1 \ldots S_m | = | N |$, $S_1 \ldots S_m = N$ □

**Theorem 1.20** *If G is an SE-group and G splits over a normal subgroup N, then G/N is an SE-group.*

Proof. Let { $N_1/N, \ldots, N_r/N$ } and { $H_1/N, \ldots, H_s/N$ } be two splitting systems for G/N. In addition, let B be a complement to N in G and let $\Sigma$ = { $S_1, S_2, \ldots, S_n$ } be a splitting system for G. By Theorem 1.19, for some $i$, $1 \leq i \leq n$, N = $S_1 \ldots S_i$. By Lemma 1.11, { $S_1, \ldots, S_i, N_1 \cap B, \ldots, N_r \cap B$ } and { $S_1, \ldots, S_i, H_1 \cap B, \ldots, H_s \cap B$ } are both splitting systems for G. Therefore r = s and there is a $\hat{g} = (g_1, \ldots, g_n) \in G \times \ldots \times G$ such that $(N_j \cap B)^{g_{i+j}} = H_j \cap B$ where $1 \leq j \leq r$. Thus $N(N_j \cap B)^{g_{i+j}} = N(H_j \cap B)$ which implies $(N_j/N)^{g_{i+j}N} = (N(N_j \cap B)/N)^{g_{i+j}N} = (N(N_j \cap B))^{g_{i+j}}/N = N(H_j \cap B)/N = H_j/N$ □

The connection between SE-groups and multiprimitive groups is now established.

**Theorem 1.21** *Let G be a solvable nC-group. If G is an SE-group, then G is multiprimitive.*

Proof. By Theorem 1.19, the normal subgroups $N_i$ over which G split can be ordered G = $N_0 \supset N_1 \supset \ldots \supset N_t \supset N_{t+1} = 1$ via containment. Since G is a solvable nC-group, for each normal subgroup N of G, N = $N_i$ for some $i$, $1 \leq i \leq t$. Thus $\{ 1 \} \lhd N_t \lhd N_{t-1} \lhd \ldots \lhd N_1 \lhd$ G is the unique chief series for G with all of the chief factors complemented. Thus by Theorem 1.1, G is multiprimitive □

**Theorem 1.22** *Suppose G is a solvable SE-group such that each subgroup in each splitting system is of prime order. If the Sylow p-subgroups of G are abelian, then G is multiprimitive.*

Proof. Let G = $G^0 \rhd G^1 \rhd \ldots \rhd G^{m-1} \rhd G^m$ = { 1 } be the derived series for G and let $\Sigma$ = { $S_1, S_2, \ldots, S_n$ } be a splitting system for $G^{m-1}$. If $m = 1$, G is nilpotent. By Corollary 1.18, G is inseparable. Therefore $n = 1$ and G = $S_1$. Hence | G | = $p$, a prime, and G is multiprimitive. Suppose $m \geq 2$. Proceed by induction on $m$.

Consider $G/G^{m-1}$. By 2.4 of [33], $G^{m-1}$ is complemented in G. Therefore, by Theorem 1.20, $G/G^{m-1}$ is an SE-group. Let $\Sigma_0$ = { $T_1/G^{m-1}, \ldots, T_r/G^{m-1}$ } be a splitting system for G/$G^{m-1}$ and B be a complement to $G^{m-1}$ in G. By Lemma 1.11, $\Sigma_0' = \{S_1, \ldots, S_n, T_1 \cap B, \ldots, T_r \cap B\}$ is a splitting system for G. For each $i$, $1 \leq i \leq r$, $T_i \cap B$ is of prime order and $T_i/G^{m-1} = (T_i \cap G^{m-1}B)/G^{m-1} = G^{m-1}(T_i \cap B)/G^{m-1} \cong T_i \cap B$. Therefore, each element of $\Sigma_0$ is of prime order. Since G has abelian Sylow p-subgroups, G/$G^{m-1}$ has abelian Sylow p-subgroups. Furthermore, G/$G^{m-1}$ is of derived length $m-1$ and G/$G^{m-1}$ is multiprimitive.

Now consider $G^{m-1}$. Since $G^{m-1}$ is abelian, by Lemma 1.7, for each $i$, $1 \leq i \leq n$, $\Sigma_i$ = $\{ S_i, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n \}$ is a splitting system for $G^{m-1}$. G splits over $G^{m-1}$. By Lemma 1.9, $\Sigma$ and $\Sigma_i$ can be extended to splitting systems $\Sigma' = \{S_1, \ldots, S_n, R_1, \ldots, R_t\}$ and $\Sigma_i' = \{S_i, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n, R_1, \ldots, R_t\}$ for G respectively. G is an SE-group, so there is an element $g_i \in$ G, such that $S_1^{g_i} = S_i$. Therefore, since $S_1$ is of prime order $p$, $G^{m-1}$ is an elementary abelian $p$-group.

Suppose there is a non-trivial subgroup H $\lhd$ G such that 1 $\lhd$ H $\lhd$ $G^{m-1}$. Since $G^{m-1}$ is an elementary abelian p-group, $G^{m-1}$ = H $\times$ B for some non-trivial subgroup B of $G^{m-1}$. Thus there is a splitting system $\Sigma_H = \{ R_1, \ldots, R_s \}$ for $G^{m-1}$ such that for some $l$, $1 \leq l \leq s-1, R_1 \ldots R_l = H$. $G^{m-1}$ is complemented in G and by Lemma 1.9, $\Sigma_H$ can be extended to a splitting system $\Sigma_H' = \{ R_1, \ldots, R_s, R_{s+1}, \ldots, R_u \}$ for G. By Lemma 1.7, $\Sigma_H'' = \{ R_s, R_1, \ldots, R_{s-1}, R_{r+1}, \ldots, R_u \}$ is also a splitting system for G. G is an SE-group and there is an element $g \in$ G such that $T_1^g = T_s$. This is a contradiction since H is normal in G. Thus H = $G^{m-1}$.

G/$G^{m-1}$ is multiprimitive and G/$G^{m-1} \rhd G^1/G^{m-1} \rhd \ldots \rhd G^{m-2}/G^{m-1} \rhd \{1_{G/G^{m-1}}\}$ is the unique chief series for G/$G^{m-1}$. Since $G^{m-1}$ is a chief factor in G, the derived series for G is a chief series. Let W be a prefrattini subgroup of G. By 6.1 of [12], W avoids each complemented chief factor of G. By 2.4 of [33], each subgroup in the derived series is complemented in G. Since the derived series for G is a chief series, W = { 1 }. Thus by 6.6 of [12], G is an nC-group. Hence by Theorem 1.20, G is multiprimitive □

Not every multiprimitive group is an SE-group. Consider the following multiprimitive group G = $[Z_5 \times Z_5]Z_3$ = ⟨ a,b,c | $a^5 = b^5 = c^3 = 1$, $a^b = a, a^c = a^3b$ and $b^c = a^2b$ ⟩. G has splitting systems { $\langle a \rangle$ , $\langle b \rangle$ , $\langle c \rangle$ } and { $\langle b \rangle$ , $\langle a \rangle$ , $\langle c \rangle$ }, but there is no element $g \in$ G such that $\langle a \rangle^g = \langle b \rangle$. Therefore, G is not an SE-group. The question arises as to what conditions must be placed on a multiprimitive G for it to be an SE-group. G is not an SE-group since the action of $\langle c \rangle$ on $\langle a, b \rangle$ does not act transitively on the collection of 1-dimensional subspaces of $\langle a, b \rangle$.

# Chapter 2

# Group Geometry Through Splitting Systems

This investigation was motivated by the study of multiprimitive groups and their structure. It is based on the induced geometric structure of elementary abelian p-groups.

## 2.1 Introduction

Suzuki gives a concise presentation of the geometry of linear groups in [29]. To begin, a brief overview of his treatise is given.

**Definition 2.1** *(3.3.1 of [29]) Let V be an n-dimensional vector space over the field F. Let $\mathcal{P}$ be the set of all 1-dimensional subspaces of V. A subset $\alpha$ of $\mathcal{P}$ is said to be a* **subspace** *of $\mathcal{P}$ if there is a subspace U of V such that $\alpha$ coincides with the set of 1-dimensional subspaces of U. In this case, we define dim $\alpha$ = d if the subspace U has dimension d + 1. The set $\mathcal{P}$ on which the concept of subspace is defined is called the (n - 1) dimensional* **projective space** *on V or a projective space defined over F.*

The 1-dimensional subspaces of V are called the **points** of $\mathcal{P}$. Let d = dimension of $\mathcal{P}$. A collection of d + 1 points $\{ P_0, P_1, \ldots, P_d \}$ of $\mathcal{P}$ is said to be **independent** if the subset $\alpha$ generated by $P_0, P_1, \ldots P_d$ is V.

**Definition 2.2** *(3.3.4 of [29]) Let $\mathcal{P}$ be a d-dimensional projective space. A set $\Sigma$ of $d$ + 1 points $P_0, P_1, \ldots, P_d$ is said to be a* **frame** *of $\mathcal{P}$ if $\{ P_0, P_1, \ldots, P_d \}$ is independent.*

Once the concept of a subspace of $\mathcal{P}$ is defined, the concept of a flag and flag space can be defined.

**Definition 2.3** *(3.3.7 of [29]) A nested sequence of non-trivial subspaces $\alpha_i$ of $\mathcal{P}$ A: $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_k$ is said to be a* **flag** *of* **rank k**, *if $\alpha_i \neq \alpha_{i+1}$ for every i= 1,2, ..., k - 1. Let B be another flag. If A is a refinement of B(that is, if the subspaces appearing in the flag B are all members of the sequence A), we say that the flag B is* **contained** *in the flag A and write $B \subset A$. The partially ordered set consisting of all the flags in $\mathcal{P}$ is called the* **flag space** *and is denoted by $\Delta(\mathcal{P})$.*

Let G be an elementary abelian p-group. Then G = $\langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle$ where $| a_i | = p$ for each $i$, $1 \leq i \leq n$. Hence each $\langle a_i \rangle$ is inseparable and $\Sigma = \{ \langle a_1 \rangle, \ldots, \langle a_n \rangle \}$ is a splitting system for G. Considering G as an n-dimensional vector space over the field of characteristic p, each $\langle a_i \rangle$ is a point in $\mathcal{P}$. Therefore, $\Sigma$ is also a frame for G.

For a separable group G, let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a splitting system for G. Then the subgroups in $\Sigma$ form a basis for G in the sense that every element $g \in$ G can be uniquely written $g = s_1 s_2 \ldots s_n$ where $s_i \in S_i$ and $S_i \cap S_j = 1$ for $i \neq j$.

Since each finite group has at least one splitting system associated with it, the concept of a frame can be generalized through the use of splitting systems. As a result, splitting systems can be used to induce a geometry on an arbitrary finite group.

## 2.2 Defining $\triangle(\mathcal{P})$ For A Finite Group

**Definition 2.4** *Each splitting system* $\Sigma = \{ S_1, \ldots, S_n \}$ *of a group* $G$ *is called a* **frame** *for* $G$.

Each splitting system for G is now defined to be a frame for G and all the properties proven in Chapter 1 for splitting systems also hold for frames .

**Definition 2.5** *A chain* $A$*:* $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ *of subgroups of a group* $G$ *is called a* **flag** *provided there is a frame* $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ *for* $G$ *such that* $\alpha_1 = S_1 \ldots S_{j_1}, \alpha_2 = S_1 \ldots S_{j_2}, \ldots,$ *and* $\alpha_d = S_1 \ldots S_{j_d}$ *where* $1 \leq j_1 < j_2 < \ldots < j_d < n$. $\Sigma$ *is said to* **support** $A$ *and the number of subgroups in the flag* $A$ *is called the* **rank** *of* $A$. $\{ 1_G \}$ *is the* **trivial flag** *of* $G$ *of rank 0 and is supported by every frame of* $G$.

This definition presents a flag for a group G to be a sequence of subgroups of G. A flag defined for a finite dimensional vector space V over a field of characteristic p is a sequence of subspaces of $\mathcal{P}$, where each subspace $\alpha$ of $\mathcal{P}$ coinicides with the set of 1-dimensional subspaces of a subspace U of V. The definition of a subspace of $\mathcal{P}$ creates a correspondence between the subspaces of $\mathcal{P}$ and the subspaces of V. This correspondence and the fact that elementary abelian p-groups split over each subgroup indicate, that without loss of generality, the definition presented here of a flag reduces to the definition of flag associated with a finite dimensional vector space over the field of characteristic p when G is an elementary abelian p-group.

**Remarks:**

**i)** When Suzuki defines a flag, the subspaces are ordered with respect to containment beginning with the subgroup of largest order. Here, the subgroups are ordered with respect to containment, but beginning with the subgroup of least order. There is no loss of generality and this is done since splitting systems are used to define frames and flags .

**ii)** Given a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, each subgroup A = $S_1 S_2 \ldots S_t$ for $1 \leq t \leq$ n-1 is a flag of rank 1.

**iii)** If G is inseparable, then G only has the trivial flag { 1 } associated with it.

**iv)** Notice that given a non-trivial flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ that $\alpha_1 \neq \{1\}$ and $\alpha_d \neq$ G.

**v)** Each subgroup in a flag is a subnormal subgroup in G.

**vi)** A non-trivial flag may be supported by more than one frame. In the group G = $S_4$ = $\langle x, y, a, b \mid x^2 = y^2 = a^3 = b^2 = 1$ and $x^y = x, x^a = y, x^b = x, y^a = xy, y^b = xy, a^b = xya^2\rangle$ the flag A: $\langle x\rangle \subset \langle x, y\rangle \subset \langle x, y, a\rangle$ is supported by the frames $\Sigma = \{ \langle x\rangle, \langle y\rangle, \langle a\rangle, \langle b\rangle\}$ and $\Sigma' = \{ \langle x\rangle, \langle xy\rangle, \langle xya\rangle, \langle b\rangle\}$.

**vii)** By definition, every flag of G is supported by at least one frame.

**Definition 2.6** *If A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ are non-trivial flags of a group G of ranks d and e respectively, then B is a* **refinement** *of A if $e \geq d$ and for each i, $1 \leq i \leq d$, $\alpha_i = \beta_{j_i}$ for $1 \leq j_i \leq e$. A is said to be* **contained** *in B or $A \subseteq B$. The trivial flag $\{ 1_G \}$ is said to be contained in every flag.*

**Definition 2.7** *Let A and B be flags of a group G.* **A** $\wedge$ **B** *is defined to be the flag C which is maximal with respect to containment in both A and B with the understanding that C might be $\{ 1_G \}$.*

**Theorem 2.8** *Let A and B be flags of a group G.*

*i) A $\wedge$ B is well-defined.*

*ii) If the frames $\Sigma_A$ and $\Sigma_B$ support A and B respectively, then the flag $C = A \wedge B$ is supported by both $\Sigma_A$ and $\Sigma_B$.*

Proof. i) A $\wedge$ B is well-defined, if A $\wedge$ B = C is unique. If there does not exist a non-trivial flag which is contained in both A and B, then C = $\{ 1_G \}$ is unique. Suppose that C is non-trivial and that there is another flag D of G such that D $\neq$ C and D is maximal with respect to containment in A and B. If C $\subset$ D or D $\subset$ C, then the maximality of C or D is contradicted. Thus C $\not\subseteq$ D and D $\not\subseteq$ C.

Since A and B are flags of G, there exist frames $\Sigma_A = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma_B = \{ T_1, T_2, \ldots, T_m \}$ which support them respectively. The flag C: $\gamma_1 \subset \gamma_2 \subset \ldots \subset \gamma_s$ is contained in both A and B, so $\gamma_1 = S_1 \ldots S_{i_1} = T_1 \ldots T_{k_1}$, $\gamma_2 = S_1 \ldots S_{i_2} = T_1 \ldots T_{k_2}, \ldots$, and $\gamma_s = S_1 \ldots S_{i_s} = T_1 \ldots T_{k_s}$ for $1 \leq i_1 < i_2 < \ldots < i_s \leq n-1$ and $1 \leq k_1 < k_2 < \ldots < k_s \leq m-1$. The flag D: $\delta_1 \subset \delta_2 \subset \ldots \subset \delta_t$ is also contained in A and B, so $\delta_1 = S_1 \ldots S_{j_1} = T_1 \ldots T_{l_1}$, $\delta_2 = S_1 \ldots S_{j_2} = T_1 \ldots T_{l_2}, \ldots$, and $\delta_t = S_1 \ldots S_{j_t} = T_1 \ldots T_{l_t}$ for $1 \leq j_1 < j_2 < \ldots < j_t \leq n-1$ and $1 \leq l_1 < l_2 < \ldots < l_t \leq m-1$. Let $\pi' = \{i_1, \ldots, i_s, j_1, \ldots, j_t \}$ and let $\pi = \{ r_1, \ldots, r_v \}$ be the ordered set of distinct whole numbers in $\pi'$. Form the flag E: $\epsilon_1 \subset \epsilon_2 \subset \ldots \subset \epsilon_v$ by letting $\epsilon_1 = S_1 \ldots S_{r_1}$, $\epsilon_2 = S_1 \ldots S_{r_2}, \ldots$, and $\epsilon_v = S_1 \ldots S_{r_v}$. By definition E is contained in A and B and contains C. Since C is maximal with respect to containment in A and B, E = C. By definition of E, D $\subseteq$ E. Thus D $\subseteq$ C, which contradicts our assumption. Hence C = D.

ii) Since C $\subseteq$ A , A is a refinement of C and hence $\Sigma_A$ supports C. Furthermore, C $\subseteq$ B, so $\Sigma_B$ also supports C □

**Examples:**

**2.1)** In the group $S_4$, the flags A: $\langle x \rangle \subset \langle x, y \rangle \subset \langle x, y, a \rangle$ and B: $\langle y \rangle \subset \langle x, y \rangle \subset \langle x, y, a \rangle$ are supported by the frames $\Sigma_A = \{ \langle x \rangle, \langle y \rangle, \langle a \rangle, \langle b \rangle \}$ and $\Sigma_B = \{ \langle y \rangle, \langle xy \rangle, \langle a \rangle, \langle b \rangle \}$

respectively. A $\wedge$ B = C: $\langle x, y\rangle \subset \langle x, y, a\rangle$, which is supported by both $\Sigma_A$ and $\Sigma_B$.

**2.2)** In the group $G = S_3 \times S_3 = \langle a, b, c, d \mid a^3 = b^2 = c^3 = d^2 = 1, a^b = a^2 \text{ and } c^d = c^2\rangle$, there are flags A: $(\langle a\rangle, 1) \subset (\langle a\rangle, \langle a\rangle) \subset (S_3, \langle a\rangle)$ and B: $(1, \langle a\rangle) \subset (1, S_3) \subset (\langle a\rangle, S_3)$. In this example, A $\wedge$ B = $\{1\}$.

**Definition 2.9** *The collection $\{ \Delta(\mathcal{P}), \wedge \}$ consisting of all flags of a group G of rank $\geq$ 0 together with the operation of $\wedge$ is the* **flag space** *of G.*

The definition of containment of a flag is reflexive and transitive. Hence $\Delta(\mathcal{P})$ is a partially ordered set. From here on, $\{ \Delta(\mathcal{P}), \wedge \}$ will simply be denoted by $\Delta(\mathcal{P})$.

**Definition 2.10** *Let G be a group and $\Delta(\mathcal{P})$ its associated flag space. A flag in $\Delta(\mathcal{P})$ is called a* **maximal flag** *if it is not properly contained in any refinement.*

In Examples 2.1) and 2.2), the flags A and B are maximal. If G is a group of order one, then the associated flag space has no maximal flag. If G is an inseparable group of order greater than one, then the trivial flag $\{ 1_G \}$ is the unique maximal flag of G.

**Theorem 2.11** *Let G be a separable group.*

*i) Every frame supports a unique maximal flag.*

*ii) Every flag is contained in a maximal flag.*

*iii) Every frame of G supports every maximal flag of $\Delta(\mathcal{P})$ if and only if there is a unique maximal flag in $\Delta(\mathcal{P})$.*

Proof. i) and ii) follow directly from the definitions.

iii) From i), every frame supports only one maximal flag in $\Delta(\mathcal{P})$. Therefore, if every frame supports every maximal flag, there must be a unique maximal flag in $\Delta(\mathcal{P})$.

Conversely, suppose there is only one maximal flag in $\Delta(\mathcal{P})$. By i), every frame must support it □.

**Remarks:**

**i)** In a flag space $\Delta(\mathcal{P})$ of a group G, not all the maximal flags need to be of the same rank. In the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$, both A: $\langle a \rangle$ and B: $\langle b \rangle \subset \langle b, a^2 \rangle$ are maximal flags.

**ii)** A flag A can be contained in two different maximal flags. Consider $G = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, a^b = a^3, a^c = a^3, b^c = a^2 b \rangle = [D_8]\langle c \rangle$. Let A: $\langle a, b \rangle \cong D_8$ be a flag of $\Delta(\mathcal{P})$. A is contained in the flags C: $\langle a \rangle \subset \langle a, b \rangle$ and D: $\langle a^2 \rangle \subset \langle a^2, b \rangle \subset \langle a, b \rangle$. Both C and D are maximal flags of $\Delta(\mathcal{P})$.

**iii)** Let $\mathcal{S}$ be the collection of all the frames for a group G and $\mathcal{S}_A$ be the collection of all frames which support the maximal flag A. Since every frame supports a maximal flag, the set of $\mathcal{S}_{A_i}$, where the $A_i$'s are the maximal flags of G, partition $\mathcal{S}$. Futhermore, this defines an equivalence relation on $\mathcal{S}$.

Given an elementary abelian p-group G, each 1-dimensional subspace is a point in projective space as defined by Suzuki. Let S be a 1-dimensional subspace of G. Then there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, such that for some $i$, $1 \leq i \leq n$, $S_i$ = S. Since G is abelian, by Lemma 1.7, $\Sigma' = \{ S_i, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n \}$ is also a frame for G. Thus the maximal flag A: $\alpha_1 \subset \ldots \subset \alpha_{n-1}$ in $\Delta(\mathcal{P})$ supported by $\Sigma'$ satisfies $\alpha_1 = S_i$.

For non-abelian groups, this is not always valid. Consider $A_4 = \langle a, b, c \mid a^2 = b^2 = c^3 = 1, ab = ba, a^c = b, b^c = ab \rangle$. $\Sigma = \{ \langle a \rangle, \langle b \rangle, \langle c \rangle \}$ is a frame for G, yet there is no frame $\Sigma' = \{ S_1, S_2, S_3 \}$ for G such that $S_1 = \langle c \rangle$. Hence there is no maximal flag A: $\alpha_1 \subset \alpha_2$ for G such that $\alpha_1 = \langle c \rangle$. This leads to the following definitions.

**Definition 2.12** *A rank 1 flag $\alpha$ in $\Delta(\mathcal{P})$ is of* **type i** *if there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$ such that $\alpha_i = \alpha$.*

**Definition 2.13** *A rank 1 flag $\alpha$ in $\Delta(\mathcal{P})$ is a* **point** *in $\Delta(\mathcal{P})$ if $\alpha$ is a rank 1 flag of type 1. Let $\mathcal{P}$ denote the collection of all points in $\Delta(\mathcal{P})$.*

Let S be in some frame for G. S is a point in $\Delta(\mathcal{P})$ if and only if S is the first element in some frame for G.

**Theorem 2.14** *Let G be an abelian group. Then every subgroup of each frame for G is a point in $\Delta(\mathcal{P})$.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G and $S_i$, $1 \leq i \leq n$, a subgroup in $\Sigma$. Since G is abelian, by Lemma 1.7, $\Sigma' = \{ S_i, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n \}$ is also a frame for G. $\Sigma'$ supports a maximal flag in $\Delta(\mathcal{P})$. Therefore $S_i$ is a rank 1 flag of type 1 in $\Delta(\mathcal{P})$ and hence a point in $\Delta(\mathcal{P})$ □

If G is an elementary abelian p-group, the standard definition and Definition 2.13 are equivalent.

**Corollary 2.15** *If G splits over an abelian subgroup N, then each element in each frame for N is a point in $\Delta(\mathcal{P})$.*

Proof. Let S be an element of a frame for N. By Theorem 2.14, there is a frame $\Sigma = \{S_1, S_2, \ldots, S_n\}$ for N such that $S_1 = $ S. By Lemma 1.9, the frame $\Sigma$ can be extended to a frame $\{ S_1, \ldots, S_n, S_{n+1}, \ldots, S_m \}$ for G. Therefore S is a point in $\Delta(\mathcal{P})$ □

For a non-abelian group G, not every subgroup of each frame for G need be a point in $\Delta(\mathcal{P})$. It might be assumed that if a group had the property that each element in each frame for G was a point in $\Delta(\mathcal{P})$, then G would be abelian. This is not the case. The group $G = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ab = ba, ac = ca, b^c = ab \rangle$ is a counter example.

**Theorem 2.16** *If each subgroup of each frame for G is nilpotent and in $\mathcal{P}$, then G is nilpotent.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. Since each $S_i$, $1 \le i \le n$, is nilpotent, there is an $m_i$ such that $G^{m_i} \le S_1 \ldots S_i$. Therefore G is solvable and F(G) exists non-trivially in G. For each $i$, $1 \le i \le n$, there is a maximal flag $A_i$: $\alpha_{i,1} \subset \ldots \subset \alpha_{i,d_i}$ in $\Delta(\mathcal{P})$, such that $\alpha_{i,1} = S_i$. Thus $S_i$ is subnormal in G and by 5.2.5 of [2], $S_i \le$ F(G). Thus F(G) = G and G is nilpotent □

## 2.3 Basic Results

Let N be a normal subgroup of a group G, such that N is a rank 1 flag in $\Delta(\mathcal{P})$. The relation between the flag space $\Delta(\mathcal{P})$ of G and the flag spaces $\Delta(\mathcal{P}_{G/N})$ of G/N and $\Delta(\mathcal{P}_N)$ of N are investigated.

The relation between $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P}_N)$ is discussed first.

**Lemma 2.17** *If $N \lhd G$ and N is a rank 1 flag in $\Delta(\mathcal{P})$, then each frame of N can be extended to a frame of G and each flag in $\Delta(\mathcal{P}_N)$ can be extended to a flag in $\Delta(\mathcal{P})$.*

Proof. Let $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ be a frame for N. Since N is a rank 1 flag in $\Delta(\mathcal{P})$, there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G such that for some $i$, $1 \le i \le$ n, N = $S_1 \ldots S_i$. Therefore, $\{ T_1, \ldots, T_m, S_{i+1}, \ldots, S_n \}$ is a frame for G.

Let $A'$: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_l$ be a flag in $\Delta(\mathcal{P}_N)$ supported by the frame $\Sigma' = \{T_1, T_2, \ldots, T_m\}$. Since $\Sigma'$ can be extended to a frame $\Sigma = \{T_1, \ldots, T_m, T_{m+1}, \ldots, T_t\}$ for G, A: $\alpha_1 \subset \ldots \subset \alpha_l$ is a flag in $\Delta(\mathcal{P})$ supported by $\Sigma$ □

**Lemma 2.18** *Let $N \lhd G$ and N be a rank 1 flag in $\Delta(\mathcal{P})$. If A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a flag in $\Delta(\mathcal{P})$, such that $N = \alpha_{i+1}$ for some i, $1 \le i \le d-1$, then $A' : \alpha_1 \subset \ldots \subset \alpha_i$ is a* *flag in* $\Delta(\mathcal{P}_N)$*.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G which supports A. Therefore, for some $j, 1 \leq j \leq n$, N = $S_1 \ldots S_j$. Thus $\Sigma' = \{ S_1, \ldots, S_j \}$ is a frame for N and $A' : \alpha_1 \subset \ldots \subset \alpha_i$ is a flag of N supported by $\Sigma'$ □

**Lemma 2.19** *Let* $N \lhd G$ *and* $N$ *be a rank 1 flag in* $\Delta(\mathcal{P})$*.*

*i) Let B:* $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ *be a maximal flag in* $\Delta(\mathcal{P}_N)$*. Then for any maximal flag A:* $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ *in* $\Delta(\mathcal{P})$*, where* $N = \alpha_i$ *for some* $i$*,* $1 \leq i \leq d$*.* $A' : \beta_1 \subset \ldots \subset \beta_e \subset \alpha_i \subset \ldots \subset \alpha_d$ *is a maximal flag in* $\Delta(\mathcal{P})$*.*

*ii) If A:* $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ *is a maximal flag in* $\Delta(\mathcal{P})$*, such that* $N = \alpha_{i+1}$ *for some* $i$ *,* $1 \leq i \leq d-1$*, then* $A' : \alpha_1 \subset \ldots \subset \alpha_i$ *is a maximal flag in* $\Delta(\mathcal{P}_N)$*.*

Proof. i) Let $\Sigma' = \{ T_1, \ldots, T_{e+1} \}$ be a frame for N which supports the flag B and let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ be a frame for G which supports A, such that N = $S_1 \ldots S_i$. Therefore $\Sigma'' = \{ T_1, \ldots, T_{e+1}, S_{i+1}, \ldots, S_{d+1} \}$ is a frame for G. By Theorem 2.11 i), $\Sigma''$ supports the maximal flag $T_1 \subset \ldots \subset T_1 \ldots T_{e+1} \subset T_1 \ldots T_{e+1} S_{i+1} \subset \ldots \subset T_1 \ldots S_d = \beta_1 \subset \ldots \subset \beta_e \subset \alpha_i \subset \ldots \subset \alpha_d$.

ii) Suppose A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a maximal flag of G such that N = $\alpha_{i+1}$. By Lemma 2.18, $A' : \alpha_1 \subset \ldots \subset \alpha_i$ is a flag in $\Delta(\mathcal{P}_N)$. Suppose that it is not maximal. Then there is a maximal flag B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ of N which contains it. By i), $\beta_1 \subset \ldots \subset \beta_e \subset \alpha_{i+1} \subset \ldots \subset \alpha_d$ is a maximal flag of G. But A ⊂ C, which contradicts the maximality of A. Thus $A'$ is a maximal flag in N □

Next the relation between $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P}_{G/N})$ is considered.

**Lemma 2.20** *Let* $N \lhd G$ *and* $N$ *be a rank 1 flag in* $\Delta(\mathcal{P})$*.*

*i) Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$, such that for some i, $1 \leq i \leq d$, $N = S_1 \ldots S_i$. Then $A'$: $\alpha_l/N \subset \ldots \subset \alpha_d/N$ is a flag in $\Delta(\mathcal{P}_{G/N})$ supported by the frame $\Sigma' = \{ S_{i+1}N/N, \ldots, S_nN/N \}$ where l is the smallest number such that $N \subset \alpha_l$.*

*ii) If G = [N]H and $A'$: $\alpha_1/N \subset \ldots \subset \alpha_d/N$ is a flag in $\Delta(\mathcal{P}_{G/N})$, then A: $\alpha_1 \subset \ldots \subset \alpha_d$ is a flag in $\Delta(\mathcal{P})$.*

Proof. i) $\Sigma$ supports A and there integers $1 \leq j_1 < j_2 < \ldots < j_d \leq n-1$, such that $\alpha_l = S_1 \ldots S_{j_l}$ for $1 \leq l \leq d$. By Lemma 1.10, $\Sigma' = \{ S_{i+1}N/N, \ldots, S_nN/N \}$ is a frame for G/N. Furthermore, for $i+1 \leq l \leq d$, $S_{i+1}NS_{i+2}N \ldots S_nN = S_{i+1}S_{i+2} \ldots S_nN$ since N $\lhd$ G. Therefore, for $1 \leq k \leq d$, $\alpha_k/N = S_{i+1} \ldots S_{j_k}N/N = S_{i+1}N/N \ldots S_{j_k}N/N$ and $A'$ is a flag in $\Delta(\mathcal{P}_{G/N})$ supported by $\Sigma'$.

ii) Let $\Sigma' = \{ R_1/N, \ldots, R_m/N \}$ support the flag $A'$: $\alpha_1/N \subset \ldots \subset \alpha_d/N$. Therefore, there are integers $1 \leq j_1 < j_2 < \ldots < j_d \leq m-1$, such that $\alpha_k = R_1N/N \ldots R_{j_k}N/N$ for $1 \leq k \leq d$. By Lemma 1.8, there is a flag $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, such that for some i, N = $S_1 \ldots S_i$. By Lemma 1.11, $\Sigma'' = \{ S_1, \ldots, S_i, R_1 \cap H, \ldots, R_m \cap H \}$ is a frame for G.

Let $1 \leq l \leq m$ and consider $S_1 \ldots S_i(R_1 \cap H) \ldots (R_l \cap H)$. For $l = 1$, $S_1 \ldots S_i(R_1 \cap H) = N(R_1 \cap H) = R_1 \cap NH = R_1$. Proceed by induction on $l$. Thus, $S_1 \ldots S_i(R_1 \cap H) \ldots (R_l \cap H) = R_1 \ldots R_{l-1}(R_l \cap H)$. Since N $\leq R_{l-1}$, $R_1 \ldots R_{l-1}(R_l \cap H) = R_1 \ldots R_{l-1}N(R_l \cap H) = R_1 \ldots R_{l-1}(R_l \cap NH) = R_1 \ldots R_{l-1}R_l$. Therefore, $S_1 \ldots S_i(R_1 \cap H) \ldots (R_l \cap H) = R_1 \ldots R_l$. Consider $\alpha_k$, for $1 \leq k \leq d$. Then $\alpha_k = R_1 \ldots R_k = S_1 \ldots S_i(R_1 \cap H) \ldots (R_k \cap H)$ and A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a flag in $\Delta(\mathcal{P})$ supported by $\Sigma''$ □

**Lemma 2.21** *If G = [N]K, A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a maximal flag in $\Delta(\mathcal{P}_N)$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_r$ is a maximal flag in $\Delta(\mathcal{P}_K)$, then $\alpha_1 \subset \ldots \subset \alpha_d \subset N \subset N\beta_1 \subset \ldots \subset N\beta_r$ is a maximal flag in $\Delta(\mathcal{P})$.*

Proof. There is a frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ for N and $\Sigma' = \{ T_1, T_2, \ldots, T_{r+1} \}$ for K which support A and B respectively. Since G = [N]K, $\Sigma'' = \{ S_1, \ldots, S_{d+1}, T_1, \ldots, T_{r+1} \}$ is a frame for G. Therefore $\Sigma''$ supports the maximal flag $S_1 \subset \ldots \subset S_1 \ldots S_d \subset S_1 \ldots S_d S_{d+1} \subset S_1 \ldots S_{d+1} T_1 \subset \ldots \subset S_1 \ldots S_{d+1} T_1 \ldots T_r = \alpha_1 \subset \ldots \subset \alpha_d \subset N \subset N\beta_1 \subset \ldots \subset N\beta_r$ □

**Lemma 2.22** *Let $N \lhd G$ and N be a rank 1 flag in $\Delta(\mathcal{P})$.*

*i) If A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a maximal flag in $\Delta(\mathcal{P})$ supported by frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ where $N = S_1 \ldots S_i$, $1 \le i \le n-1$, then $A'$: $\alpha_{i+1}/N \subset \ldots \subset \alpha_d/N$ is a maximal flag in $\Delta(\mathcal{P}_{G/N})$ supported by $\Sigma' = \{ S_{i+1}N/N, \ldots, S_n N/N \}$.*

*ii) Let B: $\beta_1/N \subset \ldots \beta_e/N$ be a maximal flag in $\Delta(\mathcal{P}_{G/N})$. If G = [N]H and A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ is a maximal flag in $\Delta(\mathcal{P})$ such that $N = \alpha_i$ for some i, $1 \le i \le d$, then $A'$ : $\alpha_1 \subset \ldots \subset \alpha_i \subset \beta_1 \subset \ldots \subset \beta_e$ is a maximal flag in $\Delta(\mathcal{P})$.*

Proof. By Lemma 2.20 i), $A'$: $\alpha_{i+1}/N \subset \ldots \subset \alpha_d/N$ is a flag in $\Delta(\mathcal{P}_{G/N})$ supported by the frame $\{ S_{i+1}N/N, \ldots, S_{d+1}N/N \}$. For $i+1 \le j \le d$, $\alpha_j = S_{i+1}N \ldots S_j N = NS_{i+1} \ldots S_j$. Therefore, by Theorem 2.11 i), $A'$ is maximal.

ii) There are frames $\Sigma' = \{ T_1/N, \ldots, T_{e+1}/N \}$ for G/N and $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ for G which support B and A respectively. Since A is maximal, $S_1 \ldots S_i = \alpha_i =$ N. By Lemma 1.11, $\Sigma'' = \{ S_1, \ldots, S_i, T_1 \cap H, \ldots, T_{e+1} \cap H \}$ is a frame for G. As shown in the proof of 2.20 ii), for $l$, $1 \le l \le e+1$, $S_1 \ldots S_i (T_1 \cap H) \ldots (T_l \cap H) = T_1 \ldots T_l$. Therefore, $\Sigma''$ supports the maximal flag $S_1 \subset \ldots \subset S_1 \ldots S_i \subset S_1 \ldots S_i (T_1 \cap H) \subset \ldots \subset S_1 \ldots S_i (T_1 \cap H) \ldots (T_e \cap H) = \alpha_1 \subset \ldots \subset \alpha_i \subset \beta_1 \subset \ldots \subset \beta_e$ □

From this point on, unless stated otherwise, when referring to a group G, it will be assumed to be separable. When referring to a flag, it will be assumed to be non-trivial.

**Definition 2.23** *A flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ of rank d is called a* **characteristic(normal) flag** *if for each i, $1 \le i \le d$, $\alpha_i$ is characteristic(normal) in G.*

**Definition 2.24** *The flag $B = \wedge \{ A \mid A \in \Delta(\mathcal{P})$ and A is maximal of rank d $\}$ is called the* **d-primary flag** *of G. If all the maximal flags of G are of the same rank, then B is the* **primary flag** *of G.*

**Theorem 2.25** *Let B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_r$ be the d-primary flag of G of rank r. The following conditions hold:*

*i) Every frame of length d+1 supports B;*

*ii) B is a characteristic flag;*

*iii) If r = d, B is the unique maximal flag of $\Delta(\mathcal{P})$ of rank d;*

*iv) If G is abelian, then B is trivial.*

Proof. i) Since every maximal flag of rank d is supported by a frame of length d+1, i) follows from Theorem 2.11 i).

ii) Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ be a frame of length $d+1$ and $\sigma \in \mathrm{Aut}(G)$. By i), $\Sigma$ supports B. By Lemma 1.4, $\Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \ldots, S_{d+1}^\sigma \}$ is a frame for G and by i), it also supports B. Therefore, for each $j$, $1 \le j \le e$, $\beta_j = S_1 \ldots S_{i_j} = (S_1 \ldots S_{i_j})^\sigma$ for $1 \le i_j \le d$. Thus B is a characteristic flag.

iii) If A is a maximal flag of rank d, then by definition of B, B $\subseteq$ A. Since B is maximal, A = B.

iv) Suppose that B is non-trivial and let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ be a frame of G. $\Sigma$ supports B and there is an $i$, $1 \le i \le d$, such that $S_1 S_2 \ldots S_i = \beta_1$. Since G is abelian, by Lemma 1.7, $\Sigma' = \{S_1, \ldots, S_{i-1}, S_{i+1}, S_i, \ldots, S_{d+1} \}$ is also a frame for G. Hence, $\Sigma'$ supports a maximal flag which does not contain B. This is a contradiction and B is trivial□

## 2.4 Collineations

**Definition 2.26** *A* **collineation** $\sigma$ *is a bijection of* $\Delta(\mathcal{P})$ *onto* $\Delta(\mathcal{P})$ *that preserves* $\wedge$ *and* $A^\sigma \cong A$ *for each* $A$ *in* $\Delta(\mathcal{P})$.

Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two non-trivial flags in $\Delta(\mathcal{P})$. If A $\cong$ B, $d = e$ and $\alpha_i \cong \beta_i$ for $i$ = 1,2, ... , $d$. If $\sigma$ preserves $\wedge$, $(A \wedge B)^\sigma = A^\sigma \wedge B^\sigma$.

Let V be a vector space over the field of characteristic p and $\mathcal{P}$ be the projective space associated with V. The usual definition of a collineation $\sigma$ is a bijection of the subspaces of $\mathcal{P}$ such that $\alpha \subset \beta$ if and only if $\alpha^\sigma \subset \beta^\sigma$ for subspace $\alpha$ and $\beta$ of $\mathcal{P}$. By definition, each subspace $\alpha$ of $\mathcal{P}$ is the set of 1-dimensional subspaces of a subspace U of V. As a result of this correspondence, a collineation induces a bijection on the subspaces of V. Since each element in each flag in $\Delta(\mathcal{P})$ associated with a group G is a subgroup of G, the definition presented here of collineation reduces to the usual one when restricted to a finite dimensional vector space over the field of characteristic p. There is no loss of interpretation from this generalized setting.

**Examples:**

**2.3)** Consider the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$. For $D_8$, $\Delta(\mathcal{P}) =$ $\{\ \{\ 1\ \}, \langle a\rangle, \langle a^2\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^2\rangle \times \langle b\rangle, \langle a^2\rangle \times \langle ab\rangle, \langle a^2\rangle \subset \langle a^2\rangle \times \langle b\rangle, \langle a^2\rangle \subset \langle a^2\rangle \times \langle ab\rangle, \langle b\rangle \subset \langle a^2\rangle \times \langle b\rangle, \langle ab\rangle \subset \langle a^2\rangle \times \langle ab\rangle, \langle a^2b\rangle \subset \langle a^2\rangle \times \langle b\rangle, \langle a^3b\rangle \subset \langle a^2\rangle \times \langle ab\rangle\}$.

The following is an example of a collineation :

$$
\begin{aligned}
\{1\} &\mapsto \{1\} \\
\langle a \rangle &\mapsto \langle a \rangle \\
\langle a^2 \rangle &\mapsto \langle a^2 \rangle \\
\langle b \rangle &\mapsto \langle ab \rangle \\
\langle ab \rangle &\mapsto \langle b \rangle \\
\langle a^2 b \rangle &\mapsto \langle a^3 b \rangle \\
\langle a^3 b \rangle &\mapsto \langle a^2 b \rangle \\
\langle a^2 \rangle \times \langle b \rangle &\mapsto \langle a^2 \rangle \times \langle ab \rangle \\
\langle a^2 \rangle \times \langle ab \rangle &\mapsto \langle a^2 \rangle \times \langle b \rangle \\
\langle a^2 \rangle \subset \langle a^2 \rangle \times \langle b \rangle &\mapsto \langle a^2 \rangle \subset \langle a^2 \rangle \times \langle ab \rangle \\
\langle a^2 \rangle \subset \langle a^2 \rangle \times \langle ab \rangle &\mapsto \langle a^2 \rangle \subset \langle a^2 \rangle \times \langle b \rangle \\
\langle b \rangle \subset \langle a^2 \rangle \times \langle b \rangle &\mapsto \langle ab \rangle \subset \langle a^2 \rangle \times \langle ab \rangle \\
\langle ab \rangle \subset \langle a^2 \rangle \times \langle ab \rangle &\mapsto \langle b \rangle \subset \langle a^2 \rangle \times \langle b \rangle \\
\langle a^2 b \rangle \subset \langle a^2 \rangle \times \langle a^2 b \rangle &\mapsto \langle a^3 b \rangle \subset \langle a^2 \rangle \times \langle ab \rangle \\
\langle a^3 b \rangle \subset \langle a^2 \rangle \times \langle ab \rangle &\mapsto \langle a^2 b \rangle \subset \langle a^2 \rangle \times \langle b \rangle
\end{aligned}
$$

**2.4)** Consider the group $G = [Z_2 \times Z_2 \times Z_2]Z_3 = \langle a, b, c, x \mid a^2 = b^2 = c^2 = x^3 = 1, a^x = b, b^x = c, c^x = a \rangle$. The flag space associated with G is too large to list here, but consider the following collineation $\phi$ defined as follows:

$$
\begin{aligned}
(\langle a \rangle)^\phi &= \langle abc \rangle \\
(\langle abc \rangle)^\phi &= \langle a \rangle \\
(\langle b \rangle)^\phi &= \langle c \rangle \\
(\langle c \rangle)^\phi &= \langle b \rangle
\end{aligned}
$$

$$(\langle ab\rangle)^{\phi} = \langle ab\rangle$$

$$(\langle ac\rangle)^{\phi} = \langle ac\rangle$$

$$(\langle bc\rangle)^{\phi} = \langle bc\rangle$$

$$(\langle x\rangle)^{\phi} = \langle x\rangle$$

It maps $\langle abc\rangle$ into $\langle a\rangle$, where Z(G) = $\langle abc\rangle$. Thus $\phi$ is not induced by any automorphism of G.

**Theorem 2.27** *The collection of collineations form a group denoted by Col($\Delta(\mathcal{P})$).*

Proof. The binary operation defined on the collection of collineations is composition of maps. Let $\sigma$ and $\gamma$ be collineations. Since $\sigma$ and $\gamma$ are bijections of a finite set, the composition $\sigma\gamma$ is also a bijection.

Let A and B be flags in $\Delta(\mathcal{P})$. Then $A^{\sigma\gamma} \wedge B^{\sigma\gamma} = (A^{\sigma} \wedge B^{\sigma})^{\gamma} = (A \wedge B)^{\sigma\gamma}$ since $\sigma$ and $\gamma$ preserve $\wedge$. Therefore, $\sigma\gamma$ preserves $\wedge$. Finally, $A^{\sigma\gamma} \cong A$ since $A \cong A^{\sigma}$ and $A^{\sigma} \cong A^{\sigma\gamma}$. Thus the operation is closed. The composition of maps on a finite set is associative, so if $\delta$ is another collineation, then $(\sigma\gamma)\delta = \sigma(\gamma\delta)$. The collineation which sends every element $A \in \Delta(\mathcal{P})$ to itself is the identity collineation, denoted by $1_{\Delta(\mathcal{P})}$. By definition, $1_{\Delta(\mathcal{P})}\sigma = \sigma 1_{\Delta(\mathcal{P})} = \sigma$.

If $\sigma$ is a collineation, then for each A in $\Delta(\mathcal{P})$, define $\sigma^{-1}$ to be the map which sends the flag A to the flag B such that $B^{\sigma} = A$. This map is well-defined and a bijection since $\sigma$ is a bijection. By definition, $\sigma^{-1}\sigma = \sigma\sigma^{-1} = \{ 1_{\Delta(\mathcal{P})} \}$. Since $B \cong B^{\sigma} \cong A$, $A \cong A^{\sigma^{-1}}$.

Finally, it is shown that $\sigma^{-1}$ preserves $\wedge$. Let A and B be flags in $\Delta(\mathcal{P})$. Let $A^{\sigma^{-1}} \wedge B^{\sigma^{-1}} = C$. Then $(A^{\sigma^{-1}} \wedge B^{\sigma^{-1}})^{\sigma} = A^{\sigma^{-1}\sigma} \wedge B^{\sigma^{-1}\sigma} = A \wedge B = C^{\sigma}$. Therefore, $(A \wedge B)^{\sigma^{-1}} = (C^{\sigma})^{\sigma^{-1}} = C$ and $(A \wedge B)^{\sigma^{-1}} = C = A^{\sigma^{-1}} \wedge B^{\sigma^{-1}}$. Thus Col($\Delta(\mathcal{P})$) is a group □

**Theorem 2.28** *Let $\sigma$ be a collineation and $A$, $B \in \Delta(\mathcal{P})$. Then $A \subseteq B$ if and only if $A^\sigma \subseteq B^\sigma$.*

Proof. If $A \subseteq B$, then $A \wedge B = A$. Therefore, $A^\sigma = (A \wedge B)^\sigma = A^\sigma \wedge B^\sigma$ and $A^\sigma \subseteq B^\sigma$. If $A^\sigma \subseteq B^\sigma$, then $A^\sigma \wedge B^\sigma = A^\sigma$. Therefore $A = (A^\sigma)^{\sigma^{-1}} = (A^\sigma \wedge B^\sigma)^{\sigma^{-1}} = (A^\sigma)^{\sigma^{-1}} \wedge (B^\sigma)^{\sigma^{-1}} = A \wedge B$. Thus $A \subseteq B$ □.

**Theorem 2.29** *Let $\sigma$ be a collineation and $A$: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ a flag in $\Delta(\mathcal{P})$. Then $A^\sigma$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma$.*

Proof. Suppose $d = 1$. Then $A^\sigma = \alpha_1^\sigma$. Suppose $d \geq 2$ and proceed by induction on $d$. Let $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a flag of rank d. Then

$$(\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_{d-1}) \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d) = \alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_{d-1}$$

and

$$\begin{aligned}(\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_{d-1})^\sigma &= ((\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_{d-1}) \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d))^\sigma \\ &= (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_{d-1})^\sigma \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d)^\sigma \\ &= \alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_{d-1}^\sigma \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d)^\sigma.\end{aligned}$$

For $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_{d-1}^\sigma \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d)^\sigma$ to equal $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_{d-1}^\sigma$, $(\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d)^\sigma = \alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_{d-1}^\sigma \subset \beta$ where $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_{d-1}^\sigma \subset \beta$ is a flag of rank d. But since $\alpha_d \wedge (\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d) = \alpha_d$, $\beta = \alpha_d^\sigma$ □

This theorem tells us that the bijective map which a collineation $\sigma$ induces on the flags of rank 1 determines the rest of the map $\sigma$.

**Theorem 2.30** *Let $G$ be a group and $\Delta(\mathcal{P})$ its associated flag space.*

*i) Each $\sigma \in Aut(G)$ induces a collineation $\sigma_{\Delta(\mathcal{P})}$ on $\Delta(\mathcal{P})$.*

*ii) There exists a homomorphism $\kappa$ from $Aut(G)$ into $Col(\Delta(\mathcal{P}))$.*

*iii) $Ker(\kappa) \cap Inn(G)$ is the collection of those elements $t \in G$ such that $t$ normalizes each element in $\Delta(\mathcal{P})$.*

Proof. i) Let $\sigma \in$ Aut(G) and A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ . Define $\sigma_{\Delta(\mathcal{P})}$ to be the map which sends A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ to $A^{\sigma_{\Delta(\mathcal{P})}}$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma$. This map is well-defined since $A^{\sigma_{\Delta(\mathcal{P})}}$ is supported by the frame $\Sigma^\sigma = \{ S_1^\sigma, S_2^\sigma, \ldots, S_n^\sigma \}$ . Since $\sigma$ induces a bijection on the subgroups of G, $\sigma_{\Delta(\mathcal{P})}$ induces a bijection of the flags in $\Delta(\mathcal{P})$. Futhermore, $A^{\sigma_{\Delta(\mathcal{P})}} \cong A$ since $\alpha_i \cong \alpha_i^\sigma$ for $i$ = 1,2, ... , $d$.

Let A $\wedge$ B = C. Then C $\subseteq$ A and C $\subseteq$ B. Since $\sigma \in$ Aut(G), $C^{\sigma_{\Delta(\mathcal{P})}} \subseteq A^{\sigma_{\Delta(\mathcal{P})}}$ and $C^{\sigma_{\Delta(\mathcal{P})}} \subseteq B^{\sigma_{\Delta(\mathcal{P})}}$. Therefore, $C^{\sigma_{\Delta(\mathcal{P})}} \subseteq A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}}$ and $(A \wedge B)^{\sigma_{\Delta(\mathcal{P})}} \subseteq A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}}$ . Consider $A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}}$ and let $A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}} = D$. Therefore $D \subseteq A^{\sigma_{\Delta(\mathcal{P})}}$ and $D \subseteq B^{\sigma_{\Delta(\mathcal{P})}}$. Since $\sigma_{\Delta(\mathcal{P})}$ is a bijection, there is a flag E such that $E^{\sigma_{\Delta(\mathcal{P})}} = D$. By definition of $\sigma_{\Delta(\mathcal{P})}$ and the fact that $\sigma \in$ Aut(G), E $\subseteq$ A and E $\subseteq$ B. Thus, E $\subseteq$ A $\wedge$ B and $D = E^{\sigma_{\Delta(\mathcal{P})}} \subseteq (A \wedge B)^{\sigma_{\Delta(\mathcal{P})}}$. Therefore, $A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}} \subseteq (A \wedge B)^{\sigma_{\Delta(\mathcal{P})}}$ and $A^{\sigma_{\Delta(\mathcal{P})}} \wedge B^{\sigma_{\Delta(\mathcal{P})}} = (A \wedge B)^{\sigma_{\Delta(\mathcal{P})}}$.

ii) Let $\kappa$ be a map from Aut(G) into Col($\Delta(\mathcal{P})$) which sends $\sigma \in$ Aut(G) to $\sigma_{\Delta_{\mathcal{P}}} \in$ Col($\Delta(\mathcal{P})$). By i), this map is well-defined. Let $\sigma$ and $\gamma$ be elements of Aut(G) and consider $(\sigma\gamma)_{\Delta(\mathcal{P})}$. For A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, $A^{\sigma_{\Delta(\mathcal{P})}\gamma_{\Delta(\mathcal{P})}} = (\alpha_1^\sigma \subset \ldots \subset \alpha_d^\sigma)^{\gamma_{\Delta(\mathcal{P})}} = \alpha_1^{\sigma\gamma} \subset \ldots \subset \alpha_d^{\sigma\gamma} = A^{(\sigma\gamma)_{\Delta(\mathcal{P})}}$.

iii) Let $t \in$ ker($\kappa$) $\cap$ Inn(G). Then for each A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, $A^t$: $\alpha_1^t \subset \alpha_2^t \subset \ldots \subset \alpha_d^t$ = A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ . Therefore, $t$ normalizes each flag in $\Delta(\mathcal{P})$.

Let $g \in G$ and suppose $A^g = A$ for each A in $\Delta(\mathcal{P})$. Then $g \in \ker \kappa \cap \mathrm{Inn}(G)$. □

**Example.**

**2.5)** Consider the dihedral group $D_8$ and its associated flag space. Then $\ker(\kappa) \cap \mathrm{Inn}(D_8)$ $= \{ 1 \}$ since $Z(D_8) = \langle a^2 \rangle$. Since $\mathrm{Aut}(D_8) \cong D_8$, $\ker(\kappa) = \{ 1 \}$ or $\mathrm{Aut}(D_8)$ $= [\mathrm{Inn}(D_8)]\ker(\kappa)$. But $\ker(\kappa) \lhd \mathrm{Aut}(D_8)$, so if $\ker(\kappa) \neq \{ 1 \}$, then $\mathrm{Aut}(D_8) \cong$ $\mathrm{Inn}(D_8) \times \ker(\kappa) \ncong D_8$. Therefore, $\ker(\kappa) = \{ 1 \}$ and $\mathrm{Aut}(D_8)$ is embedded in $\mathrm{Col}(\Delta(\mathcal{P}))$. In this case, this embedding is an equality.

Resulting from Theorem 2.30 ii), the collection of collineations induced by Aut(G) is a subgroup of $\mathrm{Col}(\Delta(\mathcal{P}))$, denoted by $Aut_G(\Delta(\mathcal{P}))$. Given an n-dimensional vector space over the field of characteristic p, PGL(n,p) denotes the collection of all collineations induced by GL(n,p). Thus $Aut_G(\Delta(\mathcal{P}))$ is a generalization of PGL(n,p) and when G is an elementary abelian p-group, $Aut_G(\Delta(\mathcal{P})) = \mathrm{PGL}(n,p)$.

It might be thought that $Aut_G(\Delta(\mathcal{P}))$ is normal in $\mathrm{Col}(\Delta(\mathcal{P}))$ , but this is not true. Consider the counter-example given by example 2.4). Let $\phi_x$ be the inner automorphism induced by conjugation by $x$. The collineation $\phi^{-1}\phi_x\phi$ sends the rank 1 flag $\langle abc \rangle$ to the rank 1 flag $\langle c \rangle$. But $Z(G) = \langle abc \rangle$ and is characteristice in G. Hence $\phi^{-1}\phi_x\phi$ is not in $Aut_G(\Delta(\mathcal{P}))$ .

## 2.5 Connected Flag Spaces

### 2.5.1 Definitions

**Definition 2.31** *Two non-trivial flags A and B in $\Delta(\mathcal{P})$ are* **adjacent** *if they are of the same rank d, $d \geq 1$, and $A \cap B$ is a flag C of rank d-1. Two non-trivial flags A and B are* **connected** *if there is a sequence $\{C_1, C_2, \ldots, C_m\}$ of maximal flags such that $A \subseteq C_1$, B*

$\subseteq C_m$, *and* $C_i$ *and* $C_{i+1}$ *are adjacent for* $1 \leq i \leq m-1$.

By definition, every pair of flags of rank 1 are adjacent.

**Theorem 2.32** *Let A and B be two flags of* $\Delta(\mathcal{P})$ *and* $\sigma \in Col(\Delta(\mathcal{P}))$.

*i) If A and B are adjacent, then* $A^\sigma$ *and* $B^\sigma$ *are adjacent.*

*ii) If A and B are connected, then* $A^\sigma$ *and* $B^\sigma$ *are connected.*

*iii) Connectedness defines an equivalence relation on the maximal flags of* $\Delta(\mathcal{P})$.

Proof. i) Since A and B are adjacent, there is a flag C of rank $d-1$ contained in A and B. By Theorem 2.28, $C^\sigma \subset A^\sigma$ and $C^\sigma \subset B^\sigma$. Since $\sigma$ is a collineation, $C^\sigma$ is of rank $d-1$.

ii) By definition, there is a set of maximal flags $\{ C_1, \ldots, C_m \}$ such that A $\subseteq C_1$, B $\subseteq C_m$ and for $1 \leq i \leq m-1$ $C_i$ and $C_{i+1}$ are adjacent. By Theorem 2.28, $A^\sigma \subseteq C_1^\sigma$ and $B^\sigma \subseteq C_m^\sigma$, and by i), $C_i^\sigma, C_{i+1}^\sigma$ are adjacent. Thus $B^\sigma$ and $A^\sigma$ are connected by the set $\{C_1^\sigma$, $\ldots$, $C_m^\sigma\}$

iii) The relation is reflexive as every flag is contained in a maximal one. The relation is symmetric, for if $\{ C_1, C_2, \ldots, C_n \}$ is a sequence of maximal flags which connect the maximal flags A and B, then $\{ C_n, C_{n-1}, \ldots, C_1 \}$ connects B and A. If the maximal flags A and B are connected by $\{ D_1, D_2, \ldots, D_n \}$ and the maximal flags B and C are connected by $\{ E_1, E_2, \ldots, E_m \}$, then A and C are connected by $\{ D_1, D_2, \ldots, D_n, E_2, \ldots, E_m \}$ and the relation is transitive. This set works since B $= D_n = E_1$ □

If G is an elementary abelian p-group, there is only one equivalence class of maximal flags. However, there do exist groups for which there is not a unique equivalence class. In the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1,\ a^b = a^3 \rangle$, the maximal flags $\langle a \rangle$ and $\langle a^2 \rangle \subset \langle a^2, b \rangle$ are not connected.

Connectedness does not define an equivalence relation on all the flags in $\Delta(\mathcal{P})$. The reflexive and symmetric properties will hold, but the transitive property may not. Consider the group $G = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, a^b = a^3, a^c = a^3, b^c = a^2b \rangle = [D_8]Z_2$. The flags $\langle a \rangle$ and $\langle a, b \rangle$ are connected by the maximal flag $\langle a \rangle \subset \langle a, b \rangle$ and the flags $\langle a, b \rangle$ and $\langle a^2 \rangle$ are connected by the maximal flag $\langle a^2 \rangle \subset \langle a^2, b \rangle \subset \langle a, b \rangle$, but $\langle a \rangle$ and $\langle a^2 \rangle$ are not connected. The problem is that a flag can be contained in two different maximal flags of differing ranks.

**Theorem 2.33** *Connectedness defines an equivalence relation on $\Delta(\mathcal{P})$ if and only if the following conditions hold:*

*i) Every flag in $\Delta(\mathcal{P})$ cannot be contained in two different maximal flags of different ranks;*

*ii) If a flag is contained in two different maximal flags of the same rank, those maximal flags are connected.*

**Proof.** Suppose that a flag A is contained in maximal flags B and C, where rank(B) $= n$, rank(C) $= m$, and $n \neq m$. Since A $\subseteq$ B, B and A are connected. Since A $\subseteq$ C, A and C are connected. Connectedness defines an equivalence relation on $\Delta(\mathcal{P})$ and B and C are connected. But this is a contradiction since $n \neq m$. Thus $n = m$ and B and C are connected.

Conversely, suppose that conditions (i) and (ii) hold. Each flag in $\Delta(\mathcal{P})$ is contained in a maximal one and is connnected to itself. If A is connected to B by the collection of maximal flags $\{ C_1, \ldots, C_t \}$, then B is connected to A by the collection of maximal flags $\{C_t, \ldots, C_1 \}$. Thus connectedness is reflexive and symmetric.

Let A, B, and C be flags in $\Delta(\mathcal{P})$ such that A and B are connected by the sequence of maximal flags $\{ D_1, \ldots, D_s \}$ and B and C are connected by the sequence of maximal flags $\{ E_1, \ldots, E_t \}$. Since B $\subseteq D_s$ and B $\subseteq E_1$, by (i), $D_s$ and $E_1$ are of the same rank.

Thus by (ii), there is a sequence of maximal flags $\{ F_1, \ldots F_r \}$ which connect $D_s$ and $E_1$. Therefore the sequence of maximal flags $\{ D_1, \ldots D_{s-1}, F_1, \ldots, F_r, E_2, \ldots E_t \}$ connect A and B □

**Note:** These two conditions do not imply that all the maximal flags in $\Delta(\mathcal{P})$ are of the same rank. Connectedness defines an equivalence relation on the flag space of $D_8$, yet $D_8$ contains maximal flags of rank 1 and 2.

**Definition 2.34** *Let $\Delta(\mathcal{P})$ be a flag space. If each pair of flags in $\Delta(\mathcal{P})$ are connected,* **the flag space** $\Delta(\mathcal{P})$ **is** *said to be* **connected.**

**Lemma 2.35** *If $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma' = \{ S_1, \ldots, S_{i-1}, S_{i+1}, S_i, \ldots S_n \}$ are both frames for G, then the maximal flags they support are adjacent.*

Proof. The frames $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma' = \{ S_1, \ldots, S_{i-1}, S_{i+1}, S_i, \ldots S_n \}$ support the flags A: $S_1 \subset S_1 S_2 \subset \ldots \subset S_1 \ldots S_{n-1}$ and B: $S_1 \subset \ldots \subset S_1 \ldots S_{i-1} \subset S_1 \ldots S_{i-1} S_{i+1} \subset S_1 \ldots S_{i-1} S_{i+1} S_i \subset \ldots \subset S_1 \ldots S_{n-1}$ respectively of rank $n-1$. The only place where A and B differ is at the subnormal subgroups $S_1 \ldots S_{i-1} S_i$ and $S_1 \ldots S_{i-1} S_{i+1}$ of A and B respectively. Therefore A ∩ B is a flag of rank $n-2$ and the flags A and B are adjacent □

**Theorem 2.36** *If G is abelian, then $\Delta(\mathcal{P})$ is connected.*

Proof. By The Fundamental Theorem of Abelian Groups, G $\cong Z_{p_1^{r_1}} \oplus \ldots \oplus Z_{p_m^{r_m}}$. Thus any frame $\Sigma = \{ S_1, \ldots, S_m \}$ for G is of length $m$ and there is a $\pi \in \mathrm{Sym}(m)$, such that for each $i$, $1 \leq i \leq m$, $S_i \cong Z_{p_{\pi(i)}^{r_{\pi(i)}}}$. Proceed by induction on the length of the maximal flags for G.

Let A: $\alpha_1 \subset \ldots \subset \alpha_{m-1}$ and B: $\beta_1 \subset \ldots \subset \beta_{m-1}$ be two maximal flags in G. Then there are frames $\Sigma_A = \{ S_1, \ldots, S_m \}$ and $\Sigma_B = \{ T_1, \ldots, T_m \}$ which support A and B respectively. There are two cases.

Case 1). $S_1 = T_1$.

If $S_1 = T_1$, then $\alpha_1 = \beta_1$. By Lemma 2.22 i), $A' : \alpha_2/\alpha_1 \subset \ldots \subset \alpha_{m-1}/\alpha_1$ and $B' : \beta_2/\alpha_1 \subset \ldots \subset \beta_{m-1}/\alpha_1$ are maximal flags in the flag space of the group $G/\alpha_1$. By the induction hypothesis, there is a sequence of maximal flags $\{ C_1/\alpha_1, \ldots, C_n/\alpha_1 \}$ in $\Delta(\mathcal{P}_{G/\alpha_1})$, such that $A' = C_1/\alpha_1, B' = C_m/\alpha_1$, and $C_i/\alpha_1$ and $C_{i+1}/\alpha_1$ are adjacent. By Lemma 2.22 ii), $\{ \alpha_1 \subset C_1, \ldots, \alpha_1 \subset C_n \}$ are a collection of maximal flags of G such that $\alpha_1 \subset C_i$ and $\alpha_1 \subset C_{i+1}$ are adjacent. Since A $= \alpha_1 \subset C_1$ and B $= \alpha_1 \subset C_n$, A and B are connected.

Case 2). $S_1 \neq T_1$.

For some $u$, $1 \leq u \leq m$, there is an $S_u$ such that $| S_u |=| T_1 |= p^r$ and $S_u \cap T_2 \ldots T_m = \{1\}$. Let T $= T_2 \ldots T_m$ and consider $T_1 = \langle t_1 \rangle$.

Suppose for all $S \in \Sigma$, such that $| S |= p^r$, that $S \leq T$. Since $\Sigma$ is a frame for G, $t_1 = xyz$, where $x = s_{i_1}^{a_{i_1}} \ldots s_{i_{n_1}}^{a_{i_{n_1}}}$ such that $| S_{i_l} |=| \langle s_{i_l} \rangle |< p^r$ for $1 \leq l \leq n_1$, $y = s_{j_1}^{a_{j_1}} \ldots s_{j_{n_2}}^{a_{j_{n_2}}}$ such that $| S_{j_l} |=| \langle s_{j_l} \rangle |= p^r$ for $1 \leq l \leq n_2$ and $z = s_{k_1}^{a_{k_1}} \ldots s_{k_{n_3}}^{a_{k_{n_3}}}$ such that $| S_{k_l} |=| \langle s_{k_l} \rangle |> p^r$ for $1 \leq l \leq n_3$. Now consider G/T $= \langle t_1 T \rangle$. By assumption, $t_1 T = xzT$. Since $| t_1 |= p^r, z \in \Phi(G)$. $\Phi(G) = \Phi(T_1) \oplus \ldots \oplus \Phi(T_m)$, so $z = g_1 \ldots g_m$, where $g_l \in \Phi(T_l)$ for $1 \leq l \leq$ m. Therefore, $t_1 T = xzT = xg_1 \ldots g_m T = xg_1 T$. Since $g_1 \in \Phi(T_1), g_1 = t_1^{p^a}$, where $1 \leq a <$ r. Therefore, $t_1 T = xt_1^{p^a} T$, which implies that $xT = (t_1^{-1})^{p^a} t_1 T$ or that $xT$ has order $p^r$. This is a contradiction. Thus there must be a $s_{j_l}^{a_{j_l}}$ for $1 \leq l \leq n_2$, such that $a_{j_l} \not\equiv 0 \pmod p$ and that $s_{j_l}^{\alpha_{j_l}} \notin$ T. Therefore $s_{j_l} \notin T$ and $\langle s_{j_l} \rangle \cap T = S_{j_l} \cap T = \{1\}$. Let $S_u = S_{j_l}$.

Since $S_u \cap T = \{1\}$, G $= [S_u]$T and $\Sigma' = \{ S_u, T_2, \ldots, T_m \}$ is a frame for G. By $u - 1$ iterations of Lemma 1.7, $\Sigma'' = \{ S_u, S_1, \ldots, S_{u-1}, S_{u+1}, \ldots, S_m \}$ is also a frame for G. Let

$A'$ and $A''$ be the maximal flags supported by $\Sigma'$ and $\Sigma''$ respectively. By $u-1$ iterations of Lemma 2.35, A and $A''$ are connected. So there are maximal flags $\{ C_1, \ldots, C_{m_1} \}$ in $\Delta(\mathcal{P})$ such that $A = C_1, A'' = C_{m_1}$ and for $1 \leq i \leq m_1, C_i$ and $C_{i+1}$ are adjacent.

By case 1), $A''$ and $A'$ are connected, so there are maximal flags $\{ D_1, \ldots, D_{m_2} \}$ such that $A'' = D_1, A' = D_{m_2}$ and for $1 \leq i \leq m_2, D_i$ and $D_{i+1}$ are adjacent. Finally, since $A'$ and B are adjacent, the set of maximal flags $\{ C_1, \ldots, C_{m_1}, D_2, \ldots, D_{m_2}, B \}$ connect A and B. Since every flag is contained in a maximal flag, $\Delta(\mathcal{P})$ is connected □

### 2.5.2 Conditions For Connectedness

**Definition 2.37** *A rank 1 flag $\alpha$ in $\Delta(\mathcal{P})$ is* **complemented** *in $G$ if there is a subgroup $B$ of $G$, such that $G = \alpha B$ and $\alpha \cap B = \{ 1 \}$.*

**Definition 2.38** *A flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ of rank d is called a* **complemented flag** *if each $\alpha_i$ , $1 \leq i \leq d$, is complemented in G.*

**Theorem 2.39** *Suppose B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_r$ is a normal complemented flag in $\Delta(\mathcal{P})$, such that the flag space associated with each quotient group $\beta_i/\beta_{i-1}$, for $1 \leq i \leq r+1$, where $\beta_0 = \{ 1 \}$ and $\beta_{r+1}$, is connected. If A and $A'$ are two maximal flags in $\Delta(\mathcal{P})$ such that $B \subseteq A$ and $B \subseteq A'$, then A and $A'$ are connected.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and $A'$: $\alpha'_1 \subset \ldots \subset \alpha'_e$ be two maximal flags in $\Delta(\mathcal{P})$ which contain B. Let $r+1$ be the number of quotient groups determined by B. If $r+1 = 1$, then $r = 0$ and G has a connected flag space. Proceed by induction on $r+1$. Suppose that B has two or more factors.

B is contained in A and $A'$ and $\alpha_{l_1} = \alpha'_{l'_1} = \beta_1, \ldots, \alpha_{l_r} = \alpha'_{l'_r} = \beta_r$ where $1 \leq l_1 < \ldots < l_r \leq$ d and $1 \leq l'_1 < \ldots < l'_r \leq$ e. Since $\alpha_{l_r} = \alpha'_{l'_r} = \beta_r$, by Lemma 2.19 ii), $A_1$:

$\alpha_1 \subset \ldots \subset \alpha_{l_r-1}$ and $A_1'$: $\alpha_1' \subset \ldots \subset \alpha'_{l_r'-1}$ are maximal flags in $\Delta(\mathcal{P}_{\beta_r})$. Furthermore, $B_1$: $\beta_1 \subset \ldots \subset \beta_{r-1}$ is a normal complemented flag in $\Delta(\mathcal{P}_{\beta_r})$, such that $B_1 \subsetneq A_1$, and $B_1 \subsetneq A_1'$. By the induction hypothesis, there is a sequence of maximal flags $\{ C_1, \ldots, C_n \}$ in $\Delta(\mathcal{P}_{\beta_r})$ which connect $A_1$ and $A_1'$. Thus $A_1 = C_1$, $A_1' = C_n$, and for $1 \leq i \leq n-1$, $C_i$ and $C_{i+1}$ are adjacent. By Lemma 2.19 i), each $C_i$ can be extended to maximal flags $C_i'$: $C_i \subset \beta_r \subset \alpha_{l_r+1} \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$.

Let $A_0$ and $A_0'$ denote the flags $\alpha_{l_r+1}/\beta_r \subset \alpha_{l_r+2}/\beta_r \subset \ldots \subset \alpha_d/\beta_r$ and $\alpha'_{l_r'+1}/\beta_r \subset \alpha'_{l_r'+2}/\beta_r \subset \ldots \subset \alpha_e'/\beta_r$ of $G/\beta_r$ respectively. By Lemma 2.22 i), $A_0$ and $A_0'$ are maximal flags in the flag space of $G/\beta_r$, which is connected. Thus there are maximal flags $\{D_1, \ldots, D_m\}$ in the flag space of $G/\beta_r$ such that $A_0 = D_1, A_0' = D_m$ and for $1 \leq j \leq m-1$, $D_j$ and $D_{j+1}$ are adjacent.

For each j, $D_j$: $\delta_{j,1}/\beta_r \subset \ldots \subset \delta_{j,t}/\beta_r$ is a maximal flag in the flag space of $G/\beta_r$. By Lemma 2.22 ii), $\alpha_1' \subset \ldots \subset \alpha'_{l_r'} \subset \delta_{j,1} \subset \ldots \subset \delta_{j,t}$ is a flag in $\Delta(\mathcal{P})$.

Define the flags $\{ F_1, \ldots, F_{n+m} \}$ in $\Delta(\mathcal{P})$ in the following manner:

For $1 \leq e \leq n$, define $F_e = C_e'$;

For n+1 $\leq e \leq n+m$, define $F_e = \alpha_1' \subset \ldots \subset \alpha'_{l_r'} \subset \delta_{e-n,1} \subset \ldots \subset \delta_{e-n,t}$.

By Lemmas 2.19 i) and 2.22 ii) they are maximal. By definition, the collection of maximal flags $\{ F_1, \ldots, F_{n-1}, F_{n+1}, \ldots, F_{n+m} \}$ connect A and $A'$ since $A = F_1$, $A' = F_{n+m}$ and for $1 \leq i \leq n-2$ and $n+1 \leq i \leq n+m-1$, $F_i$ and $F_{i+1}$ are adjacent. Futhermore, $C_n = D_1$, so $F_{n-1}$ and $F_{n+1}$ are adjacent □

**Corollary 2.40** *Suppose A and $A'$ are two flags in $\Delta(\mathcal{P})$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_r$ is a normal complemented flag in $\Delta(\mathcal{P})$, such that the flag space associated with each quotient group $\beta_i/\beta_{i-1}$, for $1 \leq i \leq r+1$, where $\beta_0 = \{ 1 \}$ and $\beta_{r+1} = G$, is connected. If C and*

*$C'$ are maximal flags in $\Delta(\mathcal{P})$ containing A and $A'$ respectively, such that $B \subseteq C$ and $B \subseteq C'$, then A and $A'$ are connected.*

Proof. A and $A'$ are contained in the maximal flags C and $C'$ respectively. By Theorem 2.39, C and $C'$ are connected and thus A and $A'$ are connected □

**Corollary 2.41** *Suppose B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_r$ is a d-primary complemented flag in $\Delta(\mathcal{P})$, such that the flag space associated with each quotient group $\beta_i/\beta_{i-1}$, for $1 \leq i \leq r+1$, where $\beta_0 = \{ 1 \}$ and $\beta_{r+1} = G$, is connected. Then all the maximal flags of rank d are connected.*

Proof. Let A and C be maximal flags of rank d. Then B is contained in both A and C. By Theorem 2.39, A and C are connected □

### 2.5.3 Solvable nC-Groups

**Theorem 2.42** *If G is a solvable nC-group, then for each frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, $S_i$ is of prime order for $1 \leq i \leq n$.*

Proof. By the definition of frame, $\{1\} \lhd S_1 \lhd S_1S_2 \lhd \ldots \lhd S_1 \ldots S_n = G$ is a normal series for G. $S_1 \ldots S_j \lhd S_1 \ldots S_{j+1}$ for $1 \leq j \leq n-1$ and since normal subgroups of nC-groups are nC-groups (3.5 of [10]), for each $i$, $1 \leq i \leq n$, $S_1 \ldots S_i$ is an nC-group. Furthermore, since the class of solvable nC-groups is a formation (1.3 of [3]), $S_1 \ldots S_i/S_1 \ldots S_{i-1}$ is a solvable nC-group for $1 \leq i \leq n$, where $S_0 = \{ 1 \}$. Since $S_1 \ldots S_i/S_1 \ldots S_{i-1} \cong S_i$, $S_i$ is an inseparable, solvable nC-group. Thus $S_i$ is simple. Since $S_i$ is solvable, $S_i$ is cyclic of prime order □

**Theorem 2.43** *Let G be a solvable nC-group. An inseparable subgroup H of G is a point in $\Delta(\mathcal{P})$ if and only if $H \leq F(G)$.*

Proof. Suppose H is a point in $\Delta(\mathcal{P})$. By definition, there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, such that H = $\alpha_1$. Thus H is subnormal in G. By Theorem 2.42, $|H| = p$ and H is nilpotent. Therefore, by Theorem 5.2.5 of [3], H $\leq$ F(G).

Conversely, suppose H $\leq$ F(G). Since $\Phi(G) = \{ 1 \}$, F(G) is elementary abelian and $| H | = p$. Furthermore, F(G) splits over H. Given that H is inseparable, there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, such that $\alpha_1$ = H. Thus H is a point in $\Delta(\mathcal{P})$ □

Let $r + 1$ be the Fitting length of a solvable group G. If G is an nC-group, each proper, non-trivial subgroup in the Fitting series $\{ 1 \} = F_0(G) \lhd F_1(G) \lhd F_2(G) \lhd \ldots \lhd F_r(G) \lhd F_{r+1}(G) = G$ is complemented in G.

**Definition 2.44** *Let G be a solvable group of Fitting length $r$ + 1, such that G splits over each subgroup in the Fitting series. Then the normal complemented flag $F_C$: $\gamma_1 \subset \ldots \subset \gamma_r$, is called the* **Fitting flag** *of $\Delta(\mathcal{P})$, where $\gamma_i = F_i(G)$.*

**Theorem 2.45** *Let G be a solvable nC-group and let A and B be two maximal flags in $\Delta(\mathcal{P})$. If A and B contain the flag $F_C$, then A and B are connected.*

Proof. Since G is a solvable nC-group, each quotient group of $F_C$ is abelian. By Theorem 2.36, each quotient group has a connected flag space. Since $F_C$ is a normal, complemented flag in $\Delta(\mathcal{P})$, by Theorem 2.39, A and B are connected □

**Theorem 2.46** *Let G be a solvable nC-group and let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$. Then A is connected to a maximal flag B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ such that B contains $F_C$: $\gamma_1 \subset \ldots \subset \gamma_r$, the Fitting flag in $\Delta(\mathcal{P})$.*

Proof. If $r$ = 1, F(G) = G. Then G is abelian and by Theorem 2.36, A is connected to every maximal flag . Proceed by induction on $r$.

Part I.

There is a frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ which supports A and has the property that there are integers $1 \leq i_1 < i_2 < \ldots < i_t \leq d+1$ such that $F(G) = S_{i_1} S_{i_2} \ldots S_{i_t}$.

Let $\Sigma' = \{ T_1, \ldots, T_m \}$ be a frame which supports A. If $\Sigma'$ has the desired property, let $\Sigma = \Sigma'$. Assume otherwise. By Theorem 2.43, $T_1 \leq F(G)$. Thus there is an $l$, $1 \leq l \leq m-1$, such that $T_1 \ldots T_l \leq F(G)$ and $T_1 \ldots T_{l+1} \not\leq F(G)$. Since $T_{l+1} \not\leq F(G)$, there is a $k_1$, $l+2 \leq k_1 \leq m$, such that $T_1 \ldots T_l = F(G) \cap T_1 \ldots T_{k_1 - 1}$ and $T_1 \ldots T_l < F(G) \cap T_1 \ldots T_{k_1}$. By Theorem 2.42, $T_{k_1}$ is of prime order $p$. Hence there is an $x_1 \in F(G) \cap T_1 \ldots T_{k_1}$ such that $| x_1 | = p$ and $x_1 \notin T_1 \ldots T_{k_1 - 1}$. Since $| x_1 | = p$, $\langle x_1 \rangle \cap T_1 \ldots T_{k_1-1} = \{ 1 \}$ and $T_1 \ldots T_{k_1} = [T_1 \ldots T_{k_1 - 1}]\langle x_1 \rangle$. Therefore, $\Sigma_1' = \{ T_1, \ldots, T_{k_1-1}, \langle x_1 \rangle, T_{k_1+1}, \ldots, T_m \}$ is a frame for G which also supports A. If $T_1 \ldots T_l \langle x_1 \rangle = F(G)$, let $\Sigma = \Sigma_1'$. If not, proceed as above to obtain a sequence of integers $l+1 \leq k_1 < k_2 < \ldots < k_s \leq m$, such that $\Sigma_s' = \{ T_1, \ldots, T_{k_1-1}, \langle x_1 \rangle, T_{k_1+1}, \ldots, T_{k_2-1}, \langle x_2 \rangle, T_{k_2+1}, \ldots, T_{k_s-1}, \langle x_s \rangle, T_{k_s+1}, \ldots, T_m \}$ is a frame for G which supports A and $F(G) = T_1 \ldots T_l \langle x_1 \rangle \ldots \langle x_s \rangle$. Let $\Sigma = \Sigma_s'$.

Part II.

$\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ is a frame which supports A and for some $l$, $1 \leq l \leq d$, $S_1 \ldots S_l \leq \gamma_1$ and $S_1 \ldots S_{l+1} \not\leq \gamma_1$.

Suppose that $S_1 \ldots S_l = \gamma_1$. By Lemma 2.22 i), D: $\alpha_{l+1}/\gamma_1 \subset \ldots \subset \alpha_d/\gamma_1$ is a maximal flag in the flag space $\Delta(\mathcal{P}_{G/\gamma_1})$. By the induction hypothesis, there is a maximal flag E: $\epsilon_1/\gamma_1 \subset \ldots \subset \epsilon_{d-l}/\gamma_1$ in $\Delta(\mathcal{P}_{G/\gamma_1})$ which contains $F_C'$: $\gamma_2/\gamma_1 \subset \ldots \subset \gamma_r/\gamma_1$, the Fitting flag of $\Delta(\mathcal{P}_{G/\gamma_1})$ and is connected to D. Thus there is a sequence of maximal flags $\{F_1, \ldots, F_n\}$ in $\Delta(\mathcal{P}_{G/\gamma_1})$ which connect D and E. Let $F_i = \delta_1^i/\gamma_1 \subset \ldots \subset \delta_m^i/\gamma_1$. By Lemma 2.22 ii), $E'$: $\alpha_1 \subset \ldots \subset \alpha_l \subset \epsilon_1 \subset \ldots \subset \epsilon_{d-l}$ and $F_i'$: $\alpha_1 \subset \ldots \subset \alpha_l \subset \delta_1^i \subset \ldots \subset \delta_m^i$ for $1 \leq i \leq n$ are maximal flags in $\Delta(\mathcal{P})$. Let B $= E'$. By definition, $F_C \subset$ B and $\{ F_1', \ldots, F_n' \}$ is a sequence of adjacent flags such that A = $F_1'$ and B = $F_n'$. Hence A and B are connected.

Part III.

Suppose that $S_1 \ldots S_l < \gamma_1$. There is a $t_1$, $l+2 \leq t_1 \leq d$, such that $t_1$ is the smallest integer where $S_{t_1} < \gamma_1$. Let T = $S_1 \ldots S_{t_1}$, S = $S_1 \ldots S_l$, and $\Sigma_1 = \{ S_1, \ldots, S_{t_1} \}$ be a frame for T. Let $x \in S_i$, where $l+1 \leq i \leq t_1 - 1$.

Let $S_{t_1} = \langle s_{t_1} \rangle$. Since $SS_{t_1} \lhd$ T (if not, then $\gamma_1$ is not normal in G), $s_{t_1}^x = ss_{t_1}^\alpha$ where $s \in S$. Therefore, $x^{-1}s_{t_1}x = ss_{t_1}^\alpha$ and $s_{t_1}x = xss_{t_1}^\alpha$. Now $S_1 \ldots S_{t_1-1} \lhd S_1 \ldots S_{t_1}$, hence $x^{s_{t_1}^{-1}} = y$ where $y \in S_1 \ldots S_{t_1-1}$. Therefore, $s_{t_1}xs_{t_1}^{-1} = y$ and $s_{t_1}x = ys_{t_1}$. Hence, $ys_{t_1} = xss_{t_1}^\alpha$ and $s^{-1}x^{-1}y = s_{t_1}^{\alpha-1}$. Since $S_1 \ldots S_{t_1-1} \cap S_{t_1} = 1$, $\alpha = 1$ and $s_{t_1}^x = ss_{t_1}$.

Therefore

$$\begin{aligned}
x^{-1}s_{t_1}x &= ss_{t_1} \\
xs_{t_1}x &= x^2ss_{t_1} \\
s_{t_1}^{-1}xs_{t_1}x &= s_{t_1}^{-1}x^2ss_{t_1} \\
x^{s_{t_1}} &= s_{t_1}^{-1}x^2ss_{t_1}x^{-1}.
\end{aligned}$$

Since S $\lhd S_1 \ldots S_{t_1-1}$, $x^2s = s'x^2$ where $s' \in$ S. Thus $x^{s_{t_1}} = s_{t_1}^{-1}s'x^2s_{t_1}x^{-1}$. Since $s_{t_1}^x = ss_{t_1}$ and $SS_{t_1} \lhd$ T, $s_{t_1}^{x^{-2}} = s''s_{t_1}$ where $s'' \in$ S. Hence $x^2s_{t_1} = s''s_{t_1}x^2$ and $x^{s_{t_1}} = s_{t_1}^{-1}s'x^2s_{t_1}x^{-1} = s_{t_1}^{-1}s's''s_{t_1}x$. Furthermore, $SS_{t_1}$ is abelian, so $x^{s_{t_1}} = s's''x$.

Let $\Sigma_2 = \{ S_1, \ldots, S_{t_1-2}, S_{t_1}, S_{t_1-1} \}$. For $x \in S_j$, $l+1 \leq j \leq t_1 - 2$, $x^{s_{t_1}} = s'x$ where $s' \in$ S. Hence, $S_1 \ldots S_{t_1-2}S_{t_1}$ is a subgroup of T. Since $s_{t_1}^y = ss_{t_1}$ where $s \in$ S and $y \in S_{t_1-1}$, $S_1 \ldots S_{t_1-2}S_{t_1} \lhd$ T. It is clear that $S_1 \ldots S_{t_1-2}S_{t_1} \cap S_{t_1-1} = \{ 1 \}$. Hence $\Sigma_2$ is a frame for T.

Now consider $\Sigma_3 = \{ S_1, \ldots, S_{t_1-3}, S_{t_1}, S_{t_1-2}, S_{t_1-1} \}$. By a method identical to the one used in the previous paragraph, it can be shown that $\Sigma_3$ is also a frame for T. Continuing in this manner, a sequence of frames $\{ \Sigma_1, \ldots, \Sigma_{t_1-l} \}$ for T is obtained, such that the frame

$\Sigma_i = \{ S_1, \ldots, S_l, \ldots S_{t_1-i}, S_{t_1}, S_{t_1-i+1}, \ldots, S_{t_1-1} \}$. These frames can be extended to a sequence $\{ \Sigma'_1, \ldots \Sigma'_{t_1-l} \}$ of frames for G where $\Sigma'_i = \{ S_1, \ldots, S_l, \ldots S_{t_1-i}, S_{t_1}, S_{t_1-i+1}, \ldots, S_{t_1-1}, S_{t_1+1}, \ldots, S_{d+1} \}$. Thus by Lemma 2.35, there is a sequence $\{ U_1, \ldots, U_{t_1-l} \}$ of flags in $\Delta(\mathcal{P})$, such that $U_i$ is supported by $\Sigma'_i$ and $U_i$ and $U_{i+1}$ are adjacent.

Part IV.

If $S_1 \ldots S_l S_{t_1} = \gamma_1$, by the induction step shown in part II, there is a sequence of maximal flags $\{ F_1, \ldots, F_n \}$ in $\Delta(\mathcal{P})$ which connect $U_{t_1-l}$ to a maximal flag B which contains $F_C$. Since A $= U_1$, $U_{t_1-l} = F_1$, B $= F_n$, and $U_i$ and $U_{i+1}$ are adjacent maximal flags , the sequence $\{ U_1, \ldots, U_{t_1-l}, F_2, \ldots, F_n \}$ of flags connects A and B.

Part V.

Suppose $S_1 \ldots S_l S_{t_1} \neq \gamma_1$. Then there is a $t_2$, $t_1 < t_2 \leq d$, such that $t_2$ is the smallest integer where $S_{t_2} < \gamma_1$. Since $S_1 \ldots S_l S_{t_1}$ is normal in $S_1 \ldots S_{t_2}$ and $S_1 \ldots S_{t_2-1}$ is normal in $S_1 \ldots S_{t_2}$, proceed as in part III to obtain a sequence of adjacent maximal flags $\{ U_{t_1-l}, \ldots, U_{t_2-(l+1)} \}$ where $U_{t_2-(l+1)} = S_1 \subset S_1S_2 \subset \ldots \subset S_1 \ldots S_l \subset S_1 \ldots S_l S_{t_1} \subset S_1 \ldots S_l S_{t_1} S_{t_2} \subset S_1 \ldots S_l S_{t_1} S_{t_2} S_{l+1} \subset \ldots \subset S_1 \ldots S_l S_{t_1} S_{t_2} S_{l+1} \ldots S_{t_2-1} S_{t_2+1} \subset \ldots \subset S_1 \ldots S_d$. If $S_1 \ldots S_{t_1} S_{t_2} = \gamma_1$, there is a sequence $\{ U_1, \ldots, U_{t_1-l}, U_{t_1-l+1}, \ldots, U_{t_2-(l+1)} \}$ of maximal flags which connect A and $U_{t_2-(l+1)}$. Then proceed as in part IV.

If $S_1 \ldots S_l S_{t_1} S_{t_2} \neq \gamma_1$, repeat the procedure given above until a sequence of adjacent maximal flags $\{ U_1, \ldots, U_{t_1-l}, \ldots, U_{t_2-(l+1)}, \ldots, U_{t_3-(l+2)}, \ldots, U_{t_n-(l+n-1)} \}$ is obtained, such that A $= U_1$ and $U_{t_n-(l+n-1)} = S_1 \subset \ldots \subset S_1 \ldots S_l \subset S_1 \ldots S_l S_{t_1} \subset \ldots \subset S_1 \ldots S_{t_1} \ldots S_{t_n} \subset S_1 \ldots S_{t_1} \ldots S_{t_n} S_{l+1} \subset \ldots \subset S_1 \ldots S_{t_1} \ldots S_{t_n} S_{l+1} \ldots S_{t_1-1} \subset \ldots \subset S_1 \ldots S_{t_1} \ldots S_{t_n} S_{l+1} \ldots S_{t_1-1} S_{t_1+1} \subset \ldots \subset S_1 \ldots S_d$. Since $S_1 \ldots S_{t_1} \ldots S_{t_n} = \gamma_1$, proceed as in part II to obtain a sequence of maximal flags $\{ F_1, \ldots, F_m \}$ which connect $U_{t_n-(l+n-1)}$ to a maximal flag B which contains $F_C$. Hence $\{ U_1, \ldots, U_{t_n-(l+n-1)}, F_2, \ldots, F_m \}$ connects A and B □

**Theorem 2.47** *If G is a solvable nC-group, then $\Delta(\mathcal{P})$ is connected.*

Proof. Let A and B be two flags in $\Delta(\mathcal{P})$. Then A and B are contained in maximal flags $A'$ and $B'$ respectively. By Theorem 2.46, $A'$ and $B'$ are connected to maximal flags C and D respectively, such that $F_C \subset$ C and $F_C \subset$ D. By Theorem 2.45, C and D are connected. Since connectedness defines an equivalence relation on the maximal flags in $\Delta(\mathcal{P})$, $A'$ and $B'$ are connected. Since A $\subset A'$ and B $\subset B'$, A and B are connected □

**Corollary 2.48** *A group G, which is the direct product of groups of square-free order, has $\Delta(\mathcal{P})$ connected.*

Proof. By Theorem 1 of [14], G is a solvable nC-group. Therefore, by Theorem 2.47, $\Delta(\mathcal{P})$ is connected □

# Chapter 3

# Point Transitive and Maximal Flag Transitive Spaces

## 3.1 Point Transitive Spaces

**Definition 3.1** *Col($\Delta(\mathcal{P})$) acts* **tranisitively on the points** *in $\Delta(\mathcal{P})$ if for each pair of points $\alpha$ and $\beta$ in $\Delta(\mathcal{P})$, there is a $\sigma \in$ Col($\Delta(\mathcal{P})$) such that $\alpha^{\sigma} = \beta$.*

**Lemma 3.2** *If Col($\Delta(\mathcal{P})$) acts transitively on the points in $\Delta(\mathcal{P})$, then for each pair of points $\alpha$ and $\beta$ in $\Delta(\mathcal{P})$, $\alpha \cong \beta$.*

Proof. By definition, there is a $\sigma \in$ Col($\Delta(\mathcal{P})$) such that $\alpha^{\sigma} = \beta$. Hence $\alpha \cong \beta$ □

**Theorem 3.3** *Let G be an abelian group. Col($\Delta(\mathcal{P})$) acts transitively on the points in $\Delta(\mathcal{P})$ if and only if G is a homocyclic p-group.*

Proof. Let $\Sigma = \{ \langle a_1 \rangle, \ldots \langle a_n \rangle \}$ be a frame for G. By Theorem 2.14, each $\langle a_i \rangle$, for $1 \leq i \leq n$, is a point in $\Delta(\mathcal{P})$. Thus by Lemma 3.2, for each $j$, $1 \leq j \leq n$, $\langle a_i \rangle \cong \langle a_j \rangle$. Hence G is a homocyclic p-group.

Conversely, let $\alpha$ and $\beta$ be two points in $\Delta(\mathcal{P})$. Then there are two frames $\Sigma_1 = \{\alpha, S_2, \ldots, S_n \}$ and $\Sigma_2 = \{ \beta, T_2, \ldots, T_n \}$ for G. Since G is homocyclic, $\alpha = \langle a \rangle$, $\beta = \langle b \rangle$,

$S_i = \langle s_i \rangle$, and $T_j = \langle t_j \rangle$ where $\alpha \cong \beta \cong S_i \cong T_j$. Define a map of G as follows:

$$\begin{aligned} \Psi : \quad a &\mapsto b \\ s_i &\mapsto t_i \quad 2 \leq i \leq n. \end{aligned}$$

Since G is homocyclic, $\Psi$ is an automorphism of G. Thus $\Psi$ induces a collineation which maps $\alpha$ to $\beta$ □

**Theorem 3.4** *Suppose that $G = [A]B$ where $A$ is an elementary abelian p-group and $B$ is an elementary abelian q-group with $p \neq q$. If $Col(\Delta(\mathcal{P}))$ acts transitively on the points in $\Delta(\mathcal{P})$, then the commutator subgroup $G' = A$.*

Proof. If G is abelian, then by Theorem 3.3, G is a homocyclic p-group. This is a contradiction as $p \neq q$. Thus G is non-abelian and $G'$ is non-trivial.

Since A and B are abelian, $G'$ = [A,B]. Suppose $G' <$ A. By 5.2.3 of [13], A = [A,B] $\oplus$ $C_A$(B), where $C_A(\text{B}) = \langle a \in \text{A} \mid ab = ba \text{ for all } b \in \text{B} \rangle$. Thus $C_A(\text{B}) \neq \{ 1 \}$. Furthermore, since G splits over A and A is elementary abelian, each 1-dimensional subspace of A is a point in $\Delta(\mathcal{P})$.

Let $G' = \langle a_1, \ldots, a_n \rangle$ and $b \in$ B. Then $\langle a_1, \ldots, a_n, b \rangle$ is a normal subgroup in G. G has elementary abelian Sylow subgroups, so by 2.3 of [1], G is an nC-group. Thus G splits over $\langle a_1, \ldots, a_n, b \rangle$ and $\langle a_1, \ldots, a_n, b \rangle$ is a rank 1 flag in $\Delta(\mathcal{P})$. Let $\langle a \rangle \subseteq C_A(\text{B})$. Since $\langle a \rangle$ and $\langle a_1 \rangle$ are points in $\Delta(\mathcal{P})$, there is a $\sigma \in \text{Col}(\Delta(\mathcal{P}))$ such that $\langle a_1 \rangle^\sigma = \langle a \rangle$. Since $\langle a_1 \rangle \subseteq \langle a_1, \ldots, a_n, b \rangle$, $\langle a_1 \rangle^\sigma = \langle a \rangle \subseteq \langle a_1, \ldots, a_n, b \rangle^\sigma$.

Now $\langle a_1, \ldots, a_n, b \rangle^\sigma \cong \langle a_1, \ldots, a_n, b \rangle$. Therefore, there is an element of order $q$ in $\langle a_1, \ldots, a_n, b \rangle^\sigma$. Let $a'b'$ be that element. Since $\langle a \rangle \subseteq \langle a_1, \ldots, a_n, b \rangle^\sigma$ and $\langle a \rangle \subseteq C_A(\text{B})$, $a^{a'b'} = a$. Thus $\text{Z}(\langle a_1, \ldots, a_n, b \rangle^\sigma) \neq \{ 1 \}$. But $G' = \langle a_1, \ldots, a_n \rangle$ and $\text{Z}(\langle a_1, \ldots, a_n, b \rangle) = \{ 1 \}$. This is a contradiction since $\langle a_1, \ldots, a_n, b \rangle^\sigma \cong \langle a_1, \ldots, a_n, b \rangle$. Thus $C_A(\text{B}) = \{ 1 \}$

and $G' = [A.B] = A$ □

**Theorem 3.5** *Let G be a solvable nC-group of derived length m. If Col(Δ(𝒫)) acts transitively on the points in Δ(𝒫), then F(G) =* $G^{m-1}$.

Proof. By Theorem 2.43, each inseparable subgroup of F(G) is a point in G. By Lemma 3.2, all the points in G are isomorphic. Therefore, F(G) is an elementary abelian p-group.

If G is abelian, then F(G) = G and m = 1. Thus F(G) = $G^{m-1}$ = G. Suppose that G is non-abelian. Then the Fitting length of G is greater than 1. Let N/F(G) be a minimal normal subgroup of G/F(G). Suppose that | N/F(G) | = $p^{\alpha}$. Since N is normal in G and Φ(G) = { 1 }, Φ(N) = { 1 }. This implies F(G) = N, a contradiction. Thus for each minimal normal subgroup N/F(G) of G/F(G), | N/F(G) | = $q^{\alpha}$ with $p \neq q$.

$G^{m-1}$ is abelian and hence nilpotent, so $G^{m-1} \leq$ F(G). Suppose $G^{m-1} <$ F(G). Since F(G) is abelian, F(G) ⊂ $G^{m-2}$. Since $G^{m-2}$ is a solvable nC-group (by 3.5 of [10]), $G^{m-2}$ = [F(G)]B. Let N/F(G) be a minimal normal subgroup in G/F(G) such that N/F(G) ≤ $G^{m-2}$/F(G). Then N = F(G)(B ∩ N), where | $B \cap N$ | = $q^{\alpha}$ with $p \neq q$. Since $G^{m-1} <$ F(G), [F(G),B ∩ N] ≤ $G^{m-1}$. By a method similar to the one used in the proof of Theorem 3.4, it is proven that $G^{m-1}$ = F(G) □

## 3.2 Maximal Flag Transitive Spaces

Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in Δ(𝒫). Associated with A is at least one ordered pair (A,Σ) where Σ is a frame which supports A.

**Definition 3.6** *Let G be a group and Δ(𝒫) be its associated flag space. Col(Δ(𝒫)) acts* **transitively** *on the maximal flags in Δ(𝒫) if for each pair of ordered pairs (A,Σ) and (B,Σ′), there is a σ ∈ Col(Δ(𝒫)) such that* $A^{\sigma} = B$.

**Lemma 3.7** *If $Col(\Delta(\mathcal{P}))$ acts transitively on the maximal flags in $\Delta(\mathcal{P})$, then*

*i) All the maximal flags are of the same rank;*

*ii) If A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ are two maximal flags in $\Delta(\mathcal{P})$, then for each $i$, $1 \leq i \leq d$, $\alpha_i \cong \beta_i$.*

Proof. i) This follows from the fact that if A is a flag in $\Delta(\mathcal{P})$ and $\sigma \in$ Col($\Delta(\mathcal{P})$), then A and $A^\sigma$ have the same rank.

ii) Since A and B are maximal in $\Delta(\mathcal{P})$, there is a $\sigma \in$ Col($\Delta(\mathcal{P})$) such that $A^\sigma$ = B. Thus $A^\sigma = (\alpha_1 \subset \ldots \subset \alpha_d)^\sigma = \alpha_1^\sigma \subset \ldots \subset \alpha_d^\sigma = \beta_1 \subset \ldots \subset \beta_d$. Hence $\alpha_i \cong \beta_i$ □

The converse to Lemma 3.7 is not true. Consider the group G = $[Z_3 \times Z_3]Z_3 = \langle a, b, c \mid ab = ba, a^c = a, b^c = ab \rangle$. If G = [N]K, where K is inseparable, then $\mid N \mid = p^2$ and N is elementary abelian. Therefore, each maximal flag in $\Delta(\mathcal{P})$ is of rank 2. For each pair A: $\alpha_1 \subset \alpha_2$ and B: $\beta_1 \subset \beta_2$, of maximal flags in $\Delta(\mathcal{P})$, $\alpha_1 \cong \beta_1$ and $\alpha_2 \cong \beta_2$. Conditions i) and ii) of Lemma 3.7 hold, yet Col($\Delta(\mathcal{P})$) does not act transitively on $\Delta(\mathcal{P})$.

**Note:** Let (A,$\Sigma$) and (B,$\Sigma'$) be two maximal flag, frame pairs for a group G with $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$. Let $\sigma \in$ Col($\Delta(\mathcal{P})$) be a collineation, such that $A^\sigma$ = B. Then $A^\sigma = (S_1 \subset S_1S_2 \subset \ldots \subset S_1S_2\ldots S_d)^\sigma = S_1^\sigma \subset (S_1S_2)^\sigma \ldots \subset (S_1S_2\ldots S_d)^\sigma = T_1 \subset T_1T_2 \subset \ldots \subset T_1T_2\ldots T_d$. Therefore, $S_1 \cong T_1$, and $\mid S_1 \mid = \mid T_1 \mid$. Since $S_1S_2 \cong T_1T_2$ and $\mid S_1 \mid = \mid T_1 \mid$, $\mid S_2 \mid = \mid T_2 \mid$. Continuing in this manner results in $\mid S_i \mid = \mid T_i \mid$ for $1 \leq i \leq d+1$.

**Theorem 3.8** *Let G be an abelian group. $Col(\Delta(\mathcal{P}))$ acts transitively on the maximal flags in $\Delta(\mathcal{P})$ if and only if G is a homocyclic p-group.*

Proof. Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ be a frame for G. Since G is abelian, for each $i$ and $j$, $1 \leq i, j \leq d+1$, $\Sigma_i = \{ S_i, S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{d+1} \}$ and $\Sigma_j = \{ S_j, S_1, \ldots, S_{j-1},$

$S_{j+1}, \ldots, S_{d+1}$ } are frames for G. $\Sigma_i$ and $\Sigma_j$ support maximal flags in $\Delta(\mathcal{P})$. Therefore, $S_i \cong S_j$ and G is a homocyclic p-group.

Conversely, let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ be two maximal flags in $\Delta(\mathcal{P})$. Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$ be frames for G which support them respectively. Let $S_i = \langle s_i \rangle$ and $T_j = \langle t_j \rangle$ where $1 \leq i, j \leq d+1$ and $| s_i | = | t_j | = p^n$. Define the map $\sigma$, such that $\sigma(s_i) = t_i$. Since G is homocylic, $\sigma$ is an automorphism of G. Therefore $S_i^\sigma = T_i$ and $A^\sigma$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma = S_1^\sigma \subset (S_1S_2)^\sigma \subset \ldots \subset (S_1 \ldots S_d)^\sigma = T_1 \subset T_1T_2 \subset \ldots \subset T_1 \ldots T_d = B$ □

**Theorem 3.9** *Let A be a maximal flag in a flag space $\Delta(\mathcal{P})$ and let C be the subgroup of Col($\Delta(\mathcal{P})$) which fixes A. If $Aut_G(\Delta(\mathcal{P}))$ acts transitively on the maximal flags in $\Delta(\mathcal{P})$, then Col($\Delta(\mathcal{P})$) = C $Aut_G(\Delta(\mathcal{P}))$.*

Proof. Let $\phi \in$ Col($\Delta(\mathcal{P})$). Then $A^\phi$ = B is a maximal flag in $\Delta(\mathcal{P})$. Since $Aut_G(\Delta(\mathcal{P}))$ acts transitively on the maximal flags of $\Delta(\mathcal{P})$, there is a $\sigma \in Aut_G(\Delta(\mathcal{P}))$ such that $A^\sigma$ = B. Therefore, $A^\phi = A^\sigma$ and $A^{\phi\sigma^{-1}}$ = A. Thus $\phi\sigma^{-1} = c \in$ C. Hence $\phi = c\sigma$ and Col($\Delta(\mathcal{P})$) = C $Aut_G(\Delta(\mathcal{P}))$ □

**Corollary 3.10** *Let G be an elementary abelian p-group of rank n. If A is a maximal flag in $\Delta(\mathcal{P})$ and C is the subgroup of Col($\Delta(\mathcal{P})$) which fixes A, then Col($\Delta(\mathcal{P})$) = C PGL(n,p).*

Proof. By 3.3.11 of [29], PGL(n,p) acts transitively on the maximal flags in $\Delta(\mathcal{P})$. By Theorem 3.9, Col($\Delta(\mathcal{P})$) = C PGL(n,p) □

**Lemma 3.11** *Let G be a group such that all the maximal flags in $\Delta(\mathcal{P})$ are of the same rank. Let B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be the primary flag of $\Delta(\mathcal{P})$ and $\Phi \in$ Col($\Delta(\mathcal{P})$). If $\beta_i$ is complemented in G for $1 \leq i \leq e$, then $\Phi$ induces a collineation of $\Delta(\mathcal{P}_{G/\beta_i})$.*

Proof. Let A: $\alpha_1/\beta_i \subset \ldots \subset \alpha_d/\beta_i \in \Delta(\mathcal{P}_{G/\beta_i})$. By Lemma 2.20 ii), $A_o$: $\alpha_1 \subset \ldots \subset \alpha_d$ is a flag in $\Delta(\mathcal{P})$. Let $(\alpha_1 \subset \ldots \subset \alpha_d)^{\Phi} = \alpha_1' \subset \ldots \subset \alpha_d'$ and define $A^{\Phi_{G/\beta_i}}$ to be $\alpha_1'/\beta_i \subset \ldots \subset \alpha_d'/\beta_i$.

Since $(\beta_i)^{\Phi} = \beta_i$, $\Phi_{G/\beta_i}$ induces a bijection on the flags of $\Delta(\mathcal{P}_{G/\beta_i})$. Let A: $\alpha_1/\beta_i \subset \ldots \subset \alpha_d/\beta_i$ and C: $\gamma_1/\beta_i \subset \ldots \subset \gamma_f/\beta_i$ be flags in $\Delta(\mathcal{P}_{G/\beta_i})$. By Lemma 2.20 ii), $A_o$: $\alpha_1 \subset \ldots \subset \alpha_d$ and $C_o$: $\gamma_1 \subset \ldots \subset \gamma_f$ are flags in $\Delta(\mathcal{P})$ and $(A_o \wedge C_o)^{\Phi} = A_o^{\Phi} \wedge C_o^{\Phi}$. Let E: $\varepsilon_1/\beta_i \subset \ldots \subset \varepsilon_g/\beta_i$ = A $\wedge$ C. Then $E_o : \varepsilon_1 \subset \ldots \subset \varepsilon_g$ is contained in $A_o$ and $C_o$, and $E_o \subseteq A_o \wedge C_o$. If $E_o \subset D_o = \delta_1 \subset \ldots \subset \delta_h = A_o \wedge C_o$, then D: $\delta_1/\beta_i \subset \ldots \subset \delta_h/\beta_i \subsetneq$ A $\wedge$ C. This is a contradiction and $E_o = A_o \wedge C_o$. Therefore, $(E_o)^{\Phi} = A_o^{\Phi} \wedge C_o^{\Phi}$. By definition, $(\mathrm{A} \wedge \mathrm{C})^{\Phi_{G/\beta_i}} = \mathrm{A}^{\Phi_{G/\beta_i}} \wedge \mathrm{B}^{\Phi_{G/\beta_i}}$ □

**Theorem 3.12** *Suppose that Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$ and that B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ is the primary flag in $\Delta(\mathcal{P})$. If $\beta_i$ is complemented for some i, $1 \leq i \leq e$, then Aut($\Delta(\mathcal{P}_{G/\beta_i})$) acts transitively on $\Delta(\mathcal{P}_{G/\beta_i})$.*

Proof. Any frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ of G supports B. Thus for some $j$, $1 \leq j \leq n-1$, $S_1S_2\ldots S_j = \beta_i$.

Let A: $\alpha_1/\beta_i \subset \ldots \subset \alpha_s/\beta_i$ and $A'$: $\alpha_1'/\beta_i \subset \ldots \subset \alpha_t'/\beta_i$ be two maximal flags in $\Delta(\mathcal{P}_{G/\beta_i})$. By Lemma 2.22 ii), $A_o$: $S_1 \subset S_1S_2 \subset \ldots \subset \beta_i \subset \alpha_1 \subset \ldots \subset \alpha_s$ and $A_o'$: $S_1 \subset S_1S_2 \subset \ldots \beta_i \subset \alpha_1' \subset \ldots \subset \alpha_t'$ are maximal flags in $\Delta(\mathcal{P})$. Hence there is a $\Phi \in$ Col($\Delta(\mathcal{P})$) such that $A_o^{\Phi} = A_o'$. By Lemma 3.11, $\Phi$ induces a collineation of $\Delta(\mathcal{P}_{G/\beta_i})$ which maps $\alpha_1/\beta_i \subset \ldots \subset \alpha_s/\beta_i$ to $\alpha_1'/\beta_i \subset \ldots \subset \alpha_t'/\beta_i$. Thus $s = t$ and Aut($\Delta(\mathcal{P}_{G/\beta_i})$) acts transitively on the maximal flags in $\Delta(\mathcal{P}_{G/\beta_i})$ □

**Lemma 3.13** *Let $\alpha$ be a rank 1 flag in $\Delta(\mathcal{P})$ and $\Psi \in$ Col($\Delta(\mathcal{P})$), such that $\Psi(\alpha) = \alpha$. Then $\Psi$ induces a collineation of $\Delta(\mathcal{P}_{\alpha})$.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a flag in $\Delta(\mathcal{P}_\alpha)$. By Lemma 2.17, A is a flag in $\Delta(\mathcal{P})$. Then $A^\Psi$ = B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ is a flag in $\Delta(\mathcal{P})$, such that $\beta_d \subset \alpha$. Therefore, there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G which supports B, such that for some $i$, $1 \leq i \leq n$, $S_1 \ldots S_i = \alpha$. Hence $\Sigma' = \{ S_1, \ldots, S_i \}$ is a frame for $\alpha$ which supports B and B is a flag in $\Delta(\mathcal{P}_\alpha)$. This results in $\Psi$ inducing a map on $\Delta(\mathcal{P}_\alpha)$ denoted by $\Psi_\alpha$

Since $\alpha^\Psi = \alpha$, $\Psi_\alpha$ induces a bijection on the flags in $\Delta(\mathcal{P}_\alpha)$. Let A and B be two flags in $\Delta(\mathcal{P}_\alpha)$. By Lemma 2.17, A and B are flags in $\Delta(\mathcal{P})$ and $(A \wedge B)^\Psi = A^\Psi \wedge B^\Psi$. From the definition of $\Psi_\alpha$, $(A \wedge B)^{\Psi_\alpha} = A^{\Psi_\alpha} \wedge B^{\Psi_\alpha}$ and $\Psi_\alpha$ is a collineation of $\Delta(\mathcal{P}_\alpha)$ □

**Theorem 3.14** *Let G be a group and $\Delta(\mathcal{P})$ its flag space. Let $\alpha$ be a rank 1 flag in $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P}_\alpha)$ its associated flag space. If Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$, then Aut($\Delta(\mathcal{P}_\alpha)$) acts transitively on the maximal flags in $\Delta(\mathcal{P}_\alpha)$.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P}_\alpha)$. Then A and B can be extented to two maximal flags $A_o$: $\alpha_1 \subset \ldots \subset \alpha_d \subset \alpha \subset \gamma_1 \subset \ldots \subset \gamma_n$ and $B_o$: $\beta_1 \subset \ldots \subset \beta_e \subset \alpha \subset \gamma_1 \subset \ldots \subset \gamma_n$ in $\Delta(\mathcal{P})$. Therefore there is a $\Psi \in$ Col($\Delta(\mathcal{P})$), such that $A_o^\Psi = B_o$. Consequently, d = e and by Lemma 3.13, $\Psi_\alpha \in$ Aut($\Delta(\mathcal{P}_\alpha)$). Since $A_o^\Psi = B_o$, $A^{\Psi_\alpha} = B$ □

The converse to Theorem 3.14 is not valid. A flag space $\Delta(\mathcal{P})$, in which each rank 1 flag of $\Delta(\mathcal{P})$ has a maximal flag transitive flag space associated with it, is not necessarily maximal flag transitive. Consider the dihedral group $D_8$. In $\Delta(\mathcal{P}_{D_8})$, the rank 1 flags are either elementary abelian 2-groups or cyclic of order 4. Thus by Theorem 3.8, each has a transitive flag space. However, the $\Delta(\mathcal{P}_{D_8})$ has two maximal flags of different ranks and thus Aut($\Delta(\mathcal{P}_{D_8})$) can't act transitively on the maximal flags in $\Delta(\mathcal{P}_{D_8})$.

By Theorem 3.8, if G is an elementary abelian p-group, then Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$. This result is extended to groups G = [A]B where A is an elementary abelian p-group and B is an elementary abelian q-group with p $\neq$ q.

**Theorem 3.15** *Let $G = [A]B$, where $A$ is an elementary abelian p-group and $B$ is an elementary abelian q-group with $p \neq q$. Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$ if and only if $B$ acts fixed point freely on $A$.*

Proof. Let $\mid A \mid = p^{\alpha}$ and $\mid B \mid = q^{\beta}$. A is the unique Sylow p-subgroup of G since A is normal in G with $p \neq q$. G has elementary abelian Sylow subgroups, so by Theorem 2.3 of [1], G is an nC-group. By Theorem 2.42, each element of each splitting system is of prime order. Thus all the frames for G are of the same length or else G has two different prime factorizations. It follows that all the maximal flags in $\Delta(\mathcal{P})$ are of the same rank.

Suppose that Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$ and that there is a $b \in$ B, such that $C_A(\mathrm{b}) = \langle a \in \mathrm{A} \mid ab = ba \rangle \neq \{ 1 \}$. Let $A_1 = C_A(\mathrm{b})$. If there is a $b' \in$ B such that $b' \notin \langle b \rangle$ and for some $a_1 \in A_1$, $(a_1)^{b'} \notin A_1$, then $(a_1)^{b'} = xy$ where $x \notin A_1$ and $y \in A_1$. Since $bb' = b'b$,

$$\begin{aligned}
(a_1)^{bb'} &= (a_1)^{b'b} \\
(a')^{b'} &= (xy)^b \\
xy &= x^b y^b \\
xy &= x^b y \\
x &= x^b.
\end{aligned}$$

This is a contradiction unless $x = 1$. Thus $A_1$ is normal in G.

Since G is an nC-group, G = $[A_1]$K for some subgroup K of G. A is the unique Sylow p-subgroup of G, hence A $\cap$ K is the unique Sylow p-subgroup of K. Since G = $[A_1]$K and A $\cap$ K is normal in A, A = (A $\cap$ K) $\times A'$. Thus G = [(A $\cap$ K)] $\times A'$]B and there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G such that A = $S_1 S_2 \ldots S_\alpha$, A $\cap$ K = $S_1 \ldots S_{i-1}$, $A' = S_i \ldots S_\alpha$, for $1 \leq i \leq \alpha - 1$, and $S_{\alpha+1} = \langle b \rangle$.

Now $\langle b \rangle$ commutes elementwise with $S_\alpha$. Since $A \cap K \lhd G$, then $S_1 \ldots S_{\alpha-1} \lhd S_1 \ldots S_{\alpha+1}$. By Lemma 1.7, $\Sigma' = \{ S_1, \ldots, S_{\alpha-1}, \langle b \rangle, S_\alpha, \ldots, S_n \}$ is also a frame for G. Therefore C: $S_1 \subset S_1 S_2 \subset \ldots \subset S_1 \ldots S_{n-1}$ and D: $S_1 \subset \ldots \subset S_1 \ldots S_{\alpha-1} \subset S_1 \ldots S_{\alpha-1}\langle b \rangle \subset S_1 \ldots S_{\alpha-1}\langle b \rangle S_{\alpha+1} \subset \ldots \subset S_1 \ldots S_{n-1}$ are maximal flags in $\Delta(\mathcal{P})$. Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$, so there is a $\sigma \in$ Col($\Delta(\mathcal{P})$) such that $C^\sigma =$ D. This implies that $p^\alpha = \mid S_1 \ldots S_\alpha \mid = \mid S_1 \ldots S_{\alpha-1}\langle b \rangle \mid = p^{\alpha-1}q$. This is a contradiction and thus B acts fixed point freely on A.

Conversely, suppose that B acts fixed point freely on A. If B is non-cyclic, then by 5.3.16 of [13], $A = \Pi_{b \in B^*} C_A(b)$ where $B^*$ is the collection non-identity elements of B. By our hypothesis, $C_A(b) = \{ 1 \}$ for each $b \in$ B, therefore B must be cyclic and $\beta = 1$. Let B $= \langle b \rangle$.

Suppose there is an $a \in$ A such that $b^a = b^\gamma$ where $2 \leq \gamma \leq q - 1$. Then

$$\begin{aligned} a^{-1}ba &= b^\gamma \\ aba &= a^2 b^\gamma \\ b^{-1}aba &= b^{-1}a^2 b^\gamma \\ a^b &= b^{-1}a^2 b^\gamma a^{-1}. \end{aligned}$$

Since G = AB and A $\lhd$ G, then $b^{-1}a^2 = a'b^{-1}$ where $a' \in$ A. Therefore, $a^b = a'b^{-1}b^\gamma a^{-1} = a^{-1}b^{\gamma-1}a^{-1}$. By the same reasoning, $b^{\gamma-1}a^{-1} = a''b^{\gamma-1}$ where $a'' \in$ A. Thus $a^b = a'a''b^{\gamma-1}$. But A is normal in G, so $\gamma - 1 = 0$ and $\gamma = 1$. This implies that $b^a = b$ or that $a^b = a$. This contradicts the fact that B acts fixed point freely on A. Hence for each $a \in$ A, $a \neq 1$, $b^a = a_1 b^\gamma \notin \langle b \rangle$ where $a_1 \in$ A.

Let $\langle x\rangle$ be a Sylow $q$-subgroup of G. Then $\langle x\rangle$ is a complement to A in G. By an argument similar to the one presented in the previous paragraph, $x^a \notin \langle x\rangle$ for all $a \in$ A.

Now suppose there is a maximal flag C: $\alpha_1 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, such that for some $l$, $1 \leq l \leq d-1$, $|\ \alpha_l\ | = p^\delta$, $1 \leq \delta < \alpha$ and $|\ \alpha_{l+1}\ | = p^\delta q$. Then there is a frame $\Sigma = \{\ S_1, S_2, \ldots, S_{d+1}\ \}$ which supports C such that $|\ S_1 \ldots S_l\ | = p^\delta$ and $|\ S_{l+1}\ | = q$. Let $S_{l+1} = \langle x\rangle$, which is a Sylow q-subgroup of $S_1 \ldots S_{l+1}$. Since $S_1 \ldots S_{l+1} \lhd S_1 \ldots S_{l+2}$, for each $s \in S_{l+2}$, $\langle x\rangle^s$ is a Sylow q-subgroup of $S_1 \ldots S_{l+1}$. Therefore, for some $s' \in S_1 \ldots S_{l-1}$, $\langle x\rangle^s = \langle x\rangle^{s'}$ and $\langle x\rangle^{s's^{-1}} = \langle x\rangle$. This implies there is a non-trivial element $a = s's^{-1} \in$ A such that $x^a \in \langle x\rangle$. This is a contradiction.

Therefore, for each maximal flag C: $\alpha_1 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, $|\ \alpha_d\ | = p^\alpha$. If C: $\alpha_1 \subset \ldots \subset \alpha_d$ and D: $\beta_1 \subset \ldots \subset \beta_d$ are two maximal flags in $\Delta(\mathcal{P})$, there are two frames $\Sigma = \{\ S_1, S_2, \ldots, S_{d+1}\ \}$ and $\Sigma' = \{\ T_1, T_2, \ldots, T_{d+1}\ \}$ which support C and D respectively. Since $|\ S_1 \ldots S_d\ | = |\ T_1 \ldots T_d\ | = p^\alpha$, $\{\ s_1, \ldots, s_d\ \}$ and $\{\ t_1, \ldots, t_d\ \}$ where $S_i = \langle s_i\rangle$ for $1 \leq i \leq d$ and $T_j = \langle t_j\rangle$ for $1 \leq j \leq d$ are bases for A. A is an elementary abelian p-group and there is a $\sigma \in$ Aut(A) such that $(s_i)^\sigma = t_i$ for $1 \leq i \leq d$. Thus $C^\sigma =$ D. Since A is the unique subgroup over which G splits such that G/A is inseparable, $A^\sigma =$ A and $\sigma$ induces an collineation on $\Delta(\mathcal{P})$. Thus Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in the flag space $\Delta(\mathcal{P})$ □

**Theorem 3.16** *Let $G = [A]B$ where $A$ is an elementary abelian p-group and $B$ is an elementary abelian q-group with $p \neq q$. If Col($\Delta(\mathcal{P})$) acts transitively on $\Delta(\mathcal{P})$, then $G \neq H \times K$ for proper subgroups $H$ and $K$.*

Proof. Suppose G = H×K where $|\ H\ | = p^{\alpha_1} q^{\beta_1}$ and $|\ K\ | = p^{\alpha_2} q^{\beta_2}$.

**Case 1)** $\beta_1$ or $\beta_2 = 0$. Assume without loss of generality that $\beta_1 = 0$. Then B $\subseteq$ K and for all $b \in$ B, $h^b = h$ for each $h \in$ H. But A $\cap$ H $\neq \{ 1 \}$, so for each $b \in$ B, $C_A(b) \neq \{ 1 \}$. This contradicts Theorem 3.15.

**Case 2)** $\alpha_1$ or $\alpha_2 = 0$. Assume w.l.o.g. that $\alpha_1 = 0$, then A $\subseteq$ K. Let $h \in$ H. Since H is a q-group, H $\subset$ Q, a Sylow q-subgroup of G. Then there is an $x \in$ A such that $B^x =$ Q. Thus for some $b \in$ B, $b^x = h$. Since A $\subseteq$ K, for each $a \in$ A, $a^h = a$ and $a^{b^x} = a$. This implies $a^{x^{-1}b} = a^{x^{-1}}$ and $a^b = a$. Thus $C_A(b) =$ A, which again contradicts Theorem 3.15.

**Case 3)** $\alpha_1, \alpha_2, \beta_1$, and $\beta_2 \neq 0$. Let $h \in$ H. Then $h = ab$ where $a \in$ A and $b \in$ B. Since $\alpha_2 \neq 0$, A $\cap$ K $\neq \{ 1 \}$. Let $x \in$ A $\cap$ K. Then $x^h = x$ and $x^{ab} = x$. This implies $x^b = x$. Therefore, $C_A(b) \neq \{ 1 \}$. This also contradicts Theorem 3.15 □

## 3.3 Maximal Flag Transitivity for all Subgroups

If G is an elementary abelian p-group, then Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$. Furthermore, for any subgroup H of G, Col($\Delta(\mathcal{P}_H)$) acts transitively on $\Delta(\mathcal{P}_H)$. Groups are now studied which have this property. In this section, it will be said that a flag space is transitive or that Col($\Delta(\mathcal{P})$) acts transitively on $\Delta(\mathcal{P})$ to mean that Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$.

**Theorem 3.17** *Let G be a group and $\Delta(\mathcal{P})$ its flag space. Suppose for each subgroup H of G, Aut($\Delta(\mathcal{P}_H)$) acts transitively on the maximal flags in $\Delta(\mathcal{P}_H)$. Then each Sylow p-subgroup P of G is of one of the following types:*

*I) P is elementary abelian;*

*II) P is cyclic;*

*III) P is the generalized quaternion group;*

*IV) P contains a maximal normal abelian subgroup N which is elementary abelian of rank $\geq 2$ and characteristic in P. Furthermore, $Exp(P) = p^2$ and for elements $x$ and $y$ not in N, $|x| = |y| = p^2$ and $x^p = y^p \in N$.*

**Proof.** Let P be a Sylow p-subgroup of G. Assume that it is not elementary abelian. Suppose each maximal abelian normal subgroup of P is cyclic. If $p \geq 3$, then by III.7.6 of [16], P is cyclic. If $p = 2$, then P is cyclic or has a cyclic subgroup N of index 2 in P. Suppose P is non-cyclic and that N is complemented in P. Then there is an $x \in$ P such that $|x| = 2$ and P = $[N]_\theta \langle x \rangle$.

Suppose that $\theta \neq$ Id. If $|N| = 4$, then P $\cong D_8$. The flag space of $D_8$ is not transitive, hence $|N| \geq 2^3$. Since $\theta \neq$ Id, there is a normal subgroup M of P such that M < N, $|M| = 4$, M is cyclic, and M$\langle x \rangle$ is isomorphic to $D_8$ or the abelian group $Z_4 \times Z_2$. The flag space of both groups is not transitive. This is a contradiction. Thus $\theta =$ Id. N and $\langle x \rangle$ are inseparable, so A: N and B: $\langle x \rangle$ are maximal flags in $\Delta(\mathcal{P}_P)$. Since $|N| \neq |\langle x \rangle|$, Aut($\Delta(\mathcal{P}_P)$) does not act transitively on $\Delta(\mathcal{P}_P)$. This is also a contradiction.

Hence there is an element $x \notin$ N such that $|x| = 4$. Thus $x^2 \in$ N. This implies that P has only one element of order 2. By III.8.2 b) of [16], G is cyclic or the generalized quaternion group.

Now suppose there is a maximal normal abelian subgroup N of P which is non-cyclic. N is an abelian p-group whose flag space is transitive. By Theorem 3.8, N is a homocyclic p-group. Thus N $\cong Z_{p^r} \oplus \ldots \oplus Z_{p^r} = \langle n_1, \ldots, n_t \mid n_i^{p^r} = 1 \text{ for } 1 \leq i \leq t \text{ and } n_i n_j = n_j n_i \text{ for } 1 \leq i, j \leq t \rangle$. N is non-cyclic and $t \geq 2$. Suppose that $r \geq 2$. Then let L = $\langle n_1 \rangle \times \langle n_2^p \rangle$. The subgroups $\langle n_1 \rangle$ and $\langle n_2^p \rangle$ are inseparable, and $\langle n_1 \rangle$ and $\langle n_2^p \rangle$ are maximal flags in $\Delta(\mathcal{P}_L)$. But $|\langle n_1 \rangle| \neq |\langle n_2^p \rangle|$ and Col($\Delta(\mathcal{P}_L)$) does not act transitively on $\Delta(\mathcal{P}_L)$. This is a contradiction. Hence r = 1 and N is elementary abelian.

Suppose there is an $x \in$ P such that $|\, x \,| = p$ and $x \notin$ N. Then L = N$\langle x \rangle$ is a subgroup of P. By III.7.3 of [16], $\langle x \rangle \notin C_P(\mathrm{N}) = \langle y \in \mathrm{P} \mid yn = ny$ for each element $n \in$ N $\rangle$. Let $1 \lhd N_1 \lhd N_2 \lhd \ldots \lhd N_{t-1} \lhd N \lhd L$ be an invariant series for L. The subgroup $\langle x \rangle$ acts on $N_1$ and $|\, N_1 \,| = |\, \langle x \rangle \,| = p$. Therefore $N_1$ and $\langle x \rangle$ commute element-wise. Thus there is an $i$, $1 \leq i \leq t-1$ such that $N_i$ and $\langle x \rangle$ commute element-wise and $N_{i+1}$ and $\langle x \rangle$ do not. Since $N_{i+1}$ is elementary abelian, $N_{i+1} = N_i \times \langle y \rangle$ where $y \in N_{i+1}$ and $\langle y \rangle$ and $\langle x \rangle$ do not commute element-wise. Thus $y^x = ny$ where $n \in N_i$. Consider the group K = [ $\langle n, y \rangle$ ] $\langle x \rangle$. If $p = 2$, then K $\cong D_8$. The flag space associated with $D_8$ is not transitive and this is a contradiction. Assume that $p \geq 3$.

Consider the following two rank 1 flags $\langle n, y \rangle$ and $\langle n, x \rangle$ in $\Delta(\mathcal{P}_K)$. Each has order $p^2$. Now consider the two maximal flags $\langle n \rangle \subset \langle n, x \rangle$ and $\langle x \rangle \subset \langle n, x \rangle$. K < G and Aut($\Delta(\mathcal{P}_K)$) acts transitively on $\Delta(\mathcal{P}_K)$. Thus there is a $\Psi \in$ Aut($\Delta(\mathcal{P}_K)$) such that $(\langle n \rangle \subset \langle n, x \rangle)^\Psi = \langle x \rangle \subset \langle n, x \rangle$. By Theorem 2.29, $(\langle n \rangle)^\Psi = \langle x \rangle$ and $(\langle n, x \rangle)^\Psi = \langle n, x \rangle$. Therefore, $(\langle n \rangle \subset \langle n, y \rangle)^\Psi = \langle x \rangle \subset (\langle n, y \rangle)^\Psi$. But the only rank 1 flag of K which contains $\langle x \rangle$ is $\langle n, x \rangle$. This implies that $(\langle n, y \rangle)^\Psi = \langle n, x \rangle$, a contradiction. Thus Aut($\Delta(\mathcal{P}_K)$) does not act transitively on $\Delta(\mathcal{P}_K)$. This again is a contradiction. Hence for all $x \in$ P, where $|\, x \,| = p$, $x \in$ N.

If P is of exponent $p$, then N = P and P is elementary abelian. P is assumed not to be elementary abelian. Therefore, there is an $x \notin$ N such that $|\, x \,| = p^m$ with $m \geq 2$. Let $\{ 1 \} \lhd N_1 \lhd \ldots \lhd N_t \lhd \ldots \lhd N_s \lhd N_{s+1} =$ P be an invariant series for P such that $N_t =$ N. $|\, N_1 \,| = p$ and let $N_1 = \langle n_1 \rangle$. Therefore, $n_1^x = n_1$. If $n_1 \notin \langle x \rangle$, then L = $\langle n_1 \rangle \times \langle x \rangle$ is an abelian, non-homocyclic subgroup of P. By Theorem 3.8, for L to have a transitive flag space, L must be homocyclic. This is a contradiction and $n_1 \in \langle x \rangle$. Thus $x^{p^{m-1}} = n_1$. Now consider $N_2$. $|\, N_2 \,| = p^2$ and let $N_2 = \langle n_1, n_2 \rangle$. The element $x$ commutes with $n_1$, so $n_2^x = n_1^u n_2$

where $1 \leq u \leq p$ and $n_2^{x^p} = n_2$. If $m \geq 3$, then L $= \langle n_2, x^p \rangle$ is a non-homocyclic, abelian subgroup of P. By Theorem 3.8, this is a contradiction. Therefore, $m = 2$. Furthermore, for any other element $y \notin$ N and $x \neq y$, a similar argument shows that $| y | = p^2$ and $y^p = n_1 = x^p$. Finally, since each element of order $p$ is in P is in N, $\Omega_1(P)$ = N and N is characteristic in P □

**Theorem 3.18** *Let $G$ be a solvable group and $\Delta(\mathcal{P})$ its flag space. If for each subgroup $H$ of $G$, $Aut(\Delta(\mathcal{P}_H))$ acts transitively on the maximal flags in $\Delta(\mathcal{P}_H)$, then $| G |$ is divisible by at most two distinct primes.*

Proof. Let $\{ 1 \} = H_0 \lhd H_1 \lhd \ldots \lhd H_n$ = G be a chief series for G. Suppose that the order of G is divisible by three or more distinct primes. Let $| H_1 | = p^t$ and let $i$ be the largest integer $2 \leq i \leq n-1$, such that $| H_i | = p^\alpha q^\beta$ with $p \neq q$ for primes $p$ and $q$. The order of G is divisible by three or more primes and there is an element $x \in$ G such that $| x |$ $= r$, where $r$ is a prime number different from $p$ and $q$. Since $H_i$ is normal in G, conjugation by $x$ induces an automorphism of $H_i$. If $x$ does not act fix-point-freely on $H_i$, there is a $y \in H_i$ such that $yx = xy$. Form the group L $= \langle x \rangle \times \langle y \rangle$. L is abelian and by Theorem 3.8, Col($\Delta(\mathcal{P}_L)$) acts transitively on $\Delta(\mathcal{P}_L)$ if and only if G is a homocylic p-group. L is not a homocylic p-group, which is a contradiction. Thus $x$ acts fixed-point-freely on $H_i$.

The element $x$ is of prime order and $x$ acts fixed-point-freely on $H_i$. By Theorem 1 of [31], $H_i$ is nilpotent. Thus there are elements $y$ and $z$ of $H_i$ such that $| y | = p$, $| z | = q$, and $yz = zy$. From the subgroup L $= \langle y \rangle \times \langle z \rangle$. By Theorem 3.8, Aut($\Delta(\mathcal{P}_L)$) acts transitively on $\Delta(\mathcal{P}_L)$ if and only if L is a homocylic p-group. This is a contradiction and the order of G can be divisible by at most two primes □

**Theorem 3.19** *Let $G$ be solvable and $\Delta(\mathcal{P})$ its flag space. Suppose that the order of $G$ is divisible by two distinct primes. If for each subgroup $H$ of $G$, $Aut(\Delta(\mathcal{P}_H))$ acts transitively on the maximal flags in $\Delta(\mathcal{P}_H)$ and no Sylow subgroup of $G$ is of Type IV), as mentioned in Theorem 3.17, then $G = [P]Q$ where $P$ and $Q$ are Sylow subgroups of $G$.*

Proof. By Theorem 3.17, P and Q are either elementary abelian, cyclic, or the generalized quaternion group. Since $\mid P \mid \neq \mid Q \mid$, only one can be the generalized quaternion group. The proof is now broken down into cases.

Case 1) P and Q are cyclic.

By Corollary 8.2.5 of [2], G is metacyclic and either P or Q is normal in G. If they are both normal in G, then by Theorem 3.8, Col($\Delta(\mathcal{P})$) does not act transitively on $\Delta(\mathcal{P})$. Thus without loss of generality, P $\lhd$ G and G = [P]Q.

Case 2) P is cyclic and Q is the generalized quaternion group.

Since P is abelian, by 4.5.15 of [30], G has an invariant series $\{ 1 \} \lhd K \lhd H \lhd$ G such that both H and K are characteristic in G, H/K $\cong$ P, and K is a 2-group.

Suppose K $\neq \{ 1 \}$. G is solvable and K is characteristic in G, so there is a minimal normal subgroup N of G such that $\mid N \mid = 2^s$. Since Q is the generalized quaternion group, $\mid N \mid = 2$. Let N = $\langle x \rangle$. Then for all $y \in$ P, $x^y = x$. Let $y_1 \in$ P and L = $\langle x \rangle \times \langle y_1 \rangle$. L is an abelian group. By Theorem 3.8, L is a homocylic p-group. This is a contradiction and hence K = $\{ 1 \}$. Thus G = [P]Q.

Case 3) P is cyclic and Q is elementary abelian.

Q is abelian, so by 4.5.15 of [30], G has a invariant series $\{ 1 \} \lhd K \lhd H \lhd$ G such that H and K are characteristic in G, H/K $\cong$ Q and K is a p-group. If K = $\{ 1 \}$, then Q is normal in G and G = [Q]P. Assume that K $\neq \{ 1 \}$. Then G has a minimal normal subgroup N such that N is a p-group and Q acts on N. Let $| Q | = q^{\beta}$. If $\beta \geq 2$, then by 5.3.16 of [13], $N = \Pi_{g \in Q^*} C_N(g)$ where $C_N(g) = \langle n \in N \mid gn = ng \rangle$ and $Q^*$ is the non-identity elements of Q. Thus there is an $x \in N$ and a $y \in Q$ such that $| x | = p$, $| y | = q$, and $yx = xy$. Let $L = \langle x \rangle \times \langle y \rangle$. L is abelian, so by Theorem 3.8, $\mathrm{Aut}(\Delta(\mathcal{P}_L))$ acts transitively on $\Delta(\mathcal{P}_L)$ if and only if L is homocylic. This is a contradiction and thus $\beta = 1$. Therefore both P and Q are cyclic and by Corollary 8.2.5 of [2], G is metacyclic. Without loss of generality, G = [P]Q.

Case 4) Q is elementary abelian and P is the generalized quaternion group.

This case is similar to case 2).

Case 5) P and Q are elementary abelian.

By 4.5.15 of [30], G has two invariant series $\{ 1 \} \lhd K_1 \lhd H_1 \lhd G$ and $\{ 1 \} \lhd K_2 \lhd H_2 \lhd G$ where $H_1/K_1 \cong P$, $K_1$ is a q-group and $K_1$ is normal in G; and $H_2/K_2 \cong Q$, $K_2$ is a p-group and $K_2$ is normal in G. If $K_1 = \{ 1 \}$, then G = [P]Q. Assume $K_1 \neq \{ 1 \}$. Let $| P | = p^{\alpha}$ and suppose $\alpha \geq 2$. Since $K_1$ is characteristic in G, G has a minimal normal subgroup N such that N is a q-group. Thus by 5.3.16 of [13], $N = \Pi_{r \in P^*} C_N(r)$ where $P^*$ is the collection of non-identity elements of P. Therefore there is an $x \in N$ and a $y \in P$ such that $| x | = q$, $| y | = p$ and $xy = yx$. The group $L = \langle x \rangle \times \langle y \rangle$ is abelian and by Theorem 3.8, it must be homocylic. This is a contradiction and hence $\alpha = 1$.

If $K_2 = \{ 1 \}$, G = [Q]P. Suppose that $K_2 \neq \{ 1 \}$. Since $\alpha = 1$, $K_2 = P$ and G = [P]Q

□

The condition that no Sylow subgroup of G is of type IV), as mentioned in Theorem 3.17, is necessary. Consider the group $G = \langle a, b, c, x, t \mid a^3 = b^3 = c^3 = x^{13} = t^9 = 1, t^3 = a, a^x = abc, b^x = ab^2c, c^x = ab^2c^2, a^t = a, b^t = ab, c^t = ab^2c, x^t = x^3 \rangle$. The subgroups $P = \langle a, b, c, x \rangle$ and $Q = \langle x \rangle$ are Sylow subgroups of G. P is of type IV) and each subgroup H of

G has a maximal flag transitive flag space associated with it. However, neither P nor Q is normal in G.

All solvable groups with the property that for each subgroup H of G, $\mathrm{Aut}(\Delta(\mathcal{P}_H))$ acts transitively on $\Delta(\mathcal{P}_H)$ are classified.

Type I) G is elementary abelian.

Type II) G is cyclic and hence inseparable.

Type III) G is the generalized quaternion group, which is also inseparable.

Type IV) P contains a maximal normal abelian subgroup N which is elementary abelian of rank $\geq 2$ and characteristic in P. Furthermore, $\mathrm{Exp}(P) = p^2$ and for elements x and y not in N, $|x| = |y| = p^2$ and $x^p = y^p \in N$.

Type V) The order of G is divisible by two primes and each Sylow subgroup is of one of the types I) - IV). Neither Sylow subgroup is normal in G.

From this point on, assume the order of G is divisible by two distinct primes and that the Sylow subgroups are of one of the types I) - III). By Theorem 3.19, G = [P]Q where P and Q are Sylow subgroups of G.

Type VI) P and Q are elementary abelian.

Let $|Q| = q^\beta$. Suppose that $\beta \geq 2$. Then by Theorem 5.3.16 of [13], $P = \Pi_{y \in Q^*} C_P(y)$ where $C_P(y) = \langle x \in P \mid xy = yx \rangle$ and $Q^*$ is the collection of non-identity elements in Q. Thus there is an $x \in P$ and a $y \in Q$ such that $yx = xy$. Let $L = \langle x \rangle \times \langle y \rangle$. By Theorem 3.8, L is homocyclic, which is a contradiction. Thus $|Q| = q$. A similar argument shows that Q must act faithfully on P. If Q acts faithfully and irreducibly on P, then G is multiprimitive of derived length 2.

Type VII) P and Q are cyclic.

Let P = $\langle x \rangle$ and Q = $\langle y \rangle$ where | P | = $p^\alpha$ and | Q | = $q^\beta$. Q acts faithfully on P. If not, there is an $n$, $1 \leq n \leq \beta - 1$, such that $x^{y^{q^\beta}} = x$. Therfore, L = $\langle x \rangle \times \langle y^{q^\beta} \rangle$ is an abelian subgroup of G and by Theorem 3.8, L must be homocyclic. This is a contradiction.

Type VIII) P is elementary abelian (non-cyclic) and Q is the generalized quaternion group.

Let | P | = $p^\alpha$. Q acts faithfully on P. If not, there is an element $y \in$ Q and $x \in$ P, such that $x^y = x$. Therefore, L = $\langle x \rangle \times \langle y \rangle$ is an abelian subgroup of G and by Theorem 3.8, L is homocyclic. This is a contradiction.

**Remark.** There are no other types. Suppose P is the generalized quaternion group and Q is cyclic. If the action of Q on P is unfaithful, then there is a $y \in$ Q and an $x \in$ P such that $xy = yx$. Thus L = $\langle x \rangle \times \langle y \rangle$ is an abelian subgroup of G. By Theorem 3.8, L is homocyclic, which is a contradiction. Thus Q acts faithfully on P. Now $\Phi$(P) is cyclic and characteristic in P. Thus $\Phi$(P) is normal in G. Since $\Phi$(P) is cyclic of order a power of 2, $\Phi$(P) and Q must commute element-wise. Thus $L_1 = \Phi(P) \times Q$ is abelian and by Theorem 3.8, $L_1$ is homocyclic. This is a contradiction and thus P can't be the generalized quaternion group. A similar argument shows that there can't be a type where P is the generalized quaternion group and Q is elementary abelian.

Suppose P is cyclic and Q is the generalized quaternion group. By 5.7.12 of [27], Aut(P) is cyclic. Thus Q can't act faithfully on P. Using a method similar to the one presented above, a contradiction is obtained.

**Examples:**

**Type IV)**

a) G = $\langle a, b \mid a^9 = b^3 = 1, a^b = a \rangle$.

b) $G = \langle a, b, c, x \mid a^5 = b^5 = c^5 = x^{25} = 1, a = x^5, ab = ba, ac = ca, bc = cb, b^x = ab, c^x = bc \rangle$.

**Type V)**

a) $G = \langle a, b, c, x, t \mid a^3 = b^3 = c^3 = x^{13} = t^9 = 1, t^3 = a, a^x = abc, b^x = ab^2c, c^x = ab^2c^2, a^t = a, b^t = ab, c^t = ab^2c, x^t = x^3 \rangle$.

**Type VI)**

a) The symmetric group $S_3$.

b) The alternating group $A_4$.

c) $G = [Z_2 \times Z_2 \times Z_2 \times Z_2]Z_3 = \langle a, b, c, d, x \mid a^2 = b^2 = c^2 = d^2 = x^3 = 1, ab = ba, ac = ca, ad = da, bc = cb, bd = db, cd = dc, a^x = b, b^x = ab, c^x = d, d^x = cd \rangle$.

**Type VII)**

a) $G = [Z_{13}]Z_4 = \langle x, y \mid x^{13} = y^4 = 1, x^y = x^5 \rangle$.

b) $G = [Z_{25}]Z_4 = \langle x, y \mid x^{25} = y^4 = 1, x^y = x^7 \rangle$.

c) $G = [Z_9]Z_2 = \langle x, y \mid x^9 = y^2 = 1, x^y = x^8 \rangle$.

**Type VIII)**

a) $G = [Z_7 \times Z_7]Q_8 = \langle a, b, x, y \mid a^7 = b^7 = x^4 = y^4 = 1, x^2 = y^2, ab = ba, a^x = ab^5, b^x = ab^6, a^y = a^2b^4, b^y = a^4b^5 \rangle$

# Chapter 4

# Maximal Flag Equivalence

Let G be an elementary abelian p-group and A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P})$. Each element of each frame for G is of order $p$, which implies $\alpha_i/\alpha_{i-1} \cong \beta_i/\beta_{i-1}$ for $1 \leq i \leq d+1$ where $\alpha_0 = \{ 1 \}$ and $\alpha_{d+1} = G$. This gives rise to the following question: Given a group G and two maximal flags A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ in $\Delta(\mathcal{P})$, when does $d = e$ and there exist a $\pi \in \mathrm{Sym}(d+1)$ such that $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$, where $\alpha_0 = \beta_0 = \{ 1 \}$ and $\alpha_{d+1} = \beta_{d+1} = g$. In other words, when are each pair of maximal flags in $\Delta(\mathcal{P})$ equivalent. In general, this is not the case. Consider the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$. Both A: $\langle a^2 \rangle \subset \langle a^2, b \rangle$ and B: $\langle a \rangle$ are maximal flags in $\Delta(\mathcal{P})$, yet they are not equivalent. In this chapter, when given a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ , $\alpha_0 = \{ 1 \}$ and $\alpha_{d+1} = G$.

## 4.1 Basic Results

**Definition 4.1** *Two maximal flags A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ in $\Delta(\mathcal{P})$ are* **equivalent** *if $d = e$ and there is a $\pi \in Sym(d+1)$ such that $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$.*

Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ be two equivalent maximal flags in $\Delta(\mathcal{P})$ supported by the frames $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$

respectively. Then there is a $\pi \in \mathrm{Sym}(d+1)$ such that $S_i \cong \alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1} \cong T_{\pi(i)}$ and $S_i \cong T_{\pi(i)}$. This condition will be used interchangable with the condition presented in the definition.

**Theorem 4.2** Equivalence *of maximal flags defines an equivalence relation on the maximal flags in $\Delta(\mathcal{P})$.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ , B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ , and C: $\gamma_1 \subset \gamma_2 \subset \ldots \subset \gamma_f$ be maximal flags in $\Delta(\mathcal{P})$. Clearly A is equivalent to itself.

Suppose A is equivalent to B. Then $d = e$ and there is a $\pi \in \mathrm{Sym}(d+1)$ such that $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$ for $1 \leq i \leq d+1$. Thus $\beta_i/\beta_{i-1} \cong \alpha_{\pi^{-1}(i)}/\alpha_{\pi^{-1}(i)-1}$ and B is equivalent to A.

Lastly suppose that A is equivalent to B and B is equivalent to C. Then $d = e$ and there is a $\pi \in \mathrm{Sym}(d+1)$ such that $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$. Furthermore, $e = f$ and there is a $\lambda \in \mathrm{Sym}(e+1)$ such that $\beta_j/\beta_{j-1} \cong \gamma_{\lambda(j)}/\gamma_{\lambda(j)-1}$. Hence $d = f$ and consider $\lambda\pi \in \mathrm{Sym}(d+1)$. Thus $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1} \cong \gamma_{\lambda\pi(i)}/\gamma_{\lambda\pi(i)-1}$ and A is equivalent to C □

**Definition 4.3** *A group G is called an* **FE-group** *if each pair of maximal flags in $\Delta(\mathcal{P})$ are equivalent.*

**Theorem 4.4** *Let N be a normal rank 1 flag in $\Delta(\mathcal{P})$. If N and G/N are FE-groups, then all the maximal flags in $\Delta(\mathcal{P})$, which contain N, are equivalent.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in$\Delta(\mathcal{P})$ such that N $\subseteq$ A and N $\subseteq$ B. Thus for some $l$, $1 \leq l \leq d$, $\alpha_l$ = N and for some $k$, 1 $\leq k \leq e$, $\beta_k$ = N.

By Lemma 2.19 ii), $A_N$: $\alpha_1 \subset \ldots \subset \alpha_{l-1}$ and $B_N$: $\beta_1 \subset \ldots \subset \beta_{k-1}$ are maximal flags in $\Delta(\mathcal{P})$. Thus, $l - 1 = k - 1$ and for some $\pi \in \mathrm{Sym}(l)$, $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$ for $1 \leq i \leq l$.

By Lemma 2.22 i), $A_{G/N}$: $\alpha_{l+1}/N \subset \ldots \subset \alpha_d/N$ and $B_{G/N}$: $\beta_{k+1}/N \subset \ldots \subset \beta_e/N$ are maximal flags in $\Delta(\mathcal{P}_{G/N})$, Then $d - l = e - k$ and there is an $\lambda \in \mathrm{Sym}(d-l)$, such that $(\alpha_{l+j}/N)/(\alpha_{l+j-1}/N) \cong (\beta_{k+\lambda(j)}/N)/(\beta_{k+\lambda(j)-1}/N)$ for $1 \leq j \leq d-l+1$. Therefore, $\alpha_{l+j}/\alpha_{l+j-1} \cong \beta_{k+\lambda(j)}/\beta_{k+\lambda(j)-1}$ for $1 \leq j \leq d-l+1$.

Since $k - 1 = l - 1$ and $d - l = e - k$, $l = k$ and $d = e$. Therefore, for $1 \leq i \leq l$, $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$ and for $l+1 \leq i \leq d+1$, $\alpha_i/\alpha_{i-1} \cong \beta_{l+\lambda(i-l)}/\beta_{l+\lambda(i-l)-1}$. Hence A is equivalent to B □

**Theorem 4.5** *An SE-group is an FE-group.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P})$ supported by the frames $\Sigma_A = \{ S_1, \ldots, S_{d+1} \}$ and $\Sigma_B = \{ T_1, \ldots, T_{e+1} \}$ respectively. Since G is an SE-group, $d + 1 = e + 1$ and there are elements $g_i \in$ G, $1 \leq i \leq d+1$, such that $S_i^{g_i} = T_i$. Hence $d = e$ and $\alpha_i/\alpha_{i-1} \cong S_i \cong T_i \cong \beta_i/\beta_{i-1}$ for $1 \leq i \leq d+1$ □

**Theorem 4.6** *An abelian group is an FE-group.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P})$ supported by $\Sigma_A = \{ S_1, \ldots, S_{d+1} \}$ and $\Sigma_B = \{ T_1, \ldots, T_{e+1} \}$ respectively. By the Fundamental Theorem of Abelian Groups, $d+1 = e+1$ and there is a $\pi \in \mathrm{Sym}(d+1)$ such that $S_i \cong T_{\pi(i)}$. Hence $d = e$ and $\alpha_i/\alpha_{i-1} \cong S_i \cong T_{\pi(i)} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$ for $1 \leq i \leq d+1$ □

**Theorem 4.7** *A solvable nC-group is an FE-group.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P})$ supported by the frames $\Sigma_A = \{ S_1, \ldots, S_{d+1} \}$ and $\Sigma_B = \{ T_1, \ldots, T_{e+1} \}$ respectively. By Theorem 2.42, for $1 \leq i \leq d+1$ and $1 \leq j \leq e+1$, each $S_i$ and $T_j$ are of prime order. Thus $d + 1 = e + 1$ or else the order of G would have two different prime factorizations. Hence $d = e$.

Let $\mid G \mid = p_1^{r_1} \ldots p_n^{r_n}$. For each $i$, $1 \leq i \leq n$, there are $r_i$ elements of $\Sigma_A$ and $\Sigma_B$ of order $p_i$. Thus there is a $\pi \in \mathrm{Sym}(d+1)$ such that $\alpha_i/\alpha_{i-1} \cong S_i \cong T_{\pi(i)} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$ □

The following question arises: If $G = S_1 \times \ldots \times S_n$ is a direct product of inseparable groups, does this imply that if $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ is a frame for G that $G = T_1 \times \ldots \times T_m$ and n = m.

The answer to the first part of the question is no. Consider the group $G \cong Q_8 \times Q_8 = \langle a, b, c, d \mid a^4 = b^4 = c^4 = d^4 = 1, a^2 = b^2, c^2 = d^2, a^b = a^3, c^d = c^3, ac = ca, bc = cb, ad = da, bd = db \rangle$ Consider the two frames $\Sigma_1 = \{ \langle a, b\rangle, \langle c, d\rangle \}$ and $\Sigma_2 = \{ \langle a, bc^2\rangle, \langle ac, bcd\rangle \}$ for G. Since every element of $\langle a, b\rangle$ commutes with every element of $\langle c, d\rangle$, $G = \langle a, b\rangle \times \langle c, d\rangle$. However, $a^{bcd} = a^3$ and $\langle a, bc^2\rangle$ does not commute elementwise with $\langle ac, bcd\rangle$ Hence G $\neq \langle a, bc^2\rangle \times \langle ac, bcd\rangle$.

The answer to the second part of the question is yes. In fact, a much stronger statement can be proven.

**Theorem 4.8** *Suppose $G = H_1 \times \ldots \times H_n$, where for each $i$, $1 \leq i \leq n$, $H_i$ is an inseparable subgroup of G. Then $\Sigma = \{ H_1, \ldots, H_n \}$ is a frame for G and if $\Sigma' = \{ S_1, \ldots, S_m \}$ is another frame for G, then $n = m$ and for some $\pi \in Sym(n)$, $S_i \cong H_{\pi(i)}$.*

Proof. For each $i$, $1 \leq i \leq n-1$, $H_1 \times \ldots \times H_i$ is normal in $H_1 \times \ldots \times H_{i+1}$ and complemented by $H_{i+1}$. Therefore $\Sigma = \{ H_1, \ldots, H_n \}$ is a frame for G. Proceed by induction on $n$.

Let $S = S_1 \ldots S_{m-1}$. Then $G = [S]S_m$. Consider the maps $\alpha_i = proj_{H_i}(G)$ for $1 \leq i \leq n$ and $\gamma = proj_{S_m}(G)$. Each $\alpha_i$, for $1 \leq i \leq n$, and $\beta = \alpha_1 + \ldots + \alpha_{n-1}$ are endomorphisms of G since $G = H_1 \times \ldots \times H_n$. By definition, $\beta + \alpha_n = \alpha_1 + \ldots + \alpha_{n-1} + \alpha_n = id_G$.

Consider $\gamma = proj_{S_m}(G)$. Let $g, h \in G$. Then $g = ss_m$ and $h = tt_m$ where $s, t \in S$ and $s_m, t_m \in S_m$. Then

$$
\begin{aligned}
(gh)^{\gamma} &= (ss_m tt_m)^{\gamma} \\
&= (ss_m ts_m^{-1} s_m t_m)^{\gamma} \\
&= (st' s_m t_m)^{\gamma} \text{ where } t' = s_m t s_m^{-1} \in S \\
&= s_m t_m \\
&= (ss_m)^{\gamma}(tt_m)^{\gamma} \\
&= g^{\gamma} h^{\gamma}.
\end{aligned}
$$

Therefore, $\gamma \in \text{End(G)}$ and $\gamma|_{S_m} \in \text{End}(S_m)$. Furthermore, $\gamma|_{S_m} = id_{S_m}$.

Since $\beta + \alpha_n = id_G$, $(\beta + \alpha_n)\gamma = \gamma$. In addition, by 4.5.2 of [27], $(\beta + \alpha_n)\gamma = \beta\gamma + \alpha_n\gamma$. Let $s_m \in S_m$. Then $s_m = h_1 \ldots h_n$ where $h_i \in H_i$ for $1 \leq i \leq n$.

$$
\begin{aligned}
s_m^{\beta\gamma} &= s_m^{(\alpha_1 + \ldots + \alpha_{n-1})\gamma} \\
&= (s_m^{\alpha_1} \ldots s_m^{\alpha_{n-1}})^{\gamma} \\
&= (h_1 \ldots h_{n-1})^{\gamma}
\end{aligned}
$$

The element $h_1 \ldots h_{n-1} = tt_m$ where $t \in$ S and $t_m \in S_m$. Therefore, $s_m^{\beta\gamma} = (h_1 \ldots h_{n-1})^{\gamma}$ $= (tt_m)^{\gamma} = t_m$ and $\beta\gamma$ maps $S_m$ to $S_m$. Since End(G) is closed under multiplication and $\beta, \gamma \in$ End(G), $\beta\gamma \in$ End(G). Consequently, $\beta\gamma|_{S_m} \in \text{End}(S_m)$. A similar argument shows that $\alpha_n\gamma|_{S_m} \in \text{End}(S_m)$.

Since $S_m$ is inseparable, by 4.5.5 of [27], $\beta\gamma|_{S_m}$ and $\alpha_n\gamma|_{S_m}$ are either nilpotent or automorphisms of $S_m$. Suppose both $\beta\gamma|_{S_m}$ and $\alpha_n\gamma|_{S_m}$ are nilpotent. From this point on, unless otherwise stated, all mentioned endomorphisms will be considered to be in $\text{End}(S_m)$.

Given that $\beta\gamma + \alpha_n\gamma = id_{S_m}$,

$$\begin{aligned}
\alpha_n\gamma(\beta\gamma+\alpha_n\gamma) &= \alpha_n\gamma(id_{S_m})\\
\alpha_n\gamma\beta\gamma+\alpha_n\gamma\alpha_n\gamma &= (id_{S_m})\alpha_n\gamma\\
\alpha_n\gamma\beta\gamma+\alpha_n\gamma\alpha_n\gamma &= (\beta\gamma+\alpha_n\gamma)\alpha_n\gamma\\
\alpha_n\gamma\beta\gamma+\alpha_n\gamma\alpha_n\gamma &= \beta\gamma\alpha_n\gamma+\alpha_n\gamma\alpha_n\gamma\\
\alpha_n\gamma\beta\gamma &= \beta\gamma\alpha_n\gamma.
\end{aligned}$$

Therefore, by 4.5.4 of [27], $\beta\gamma + \alpha_n\gamma$ is nilpotent. This is a contradiction. Hence $\beta\gamma$ or $\alpha_n\gamma \in \mathrm{Aut}(S_m)$.

Suppose that $\alpha_n\gamma \in \mathrm{Aut}(S_m)$. Given that $\alpha_n = proj_{H_n}(\mathrm{G})$, $\alpha_n|_{S_m}$ is a monomorphism form $S_m$ to $H_n$. Furthermore, given that $\gamma = proj_{S_m}(\mathrm{G})$, $\gamma|_{H_n}$ is an epimorphism from $H_n$ into $S_m$. Consider the endomorphisms $\alpha_n$ and $\gamma$ of G. Since End(G) is closed under multiplication, $\gamma\alpha_n \in \mathrm{End}(\mathrm{G})$. Let $h_n \in H_n$. Then $h_n = ss_m$ where $s \in \mathrm{S}$ and $s_m \in S_m$. Therefore, $h_n^{\gamma\alpha_n} = (ss_m)^{\gamma\alpha_n} = s_m^{\alpha_n}$. The element $s_m = k_1 \ldots k_n$ where $k_i \in H_i$ for $1 \leq i \leq n$. Hence, $h_n^{\gamma\alpha_n} = s_m^{\alpha_n} = k_n$ and $\gamma\alpha_n|_{H_n} \in \mathrm{End}(H_n)$.

$H_n$ is inseparable, so by Theorem 4.5.5 of [27], $\gamma\alpha_n|_{H_n}$ is nilpotent or an automorphism of $H_n$. If $\gamma\alpha_n|_{H_n}$ is nilpotent, there is a positive integer $r$ such that $(\gamma\alpha_n|_{H_n})^r = 0$. Thus $(\alpha_n\gamma|_{S_m})^{r+1} = \alpha_n(\gamma\alpha_n|_{H_n})^r\gamma = \alpha_n\ 0\ \gamma = 0$. This implies that $\alpha_n\gamma|_{S_m}$ is nilpotent, a contradiction. Hence $\gamma\alpha_n|_{H_n} \in \mathrm{Aut}(H_n)$. Given that $\gamma = proj_{S_m}(\mathrm{G})$, $\gamma|_{H_n}$ is a monomorphism from $H_n$ into $S_m$. Furthermore, given that $\alpha_n = proj_{H_n}(\mathrm{G})$, $\alpha_n|_{S_m}$ is an epimorphism from $S_m$ onto $H_n$. Consequently, $\gamma|_{H_n}$ is an isomorphism from $H_n$ into $S_m$ and $\alpha_n|_{S_m}$ is an isomorphism from $S_m$ into $H_n$.

Consider $\gamma\alpha_n$ as an endomorphism of G. Since $\alpha_n|_{S_m}$ is an isomorphism from $S_m$ into $H_n$, $\mathrm{G}^{\gamma\alpha_n} = H_n$. Therefore, $\mathrm{G}^{(\gamma\alpha_n)^2} = (\mathrm{G}^{\gamma\alpha_n})^{\gamma\alpha_n} = H_n^{\gamma\alpha_n} = H_n$. Hence, $\gamma\alpha_n$ has final image $H_n$. By 4.5.5 of [27], G = $[\ker(\alpha_n\gamma)]H_n = \ker(\alpha_n\gamma) \times H_n$ as $H_n \lhd$ G. Let $g = ss_m \in \ker(\alpha_n\gamma)$, where $s \in$ S and $s_m \in S_m$. Then $g^{\gamma\alpha_n} = (ss_m)^{\gamma\alpha_n} = s_m^{\alpha_n} = 1$. Since $\alpha_n|_{S_m}$ is an isomorphism from $S_m$ into $H_n$, $s_m = 1$ and $\ker(\gamma\alpha_n) =$ S.

Therefore, G = S $\times H_n = S_1 \ldots S_{m-1} \times H_n$ where $H_n \cong S_m$. Furthermore, $S_1 \ldots S_{m-1} \cong H_1 \times \ldots \times H_{n-1}$. Without loss of generality, $H_1 \times \ldots \times H_{n-1} = S'_1 \ldots S'_{m-1}$, where $S'_j \cong S_j$ for $1 \leq j \leq m-1$ and $\{ S'_1, \ldots, S'_{m-1} \}$ is a frame for $H_1 \times \ldots \times H_{n-1}$. By the induction hypothesis, $m - 1 = n - 1$ and there is an $\omega \in$ Sym(n-1) such that $S'_j \cong H_{\omega(j)}$ for $1 \leq j \leq m-1$. Therefore, $n = m$ and define $\pi \in$ Sym(n) such that for $1 \leq j \leq n-1$ $\pi$(j) = $\omega$(j) and $\pi$(n) = n. For $1 \leq i \leq n-1$, $S_i \cong S'_i \cong H_{\pi(i)}$ and for $i = n$, $S_n \cong H_n$.

Now suppose $\beta\gamma|_{S_m} \in \mathrm{Aut}(S_m)$. Since each $\alpha_i$, for $1 \leq i \leq n-1$, and $\gamma$ are endomorphisms of G, $\beta\gamma = (\alpha_1 + \ldots + \alpha_{n-1})\gamma = \alpha_1\gamma + \ldots + \alpha_{n-1}\gamma$. Furthermore, since End(G) is closed under multiplication, $\alpha_i\gamma \in$ End(G) for $1 \leq i \leq n-1$. Let $s_m \in S_m$. Then $s_m = h_1 \ldots h_n$ where $h_i \in H_i$ for $1 \leq i \leq n$. Hence, $s_m^{\alpha_i\gamma} = h_i^{\gamma}$. Since $h_i = aa_m$ where $a \in$ S and $a_m \in S_m$, $s_m^{\alpha_i\gamma} = h_i^{\gamma} = (aa_m)^{\gamma} = a_m$. Consequently, $\alpha_i\gamma|_{S_m} \in \mathrm{End}(S_m)$. From this point on, any endomorphism will be considered to be in $\mathrm{End}(S_m)$ unless otherwise stated.

$S_m$ is inseparable, so by 4.5.5 of [27], for each $i$, $1 \leq i \leq n-1$, $\alpha_i\gamma$ is either nilpotent or an automorphism of $S_m$. Suppose for each $i$, $1 \leq i \leq n-1$, that $\alpha_i\gamma$ is nilpotent. Since $\beta\gamma = \alpha_1\gamma + \ldots + \alpha_{n-1}\gamma \in \mathrm{Aut}(S_m)$, $\alpha_1\gamma + \ldots + \alpha_{n-1}\gamma = \delta \in \mathrm{Aut}(S_m)$. Therefore, $(\alpha_1\gamma + \ldots + \alpha_{n-1}\gamma)\delta^{-1} = \alpha_1\gamma\delta^{-1} + \ldots + \alpha_{n-1}\gamma\delta^{-1} = id_{S_m}$. Since $\delta^{-1} \in \mathrm{Aut}(S_m)$ and $S_m$ is inseparable, for each $i$, $1 \leq i \leq n-1$, $\alpha_i\gamma\delta^{-1}$ is nilpotent.

Suppose $n-1 = 2$. Then $\alpha_1\gamma\delta^{-1} + \alpha_2\gamma\delta^{-1} = id_{S_m}$ and $\alpha_1\gamma\delta^{-1}$ and $\alpha_2\gamma\delta^{-1}$ are nilpotent, an argument similar to the presented earlier shows that $\alpha_1\gamma\delta^{-1} + \gamma\alpha_2\delta^{-1}$ is nilpotent. Proceed by induction on $n-1$. Let $\tau = \alpha_1\gamma\delta^{-1} + \ldots + \alpha_{n-2}\gamma\delta^{-1}$ be nilpotent. Hence

$$\begin{aligned}
\tau + \alpha_{n-1}\gamma\delta^{-1} &= id_{S_m} \\
\alpha_{n-1}\gamma\delta^{-1}(\tau + \alpha_{n-1}\gamma\delta^{-1}) &= \alpha_{n-1}\gamma\delta^{-1}(id_{S_m}) \\
\alpha_{n-1}\gamma\delta^{-1}(\tau) + (\alpha_{n-1}\gamma\delta^{-1})^2 &= (id_{S_m})\alpha_{n-1}\gamma\delta^{-1} \\
\alpha_{n-1}\gamma\delta^{-1}(\tau) + (\alpha_{n-1}\gamma\delta^{-1})^2 &= (\tau + \alpha_{n-1}\gamma\delta^{-1})\alpha_{n-1}\gamma\delta^{-1} \\
\alpha_{n-1}\gamma\delta^{-1}(\tau) + (\alpha_{n-1}\gamma\delta^{-1})^2 &= (\tau)\alpha_{n-1}\gamma\delta^{-1} + (\alpha_{n-1}\gamma\delta^{-1})^2 \\
\alpha_{n-1}\gamma\delta^{-1}(\tau) &= (\tau)\alpha_{n-1}\gamma\delta^{-1}.
\end{aligned}$$

By 4.5.4 of [27], $\tau+\alpha_n\gamma\delta^{-1} = \alpha_1\gamma\delta^{-1}+\ldots+\alpha_{n-1}\gamma\delta^{-1}$ is nilpotent. This is a contradiction. Therefore, for some $j$, $1 \leq j \leq n-1$, $\alpha_j\gamma\delta^{-1} \in \mathrm{Aut}(S_m)$. Since $\delta^{-1} \in \mathrm{Aut}(S_m)$, $\alpha_j\gamma \in \mathrm{Aut}(S_m)$.

Proceed in a manner similar to the presented above to obtain $S_m \cong H_j$ and G = $S_1 \ldots S_{m-1} \times H_j$. Therefore, $S_1 \ldots S_{m-1} \cong H_1 \times \ldots \times H_{j-1} \times H_{j+1} \times \ldots \times H_n$ results in the same conclusion □

**Lemma 4.9** *Let G be a non-abelian, separable metacyclic p-group with $p > 2$. Then*

*i)* $G = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^k \rangle$;

*ii) If* $G = [N]H$, $N \cong \langle a \rangle$ *and* $H \cong \langle b \rangle$.

Proof. i) By III.11.2 of [16], G $= \langle a, b \mid a^{p^n} = 1, b^{p^m} = a^{p^t}, a^b = a^k \rangle$. Since G is separable, $t = 0$ and G $= \langle a, b \mid a^{p^n} = 1, b^{p^m} = 1, a^b = a^k \rangle$.

2) Suppose G = [N]H. By III.11.4 of [16], $\mid G/\mho_1(G) \mid = \mid \Omega_1(G) \mid = p^2$. Consider $\Omega_1(N)$ and $\Omega_1(H)$. Since $\Omega_1(N)$ is characteristic in N, $\Omega_1(N)$ is normal in G. Hence $\Omega_1(N)\Omega_1(H)$ is a subgroup of G contained in $\Omega_1(G)$. If $\Omega_1(N)\Omega_1(H) < \Omega_1(G)$, then $\Omega_1(N) = \{ 1 \}$ or $\Omega_1(H) = \{ 1 \}$. This is a contradiction. Therefore, $\Omega_1(N)\Omega_1(H) = \Omega_1(G)$ resulting in $\mid \Omega_1(N) \mid = p$ and $\mid \Omega_1(H) \mid = p$. Hence N and H each have only one subgroup of order $p$. By III.8.2 of

[16], N and H are cyclic. Since G = $[\langle a\rangle]\langle b\rangle$, by Theorem 4.4 of [18], N $\cong \langle a\rangle$ and H $\cong \langle b\rangle$ □

**Theorem 4.10** *Let G be a separable metacyclic p-group with $p > 2$. Then G is an FE-group.*

Proof. If G is abelian, the result follows from Theorem 4.8. Suppose G is non-abelian. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{S_1, S_2, \ldots, S_{d+1}\}$. By Theorem 4.9, $d+1 = 2$, $S_1 \cong \langle a\rangle$, and $S_2 \cong \langle b\rangle$. Hence $\alpha_1 \cong S_1 \cong \langle a\rangle$ and $G/\alpha_1 \cong S_2 \cong \langle b\rangle$. Thus G is an FE-group □

If $p = 2$, the theorem is not valid. The dihedral group $D_8$ is a metacyclic 2-group, but is not an FE-group.

**Lemma 4.11** *Suppose G is a non-abelian, metacyclic group with $G = [\langle a\rangle]\langle b\rangle$ where $| a | = p^n$, $| b | = q^m$, $p \neq q$, and $a^b = a^j$. Then $\langle a\rangle$ is the only subgroup of G over which it splits.*

Proof. Let G = [N]K where K is inseparable. Suppose $\langle a\rangle \cap N \neq \{ 1 \}$. Then by 4.2.7 of [2], $\langle a\rangle = [N \cap \langle a\rangle]K \cap \langle a\rangle$. Since $\langle a\rangle$ is inseparable, this is a contradiction unless N = $\langle a\rangle$.

Suppose $\langle a\rangle \cap N = \{ 1 \}$. Then $| N | = q^t$, $1 \leq t \leq m$. If $t < m$, then by Theorem 4.2.7 of [2], for some Sylow q-subgroup Q of G, Q = [N ∩ Q]K ∩ Q where N ∩ Q ≠ Q. Since Q is inseparable, this is a contradiction. Thus $| N | = q^m$. Since N ◁ G, G = N $\times \langle a\rangle$ and G is abelian. This is a contradiction and N = $\langle a\rangle$ □

**Remark:** This theorem does not hold if the condition is relaxed to say that G = $[\langle a\rangle]\langle b\rangle$ where $| a | = n$, $| b | = m$, and $(n, m) = 1$. Consider G = $\langle a, b \mid a^{14} = b^3 = 1 \text{ and } a^b = a^9\rangle$. G = $[\langle a\rangle]\langle b\rangle$ and G = $[\langle a^2, b\rangle]\langle a^7\rangle$. Hence $\langle a\rangle$ is not the unique subgroup over which G splits.

**Theorem 4.12** *Suppose $G$ is a metacyclic group with $G = [\langle a\rangle]\langle b\rangle$ where $\mid a \mid = p^n$, $\mid b \mid = q^m$, $p \neq q$, and $a^b = a^j$. Then $G$ is an FE-group.*

Proof. If G is abelian, the result follows from Theorem 4.8. Suppose G is non-abelian. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$. By Lemma 4.11, $\langle a\rangle$ is the unique group over which G splits. Since the complement to $\langle a\rangle$ in G is a cyclic Sylow q-subgroup, $d = 1$, $\alpha_1 \cong \langle a\rangle$, and $G/\alpha_1 \cong \langle b\rangle$ □

**Lemma 4.13** *Let $G$ be a separable nilpotent group. Then for each frame $\Sigma = \{S_1, S_2, \ldots, S_n\}$ for $G$, each $S_i$, $1 \leq i \leq n$, is of prime power order.*

Proof. Suppose there is a $j$, $1 \leq j \leq n$, such that $\mid S_j \mid$ is divisible by two or more primes. $S_j$ is nipotent, so it is the direct product of its Sylow subgroups. This contradicts $S_j$ being separable □

A necessary and sufficient condition is now given for when a nilpotent group is an FE-group.

**Theorem 4.14** *Let $G$ be a separable, nilpotent group. $G$ is an FE-group if and only if each Sylow p-subgroup of $G$ is an FE-group.*

Proof. Let P be a Sylow p-subgroup of G and A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be two maximal flags in $\Delta(\mathcal{P}_P)$. G splits over P, so by Lemma 2.19 i), A and B can be extended to maximal flags $A'$: $\alpha_1 \subset \ldots \subset \alpha_d \subset \alpha_{d+1} \subset \ldots \subset \alpha_l$ and $B'$: $\beta_1 \subset \ldots \subset \beta_e \subset \beta_{e+1} \subset \ldots \subset \beta_k$ in $\Delta(\mathcal{P})$ , where $\alpha_{d+1} = \beta_{e+1} =$ P. G is an FE-group, so $k = l$ and there is a $\pi \in \mathrm{Sym}(l+1)$ such that $\alpha_i/\alpha_{i-1} \cong \beta_{\pi(i)}/\beta_{\pi(i)-1}$. Since $\alpha_{d+1} = \beta_{e+1}$ = P and P is unique, for all $i$, where $1 \leq i \leq d+1$, $1 \leq \pi(i) \leq e+1$. Hence $e+1 = d+1$ and for $1 \leq j \leq d+1$, $\alpha_j/\alpha_{j-1} \cong \beta_{\pi(j)}/\beta_{\pi(j)-1}$. Hence A and B are equivalent and P is an FE-group.

Conversely, let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$ supported by $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ . G is nilpotent, so G $= P_1 \times \ldots \times P_n$ for Sylow $p_i$-subgroups of G. For each $j$, $1 \leq j \leq n$, G splits over $P_1 \times \ldots \times P_j$. Hence there is a maximal flag B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ in $\Delta(\mathcal{P})$ supported by $\Sigma' = \{ T_1, T_2, \ldots, T_{e+1} \}$ such that $T_{i_{l-1}+1} \ldots T_{i_l} = P_l$ for $1 \leq l \leq n$ and $1 = i_0 \leq i_1 < i_2 < \ldots < i_n = e + 1$.

If $n = 1$, G is a p-group and A and B are equivalent. Proceed by induction on $n$. By Lemma 4.13, for each $k$, $1 \leq k \leq d+1$, $S_k$ is of prime power order. Let $j_1$ be the smallest integer, $1 \leq j_1 \leq d+1$, such that $p_1$ divides $| S_{j_1} |$. For each $h$, $1 \leq h \leq j_1 - 1$, $(| S_h |, | S_{j_1} |) = 1$. Therefore, $S_h$ and $S_{j_1}$ commute element-wise and by Lemma 1.7, $\Sigma_1 = \{S_{j_1}, S_1, \ldots, S_{j_1-1}, S_{j_1+1}, \ldots, S_{d+1}\}$ is a frame for G. Continue in this manner to find integers $1 \leq j_1 < j_2 < \ldots < j_m \leq d+1$ such that $\Sigma_m = \{S_{j_1}, S_{j_2}, \ldots, S_{j_m}, \ldots, S_{j_1-1}, S_{j_1+1}, \ldots, S_{d+1}\}$ is a frame for G where $P_1 = S_{j_1} \ldots S_{j_m}$. Let $\Sigma_m = \{ R_1, \ldots, R_{d+1} \}$. Since $\Sigma_m$ is just a permutation of the elements of $\Sigma$, there is $\tau \in \mathrm{Sym}(d+1)$, such that for $1 \leq u \leq d+1$, $S_u = R_{\tau(u)}$.

Since $\{ R_1, \ldots, R_m \}$ and $\{ T_1, \ldots T_{i_1} \}$ are frames for $P_1$, $m = i_1$ and there is a $\lambda \in \mathrm{Sym}(m)$ such that $R_v \cong T_{\lambda(v)}$ for $1 \leq v \leq m$. By Lemma 1.10, $\{R_{m+1}P_1/P_1, \ldots, R_{d+1}P_1/P_1\}$ and $\{T_{m+1}P_1/P_1, \ldots, T_{e+1}P_1/P_1\}$ are frames for $G/P_1$. By induction, $d+1-m = e+1-m$ and there is an $\xi \in \mathrm{Sym}(d+1-m)$ such that $R_{m+w}P_1/P_1 \cong T_{m+\xi(w)}P_1/P_1$ for $1 \leq w \leq d+1-m$. Since $R_{m+w} \cap P_1 = T_{m+\xi(w)} \cap P_1 = \{ 1 \}$, $R_{m+w} \cong T_{m+\xi(w)}$.

Given that $d+1-m = e+1-m$, $d+1 = e+1$. Define $\pi \in$ Sym(d+1) as follows:

$$\pi(i) = \lambda(i) \text{ for } 1 \leq i \leq m$$

$$\pi(i) = m + \xi(i-m) \text{ for } m+1 \leq i \leq d+1.$$

Therefore $\alpha_1/\alpha_{i-1} \cong S_i \cong R_{r(i)} \cong T_{\pi(r(i))} \cong \beta_{\pi(r(i))}/\beta_{\pi(r(i))-1}$. Hence each maximal flag in $\Delta(\mathcal{P})$ is equivalent to B and G is an FE-group □

## 4.2 f-primary Groups

**Definition 4.15** *A group G is called an* **f-primary group of type S**, *where S is an inseparable group, if for each maximal flag A:* $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ *in* $\Delta(\mathcal{P})$, $\alpha_i/\alpha_{i-1} \cong S$ *for* $1 \le i \le d+1$ *where* $\alpha_0 = \{ 1 \}$ *and* $\alpha_{d+1} = G$.

Every f-primary group G of type S is an FE-group. If $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ is a frame for G, then $S_i \cong$ S for each $i$, $1 \le i \le n$.

**Theorem 4.16** *Let G be an f-primary group of type S and* $N \in \Delta(\mathcal{P})$.

*i) If N is complemented in G, then G/N is an f-primary group of type S.*

*ii) N is an f-primary group of type S.*

Proof. i) Let A: $\alpha_1/N \subset \ldots \subset \alpha_{d+1}/N$ be a maximal flag in $\Delta(\mathcal{P}_{G/N})$. By Lemma 2.22 ii), A can be extended to a maximal flag $A'$: $\gamma_1 \subset \ldots \subset \gamma_e \subset N \subset \alpha_1 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$. Thus for each $i$, $1 \le i \le d+1$, $\alpha_i/\alpha_{i-1} \cong$ S. Therefore, $(\alpha_i/N)/(\alpha_{i-1}/N) \cong \alpha_i/\alpha_{i-1} \cong$ S.

ii) Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P}_N)$. By Lemma 2.19 i), A can be extended to maximal flag $A'$: $\alpha_1 \subset \ldots \subset \alpha_d \subset N \subset \gamma_1 \subset \ldots \subset \gamma_e$ in $\Delta(\mathcal{P})$. Thus for each $i$, $1 \le i \le d+1$, $\alpha_i/\alpha_{i-1} \cong$ S □

**Theorem 4.17** *If G is the direct product of isomorphic, inseparable groups, then G is an f-primary group of type S.*

Proof. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$. G = $H_1 \times \ldots \times H_n$, where $H_i \cong H_j$ for $1 \le i, j \le n$ and $H_i$ inseparable. By Theorem 4.8, for each $l$, $1 \le l \le d+1$, $S_l \cong H_1$. Thus $\alpha_l/\alpha_{l-1} \cong H_1$ □

**Theorem 4.18** *G is an f-primary group of type S, where S is of prime order, if and only if G is a p-group of exponent p.*

Proof. Suppose that G is an f-primary group of type S, where S is of prime order $p$. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G and suppose G is a counter-example of minimal order. Then $\mid G \mid \geq p^3$ and there is an element $x \in$ G, such that $\mid x \mid > p$. Let M be a maximal subgroup of G which contains $\langle x \rangle$. If not, $\langle x \rangle$ = G, which implies that G is cyclic and inseparable. This is a contradiction.

If for all $i$, $1 \leq i \leq n$, $S_i \leq$ M, then G = M which is a contradiction. Thus there is a $j$, $1 \leq j \leq n$, such that $S_j \not\leq$ M. Since $\mid S_j \mid = p$, M $\cap S_j = \{ 1 \}$. M is normal and maximal in G, so G = [M]$S_j$. By Theorem 2.17, any frame for M can be extended to a frame for G. Hence M is an f-primary group of type S. Since | M | < G, M is of exponent $p$. Thus $\mid x \mid = p$ and a contradiction is reached.

Conversely, suppose G is a p-group of exponent p and a counter-example of minimal order. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. $\Sigma' = \{ S_1, \ldots, S_{n-1} \}$ is a frame for $S_1 \ldots S_{n-1}$, such that $\mid S_1 \ldots S_{n-1} \mid < \mid G \mid$. Therefore, for $1 \leq i \leq n-1$, $\mid S_i \mid = p$. Futhermore, $\mid S_n \mid < \mid G \mid$. Since $S_n$ is inseparable, $\mid S_n \mid = p$ □

## 4.3 Geometric Similarities Between Groups

Groups are studied whose induced geometry have properties similar to the induced geometry of elementary abelian p-groups. Elementary abelian p-groups satisfy the following two properties.

**P**$_1$: G splits over every rank 1 flag in $\Delta(\mathcal{P})$.

**P**$_2$: Each element of each frame is of prime order.

**Lemma 4.19** *If G is a group satisfies property* $\mathbf{P}_1$ *and N is a normal rank 1 flag in* $\Delta(\mathcal{P})$*, then N and G/N also satisfy* $\mathbf{P}_1$.

Proof. Let $\alpha$ be a rank 1 flag in $\Delta(\mathcal{P}_N)$. By Lemma 2.17, $\alpha$ is a rank 1 flag of $\Delta(\mathcal{P})$. Therefore G splits over $\alpha$ and G = $[\alpha]$H for H < G. Since $\alpha <$ N, N = N $\cap$ G = N $\cap$ $\alpha$H = $\alpha$(H $\cap$ N). Thus $\alpha$ is complemented in N.

Let $\alpha$/N be a rank 1 flag in $\Delta(\mathcal{P}_{G/N})$. Since N is complemented in G, by Lemma 2.20 ii), $\alpha$ is a rank 1 flag in $\Delta(\mathcal{P})$. Therefore G = $[\alpha]$H. Since N $\subset \alpha$, G/N = $[\alpha/\text{N}]$HN/N □

**Lemma 4.20** *Let G be a group satisfying* $\mathbf{P}_2$.

*i) Any rank 1 flag of G also satisfies* $\mathbf{P}_2$.

*ii) If N is a complemented rank 1 flag for G, then G/N satisfies* $\mathbf{P}_2$.

*iii) G is an FE-group.*

*iv) G is solvable.*

*v) If G satisfies* $\mathbf{P}_1$*, then G is supersolvable.*

Proof. i) Let N be a rank 1 flag in $\Delta(\mathcal{P})$ and let $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ be a frame for N. By Lemma 2.17, $\Sigma'$ can be extended to a frame for G. Therefore for $1 \leq i \leq m$, $T_i$ is of prime order.

ii) Let $\{ S_1/\text{N}, \ldots, S_n/\text{N} \}$ be a frame for G/N and let $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ be a frame for N. Let H be a complement to N in G. By Lemma 1.11 $\{ T_1, \ldots, T_m, S_1 \cap \text{H}, \ldots, S_n \cap \text{H} \}$ is a frame for G. Therefore, for each $i$, $1 \leq i \leq n$, $S_i \cap H$ is of prime order. Since H $\cap$ N = { 1 }, $S_i/\text{N} = (S_i \cap \text{NH})/\text{N} = (S_i \cap \text{H})\text{N}/\text{N} \cong (S_i \cap \text{H})/(\text{N} \cap S_i \cap \text{H}) \cong S_i \cap \text{H}$ for $1 \leq i \leq n$ and $S_i/\text{N}$ is of prime order.

iii) Suppose that $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_m \}$ are two frames for G. Each element of each frame $\Sigma$ and $\Sigma'$ is of prime order. Therefore $n = m$ or else the order of G has two different prime factorizations. For each prime $p$ dividing the order of G, there are an equal number of subgroups in $\Sigma$ and $\Sigma'$ of order $p$. Thus there is a $\pi \in \mathrm{Sym}(n)$, such that for $1 \leq i \leq n$, $S_i \cong T_{\pi(i)}$ and G is an FE-group.

iv) Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. Since $G/(S_1 \ldots S_{n-1})$ is of prime order, $G' \subseteq S_1 \ldots S_{n-1}$. $S_1 \ldots S_{n-1}/S_1 \ldots S_{n-2}$ is of prime order, so $G^2 \subseteq S_1 \ldots S_{n-2}$. Continuing in this manner, $G^{n-1} \subseteq S_1$ and $G^n = \{ 1 \}$. Therefore, G is solvable.

v) Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. Then $A{:}S_1 \subset S_1S_2 \subset \ldots \subset S_1 \ldots S_{n-1}$ is a maximal flag in $\Delta(\mathcal{P})$. G satisfies $\mathbf{P}_1$ and for each $i$, $1 \leq i \leq n-1$, $S_1 \ldots S_i$ is normal in G. Hence $\{ 1 \} \lhd S_1 \lhd S_1S_2 \lhd \ldots \lhd S_1 \ldots S_{n-1} \lhd$ G is a chief series for G. Since $S_i$ is of prime order for each $i$, $1 \leq i \leq n$, G is supersolvable □

Consider a group G which has every subgroup in each frame of prime order. Theorem 4.18 might lead one to believe that each element of G is of prime order. This is not true. Consider the group $S_4$. $S_4$ has each subgroup in each frame of prime order, but contains an element of order 4. However, a weaker statement can be proven.

**Definition 4.21** *(Deaconescu [11]) A group G is called a* $\mathcal{P}$-**group** *if each element of G is of prime order.*

**Theorem 4.22** *If a separable group G is a $\mathcal{P}$-group, then each subgroup of each frame for G is of prime order.*

Proof. Theorem 4.18 has shown this to be true when G is a p-group. Assume that G is not a p-group.

By M. Deaconescu (Main Theorem of [11]), G must be a group of the following type:

a) $|\, G \,| = p^a q$, $3 \leq p < q, a \geq 3$, $|\, F(G) \,| = p^{a-1}$, $|\, G/G' \,| = p$;

b) $|\, G \,| = p^a q$, $3 \leq q < p, a \geq 1$, $|\, F(G) \,| = |\, G' \,| = p^a$;

c) $\mid G \mid = 2^a p$, $p \geq 3$, $a \geq 2$, $\mid F(G) \mid = \mid G' \mid = 2^a$;

d) $\mid G \mid = 2p^a$, $p \geq 3$, $a \geq 1$, $\mid F(G) \mid = \mid G' \mid = p^a$ and F(G) is elementary abelian.

Case 1) G is of type a).

Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G and suppose for some $i$, $1 \leq i \leq n$, that $\mid S_i \mid = p^\alpha q^\beta$ where $1 \leq \alpha \leq a$ and $\beta = 0$ or 1. If $\beta = 0$, then by Theorem 4.18 and the fact that every subgroup of a $\mathcal{P}$-group is a $\mathcal{P}$-group, $\alpha = 1$. Therefore $\beta = 1$. Since $S_i$ is a $\mathcal{P}$-group and $\beta = 1$, $S_i$ is of type a) or b).

If $S_i$ is of type a), $\mid S_i/S_i' \mid$ = p. Therefore, there is an element $x \in S_i$ such that $\mid x \mid = p$ and $x \notin S_i'$. Hence, $S_i = [S_i']\langle x \rangle$ which contradicts the fact that $S_i$ is inseparable. Thus, $\mid S_i \mid = p$ or $q$.

If $S_i$ is of type b), then $\mid F(S_i) \mid = p^\alpha$ and there is an element $y$ of order $q$ such that $S_i = [F(S_i)]\langle y \rangle$. This again contradicts the fact that $S_i$ is inseparable and therefore $\mid S_i \mid = p$ or $q$.

Case 2) G is of type b).

An argument similar to the the one given in Case 1) shows G has each element of each frame of prime order.

Case 3) G is of type c).

Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G and suppose for some $i$, $1 \leq i \leq n$, that $\mid S_i \mid = 2^\alpha p^\beta$ where $1 \leq \alpha \leq a$ and $\beta = 0$ or 1. If $\beta = 0$, then by Theorem 4.18 and the fact that $S_i$ is a $\mathcal{P}$-group, $\alpha = 1$. Therefore $\beta = 1$ and $S_i$ is of type c) or d).

If $S_i$ is of type c), then $\mid F(S_i) \mid = 2^\alpha$ and there is an element $x \in S_i$ such that it has order $p$ and $S_i = [F(S_i)]\langle x \rangle$. This contradicts the fact that $S_i$ is inseparable.

If $S_i$ is of type d), then $|$ F($S_i$) $| = p$ and there is an element $y$ of order 2 such that $S_i$ = [F($S_i$)]$\langle y\rangle$. This contradicts the fact that $S_i$ is inseparable.

Case 4) G is of type d).

The argument here is similar to the one in Case 3) □.

**Theorem 4.23** *If G satisfies properties* $\mathbf{P}_1$ *and* $\mathbf{P}_2$*, then* $\Phi(G) = \{ 1 \}$.

Proof. Proceed by induction on the order of G. Therefore $|$ G $| > pq$ where $p$ and $q$ are primes. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. The subgroup $S_1$ is a rank 1 flag in $\Delta(\mathcal{P})$ and G splits over it. By Lemmas 4.19 and 4.20, G/$S_1$ satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$. Hence $\Phi(G/S_1) = \{ 1_{G/S_1} \}$ and $\Phi(G) \leq S_1$. The subgroup $S_1$ is of prime order and $S_1 \lhd G$, therefore $\Phi(G) = \{ 1 \}$ □

**Definition 4.24** *A group G is* **complemented** *if for each subgroup H of G, there is a subgroup K of G such that G = HK and* $H \cap K = \{ 1 \}$.

**Theorem 4.25** *If a group G has properties* $\mathbf{P}_1$ *and* $\mathbf{P}_2$*, then G is complemented.*

Proof. By Lemma 4.20 iv), G is solvable. Let $1 = F_0(G) \lhd F_1(G) \lhd F_2(G) \lhd \ldots \lhd F_{r+1}(G)$ = G be the Fitting series for G. By Theorem 4.23, $\Phi(G) = \{ 1 \}$ and G splits over $F_1$(G). By Lemmas 4.19 and 4.20, G/$F_1$(G) satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$ and $\Phi(G/F_1(G)) = \{ 1_{G/F_1(G)} \}$. Proceed by induction on $r+1$. Let $2 \leq i \leq r+1$. G/$F_{i-1}$(G) satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$, so by Theorem 4.23, $\Phi(G/F_{i-1}(G)) = \{ 1_{G/F_{i-1}(G)} \}$ and G/$F_{i-1}$(G) splits over $F_i(G)/F_{i-1}(G)$. By Lemmas 4.19 and 4.20, $(G/F_{i-1}(G))/(F_i(G)/F_{i-1}(G)) \cong G/F_i(G)$ satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$. Therefore, $\Phi(G/F_i(G)) = \{ 1_{G/F_i(G)} \}$. By I.23 of [28], G is a K-group.

By Lemma 4.20 v), G is supersolvable. Therefore, by I.24 of [28], G is complemented □

The converse to this theorem is not true. Consider the group $G = [Z_7 \times Z_7 \times Z_7]Z_3 = \langle a, b, c, t \mid a^7 = b^7 = c^7 = t^3 = 1, ab = ba, ac = ca, bc = cb, a^t = b, b^t = c, c^t = a\rangle$. G =

$\langle a^2b^4c\rangle \times \langle a^4b^2c\rangle \times \langle abc, t\rangle$ and by Theorem 2 of [14], G is complemented. However, $\langle a\rangle$ is a rank 1 flag in $\Delta(\mathcal{P})$ and it is not normal in G.

**Theorem 4.26** *If G is a group satisfying* $\mathbf{P}_1$ *and* $\mathbf{P}_2$*, then each inseparable subgroup of G is contained in frame for G.*

Remark. This is a property of all elementary abelian p-groups.

Proof. Let H be an inseparable subgroup of G. By Theorem 4.24, G is complemented. Since H is a subgroup of G, H is also complemented. By Theorem 2 of [14], H is supersolvable and F(H) $\neq$ { 1 }. If F(H) $\neq$ H, then H splits over F(H), a contradiction. Thus F(H) = H and H is nilpotent. Since H is inseparable, H must be a p-group of order p.

Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. If $S_1 \ldots S_{n-1} \cap$ H = { 1 }, G = $[S_1 \ldots S_{n-1}]$H and { $S_1, \ldots, S_{n-1}$, H } is a frame for G. If H $\cap$ $S_1 \ldots S_{n-1} \neq$ { 1 }, H $\leq S_1 \ldots S_{n-1}$. If H $\cap S_1 \ldots S_{n-2}$ = { 1 }, then $S_1 \ldots S_{n-1} = [S_1 \ldots S_{n-2}]$H and { $S_1, \ldots, S_{n-2}$, H , $S_n$ } is a frame for G. If H $\cap$ $S_1 \ldots S_{n-2} \neq$ { 1 }, then H $\leq S_1 \ldots S_{n-2}$. Continue in this manner until for some $i$, $1 \leq i \leq n-2$, $\Sigma' = \{ S_1, \ldots, S_i, H, S_{i+2}, \ldots, S_n \}$ is a frame for G or H = $S_1$ □

Groups are now presented whose induced geometry is similar to the induced geometry of elementary p-groups.

**Case 1)** Each subgroup N of G is normal and complemented in G.

Each Sylow p-subgroup of G is normal and hence G is nilpotent. Futhermore, G is a solvable nC-group and by Theorem 2.42 satisfies $\mathbf{P}_2$. Since each rank 1 flag is normal in G, it is complemented in G. Therefore, G satisfies $\mathbf{P}_1$. By Theorem 4.25, G is complemented and by Theorem 2 of [14], each Sylow p-subgroup is elementary abelian. Since G is nilpotent, G is the direct product of its Sylow p-subgroups. Hence G is abelian with $\Phi$(G) = { 1 }.

An example of this type of group G $\cong Z_2 \times Z_3 \times Z_3 \times Z_5 \times Z_5$.

**Case 2)** Each element of each splitting system is of prime order and G splits over each rank 1 flag in $\Delta(\mathcal{P})$.

G satisfies $\mathbf{P}_1$ and $\mathbf{P}_2$, so by Theorem 4.25, G is complemented. By Theorem 1 in [14], G is the direct product of groups of square-free order.

The next theorem classifies complemented groups through their induced geometry.

**Theorem 4.27** *G is complemented if and only if each element of each frame is of prime order and each rank 1 flag in $\Delta(\mathcal{P})$ is complemented in G.*

Proof. If G is complemented, then G is a solvable nC-group. By Theorem 2.42, each element of each frame is of prime order. Let $\alpha$ be a rank 1 flag in $\Delta(\mathcal{P})$. Since $\alpha$ is a subgroup of G, $\alpha$ is complemented in G.

Conversely, let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a maximal flag in $\Delta(\mathcal{P})$. Then G is supported by a frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ such that each element in $\Sigma$ is of prime order. By Lemma 4.20 iv), G is solvable.

The rank 1 flag $\alpha_1$ is of prime order and $\Phi(\alpha_1) = \{ 1 \}$. Suppose there is an $i$, $2 \leq i \leq d+1$, such that $\Phi(\alpha_i) \neq \{ 1 \}$ and $\Phi(\alpha_{i-1}) = \{ 1 \}$ where $\alpha_{d+1} =$ G. Since $\alpha_i/\alpha_{i-1}$ is prime cyclic, $\Phi(\alpha_i) \leq \alpha_{i-1}$. The subgroup $\alpha_i$ splits over $\alpha_{i-1}$, so $\Phi(\alpha_i) < \alpha_{i-1}$. $\Phi(\alpha_i)$ is normal in $\alpha_{i-1}$ and let M be a minimal normal subgroup of $\alpha_{i-1}$ contained in $\Phi(\alpha_i)$. $\Phi(\alpha_{i-1}) = \{1\}$, so $\alpha_{i-1}$ splits over M. Hence M is in the flag space associated with $\alpha_{i-1}$. $\alpha_{i-1}$ is a flag in $\Delta(\mathcal{P})$, so M is a flag in $\Delta(\mathcal{P})$. Thus M is complemented in G. Hence G = MK, where K $<$ G and M $\cap$ K = $\{ 1 \}$. Therefore, $\alpha_i = \alpha_i \cap$ G $= \alpha_i \cap$ MK = M$(\alpha_i \cap$ K), which implies M is complemented in $\alpha_i$. This is a contradiction since M $\leq \Phi(\alpha_i)$. Thus $\Phi(\alpha_i) = \{ 1 \}$. Hence $\Phi$(G) $= \{ 1 \}$.

Let $1 = F_0(G) \lhd F_1(G) \lhd F_2(G) \lhd \ldots \lhd F_{r+1}(G) = G$ be the Fitting series for G. Since $\Phi(G) = \{ 1 \}$, G splits over $F_1(G)$. Therefore $F_1(G) \in \Delta(\mathcal{P})$ and $G/F_1(G)$ satisifies the hypothesis. Hence $\Phi(G/F_1(G)) = \{ 1_{G/F_1(G)} \}$. Proceed by induction on $r+1$. Let $2 \leq i \leq r+1$. $\Phi(G/F_{i-1}(G)) = \{ 1_{G/F_{i-1}(G)} \}$ and $G/F_{i-1}(G)$ splits over $F_i(G)/F_{i-1}(G)$. Since $F_i(G)/F_{i-1}(G)$ is in the flag space associated with $G/F_{i-1}(G)$, $(G/F_{i-1}(G))/(F_i(G)/F_{i-1}(G)) \cong G/F_i(G)$ satisfies the hypothesis. Therefore, $\Phi(G/F_i(G)) = \{ 1_{G/F_i(G)} \}$. By I.23 of [28], G is a K-group.

Let $1 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_l = G$ be a chief series for G. Suppose that $H_1$ is not of prime order. Then there is a subgroup N of G such that $1 < N < H_1$. $H_1$ is elementary abelian and $H_1$ splits over N. G splits over $H_1$, so N is a rank 1 flag in $\Delta(\mathcal{P})$. Therefore, G = NK, where K < G and N ∩ K = { 1 }. Consider $H_1 \cap$ K. $H_1 = H_1 \cap G = H_1 \cap NK = N(H_1 \cap K)$. Thus $H_1 \cap K \neq \{ 1 \}$ and $H_1 = N \times (H_1 \cap K)$. $H_1 \cap$ K is clearly normal in $H_1$. Let $x \in H_1 \cap K$ and $k \in K$. Thus $x^k \in H_1 \cap K$ and $H_1 \cap$ K is normal in K. $G = H_1K$, so $H_1 \cap$ K is normal in G. This contradicts $H_1$ being minimal normal in G. Thus $H_1$ is of prime order. Since G splits over $H_1$, $G/H_1$ satisfies both conditions in the hypothesis and $G/H_1$ is supersolvable. Since $H_2/H_1 \lhd \ldots \lhd H_r/H_1 = G/H_1$ is a chief series for $G/H_1$, $| H_i/H_1 / H_{i-1}/H_1 |$ is of prime order. Hence $H_i/H_1 / H_{i-1}/H_1 \cong H_i/H_{i-1}$ and G is supersolvable. By Theorem I.24 of [28], G is complemented □

**Case 3)** Each element of each frame is of prime order and there is a complemented flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$ such that $\{ 1 \} \lhd \alpha_1 \lhd \ldots \lhd \alpha_d \lhd G$ is an invariant series for G.

By Theorem 4.20 iv), G is solvable. Proceed by induction on $d$ to show G is a solvable nC-group. Since $\alpha_1 \lhd G$, $A'$: $\alpha_2/\alpha_1 \subset \ldots \subset \alpha_d/\alpha_1$ is a complemented flag in $\Delta(\mathcal{P}_{G/\alpha_1})$ and $\{ 1_{G/\alpha_1} \} \lhd \alpha_2/\alpha_1 \subset \ldots \subset \alpha_d/\alpha_1 \lhd G/\alpha_1$ is an invariant series for $G/\alpha_1$. Futhermore, by Lemma 4.20 ii), each element of each frame for $G/\alpha_1$ is of prime order. Thus G is a solvable nC-group. Hence $\Phi(G) \leq \alpha_1$. $\alpha_1$ is minimal normal in G, so $\Phi(G) = \{ 1 \}$. Thus by Theorem 6.1.4 of [2], F(G) is complemented in G. Hence by Theorem I.22 of [28], G is a solvable nC-group. This is essentially Theorem 5 of [6].

**Case 4)** Each subgroup of G is in some flag in $\Delta(\mathcal{P})$.

Thus each subgroup of G is subnormal in G and hence G is nilpotent. Furthermore, each element of each frame for G is of prime order. Thus each Sylow p-subgroup is of exponent $p$. An example of this type of group is $G = [Z_3 \times Z_3]Z_3 = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ab = ba, a^c = a, b^c = ab \rangle$.

# Chapter 5

# Col($\Delta(\mathcal{P})$)

By Theorem 2.30 ii), there is a homomorphism $\kappa$ from Aut(G) into Col($\Delta(\mathcal{P})$). If G is an elementary abelian p-group, the image of GL(n,p) under $\kappa$ in Col($\Delta(\mathcal{P})$) is the projective linear group PGL(n,p). Furthermore, PGL(n,p) $\cong$ GL(n,p)/Z(GL(n,p)) where Z(GL(n,p)) = ker($\kappa$). The following questions arise:

1) When is $\kappa$ an epimorphism and Col($\Delta(\mathcal{P})$) $\cong$ Aut(G)/ker($\kappa$)?

2) When is ker($\kappa$) = Z(Aut(G))?

## 5.1 Aut(G)/ker($\kappa$)

This investigation is begun by examining when Col($\Delta(\mathcal{P})$) $\cong$ Aut(G)/ker($\kappa$) for elementary abelian p-groups G.

By Definition 2.12, a rank 1 flag $\alpha$ of $\Delta(\mathcal{P})$ is called a rank 1 flag of type i if there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$ such that $\alpha_i = \alpha$.

**Lemma 5.1** *Let $\sigma \in$ Col($\Delta(\mathcal{P})$) and $\alpha$ be a rank 1 flag of type i.*

*a) $\alpha^\sigma$ is a rank 1 flag of type i.*

*b) If all the maximal flags of $\Delta(\mathcal{P})$ are of the same rank, $\alpha$ is only of one type.*

Proof. a) There is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, such that for some $i$, $1 \leq i \leq d$, $\alpha_i = \alpha$. Since $A^\sigma$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma$ is also a maximal flag in $\Delta(\mathcal{P})$, $\alpha^\sigma$ is of type $i$.

b) Suppose $\alpha$ is of type $i$ and of type $j$, where $i \neq j$. Then there are maximal flags A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ in $\Delta(\mathcal{P})$, such that $\alpha = \alpha_i$ and $\alpha = \beta_j$. Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$ be frames for G which support A and B respectively. Then $\alpha = S_1 \ldots S_i$ and $\Sigma'' = \{ S_1, \ldots, S_i, T_{j+1}, \ldots, T_{d+1} \}$ is a frame of length $d + 1 - j + i$. Since $i \neq j$, $\Sigma''$ supports a maximal flag which is not of rank $d$. This is a contradiction and $i = j$ □

**Example:** Consider G $= [D_8]Z_2 = \langle a, b, c, | \ a^4 = b^2 = c^2 = 1, a^b = a^3, a^c = a^3, b^c = a^2 b \rangle$. Both $\langle a \rangle \subset \langle a, b \rangle$ and $\langle a^2 \rangle \subset \langle a^2, b \rangle \subset \langle a, b \rangle$ are maximal flags in $\Delta(\mathcal{P})$. Thus $\langle a, b \rangle$ is a rank 1 flag of type 2 and 3. However, a rank 1 flag could never be of type 1 and type $i$, where $i \geq 2$. This would imply that the subgroup would be inseparable and separable at the same time.

**Lemma 5.2** *Let G be an elementary abelian p-group of dimension n and let $\sigma \in Col(\Delta(\mathcal{P}))$. Then $\sigma$ is determined by the bijection it induces on the rank 1 flags of type 1 in $\Delta(\mathcal{P})$.*

Proof. Let $\langle a_1 \rangle, \ldots, \langle a_m \rangle$ be the rank 1 flags in $\Delta(\mathcal{P})$ of type 1. Let $\alpha$ be a rank 1 flag in $\Delta(\mathcal{P})$. Then $\alpha = \langle a_{i_1}, \ldots, a_{i_t} \rangle$ for $1 \leq i_1 < i_2 < \ldots < i_t \leq m$ and $1 \leq t \leq n$. For each $j$, $1 \leq j \leq t$, $\langle a_{i_j} \rangle \subseteq \langle a_{i_1}, \ldots, a_{i_t} \rangle$, so by Theorem 2.28, $\langle a_{i_j} \rangle^\sigma \subseteq \langle a_{i_1}, \ldots, a_{i_t} \rangle^\sigma$. Thus $\langle a_{i_1}, \ldots, a_{i_t} \rangle^\sigma = \langle \langle a_{i_1} \rangle^\sigma, \ldots, \langle a_{i_t} \rangle^\sigma \rangle$ □

**Remark:** Not every bijection of the rank 1 flags of type 1 induces a collineation. Consider G $= Z_2 \times Z_2 \times Z_2 = \langle a, b, c \rangle$ and the following bijection $\pi$ of the rank 1 flags of type 1.

$$
\begin{array}{rrcl}
\pi: & \langle a\rangle & \mapsto & \langle c\rangle \\
& \langle b\rangle & \mapsto & \langle b\rangle \\
& \langle c\rangle & \mapsto & \langle a\rangle \\
& \langle ab\rangle & \mapsto & \langle ab\rangle \\
& \langle ac\rangle & \mapsto & \langle bc\rangle \\
& \langle bc\rangle & \mapsto & \langle ac\rangle \\
& \langle abc\rangle & \mapsto & \langle abc\rangle
\end{array}
$$

Since $\langle b\rangle \subset \langle a,b\rangle$ and $\langle b\rangle^{\pi} = \langle b\rangle$, the collineation induced by $\pi$ must send $\langle a,b\rangle$ to $\langle a,b\rangle, \langle b,ac\rangle$, or $\langle b,c\rangle$. $\langle ab\rangle^{\pi} = \langle ab\rangle$, so the collineation induced by $\pi$ must send $\langle a,b\rangle$ to $\langle a,b\rangle, \langle ab,c\rangle$, or $\langle ab,bc\rangle$. The only subgroup in common is $\langle a,b\rangle$. However, $\langle a\rangle^{\pi} = \langle c\rangle$ and $\langle c\rangle \not\subset \langle a,b\rangle$. Thus $\pi$ does not induce a collineation of $\Delta(\mathcal{P})$.

**Lemma 5.3** *Suppose G is an elementary abelian 3-group of dimension n. Let* $G = \langle x_1\rangle \times \ldots \times \langle x_n\rangle$ *and* $\sigma \in Col(\Delta(\mathcal{P}))$. *If for each t,* $1 \le t \le n$ *and* $1 \le i_1 < i_2 < \ldots < i_t \le n$, $\langle x_{i_1}x_{i_2}\ldots x_{i_t}\rangle^{\sigma} = \langle (x_{i_1}x_{i_2}\ldots x_{i_t})^{\Psi}\rangle$ *for some fixed* $\Psi$ *in GL(n,3), then* $\sigma$ *is induced by* $\Psi$.

Proof. Let $\langle x_{i_1}^{\alpha_{i_1}}\ldots x_{i_t}^{\alpha_{i_t}}\rangle$ be a 1-dimensional subspace of G such that for at least one $\alpha_{i_j}$, $1 \le j \le t$, $\alpha_{i_j} = 2$. Then without loss of generality, there is an $l$, $1 \le l \le t$ such that $\alpha_{i_1} = \ldots = \alpha_{i_l} = 2$ and $\alpha_{i_{l+1}} = \ldots = \alpha_{i_t} = 1$. Thus $x_{i_1}^{\alpha_{i_1}}\ldots x_{i_t}^{\alpha_{i_t}} = (x_{i_1}\ldots x_{i_l})(x_{i_1}\ldots x_{i_t})$. Consider the rank 1 flag $\alpha = \langle x_{i_1}\ldots x_{i_l}, x_{i_1}\ldots x_{i_t}\rangle$. Since $\sigma$ is a collineation, $\langle x_{i_1}\ldots x_{i_l}, x_{i_1}\ldots x_{i_t}\rangle^{\sigma} = \langle \langle x_{i_1}\ldots x_{i_l}\rangle^{\sigma}, \langle x_{i_1}\ldots x_{i_t}\rangle^{\sigma}\rangle = \langle\langle (x_{i_1}\ldots x_{i_l})^{\Psi}\rangle, \langle (x_{i_1}\ldots x_{i_t})^{\Psi}\rangle\rangle$ for some $\Psi$ in GL(n,3).

The rank 1 flag $\alpha$ contains four subgroups: $\langle x_{i_1}\ldots x_{i_l}\rangle, \langle x_{i_1}\ldots x_{i_t}\rangle, \langle x_{i_{l+1}}\ldots x_{i_t}\rangle$, and $\langle x_{i_1}^2\ldots x_{i_l}^2 x_{i_{l+1}}\ldots x_{i_t}\rangle$. By our hypothesis:

$$\begin{aligned} \langle x_{i_1} \ldots x_{i_l} \rangle^{\sigma} &= \langle (x_{i_1} \ldots x_{i_l})^{\Psi} \rangle \\ \langle x_{i_1} \ldots x_{i_t} \rangle^{\sigma} &= \langle (x_{i_1} \ldots x_{i_t})^{\Psi} \rangle \\ \langle x_{i_{l+1}} \ldots x_{i_t} \rangle^{\sigma} &= \langle (x_{i_{l+1}} \ldots x_{i_t})^{\Psi} \rangle \end{aligned}$$

The subgroup $\alpha^{\sigma} = \langle \langle (x_{i_1} \ldots x_{i_l})^{\Psi} \rangle, \langle (x_{i_1} \ldots x_{i_t})^{\Psi} \rangle \rangle$ has subgroups

$$\langle (x_{i_1} \ldots x_{i_l})^{\Psi} \rangle,$$

$$\langle \ (x_{i_1} \ \ldots \ x_{i_t})^{\Psi} \rangle,$$

$$\langle \ (x_{i_{l+1}} \ \ldots \ x_{i_t})^{\Psi} \rangle, \text{ and}$$

$$\langle \ (x_{i_1}^2 \ \ldots \ x_{i_l}^2 x_{i_{l+1}} \ \ldots \ x_{i_t})^{\Psi} \rangle.$$

Hence $\langle x_{i_1}^2 \ldots x_{i_l}^2 x_{i_{l+1}} \ldots x_{i_t} \rangle^{\sigma} = \langle (x_{i_1}^2 \ldots x_{i_l}^2 x_{i_{l+1}} \ldots x_{i_t})^{\Psi} \rangle$. Thus for each 1-dimensional subspace $\langle a \rangle$ of G, $\langle a \rangle^{\sigma} = \langle a^{\Psi} \rangle$. Since each 1-dimensional subspace of G is a rank 1 flag of type 1, by Lemma 5.3, $\sigma$ is induced by $\Psi$ □

**Theorem 5.4** *If G is an elementary abelian p-group of order $p^n$, then Col($\Delta(\mathcal{P})$) $\cong$ GL(n,p)/Z(GL(n,p)) $\cong$ PGL(n,p) only if $p = 2$ or 3.*

Proof. Suppose $p = 2$ and $\sigma \in$ Col($\Delta(\mathcal{P})$). Let $\{ a_1, \ldots, a_n \}$ be a basis for G. Since $\sigma \in$ Col($\Delta(\mathcal{P})$) , $\langle a_1 \rangle^{\sigma} = \langle b_1 \rangle, \ldots, \langle a_n \rangle^{\sigma} = \langle b_n \rangle$, where $\langle b_i \rangle, 1 \leq i \leq n$, are 1-dimensional subspaces of G and $\{ b_1, \ldots, b_n \}$ is also a basis for G.

It is proven that $\langle a_{i_1} \ldots a_{i_l} \rangle^{\sigma} = \langle b_{i_1} \ldots b_{i_l} \rangle$ for $1 \leq i_1 < \ldots < i_l \leq n$ and $1 \leq l \leq n$. Suppose $l = 2$. Since $\langle a_{i_1} \rangle^{\sigma} = \langle b_{i_1} \rangle, \langle a_{i_2} \rangle^{\sigma} = \langle b_{i_2} \rangle$, and $\langle a_{i_1}, a_{i_2} \rangle^{\sigma} = \langle b_{i_1}, b_{i_2} \rangle$, $\langle a_{i_1} a_{i_2} \rangle^{\sigma}$ must equal $\langle b_{i_1} b_{i_2} \rangle$ as $p = 2$ and $\langle a_{i_1} a_{i_2} \rangle$ and $\langle b_{i_1} b_{i_2} \rangle$ are the only other 1-dimensional subspaces of $\langle a_{i_1}, a_{i_2} \rangle$ and $\langle b_{i_1}, b_{i_2} \rangle$ respectively. Proceed by induction on $l$. Therefore, $\langle a_{i_1} \ldots a_{i_{l-1}} \rangle^{\sigma} = \langle b_{i_1} \ldots b_{i_{l-1}} \rangle$ and $\langle a_{i_l} \rangle^{\sigma} = \langle b_{i_l} \rangle$. Hence $\langle a_{i_1} \ldots a_{i_{l-1}}, a_{i_l} \rangle^{\sigma} = \langle b_{i_1} \ldots b_{i_{l-1}}, b_{i_l} \rangle$

and since $p = 2$, $\langle a_{i_1} \ldots a_{i_l} \rangle^\sigma = \langle b_{i_1} \ldots b_{i_l} \rangle$.

Since $\{ b_1, \ldots, b_n \}$ is a basis for G, there is a $\Psi \in$ GL(n,2) such that $a_i^\Psi = b_i$. Therefore, $\langle a_{i_1} \ldots a_{i_l} \rangle^\sigma = \langle b_{i_1} \ldots b_{i_l} \rangle = \langle (a_{i_1} \ldots a_{i_l})^\Psi \rangle$ for $1 \leq i_1 < \ldots < i_l \leq n$ and $1 \leq l \leq n$. By Lemma 5.2, $\sigma$ is induced by $\Psi$.

Suppose that $p = 3$. Let $\sigma \in \mathrm{Col}(\Delta(\mathcal{P}))$ and $\beta = \{ x_1, \ldots, x_n \}$ be a basis for G. For each $i$, $1 \leq i \leq n$, $\langle x_i \rangle$ is a rank 1 flag of type 1. Hence $\langle x_i \rangle^\sigma = \langle a_i \rangle$ is also a rank 1 flag of type 1. Since $\sigma \in \mathrm{Col}(\Delta(\mathcal{P}))$, $\{ a_1, \ldots, a_n \}$ is also a basis for G. Furthermore, for $1 \leq \alpha_i \leq 2$, $\{ a_1^{\alpha_1}, \ldots, a_n^{\alpha_n} \}$ is also a basis for G. Thus there is a $\Psi \in$ GL(n,3) such that $x_i^\Psi = a_i^{\alpha_i}$. It is proven that the $\alpha_i$'s can be choosen appropriately such that $\Psi$ induces $\sigma$.

For each flag $\langle x_i x_j \rangle$, where $x_i$ and $x_j \in \beta$, $1 \leq i, j \leq n$ and $i \neq j$, values for $\alpha_i$ and $\alpha_j$ can be found such that $\langle x_i x_j \rangle^\sigma = \langle (x_i x_j)^\Psi \rangle$. Consider the rank 1 flag $\langle x_i, x_j \rangle$. Since $\langle x_i \rangle^\sigma = \langle a_i \rangle$ and $\langle x_j \rangle^\sigma = \langle a_j \rangle$, $\langle x_i, x_j \rangle^\sigma = \langle a_i, a_j \rangle = \langle x_i^\Psi, x_j^\Psi \rangle$. Thus $\sigma$ induces the following maps:

$$\begin{aligned}
\langle x_i \rangle &\mapsto \langle a_i \rangle \\
\langle x_j \rangle &\mapsto \langle a_j \rangle \\
\langle x_i x_j \rangle &\mapsto \langle a_i^N a_j \rangle \\
\langle x_i^2 x_j \rangle &\mapsto \langle a_i^{3-N} a_j \rangle.
\end{aligned}$$

where $N = 1$ or 2. If $N = 1$, let $\alpha_i = \alpha_j = 1$ or $\alpha_i = \alpha_j = 2$. If $N = 2$, let $\alpha_i = 2$ and $\alpha_j = 1$ or $\alpha_i = 1$ and $\alpha_j = 2$. If $n = 2$, $\sigma$ is induced by $\Psi$ by Lemma 5.3.

Suppose that $n \geq 3$. It must be made sure that our choices for $\alpha_i$ and $\alpha_j$ will make it so for each rank 1 flag $\langle x_l x_k \rangle$, that $\langle x_l x_k \rangle^\sigma = \langle (x_l x_k)^\Psi \rangle$. This is valid if $l \neq i, j$ and $k \neq i, j$ (proceed as above). If $l = i$ or $j$ and $k = i$ or $j$, this is the same case as above. Thus assume w.l.o.g. that $l = i$ or $j$ and $k \neq i, j$. Consider the following 3 flags : $\langle x_i, x_j \rangle$, $\langle x_i, x_k \rangle$, and $\langle x_j, x_k \rangle$. $\sigma$ induces the following maps:

$$\begin{array}{rcl|rcl|rcl}
\langle x_i, x_j\rangle & \mapsto & \langle a_i, a_j\rangle & \langle x_i, x_k\rangle & \mapsto & \langle a_i, a_k\rangle & \langle x_j, x_k\rangle & \mapsto & \langle a_j, a_k\rangle \\
\langle x_i\rangle & \mapsto & \langle a_i\rangle & \langle x_i\rangle & \mapsto & \langle a_i\rangle & \langle x_j\rangle & \mapsto & \langle a_j\rangle \\
\langle x_j\rangle & \mapsto & \langle a_j\rangle & \langle x_k\rangle & \mapsto & \langle a_k\rangle & \langle x_k\rangle & \mapsto & \langle a_k\rangle \\
\langle x_i x_j\rangle & \mapsto & \langle a_i^N a_j\rangle & \langle x_i x_k\rangle & \mapsto & \langle a_i^M a_k\rangle & \langle x_j x_k\rangle & \mapsto & \langle a_j^R a_k\rangle \\
\langle x_i^2 x_j\rangle & \mapsto & \langle a_i^{3-N} a_j\rangle & \langle x_i^2 x_k\rangle & \mapsto & \langle a_i^{3-M} a_k\rangle & \langle x_j^2 x_k\rangle & \mapsto & \langle a_j^{3-R} a_k\rangle \\
N & = & 1 \text{ or } 2 & M & = & 1 \text{ or } 2 & R & = & 1 \text{ or } 2
\end{array}$$

Consider the following subgroups:$\langle x_i, x_j x_k\rangle$, $\langle x_j, x_i x_k\rangle$, and $\langle x_k, x_i x_j\rangle$. $\sigma$ induces the following maps:

$$\begin{array}{rcl|rcl}
\langle x_i, x_j x_k\rangle & \mapsto & \langle a_i, a_j^R a_k\rangle & \langle x_j, x_i x_k\rangle & \mapsto & \langle a_j, a_i^M a_k\rangle \\
\langle x_i\rangle & \mapsto & \langle a_i\rangle & \langle x_j\rangle & \mapsto & \langle a_j\rangle \\
\langle x_j x_k\rangle & \mapsto & \langle a_j^R a_k\rangle & \langle x_i x_k\rangle & \mapsto & \langle a_i^M a_k\rangle \\
\langle x_i x_j x_k\rangle & \mapsto & \langle a_i^\alpha a_j^R a_k\rangle & \langle x_i x_j x_k\rangle & \mapsto & \langle a_i^M a_j^\beta a_k\rangle \\
\alpha & = & 1 \text{ or } 2 & \beta & = & 1 \text{ or } 2
\end{array}$$

$$\begin{array}{rcl}
\langle x_k, x_i x_j\rangle & \mapsto & \langle a_k, a_i^N a_j\rangle \\
\langle x_k\rangle & \mapsto & \langle a_k\rangle \\
\langle x_i x_j\rangle & \mapsto & \langle a_i^N a_j\rangle \\
\langle x_i x_j x_k\rangle & \mapsto & \langle a_i^N a_j a_k^\gamma\rangle \\
\gamma & = & 1 \text{ or } 2
\end{array}$$

$\langle x_i x_j x_k\rangle$ must be mapped by $\sigma$ into one rank 1 flag . If $N = 1$, $\gamma = 1$ or 2. If $\gamma = 1$, $M = \beta = 1$ and $\alpha = R = 1$. Thus $\sigma$ is induced by $\Psi$ where $\alpha_i = \alpha_j = \alpha_k = 1$. If $\gamma = 2$,

$M = \beta = 2$ and $\alpha = R = 2$. Thus $\sigma$ is induced by $\Psi$ where $\alpha_i = 1, \alpha_j = 1$, and $\alpha_k = 2$. Suppose $N = 2$. Then $\gamma = 1$ or $2$. If $\gamma = 1$, $M = 2, \beta = 1, \alpha = 2$, and $R = 1$. Thus $\sigma$ is induced by $\Psi$ when $\alpha_i = 2, \alpha_j = 1$, and $\alpha_k = 1$. If $\gamma = 2$, then $M = 1, \beta = 2, \alpha = 1$, and $R = 2$. Thus $\sigma$ is induced by $\Psi$ when $\alpha_i = 2, \alpha_j = 1$, and $\alpha_k = 2$. Hence for each pair $x, y \in \beta$, $\alpha_i$'s can be found, such that $\langle xy\rangle^\sigma = \langle (xy)^\Psi\rangle$.

If $n = 3$, $\beta = \{ x_1, x_2, x_3 \}$. It was shown in the previous argument that $\langle x_1x_2x_3\rangle^\sigma = \langle (x_1x_2x_3)^\Psi\rangle$. By Lemma 5.3, $\sigma$ is induced by $\Psi$.

Let $n \geq 4$. Suppose for some $m$, $3 \leq m \leq n$, that $m$ is minimal with respect to $\langle x_{i_1} \dots x_{i_m}\rangle^\sigma \neq \langle (x_{i_1} \dots x_{i_m})^\Psi\rangle$ for any choice of the $\alpha_i$'s. Consider the following rank 1 flags : $\langle x_{i_1}, x_{i_2} \dots x_{i_m}\rangle$, $\langle x_{i_2}, x_{i_1}x_{i_3} \dots x_{i_m}\rangle$, and $\langle x_{i_1}x_{i_2}, x_{i_3} \dots x_{i_m}\rangle$. $\sigma$ induces the following maps:

$$
\begin{array}{rcl}
\langle x_{i_1}, x_{i_2} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_1}^{\alpha_{i_1}}, a_{i_2}^{\alpha_{i_2}} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\langle x_{i_1}\rangle & \mapsto & \langle a_{i_1}^{\alpha_{i_1}}\rangle \\
\langle x_{i_2} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_2}^{\alpha_{i_2}} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\langle x_{i_1} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_1}^{\alpha_{i_1}N} a_{i_2}^{\alpha_{i_2}} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\hline
\langle x_{i_2}, x_{i_1}x_{i_3} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_2}^{\alpha_{i_2}}, a_{i_1}^{\alpha_{i_1}} a_{i_3}^{\alpha_{i_3}} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\langle x_{i_2}\rangle & \mapsto & \langle a_{i_2}^{\alpha_{i_2}}\rangle \\
\langle x_{i_1}x_{i_3} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_1}^{\alpha_{i_1}} a_{i_3}^{\alpha_{i_3}} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\langle x_{i_1} \dots x_{i_m}\rangle & \mapsto & \langle a_{i_1}^{\alpha_{i_1}} a_{i_2}^{\alpha_{i_2}M} \dots a_{i_m}^{\alpha_{i_m}}\rangle \\
\hline
\end{array}
$$

$$
\begin{aligned}
\langle x_{i_1} x_{i_2}, x_{i_3} \ldots x_{i_m} \rangle &\mapsto \langle a_{i_1}^{\alpha_{i_1}} a_{i_1}^{\alpha_{i_1}}, a_{i_3}^{\alpha_{i_3}} \ldots a_{i_m}^{\alpha_{i_m}} \rangle \\
\langle x_{i_1} x_{i_2} \rangle &\mapsto \langle a_{i_1}^{\alpha_{i_1}} a_{i_2}^{\alpha_{i_2}} \rangle \\
\langle x_{i_3} \ldots x_{i_m} \rangle &\mapsto \langle a_{i_3}^{\alpha_{i_3}} \ldots a_{i_m}^{\alpha_{i_m}} \rangle \\
\langle x_{i_1} \ldots x_{i_m} \rangle &\mapsto \langle a_{i_1}^{\alpha_{i_1} R} a_{i_2}^{\alpha_{i_2} R} \ldots a_{i_m}^{\alpha_{i_m}} \rangle
\end{aligned}
$$

where $1 \leq N, M, R \leq 2$. If $N = 1$, then $\sigma$ does map $\langle x_{i_1} \ldots x_{i_m} \rangle$ to $\langle (x_{i_1} \ldots x_{i_m})^{\Psi} \rangle$. Suppose $N = 2$. Then no matter what $M$ and $R$ are, $\sigma$ will not be a collineation as $\langle x_{i_1} \ldots x_{i_m} \rangle$ can't be mapped to two different flags . Thus $N = 1$ and we have a contradiction.

Hence for each rank 1 flag $\langle x_{i_1} \ldots x_{i_l} \rangle$, $1 \leq i_1 < i_2 < \ldots < i_l \leq n$ and $1 \leq l \leq n$, $\langle x_{i_1} \ldots x_{i_l} \rangle^{\sigma} = \langle (x_{i_1} \ldots x_{i_l})^{\Psi} \rangle$. Thus by Lemma 5.3, $\sigma$ is induced by $\Psi$.

Suppose $p \geq 5$. Let G $= \langle a_1, \ldots, a_n \rangle$ and consider the following collineation $\sigma$ of $\Delta(\mathcal{P})$ :

$$
\begin{aligned}
\langle a_1 \rangle^{\sigma} &= \langle a_2 \rangle \\
\langle a_2 \rangle^{\sigma} &= \langle a_1 a_2 \rangle \\
\langle a_1 a_2 \rangle^{\sigma} &= \langle a_1 \rangle \\
\langle a_1, a_2 \rangle^{\sigma} &= \langle a_1, a_2 \rangle \\
\langle a_1, H \rangle^{\sigma} &= \langle a_2, H \rangle \text{ for H } \not\leq \langle a_1, a_2 \rangle \\
\langle a_2, K \rangle^{\sigma} &= \langle a_1 a_2, K \rangle \text{ for K } \not\leq \langle a_1, a_2 \rangle \\
\langle a_1 a_2, L \rangle^{\sigma} &= \langle a_1, L \rangle \text{ for L } \not\leq \langle a_1, a_2 \rangle \\
\langle a_1 b \rangle^{\sigma} &= \langle a_2 b \rangle \text{ for } \langle a_1 b \rangle \notin \langle a_1, a_2 \rangle \\
\langle a_2 c \rangle^{\sigma} &= \langle a_1 a_2 c \rangle \text{ for } \langle a_2 c \rangle \notin \langle a_1, a_2 \rangle
\end{aligned}
$$

$$\langle a_1 a_2 d\rangle^\sigma = \langle a_1 d\rangle \text{ for } \langle a_1 a_2 d\rangle \notin \langle a_1, a_2\rangle$$
$$M^\sigma = M \text{ for all other rank 1 flags in } \Delta(\mathcal{P})$$

where $H, K$ and $L$ are subgroups of G such that $| H | \leq p^{n-2}$ and $b, c$ and $d$ are elements of G. If $\sigma$ is induced by a $\Psi \in$ GL(n,p), then

$$a_1^\Psi = a_2^\alpha, 1 \leq \alpha \leq p-1 \text{ and}$$
$$a_2^\Psi = a_1^\beta a_2^\beta, 1 \leq \beta \leq p-1.$$

Thus $(a_1a_2)^\Psi = a_2^\alpha a_1^\beta a_2^\beta = a_1^\beta a_2^{\alpha+\beta}$. Since $\langle a_1 a_2\rangle^\sigma = \langle a_1\rangle$, $\alpha + \beta \equiv 0 \pmod p$. Furthermore, $(a_1^2a_2)^\Psi = a_2^{2\alpha} a_1^\beta a_2^\beta = a_1^\beta a_2^{2\alpha+\beta}$. Since $\langle a_1^2 a_2\rangle^\sigma = \langle a_1^2 a_2\rangle$, $a_1^\beta a_2^{2\alpha+\beta} \in \langle a_1^2 a_2\rangle$. Hence $2(2\alpha+\beta) \equiv \beta \pmod p$. Thus $4\alpha + 2\beta \equiv \beta \pmod p$ and $4\alpha+\beta \equiv 0 \pmod p$. Therefore, $4\alpha + \beta \equiv \alpha + \beta \pmod p$ or $3\alpha \equiv 0 \pmod p$. Since $p \geq 5$, $\alpha = 0$. This is a contradiction. Hence $\sigma$ can't be induced by any $\Psi \in$ GL(n,p) □

**Theorem 5.5** *Suppose that G is a separable, non-elementary, abelian p-group with $p \geq 3$. Then Col($\Delta(\mathcal{P})$) is not the image under $\kappa$ of Aut(G).*

Proof. $G = \langle a_1\rangle \times \ldots \times \langle a_n\rangle$ where $| a_i | = p^{r_i}, 1 \leq i \leq n$. Without loss of generality, assume $r_1 \geq r_2$. Define a map $\sigma \in$ Col($\Delta(\mathcal{P})$) in the following manner:

$$
\begin{aligned}
\langle a_1\rangle^\sigma &= \langle a_1 a_2\rangle \\
\langle a_1 a_2\rangle^\sigma &= \langle a_1\rangle \\
\langle a_1 b\rangle^\sigma &= \langle a_1 a_2 b\rangle \text{ for } b \notin \langle a_1, a_2\rangle \\
\langle a_1 a_2 c\rangle^\sigma &= \langle a_1 c\rangle \text{ for } c \notin \langle a_1, a_2\rangle \\
\langle a_1, a_2\rangle^\sigma &= \langle a_1, a_2\rangle \\
\langle a_1, H\rangle^\sigma &= \langle a_1 a_2, H\rangle \text{ for } H \not\leq \langle a_1, a_2\rangle \\
\langle a_1 a_2, K\rangle^\sigma &= \langle a_1, K\rangle \text{ for } K \not\leq \langle a_1, a_2\rangle \\
\gamma^\sigma &= \gamma \text{ for all other rank 1 flags in } \Delta(\mathcal{P})
\end{aligned}
$$

where $H$ and $K$ are proper subgroups of G and $b$ and $c$ are elements of G.

If $\sigma$ is induced by an element $\Psi \in \mathrm{Aut}(G)$, then $a_1^\Psi = a_1^\alpha a_2^\beta$ and $a_2^\Psi = a_1^x a_2^y$. Since $\langle a_2\rangle^\sigma = \langle a_2\rangle$, $a_2^\Psi \in \langle a_2\rangle$. Hence $x = 0$ and $y \not\equiv 0 \pmod p$. In addition, $\langle a_1 a_2\rangle^\sigma = \langle a_1\rangle$, so $(a_1 a_2)^\Psi \in \langle a_1\rangle$ and $(a_1 a_2)^\Psi = a_1^\alpha a_2^\beta a_2^y \in \langle a_1\rangle$. Hence, $\beta + y \equiv 0 \pmod{p^{r_2}}$. Lastly, $(\langle a_1\rangle)^\sigma = \langle a_1 a_2\rangle$ and $a_1^\Psi \in \langle a_1 a_2\rangle$. Hence, $a_1^\Psi = a_1^\alpha a_2^\beta \in \langle a_1 a_2\rangle$ and $\alpha \equiv \beta \pmod{p^{r_2}}$.

Since $r_1 > r_2$. There is an $r$, such that $r_1 - r = r_2$. Consider the rank 1 flag $\langle a_1^{p^r} a_2\rangle$. Thus $(\langle a_1^{p^r} a_2\rangle)^\sigma = \langle a_1^{p^r} a_2\rangle$ and

$$
\begin{aligned}
(a_1^{p^r} a_2)^\Psi &\in \langle a_1^{p^r} a_2\rangle \\
(a_1^\alpha a_2^\beta)^{p^r} a_2^y &\in \langle a_1^{p^r} a_2\rangle \\
a_1^{\alpha p^r} a_2^{\beta p^r + y} &\in \langle a_1^{p^r} a_2\rangle \\
a_1^{\alpha p^r} a_2^{\beta p^r + y} &= (a_1^{p^r} a_2)^\lambda \\
a_1^{\alpha p^r} a_2^{\beta p^r + y} &= a_1^{p^r \lambda} a_2^\lambda .
\end{aligned}
$$

Thus $\alpha p^r \equiv \lambda p^r \pmod{p^{r_1}}$ and $\alpha \equiv \lambda \pmod{p^{r_2}}$. Since $\beta p^r + y \equiv \lambda \pmod{p^{r_2}}$, $\beta p^r + y \equiv \alpha \pmod{p^{r_2}}$. Thus $\beta(p^r - 1) + \beta + y \equiv \alpha \pmod{p^{r_2}}$. Given that $\beta + y \equiv 0 \pmod{p^{r_2}}$, $\beta(p^r - 1) \equiv \alpha \pmod{p^{r_2}}$. But $\alpha \equiv \beta \pmod{p^{r_2}}$ and since $\alpha \not\equiv 0 \pmod{p}$, $p^r - 1 \equiv 1 \pmod{p^{r_2}}$. Since $p > 2$, this is a contradiction.

Suppose $r_1 = r_2$. Consider the rank 1 flag $\langle a_1 a_2^p \rangle$. Thus $(\langle a_1 a_2^p \rangle)^\sigma = \langle a_1 a_2^p \rangle$, which implies $(a_1 a_2^p)^\Psi = a_1^\alpha a_2^\beta a_2^{yp} \in \langle a_1 a_2^p \rangle$. Then

$$\begin{aligned} a_1^\alpha a_2^\beta a_2^{yp} &= (a_1 a_2^p)^\lambda \\ a_1^\alpha a_2^{\beta + yp} &= a_1^\lambda a_2^{p\lambda}. \end{aligned}$$

Thus $\alpha = \lambda \pmod{p^{r_1}}$. Furthermore, $\beta + yp \equiv p\lambda \pmod{p^{r_1}}$ and $\beta + yp \equiv p\alpha \pmod{p^{r_1}}$. Thus $\beta + y + y(p-1) \equiv p\alpha \pmod{p^{r_1}}$. Since $\beta + y \equiv 0 \pmod{p^{r_2}}$ and $p^{r_1} = p^{r_2}$, $y(p-1) \equiv p\alpha \pmod{p^{r_1}}$. This is a contradiction as $y \not\equiv 0 \pmod{p}$ □

**Theorem 5.6** *Let $d_1, \ldots, d_m$ be the different ranks of the maximal flags of $\Delta(\mathcal{P})$. Suppose for each $i$, $1 \le i \le m$, the $d_i$-primary flag $B_i$: $\beta_{i,1} \subset \ldots \subset \beta_{i,e_i}$ of $\Delta(\mathcal{P})$ satisfy the following conditions:*

*1) There is a $j$, $1 \le j \le e_i$, such that $\beta_{i,1} \subset \ldots \subset \beta_{i,j-1}$ and $\beta_{i,j+2}/\beta_{i,j+1} \subset \ldots \subset \beta_{i,e_i}/\beta_{i,j+1}$ are maximal flags in $\Delta(\mathcal{P}_{\beta_{i,j}})$ and $\Delta(\mathcal{P}_{G/\beta_{i,j+1}})$ respectively;*

*2) $\beta_{i,j+1}/\beta_{i,j}$ is an elementary abelian p-group of dimension n such that Aut(G) acts $(n-1)$-transitively on the 1-dimensional subspaces of $\beta_{i,j+1}/\beta_{i,j}$.*

*Then $\kappa$ maps Aut(G) onto Col($\Delta(\mathcal{P})$).*

Proof. Let $\sigma \in$ Col($\Delta(\mathcal{P})$) and let $A_1 \in \Delta(\mathcal{P})$. Then $A_1$ is contained in a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$. Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ be a frame which supports A. Let B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ be that $d$-primary flag of $\Delta(\mathcal{P})$. By 1), there is a $j$, $1 \le j \le e$ and $1 \le l \le d$, such that $S_1 = \beta_1$, $S_1 S_2 = \beta_2$, ..., $S_1 \ldots S_j = \beta_j, S_1 \ldots S_{j+l} = \beta_{j+1}, \ldots$,

$S_1 \ldots S_{n-1} = \beta_e$.

$A^\sigma$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma$ is a maximal flag supported by a frame $\Sigma' = \{T_1, \ldots, T_n\}$. By 1), $T_1 = \beta_1$, $T_1T_2 = \beta_2, \ldots, T_1 \ldots T_j = \beta_j, T_1 \ldots T_{j+l} = \beta_{j+1}, \ldots, T_1 \ldots T_{n-1} = \beta_e$. By Lemma 1.10, $\Sigma_o' = \{ T_{j+1}\beta_j/\beta_j, \ldots, T_{j+l}\beta_j/\beta_j \}$ and $\Sigma_o = \{ S_{j+1}\beta_j/\beta_j, \ldots, S_{j+l}\beta_j/\beta_j \}$ are frames for $\beta_{j+1}/\beta_j$. By 2), there is a $\Psi \in \mathrm{Aut(G)}$ such that $(S_{j+k}\beta_j/\beta_j)^\Psi = (T_{j+k}\beta_j/\beta_j)$ where $1 \le k \le l-1$. Since $\beta_1, \ldots, \beta_e$ are characteristic in G, for $1 \le i \le d$, $(S_1 \ldots S_i)^\sigma = T_1 \ldots T_i = (S_1 \ldots S_i)^\Psi$. Hence $A^\sigma = A^\Psi$. Since $A_1 \subseteq A$ and $A_1^\sigma \subseteq A^\sigma$, $A_1^\sigma = A_1^\Psi$ □

**Example.** Consider the symmetric group $S_4 = \langle x, y, a, b \mid x^2 = y^2 = a^3 = b^2 = 1, xy = yx, x^a = xy, x^b = x, y^a = xy, y^b = xy, a^b = xya^2 \rangle$. All maximal flags in $\Delta(\mathcal{P})$ are of rank 3 and the 3-primary flag is $\langle x, y \rangle \subset \langle x, y, a \rangle$. $\langle x, y \rangle$ is elementary abelian with $x^a = y$, $y^a = xy$, and $(xy)^a = x$. By Theorem 5.6, $\mathrm{Aut(G)/ker}(\kappa) \cong \mathrm{Col}(\Delta(\mathcal{P}))$.

## 5.2 Ker($\kappa$)

For an elementary abelian p-group, $\ker(\kappa) = Z(\mathrm{Aut(G)})$. This result is generalized here.

**Theorem 5.7** *Let G be an abelian group and $\sigma \in Aut(G)$. Then $\sigma \in ker(\kappa)$ if and only if $\sigma \in Z(Aut(G))$.*

Proof. By the Fundamental Theorem of Abelian Groups, $G \cong Z_{p_1^{r_1}} \oplus \ldots \oplus Z_{p_n^{r_n}}$. Hence for any frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, there is a $\pi \in \mathrm{Sym}(n)$ such that $S_{\pi(i)} \cong Z_{p_i^{r_i}}$ for $1 \le i \le n$. Let $S_i = \langle a_i \rangle$.

Suppose $\sigma \in \ker(\kappa)$. Then $(\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \ldots \subset \langle a_1, a_2, \ldots, a_{n-1} \rangle)^\sigma = \langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \ldots \subset \langle a_1, a_2, \ldots, a_{n-1} \rangle$ and $\langle a_1 \rangle^\sigma = \langle a_1 \rangle$. G is abelian, so by Lemma 1.7, for each $i$, $1 \le i \le n$, $\Sigma_i = \{ \langle a_i \rangle, \langle a_1 \rangle, \ldots, \langle a_{i-1} \rangle, \langle a_{i+1} \rangle, \ldots, \langle a_n \rangle \}$ is also a frame for G. Thus $\langle a_i \rangle^\sigma = \langle a_i \rangle$ and $a_i^\sigma = a_i^{t_i}$.

Let $1 \leq i \leq n$ and consider $\langle a_i \rangle$. Then $|\, a_i \,| = p_i^{r_i}$ and let $P_i$ be the Sylow $p_i$-subgroup of G. Thus $P_i = \langle a_{j_1} \rangle \oplus \ldots \oplus \langle a_{j_m} \rangle$ where $|\, a_{j_l} \,| = p_i^{r_{j_l}}$. For some $k$, $1 \leq k \leq m$, $\langle a_i \rangle = \langle a_{j_k} \rangle$. To simplify the notation, let $p = p_i$ and $P = P_i$.

Let $\Psi \in \mathrm{Aut}(G)$. Since $P$ is characteristic in G, $a_i^{\Psi} = a_{j_1}^{\alpha_{i,j_1}} \ldots a_{j_m}^{\alpha_{i,j_m}}$. Now $a_i^{\sigma\Psi} = (a_i^{t_i})^{\Psi}$ $= (a_i^{\Psi})^{t_i} = a_{j_1}^{t_i\alpha_{i,j_1}} \ldots a_{j_m}^{t_i\alpha_{i,j_m}}$ and $a_i^{\Psi\sigma} = (a_{j_1}^{\alpha_{i,j_1}} \ldots a_{j_m}^{\alpha_{i,j_m}})^{\sigma} = a_{j_1}^{t_{j_1}\alpha_{i,j_1}} \ldots a_{j_m}^{t_{j_m}\alpha_{i,j_m}}$. Thus for $a_i^{\sigma\Psi}$ to equal $a_i^{\Psi\sigma}$, $t_i\alpha_{i,j_l} \equiv t_{j_l}\alpha_{i,j_l} \pmod{p^{r_{j_l}}}$ for each $l$, $1 \leq l \leq m$. This is trivially true when $l = k$.

Case 1) $|\, a_{j_l} \,| \leq |\, a_i \,|$ for $1 \leq l \leq m$, $l \neq k$.

$\langle a_{j_l} a_i \rangle$ is a subgroup for G and in some frame for G. Hence $a_{j_l}^{t_{j_l}} a_i^{t_i} = (a_{j_l} a_i)^{\sigma} = (a_{j_l} a_i)^{s}$ $= a_{j_l}^{s} a_i^{s}$, where $1 \leq s \leq p^{r_i} - 1$. Therefore $s \equiv t_{j_l} \pmod{p^{r_{j_l}}}$ and $s \equiv t_i \pmod{p^{r_i}}$. Hence, $s - t_{j_l} = \lambda_1 p^{r_{j_l}}$ and $s - t_i = \lambda_2 p^{r_i}$. This implies $s = \lambda_1 p^{r_{j_l}} + t_{j_l}$ and $s = \lambda_2 p^{r_i} + t_i$. Therefore, $\lambda_1 p^{r_{j_l}} + t_{j_l} = \lambda_2 p^{r_i} + t_i$ and $t_{j_l} - t_i = \lambda_2 p^{r_i} - \lambda_1 p^{r_{j_l}}$. Since $|\, a_{j_l} \,| \leq |\, a_i \,|$, $r_i = r_{j_l} + r$ and $t_{j_l} - t_i = \lambda_2 p^{r_{j_l}+r} - \lambda_1 p^{r_{j_l}}$. Thus $t_{j_l} - t_i = (\lambda_2 p^{r} - \lambda_1) p^{r_{j_l}}$ and $t_i \equiv t_{j_l} \pmod{p^{r_{j_l}}}$. Therefore, $t_i\alpha_{i,j_l} \equiv t_{j_l}\alpha_{i,j_l} \pmod{p^{r_{j_l}}}$.

Case 2) $|\, a_{j_l} \,| > |\, a_i \,|$ for $1 \leq l \leq m$, $l \neq k$.

Since $p^{r_i} = |\, a_i \,|$, $p^{r_i} = |\, a_i^{\Psi} \,| = |\, a_{j_1}^{\alpha_{i,j_1}} \ldots a_{j_m}^{\alpha_{i,j_m}} \,|$. Thus for each $l$, $1 \leq l \leq m$, $(a_{j_l}^{\alpha_{i,j_l}})^{p^{r_i}}$ $= 1$. This implies that $a_{j_l}^{\alpha_{i,j_l} p^{r_i}} = 1$ and that $\alpha_{i,j_l} p^{r_i} \equiv 0 \pmod{p_i^{r_{j_l}}}$. Therefore $\alpha_{i,j_l} p^{r_i} =$ $\lambda p^{r_{j_l}}$ and $\alpha_{i,j_l} = \lambda p^{r_{j_l} - r_i}$. In a procedure similar to the one presented in case 1), it can be shown that $t_i \equiv t_{j_l} \pmod{p^{r_i}}$. Then

$$\begin{aligned}
t_i - t_{j_l} &= \lambda' p^{r_i} \\
(t_i - t_{j_l})\alpha_{i,j_l} &= \lambda' p^{r_i}\alpha_{i,j_l} \\
t_i\alpha_{i,j_l} - t_{j_l}\alpha_{i,j_l} &= \lambda' p^{r_i} \lambda p^{r_{j_l} - r_i} \\
t_i\alpha_{i,j_l} - t_{j_l}\alpha_{i,j_l} &= \lambda' \lambda p^{r_{j_l}}.
\end{aligned}$$

Hence $t_i\alpha_{i,j_l} \equiv t_{j_l}\alpha_{i,j_l} \pmod{p^{r_{j_l}}}$ and $\sigma \in Z(\text{Aut}(G))$.

Conversely, let $\sigma \in Z(\text{Aut}(G))$. To show that $\sigma \in \ker(\kappa)$, it is shown that $\langle a_i\rangle^\sigma = \langle a_i\rangle$. Let $1 \le i \le n$ and $|\langle a_i\rangle| = p^{r_i}$. If $\langle a_i\rangle$ is the Sylow p-subgroup of G, then $a_i^\sigma \in \langle a_i\rangle$. If $\langle a_i\rangle$ is not the Sylow p-subgroup of G, there is a $j$, $1 \le j \le n$ and $j \ne i$ such that $|\langle a_j\rangle| = p^{r_j}$.

Case 1) $r_i \le r_j$.

Let $H = \langle a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n\rangle$. Then $G = [H]\langle a_j\rangle$ and define a map $f$ as follows:

$$f(a) = a \text{ for all } a \in H$$

$$f(a_j) = a_i a_j.$$

This map is well defined and $f$ is an automorphism of G. Let $\Psi$ be the map which sends each element in G to its inverse. Since G is abelian, $\Psi$ is an automorphism of G. Furthermore, the collection of endomorphisms, End(G), of G is a ring. Since $f, \Psi \in \text{Aut}(G)$, $f, \Psi \in \text{End}(G)$ and $f + \Psi \in \text{End}(G)$. Clearly, $\ker(f + \Psi) = H$ and $\text{im}(f + \Psi) = \langle a_i\rangle$ $(a_j^{f+\Psi} = a_j^f a_j^\Psi = a_i a_j a_j^{-1} = a_i)$. Thus $\langle a_i\rangle = G^{f+\Psi} = G^{(f+\Psi)^\sigma}$ since $\sigma \in Z(\text{Aut}(G))$. Hence $\langle a_i\rangle = G^{\sigma^{-1}(f+\Psi)\sigma} = G^{(f+\Psi)\sigma} = \langle a_i\rangle^\sigma$.

Case 2) $r_i > r_j$.

$P = \langle a_i\rangle \times \langle b_1\rangle \times \ldots \times \langle b_m\rangle$ where for each $k$, $1 \le k \le m$, $|b_k| < |a_i|$. If $a_i^\sigma \notin \langle a_i\rangle$, then $a_i^\sigma = a_i^\alpha b_1^{\alpha_1} \ldots b_m^{\alpha_m}$ where for some $l$, $1 \le l \le m$, $\alpha_l \ne 0$. Since $r_i > r_l$, there is an $r$ such that $r_l + r = r_i$. Consider the element $a_i^{p^r}$. Then $(a_i^{p^r})^{p^{r_l}} = a_i^{p^r p^{r_l}} = a_i^{p^{r+r_l}} = a_i^{p^{r_i}} = 1$. Thus $|a_i^{p^r}| = p^{r_l}$. Define a map $\Psi$ as follows:

$$\begin{aligned} \Psi(a) &= a \text{ if } |a| = x \text{ and } (x,p) = 1 \\ \Psi(a_i) &= a_i \\ \Psi(b_j) &= b_j \text{ for } j \ne l \\ \Psi(b_l) &= a_i^{p^r} b_l. \end{aligned}$$

$\Psi$ is an automorphism of G and

$$\begin{aligned} a_i^{\sigma\Psi} &= a_i^{\Psi\sigma} \\ (a_i^{\alpha} b_1^{\alpha_1} \dots b_m^{\alpha_m})^{\Psi} &= a_i^{\sigma} \\ a_i^{\alpha} b_1^{\alpha_1} \dots (a_i^{p^r} b_l)^{\alpha_l} \dots b_m^{\alpha_m} &= a_i^{\alpha} b_1^{\alpha_1} \dots b_m^{\alpha_m} \\ a_i^{\alpha+\alpha_l p^r} b_1^{\alpha_1} \dots b_m^{\alpha_m} &= a_i^{\alpha} b_1^{\alpha_1} \dots b_m^{\alpha_m}. \end{aligned}$$

Thus $\alpha + \alpha_l p^r \equiv \alpha \pmod{p^{r_i}}$ and $\alpha_l p^r \equiv 0 \pmod{p^{r_i}}$. If $\alpha_l \not\equiv 0 \pmod{p}$, then $p^r \equiv 0 \pmod{p^{r_i}}$. This is a contradiction since $r < r_i$. Thus $\alpha_l \equiv 0 \pmod{p}$. Then $\alpha_l = \lambda p^t$ where $1 \le t \le p^{r_l - 1}$ and $\lambda \not\equiv 0 \pmod{p}$. Therefore, $\lambda p^t p^r \equiv 0 \pmod{p^{r_i}}$ and as $(\lambda, p) = 1$, $p^{r+t} \equiv 0 \pmod{p^{r_i}}$. But $1 \le t \le r_l - 1$ and $r = r_i - r_l$. Thus $r + t < r_i$, which is a contradiction. Thus $\alpha_l = 0$ and $\langle a_i \rangle^{\sigma} = \langle a_i \rangle$ □

For G non-abelian, $\sigma \in$ Z(Aut(G)) does not imply that $\sigma \in \ker(\kappa)$. Consider $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$ and the automorphism $\alpha$ which maps $a \mapsto a$ and $b \mapsto a^2 b$. Now $\alpha \in$ Z(Aut(G)), but $(\langle b \rangle \subset \langle a^2 \rangle \times \langle b \rangle)^{\alpha} = \langle a^2 b \rangle \subset \langle a^2 \rangle \times \langle b \rangle$.

It was shown in the proof of Theorem 5.7, that if G is abelian and $\sigma \in \ker(\kappa)$, then for each frame $\Sigma = \{ S_1, S_2, \dots, S_n \}$ for G, $S_i^{\sigma} = S_i$ for $1 \le i \le n$. This is not the case in general. Consider $S_4 = \langle x, y, a, b, \mid x^2 = y^2 = a^3 = b^2 = 1, xy = yx, x^a = y, y^a = xy, x^b = x, y^b = xy, a^b = xya^2 \rangle$. Here, $\ker(\kappa) = \{ 1, \sigma_x, \sigma_y, \sigma_{xy} \}$ where $\sigma_g$ is the inner automorphism induced by $g$. However, $\sigma_x, \sigma_y$, and $\sigma_{xy}$ do not fix $\langle a \rangle$. This leads to the following definition.

**Definition 5.8** *The* **stabilizer** *of Aut(G) is the collection of automorphisms $\sigma \in$ Aut(G) such that for each frame $\Sigma = \{ S_1, S_2, \dots, S_n \}$ for G, $S_i^{\sigma} = S_i$ for each $i$, $1 \le i \le n$. This collection is denoted by* **St(Aut(G))**.

St(Aut(G)) is a subgroup of Aut(G). Let $\sigma_1$ and $\sigma_2 \in$ St(Aut(G)) and consider $\sigma_1 \sigma_2^{-1}$. Let $\Sigma = \{ S_1, S_2, \dots, S_n \}$ be a frame for G. By Lemma 1.4, $S_i^{\sigma_1 \sigma_2^{-1}} = T$, an element in another frame for G. Therefore, $S_i^{\sigma_1} = T^{\sigma_2}$ and $T = S_i$.

**Lemma 5.9**

*i) St(Aut(G)) is a normal subgroup of Aut(G).*

*ii)St(Aut(G)) $\subseteq$ ker($\kappa$).*

Proof. i) Let $\Psi \in$ Aut(G) and let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. By Lemma 1.4, $\Sigma^{\Psi^{-1}} = \{ S_1^{\Psi^{-1}}, \ldots, S_n^{\Psi^{-1}} \} = \{ T_1, \ldots, T_n \}$ is also a frame for G. Let $\sigma_1 \in$ St(Aut(G)) and $1 \leq i \leq n$. Then $S_i^{\Psi^{-1}\sigma_1\Psi} = T_i^{\sigma_1\Psi} = T_i^{\Psi} = S_i$ and $\Psi^{-1}\sigma_1\Psi \in$ St(Aut(G)).

ii) Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a flag in $\Delta(\mathcal{P})$. Then there is a frame $\Sigma = \{S_1, S_2, \ldots, S_n\}$ for G which supports it. Therefore, for each $i$, $1 \leq i \leq d$, there is a $j_i$, $1 \leq j_i \leq n$, such that $\alpha_i = S_1 \ldots S_{j_i}$. Let $\sigma \in$ St(Aut(G)). For each $i$, $1 \leq i \leq d$, $\alpha_i^\sigma = (S_1 \ldots S_{j_i})^\sigma = S_1^\sigma \ldots S_{j_i}^\sigma = S_1 \ldots S_{j_i} = \alpha_i$ □

**Lemma 5.10** *Let $N \in \Delta(\mathcal{P})$ and $\sigma \in$ St(Aut(G)).*

*i) If G splits over N, then the automorphism $\sigma'$ induced by $\sigma$ on G/N is in St(Aut(G/N)).*

*ii) The automorphism $\sigma'$ induced by $\sigma$ on N is in St(Aut(N)).*

Proof. i) $N^\sigma$ = N and $\sigma$ induces an automorphism $\sigma'$ on G/N given by $(gN)^{\sigma'} = g^\sigma N$. By Lemma 1.8, there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G such that for some $i$, $1 \leq i \leq n$, N $= S_1 \ldots S_i$.

Let $\Sigma_{G/N} = \{ N_1/N, \ldots, N_r/N \}$ be a frame for G/N. By Lemma 1.11, $\Sigma' = \{S_1, \ldots, S_i, N_1 \cap B, \ldots, N_r \cap B\}$ is a frame for G where B is any complement to N in G. Then for each $j$, $1 \leq j \leq r$, $(N_j \cap B)^\sigma = N_j \cap B$. Therefore, $(N_j/N)^{\sigma'} = N_j^\sigma/N = (N_j \cap NB)^\sigma/N = (N(N_j \cap B))^\sigma/N = N^\sigma(N_j \cap B)^\sigma/N = N(N_j \cap B)/N = N_j/N$. Thus $\sigma' \in$ St(Aut(G/N)).

ii) $N^\sigma$ = N and $\sigma$ induces an automorphism $\sigma'$ on N by $n^{\sigma'} = n^\sigma$ for each $n \in$ N.

Let $\Sigma = \{ S_1, \ldots, S_r \}$ be a frame for N. By Lemma 1.8, $\Sigma$ can be extended to a frame $\Sigma' = \{ S_1, \ldots, S_r, S_{r+1}, \ldots, S_t \}$ for G. For each $i$, $1 \leq i \leq r$, $S_i^{\sigma} = S_i$ and hence $S_i^{\sigma'} = S_i$. Therefore, $\sigma' \in$ St(Aut(N)) □

The following is a known result. It can be found in [19].

**Theorem 5.11** *If $Z(G) = \{ 1 \}$, then $Z(Aut(G)) = \{ 1 \}$.*

Proof. Suppose $\sigma \in$ Z(Aut(G)) and $\sigma \neq 1$. Then there is an element $g \in$ G such that $g^{\sigma} = g_1$ where $g_1 \neq g$. Let $\Psi_g$ be the inner automorphism induced by $g$. For each $h \in$ G,

$$\begin{aligned} h^{\sigma\Psi_g} &= h^{\Psi_g\sigma} \\ (h^{\sigma})^{\Psi_g} &= (g^{-1}hg)^{\sigma} \\ g^{-1}(h^{\sigma})g &= (g^{-1})^{\sigma}h^{\sigma}g^{\sigma} \\ g^{-1}(h^{\sigma})g &= (g^{\sigma})^{-1}h^{\sigma}g^{\sigma} \\ g^{-1}(h^{\sigma})g &= g_1^{-1}h^{\sigma}g_1 \\ h^{\sigma}(gg_1^{-1}) &= (gg_1^{-1})h^{\sigma}. \end{aligned}$$

Therefore, $gg_1^{-1} \in$ Z(G). This is a contradiction □

**Lemma 5.12** *Let G be a homocyclic p-group of type $p^n$ and let $\sigma \in Aut(G)$. Then $\sigma \in$ St(Aut(G)) if and only if for each subgroup $\langle a \rangle$ of G which is in some frame for G, $a^{\sigma} = a^{\lambda}$ where $1 \leq \lambda \leq p^n - 1$ and $\lambda \not\equiv 0 \pmod{p}$.*

Proof. Let $\langle a \rangle$ and $\langle b \rangle$ be two subgroups of G which are in frames for G. Since G is homocyclic, $|\langle a \rangle| = |\langle b \rangle| = p^n$. Given that $\sigma \in$ St(Aut(G)), $a^{\sigma} = a^{\lambda_1}$ and $b^{\sigma} = b^{\lambda_2}$ where $1 \leq \lambda_1, \lambda_2 \leq p^n - 1$, $\lambda_1 \not\equiv 0 \pmod{p}$, and $\lambda_2 \not\equiv 0 \pmod{p}$. The subgroup $\langle ab \rangle$ is also in some frame for G. Hence $(ab)^{\sigma} = (ab)^{\lambda} = a^{\lambda}b^{\lambda}$, where $1 \leq \lambda \leq p^n$. Therefore, $a^{\lambda}b^{\lambda} = (ab)^{\sigma} = a^{\sigma}b^{\sigma} = a^{\lambda_1}b^{\lambda_2}$ and $\lambda_1 = \lambda_2$.

The converse holds by the definition of St(Aut(G)) □

**Theorem 5.13** *Let $G$ be a solvable nC-group. If $\sigma \in St(Aut(G))$, then $\sigma \in Z(Aut(G))$.*

Proof. If the Fitting length of G is 1, then G is abelian and $\sigma \in$ Z(Aut(G)) by Theorem 5.9. Proceed by induction on the Fitting length of G. Let $\{ 1 \} = F_0(G) \lhd F_1(G) \lhd \ldots \lhd F_r(G) \lhd F_{r+1}(G) =$ G be the Fitting series for G. Since G is a solvable nC-group, for each $j$, $1 \leq j \leq r$, G splits over $F_j(G)$. Therefore, there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G and integers $1 \leq l_1 < l_2 < \ldots < l_r < n$ such that $S_1 \ldots S_{l_j} = F_j(G)$. Furthermore, by Theorem 2.42, for each $i$, $1 \leq i \leq n$, $S_i$ is of prime order. Let $S_i = \langle s_i \rangle$. Given that $\sigma \in$ St(Aut(G)), $s_i^\sigma = s_i^{\alpha_i}$.

Let $\Psi \in$ Aut(G). By Lemma 1.4, $\Sigma^\Psi = \{ S_1^\Psi, \ldots, S_n^\Psi \} = \{ T_1, \ldots, T_n \}$ is also a frame for G. For each $i$, $1 \leq i \leq n$, $T_i = \langle t_i \rangle$ and is of prime order. Futhermore, $t_i^\sigma = t_i^{\beta_i}$. Therefore, $s_i^{\sigma\Psi} = (s_i^{\alpha_i})^\Psi = t_i^{\alpha_i}$ and $s_i^{\Psi\sigma} = t_i^\sigma = t_i^{\beta_i}$. For $\sigma$ to be in Z(Aut(G)), for each $i$, $1 \leq i \leq n$, $\alpha_i$ must equal $\beta_i$.

For each $j$, $1 \leq j \leq r$, $F_j$(G) is a characteristic subgroup of G. The method used to choose the frame $\Sigma$ for G indicates there is a number $k$, $0 \leq k \leq r$ such that $\langle s_i \rangle$ and $\langle t_i \rangle \not\leq F_k$(G) and $\langle s_i \rangle$ and $\langle t_i \rangle \leq F_{k+1}$(G). If $k = 0$, then $\langle s_i \rangle$ and $\langle t_i \rangle$ are in $F_1$(G). If $\langle s_i \rangle = \langle t_i \rangle$, then trivially, $\alpha_i = \beta_i$. Otherwise, since $| s_i | = | t_i | = p$ and $F_1$(G) is elementary abelian, $\langle s_i t_i \rangle$ is in some frame for $F_1$(G). G splits over $F_1$(G), so $\langle s_i t_i \rangle$ is in some frame for G. Thus $(s_i t_i)^\sigma \in \langle s_i t_i \rangle$ and $(s_i t_i)^\sigma = (s_i t_i)^\lambda = s_i^\lambda t_i^\lambda$, for $1 \leq \lambda \leq p - 1$. Therefore, $s_i^\lambda t_i^\lambda = (s_i t_i)^\sigma = s_i^{\alpha_i} t_i^{\beta_i}$ and $\alpha_i = \beta_i$.

Suppose $k \geq 1$. By Lemma 1.10, $\Sigma_1 = \{ \langle s_{l_k+1} \rangle F_k(\text{G})/F_k(\text{G}), \ldots, \langle s_n \rangle F_k(\text{G})/F_k(\text{G}) \}$ and $\Sigma_2 = \{ \langle t_{l_k+1} \rangle F_k(\text{G})/F_k(\text{G}), \ldots, \langle t_n \rangle F_k(\text{G})/F_k(\text{G}) \}$ are frames for G$/F_k$(G). Furthermore, $\sigma$ and $\Psi$ induce automorphisms $\sigma'$ and $\Psi'$ respectively of G$/F_k$(G). The automorphism $\sigma'$ maps $s_i F_k(\text{G}) \mapsto s_i^{\alpha_i} F_k(\text{G})$ and maps $t_i F_k(\text{G}) \mapsto t_i^{\beta_i} F_k(\text{G})$. The automorphism $\Psi'$

maps $s_iF_k(G)/F_k(G) \longmapsto t_iF_k(G)/F_k(G)$.

By Lemma 5.10 i), $\sigma' \in$ St(Aut(G/$F_k$(G))). Then by the induction hypothesis, $\sigma' \in$ Z(Aut(G/$F_k$(G))). Thus

$$\begin{aligned} (s_iF_k(G))^{\sigma'\Psi'} &= (s_iF_k(G))^{\Psi'\sigma'} \\ (s_i^{\alpha_i}F_k(G))^{\Psi'} &= (t_iF_k(G))^{\sigma'} \\ t_i^{\alpha_i}F_k(G) &= t_i^{\beta_i}F_k(G). \end{aligned}$$

Since $\mid t_i \mid = \mid s_i \mid = p$ and $\langle t_i \rangle \cap F_k(G) = \langle s_i \rangle \cap F_k(G) = \{ 1 \}$, $\alpha_i = \beta_i$. Thus $\sigma \in$ Z(Aut(G)) □

**Corollary 5.14** *If G is a solvable multiprimitive group, then St(Aut(G)) = { 1 }.*

Proof. Suppose $\sigma \in$ St(Aut(G)) and $\sigma \neq 1$. G is a solvable nC-group, so by Theorem 5.13, $\sigma \in$ Z(Aut(G)). Therefore, by Theorem 5.11, Z(G) $\neq$ 1. This is a contradiction □

**Theorem 5.15** *Suppose G = [A]B where A and B are elementary abelian p and q groups respectively. If St(Aut(G)) $\neq$ { 1 }, then G is abelian.*

Proof. Suppose G is non-abelian. Then there is an element $b \in$ B, such that $C_A(b) = \langle a \in A \mid ab = ba \rangle \neq$ A. Hence there is an element $a_1 \in$ A, such that $a_1 \notin C_A(b)$. Thus $a_1^b = a_2$, $a_2^b = a_3, \ldots, a_{q-1}^b = a_q$, and $a_q^b = a_1$, where for $i \neq j$, $1 \leq i, j \leq q$, $a_i \neq a_j$. Since G splits over A and A and B are elementary abelian, for each $a \in$ A and $b \in$ B, $\langle a \rangle$ and $\langle b \rangle$ are in some frame for G. By Lemma 5.12, $a^\sigma = a^\alpha$ and $b^\sigma = b^\beta$ where $1 \leq \alpha \leq p-1$ and $1 \leq \beta \leq q-1$.

Thus

$$
\begin{aligned}
a_1^b &= a_2 \\
a_1 b &= b a_2 \\
(a_1 b)^\sigma &= (b a_2)^\sigma \\
a_1^\sigma b^\sigma &= b^\sigma a_2^\sigma \\
a_1^\alpha b^\beta &= b^\beta a_2^\alpha .
\end{aligned}
$$

Now $b^\beta a_2 = a_r b^\beta$ where $r \equiv 2-\beta \pmod q$. Thus $a_1^\alpha b^\beta = a_r^\alpha b^\beta$ and $r = 1$. Hence $1 \equiv 2-\beta \pmod q$ and $\beta = 1$.

Since $a_1 \notin C_A(b)$, $b^{a_1} = a'b$ where $a' \in$ A and $a' \neq 1$. Thus $\langle a'b \rangle$ is in some frame for G. Hence $(a'b)^\sigma \in \langle a'b \rangle$. Now $(a'b)^\sigma = (a')^\alpha b$. Since $|\ b\ | = q$, $(a')^\alpha b = a'b$ and $(a')^\alpha = a'$. Thus $\alpha = 1$ and $\sigma =$ id. This is contrary to our assumption and thus G is abelian □

**Theorem 5.16** *Let G be a p-group of exponent p. G is elementary abelian if and only if St(Aut(G)) $\neq \{ 1 \}$.*

Proof. If G is elementary abelian, St(Aut(G)) = Z(Aut(G)) $\neq \{ 1 \}$.

Conversely, suppose that G is non-abelian. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. G is of exponent $p$, so by Theorem 4.18, for each $i$, $1 \leq i \leq n$, $|\ S_i\ | = p$. Let $S_i = \langle s_i \rangle$ and let $\sigma \in$ St(Aut(G)) be non-trivial. Thus $s_i^\sigma = s_i^{\alpha_i}$ for $1 \leq \alpha_i \leq p-1$. Proceed by induction on the order of G.

Let S = $\langle s_1 \rangle \ldots \langle s_{n-1} \rangle$ and G = [S]$S_n$. If for some $i$, $1 \leq i \leq n-1$, $\alpha_i \neq 1$, then $\sigma$ induces a non-trivial automorphism $\sigma'$ of S. By Lemma 5.10 i), $\sigma' \in$ St(Aut(S)). By the induction hypothesis, S is abelian. Since S is of exponent $p$, S is elementary abelian. Therefore, by Theorem 5.15, G is abelian. This is a contradiction and for each $i$, $1 \leq i \leq n-1$, $\alpha_i = 1$.

Suppose $s_n$ commutes elementwise with each element $s \in$ S. Then $\langle s_1 s_n \rangle$ complements S in G. Therefore, $\langle s_1 s_n \rangle^\sigma = \langle s_1 s_n \rangle$ and $(s_1 s_n)^\sigma = (s_1 s_n)^\alpha = s_1^\alpha s_n^\alpha$ for $1 \leq \alpha \leq p-1$. Thus, $s_1^\alpha s_n^\alpha = (s_1 s_n)^\sigma = s_1 s_n^{\alpha_n}$ and $\alpha_n = 1$. This implies $\sigma = id$, a contradiction. Therefore, there is an $i$, $1 \leq i \leq n-1$, such that $s_i$ and $s_n$ do not commute elementwise. Let $a_1 = s_i$. Then $a_1^{s_n} = a_2$, $a_2^{s_n} = a_3, \ldots, a_{p-1}^{s_n} = a_p$, and $a_p^{s_n} = a_1$ where for $i \neq j$, $\langle a_i \rangle \cap \langle a_j \rangle = 1$. Conjugation by $s_n$ induces an automorphism of G and since $a_1 = s_i$, $\langle a_2 \rangle$ is an element of some frame for G and $\langle a_2 \rangle \leq$ S. Thus $a_2^\sigma = a_2$. Hence

$$\begin{aligned} a_1 s_n &= s_n a_2 \\ (a_1 s_n)^\sigma &= (s_n a_2)^\sigma \\ a_1^\sigma s_n^\sigma &= s_n^\sigma a_2^\sigma \\ a_1 s_n^{\alpha_n} &= s_n^{\alpha_n} a_2 \\ a_1 s_n^{\alpha_n} &= a_r s_n^{\alpha_n}. \end{aligned}$$

where $r \equiv 2 - \alpha_n \pmod{p}$. Thus $r = 1$ and $\alpha_n = 1$. This implies $\sigma = id$, a contradiction. Thus G is abelian □

**Theorem 5.17** *Suppose G is non-abelian, | G | is divisible by at least two primes, and that each proper subgroup of G is abelian. Then St(Aut(G)) = Z(Aut(G)) = { 1 }.*

Proof. By [22], G = [A]B where A is an elementary abelian p-group of order $p^n$ and B is a cyclic q-group of order $q^m$ with $p \neq q$. Let B = $\langle b \rangle$ and let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G and S = $S_1 \ldots S_{n-1}$. If $q$ divides the order of S, then $q^m$ divides the order of S as the Sylow q-subgroups of G are cyclic and inseparable. Since S $\triangleleft$ G, the number of Sylow q-subgroups in G will be less than $p^n$. By [22], the number of Sylow q-subgroups is $p^n$. This is a contradiction and $q$ does not divide | S |. Therefore, S = A.

Let $\sigma \in St(Aut(G))$ and $\Psi \in Aut(G)$. By Lemma 1.4, $\Sigma^{\Psi} = \{S_1^{\Psi}, \ldots, S_n^{\Psi}\} = \{T_1, \ldots, T_n\}$ is a frame for G, where $T_1 \ldots T_{n-1} = A$. A is characteristic in G and $\Psi|_A \in Aut(A)$. By Lemma 5.12, for each $a \in A$, $a^{\sigma} = a^{\lambda}$ where $1 \leq \lambda \leq p-1$. Therefore for each $i$, $1 \leq i \leq n-1$, $s_i^{\sigma\Psi} = (s_i^{\lambda})^{\Psi} = t_i^{\lambda}$ and $s_i^{\Psi\sigma} = (t_i^{\lambda})^{\sigma} = t_i^{\lambda}$. Hence, $s_i^{\sigma\Psi} = s_i^{\Psi\sigma}$.

Since $| s_n | = | t_n | = q^m$, $s_n = a_1 b^{r_1}$ and $t_n = a_2 b^{r_2}$ where $a_1, a_2 \in A$, $r_1 \not\equiv 0 \pmod{q}$, and $r_2 \not\equiv 0 \pmod{q}$. The subgroup $\langle b \rangle$ complements A in G and is in some frame for G. Thus $b^{\sigma} = b^t$, where $1 \leq t \leq q^m - 1$. In addition, $s_n^{\sigma} = s_n^{\alpha_n}$ and $t_n^{\sigma} = t_n^{\beta_n}$. Hence $s_n^{\sigma} = (a_1 b^{r_1})^{\sigma} = a_1^{\lambda} b^{t r_1} = (a_1 b^{r_1})^{\alpha_n}$ since $\langle s_n \rangle \in \Sigma$. Therefore, $t r_1 \equiv r_1 \alpha_n \pmod{q^m}$ and $t \equiv \alpha_n \pmod{q^m}$. Furthermore, $t_n^{\sigma} = (a_2 b^{r_2})^{\sigma} = a_2^{\lambda} b^{t r_2} = (a_2 b^{r_2})^{\beta_n}$. Hence, $t r_2 \equiv r_2 \beta_n \pmod{q^m}$ and $t \equiv \beta_n \pmod{q^m}$. Therefore, $s_n^{\sigma\Psi} = s_n^{\Psi\sigma}$ and $\sigma \in Z(Aut(G))$.

Since [A,B] = A, Z(G) = 1. By Theorem 5.11, Z(Aut(G)) = { 1 }. Since St(Aut(G)) $\subseteq$ Z(Aut(G)), St(Aut(G)) = Z(Aut(G)) = { 1 } □

# Chapter 6

# Hyperplanes and Transvections

Given a vector space V, a **hyperplane** H in V is a subspace of codimension 1. A **transvection** associated with H is an invertible linear map $\tau$ of G such that for each $h \in$ H, $h^\tau = h$ and for each $v \in$ V, $v^\tau v^{-1} \in$ H. Let G be a group and M the largest normal subgroup over which G splits. Then G = [M]B, where B is inseparable. Since B is the generalizaton of a 1-dimensional subspace of a vector space V, M can be considered as having codimension 1 and called a hyperplane in G. Any automorphism $\psi$ of G, which fixes each element of M and has the property that for all $g \in$ G, $g^\psi g^{-1} \in$ M, can be considered a transvection.

Hyperplanes and transvections are fundamental to the study of finite dimensional vector spaces over the field of characteristic p. This section generalizes hyperplanes and transvections through the use of flags and frames. Consequently, many results concerning hyperplanes and GL(n,p) are obtained.

## 6.1 Hyperplanes

**Definition 6.1** *A rank 1 flag $\alpha$ in $\Delta(\mathcal{P})$ is called a* **hyperplane** *if there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ such that $\alpha_d = \alpha$.*

If G is an elementary abelian p-group, then G splits over any maximal subgroup of G. Therefore, this definition of a hyperplane reduces to the traditional definition of a hyperplane when G is an elementary abelian p-group.

In addition, suppose $\alpha \in \Delta(\mathcal{P})$ is a hyperplane and $\sigma \in \mathrm{Col}(\Delta(\mathcal{P}))$. Then there is a maximal flag A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ in $\Delta(\mathcal{P})$, such that $\alpha_d = \alpha$. $A^\sigma$: $\alpha_1^\sigma \subset \alpha_2^\sigma \subset \ldots \subset \alpha_d^\sigma$ is also maximal in $\Delta(\mathcal{P})$ and $\alpha^\sigma = \alpha_d^\sigma$ is also a hyperplane in $\Delta(\mathcal{P})$.

**Theorem 6.2** *Let $G$ be a group such that the commutator subgroup $G' \in \Delta(\mathcal{P})$. If for each hyperplane $H$ of $\Delta(\mathcal{P})$, $G/H$ is abelian, then $G' = \cap_i H_i$, where $H_i$ is a hyperplane of $\Delta(\mathcal{P})$.*

Proof. Let H be a hyperplane of $\Delta(\mathcal{P})$. Since G/H is abelian, $G' \subseteq$ H and $G' \subseteq \cap_i H_i$.

Consider $G'$. $G' \in \Delta(\mathcal{P})$ and there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G which supports $G'$. Thus for some $l$, $1 \leq l \leq n-1$, $S_1 \ldots S_l = G'$. By Lemma 1.10, $\{S_{l+1}G'/G'$, $\ldots$ , $S_nG'/G'\}$ is a frame for $G/G'$. $G/G'$ is abelian and for each $j$, $l+1 \leq j \leq n$, $\Sigma_j = \{ S_{l+1}G'/G', \ldots , S_{j-1}G'/G', S_{j+1}G'/G', \ldots , S_nG'/G', S_jG'/G' \}$ is a frame for $G/G'$. By Lemma 1.12, $\Sigma_j' = \{ S_1, \ldots, S_l, \ldots, S_{j-1}, S_{j+1}, \ldots, S_n, S_j \}$ is a frame for G. Since each frame supports a maximal flag , $H_j = S_1 \ldots S_l \ldots S_{j-1}S_{j+1} \ldots S_n$ is a hyperplane in $\Delta(\mathcal{P})$. Clearly, $G' = S_1 \ldots S_l \subseteq \cap_j H_j$.

Let $h \in \cap_j H_j$. Then $h \in H_j$ for each $j$, $l+1 \leq j \leq n$. Thus for each $j$, $h = s_{j,1} \ldots s_{j,l}s_{j,l+1} \ldots s_{j,j-1}s_{j,j+1} \ldots s_{j,n}$ where $s_{j,i} \in S_i$ for $1 \leq i \leq n$. $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ is a frame for G and each representation of the element $h$ by $\Sigma$ is unique. Therefore, $s_{j,k} = 1$ for each $k$, $l+1 \leq k \leq n$. Thus $h \in S_1 \ldots S_l = G'$ and $\cap_j H_j = G'$.

Consequently, $\cap_i H_i \subseteq \cap_j H_j = G'$ and $G' = \cap_i H_i$ □

**Corollary 6.3** *If $G$ is a solvable nC-group, then $\cap_i H_i = G'$, where $H_i$ is a hyperplane of $\Delta(\mathcal{P})$.*

Proof. Let H be a hyperplane in $\Delta(\mathcal{P})$. By Theorem 2.42, [G:H] is a prime and G/H is abelian. G is a solvable nC-group, so G splits over $G'$ and $G' \in \Delta(\mathcal{P})$. By Theorem 6.2, $\cap_i H_i = G'$ □

**Definition 6.4** *Aut(G)* **acts transitively on the collection of hyperplanes in** $\Delta(\mathcal{P})$ *if for each pair of hyperplanes H and K in* $\Delta(\mathcal{P})$*, there is a* $\Psi \in Aut(G)$ *such that* $H^\Psi = K$.

**Theorem 6.5** *Let G be abelian. Aut(G) acts transitively on the hyperplanes in* $\Delta(\mathcal{P})$ *if and only if G is a homocyclic p-group.*

Proof. By the Fundamental Theorem of Abelian Groups, G = $\langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle$ where $\mid a_i \mid = p_i^{r_i}$ for $1 \leq i \leq n$. $\langle a_i \rangle$ is inseparable for each $i$ and $H_i = \langle a_1 \rangle \oplus \ldots \oplus \langle a_{i-1} \rangle \oplus \langle a_{i+1} \rangle \oplus \ldots \oplus \langle a_n \rangle$ is a hyperplane in $\Delta(\mathcal{P})$. Aut(G) acts transitively on the hyperplanes in $\Delta(\mathcal{P})$, so $H_i \cong H_j$, for $1 \leq i, j \leq n$. Since G is abelian and G = $H_i \oplus \langle a_i \rangle = H_j \oplus \langle a_j \rangle$, $\langle a_i \rangle \cong \langle a_j \rangle$. Therefore G is a homocyclic p-group.

Suppose G is a homocyclic p-group of type $p^m$. Let H and K be two hyperplanes in $\Delta(\mathcal{P})$. Then G = [H]$\langle a_n \rangle$ and G = [K]$\langle b_n \rangle$ where $\mid a_n \mid = \mid b_n \mid = p^m$. Since H and K are isomorphic homocyclic p-groups, H = $\langle a_1 \rangle \oplus \ldots \oplus \langle a_{n-1} \rangle$ and K = $\langle b_1 \rangle \oplus \ldots \oplus \langle b_{n-1} \rangle$, where for $1 \leq i, j \leq n-1$, $i \neq j$, $\langle a_i \rangle \cap \langle a_j \rangle = \langle b_i \rangle \cap \langle b_j \rangle = \{ 1 \}$. Hence both $\{\langle a_1 \rangle, \ldots, \langle a_n \rangle\}$ and $\{\langle b_1 \rangle, \ldots, \langle b_n \rangle\}$ are frames for G. Define the map $\Psi$, such that $a_i^\Psi = b_i$. $\Psi$ is an automorphism of G, such that $H^\Psi$ = K. Therefore, Aut(G) acts transitively on the hyperplanes in the flag space $\Delta(\mathcal{P})$ □

For abelian groups, by Theorems 6.5 and 3.8, transitivity of maximal flags and transitivity of hyperplanes are equivalent. In general, this is not true. For any multiprimitive group of derived length $\geq 4$, Aut(G) acts transitively on the hyperplanes, yet Col($\Delta(\mathcal{P})$) does not act transitively on the maximal flags in $\Delta(\mathcal{P})$.

## 6.2 Transvections

**Definition 6.6** *An automorphism $\sigma$ of a group $G$ is called a* **transvection** *if there is a hyperplane $\alpha \in \Delta(\mathcal{P})$ such that $a^\sigma = a$ for each $a \in \alpha$ and $g^\sigma g^{-1} \in \alpha$ for each $g \in G$.*

Let $\alpha$ be a hyperplane in $\Delta(\mathcal{P})$ and B be its complement in G. If $\sigma$ is a transvection associated with the hyperplane $\alpha$, then $a^\sigma = a$ for each $a \in \alpha$ and $b_i^\sigma = a_i b_i$ for each $b_i \in$ B. Since G $= [\alpha]$B, this is also a convienient way of identifying the transvection $\sigma$ associated with the hyperplane $\alpha$.

If G is an elementary abelian p-group, this definition reduces to the traditional one. Algebraically, transvections are used to study the automorphism groups of the classical groups and to study general linear groups over rings. The first geometric generalization of a transvection was done by Liebert in [20].

**Definition 6.7** *Let $L$ be a subgroup of $G$. $L$ is called a* **line** *in $G$ if there is a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for $G$ such that $L = S_i$ for some $i$, $1 \leq i \leq n$.*

In [20], Liebert defines a transvection associated with abelian p-groups G of rank $\geq 3$. He defines a **line** B for G to be a cyclic subgroup of G such that G = B $\oplus$ K for some subgroup K of G. When G is an abelian p-group of rank $\geq 3$, this definition of a line is equivalent to the definition of a line in 6.7.

**Definition 6.8** *(Liebert [20]) An automorphism $\phi \neq 1$ of an abelian p-group $G$ of rank $\geq 3$ is called a* **transvection** *if there is a hyperplane $H$ and a line $B$ satisfying:*

*1) $B \leq H$;*

*2) $x^\phi = x$ for all $x \in H$;*

*3) $x^\phi x^{-1}$ is in $B$ for all $x \in G$.*

Definition 6.6 is equivalent to this definition when G is an abelian p-group of rank $\geq 3$. Suppose H is a hyperplane in G and $\tau$ is a transvection associated with the hyperplane H. Then $G = H \oplus K$ where $K = \langle k \rangle$ is cyclic. Let $g \in G$. Then $g = h_1 k^\alpha$, where $h_1 \in H$. Hence

$$\begin{aligned} g^\tau g^{-1} &= (h_1 k^\alpha)^\tau (h_1 k^\alpha)^{-1} \\ &= h_1^\tau (k^\alpha)^\tau k^{-\alpha} h_1^{-1} \\ &= h_1 h^\alpha k^\alpha k^{-\alpha} h_1^{-1} \qquad \text{for } h \in H \\ &= h_1 h^\alpha h_1^{-1} \\ &= h^\alpha. \end{aligned}$$

Therefore, for each $g \in G$, $g^\tau g^{-1} \in \langle h \rangle$. Since $\langle h \rangle$ is cyclic and G is an abelian p-group, there is an inseparable subgroup B of H such that $\langle h \rangle \subseteq B$ and $H = A \oplus B$. Thus the two definitions are equivalent when G is an abelian p-group of rank $\geq 3$.

Given an elementary abelian p-group G of rank $\geq 2$, every hyperplane in G has a non-trivial transvection associated with it. In general, this is not valid.

**Theorem 6.9** *Let $\alpha$ be a hyperplane in $\Delta(\mathcal{P})$. For each transvection $\tau$ associated with $\alpha$, $g^\tau g^{-1} \in Z(\alpha)$ for each $g \in G$.*

**Proof.** Let $\tau$ be a transvection associated with $\alpha$. For each $g \in G$, $g^\tau g^{-1} \in \alpha$ and $g^\tau = ag$ for $a \in \alpha$. Let $a_1 \in \alpha$. There is an $a_1' \in \alpha$ such that $ga_1' = a_1 g$. Therefore

$$\begin{aligned} (ga_1')^\tau &= (g)^\tau (a_1')^\tau \\ (a_1 g)^\tau &= (g)^\tau (a_1')^\tau \\ (a_1)^\tau (g)^\tau &= (g)^\tau (a_1')^\tau \end{aligned}$$

$$a_1 a g = a g a_1'$$
$$a_1 a g = a a_1 g$$
$$a_1 a = a a_1.$$

Hence $a$ commutes with each element of $\alpha$ and $g^\tau g^{-1} = a \in Z(\alpha)$ □

**Corollary 6.10** *Let $\alpha$ be a hyperplane in $\Delta(\mathcal{P})$. If there is a non-trivial transvection associated with the hyperplane $\alpha$, then $Z(\alpha) \neq \{ 1 \}$.*

Proof. Let $\tau$ be a non-trivial transvection associated with the hyperplane $\alpha$. Then for some $g \in G$, $g^\tau g^{-1} = a \in \alpha$ and $a \neq 1$. By Theorem 6.9, $a \in Z(\alpha)$ and $Z(\alpha) \neq \{ 1 \}$ □

Corollary 6.10 implies that if H is a hyperplane in $\Delta(\mathcal{P})$ and $Z(H) = \{ 1 \}$, then H does not have a non-trivial transvection associated with it. However, $Z(H) \neq \{ 1 \}$ does not imply that H has a non-trivial transvection associated with it. Consider the abelian group $G = Z_3 \oplus Z_2 = \langle a, b \mid a^3 = b^2 = 1, ab = ba \rangle$. The subgroup $\langle a \rangle$ is a hyperplane in G and $Z(\langle a \rangle) \neq \{ 1 \}$. However, there does not exist a non-trivial transvection associated with $\langle a \rangle$. A weaker statement can be proven.

**Lemma 6.11** *Let $\alpha$ be a hyperplane in $\Delta(\mathcal{P})$. If $Z(\alpha) \neq \{ 1 \}$ and $Z(\alpha) \not\leq Z(G)$, then $\alpha$ has a non-trivial transvection associated with it.*

Proof. Consider $a \in Z(\alpha) \setminus Z(G)$. Let B be a complement to $\alpha$ in G. For each $a' \in \alpha$, $(a')^a = a'$. Since $a \notin Z(G)$, there is a $b \in B$, such that $a^b \neq a$. Hence $b^{-1}ab = a_1$ where $a_1 \in Z(\alpha)$ and $a_1 \neq a$. Then

$$
\begin{aligned}
b^{-1}ab &= a_1 \\
bab &= b^2a_1 \\
a^{-1}bab &= a^{-1}b^2a_1 \\
b^a &= a^{-1}b^2a_1b^{-1} \\
b^a &= a^{-1}a_2b \text{ where } a_2 = b^2a_1b^{-2} \in \alpha.
\end{aligned}
$$

Since $a \notin Z(G)$, conjugation by $a$ induces a non-trivial transvection associated with the hyperplane $\alpha$ □

**Definition 6.12** $\mathcal{T}(G) = \langle \tau \in Aut(G) \mid \tau$ *is a transvection in Aut(G)* $\rangle$.

If G is an elementary abelian p-group, by II.6.7 of [16], $\mathcal{T}(G)$ = SL(n,p). Hence $\mathcal{T}$ is a generalization of the Special Linear Group of a vector space.

**Lemma 6.13** *$\mathcal{T}(G) \lhd Aut(G)$.*

Proof. Let $\tau$ be a generator of $\mathcal{T}(G)$. Then there is a hyperplane $\alpha$ in $\Delta(\mathcal{P})$ such that $a^\tau = a$ for each $a \in \alpha$ and $g^\tau g^{-1} \in \alpha$ for each $g \in G$. Let $\Psi \in Aut(G)$ and consider $\Psi^{-1}\tau\Psi$.

The subgroup $\alpha_1 = \alpha^\Psi$ is a hyperplane in $\Delta(\mathcal{P})$. Let $a_1 \in \alpha_1$ and $a_1^{\Psi^{-1}} = a \in \alpha$. Hence $a_1^{\Psi^{-1}\tau\Psi} = a^{\tau\Psi} = a^\Psi = a_1$. Let $g \in G$. Then $g^{\Psi^{-1}\tau\Psi} = k^{\tau\Psi} = (a'k)^\Psi$ where $a' \in \alpha$. Therefore, $g^{\Psi^{-1}\tau\Psi} = (a'k)^\Psi = (a')^\Psi g$ where $(a')^\Psi \in \alpha_1$. Thus $\Psi^{-1}\tau\Psi$ is a transvection associated with $\alpha_1$ and in $\mathcal{T}(G)$ □

The next question is when are each pair of transvections in Aut(G) conjugate. In general, this is not true.

**Example 1)** $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3 \rangle$. The automorphisms

$$\begin{array}{rrcl} \sigma: & a & \mapsto & a \\ & b & \mapsto & ab \end{array} \quad \text{and} \quad \begin{array}{rrcl} \sigma^2: & a & \mapsto & a \\ & b & \mapsto & a^2b \end{array}$$

are transvections. Since $\sigma$ is an outer automorphism and $\sigma^2$ is an inner automorphism, they can't be conjugate.

Consider abelian groups.

**Example 2)** $G = Z_4 \oplus Z_4 \oplus Z_4 = \langle a, b, c \mid a^4 = b^4 = c^4 = 1, ab = ba, ac = ca, bc = cb \rangle$. Consider the following transvections

$$\begin{array}{rrcl} \sigma_1: & a & \mapsto & a \\ & b & \mapsto & b \\ & c & \mapsto & ac \end{array} \quad \text{and} \quad \begin{array}{rrcl} \sigma_2: & a & \mapsto & a \\ & b & \mapsto & b \\ & c & \mapsto & a^2c. \end{array}$$

If $\sigma_1$ and $\sigma_2$ are conjugate, then there is a $\Psi \in \text{Aut}(G)$ such that

$$\begin{array}{rrcl} \Psi: & a & \mapsto & a^\alpha b^\beta c^\gamma \\ & b & \mapsto & a^n b^m c^p \\ & c & \mapsto & a^r b^s c^t \end{array}$$

and $\sigma_1 \Psi = \Psi \sigma_2$. Therefore

$$\begin{aligned} c^{\sigma_1 \Psi} &= c^{\Psi \sigma_2} \\ (ac)^\Psi &= (a^r b^s c^t)^{\sigma_2} \\ a^\alpha b^\beta c^\gamma a^r b^s c^t &= a^r b^s a^{2t} c^t \\ a^{\alpha+r} b^{\beta+s} c^{\gamma+t} &= a^{r+2t} b^s c^t. \end{aligned}$$

Hence $\beta + s \equiv s \pmod 4$ and $\beta \equiv 0 \pmod 4$. Furthermore, $\alpha + r \equiv r + 2t \pmod 4$ and $\alpha \equiv 2t \pmod 4$. Finally, $\gamma + t \equiv t \pmod 4$ and $\gamma \equiv 0 \pmod 4$. Thus $a^{\Psi} = a^{\alpha}$. Since $\alpha \equiv 2t \pmod 4$, $\alpha = 0$ or $2$. Thus $a^{\Psi} = 1$ or $a^2$. Either is a contradiction given that 1 and $a^2 \in \Phi(\mathrm{G})$.

In example 2), $c^{\sigma_1}c^{-1} \notin \Phi(\mathrm{G})$ and $c^{\sigma_2}c^{-1} \in \Phi(\mathrm{G})$. Perhaps the two transvections $\sigma_1$ and $\sigma_2$ would be conjugate if for all $g \in \mathrm{G}$, $g^{\sigma_1}g^{-1} \in \Phi(\mathrm{G})$, $g^{\sigma_2}g^{-1} \in \Phi(\mathrm{G})$, and $| g^{\sigma_1}g^{-1} | = | g^{\sigma_2}g^{-1} |$.

**Example 3)** Let $\mathrm{G} = Z_8 \oplus Z_4 \oplus Z_2 = \langle a, b, c \mid a^8 = b^4 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$

Consider the following transvections:

$$\sigma_1 : \begin{array}{lcl} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & a^4c \end{array} \quad \text{and} \quad \sigma_2 : \begin{array}{lcl} a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & b^2c. \end{array}$$

If $\sigma_1$ and $\sigma_2$ are conjugate, then there is a $\Psi \in \mathrm{Aut}(\mathrm{G})$ such that

$$\Psi : \begin{array}{lcl} a & \mapsto & a^{\alpha}b^{\beta}c^{\gamma} \\ b & \mapsto & a^{n}b^{m}c^{p} \\ c & \mapsto & a^{r}b^{s}c^{t} \end{array}$$

and $\sigma_1\Psi = \Psi\sigma_2$. Therefore

$$\begin{aligned} c^{\sigma_1\Psi} &= c^{\Psi\sigma_2} \\ (a^4c)^{\Psi} &= (a^rb^sc^t)^{\sigma_2} \\ a^{4\alpha}b^{4\beta}c^{4\gamma}a^rb^sc^t &= a^rb^sb^{2t}c^t \\ a^{4\alpha+r}b^{4\beta+s}c^{4\gamma+t} &= a^rb^{s+2t}c^t. \end{aligned}$$

Hence $4\alpha + r \equiv r \pmod 8$ and $4\alpha \equiv 0 \pmod 8$. This implies $\alpha = 0$ or 2. Thus $| a^{\Psi} |$ is at most 4. This is a contradiction as $\Psi$ is an automorphism.

These examples lead to the following theorem.

**Theorem 6.14** *Let $G$ be an abelian group of rank $n \geq 2$. Let $\sigma_1$ and $\sigma_2$ be two transvections in Aut(G) associated with the hyperplanes $\alpha_1$ and $\alpha_2$ respectively, and let $G = [\alpha_1]\langle b_1\rangle$ and $G = [\alpha_2]\langle b_2\rangle$. If $\alpha_1 \cong \alpha_2$, $b_1^{\sigma_1}b_1^{-1} \notin \Phi(G)$, $b_2^{\sigma_2}b_2^{-1} \notin \Phi(G)$, and $| b_1^{\sigma_1}b_1^{-1} | = | b_2^{\sigma_2}b_2^{-1} |$, then $\sigma_1$ and $\sigma_2$ are conjugate in Aut(G).*

Proof. Since G is abelian and $\alpha_1 \cong \alpha_2$, $\langle b_1\rangle \cong \langle b_2\rangle$. Since $\sigma_1$ and $\sigma_2$ are transvections, $b_1^{\sigma_1} = a_1b_1$ and $b_2^{\sigma_2} = a_2b_2$ where $a_i \in \alpha_i$ and $a_i \notin \Phi$(G), for $i = 1, 2$. Thus $\alpha_1 = \langle a_1\rangle \oplus \langle x_1\rangle \oplus \ldots \oplus \langle x_n\rangle$ and $\alpha_2 = \langle a_2\rangle \oplus \langle y_1\rangle \oplus \ldots \oplus \langle y_n\rangle$ where $| a_1 | = | a_2 |$ and $| x_i | = | y_i |$ for $1 \leq i \leq n$. Consequently, G = $\langle a_1\rangle \oplus \langle x_1\rangle \oplus \ldots \oplus \langle x_n\rangle \oplus \langle b_1\rangle$ and G = $\langle a_2\rangle \oplus \langle y_1\rangle \oplus \ldots \oplus \langle y_n\rangle \oplus \langle b_2\rangle$. Define a map $\Psi$ as follows:

$$\begin{aligned} \Psi : a_1 &\mapsto a_2 \\ x_i &\mapsto y_i \\ b_1 &\mapsto b_2. \end{aligned}$$

$\Psi$ is an automorphism of G and consider $\Psi^{-1}\sigma_1\Psi$. Let $y \in \alpha_2$. Then $y^{\Psi^{-1}} = x \in \alpha_1$. Hence $y^{\Psi^{-1}\sigma_1\Psi} = x^{\sigma_1\Psi} = x^{\Psi} = y$. Futhermore, $b_2^{\Psi^{-1}\sigma_1\Psi} = b_1^{\sigma_1\Psi} = (a_1b_1)^{\Psi} = a_2b_2$ and $\Psi^{-1}\sigma_1\Psi = \sigma_2$ □

The following result, II.6.9 of [16], is proven as a corollary to Theorem 6.14.

**Corollary 6.15** *All transvections in GL(n,p), for $n \geq 2$, are conjugate.*

Proof. Since $\Phi$(G) = { 1 } and G is of exponent $p$, this follows from Theorem 6.14 □

Example 1 indicates that a group G can have transvections which are inner automorphisms of G.

**Lemma 6.16** *If all transvections are conjugate in Aut(G), then $\mathcal{T}(G) \subseteq$ Inn(G) or $\mathcal{T}(G) \cap$ Inn(G) = { 1 }.*

Proof. Let $\tau$ be a transvection. Suppose $\tau \in$ Inn(G) and $\tau_1$ is another transvection. There is a $\Psi \in$ Aut(G) such that $\tau^{\Psi} = \tau_1$. Hence $\tau_1 \in$ Inn(G) and $\mathcal{T}(G) \subseteq$ Inn(G). If $\tau \notin$ Inn(G), a similar argument shows that $\tau_1 \notin$ Inn(G) and $\mathcal{T}(G) \cap$ Inn(G) = { 1 } □

**Theorem 6.17** *If all the non-trivial transvections in Aut(G) are conjugate, then Aut(G) acts transitively on the hyperplanes, of prime index in G, associated with the transvections.*

Proof. Let $H_1$ and $H_2$ be hyperplanes in $\Delta(\mathcal{P})$ associated with the non-trivial transvections $\tau_1$ and $\tau_2$ respectively. Then G = $[H_1]\langle b\rangle$ and G = $[H_2]\langle c\rangle$, where $\langle b\rangle$ and $\langle c\rangle$ are cyclic of prime order. $\tau_1$ and $\tau_2$ induce the following automorphisms:

$$\tau_1 : \begin{array}{ccc} h_1 & \mapsto & h_1 \\ b & \mapsto & kb^r \end{array} \quad \text{and} \quad \tau_2 : \begin{array}{ccc} h_2 & \mapsto & h_2 \\ c & \mapsto & lc^t \end{array}$$

where $h_1, k \in H_1$ and $h_2, l \in H_2$.

There is a $\Psi \in$ Aut(G), such that $\tau_1^{\Psi} = \tau_2$. Let $h_2 \in H_2$ and suppose $h_2^{\Psi^{-1}} = h_1 b^s$ where $h_1 \in H_1$ and $s \neq 0$. Then

$$\begin{aligned}
h_2 &= h_2^{\tau_2} \\
&= h_2^{\Psi^{-1}\tau_1\Psi} \\
&= (h_1 b^s)^{\tau_1\Psi} \\
&= (h_1 k_1 b^s)^{\Psi} \text{ for } k_1 \in H_1 \\
&= (k_1 h_1 b^s)^{\Psi} \text{ since } k_1 \in Z(H_1) \\
&= k_1^{\Psi}(h_1 b^s)^{\Psi} \\
&= k_1^{\Psi} h_2
\end{aligned}$$

Therefore, $k_1^{\Psi} = 1$ and $k_1 = 1$. This implies $(b^s)^{\tau_1} = b^s$. Since G = $H_1\langle b\rangle$, $\tau_1 = id_G$. This is a contradiction. Thus $H_2^{\Psi^{-1}} = H_1$ □

**Corollary 6.18** *Suppose each hyperplane in $\Delta(\mathcal{P})$ has a non-trivial transvection associated with it. If all the transvections are conjugate in Aut(G) and each hyperplane has prime index in G, then Aut(G) acts transitively on all the hyperplanes in $\Delta(\mathcal{P})$.*

**Theorem 6.19** *Let G be a p-group and H a hyperplane in $\Delta(\mathcal{P})$. If H is complemented by a cyclic group, then H has a non-trivial transvection associated with it.*

Proof. If Z(H) $\not\leq$ Z(G), then there is a non-trivial transvection associated with H by Lemma 6.11. Suppose Z(H) $\leq$ Z(G). Let G = [H]B where B = $\langle b\rangle$ is cyclic. Let $a_1 \in$ H such that $a_1 \in$ Z(H) and $|\, a_1 \,| = p$. Define the map $\tau$ as follows:

$$\begin{aligned}
a &\mapsto a \qquad \text{for all } a \in \text{H} \\
b &\mapsto a_1 b.
\end{aligned}$$

If $\tau$ is an automorphism, $\tau$ is a transvection. Let $g, k \in$ G. Then $g = h_1 b^{r_1}$ and $k = h_2 b^{r_2}$

where $h_1, h_2 \in$ H. Suppose $g^\tau = k^\tau$. Then

$$\begin{aligned} g^\tau &= k^\tau \\ (h_1 b^{r_1})^\tau &= (h_2 b^{r_2})^\tau \\ h_1 a_1^{r_1} b^{r_1} &= h_2 a_1^{r_2} b^{r_2} \\ h_2^{-1} h_1 a_1^{r_1 - r_2} &= b^{r_2 - r_1}. \end{aligned}$$

Since H $\cap$ B = { 1 }, $r_1 = r_2$ and $h_1 = h_2$. Hence $g = k$ and $\tau$ is one-to-one.

Let $g \in$ G. Then $g = hb^r$ where $h \in$ H. Let $t \equiv r \pmod p$ and $s = p - t$. Consider $k = ha_1^{p-s}b^r$. Then $k^\tau = (ha_1^{p-s}b^r)^\tau = ha_1^{p-s}a_1^r b^r = ha_1^{p-s}a_1^t b^r = ha_1^{p-s+t}b^r = hb^r = g$. Thus $\tau$ is onto.

Let $g_1$ and $g_2 \in$ G. Then $g_1 = h_1 b^{r_1}$ and $g_2 = h_2 b^{r_2}$ where $h_1$ and $h_2$ are in H. Since H $\lhd$ G, $b^{r_1} h_2 = h_2' b^{r_1}$ where $h_2' \in$ H. Then

$$\begin{aligned} (g_1 g_2)^\tau &= (h_1 b^{r_1} h_2 b^{r_2})^\tau \\ &= (h_1 h_2' b^{r_1 + r_2})^\tau \\ &= h_1 h_2' a_1^{r_1 + r_2} b^{r_1 + r_2} \end{aligned}$$

and

$$\begin{aligned} (g_1)^\tau (g_2)^\tau &= (h_1 b^{r_1})^\tau (h_2 b^{r_2})^\tau \\ &= h_1 a_1^{r_1} b^{r_1} h_2 a_1^{r_2} b^{r_2} \\ &= h_1 a_1^{r_1} h_2' b^{r_1} a_1^{r_2} b^{r_2} \\ &= h_1 h_2' a_1^{r_1} b^{r_1} a_1^{r_2} b^{r_2} \\ &= h_1 h_2' a_1^{r_1 + r_2} b^{r_1 + r_2}. \end{aligned}$$

Thus $(g_1 g_2)^\tau = g_1^\tau g_2^\tau$ and $\tau$ is an automorphism □

**Corollary 6.20** *If G is a p-group of exponent p, then each hyperplane has a non-trivial transvection associated with it.*

Proof. By Theorem 4.18, each element of each frame for G is of prime order. If H is a hyperplane in $\Delta(\mathcal{P})$, then H is complemented by a cyclic subgroup of G. Thus H has a non-trivial transvection associated with it by Theorem 6.19 □

The following result, II.6.8 of [16], is proven as a corollary to Theorem 6.19.

**Corollary 6.21** *GL(n,p) acts transitively on the hyperplanes in a finite dimensional vector space V of characteristic p.*

Proof. By Corollary 6.20, each hyperplane in V has a non-trivial transvection associated with it. By Corollary 6.15, all the transvections in GL(n,p) are conjugate. Therefore GL(n,p) acts transitively on the hyperplanes in V by Corollary 6.18 □

**Corollary 6.22** *If G is a metacyclic p-group, then each hyperplane has a non-trivial transvection associated with it.*

Proof. Let H be a hyperplane in $\Delta(\mathcal{P})$. H is complemented by a cyclic group and by Theorem 6.19, H has a nontrivial transvection associated with it □

## 6.3 Primitive Transvections

In this section, some of the ideas and results concerning transvections from Liebert in [20] are generalized.

Let G be an abelian p-group of rank $\geq 3$. Earlier, it was shown that for each transvection $\tau$ in Aut(G) and each $g \in$ G, $g^\tau g^{-1} \in$ L $<$ G, where L is a line of G. This is investigated for groups in general.

**Definition 6.23** *A transvection* $\tau$ *is* **associated with a hyperplane H and a line L** *if for all* $g \in G$, $g^{\tau}g^{-1} \in L \leq H$ *and* $L$ *is a line in* $H$. *The subgroup of* $Aut(G)$ *generated by the transvections associated with a hyperplane* $H$ *and a line* $L$ *will be denoted by* $\Lambda_{H,L}$.

**Lemma 6.24** *Any* $\sigma \in \Lambda_{H,L}$ *is a transvection associated with the hyperplane* $H$ *and the line* $L$.

Proof. $\sigma = \tau_1^{r_1} \ldots \tau_n^{r_n}$ where for each $i$, $1 \leq i \leq n$, $\tau_i$ is a transvection associated with the hyperplane H and the line L. Let $h \in$ H. Then $h^{\sigma} = h^{\tau_1^{r_1} \ldots \tau_n^{r_n}} = h$.

Let $g \in$ G. For each $i$, $g^{\tau_i} = l_i g$, where $l_i \in$ L. Therefore

$$\begin{aligned} g^{\sigma} &= g^{\tau_1^{r_1} \ldots \tau_n^{r_n}} \\ &= (l_1 g)^{\tau_1^{r_1 - 1} \ldots \tau_n^{r_n}} \\ &= (l_1^2 g)^{\tau_1^{r_1 - 2} \ldots \tau_n^{r_n}} \\ &= (l_1^{r_1} g)^{\tau_2^{r_2} \ldots \tau_n^{r_n}} \\ &= l_1^{r_1} l_2^{r_2} \ldots l_n^{r_n} g. \end{aligned}$$

Since $l_1^{r_1} \ldots l_n^{r_n} \in$ L, $\sigma$ is a transvection associated with hyperplane H and line L □

**Theorem 6.25** $\Lambda_{H,L}$ *is abelian.*

Proof. Let $\tau_1$ and $\tau_2 \in \Lambda_{H,L}$. By Lemma 6.24, $\tau_1$ and $\tau_2$ are both transvections associated with the hyperplane H and the line L. Then for each $g \in$ G, $g^{\tau_1}g^{-1} = h_1$ and $g^{\tau_2}g^{-1} = h_2$ where $h_1$ and $h_2$ are in H. By Lemma 6.9, $h_1$ and $h_2$ are in Z(H). Thus $g^{\tau_1\tau_2} = (h_1 g)^{\tau_2} = h_1 h_2 g$ and $g^{\tau_2\tau_1} = (h_2 g)^{\tau_1} = h_2 h_1 g = h_1 h_2 g$. Therefore $\tau_1\tau_2 = \tau_2\tau_1$ and $\Lambda_{H,L}$ is abelian □

**Definition 6.26** *A transvection* $\tau \in \Lambda_{H,L}$ *is called* **primitive** *if* $\tau \notin \Phi(\Lambda_{H,L})$.

When G is an abelian p-group or rank $\geq 3$, this definition of a primitive transvection coincides with Liebert's definition, which states that $\tau$ in $\Lambda_{H,L}$ is primitive if $\Lambda_{H,L} = \langle \tau \rangle$.

If G is an abelian p-group, then each complement of H in G is cyclic. Then $\Lambda_{H,L}$ is cyclic and if $\tau \notin \Phi(\Lambda_{H,L})$, $\Lambda_{H,L} = \langle \tau \rangle$.

By Lemma 6.24, each $\tau \in \Lambda_{H,L}$ is a transvection associated with the hyperplane H and the line L.

**Definition 6.27** *An element $\tau \in \Lambda_{H,L}$ is an* **L-generator** *if $\langle g^\tau g^{-1} \mid g \in G \rangle = L$.*

**Lemma 6.28** *Let $B = \langle g^\tau g^{-1} \mid g \in G$ and $\tau \in \Lambda_{H,L} \rangle$. Then $B \lhd G$.*

Proof. Let $b \in$ B be a generator of B. By definition, $b = g^\tau g^{-1}$ for some $g \in$ G and $\tau \in \Lambda_{H,L}$. Let $f$ be an element of G. Then $f^\tau = b_1 f$ and $b_1 f b g = f^\tau g^\tau = (fg)^\tau = b_2 f g$ where $b_1, b_2 \in$ B. Therefore, $b_1 f b = b_2 f$ and $f b f^{-1} = b_1^{-1} b_2 \in$ B. Hence B $\lhd$ G □

**Corollary 6.29** *If $\tau$ is an L-generator for a line L in a hyperplane H, then $L \lhd G$.*

Proof. Since $\tau$ is an L-generator, B $= \langle g^\sigma g^{-1} \mid g \in$ G and $\sigma \in \Lambda_{H,L} \rangle =$ L. Hence by Lemma 6.28, L $\lhd$ G □

**Lemma 6.30** *Let $\tau \in \Lambda_{H,L}$ be an L-generator. Then $L \leq Z(H)$.*

Proof. By Lemma 6.9, for each $g \in$ G, $g^\tau g^{-1} \in$ Z(H). Therefore L $\leq$ Z(H) □

By Corollary 6.28, L is normal in G. By definition, L is in some frame for H. This, along with Lemma 6.30, leads one to believe that if H is separable, H = L $\oplus$ K, for K $<$ H. However, this is not the case. Consider the following example. G $= [Z_2 \oplus Z_2] Z_2 \oplus Z_2 = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, ab = ba, cd = dc, a^c = b, b^c = a, a^d = b, b^d = a \rangle$. $\Sigma = \{ \langle a \rangle, \langle ab \rangle, \langle c \rangle, \langle d \rangle \}$ is a frame for G and H $= [\langle a \rangle \oplus \langle ab \rangle] \langle c \rangle$ is a hyperplane in $\Delta(\mathcal{P})$. The map

$$\begin{aligned} \tau: a &\mapsto a \\ b &\mapsto b \\ c &\mapsto c \\ d &\mapsto abd \end{aligned}$$

is a transvection associated with the hyperplane H and the line $\langle ab \rangle$. The transvection $\tau$ is an $\langle ab \rangle$-generator. However, there is no subgroup K of H such that H $= \langle ab \rangle \oplus$ K.

**Theorem 6.31** *Let* $\tau \in \Lambda_{H,L}$.

*1) If* $\tau$ *is an L-generator, then L is cyclic of prime power order.*

*2) If for each hyperplane H in* $\Delta(\mathcal{P})$ *and each line L in H,* $\Lambda_{H,L}$ *contains an L-generator, then each hyperplane H in* $\Delta(\mathcal{P})$ *is abelian.*

*3) If the line L in the hyperplane H is a cyclic p-group, then* $\Lambda_{H,L}$ *is an abelian p-group.*

Proof. 1) By Lemma 6.30, L $\leq$ Z(H) and L is abelian. Since L is inseparable, L must be cyclic of prime power order.

2) Let H be a hyperplane in G and $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G which supports H. Then for each $S_i$, $1 \leq i \leq n-1$, there is an $\tau_i \in \Lambda_{H,S_i}$ which is an $S_i$-generator. By Lemma 6.30, $S_i \leq$ Z(H). Furthermore, by (1), $S_i$ is cyclic of prime power order. Therefore, H $= S_1 \oplus \ldots \oplus S_{n-1}$ and is abelian.

3) By Theorem 6.25, $\Lambda_{H,L}$ is abelian. Let $\tau \in \Lambda_{H,L}$ and L $= \langle l \rangle$ with $|$ L $| = p^n$. For each $g \in$ G, $g^\tau = l^{\alpha_g} g$, where $1 \leq \alpha_g \leq p^n$. Let $\alpha$ be minimal such that $\alpha\alpha_g \equiv 0 \pmod{p^n}$ for each $g \in$ G. Since $|$ L $| = p^n$, $\alpha = p^m$ where $1 \leq m \leq n$.

For each $g \in$ G,

$$
\begin{aligned}
g^{\tau^{\alpha}} &= (l^{\alpha_g} g)^{\tau^{\alpha-1}} \\
&= (l^{\alpha_g} l^{\alpha_g} g)^{\tau^{\alpha-2}} \\
&= (l^{2\alpha_g} g)^{\tau^{\alpha-2}} \\
&= l^{\alpha\alpha_g} g \\
&= g
\end{aligned}
$$

Since $\alpha$ is minimal, $|\ \tau\ | = \alpha = p^m$. Hence $\Lambda_{H,L}$ is a p-group □

**Theorem 6.32** *If $\tau$ is an L-generator, then $\tau$ is primitive.*

Proof. By Theorem 6.31 1), L = $\langle l \rangle$ is cyclic. Since L = $\langle g^{\tau}g^{-1} \mid g \in$ G $\rangle$, there is a $g \in$ G such that $g^{\tau} = l^{\alpha}g$ where L = $\langle l^{\alpha} \rangle$. Let | L | = $p^n$. Given that L = $\langle l^{\alpha} \rangle$, $|\ \tau\ | = p^n$.

Let $\sigma$ be any element of $\Lambda_{H,L}$. Then for each $g \in$ G, $g^{\tau} = l^{\alpha_g}g$. By Theorem 6.31 3), $|\ \sigma\ | = p^m$. If $m > n$, there is a $g_1 \in$ G such that $g_1^{\tau} = l^{\alpha_{g_1}}g_1$ where $|\ l^{\alpha_{g_1}}\ | = p^m$. But | L | = $p^n$, which is a contradiction. Hence $m \leq n$. Since $\Lambda_{H,L}$ is an abelian p-group of exponent $p^n$, $\tau \notin \Phi$(G). Thus $\tau$ is primitive □

**Remark.** The converse of this theorem is not valid. Consider the group G = $Z_8 \oplus Z_4 \oplus Z_2$ = $\langle a, b, c \mid a^8 = b^4 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$. Let $\Lambda_{H,L}$ be generated by the transvections associated with the hyperplane H = $\langle a, b \rangle$ and the line L = $\langle a \rangle$. Since [G:H] = 2, any $\tau \in \Lambda_{H,L}$ can have at most order 2. Thus B = $\langle g^{\tau}g^{-1} \mid g \in$ G and $\tau \in \Lambda_{H,L} \rangle$ = $\langle a^4 \rangle$. $\Lambda_{H,L}$ clearly contains a primitive transvection, but it is not an L-generator.

**Theorem 6.33** *Let G be an abelian p-group. If each transvection in Aut(G) is an L-generator for some line L in G, then G is elementary abelian.*

Proof. By the Fundamental Theorem of Abelian Groups, there is a frame $\Sigma$ = { $\langle a_1 \rangle$,

..., $\langle a_n \rangle$ } for G such that $| a_i | = p^{r_i}$ where without loss of generality, $r_1 \geq r_2 \geq \ldots \geq r_n$. Suppose G is not elementary abelian: Then $r_1 \geq 2$. Consider the following map:

$$\begin{aligned} \tau : \quad a_i &\longmapsto a_i \qquad\qquad \text{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto a_1^{p^{r_1-1}} a_n \end{aligned}$$

$\tau$ is a transvection associated with the hyperplane H $= \langle a_1 \rangle \oplus \ldots \oplus \langle a_{n-1} \rangle$ and the line L $= \langle a_1 \rangle$. However $\langle g^\tau g^{-1} \mid g \in \text{G} \rangle = \langle a_1^{p^{r_1-1}} \rangle \neq \langle a_1 \rangle$. Thus $\tau$ is not an L-generator. This is a contradiction and $r_1 = 1$ □

## 6.4 The Quotient Group Aut(G)/$\mathcal{T}$(G)

If G is an elementary abelian p-group, by II.6.7 of [16], $\mathcal{T}$(G) = SL(n,p). Furthermore, by I.9.3 of [29], GL(n,p)/SL(n,p) is isomorphic to the multiplicative group $Z_p^*$, made up of the non-zero elements of $Z_p$ and by 8.12 in [23], that SL(n,p) is complemented in GL(n,p). In this section, the factor group Aut(G)/$\mathcal{T}$(G) and the complementation of $\mathcal{T}$(G) in Aut(G) are studied.

### 6.4.1 Homocyclic p-groups

Let G be a homocyclic p-groups of type $p^m$ and rank $n$. Then G $\cong Z_{p^m} \oplus \ldots \oplus Z_{p^m}$ $n$-times. In this section, additive notation will be used. In addition, when referring to a group G, G is a **homocyclic p-group of type $p^m$ and rank $n$.**

Consider $g \in$ G. G is of exponent $p^m$ and for any $\alpha$, $1 \leq \alpha \leq p^m$, $\alpha g \in$ G. Since $Z_{p^m}$ is a commutative ring, G is a $Z_{p^m}$-module. The following three definitions are from [17] and are included here for completeness.

**Definition 6.34** *(p.80 of [17]) Let M be a module of a ring A and let S be a subset of*

*M. By a* **linear combination** *of elements of S(with coefficients in A) one means* $\Sigma_{x \in S} a_x.x$ *where* { $a_x$ } *is a set of elements of A, for which at most a finite number differ from zero.*

Let N denote the linear combinations of elements in S. Then N is a submodule of M and S is said to **generate** M if N = M.

**Definition 6.35** *(p.81 of [17]) Let M be a module of a ring A and let S be a subset of M. The subset S is* **linearly independent** *(over A) if whenever we have a linear combination* $\Sigma_{x \in S} a_x.x$ *which equals 0, then* $a_x = 0$ *for all* $x \in S$.

If S is linearly independent and two linear combinations $\Sigma_{x \in S} a_x x$ and $\Sigma_{x \in S} b_x.x$ are equal, then $a_x = b_x$ for all $x$. Hence each representation of an element $n \in$ N is unique.

**Definition 6.36** *(p.84 of [17]) Let M be a module over a ring A and S a subset of M. S is a* **basis** *of M if S generates M and S is linearly independent.*

**Lemma 6.37** *Let* $\beta = \{ a_1, \ldots, a_t \}$ *be a basis for G. Then*

*1)* $\langle a_i \rangle \cap \langle a_j \rangle = \{ 1 \}$ *for* $1 \leq i, j \leq t,\ i \neq j$;

*2)* $| a_i | = p^m$;

*3)* $t = n$.

Proof. 1) Suppose there are $i, j$, $1 \leq i, j \leq t$ and $i \neq j$, such that $\langle a_i \rangle \cap \langle a_j \rangle \neq \{1\}$. Then $\langle a_i \rangle \cap \langle a_j \rangle = \langle p^{l_1} a_i \rangle = \langle p^{l_2} a_j \rangle$ where $1 \leq p^{l_1} \leq | a_i |$ and $1 \leq p^{l_2} \leq | a_j |$. Thus for some $l$, $lp^{l_1} a_i = p^{l_2} a_j$. Hence $lp^{l_1} a_i$ - $p^{l_2} a_j = 0$. This implies $lp^{l_1} = p^{l_2} = 0$. This is a contradiction and $\langle a_i \rangle \cap \langle a_j \rangle = \{ 1 \}$.

2) For each $j$, $1 \leq j \leq t$, $| a_j | \leq p^m$ since G is of exponent $p^m$. Suppose there is an $i$, $1 \leq i \leq t$, such that $| a_i | < p^m$. Then there is an $a \in$ G such that $\langle a_i \rangle \subset \langle a \rangle$ with $| a | = p^m$. Now $a = \alpha_1 a_1 + \ldots + \alpha_t a_t$ where at least one $\alpha_j \neq 0$. Since $\langle a_i \rangle \subset \langle a \rangle$, $a_i = p^r a$ and $a_i$

$= p^r a = p^r \alpha_1 a_1 + \ldots + p^r \alpha_t a_t$. But $a_i = 1a_i$ and $a_i$ has two different representations with respect to the basis $\beta$. This is a contradiction and $\mid a_i \mid = p^m$.

3) For each $i, j$, $1 \leq i, j \leq t$, $i \neq j$, $\langle a_i \rangle \cap \langle a_j \rangle = \{ 1 \}$ and $\mid a_i \mid = p^m$. Since $\mid$ G $\mid$ = $np^m$, $t = n$ □

**Theorem 6.38** $\Sigma = \{ \langle a_1 \rangle, \ldots, \langle a_n \rangle \}$ *is a frame for G if and only if* $\beta = \{ a_1, \ldots, a_n \}$ *is a basis for G.*

Proof. Each element $g \in$ G can be uniquely written as $g = \alpha_1 a_1 + \ldots + \alpha_n a_n$ where $1 \leq \alpha_i \leq p^m$ and $1 \leq i \leq n$. Thus $\beta = \{ a_1, \ldots, a_n \}$ is linearly independent. G = $\langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle$ and $\beta$ generates G. Hence $\beta$ is a basis for G.

Conversely, let $\beta = \{ a_1, \ldots, a_n \}$ be a basis for G. By Lemma 6.37, G = $\langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle$. For each $i$, $1 \leq i \leq n$, $\langle a_i \rangle$ is inseparable. Therefore $\Sigma = \{ \langle a_1 \rangle, \ldots, \langle a_n \rangle \}$ is a frame for the group G □

Lemma 6.38 implies that G is a free $Z_{p^m}$-module.

Let $\phi \in$ Aut(G) and $\beta = \{ a_1, \ldots, a_n \}$ be a basis for G. Then

$$\begin{aligned} \phi(a_1) &= \alpha_{1,1} a_1 + \alpha_{1,2} a_2 + \ldots + \alpha_{1,n} a_n \\ \phi(a_2) &= \alpha_{2,1} a_1 + \alpha_{2,2} a_2 + \ldots + \alpha_{2,n} a_n \\ &\vdots \\ \phi(a_n) &= \alpha_{n,1} a_1 + \alpha_{n,2} a_2 + \ldots + \alpha_{n,n} a_n \end{aligned}$$

where $\alpha_{i,j} \in Z_{p^m}$, for $1 \leq i, j \leq n$. Thus $\phi$ can be represented as an $n \times n$ matrix over the commutative ring $Z_{p^m}$. Let

$$\phi_{M_\beta} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{bmatrix}$$

denote the matrix representation of $\phi$ with respect to $\beta$.

The following three lemmas present known results and are presented here for completeness. They can be found in [17] and are given without proof.

**Lemma 6.39** *Let $\phi \in Aut(G)$ and $\beta = \{ a_1, \ldots, a_n \}$ be a basis for G. Then $\phi_{M_\beta}$ is invertible.*

**Definition 6.40** *Let $GL_n(Z_{p^m})$ denote the invertible $n \times n$ matrices over the ring $Z_{p^m}$. Let $SL_n(Z_{p^m})$ denote the invertible matrices of determinant 1.*

**Lemma 6.41** *$Aut(G) \cong GL_n(Z_{p^m})$.*

**Remark.** Given a basis $\beta$ for G, the map which sends $\phi \in$ Aut(G) to its matrix representation $\phi_{M_\beta}$ is an isomorphism from Aut(G) into $GL_n(Z_{p^m})$.

**Lemma 6.42** *Let $\phi \in Aut(G)$ and let $\beta_1 = \{ a_1, \ldots, a_n \}$ and $\beta_2 = \{ b_1, \ldots, b_n \}$ be two bases for G. Then $det(\phi_{M_{\beta_1}}) = det(\phi_{M_{\beta_2}})$.*

**Lemma 6.43** *Let $\mathcal{T}(G)$ be the normal subgroup of Aut(G) generated by the transvections in Aut(G). Regardless of choice of basis, for each $\tau \in \mathcal{T}(G)$ and basis $\beta$, $det(\tau_{M_\beta}) = 1$.*

Proof. Let $\tau \in \mathcal{T}(G)$. Then $\tau = \tau_1^{r_1} \ldots \tau_l^{r_l}$, where for each $i$, $1 \leq i \leq l$, $\tau_i$ is a transvection in Aut(G). Let $H_i$ be a hyperplane associated with the transvection $\tau_i$. Then $G = H_i \oplus K_i$

where $(h_i)^{\tau_i} = h_i$ for all $h_i \in H_i$ and $(k_i)^{\tau_i} k_i^{-1} \in H_i$ for all $k_i \in K_i$. Since $K_i$ is cyclic, there is a frame $\Sigma_i = \{ \langle a_{i,1} \rangle, \langle a_{i,2} \rangle, \ldots, \langle a_{i,n} \rangle \}$ for G such that $H_i = \langle a_{i,1} \rangle \oplus \ldots \oplus \langle a_{i,n-1} \rangle$ and $K_i = \langle a_{i,n} \rangle$. By Theorem 6.38, $\beta_i = \{ a_{i,1}, \ldots, a_{i,n} \}$ is a basis for G. Hence for $1 \leq j \leq n-1$, $(a_{i,j})^{\tau_i} = a_{i,j}$ and for $j = n$, $(a_{i,n})^{\tau_i} = \alpha_{i,1} a_{i,1} + \alpha_{i,2} a_{1,2} + \ldots + \alpha_{i,n-1} a_{i,n-1} + a_{i,n}$. Then $\tau_{i M_{\beta_i}}$ has the matrix representation

$$\tau_{i M_{\beta_i}} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{i,1} & \alpha_{i,2} & \ldots & \alpha_{i,n-1} & 1 \end{bmatrix}$$

and $\det(\tau_{i M_{\beta_i}}) = 1$.

Let $\beta$ be any basis for G. Since $\det(\tau_{i M_{\beta_i}}) = 1$, by Lemma 6.42, $\det(\tau_{i M_\beta}) = 1$ for each $i$, $1 \leq i \leq l$. By Lemma 6.41,

$$\tau_{M_\beta} = (\tau_{1 M_\beta})^{r_1} \ldots (\tau_{l M_\beta})^{r_l}$$

and

$$\begin{aligned} det((\tau_{M_\beta}) &= det((\tau_{1 M_\beta})^{r_1} \ldots (\tau_{l M_\beta})^{r_l}) \\ &= det((\tau_{1 M_\beta})^{r_1}) \ldots det((\tau_{l M_\beta})^{r_l}) \\ &= (det(\tau_{1 M_\beta}))^{r_1} \ldots (det(\tau_{l M_\beta}))^{r_l} \\ &= 1^{r_1} \ldots 1^{r_l} \\ &= 1 \end{aligned}$$

□

Consider an $n \times n$-matrix A over $Z_{p^m}$. Adding a multiple $\lambda$ of the $i^{th}$ row to the $j^{th}$ row, where $i \neq j$, can be represented by multiplying A by the elementary matrix

$$T_{i,j}(\lambda) = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & \lambda_{i,j} & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & 1 \end{bmatrix}.$$

Clearly, $\det(T_{i,j}(\lambda)) = 1$. Hence $T_{i,j}(\lambda)$ is invertible and $T_{i,j}(\lambda) \in SL_n(Z_{p^m})$. Let $\beta = \{ a_1, \ldots, a_n \}$ be a basis for G. Then

$$T_{i,j}(\lambda) \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_j + \lambda a_i \\ \vdots \\ a_n \end{bmatrix}.$$

is also a basis for G since $T_{i,j}(\lambda)$ is invertible.

By Lemma 6.41, there is a $\phi \in$ Aut(G), such that $(a_l)^\phi = a_l$ for $l \neq j$ and $(a_j)^\phi = \lambda a_i + a_j$. Since G $= \langle a_1 \rangle \oplus \ldots \oplus \langle a_{j-1} \rangle \oplus \langle a_{j+1} \rangle \oplus \ldots \oplus \langle a_n \rangle \oplus \langle a_j \rangle$, $\phi$ is a transvection in Aut(G). This result is stated in the next lemma.

**Lemma 6.44** *Each elementary matrix $T_{i,j}(\lambda)$ in $GL_n(Z_{p^m})$ of the type described above is the image in $GL_n(Z_{p^m})$ of some transvection $\tau \in Aut(G)$ with respect to any basis $\alpha$.*

**Theorem 6.45** *Let $T(G)$ denote the normal subgroup of $Aut(G)$ generated by the transvections. Then the image of $T(G)$ in $GL_n(Z_{p^m})$ is $SL_n(Z_{p^m})$.*

Proof. Let $\beta$ be a basis for G and consider the isomorphism $\phi$:Aut(G) $\mapsto GL_n(Z_{p^m})$ with respect to basis $\beta$. By Lemma 6.43, for each $\tau \in \mathcal{T}(G)$, $\det(\tau_{M_\beta}) = 1$ and $(\mathcal{T}(G))^\phi \leq SL_n(Z_{p^m})$.

Let $\Im$ be generated by the collection of matrices $T_{i,j}(\lambda)$. By Lemma 6.44, for each $T_{i,j}(\lambda)$ in $\Im$, there is a transvection $\tau \in \mathcal{T}(G)$ such that $\tau_{M_\beta} = T_{i,j}(\lambda)$. By Theorem 9 of [21], $\Im = SL_n(Z_{p^m})$. Thus each matrix A in $SL_n(Z_{p^m})$ is the image under $\phi$ of some element in $\mathcal{T}(G)$ and $(\mathcal{T}(G))^\phi = SL_n(Z_{p^m})$ □

**Definition 6.46** *Consider the commutative ring $Z_{p^m}$. $Z^*_{p^m}$ denotes the collection of units of $Z_{p^m}$.*

$Z^*_{p^m}$ is a multiplicative group.

**Theorem 6.47** *Let G be a homocyclic p-group of type $p^m$. Then $Aut(G)/\mathcal{T}(G) \cong Z^*_{p^m}$.*

Proof. Let $\beta$ be a basis for G. By Lemma 6.41, the map $\lambda_\beta$: $\phi \mapsto \phi_{M_\beta}$ where $\phi \in$ Aut(G) is an isomorphism. Define a map $f$: $GL_n(Z_{p^m}) \mapsto Z^*_{p^m}$ by $f(M) = \det(M)$. This map is well-defined since $GL_n(Z_{p^m})$ is the collection of $n \times n$ invertible matrices over $Z_{p^m}$. Furthermore, $f$ is a homomorphism. Thus $\lambda_\beta f$ is a homomorphism from Aut(G) into $Z^*_{p^m}$ Since $\lambda_\beta$ and $f$ are surjective, $\lambda_\beta f$ is surjective and $\lambda_\beta f$ is an epimorphism.

By the Homomorphism Theorem, Aut(G)/ker($\lambda_\beta f$) $\cong Z^*_{p^m}$. By Theorem 6.45, ker($\lambda_\beta f$) $= \mathcal{T}(G)$. Thus Aut(G)/$\mathcal{T}$(G) $\cong Z^*_{p^m}$ □

The following result, I.9.3 of [29], is proven as a corollary to Theorem 6.47.

**Corollary 6.48** *Let $V = V(n,p)$ be a vector space of dimension n over the field of p elements. Then $GL(n,p)/SL(n,p) \cong Z^*_p$.*

**Theorem 6.49** *Let G be a homocyclic p-group of type $p^m$ and rank n. Then $\mathcal{T}(G)$ is complemented in Aut(G).*

Proof. By Theorem 6.47, $\mathrm{Aut(G)}/\mathcal{T}(\mathrm{G}) \cong Z^*_{p^m}$. Let $\mathrm{G} = \langle a_1\rangle \oplus \ldots \oplus \langle a_n\rangle$. By Theorem 6.38, $\beta = \{ a_1, \ldots, a_n \}$ is a basis for G. Since $\langle a_1\rangle \cong Z_{p^m}$, by Theorem 5.7.11 of [27], $\mathrm{Aut}(\langle a_1\rangle) \cong Z^*_{p^m}$. Let $\sigma_i \in \mathrm{Aut}(\langle a_1\rangle)$. Then $(a_1)^{\sigma_i} = a_1^{\alpha_i}$, for $1 \leq \alpha_i \leq p^m$ and $\alpha_i \not\equiv 0 \pmod p$. The automorphism $\sigma_i$ can be extended to an automorphism $\sigma_i'$ of G by

$$\begin{aligned} \sigma_i' : \quad a_1 &\mapsto a_1^{\sigma_i} \\ a_j &\mapsto a_j \quad 2 \leq j \leq n. \end{aligned}$$

Let $\mathrm{K} = \langle \sigma_i' \mid \sigma_i' \text{ is induced by a } \sigma_i \in \mathrm{Aut}(\langle a_1\rangle)\rangle$. Let $\sigma'_{iM}$ be the matrix representation of $\sigma_i'$ with repect to the basis $\beta$. Then $\det(\mathrm{M}) = \alpha_i$ and $\mathrm{K} \cap \mathcal{T}(\mathrm{G}) = \{ 1 \}$. Since $\mathrm{K} \cong Z^*_{p^m}$, $\mathrm{Aut(G)} = [\mathcal{T}(\mathrm{G})]\mathrm{K}$ □

**Theorem 6.50** *Let G be a homocyclic p-group of type $p^m$ and rank $n \geq 3$. Each pair of transvections $\tau_1$ and $\tau_2$ in Aut(G), such that for some $g, k \in G$, $g^{\tau_1}g^{-1} \notin \Phi(G)$ and $k^{\tau_2}k^{-1} \notin \Phi(G)$, are conjugate in $\mathcal{T}(G)$.*

Proof. Let $H_1$ and $H_2$ be hyperplanes associated with $\tau_1$ and $\tau_2$ respectively. Then G $= [H_1]\langle b_1\rangle$ and G $= [H_2]\langle b_2\rangle$ where $b_1^{\tau_1} = a_1 b_1$ and $b_2^{\tau_2} = a_2 b_2$ for $a_1 \in H_1$, $a_2 \in H_2$, and $a_1, a_2 \notin \Phi(\mathrm{G})$. Hence $H_1 = \langle a_1\rangle \oplus \langle x_1\rangle \oplus \ldots \oplus \langle x_t\rangle$ and $H_2 = \langle a_2\rangle \oplus \langle y_1\rangle \oplus \ldots \oplus \langle y_t\rangle$.

Define a map $\phi$ as follows:

$$\begin{aligned} \phi : \quad a_1 &\mapsto a_2 \\ x_1 &\mapsto c y_1 \quad \text{where } 1 \leq c \leq p^m - 1 \text{ and } c \not\equiv 0 \pmod p \\ x_i &\mapsto y_i \quad 2 \leq i \leq t \\ b_1 &\mapsto b_2. \end{aligned}$$

$\beta_1 = \{ a_1, x_1, \ldots, x_t, b_1 \}$, $\beta_2 = \{ a_2, y_1, \ldots, y_t, b_2 \}$, and $\beta_2' = \{ a_2, c y_1, y_2, \ldots, y_t, b_2 \}$

are bases for G. Therefore $\phi$ is an automorphism of G. Consider $\phi^{-1}\tau_1\phi$.

$$\begin{aligned}
a_2^{\phi^{-1}\tau_1\phi} &= a_1^{\tau_1\phi} &&= a_1^{\phi} &&= a_2 \\
y_1^{\phi^{-1}\tau_1\phi} &= (-cx_1)^{\tau_1\phi} &&= (-cx_1)^{\phi} &&= y_1 \\
y_i^{\phi^{-1}\tau_1\phi} &= x_i^{\tau_1\phi} &&= x_i^{\phi} &&= y_i \\
b_2^{\phi^{-1}\tau_1\phi} &= b_1^{\tau_1\phi} &&= (a_1b_1)^{\phi} &&= a_2b_2
\end{aligned}$$

Therefore $\phi^{-1}\tau_1\phi = \tau_2$. Since $\beta_1$ and $\beta_2$ are bases for G, there is an invertible matrix $\Lambda \in GL_n(Z_{p^m})$ such that $\beta_2 = \Lambda\beta_1$. Since $\phi$ maps $x_1$ to $cy_1$, $\det(\phi_{M_\alpha}) = c\det(\Lambda)$. Furthermore, $\det(\Lambda) \not\equiv 0 \pmod p$ and a $c$ can be found, such that $c\det(\Lambda) = 1$. Therefore $\det(\phi_{M_\alpha}) = 1$ and by Theorem 6.45, $\phi \in \mathcal{T}(G)$. Hence $\tau_1$ and $\tau_2$ are conjugate in $\mathcal{T}(G)$ □

In Theorem 6.50, $n \geq 3$ is a necessity. If $n = 1$, G is cyclic and Aut(G) is abelian. Suppose $n = 2$. Consider the group $G = Z_4 \times Z_4 = \langle a, b \mid a^4 = b^4 = 1, ab = ba\rangle$. $\langle a\rangle$ is a hyperplane in G and consider the following two transvections $\tau_1$ and $\tau_2$ associated with $\langle a\rangle$:

$$\begin{aligned}
\tau_1 &: a \mapsto a & \qquad \tau_2 &: a \mapsto a \\
& \quad b \mapsto ab & & \quad a \mapsto a^3b
\end{aligned}$$

Any automorphism in Aut(G) which conjugates $\tau_1$ and $\tau_2$ is not in $\mathcal{T}(G)$.

### 6.4.2 Abelian and Metacyclic Groups

**Theorem 6.51** *Let G be an abelian group. Then $\mathcal{T}(G) = \{ 1 \}$ if and only if each Sylow p-subgroup of G is cyclic.*

**Proof.** Assume that the rank $G \geq 2$, since rank $G = 1$ makes the theorem trivial.

Let $\Sigma = \{ \langle a_1\rangle, \ldots, \langle a_n\rangle \}$ be frame for G, with $| a_i | = p_i^{r_i}$ for primes $p_i$. Suppose that for some $l$ and $k$, $1 \leq l, k \leq n$ and $l \neq k$, $p_l = p_k$. Without loss of generality, assume $r_l \geq r_k$.

$H = \langle a_1 \rangle \oplus \ldots \oplus \langle a_{l-1} \rangle \oplus \langle a_{l+1} \rangle \oplus \ldots \oplus \langle a_n \rangle$ is a hyperplane in G and is complemented in G by $\langle a_l \rangle$. Define a map $\tau$ as follows:

$$\begin{aligned} \tau : \; a_i &\longmapsto a_i \qquad \text{for } i \neq l \\ a_l &\longmapsto a_k a_l. \end{aligned}$$

$\tau$ is a non-trivial transvection associated with H. However $\mathcal{T}(G) = \{ 1 \}$, which implies that $\tau$ is trivial. This is a contradiction. Thus $(p_l, p_k) = 1$ for all $1 \leq l, k \leq n$, $l \neq k$ and each Sylow p-subgroup is cyclic.

Conversely, suppose that each Sylow p-subgroup of G is cyclic and $\tau$ is a non-trivial transvection in Aut(G). Let H be a hyperplane associated with $\tau$. Then there is a frame $\Sigma = \{ \langle a_1 \rangle, \ldots, \langle a_n \rangle \}$ for G such that $H = \langle a_1 \rangle \oplus \ldots \oplus \langle a_{n-1} \rangle$. Let $| \, a_i \, | = p_i^{r_i}$, $1 \leq i \leq n$. Since for all $h \in H$, $h^\tau = h$ and $\langle a_n \rangle$ is cyclic, $\tau$ is determined by $a_n^\tau$.

$\tau$ is non-trivial, so $a_n^\tau = k a_n$ for some $k \in H$, $k \neq 1$, where $| \, a_n \, | = | \, k a_n \, |$. Since G is abelian, $| \, k \, |$ divides $| \, a_n \, |$. But each Sylow p-subgroup for G is cyclic and $(| \, H \, |, p_n) = 1$. This is a contradiction. Hence $\mathcal{T}(G) = \{ 1 \}$ □

**Theorem 6.52** *Suppose G is a non-abelian, metacyclic group where $G = [\langle a \rangle]\langle b \rangle$ with $| \, a \, | = p^n$, $| \, b \, | = q^m$, $a^b = a^j$, and $p \neq q$. Then $\mathcal{T}(G)$ is complemented in Aut(G).*

Proof. By Lemma 4.11, $\langle a \rangle$ is the unique hyperplane in $\Delta(\mathcal{P})$. Thus $\mathcal{T}(G)$ is generated by the collection of transvections associated with $\langle a \rangle$. Let $1 \leq \alpha \leq p^n$ and $\tau_\alpha$ be the map which sends $a \mapsto a$ and $b \mapsto a^\alpha b$ where $| \, b \, | = | \, a^\alpha b \, |$. The map $\tau_\alpha$ is a transvection associated with $\langle a \rangle$.

Let $t$ be the smallest integer, $1 \leq t \leq m$, such that $a^{b^{q^t}} = a$. Therefore, $q^t$ is the smallest integer such that $j^{q^t} \equiv 1 \pmod{p^n}$. Note that since $a^b = a^j$, $a^{b^l} = a^{j^l}$ and $b^{q^m - l} a = a^{j^l} b^{q^m - l}$ for $1 \leq l \leq q^m$. In addition, $a^{b^{q^m-k}} = a^{j^{q^m-k}}$ and $b^k a = a^{j^{q^m-k}} b^k$ for

$1 \leq k \leq q^m$.

Case 1) $t = m$.

Let $\Psi \in \mathrm{Aut(G)}$. Then

$$\begin{aligned} \Psi : \quad a &\mapsto a^r \qquad r \not\equiv 0 \pmod{p} \\ b &\mapsto a^x b^y. \end{aligned}$$

Then $(ba)^\Psi = b^\Psi a^\Psi$. Evaluating both sides of the equality gives $(ba)^\Psi = (a^{j^{q^m-1}}b)^\Psi = a^{rj^{q^m-1}}a^x b^y$ and $b^\Psi a^\Psi = a^x b^y a^r = a^x a^{rj^{q^m-y}} b^y$. Therefore, $a^{rj^{q^m-1}} = a^{rj^{q^m-y}}$. Thus

$$\begin{aligned} rj^{q^m-1} &\equiv rj^{q^m-y} \pmod{p^n} \\ j^{q^m-1} &\equiv j^{q^m-y} \pmod{p^n} \qquad \text{since } (r, p^n) = 1. \end{aligned}$$

Suppose $y \geq 2$. Then there is $\lambda$, $1 \leq \lambda \leq q^m - 2$, such that $q^m - 1 = \lambda + q^m - y$. Since $j^{q^m-1} \equiv j^{q^m-y} \pmod{p^n}$, $j^{q^m-y}j^\lambda \equiv j^{q^m-y} \pmod{p^n}$. Conjugation by $b$ sends a generator of $\langle a \rangle$ to a generator of $\langle a \rangle$. Thus $(j^{q^m-y}, p^n) = 1$ and

$$\begin{aligned} j^{q^m-y}j^\lambda &\equiv j^{q^m-y} \pmod{p^n} \\ j^\lambda &\equiv 1 \pmod{p^n}. \end{aligned}$$

But $q^m$ is the smallest such that $j^{q^m} \equiv 1 \pmod{p^n}$. This is a contradiction and $\lambda = 0$. Thus $y = 1$ and

$$\begin{aligned} \Psi : \quad a &\mapsto a^r \\ b &\mapsto a^x b. \end{aligned}$$

Since $(r, p) = 1$, there is an automorphism $\sigma_r$ of G such that

$$\sigma_r : a \mapsto a^r$$
$$b \mapsto b.$$

Furthermore, $| b | = | a^x b |$ and there is an automorphism

$$\tau_x : a \mapsto a$$
$$b \mapsto a^x b.$$

Hence $\Psi = \sigma_r \tau_x$. Since $\langle a \rangle$ is the unique hyperplane in $\Delta(\mathcal{P})$, $\mathcal{T}(\mathrm{G}) = \langle \tau_\alpha$ | where $a^{\tau_\alpha} = a$ and $b^{\tau_\alpha} = a^\alpha b$, with $| b | = | a^\alpha b | \rangle$. Let $\mathrm{H} = \langle \sigma_r$ | where $a^{\sigma_r} = a^r$, for $r \not\equiv 0 \pmod p$ and $b^{\sigma_r} = b \rangle$. Hence $\mathrm{Aut(G)} = [\mathcal{T}(\mathrm{G})]\mathrm{H}$.

Case 2) $t < m$.

Let $\Psi \in \mathrm{Aut(G)}$. Then

$$\Psi : a \mapsto a^r \qquad r \not\equiv 0 \pmod p$$
$$b \mapsto a^x b^y.$$

$\Psi$ is a homorphism and $(ba)^\Psi = b^\Psi a^\Psi$. Evaluating both sides of the equality gives $(ba)^\Psi = (a^{j^{q^m-1}} b)^\Psi = a^{rj^{q^m-1}} a^x b^y$ and $b^\Psi a^\Psi = a^x b^y a^r = a^x a^{rj^{q^m-y}} b^y$. Therefore, $a^{rj^{q^m-1}} = a^{rj^{q^m-y}}$. Hence

$$rj^{q^m-1} \equiv rj^{q^m-y} \pmod{p^n}$$
$$j^{q^m-1} \equiv j^{q^m-y} \pmod{p^n} \qquad \text{since } (r, p^n) = 1.$$

Suppose $y \geq 2$. Then there is $\lambda$, $1 \leq \lambda \leq q^m - 2$, such that $q^m - 1 = \lambda + q^m - y$. Since $j^{q^m-1} \equiv j^{q^m-y} \pmod{p^n}$, $j^{q^m-y} j^\lambda \equiv j^{q^m-y} \pmod{p^n}$. Conjugation by $b$ sends a generator of $\langle a \rangle$ to a generator of $\langle a \rangle$ and $(j^{q^m-y}, p^n) = 1$. Therefore

$$\begin{aligned} j^{q^m-y}j^{\lambda} &\equiv j^{q^m-y} \pmod{p^n} \\ j^{\lambda} &\equiv 1 \pmod{p^n}. \end{aligned}$$

Since $q^t$ is the smallest integer such that $j^{q^t} \equiv 1 \pmod{p^n}$, $\lambda \equiv 0 \pmod{q^t}$ and $\lambda = zq^t$. Thus

$$\begin{aligned} q^m - 1 &= \lambda + q^m - y \\ q^m - 1 &= zq^t + q^m - y \\ y - 1 &= zq^t \\ y &\equiv 1 \pmod{q^t}. \end{aligned}$$

Therefore, the following map is an automorphism of G.

$$\begin{aligned} \delta_k : \ a &\mapsto a \\ b &\mapsto b^k \text{ where } k \equiv 1 \pmod{q^t} \end{aligned}$$

Consider the automorphisms

$$\begin{aligned} \sigma_r : \ a &\mapsto a^r \quad r \not\equiv 0 \pmod{p} \quad \text{and} \quad & \delta_y : \ a &\mapsto a \\ b &\mapsto b & b &\mapsto b^y \quad y \equiv 1 \pmod{q^t} \end{aligned}$$

Since $| a^x b^y | = q^m$ and $\langle a \rangle \lhd$ G, there is an $s$, $1 \leq s \leq q^m - 1$ such that $(a^x b^y)^s = a^{x_1} b^{ys}$ where $sy \equiv 1 \pmod{q^m}$. Thus $(a^x b^y)^s = a^{x_1} b$. Hence $\langle a^x b^y \rangle = \langle a^{x_1} b \rangle$. Furthermore, $(a^{x_1} b)^y = a^x b^y$. The following map

$$\tau_{x_1}: \quad a \mapsto a$$
$$b \mapsto a^{x_1} b$$

is a transvection associated with $\langle a \rangle$. Then

$$a^{\sigma_r \delta_y \tau_{x_1}} = (a^r)^{\delta_y \tau_{x_1}} = (a^r)^{\tau_{x_1}} = a^r$$
$$b^{\sigma_r \delta_y \tau_{x_1}} = b^{\delta_y \tau_{x_1}} = (b^y)^{\tau_{x_1}} = (a^{x_1} b)^y = a^x b^y$$

and $\Psi = \sigma_r \delta_y \tau_{x_1}$. Since $\langle a \rangle$ is the unique hyperplane, $\mathcal{T}(\mathrm{G}) = \langle \tau_\alpha \mid$ where $a^{\tau_\alpha} = a$ and $b^{\tau_\alpha} = a^\alpha b$, with $\mid b \mid = \mid a^\alpha b \mid \rangle$. Let $\mathrm{H} = \langle \sigma_i, \delta_j \mid \sigma_i$ maps $a \mapsto a^i$ with $(i, p) = 1$ and $b \mapsto b$; and $\delta_j$ maps $a \mapsto a$ and $b \mapsto b^j$ where $j \equiv 1 \pmod{q^t} \rangle$. Then $\mathrm{Aut(G)} = [\mathcal{T}(\mathrm{G})]\mathrm{H}$ □

# Chapter 7

# Multiprimitive Groups

In this section, the question as to when a multiprimitive group is an SE-group is answered. In addition, all solvable multiprimitive groups will be classified through their induced geometry. In this chapter, all multiprimitive groups will be considered to be solvable.

## 7.1 Algebraic Structure of Multiprimitive Groups

**Lemma 7.1** *Let G be a multiprimitive group and $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame for G. Then for each i, $1 \leq i \leq n$, $S_i$ is of prime order. Futhermore, $S_1 \leq M$, the unique minimal normal subgroup of G.*

Proof. G is a solvable nC-group, so by Theorem 2.42, for each $i$, $1 \leq i \leq n$, $S_i$ is of prime order.

$S_1$ is a subnormal nilpotent subgroup of G. By 5.2.5 of [2], there is a normal nilpotent subgroup of G which contains $S_1$. The only nilpotent normal subgroup of G is M. Thus $S_1 \leq$ M □

Multiprimitive groups have a unique maximal normal complemented flag, the Fitting flag $F_C$: $\gamma_1 \subset \ldots \subset \gamma_r$. The only normal subgroups of G are those in $F_C$ and each of them is complemented in G. Therefore, given a frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G, $S_1 \ldots S_{n-1} = \gamma_r$. By Lemma 7.1, $S_1 \leq \gamma_1$. This suggests there are integers $1 \leq i_1 < i_2 < \ldots < i_r \leq n-1$, such that for $1 \leq j \leq r$, $\gamma_j = S_1 \ldots S_{i_j}$. Therefore, $F_C$ would be supported by each frame for G and $F_C$ would be the primary flag in $\Delta(\mathcal{P})$. By Definition 2.24, a primary flag of a flag space $\Delta(\mathcal{P})$ is the intersection of all the maximal flags in $\Delta(\mathcal{P})$. However, this is not the case. Consider the following multiprimitive group.

$$G = \langle x_1, x_2, x_3, y_1, y_2, y_3, a_1, a_2, a_3, w, z, c, d \mid x_i^2 = y_i^2 = a_i^3 = w^2 = z^2 = c^3 = d^2 = 1 \rangle$$

Listed below are the defining relations:

| | | | |
|---|---|---|---|
| $x_i^{y_i} = x_i$ | $y_i^{a_i} = x_i y_i$ | $y_1^d = y_1$ | $a_3^c = a_1$ |
| $x_i^{a_i} = y_i$ | $y_i^{a_j} = y_i, i \neq j$ | $y_2^d = x_3 y_3$ | $a_1^d = a_1$ |
| $x_i^{a_j} = x_i, i \neq j$ | $y_1^w = y_1$ | $y_3^d = x_2 y_2$ | $a_2^d = x_3 y_3 a_3^2$ |
| $x_i^w = x_i$ | $y_2^w = x_2 y_2$ | $a_1^w = a_1$ | $a_3^d = x_2 y_2 a_2^2$ |
| $x_i^z = x_i$ | $y_3^w = x_3 y_3$ | $a_2^w = x_1 y_2 a_2^2$ | $a_i^{a_j} = a_i, i \neq j$ |
| $x_1^c = x_2$ | $y_1^z = x_1 y_1$ | $a_3^w = x_3 y_3 a_3^2$ | $w^z = w$ |
| $x_2^c = x_3$ | $y_2^z = y_2$ | $a_1^z = x_1 y_1 a_1^2$ | $w^c = z$ |
| $x_3^c = x_1$ | $y_3^z = x_3 y_3$ | $a_2^z = a_2$ | $z^c = wz$ |
| $x_1^d = x_1$ | $y_1^c = y_2$ | $a_3^z = x_3 y_3 a_3^2$ | $w^d = w$ |
| $x_2^d = x_3$ | $y_2^c = y_3$ | $a_1^c = a_2$ | $z^d = wz$ |
| $x_3^d = x_2$ | $y_3^c = y_1$ | $a_2^c = a_3$ | $c^d = wzc^2$ |

G was constructed by taking the twisted wreath product of the base group $A_4 = \langle x, y, a \rangle$ with $S_4 = \langle w, z, c, d \rangle$ for (M, $\alpha$), where M $= \langle w, z, d \rangle$ is a Sylow 2-subgroup of $S_4$ and $\alpha$ is the following homomorphism from M into Aut($A_4$):

$$
\begin{aligned}
w &\mapsto Id \\
z &\mapsto x \mapsto x \\
&\quad\;\; y \mapsto xy \\
&\quad\;\; a \mapsto xya^2 \\
d &\mapsto Id
\end{aligned}
$$

This construction is based on 2.5 of [15].

G = $[[[[\langle x_1, x_2, x_3, y_1, y_2, y_3\rangle]\langle a_1, a_2, a_3\rangle]\langle w, z\rangle]\langle c\rangle]\langle d\rangle$. G is of Fitting length 5 and F(G) = $\langle x_1, x_2, x_3, y_1, y_2, y_3\rangle$. $\Sigma = \{ \langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, \langle y_1\rangle, \langle y_2\rangle, \langle y_3\rangle, \langle a_1\rangle, \langle a_2\rangle, \langle a_3\rangle, \langle w\rangle, \langle z\rangle, \langle c\rangle, \langle d\rangle \}$ is a frame for G. Since $\langle x_1\rangle\langle x_2\rangle\langle x_3\rangle\langle y_1\rangle\langle y_2\rangle \lhd \langle x_1\rangle\langle x_2\rangle\langle x_3\rangle\langle y_1\rangle\langle y_2\rangle\langle y_3\rangle\langle a_1\rangle$ and $\langle y_3\rangle$ and $\langle a_1\rangle$ commute elementwise, by Lemma 1.7, $\Sigma' = \{ \langle x_1\rangle, \langle x_2\rangle, \langle x_3\rangle, \langle y_1\rangle, \langle y_2\rangle, \langle a_1\rangle, \langle y_3\rangle, \langle a_2\rangle, \langle a_3\rangle, \langle w\rangle, \langle z\rangle, \langle c\rangle, \langle d\rangle \}$ is also a frame for G. There does not exist a number $m$, $1 \leq m \leq n-1$ such that the subgroup generated by the first $m$ subgroups in $\Sigma'$ equals F(G).

**Theorem 7.2** *Let G be a multiprimitive group. $F_C$ is the primary flag of $\Delta(\mathcal{P})$ if and only if G is supersolvable or G/M is supersolvable where M is the unique minimal normal subgroup of G.*

Proof. A supersolvable multiprimitive group is of derived length 2. Thus $\mid G/M \mid = q$ and $\mid M \mid = p$, for primes $p$ and $q$ with $p \neq q$. Since M is inseparable and the largest normal subgroup over which G splits, every frame for G is of length 2 and supports M. Therefore, $F_C$: M is the primary flag in $\Delta(\mathcal{P})$. Suppose G is not supersolvable.

Let G = $G^0 \rhd G^1 \rhd \ldots \rhd G^{n-1} \rhd G^n = \{ 1 \}$ be the derived series of G. Then $\mid G^{n-1} \mid = p^{\alpha}$ with $\alpha \geq 2$. Since G is multiprimitive, $\mid G^{n-2}/G^{n-1} \mid = q^{\beta}$ where $q \neq p$. Furthermore, $G^{n-2}$

$= [G^{n-1}]B$ where $B \cong G^{n-2}/G^{n-1}$.

Suppose that $\beta \geq 2$. By 5.3.16 of [13], $G^{n-1} = \Pi_{b \in B^*} C_{G^{n-1}}(b)$ where $C_{G^{n-1}}(b) = \langle a \in G^{n-1} \mid ab = ba \rangle$ and $B^*$ is the set of non-identity elements in B. Thus there is a frame $\Sigma' = \{ \langle b_1 \rangle, \ldots, \langle b_\beta \rangle \}$ for B such that $C_{G^{n-1}}(b_1) \neq \{ 1 \}$. By 5.2.3 of [13], $G^{n-1} = [G^{n-1}, \langle b_1 \rangle] \times C_{G^{n-1}}(b_1)$. Since $C_{G^{n-1}}(b_1) \neq \{ 1 \}$, there is a frame $\Sigma'' = \{ \langle a_1 \rangle, \ldots, \langle a_\alpha \rangle \}$ for $G^{n-1}$ such that for some $j$, $1 \leq j \leq \alpha - 1$, $\langle a_1 \rangle \times \ldots \times \langle a_j \rangle = [G^{n-1}, \langle b_1 \rangle]$ and $\langle a_{j+1} \rangle \times \ldots \times \langle a_\alpha \rangle = C_{G^{n-1}}(b_1)$.

By Lemma 1.8, $\Sigma = \{ \langle a_1 \rangle, \ldots, \langle a_\alpha \rangle, \langle b_1 \rangle, \ldots, \langle b_\beta \rangle \}$ is a frame for $G^{n-2}$. Since the subgroup $\langle a_1 \rangle \ldots \langle a_{\alpha-1} \rangle \lhd \langle a_1 \rangle \ldots \langle a_\alpha \rangle \langle b_1 \rangle$ and $\langle a_\alpha \rangle$ and $\langle b_1 \rangle$ commute elementwise, by Lemma 1.7, $\Sigma_1 = \{ \langle a_1 \rangle, \ldots, \langle a_{\alpha-1} \rangle, \langle b_1 \rangle, \langle a_\alpha \rangle, \langle b_2 \rangle, \ldots, \langle b_\beta \rangle \}$ is also a frame for $G^{n-2}$. G splits over $G^{n-2}$, so $\Sigma_1$ can be extended to a frame $\Sigma_2 = \{ \langle a_1 \rangle, \ldots, \langle a_{\alpha-1} \rangle, \langle b_1 \rangle, \langle a_\alpha \rangle, \langle b_2 \rangle, \ldots, \langle b_\beta \rangle, S_1, \ldots, S_m \}$ for G. $F(G) = \langle a_1, \ldots a_\alpha \rangle$ and the maximal flag supported by $\Sigma_2$ does not contain $F_C$. This is a contradiction. Hence $\beta = 1$.

Since $G/G^{n-1}$ is also multiprimitive with $\mid G^{n-2}/G^{n-1} \mid = q$, the derived length of G can be at most 3. If the derived length is 2, then G/M is cyclic and hence supersolvable. If the derived length of G is 3, then $\mid G^1/M \mid = q$. Since G/M is multiprimitive, $G/G^1$ must also be cyclic of prime order. Thus G/M is supersolvable.

Conversely, suppose G or G/M is supersolvable. If G is supersolvable, then $\Delta(\mathcal{P})$ contains only one non-trivial flag and the result follows. Suppose that G/M is supersolvable and that there is a maximal flag A in $\Delta(\mathcal{P})$ which does not contain $F_C$.

Since G/M is supersolvable, G is of derived length 2 or 3. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ be a frame which supports A and let $S_i = \langle s_i \rangle$ for $1 \leq i \leq n$. Suppose G is of derived length 2. Then $G^1 = M$ and $F_C$: $G^1$ is the Fitting flag of $\Delta(\mathcal{P})$. Since $F_C \not\subseteq A$, $S_1 \ldots S_{n-1} \neq G^1$. This is a contradiction since $G^1$ is the only normal subgroup in G. Thus, $S_1 \ldots S_{n-1} = G^1$

and $F_C \subset$ A.

Suppose that G is of derived length 3. Then $F_C : G^2 \subset G^1$ is the Fitting flag where $|\, G^2 \,| = p^\alpha$ and $|\, G^1/G^2 \,| = q$ with $p \neq q$. Since $F_C \not\subset$ A and $S_1 \ldots S_{n-1} = G^1$, $|\, S_{n-1} \,| = p$. By Lemma 7.1, $|\, S_1 \,| = p$, so for some $t$, $2 \leq t \leq n-2$, $|\, S_t \,| = q$. The subgroup $\langle s_t \rangle^{s_{t+1}}$ is a Sylow q-subgroup of $S_1 \ldots S_t$. All the Sylow q-subgroups of $S_1 \ldots S_t$ are conjugate and there is an $s \in S_1 \ldots S_{t-1}$ such that $\langle s_t \rangle^s = \langle s_t \rangle^{s_{t+1}}$. Then $\langle s_t \rangle^{ss_{t+1}^{-1}} = \langle s_t \rangle$ and $s_t^{ss_{t+1}^{-1}} = s_t^\gamma$, for $1 \leq \gamma \leq q-1$.

Let $ss_{t+1}^{-1} = g \in G^2$. Since $G^2 \lhd G^1$, $g^{s_t^{-1}} = h \in G^2$ and $s_t g = h s_t$. Since $s_t g = g s_t^\gamma$, $h s_t = g s_t^\gamma$ and $g^{-1}h = s_t^{\gamma-1}$. But $g^{-1}h \in G^2$ and $s_t \notin G^2$, therefore $g^{-1}h = s_t^{\gamma-1} = 1$ and $\gamma = 1$. Therefore, $s_t^g = s_t$ and $C_{G^2}(s_t) = \langle x \in G^2 \mid x s_t = s_t x \rangle \neq \{\, 1 \,\}$. This implies $(G^1)' < G^2$, a contradiction. Hence $t = n-1$ and $F_C \subset$ A □

There are many groups of the type listed in Theorem 7.2.

**Examples:**

**7.1)** The alternating group $A_4$

**7.2)** The symmetric group $S_4$

**7.3)** $G = \langle x, y, a, b \mid x^5 = y^5 = a^3 = b^2 = 1, x^a = x^3 y, y^a = x^2 y, x^b = x^4, y^b = x^2 y, a^b = a^2 \rangle$. $F_C = \langle x, y \rangle \subset \langle x, y, a \rangle$ .

**7.4)** $G = \langle x, y, z, a, b \mid x^2 = y^2 = z^2 = a^7 = b^3 = 1, x^a = xyz, y^a = xy, z^a = yz, x^b = xyz, y^b = yz, z^b = y, a^b = a^2 \rangle$. $F_C = \langle x, y, z \rangle \subset \langle x, y, z, a \rangle$ .

Let G be a multiprimitive group and M the unique minimal normal subgroup of G. G is a solvable nC-group with F(G) = M. By Theorem 2.43, the points in $\Delta(\mathcal{P})$ correspond to the 1-dimensional subspaces of M.

All multiprimitive groups of derived length 2 and 3 are now classified and a method will be given for the classification of multiprimitive groups of derived length $\geq 4$.

Multiprimitive Groups of Derived Length 2.

From Definition 4.21, a $\mathcal{P}$-group is a group in which each element is of prime order.

**Theorem 7.3** *If $G$ is a multiprimitive group of derived length 2, then $G$ is a $\mathcal{P}$-group.*

Proof. G = [A]B where A is an elementary abelian p-group and B is a cyclic group of order q with $p \neq q$. Suppose there is an element $x \in$ G which is not of prime order. Then $x \notin$ A and $|\, x \,| = p^\alpha q$ with $\alpha \neq 0$. P = A $\cap$ $\langle x \rangle$ is the normal Sylow p-subgroup of $\langle x \rangle$. Since $x = ab$ with $a \in$ A and $b \in$ B, $b \neq 1$, $P = P^x = P^{ab} = P^b$. Thus $P^b = P$ and P is normal in G. This implies A = P and that $\langle x \rangle$ = G. This is a contradiction □

There are three types of multiprimitive groups of derived length 2. $G'$ is the commutator subgroup of G.

**Type I.** G is supersolvable, where $|\, G' \,| = p$, $|\, G/G' \,| = q$, with $q \neq p$, and $q$ divides $p - 1$.

An example of a group of this type is the symmetric group $S_3$.

**Type II.** $|\, G' \,| = p^n$ where $n \geq 2$ and $|\, G/G' \,| = q$, with $p \neq q$. Futhermore, $q = (p^n - 1)/(p-1)$, the number of 1-dimensional subspaces of $G'$, and G acts transitively on all the points in $\Delta(\mathcal{P})$.

Examples of groups of this type are the alternating group $A_4$,

G $= [Z_2 \times Z_2 \times Z_2]Z_7 = \langle x, y, z, a \mid x^2 = y^2 = z^2 = a^7 = 1, x^y = x, x^z = z, y^z = y, x^a = xyz, y^a = xy, z^a = yz \rangle$, and

G $= [Z_3 \times Z_3 \times Z_3]Z_{13} = \langle x, y, z, a \mid x^y = x, x^z = x, y^z = y, x^a = xyz, y^a = xy, z^a = x^2z^2 \rangle$.

**Type III.** $| G' | = p^n$ where $n \geq 2$ and $| G/G' | = q$ with $p \neq q$. Here $q \neq (p^n - 1)/(p - 1)$ and G does not act transitively on all the points of $\Delta(\mathcal{P})$.

An example of a group of this type is $G = [Z_5 \times Z_5]Z_3 = \langle x, y, a \mid x^5 = y^5 = a^3 = 1, x^y = x, x^a = x^3y, y^a = x^2y \rangle$.

Multiprimitive Groups of Derived Length 3.

There are four types of multiprimitive groups of derived length 3. Let $G = G^0 \rhd G^1 \rhd G^2 \rhd G^3 = \{ 1 \}$ be the derived series of G, where $| G/G^1 | = r, | G^1/G^2 | = q^m$, and $| G^2 | = p^n$ for primes $p$, $q$, and $r$.

**Type I.** $G/G^2$ is supersolvable(m=1) and G acts transitively on all the points in $\Delta(\mathcal{P})$.

Examples of this group are $S_4$ and example 7.4).

**Type II.** $G/G^2$ is supersolvable(m=1) and G does not act transitively on the points in $\Delta(\mathcal{P})$.

An example of a group of this type is example 7.3).

**Type III.** $G/G^2$ is a Type II multiprimitive group of derived length 2 and G does not act transitively on the points in $\Delta(\mathcal{P})$.

An example of a group of this type is $G = [[Z_3 \times Z_3 \times Z_3]Z_2 \times Z_2]Z_3 = \langle a, b, c, x, y, t \mid a^3 = b^3 = c^3 = x^2 = y^2 = t^3 = 1, ab = ba, ac = ca, bc = cb, xy = yx, a^x = a^2, b^x = b, c^x = c^2, a^y = a^2, b^y = b^2, c^y = c, a^t = b, b^t = c, c^t = a, x^t = y, y^t = xy \rangle$.

**Type IV.** $G/G^2$ is a Type III multiprimitive group of derived length 2 and G does not act transitively on the points in $\Delta(\mathcal{P})$.

An example of a group of this type is $G = [[Z_3 \times Z_3 \times Z_3 \times Z_3 \times Z_3]Z_2 \times Z_2 \times Z_2 \times Z_2]Z_5 = \langle a, b, c, d, e, x, y, z, w, t \mid a^3 = b^3 = c^3 = d^3 = e^3 = x^2 = y^2 = z^2 = w^2 = t^5 = 1 \rangle$

The relations are listed below.

| | | | | | |
|---|---|---|---|---|---|
| $a^b = a$ | $c^e = c$ | $a^x = a^2$ | $d^y = d$ | $b^w = b^2$ | $e^t = a$ |
| $a^c = a$ | $d^e = d$ | $b^x = b^2$ | $e^y = e^2$ | $c^w = c^2$ | $x^t = y$ |
| $a^d = a$ | $x^y = x$ | $c^x = c$ | $a^z = a^2$ | $d^w = d^2$ | $y^t = z$ |
| $a^e = a$ | $x^z = x$ | $d^x = d^2$ | $b^z = b^2$ | $e^w = e^2$ | $z^t = w$ |
| $b^c = b$ | $x^w = x$ | $e^x = e^2$ | $c^z = c^2$ | $a^t = b$ | $w^t = xyzw$ |
| $b^d = b$ | $y^z = y$ | $a^y = a^2$ | $d^z = d^2$ | $b^t = c$ | |
| $b^e = b$ | $y^w = y$ | $b^y = b^2$ | $e^z = e$ | $c^t = d$ | |
| $c^d = c$ | $z^w = z$ | $c^y = c^2$ | $a^w = a$ | $d^t = e$ | |

A theorem is proven to show there are no other types of multiprimitive groups of derived length 3.

**Theorem 7.4** *A multiprimitive group G of derived length 3 is of type I if and only if G acts transitively on the points in $\Delta(\mathcal{P})$.*

Proof. If G is a Type I multiprimitive group of derived length 3, then by definition, G acts transitively on the points in $\Delta(\mathcal{P})$.

Conversely, suppose G is a multiprimitive group of derived length 3 and that G acts transitively on the points in $\Delta(\mathcal{P})$. There are subgroups A, B, and C of G, such that G = [A]([B]C), where A, B, and C are elementary abelian with $|$ A $| = p^n$, $|$ B $| = q^m$, and $|$ C $| = r$ for primes $p, q$, and $r$. Since G is multiprimitive, $p \neq q$, and since G/A is multiprimitive, $q \neq r$. By Theorem 2.43, the collection $\mathcal{P}$ of points in $\Delta(\mathcal{P})$ corresponds to the 1-dimensional subspaces of A. The number of 1-dimensional subspaces in A is $k = (p^n - 1)/(p-1) = p^{n-1} + \ldots + p + 1$ and the number of points in $\mathcal{P}$ is $k$. Given that A is abelian, BC acts transitively on the points in $\mathcal{P}$ and ($\mathcal{P}$, BC) is a transitive permutation group.

Suppose the action of B suffices for BC to act transitively on the points in $\mathcal{P}$. Consider the transitive permutation group ($\mathcal{P}$, B). Suppose $m \geq 2$. For each $b \in$ B, $b \neq 1$, $\langle b \rangle \lhd$ B. By 10.1.7 of [27], $\mathrm{Ch}(\langle b \rangle) = 0$, where $\mathrm{Ch}(\langle b \rangle)$ is the number of points $\langle a \rangle \in \mathcal{P}$ such that $\langle a \rangle^x = \langle a \rangle$ for each $x \in \langle b \rangle$. Since $\langle b \rangle$ is cyclic, $\mathrm{Ch}(\langle b \rangle)$ is the number of points $\langle a \rangle \in \mathcal{P}$ such that $\langle a \rangle^b = \langle a \rangle$. If $C_A(b) = \langle a \in \mathrm{A} \mid ab = ba \rangle \neq \{ 1 \}$, then $\mathrm{Ch}(\langle b \rangle) \geq 1$. This is a contradiction. Therefore, for each $b \in$ B, $b \neq 1$, $C_A(b) = \{ 1 \}$. Since $m \geq 2$, by 5.3.16 of [13],

$$\mathrm{A} = \Pi_{b \in B^*} C_A(b)$$

where $B^*$ is the set of non-identity elements of B. This implies that A $= \{ 1 \}$, a contradiction. Therefore, $m = 1$ and G is a type I multiprimitive group of derived length 3.

Now suppose that the action of B does not act transitively on the points in $\mathcal{P}$. Therefore the permutation group ($\mathcal{P}$, B) is intransitive. Let $\mathcal{P}_1, \ldots, \mathcal{P}_t$ be the orbits of ($\mathcal{P}$, B) where $t \geq 2$. Let C $= \langle c \rangle$ and $\langle a \rangle \in \mathcal{P}$. The orbits of ($\mathcal{P}$, B) partition $\mathcal{P}$ and for some $i$, $1 \leq i \leq t$, $\langle a \rangle \in \mathcal{P}_i$. The number of points in an orbit containing $\langle a \rangle$, induced by C, is $[\mathrm{C}{:}N_C(\langle a \rangle)]$. C is of prime order $r$, so $[\mathrm{C}{:}N_C(\langle a \rangle)] = 1$ or $r$. If $[\mathrm{C}{:}N_C(\langle a \rangle)] = 1$, then $\langle a \rangle^c = \langle a \rangle$. This contradicts the fact that ($\mathcal{P}$, BC) is a transitive permutation group, since there are at least two orbits of ($\mathcal{P}$, B). Thus for each $\langle a \rangle \in \mathcal{P}$, $[\mathrm{C}{:}N_C(\langle a \rangle)] = r$. Therefore, $r \mid k$, where $k$ is the number of points in $\mathcal{P}$. This implies that $r \neq p$ and that BC is a $p'$-group. Furthermore, BC acts faithfully on A or else A $\neq$ F(G) and G would not be multiprimitive. Therefore, by 3.4.4 of [13], $C_A(c) \neq \{ 1 \}$. Thus there is an element $a \in$ A such that $a^c = a$. This implies that $N_C(\langle a \rangle) = \langle c \rangle$ and that $[\mathrm{C}{:}N_C(\langle a \rangle)] = 1$. This is a contradiction. Therefore, the permutation group ($\mathcal{P}$, B) must be transitive and G is a type I multiprimitive group of derived length 3 □

Suppose G is a multiprimitive group of derived length $n$, with $n \geq 4$, and derived series $G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^{n-1} \triangleright G^n = \{ 1 \}$. $G^{n-1}$ is the unique minimal normal subgroup of G and $G/G^{n-1}$ is a multiprimitive group of derived length $n-1$. Thus G is a multiprimitive group where $G/G^{n-1}$ is of a certain type and where G may or may not act transitively on the points in $G^{n-1}$.

**Theorem 7.5** *Let G be a multiprimitive group. Col($\Delta(\mathcal{P})$) acts transitively on the maximal flags in $\Delta(\mathcal{P})$ if and only if G is supersolvable or G/M is supersolvable where M is the minimal normal subgroup of G.*

Proof. If G is supersolvable, then $\Delta(\mathcal{P})$ contains only one non-trivial flag. Hence Col($\Delta(\mathcal{P})$) = { 1 } and Col($\Delta(\mathcal{P})$) acts transitively on $\Delta(\mathcal{P})$. Assume that G is not supersolvable. Let $G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^{n-1} \triangleright G^n = \{ 1 \}$ be the derived series for G.

Suppose Col($\Delta(\mathcal{P})$) acts transitively on $\Delta(\mathcal{P})$. Let $| G^{n-1} | = p_1^s$, $| G^{n-2}/G^{n-1} | = p_2^t$, and $| G^{n-3}/G^{n-2} | = p_3^u$ for primes $p_1, p_2$, and $p_3$. Let $F_C : \gamma_1 \subset \ldots \subset \gamma_r$ be the Fitting flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$. Thus for some $i$, $1 \leq i \leq d$, $S_1 \ldots S_i = \gamma_1$.

Suppose $t \geq 2$. Then by a method identical to the one presented in the proof of Theorem 7.2, a frame $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$ for G can be constructed such that $| T_1 \ldots T_i | = p_1^{s-1} p_2$. Thus there is no collineation which sends the flag supported by $\Sigma$ to the flag supported by $\Sigma'$. This is a contradiction and $t = 1$. Since $G/G^{n-1}$ is a multiprimitive group with $| G^{n-2}/G^{n-1} | = p_2$, $G/G^{n-1}$ is supersolvable.

Conversely, suppose G/M is supersolvable. Then G is of derived length 2 or 3. Suppose G is of derived length 2. Then G = [M]B, where B is cyclic of prime order. Since G is multiprimitive, B acts fixed-point-freely on M. Thus by Theorem 3.15, Col($\Delta(\mathcal{P})$) acts transitively on $\Delta(\mathcal{P})$.

Suppose G is of derived length 3. Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_d$ be two maximal flags in $\Delta(\mathcal{P})$. By Theorem 7.2, $F_C \subset$ A and $F_C \subset$ B. Hence $G^1 = \alpha_d = \beta_d$ and $G^2 = \alpha_{d-1} = \beta_{d-1}$. Thus $A'$: $\alpha_1 \subset \ldots \subset \alpha_{d-2}$ and $B'$: $\beta_1 \subset \ldots \subset \beta_{d-2}$ are maximal flags in $\Delta(\mathcal{P}_{G^2})$. $G^2$ is an elementary abelian p-group, so by Theorem 3.8, Aut($\Delta(\mathcal{P}_{G^2})$) acts transitively on $\Delta(\mathcal{P}_{G^2})$. Thus there is a $\sigma \in$ Aut($\Delta(\mathcal{P}_{G^2})$), such that $(A')^\sigma = B'$. Since $G^1 = \alpha_d = \beta_d$ and $G^2 = \alpha_{d-1} = \beta_{d-1}$, $\sigma$ can be extended to a collineation of $\Delta(\mathcal{P})$ such that $A^\sigma$ = B □

In Chapter 1, it was shown that a solvable nC-group, which is also SE-group, is multiprimitive. The following statement improves on this result.

**Theorem 7.6** *Let G be a solvable nC-group. If G is an SE-group, then G is a multiprimitive group, such that G is supersolvable or G/M is supersolvable, where M is the minimal normal subgroup of G.*

Proof. By Theorem 1.21, G is multiprimitive. Let $F_C$: $\gamma_1 \subset \ldots \subset \gamma_r$ be the Fitting flag in $\Delta(\mathcal{P})$ supported by the frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$. Since G is an SE-group, each frame for G supports $F_C$. Hence $F_C$ is the primary flag in $\Delta(\mathcal{P})$. Thus by Theorem 7.2, G is supersolvable or G/M is supersolvable where M is the minimal normal subgroup of G □

The converse of this theorem is not true. The group G = $[Z_5 \times Z_5]Z_3 = \langle a, b, c \mid a^5 = b^5 = c^3 = 1, ab = ba, a^c = a^3b, b^c = a^2b \rangle$ is multiprimitive, but not an SE-group.

**Theorem 7.7** *Let G be a solvable nC-group. G is an SC-group if and only if G is a supersolvable multiprimitive group or a cyclic group of prime order.*

Proof. The only inseparable solvable nC-group is that which is prime cyclic. Assume G is separable. Every SC-group is an SE-group, so by Theorem 1.21, G is multiprimitive. By

Theorem 7.6. G is supersolvable or G/M is supersolvable where M is the minimal normal subgroup of G. If G is supersolvable, then any frame for G is of length 2. Therefore, the first element of each frame is normal in G and G is an SC-group. Suppose G is not supersolvable.

Case 1) G/M is cyclic.

Then G = [M]K where $| M | = p^n$ and $n \geq 2$. Let K = $\langle k \rangle$, where $| K | = q$. Since $n \geq 2$, $q \geq 3$. There are $q$ 1-dimensional subspaces $S_1, S_2, \ldots, S_q$ of M such that $S_1^k = S_2, \ldots, S_{q-1}^k = S_q$. Consider the frames $\Sigma_1 = \{ S_1, S_2, \ldots, S_n, K \}$ and $\Sigma_2 = \{ S_2, S_1, \ldots, S_n, K \}$ for G. Since $\Sigma_1$ and $\Sigma_2$ are conjugate and $S_1^k = S_2$, $k$ must conjugate $\Sigma_1$ and $\Sigma_2$. Thus $S_2^k = S_1$. This is a contradiction since $q \geq 3$.

Case 2) G/M is supersolvable, but not cyclic.

Then G = $[M]([\langle h \rangle]\langle k \rangle)$ where $| M | = p^n$ and $n \geq 2$. Furthermore, $\langle h \rangle$ and $\langle k \rangle$ are prime cyclic with and $h^k = h^\alpha$. Let $| k | = q$. If for each 1-dimensional subspace $S$ in M, $S^k = S$, then $\alpha = 1$ and G/M is not multiprimitive. Let $S_1$ be a 1-dimensional subspace of M which is not fixed by $k$. Thus $S_1^k = S_2$, where $S_1 \cap S_2 = \{ 1 \}$.

Let $S_1 = \langle s_1 \rangle$ and $S_2 = \langle s_2 \rangle$. Consider the two frames $\Sigma_1 = \{ S_1, S_2, \ldots, S_n, \langle h \rangle, \langle k \rangle \}$ and $\Sigma_2 = \{ S_2, \langle s_1 s_2 \rangle, \ldots, S_n, \langle h \rangle, \langle k \rangle \}$ for G. Since $\Sigma_1$ and $\Sigma_2$ are conjugate in G, there is a $g = mh^r k^t$ in G, with $m \in$ M, which conjugates them. Since M is normal in G, $mh^r k^t = h^r k^t n$ where $n \in$ M. Thus $\langle h \rangle = \langle h \rangle^g = \langle h \rangle^{h^r k^t n} = \langle h \rangle^{k^t n} = \langle h \rangle^n$. Given that $n \in$ M, $n = 1$ and hence $m = 1$. Now $\langle h \rangle \lhd \langle h, k \rangle$, so $h^r k^t = k^t h^{r_1}$ and $\langle k \rangle = \langle k \rangle^g = \langle k \rangle^{h^r k^t} = \langle k \rangle^{k^t h^{r_1}} = \langle k \rangle^{h^{r_1}}$. Thus $r_1 = 0$ and hence $r = 0$. Therefore $g = k^t$. Since $S_1^g = S_2$, $t = 1$. If $| k | = 2$, then $S_2^k = S_1$. If $| k | \geq 3$, then $S_2^k = S_3$ where $S_3 \cap S_1 = S_3 \cap S_2 = \{ 1 \}$. Both lead to a contradiction.

The converse is evident □

This is Theorem 6 from [6].

All multiprimitive SE-groups are now classified.

**Theorem 7.8** *Let G be a solvable nC-group. G is an SE-group if and only if G is a type I or type II multiprimitive group of derived length 2 or if G is a type I multiprimitive group of derived length 3.*

Proof. Let M be a minimal normal subgroup of G. If G is an SE-group, by Theorem 7.6, G is a multiprimitive group, where G or G/M is supersolvable and M is unique. Therefore, the derived length of G is at most 3. By Theorem 7.2, for each frame $\Sigma = \{S_1, S_2, \ldots, S_n\}$ for G supports M. Furthermore, each 1-dimensional subspace in M is point in $\Delta(\mathcal{P})$. Therefore, an SE-group G acts transitively on the points in $\Delta(\mathcal{P})$.

A supersolvable multiprimitive group is a type I multiprimitive group of derived length 2. Suppose that G is not supersolvable. G could be a type II multiprimitive group of derived length 2, since G acts transitively on the 1-dimensional subspaces of M and the complement of M in G is a Sylow p-subgroup. G can't be of type III, since a type III group does not act transitively on the points in $\Delta(\mathcal{P})$.

Suppose G is of derived length 3. G must be a type I multiprimitive group of derived length 3, since any other type of derived length 3 does not have G acting transitively on the 1-dimensional subspaces in M.

Conversely, suppose G is a multiprimitive group of derived length 2 or 3. Suppose G is of derived length 2. By Theorem 7.7, a Type I multiprimitive group of derived length 2 is an SE-group. A type II multiprimitive group of derived length 2 is also an SE-group, since G acts transitively on the 1-dimensional subspaces in M and the complement to M in G is a Sylow p-subgroup of G.

Suppose G is a type I multiprimitive group of derived length 3. Let $G = G^0 \rhd G^1 \rhd G^2 \rhd G^3 = \{ 1 \}$ be the derived series of G where $| G/G^1 | = p_1$, $| G^1/G^2 | = p_2$, and $| G^2 |$

$= p_3^{\beta}$ for primes $p_1, p_2$, and $p_3$. Since the groups G and $G/G^2$ are multiprimitive, $p_2 \neq p_3$ and $p_1 \neq p_2$. Let $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_n \}$ be two frames for G. By Theorem 7.2, $S_1 \ldots S_{n-2} = T_1 \ldots T_{n-2} = G^2$. Since G acts transitively on the points in $\Delta(\mathcal{P})$, there are $g_i \in$ G, $1 \leq i \leq n-2$, such that $S_i^{g_i} = T_i$. Since $S_{n-1}$ and $T_{n-1}$ are Sylow $p_2$-subgroups of G, there is an element $h \in$ G such that $S_{n-1}^h = T_{n-1}$. Finally, since $S_1 \ldots S_{n-1} = T_1 \ldots T_{n-1} = G^1$, by Corollary 5.6 of [10], there is a $k \in$ G such that $S_n^k = T_n$. Thus a Type I multiprimitive group of derived length 3 is an SE-group □

## 7.2 Geometric Structure of Multiprimitive Groups

**Definition 7.9** *Let A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ be a rank d flag in $\Delta(\mathcal{P})$. A is* **invariant under Col($\Delta(\mathcal{P})$)** *if for each $\sigma \in Col(\Delta(\mathcal{P}))$, $A^\sigma = A$.*

**Lemma 7.10** *If a rank 1 flag $\alpha$ is invariant under $Col(\Delta(\mathcal{P}))$, then $\alpha$ is characteristic in G.*

Proof. By Theorem 2.30 i), each $\sigma \in$ Aut(G) induces a collineation. Since $\alpha$ is invariant under Col($\Delta(\mathcal{P})$), $\alpha^\sigma = \alpha$ for each $\sigma \in$ Aut(G). Thus $\alpha$ is characteristic in G □

**Lemma 7.11** *Let N be a normal, rank 1 complemented flag in $\Delta(\mathcal{P})$ and let A: $\alpha_1/N \subset \ldots \subset \alpha_d/N$ and B: $\beta_1/N \subset \ldots \subset \beta_e/N$ be two flags in $\Delta(\mathcal{P}_{G/N})$.*

*i) $A_0$: $\alpha_1 \subset \ldots \subset \alpha_d$ and $B_0$: $\beta_1 \subset \ldots \subset \beta_e$ are flags in $\Delta(\mathcal{P})$.*

*ii) Let $C_0 = A_0 \wedge B_0 = \gamma_1 \subset \ldots \subset \gamma_f$. Then C: $\gamma_1/N \subset \ldots \subset \gamma_f/N = A \wedge B$.*

Proof. i) This follows from Lemma 2.20 ii).

ii) Since C: $\gamma_1$/N $\subset \ldots \subset \gamma_f$/N is contained in both A and B, C $\subseteq$ A $\wedge$ B. Suppose there is a rank 1 flag $\gamma$/N $\subseteq$ A $\wedge$ B such that $\gamma$/N $\not\subseteq$ C. Then $\gamma \subseteq$ A and $\gamma \subseteq$ B. Thus $\gamma \subseteq C_0 = A_0 \wedge B_0$ and $\gamma$/N $\subseteq$ C. This is a contradiction. Therefore C = A $\wedge$ B □

**Lemma 7.12** *Let N be a normal, rank 1 complemented flag in $\Delta(\mathcal{P})$ and let $\sigma \in Col(\Delta(\mathcal{P}))$. If $N^\sigma = N$, then $\sigma$ induces a collineation of $\Delta(\mathcal{P}_{G/N})$.*

Proof. Let A: $\alpha_1/N \subset \ldots \subset \alpha_d/N \in \Delta(\mathcal{P}_{G/N})$. By Lemma 2.20 ii), $A_0$: $\alpha_1 \subset \ldots \subset \alpha_d$ is a flag in $\Delta(\mathcal{P})$. Let $A_0^\sigma = \alpha_1' \subset \ldots \subset \alpha_d'$ and define $A^{\sigma_{G/N}}$: $\alpha_1'/N \subset \ldots \subset \alpha_d'/N$.

The map $\sigma_{G/N}$ is well defined since N is invariant under $\sigma$ and $\sigma_{G/N}$ induces a bijection on $\Delta(\mathcal{P}_{G/N})$. Suppose A: $\alpha_1/N \subset \ldots \subset \alpha_d/N$ and B: $\beta_1/N \subset \ldots \subset \beta_e/N$ are two flags in $\Delta(\mathcal{P}_{G/N})$. By Theorem 2.20 ii), $A_0$: $\alpha_1 \subset \ldots \subset \alpha_d$ and $B_0$: $\beta_1 \subset \ldots \subset \beta_e$ are flags in $\Delta(\mathcal{P})$. Therefore, $(A_0 \wedge B_0)^\sigma = A_0^\sigma \wedge B_0^\sigma$. Let $A_0 \wedge B_0 = \gamma_1 \subset \ldots \subset \gamma_f$ and $(A_0 \wedge B_0)^\sigma = \gamma_1' \subset \ldots \subset \gamma_f'$. The collineation $\sigma$ maps $A_0^\sigma$ into $\alpha_1' \subset \ldots \subset \alpha_d'$ and $B_0^\sigma$ into $\beta_1' \subset \ldots \subset \beta_e'$. Then $A^{\sigma_{G/N}} = \alpha_1'/N \subset \ldots \subset \alpha_d'/N$ and $B^{\sigma_{G/N}} = \beta_1'/N \subset \ldots \subset \beta_e'/N$. Since $(A_0 \wedge B_0)^\sigma = A_0^\sigma \wedge B_0^\sigma$, $\gamma_1' \subset \ldots \subset \gamma_f' = (\alpha_1' \subset \ldots \subset \alpha_d') \wedge (\beta_1' \subset \ldots \subset \beta_e')$. By Lemma 7.11, $\gamma_1'/N \subset \ldots \subset \gamma_f'/N = (\alpha_1'/N \subset \ldots \subset \alpha_d'/N) \wedge (\beta_1'/N \subset \ldots \subset \beta_e'/N)$. Hence $(A \wedge B)^{\sigma_{G/N}} = A^{\sigma_{G/N}} \wedge B^{\sigma_{G/N}}$ □

**Theorem 7.13** *Let G be a solvable nC-group and $\{1\} = F_0(G) \lhd F_1(G) \lhd \ldots \lhd F_r(G) \lhd F_{r+1}(G) = G$ be the Fitting series for G. Let $P_{i,j}/F_{i-1}(G)$ be the Sylow $p_j$-subgroup of $F_i(G)/F_{i-1}(G)$, where $1 \leq i \leq r+1$ and $1 \leq j \leq t_i$, with $t_i$ being the number of Sylow p-subgroups in $F_i(G)/F_{i-1}(G)$. Then $P_{1,1} \subset P_{1,1}P_{1,2} \subset \ldots \subset P_{1,1}\ldots P_{1,t_1-1} \subset F_1(G) \subset P_{2,1} \subset \ldots \subset P_{2,1}\ldots P_{2,t_2-1} \subset F_2(G) \subset \ldots \subset F_r(G) \subset P_{r+1,1} \subset \ldots \subset P_{r+1,1}\ldots P_{r+1,t_{r+1}-1}$ is invariant under $Col(\Delta(\mathcal{P}))$.*

Proof. Since G is a solvable nC-group, $F_1$(G) is elementary abelian. Let $P_j = \langle a_1 \rangle \times \ldots \times \langle a_n \rangle$ where for each $l$, $1 \leq l \leq n$, $a_l$ is of order $p_j$. $P_j$ is complemented in $F_1$(G) and is a rank 1 flag in $\Delta(\mathcal{P})$. Since $P_j$ is abelian, for each $l$, $1 \leq l \leq n$, $A_l$: $\langle a_l \rangle \subset \langle a_l, a_1 \rangle \subset \ldots \subset P_j$ is a flag in $\Delta(\mathcal{P})$. For each $\sigma \in$ Col($\Delta(\mathcal{P})$), $A_l^\sigma = \beta_{l,1} \subset \ldots \subset \beta_{l,n}$ is a flag in $\Delta(\mathcal{P})$, where

$\beta_{l,n} \cong P_j$ and $\beta_{l,n}$ is nilpotent. Since $\beta_{l,n}$ is subnormal in G, by 5.2.5 of [2], there is a normal nilpotent subgroup of G which contains $\beta_{l,n}$. Thus $\beta_{l,n} \leq F_1(G)$. But $P_j$ is the unique Sylow $p_j$-subgroup of $F_1(G)$, hence $\beta_{l,n} = P_j$. Therefore, $P_j^\sigma = P_j$ for each Sylow p-subgroup of $F_1(G)$. Hence $P_{1,1} \subset P_{1,1}P_{1,2} \subset \ldots \subset P_{1,t_1-1} \subset F_1(G)$ is invariant under Col($\Delta(\mathcal{P})$).

Proceed by induction $r+1$, the Fitting length of G. Suppose there is an $i$ and $j$, $i \geq 2$, such that for some $\sigma \in$ Col($\Delta(\mathcal{P})$), $(P_{i,1}\ldots P_{i,j})^\sigma \neq P_{i,1}\ldots P_{i,j}$. Let $i$ and $j$ be minimal. Since $i \geq 2$, $F_1(G) \leq P_{i,1}\ldots P_{i,j-1}$. $G/F_1(G)$ is a solvable nC-group of Fitting length $r$. Thus for all $\Psi \in$ Aut($\Delta(\mathcal{P}_{G/F_1(G)})$), $(P_{i,1}\ldots P_{i,j}/F_1(G))^\Psi = P_{i,1}\ldots P_{i,j}/F_1(G)$. By Lemma 7.12, $\sigma$ induces a collineation $\sigma'$ on $\Delta(\mathcal{P}_{G/F_1(G)})$. But by definition, $(P_{i,1}\ldots P_{i,j}/F_1(G))^{\sigma'}$ $\neq P_{i,1}\ldots P_{i,j}/F_1(G)$. This is a contradiction □

**Corollary 7.14** *Let G be a solvable nC-group. Then the Fitting flag is invariant under Col($\Delta(\mathcal{P})$).*

Corollary 7.14 leads one to believe that given a group G and a rank 1 flag $\alpha \in \Delta(\mathcal{P})$ which is invariant under Col($\Delta(\mathcal{P})$), then $\alpha$ is complemented in G. This is true for solvable nC-groups by Lemma 7.10, but not in general. Consider $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^3\rangle$. $\Phi(D_8) = \langle a^2 \rangle$ and is invariant under Col($\Delta(\mathcal{P})$), but is not complemented.

The class of complemented groups and the class of multiprimitive groups are subclasses of the class of solvable nC-groups. In [28], Suzuki proves that the class of supersolvable nC-groups is the class of complemented groups. It would be nice to obtain a similar type of classification for multiprimitive groups. The next theorem gives necessary and sufficient geometric conditions for a solvable nC-group to be multiprimitive.

**Theorem 7.15** *Let G be a solvable nC-group. G is multiprimitive if and only if each normal subgroup of G is an invariant rank 1 flag in $\Delta(\mathcal{P})$ and $\Delta(\mathcal{P})$ has a unique maximal* *invariant flag* .

Proof. If G is multiprimitive and N $\lhd$ G, then N = $F_i$(G) for some $i$, $1 \leq i \leq r$, where $r+1$ is the Fitting length of G. By Corollary 7.14, N is invariant under Col($\Delta(\mathcal{P})$). Suppose there is a rank 1 flag $\alpha \in \Delta(\mathcal{P})$ which is invariant under Col($\Delta(\mathcal{P})$) and $\alpha \neq F_i$(G) for any $i$, $1 \leq i \leq r$. By Lemma 7.10, $\alpha$ is characteristic, and hence normal, in G. Thus $\alpha$ = $F_j$(G) for some $j$, $1 \leq j \leq r$. This is a contradiction and hence the Fitting flag $F_1$(G) $\subset \ldots \subset F_r$(G) is the unique maximal invariant flag in $\Delta(\mathcal{P})$.

Conversely, suppose G is a solvable nC-group. By Theorem 7.13, G has an invariant flag $P_{1,1} \subset P_{1,1}P_{1,2} \subset \ldots \subset P_{1,1}\ldots P_{1,t_1-1} \subset F_1(G) \subset P_{2,1} \subset \ldots \subset P_{2,1}\ldots P_{2,t_2-1} \subset F_2(G) \subset \ldots \subset F_r(G) \subset P_{r+1,1} \subset \ldots \subset P_{r+1,1}\ldots P_{r+1,t_{r+1}-1}$ where $P_{i,j}/F_{i-1}$(G) is the Sylow $p_j$-subgroup of $F_i$(G)$/F_{i-1}$(G) and $t_i$, for $1 \leq i \leq r+1$, is the number of Sylow $p$-subgroups in $F_i$(G)$/F_{i-1}$(G).

Suppose there is a $t_i$, $1 \leq i \leq r$, such that $t_i \geq 2$. Then $P_{1,1} \subset \ldots \subset F_{i-1}(G) \subset P_{i,1} \subset P_{i,1}P_{i,2} \subset \ldots \subset P_{i,1}\ldots P_{i,t_i-1} \subset \ldots \subset F_r(G) \subset P_{r+1,1} \subset \ldots \subset P_{r+1,1}\ldots P_{r+1,t_{r+1}-1}$ and $P_{1,1} \subset \ldots \subset F_{i-1}(G) \subset P_{i,2} \subset P_{i,1}P_{i,2} \subset \ldots \subset P_{i,1}\ldots P_{i,t_i-1} \subset \ldots \subset F_r(G) \subset P_{r+1,1} \subset \ldots \subset P_{r+1,1}\ldots P_{r+1,t_{r+1}-1}$ are both invariant under Col($\Delta(\mathcal{P})$). This contradicts the fact that $\Delta(\mathcal{P})$ has a unique maximal invariant flag. Hence for each $i$, $1 \leq i \leq r+1$, $t_i = 1$ and $F_1$(G) $\subset \ldots \subset F_r$(G) is the unique maximal invariant flag in $\Delta(\mathcal{P})$.

Now suppose for some $i$, $1 \leq i \leq r$, that $F_i$(G)$/F_{i-1}$(G) is not minimal normal in G$/F_{i-1}$(G). Then $F_i$(G)$/F_{i-1}$(G) = $N_{i,1}/F_{i-1}$(G) $\times \ldots \times N_{i,s_i}/F_{i-1}$(G) where $s_i \geq 2$ and $N_{i,j}/F_{i-1}$(G) is minimal normal in G$/F_{i-1}$(G). Since each normal subgroup is invariant under Col($\Delta(\mathcal{P})$), both $F_1$(G) $\subset \ldots \subset F_{i-1}$(G) $\subset N_{i,1} \subset N_{i,1}N_{i,2} \subset \ldots \subset F_i$(G) $\subset \ldots \subset$ $F_r$(G) and $F_1$(G) $\subset \ldots \subset F_{i-1}$(G) $\subset N_{i,2} \subset N_{i,1}N_{i,2} \subset \ldots \subset F_i$(G) $\subset \ldots \subset F_r$(G) are invariant under Col($\Delta(\mathcal{P})$). This is a contradiction. Thus for each N $\lhd$ G, there is an $j$,

$1 \leq j \leq r$, such that $N = F_j(G)$. Hence G is multiprimitive □

Both conditions in Theorem 7.15 are necessary. Consider the group $G = [Z_2 \times Z_2 \times Z_2 \times Z_2]Z_3 = \langle a, b, c, d, x \mid a^2 = b^2 = c^2 = d^2 = x^3 = 1, ab = ba, ac = ca, ad = da, bc = cb, bd = db, cd = dc, a^x = b, b^x = ab, c^x = d, d^x = cd \rangle$. It has a unique invariant flag $\langle a, b, c, d \rangle$. Yet both $\langle a, b \rangle$ and $\langle c, d \rangle$ are normal in G and G is not multiprimitive. Also consider the group $G = [Z_7 \times Z_5]Z_2 = \langle a, b, c \mid a^7 = b^5 = c^2 = 1, ab = ba, a^c = a^6, b^c = b^4 \rangle$. Each of its normal subgroups $\langle a \rangle$, $\langle b \rangle$, and $\langle a, b \rangle$ are invariant under Col($\Delta(\mathcal{P})$), but G is not multiprimitive.

# Chapter 8

# Further Questions

There are a number of avenues of further investigation motivated by this work. The construction of the flag space $\Delta(\mathcal{P})$ for a finite dimensional vector space V over the field of characteristic $p$ was generalized to construct the flag space $\Delta(\mathcal{P})$ associated with a finite group. Given a maximal flag A in the flag space $\Delta(\mathcal{P})$ of V, the partially ordered set of all flags contained in A is a simplex (I on page 313 of [29]). A partially ordered set X is called a **simplex** of rank $d$ if X is isomorphic to the partially ordered set formed of all the subsets of a set of $d$ elements with respect to the containment relation. This does not hold for the flag space of groups in general, but groups which satisfy this condition deserve investigation. A set theoretical generalization of the concept of a flag space was done by Tits in [32], in which he developed the theory of complexes. This theory has applications in many areas of mathematics. Given a set $\Delta$ and a partially ordering defined on $\Delta$, the set $\Delta$ is said to be a **complex** if for each pair of elements A and B in $\Delta$, there is a greatest lower bound and for each element A in $\Delta$, the subset of $\Delta$, which consists of the elements contained in A, forms a simplex. Not every group has a flag space which is a complex. However, the theory of complexes may yield new results concerning the algebraic and induced geometric structure of groups whose associated flag space is a complex.

In Theorem 1.21, it was shown that if a solvable nC-group is an SE-group, then G is multiprimitive. The following counter-example was given to show that the converse is not always valid. Let G = $[Z_5 \times Z_5]Z_3 = \langle a, b, c \mid a^5 = b^5 = c^3 = 1, ab = ba, a^c = a^3b$ and $b^c = a^2b\rangle$. The group G is not an SE-group, because each 1-dimensional subspace of $\langle a, b\rangle$ is in some splitting system for G, yet the action on $\langle c\rangle$ on $\langle a, b\rangle$ does not act transitively on the 1-dimensional subspaces in $\langle a, b\rangle$. It is known(3.3.6 of [29]) that GL(2,5) acts transitively on the 1-dimensional subspaces of $Z_5 \times Z_5$. Given an n-dimensional vector space V over the field of characteristic p, this brings up the question as to what conditions must be placed on a subgroup H of GL(n,p) in order for H to act transitively on the 1-dimensional subspaces of V. The general question is: Given a finite dimensional vector space V over the field of characteristic p, what conditions must be placed on a subgroup H of GL$(n, p)$ in order for H to act $k$-transitively on the 1-dimensional subspaces of V, where $1 \leq k \leq n$.

The use of splitting systems to generalize the concept of a frame and a flag was motivated by the geometry of linear groups. Given a finite dimensional vector space V over the field of characteristic p, each subspace of each frame for V is cyclic of prime order. In Chapter 4, groups were studied in which each subgroup of each frame for the group was cyclic of prime order. This motivates the study of groups in which each subgroup of each frame for the group is of some other type of inseparable group. In particular, to study f-primary groups of type S. Recall that an f-primary group G of type S is a group in which each frame $\Sigma = \{ S_1, S_2, \ldots, S_n \}$ for G satisfies $S_i \cong$ S for $1 \leq i \leq n$. The next step in this investigation would be to study the algebraic and geometric structure of groups which have each subgroup of each frame cyclic of prime power order. Of special interest would be when each subgroup in each frame for G is simple.

Another question comes from the study of maximal flag transitive spaces. If A: $\alpha_1 \subset \alpha_2 \subset \ldots \subset \alpha_d$ and B: $\beta_1 \subset \beta_2 \subset \ldots \subset \beta_e$ are two maximal flags in a maximal flag transitive space $\Delta(\mathcal{P})$, then A and B are isomorphic. Thus $d = e$ and for each $i$, $1 \leq i \leq d$,

$\alpha_i \cong \beta_i$. Let $\Sigma = \{ S_1, S_2, \ldots, S_{d+1} \}$ and $\Sigma' = \{ T_1, T_2, \ldots, T_{d+1} \}$ be frames for G which support A and B respectively. Therefore, for each $i$, $2 \leq i \leq d$, $[S_1 \ldots S_{i-1}]S_i = \alpha_i \cong \beta_i = [T_1 \ldots T_{i-1}]T_i$. Since $S_1 \ldots S_{i-1} = \alpha_{i-1} \cong \beta_{i-1} = T_1 \ldots T_{i-1}$, this seems to imply that $S_i \cong T_i$. However, up to this point, this has not been proven. To answer this question would require answering it as posed in its simplest form. Let G be a finite group. If G = [A]B = [C]D, where A, B, C, and D are inseparable and A ≅ C, is B ≅ D? If G is the direct product of inseparable groups, then by Theorem 4.8, B ≅ D. The problem then reduces to considering G, when it is not the direct product of inseparable groups. The more general form of this question is: If G = $[A_1]A_2 = [B_1]B_2$, where $A_1, A_2, B_1$, and $B_2$ are inseparable, is there a $\pi \in$ Sym(2), such that $A_i \cong B_{\pi(i)}$ for $i$ = 1,2.

Through the flag space $\Delta(\mathcal{P})$ of a group G, the hyperplanes of G can be identified. The subgroup $\mathcal{T}$(G) of Aut(G) is generated by the transvections in Aut(G). It was proven in Theorem 6.19, that if a hyperplane H in a p-group G is complemented in G by a cyclic group, then H has a non-trivial transvection associated with it. Therefore a p-group G, which has at least one hyperplane complemented by a cyclic group, has $\mathcal{T}$(G) existing non-trivially in Aut(G). This provides a tool in the investigation of the automorphism groups of p-groups which satisfy this property. This approach has an immediate application to the classification of p-groups G of exponent p. If | G | ≥ 2, then each maximal subgroup in G is a hyperplane in G. Furthermore, by Corollary 6.20, each hyperplane H in G has a non-trivial transvection associated with it.

The structure of metacyclic groups is well known([16] and [18]), but not much is known concerning the automorphism groups of metacyclic groups. There are two types of metacyclic groups which are of special interest. Those metacyclic groups which have cyclic Sylow p-subgroups and metacyclic p-groups. J. Malone and G. Mason are currently investigating the automorphism groups of metacyclic groups with cyclic Sylow p-subgroups. The next step is to study the automorphism groups of metacyclic p-groups. If a metacyclic group G is separable, each hyperplane in G is complemented in G by a cyclic group and has a non-trivial transvection associated with it. Therefore, $\mathcal{T}$(G) exist non-trivially in Aut(G).

Given an abelian group G of rank $\geq 2$, by the Fundamental Theorem of Abelian Groups, each hyperplane in G is complemented in G by a cyclic group. Therefore, $\mathcal{T}$(G) exists non-trivially in Aut(G). In Chapter 6, $\mathcal{T}$(G) was used to study the automorphism groups of homocyclic p-groups. Continuing this line of investigation would be to use $\mathcal{T}$(G) to study the automorphism group of abelian p-groups. In particular, what could such an investigation yield concerning abelian p-groups of type $(k,1,1,\ldots,1)$, where $k \geq 2$.

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