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# Beliefs, autonomy, and mathematical knowledge

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**Rector, Judy Ann, Ph.D.**

**University of New Hampshire, 1992**

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**BELIEFS, AUTONOMY, AND  
MATHEMATICAL KNOWLEDGE**

**BY**

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**DISSERTATION**

**Submitted to the University of New Hampshire  
in Partial Fulfillment of  
the Requirements for the Degree of**

**Doctor of Philosophy  
in  
Mathematics Education**

**May, 1992**

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**ABSTRACT**

**BELIEFS, AUTONOMY, AND  
MATHEMATICAL KNOWLEDGE**

by

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University of New Hampshire, May, 1992

The purpose of this study was to investigate the apparent effects of students' beliefs about mathematics and autonomy on their learning of mathematics. The study utilized a multiple-case study design with analysis by and across cases. The cases represented six high school students enrolled in either Algebra II or Algebra II/Trigonometry. Data was collected in three phases: (a) classroom observations and assessment of the teacher's perception of her role in the learning process, (b) an assessment of students' beliefs about mathematics and autonomy, and (c) an assessment of students' newly formed mathematical constructs on functions.

The beliefs' assessment included observing and questioning students while they solved mathematics problems. Follow-up probes explored the students' rationale for their strategies and their dependency on rules and algorithms when solving problems, particularly problem-solving situations. To further corroborate the beliefs expressed by the students or those inferred from their solutions, the students also marked and discussed a mathematics topics ranking grid and vocabulary lists, graded a sample algebra test, and responded to scenarios on student's/teacher's roles. From the data, a detailed portrait of each individual's beliefs about mathematics was developed. These portraits were compared with the students' understanding of functions and the classroom expectations.

The results from this research investigation suggest three hypotheses concerning students' beliefs about mathematics, autonomy, and mathematical knowledge. First, students' beliefs about mathematics rather than being dichotomous form a continuum from strongly

conceptual in outlook to strongly procedural. Second, students' autonomy augments their beliefs about mathematics and often mediates them. Third, students' beliefs and autonomy appear to concur with their problem-solving strategies and with their knowledge of mathematics. Collectively these hypotheses suggest that students' beliefs and autonomy are an integral component of students' conception of mathematics and influence both how problems are approached and how mathematics is learned. Further study needs to be done on how and when these beliefs are formed and under what conditions these beliefs are modified and changed. Finally, the interplay among beliefs, autonomy, and learning needs to be investigated in the actual classroom context.

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## CHAPTER I

### INTRODUCTION

A recurrent theme in mathematics education has been the impact of students' attitudes on their achievement in mathematics. Evidence for this interaction is suggested by the results of the recent National Assessments of Education Progress (NAEP). In the fourth NAEP, data was collected on both students' proficiency on basic arithmetic skills, problem solving, and conceptual understanding; and on students' attitudes towards mathematics (Brown, Carpenter, Kouba, Lindquist, Silver & Swafford, 1988b; Swafford & Brown, 1989). The mathematical results indicate a strong reliance on algorithms with little conceptual underpinning to facilitate applications or problem solving. Equally alarming are the attitudinal results. A majority of the seventh and eleventh grade students surveyed responded that they perceived mathematics as merely following rules and over half felt learning mathematics is mostly memorization. Approximately 20 percent of the students agreed with the statement that mathematics is made up of unrelated topics and about 35 percent agreed that new discoveries are seldom made in mathematics and that mathematicians work with symbols and not ideas.

For Silver (1987), these beliefs about mathematics reflect a 'hidden' component in the mathematics curriculum.

These statements and others like them, reflect students' beliefs about or attitudes toward mathematics. The students' beliefs and attitudes have been shaped by their school mathematics experiences. Despite the fact that neither the authors of the mathematics curriculum nor the teachers who taught the courses had intentional curricular objectives related to students' attitudes toward and beliefs about mathematics, students emerged from their experience with the curriculum and the instruction with these attitudes and beliefs. Since the students' viewpoint represented by these statements is clearly inadequate, and potentially harmful to their future progress in mathematics, we need to focus our attention more clearly on those hidden products of the mathematics curriculum. (Silver, 1987, p. 57)

Other researchers also have noted the potential harm in these beliefs about mathematics and have referred to them as dysfunctional beliefs (Baroody & Ginsburg, 1986; Borasi, 1990; Buerk, 1981; Frank, 1985).

In his review of the literature on beliefs about mathematics, Underhill (1988) pleads that: As we know more of learners' beliefs, we are struck by the disparities between what we

believe and what they believe, what we intend to be learned and what is learned. Further study can surely help us improve mathematics instruction by providing a new type of feedback. Too many learners have no sense of empowerment; they are looking only for correct answers; they are memorizing facts and procedures. Far too few feel mathematically empowered; far too few feel in charge of their own learning, feel in charge of the growth and development of their own mathematical knowledge. (p. 66)

While the results from the NAEP and the observations made by Silver and Underhill suggest the importance of beliefs, little is known about the actual nature of the interaction between attitudes/beliefs and achievement. A review of the attitude literature reveals numerous statistical studies showing a consistently low to medium correlation (.19 to .54) between the measures of attitudes towards mathematics and achievement (Aiken, 1970a,b, 1971, 1976; Kulm, 1980; Reyes, 1984). These statistical results have been interpreted as indicating that attitudes have a secondary rather than a causal effect on achievement. One current theory suggests that attitudes through their effect on motivation, self-confidence, and anxiety have a subsequent effect on achievement.

Although numerous, the research results have proven inconclusive and fragmented. Reviewers of the literature have criticized the research on several key points: (a) no unifying definition of attitude, (b) the absence of a theoretical basis for interpreting statistical data or directing research questions, (c) little cross referencing or building on previous work, and (d) inadequate models to explain the interaction between attitude and achievement. Aiken (1976), in his review, indicates that attitude research has relied too heavily on correlation methods and indirect measures of attitudes such as questionnaires. He recommends that future research consider the distinction between the cognitive and emotional subcomponents of attitude when developing attitudinal instruments. Kulm (1980) further argues for theory development studies utilizing qualitative methods which are sensitive to nuances in beliefs, opinions, and behavior.

The recent qualitative research into students' problem-solving strategies has reopened the debate as to the effect of beliefs on students' cognitive processes. The studies of Buchanan (1984), Cobb (1985, 1986), Frank (1985), and Schoenfeld (1983, 1985) have pointed to students' beliefs about mathematics as a limiting factor in their problem-solving behavior. Beliefs by setting up expectations appear to constrain students' choice of heuristics and even restrict the type of problems students perceive as mathematics (Kouba & MacDonald, 1987, 1991). Collectively, these studies suggest that beliefs have an interactive role as students solve

problems and learn mathematics.

Skemp (1987) speculated that instrumental (procedural) and relational (conceptual) knowledge structures are linked with differing expectations about mathematics. Specifically instrumental knowledge is associated with a view of mathematics as memorized rules with results being evaluated right or wrong. In contrast, relational knowledge is associated with a view of mathematics that stresses the importance of understanding the reasons behind procedures. Skemp also differentiated between knowledge that was accepted as valid solely on the basis of the teacher's authority and knowledge that was accepted as valid because it was meaningful to the individual. Here again, these distinctions can be described by differing expectations or beliefs about the source of validity for one's mathematical knowledge.

Both the reviews of the attitude literature and the problem-solving research express the need for qualitative investigations into students' beliefs about mathematics and into the interaction between beliefs and problem-solving behavior and learning. These concerns form the basis of this research study.

In this investigation, a multiple case study design was utilized to examine six Algebra II students' beliefs about mathematics and their possible effect on learning mathematics. The research plan consisted of a pilot study and three data gathering phases: (a) classroom observations and assessment of the teacher's perception of her role in the learning process, (b) an assessment of student participants' beliefs about mathematics and autonomy with mathematics, and (c) an assessment of the students' newly formed mathematical constructs on functions.

Briefly, the rationale behind the research plan was to develop a detailed portrait of each individual's beliefs about mathematics, to observe these individuals within the social context of their mathematics classes, and finally, to carefully examine the mathematical constructs that the individuals formed from their classroom experiences. By synthesizing the information on individual students' beliefs with the classroom observations and expectations and comparing these results with the individual student's mathematical constructs, a description was developed for each student. These descriptions attempted to explain the type and depth of the mathematical constructs relative to the students' beliefs and to the classroom expectations. Finally, these individual descriptions were compared for any patterns and similarities among

students' beliefs about mathematics, students' autonomy with mathematics, and students' understanding of functions. In exploring the relationships between students' beliefs about mathematics and autonomy, and students' knowledge of functions in the classroom environment, the following questions were addressed.

#### Students' Beliefs

1. To what extent do the students hold beliefs about mathematics as conceptual or procedural in nature?
2. To what extent do the students hold the belief that they are the source of authority for their knowledge?
3. What do students perceive as their role and the teacher's role in learning mathematics?
4. How are the students' beliefs about mathematics as conceptual or procedural related to their source of authority? For example, do students with a view of mathematics as conceptual have an internal source of authority?

#### Classroom Environment

1. What are the teacher's beliefs about mathematics as a discipline?
2. What does the teacher perceive as her role and the students' role in learning mathematics?
3. In what ways does the teacher promote students' autonomy with mathematics?
4. In what ways does the teacher promote a view of mathematics as conceptual or procedural in nature? More specifically, to what extent do the teacher's questions require students to use conceptual or procedural knowledge?

#### Relationship between Beliefs and Knowledge of Functions

1. To what extent do the students reveal a conceptual or procedural understanding of functions which includes the definition of function; the concepts of domain and range; function notation; composition of functions; and linear functions?
2. How do the students' beliefs about mathematics as conceptual or procedural fit with their knowledge of functions? For example, do students who hold a view of mathematics as procedural construct a procedural or conceptual understanding of functions?

3. How does the students' source of authority fit with their knowledge of functions? For example, does a student with an internal source of authority construct a conceptual or procedural understanding of functions?

## CHAPTER II

### THEORETICAL FRAMEWORK

This chapter begins by defining and distinguishing between the terms attitude and belief. Then a theoretical foundation for beliefs will be established by outlining the formation, organization, function, and measurement of beliefs. The connection between beliefs and learning will be made through a discussion of the nature of knowledge. It will be argued here that all knowledge can be perceived as socially justified beliefs. Next it will be suggested through the theories and research of Perry (1970) and Kolb (1981) that knowledge of a discipline entails more than content knowledge. Discipline knowledge also encompasses beliefs about the discipline and about the source of authority for one's knowledge. The discussion will then focus specifically on mathematics as a discipline and how it fits with these general theories of knowledge. The discussion also will include the results and speculations from research on students' beliefs about mathematics and Skemp's theory (1987) on the structure of mathematical knowledge as instrumental (procedural) and relational (conceptual). Finally these various theories and results will be summarized and joined in a discussion of research hypotheses.

#### Definitions of Attitude and Belief

The principle difficulties in assigning a meaning to the terms attitude and belief arise from the ambiguous usage of the terms, inconsistent meaning of the terms across disciplines, and the colloquial usage. It is our intention to successively build a meaning for these terms, rather than to prescribe a definition at the outset. This tactic has the advantage of postponing the statements of definition until the theory and rationale can be developed to support their usage.

#### General Definitions of Attitude from the Psychological Literature

An examination of the psychological and social-psychological research literature revealed a multitude of definitions for the term attitude. While many definitions exist, Leder (1985) notes four crucial assumptions that the definitions seemed to share: "Attitude is learned, it predisposes to action, the action towards the object is either favourable or unfavourable, and there is response consistency (p. 18)." The definitions share another common trait. They describe or



define attitude in terms of its psychological function. That is, the definitions are less declarative in nature than descriptive. An attitude is defined or recognized by what it does to  $x$  or how it affects  $y$ .

As the researchers have studied and reflected on the meaning of attitudes, the definitions have shifted in emphasis. Current definitions attempt to refine and explain the relationships between attitude and behavior. In his review of attitude-behavior models, Liska (1984) described the state of development as:

Research mushroomed, but in a disorderly fashion. Definitions of attitudes became increasingly ambiguous, and each new study seemed to identify a new "other" variable [to explain the interaction]. There was little concern with clarifying the definition of attitude, and with identifying general categories of other variables and the causal processes by which they influence the attitude-behavior relationship. (p. 62)

One specific model of attitude-behavior surfaced in literature and received notoriety: the Fishbein-Ajzen (1975) model. This model distinguished between three subcomponents of attitude: cognition, conation, and affect. The term cognition referred to an individual's beliefs about a situation. Conation was defined as the behavior intention of an individual while affect reflected an individual's emotions or feelings about a situation. These three subcomponents were perceived as conceptually distinct but causally related. This model assumed that actual behavior was caused by behavioral intent which was influenced by affect which in turn was a manifestation of beliefs (Liska, 1985). Fishbein and Ajzen elected to use the terminology attitude for the affective subcomponent. This dual usage of the term attitude for the entire concept and for the specific subcomponent proved problematic for it is unclear which meaning subsequent authors intend.

The recognition of the distinct subcomponents of attitude and the possible influence of other variables in the attitude-behavior interaction served as a focal point for much of the research literature (Fazio, 1978; Kleinke, 1984; Liska, 1985; Norman, 1975; Rokeach, 1968). Serious consideration was and is being given to these affective variables' ability to predict behavior and to the consistency in measurements between the affective and cognitive subcomponents.

#### Definitions From Mathematics Education Literature

The difficulties in terminology that have accompanied the research in social psychology and psychology have been mirrored in the mathematics education research literature (Aiken,

1970; Reyes, 1984). Hart (1989), Kulm (1980), and Leder (1985) in their respective reviews of the mathematics education literature found not only inconsistency in terminology but a lack of any explicit definition and a reliance on operational definitions tied to the instruments used to measure attitude.

In recent years, there has been some attention given to unifying and clarifying the meaning of the terminology in the mathematics education literature (Hart, 1989; McLeod, 1987, 1989). McLeod and Hart denote the whole concept as the affective domain and highlight three terms: belief, attitude, and emotions. However, unlike the social psychologists, McLeod perceives these terms as demarking points on a continuum rather than being subcomponents of the affective domain. McLeod describes this continuum by saying: "the terms beliefs, attitudes, and emotions are listed in order of increasing affective involvement, decreasing cognitive involvement, increasing intensity, and decreasing stability (p. 171)."

Table 1

Contrasting Models for Attitude/Affective Domain

	Models	
	Fishbein/Azjen	McLeod/Hart
<b>Concept</b>	Attitude	Affective Domain
<b>Terms</b>	Belief Attitude (Affect)	Belief Attitude (Affect)
<b>Assumptions</b>	Terms denote subcomponents which are causally related behavioral intent--attitude --beliefs.	Terms denote points on a continuum that differ with respect to intensity, stability, affect, and cognition.

The terminology and definitions suggested by McLeod and Hart agree in spirit with the definition given in the sociological and psychological literature as illustrated in Table 1. Both sets of definitions acknowledge the essential cognitive function of belief. Cobb (1986) and Schoenfeld (1983) concur with this view and describe beliefs as a collection of thoughts that function by helping to organize and interpret knowledge, and by acting as a filter for the strategies employed

in solving problems.

### **Belief Systems**

To understand beliefs and belief systems four issues need to be addressed: (a) formation of beliefs, (b) description of belief systems, (c) function of beliefs, and (d) measurement of beliefs.

#### **Formation of Beliefs**

The term belief system denotes the internal structure of an individual's collection of beliefs. An individual's beliefs and associated structure are theorized to form in response to a basic need to make sense of the world and to assert control over it by predicting future events. Kelly (1963) uses the analogy of a scientist to explain this process of construct (belief) formation. Like the scientist, an individual is searching his experience and environment for patterns of commonality that explain phenomena. Once observed, a pattern is tested for its replicability and reliability in predicting future events. As in science, some personal theories or explanations are discarded over time while others become fundamental.

Kelly cautions, however, that a construct is not to be misconstrued as the experience itself but rather it represents the interpretation the individual gives that experience. In addition the interpretations or constructs that an individual develops are specific to that individual. This individuality manifests itself in three ways: the criteria used to develop a construct, the construct itself, and the organization of these constructs (Green, 1971; Kelly, 1963).

#### **Description of Belief Systems**

Green (1971) describes a belief system in terms of the internal hierarchy or structure given to the beliefs by the individual. He characterizes the hierarchy or organization by three traits: primality, centrality, and clustering. Primality refers to a belief that is not logically derived from another belief. Centrality describes the relative strength of a belief and its importance within an individual and clustering refers to how beliefs are grouped together within an individual's belief system. Thus, an individual's beliefs can be depicted as gathered into clusters with some clusters being central and vital to the individual while others are peripheral. Within the clusters, the beliefs are further organized by their logical/primality ordering.

This hierarchical structure suggests a priority ranking within the belief structure. That is, the more central beliefs receive greater preference and have a greater influence in decision

making. While viewed as an interconnected and hierarchical network, the belief structure does not necessarily imply a logically consistent system. Rokeach (1968) suggests that:

If a person acts contrary to one attitude [Rokeach's term for a cluster of beliefs organized around an object or situation] it means that he acted in accord with a second (or third or fourth) attitude that overrode the first attitude in importance. When there is a negative correlation between a given attitude and behavior there is always the possibility that some other attitude that was not measured may be congruent with the behavior. (p. 128)

Belief structures are considered dynamic systems that can be modified or changed as individuals re-evaluate the utility of their beliefs to predict future events (Kelly, 1963; Rokeach, 1968). A belief's resilience or adaptability to change is another characterization of an individual's belief structure (Green, 1971; Kelly, 1963). Green (1971) proposes that:

Belief systems of different people can be described in relation to the ease with which they may change and grow, the ease with which different clusters of beliefs can be related, the number and nature of the logically primary and psychologically central beliefs; the ease with which they may move from center to periphery and back; the correspondence, or lack of it, between the objective, logical order of beliefs and the order in which they are actually held. (p. 48).

Hence for Green, a crucial component of belief is how that belief is held. That is, on what basis an individual tests the belief's validity. How a belief is held, Green believes affects its ability to be changed, modified, or even opened for consideration. Green argues that "a person may hold a belief because it is supported by evidence, or he may accept the evidence because it happens to support a belief he already holds (p. 49)." He further asserts that before beliefs can be scrutinized or re-evaluated, that an individual needs to have a deep conviction that "beliefs can and should be rationally examined (p. 54)." This conviction enables an individual "to hold all other beliefs open to challenges, examination, and changes in the light of further evidence and fresh reason (p. 54)."

#### Function of Beliefs

Reviewing the research on various functions of beliefs, Rokeach (1965) cites a knowledge function which coincides with Kelly's analogy of the individual as scientist. In this role, beliefs function to help an individual structure the universe by aiding in the search for meaning and understanding by organizing perceptions, and by providing clarity and consistency. Since beliefs ascribe meaning or give interpretation to experience, they help an individual understand the world and consequently, help to predict future events. It is in the role of anticipating future events that beliefs are associated with actions or behaviors. For example, a

student may believe that mathematics is learned through memorization. When presented with an upcoming test, that student may prepare by memorizing homework exercises. However, the existence of a belief does not guarantee a prescribed action. For example, an individual may hold the belief that alcohol consumption is harmless, yet refrain from drinking. Although appearing inconsistent, the actions may reflect the presence of other intervening variables (in Green's terminology, a more central or dominant belief). In the above scenario, the individual may abstain in order to conform to the societal norms of the culture, to the expectations of parents, or to religious precepts. This last example, illustrates one of the difficulties in associating a direct causal relationship between beliefs and actions. In fact, much of the controversy in the psychological literature focuses on the interrelationship between beliefs and actions (Liska, 1984). It is in recognition of this difficulty, that the definitions for belief and attitude use such phrases as "a predisposition to action." A belief may suggest a course of action, it may be influential in the decision process, and yet not be accurate predictor of an individual's actions.

#### Measurement of Beliefs

Beliefs are constructs of the mind and as such often are held unconsciously. Thus, individuals, as well as outside observers, have difficulty clearly accessing beliefs, mapping the internal structure of the belief systems, and assigning causality from beliefs to behavior (Nisbett & Ross, 1980). Despite the difficulties, two basic strategies have been employed by researchers in measuring or ascertaining beliefs: (a) self-reports and (b) inference from behavior.

Self-reports in the form of paper and pencil measures (e.g. semantic differentials scales, Likert scales, inventories, and preference rankings) and clinical interviews assume that the individual is truthful and able to clearly and accurately access internal beliefs (Leder, 1985). Even with the assumption of trustworthiness, the psychological research has shown that self-reflections involve selective recall and tend to vary across time (Nisbett & Ross, 1980; Nisbett & Wilson, 1977).

The reliability of self-reports is further compromised if the underlying assumption of trustworthiness is violated. Such is the case when subjects report only what they think a researcher wishes to hear or when subjects present only those thoughts that will yield a positive self-report.

The second strategy of inferring beliefs from action is likewise inconclusive. Here the

difficulty rests in linking behavior causally to beliefs. Intervening beliefs may make this process inaccurate. In fact, the research literature on the relationship between attitudes/beliefs and behavior has repeatedly shown only a moderate correlation (Ajzen & Fishbein, 1973; Bagozzi & Burnkrant, 1979; Regan & Fazio, 1977).

While both strategies for measuring beliefs are inadequate, the psychological research literature has developed and posed several techniques for improving the reliability of these strategies. First, it is suggested that self-reports of beliefs should be taken in close proximity to the experience that generates the beliefs or to one that parallels that experience. This temporal closeness is theorized to provide the individual with clear access to beliefs and thus make the beliefs more salient to the individual (Fazio & Zanna, 1978; Kleinke, 1984; Nisbett & Ross, 1980; Nisbett & Wilson, 1977; Regan & Fazio, 1977).

It also is suggested that thinking-aloud protocols serve a similar function, in that, the reports are more immediate and less subject to an individual's reconstruction of events (Nisbett & Ross, 1980). Other researchers have argued that open-ended instruments like sentence completions, interviews, and thinking-aloud protocols permit deeper and more subtle exploration of beliefs (Head & Sutton, 1985; Leder, 1985; Mischler, 1986). The supposition is that these techniques allow for clarification of meaning between researcher and subject, and empower individuals to express beliefs in their own words rather than deciding among preselected terminology as in the case of semantic differentials and Likert scales. In addition, the results of multiple measuring techniques are more reliable than single measurements of beliefs (Bagozzi & Burnkrant, 1979). Finally inferences about beliefs from behavior are more reliable (stable and consistent) when the behaviors are observed or arise in situations salient to the individual's beliefs (Nisbett & Ross, 1980). Again the context is theorized to provide the individual with more salient and immediate information about unconscious beliefs and consequently have a greater influence on present actions.

#### Definition and Theory of Knowledge

One of the motivating questions behind this study is how do one's beliefs about a discipline interact with the classroom experience to affect content knowledge of that discipline? A discussion of this question necessitates a clear conception of how knowledge is formed, what constitutes knowledge and mathematical knowledge, in particular, and what distinguishes belief

from knowledge.

Any discussion of the nature of knowledge eventually reduces to two philosophical quandaries: first, where does knowledge lodge?; and second, what constitutes knowledge? In addressing the first quandary, Davis (1988) argues that historically knowledge has been perceived as originating from God, from objective reality, and finally from the community of practitioners. While the first premise that knowledge originates from the mind of God is no longer viewed as tenable, considerable debate still focuses on whether knowledge exists outside of the individual or whether knowledge is a creation of the individual's mind which is laid on the outside world (von Glaserfeld, 1984). Kuhn (1962) uses the evolution of science as evidence that what constitutes knowledge is often dependent upon the culture and upon the era. Toulmin (1972), as well, argues that the source of one's knowledge is intrinsically tied to the criteria used to verify or validate it and this criteria is influenced in turn by the cultural and historical era.

What concepts a man employs, what standards of rational judgement he acknowledges, how he organizes his life and interprets his experience: all these things depend--it seems--not on the characteristics of a universal 'human nature', or the intuitive self-evidence of his basic ideas alone, but also on when he happened to be born and where he happened to live. (Toulmin, 1972, p. 50)

For Toulmin then, human understanding or knowledge is conceived as an interplay between the individual and the society.

If one accepts this premise, that knowledge resides in the individual in conjunction with the society, the dilemma still remains as to what constitutes knowledge. Are all human interpretations of experience equated with knowledge? What distinguishes a belief from knowledge? The answer to this dilemma again appears to be culturally dependent. Consider the following statements:

I know that  $2 + 2 = 4$ .

I know that the sun circles the earth.

I know that God exists.

One immediate and plausible delineation among these statements is the certainty with which each of these statements is held with the stronger being equated with knowledge; the weaker, with belief (Bogdan, 1986; Malcolm, 1952). Yet to assert that one statement is more certain than another is to tacitly imply an abstract standard against which each of these statements is measured. This abstract standard represents an assertion about truth. Thus, any delineation

between beliefs about and knowledge of an object quickly reverts to a discourse on the criteria used to establish the credibility of statements. Yet, it has been argued that such criteria often resides in the culture. Thus the demarkation between belief and knowledge can be seen as a transient boundary, appealing to the consensus in the community to establish the division.

Constructivism is a philosophy of knowledge that coincides with the perspective articulated by Kuhn and Toulmin. Constructivism asserts that: "man--and man alone--is responsible for his thinking, his knowledge, and therefore also for what he does" (von Glaserfeld, 1984, p. 18). Constructivism assumes that: "knowledge does not reflect an 'objective' ontological reality, but exclusively an ordering and organization of a world constituted by our experience" (von Glaserfeld, 1984, p. 24). This distinction is at the heart of constructivism. It represents a shift in philosophy as to what it means "to know". Knowledge is no longer seen as a match to reality. Rather knowledge or to know means to have a conceptual structure that fits experience (von Glaserfeld & Cobb, 1983). One immediate consequence of this outlook on knowledge and reality is that the knower is no longer seen as a passive receptacle of sensual experience. The individual is perceived as actively involved both in the initial construction of knowledge and in its re-evaluation (Confrey, 1985; von Glaserfeld, 1984). Constructivism also assumes that the constructs are unique to the individual.

Yet the individual lives and interacts within a greater society. Confrey (1981) expands on this dualism between the individual and society by proposing that all concepts (knowledge) have both a private (individual) and public (societal) role.

In their private role concepts allow people to organize and select information and make sense of their experience; in their public role concepts must be subjected to collective scrutiny in order to judge their precision, their effectiveness, and their correspondence with the world as experienced by others. (p. 8)

The private/public roles of concepts do not simply entail dual functions, but suggest a process of movement between these two roles. (p. 10)

For Confrey the public definition of a concept represents: "The current state of evolution of a series of increasingly adequate private conceptions, or as one possibility among a variety of alternative conceptions (p. 8)."

This movement of a concept between the individual and society suggests to Confrey that education can be viewed "as bridging the private concepts of individuals with the public or collective disciplinary concepts (p. 8)." Confrey (1985) also takes the position that the



construction of a 'powerful' concept in a discipline necessitates that an individual develop both critical reflection and personal autonomy.

The most fundamental quality of a powerful construction is that students must believe it. Ironically, in most formal knowledge, students distinguish between believing and knowing. To them there is no contradiction in saying, "I know that such and such is true, but I do not believe it." To a constructivist, knowledge without belief is contradictory. Thus, I wish to assert that personal autonomy is the backbone of the process of construction. (Confrey, 1985, pp. 7-8)

For Confrey, education also entails helping individuals develop these abilities so that they can construct for themselves concepts that are consistent, integrated, and internally justified.

In her discussion on constructivism, Confrey (1985) noted the important role that autonomy and reflection play in the formation of powerful content knowledge. These observations also are confirmed by Perry's (1981) research on intellectual growth. Perry observed that students' views on education seemed to evolve through four developmental stages which he named: dualism, multiplicity, relativism, and commitment. These stages are summarized briefly below.

**Dualism.** Division of meaning into two realms--Good versus Bad, Right versus Wrong, We versus They, All that is not success is Failure, and the like. Right Answers exist somewhere for every problem, and authorities know them. Right Answers are to be memorized by hard work. Knowledge is quantitative.

**Multiplicity.** Diversity of opinion and values is recognized as legitimate in areas where right answers are not yet known. No judgement can be made among them so "everyone has a right to his own opinion."

**Relativism.** Knowledge is qualitative, dependent on contexts.

**Commitment.** An affirmation, choice, or decision (career, values, politics, personal relationship) made in the awareness of Relativism. (Perry, 1981, pp. 79-80)

Each stage is characterized by changes in students' beliefs about the source of authority for their knowledge and about the nature of that knowledge. From dualism to commitment, the voice of authority shifts from outside the student to within. Students' beliefs about the nature of knowledge also shift from an expectation that all knowledge is absolute, to a belief that all knowledge is contextualized, to a resolution that even within diversity positions can be taken and justified provided they remain amenable to future change.

In addition to these shifts in beliefs, Perry observed that these transitions were often accompanied by emotional turmoil. Students expressed confusion, anxiety, anger, and uncertainty. Although the stages were presented here as a linear progression, Perry found that

students, when faced with the uncertainty and challenge of the next stage, would often retreat or postpone movement.

Perry's scheme highlights the interconnection between intellectual growth and beliefs about autonomy/authority and beliefs about the nature of knowledge. His research suggests that these aspects are inseparable from the formation and development of content knowledge. Kolb's (1981) research suggests that individual academic disciplines also may necessitate different learning strategies. Kolb proposed that two concept dimensions, active-reflective and concrete-abstract, could be used to describe the cognitive style in various disciplines.

Kolb labelled the learning styles associated with these dimensions: divergers (concrete and reflective), accommodators (concrete and active), convergers (abstract and active), and assimilators (abstract and reflective), respectively. His research demonstrated that convergers were associated with the physical sciences, divergers with the arts, assimilators with science and mathematics, and accommodators with technical fields and business.

Kolb also proposed that disciplines can be further differentiated on the basis of the following seven criteria: inquiry strategy, dominant philosophy, theory of truth, basic inquiry question, basic unit of knowledge, how knowledge is portrayed, and typical inquiry method.

Kolb's research along with these criteria suggests that knowledge of a discipline entails more than specific content knowledge. It includes an awareness of the processes by which the content knowledge is constructed and validated, and an awareness of the disciplines' internal structure.

#### Philosophy of Mathematics

Over the centuries mathematics has been singled out as an example of a discipline divorced from social influences. The principles and precepts of mathematics can be explicitly stated to appear independent of changing social ideas. It is the view of mathematics as employing and embodying deductive reasoning that is the focal point of much of the philosophy of mathematics. Kline (1980) described its importance by saying:

Deductive reasoning, by its very nature, guarantees the truth of what is deduced if the axioms are truths. By utilizing this seemingly clear, infallible, and impeccable logic, mathematicians produced apparently indubitable and irrefutable conclusions. (p. 4)

Thus, mathematics also became associated with truth and certainty.

Yet during the early 19th century, the certitude of mathematics was shaken by the

creation of multiple geometries. The logical existence of these differing geometries which were all capable of explaining spatial experience precluded any one as being the true representation of reality (Kline, 1980). This uncertainty caused a reexamination of the structural foundations of mathematics. However, paradoxes arose within the axiomatic-deductive structure that subsequently was built up (MacLane, 1986). Together these experiences stimulated a change in how mathematics was viewed: it was no longer perceived as absolute truth. The philosophical question then arises: If mathematical statements are no longer seen as truth, what do they represent? Are they tenuous and subject to revision in the same way as scientific theories? Are these statements perceived, as in science, as socially justified beliefs? MacLane (1986) believes that the question of truth is an inappropriate one when speaking of mathematics. He prefers instead to discuss its "correctness." MacLane makes this distinction for two reasons. First, he argues that mathematics is not like science; it does not purport to represent reality. Second, since the logic and axiomatic system is incapable of resolving its own internal difficulties, mathematical statements cannot be evaluated as true or false independent of the system in which they reside. MacLane also argues that the standard of rigor in mathematics is absolute, external, and impersonal. This view, however, does not preclude the position that these standards represent the common rules for validating knowledge in that discipline. And as such, the axioms and the rules of logic serve as a common standard for the society of practitioners, mathematicians, to justify their statements.

Thus, mathematical knowledge, like scientific knowledge, can be viewed as socially justified beliefs. The society of practitioners, mathematicians, utilize a common criterion for validating statements within the discipline, namely an axiomatic and deductive structure. Mathematical statements are no longer seen as absolute truth, but as valid statements conditional upon the axiomatic system. Mathematics, however, differs from science in that its validity is not adjoined to its fit with reality. In fact, mathematical statements are argued independent of their connection to reality.

Mathematicians often distinguish between the way mathematics is created and studied, and the way it is presented. Formal mathematics involves the language of sets and the axiomatic-deductive structure. MacLane (1986) believes:

There are good reasons why Mathematicians do not usually present their proofs in fully

formal style. It is because proofs are not only a means for certainty, but also a means for understanding. Behind each substantial formal proof there lies an idea, or perhaps several ideas. The idea, initially perhaps tenuous, explains why the results hold. The idea becomes Mathematics only when it can be formally expressed, but that expression must be so couched as to reveal the idea; it will not do to bury the idea under the formalism. (p. 378)

The creation of mathematics involves informal mathematics, that is, the use of ideas, conjectures, and intuition. Halmos (1968) describes the process of creation by:

Mathematics--this may surprise you or shock you some--is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof. The conviction is not likely to come early--it usually comes after many attempts, many failures, many discouragements, many false starts. (p. 380)

For the mathematician, then, the creation of mathematical ideas involves two aspects: the intuition or insight which connects the ideas together and the presentation of these insights in the form of a proof.

To this point, the discussion of mathematics has focused primarily on the public views of the discipline: on the nature of truth and on the creation of mathematics. While a mathematical proof contains the elements of an effective argument, mathematicians still speak of personally needing to believe the argument. Davis and Hersh (1986) call this process "reestablishing meaning" and illustrate it with an anecdote from their own teaching experience.

As a professor of mathematics, most of my teaching takes place in lectures. Very often, I work at the blackboard with my back to the class. As I am working along, talking, explaining, writing strings of symbols, quite suddenly the symbols give way and become vague in my mind. They lose coherence. They lose their relation to each other and to what I have been saying. Their meaning has drained away and they stand on the blackboard in front of me as just so many strange and naked geometrical shapes. . . I found that I had to prepare notes on two levels. The first level, the formal level, contains the material that is part of the public record. The second level contains a thin sprinkling of private notes to myself, interpretable only by myself, which helps me establish the meaning of what is written on the first level. [underline added]. Armed with such a set of notes, I find I can, whenever the meaning or whenever the formal aspects weaken as I lecture, reestablish either one with reasonable ease. (pp. 295-296)

Thus, in the private capacity, public concepts also need to be internally accepted and verified at some level before they are valid to the individual. This verification can come, as described by Davis and Hersh, by reestablishing the meaning of the public proof.

Mishener (1978) elaborates on what is meant by establishing meaning or understanding in mathematics by saying:

When a mathematician says he understands a mathematical theory, he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs.

He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it and what is worth remembering. He has an intuitive feel for the subject, how it hangs together, how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them. (p. 361)

Summarizing then, mathematical meaning is developed by the individual by reestablishing the meaning of public proofs and by connecting these mathematical results to examples, and more broadly to one's conceptions of the discipline.

### Beliefs and Mathematical Activity

#### Conjectures

To this point, the discourse has focused primarily on the theoretical perspectives on belief and knowledge. In the discussions by Perry (1981) and Confrey (1985), knowledge development was closely associated with beliefs about autonomy/authority and beliefs about the nature of knowledge. These themes were also reverberated in the discussion of development of meaning in mathematics. In this section, the emphasis is shifted to the specific conjectures and suppositions about students' beliefs about mathematics and their interaction with mathematical activity.

Research suggests that students' beliefs about mathematics delimits not only how they employ their heuristics and resources but also what features of a problem students will address (Cobb, 1985; Kouba & MacDonald, 1991; Schoenfeld, 1985). Beliefs can even limit what is considered a mathematics problem (Kouba & MacDonald, 1987). Thus, beliefs appear to influence behavior by the way students anticipate and interpret mathematical activity. Furthermore, students' beliefs have been linked with the formation of their knowledge structures (Anderson, 1984; Kouba & MacDonald, 1991). In particular, it is suggested that some beliefs about mathematics may inhibit students' development of conceptual relationships within their mathematical knowledge. Thus, beliefs about mathematics also are theorized to influence how ideas are conceptually related.

Cobb (1986) summarizes and theorized about these two positions in the following statement:

Firmly held beliefs constitute, for the believer, current knowledge about the world. They are a crucial part of the assimilatory structures used to create meaning and to establish overall goals that specify general contexts. The act of formulating a goal immediately delimits possible actions; the goal, as an expression of beliefs, embodies implicit anticipations and expectations about how a situation will unfold. For example, students who have constructed instrumental beliefs about mathematics (Skemp, 1977) anticipate

that future classroom experiences will "fit" these beliefs. They intend to rely on an authority as a source of knowledge, they expect to solve tasks by employing procedures that have been explicitly taught, they expect to identify superficial cues when they read problem statements, and so forth. Alternative ways of operating do not occur to them. Consequently, an examination of the situations in which a student's expectations are corroborated and contradicted by experience provides valuable information about his or her beliefs. (p. 10-11)

#### Students' Knowledge Structures in Mathematics

As these conjectures suggest, beliefs about mathematics and about autonomy with mathematics may have either a facilitating or debilitating effect on students' ability to solve problems and on students' formulation of their knowledge structure. Skemp (1987) proposes that students' mathematical knowledge structures can be characterized as either relational or instrumental understanding. By relational, Skemp means knowledge or understanding in mathematics that is integrated--"knowing what to do and why (p. 153)", while instrumental denotes knowledge or understanding based on the execution of rules without reference to their rationale. While Skemp designates these two categories as types of understanding, they are closely associated with students' beliefs about mathematical knowledge and with students' autonomy with mathematics. In fact, Cobb (1986), Frank (1985), and Buchanan (1984) identify these categorizations with differing belief systems. For Skemp, a student with an instrumental view of mathematics is one who expects mathematics instruction to focus on rules and their execution with results being evaluated right or wrong.

The pupils just 'won't want to know' all the careful ground-work he [the teacher] gives in preparation for whatever is to be learnt next, nor his careful explanations. All they want is some kind of rule for getting the answer. [underline added] As soon as this is reached, they latch on to it and ignore the rest. (Skemp, 1987, p. 155)

Instrumental understanding also is based on the expectation that mathematical learning "necessitates memorizing which problems a method works for and which not, and also learning a different method for each new class of problems (p. 159)." "The goal of instrumental learning is to be able to give right answers, as many as possible, to questions asked by a teacher (p. 168)." Any satisfaction from this type of learning is derived mainly from pleasing others--teacher or examiner.

In contrast, students with a view of mathematics as relational expect mathematics to be rational and knowable. The students also are actively engaged in reflecting about their knowledge trying "to make them more cohesive and better organized (p. 169)." Finally, for

students with this perspective, learning is "an intrinsically satisfying goal in itself (p. 163)" and their learning is held independently and with more confidence.

Skemp perceives the development of these two conceptions of mathematics as a product of the classroom environment and students' interpretations of those experiences. Specifically, Skemp explains the development of knowledge or understanding in terms of "assimilate [ion] into an appropriate schema (p. 29)." Skemp defines a schema as a conceptual structure that an individual constructs in response to (a) experiences in the world, (b) interactions with others' ideas, and (c) internal reflections. An individual tests the validity of that schema against (a) physical events, (b) others' ideas, and (c) personal knowledge or beliefs. The schema are then utilized by the individual to (a) integrate knowledge and make predictions about future events, (b) facilitate communication with others, and (c) aid future growth and reflection. With regard to this last usage, Skemp (1987) states that:

Experiences that fit easily into our existing schemas are more readily learned, and better remembered than those that do not. They also sensitize [sic] us to experiences that we would otherwise have ignored. Our schemas thus act selectively, enlarging themselves rather than other schemas. (p. 114)

Skemp also believes that the assimilation process involves acceptance of the new information by the individual. This acceptance can arise in two ways. In the first way, "acceptance of an assertion depends on the acceptance of the teacher's authority, and acting on it partakes more of the nature of obedience than of comprehension (p. 87)." In contrast, in the second way, "the assimilation of meaningful material depends on its acceptability to the intelligence of the student. Acting on it results from, and consolidates, enlargement of the learner's schemas (p. 87)."

A careful comparison of Skemp's descriptions of how schemas are formed and utilized with the description of how beliefs are formed and used, suggests that schemas also can be identified with beliefs about the discipline and autonomy. If this identification is made, then schema or beliefs can be viewed as a structuring schema by which individuals attach or assimilate new information into existing knowledge structure. The schema or belief would direct how the information is placed into the structure--whether it is tied to other ideas either directly or by a process of reflection or held apart, and whether the validity of the information is associated within or outside the individual. If one extends this association, then relational or

instrumental understanding would result from schemas (beliefs) that view mathematics as internally related and conceptual or as isolated, rule bound, and arising from outside the individual.

Other writers and researchers also have delineated mathematical knowledge into broad categories based on the type of internal structure associated with the knowledge. Most notably Hiebert and Lefevre (1986) categorized mathematical knowledge as conceptual or procedural. Like Skemp, Hiebert and Lefevre do not identify these categories directly with any belief systems but their language is again suggestive of beliefs. While describing categories comparable to Skemp's, Hiebert and Lefevre's discussion offers insight into how these types of knowledge are observable in mathematical activity.

Hiebert and Lefevre (1986) define these categories as:

Conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. (p. 3)

Procedural knowledge is the formal language or symbol representation system of mathematics and its algorithms or rules for completing mathematical tasks. (p. 6)

Conceptual knowledge is composed of relationships which can be established at two different levels. The first level, which Hiebert and Lefevre call primary, denotes a relationship between information that "is constructed at the same level of abstraction (or at a less abstract level) than that at which the information itself is represented. That is, the relationship is no more abstract than the information it is connecting (p. 4)." The second level consists of those relationships which are constructed at higher levels of abstraction than the information being connected. These relationships are characterized by their independence from context.

With procedural knowledge, symbols are understood only at the surface level. They represent objects manipulated according to rules. This type of knowledge is associated with rote memorization. Hiebert and Lefevre (1986) indicate that:

The knowledge that results from rote learning is not linked with other knowledge and therefore does not generalize to other situations. It can be accessed and applied only in those contexts that look very much like the original. . . . Facts and propositions learned by rote are stored in memory as isolated bits of information, not linked with any conceptual network. (p. 8)

A point of clarification needs to be made. By conceptual knowledge, Hiebert and Lefevre mean only the rationale and ideas underpinning procedures but not the knowledge of how to execute the procedures themselves. This distinction is evident in the following statement:



Mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge. Students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not solve the problems or they may generate answers but not understand what they are doing. (Hiebert & Lefevre, 1986, p. 9)

Thus, to have what Skemp calls relational understanding, would entail an integration of conceptual and procedural knowledge. However, Skemp's instrumental understanding does coincide with procedural knowledge since both deal exclusively with the execution of rules and algorithms to get answers without reference to the underlying rationale.

When conceptual and procedural knowledge are joined into relational understanding, several benefits are realized (Hiebert & Lefevre, 1986). These benefits are important to note since they provide a means for observing and distinguishing relational knowledge from instrumental/procedural knowledge. First, symbols develop meaning. Second, procedures are now perceived as reasonable. Since the procedures are understood, they are more easily remembered and recalled. Third, problem solution is enhanced. This enhancement is achieved by (a) simplifying procedural demands, (b) monitoring procedure selection and execution, and (c) promoting transference. These advantages are realized since "the conceptualization of a task enables one to anticipate the consequences of possible actions. This information can be used to select and coordinate appropriate procedures (p. 12)." The transference is furthered since procedures are no longer tied to the surface context in which they were learned. As a result of this freedom from context, procedures are more readily generalized. Finally, procedural outcomes are monitored. Conceptual knowledge functions in this regard as a validating criteria, judging the reasonableness of the answer.

#### Autonomy

Autonomy with mathematics has been a recurrent theme in the various sections of this theoretical proposal. Confrey (1985) in her discussion of constructivism, argued that autonomy was the "backbone" of the constructive process. Perry's (1981) description of intellectual development included a change in authority, from external to internal, as a necessary evolution in the maturation process. Again, in the description of the development of meaning in mathematics, it was proposed that the individual needs to internally reestablish the validity of mathematics (Davis & Hersh, 1981). The manner of students' acceptance of mathematics was

also an important component of the assimilatory structure, schema, in Skemp's (1987) theory of the acquisition of mathematical knowledge. While each of these pieces suggests the importance of autonomy in the acquisition and use of knowledge, there is little explicit theory or research directed at mathematical autonomy as a cognitive trait.

Autonomy has been proposed by Fennema and Peterson (1985) as a vital link in the development of higher-level cognitive skills. Fennema and Peterson believed that in order to develop these skills individuals need to participate in autonomous learning behaviors which they described as "working independently on high-level tasks, persisting at such tasks, choosing to do, and achieving success in such task (p. 20)."

Fennema and Peterson suggest that the bibliographies of outstanding mathematicians point to independence, persistence, and choice of difficult tasks as a common and key link in strengthening these individuals' knowledge and ability in mathematics. Thus, for Fennema and Peterson, autonomy is linked to motivation and confidence.

Confrey (1985), however, emphasizes the connection between autonomy and students' source for the authority of their knowledge. For Confrey, autonomy represents an internal source for the validation of one's mathematics. That is, individuals believe that they are and should be responsible for the truthfulness or correctness of their knowledge and answers. Mathematics is valid or acceptable when it makes sense to the individual. Confrey proposes that without the acceptance of this responsibility that students will remain dependent on outside authority, teacher or text; develop knowledge that is formalized and isolated from the rest of their experience; and feel powerless with respect to their use and knowledge of mathematics.

Confrey, Fennema, and Peterson contend that autonomy is associated with one's beliefs about mathematics and that together they greatly influence how one acquires knowledge and solves problems. Under Confrey's definition autonomy represents a desire to internally validate one's mathematics knowledge, to actively strive to make sense of the disciplinary knowledge. In Skemp's (1987) terminology, this internalization means to accept and to assimilate that knowledge into one's schema. Like Skemp, Head and Sutton (1985) perceive that the assimilation process is tied to an internal acceptance. They propose that without this commitment on the part of the individual, knowledge will remain at a surface level. However, if knowledge is accepted then it becomes part of the individual's beliefs about the world.

There are a number of possible reactions a person can give to a newly encountered idea. As mentioned above, it might be ignored or relegated to a low status by a dismissive statement. A full understanding often cannot be achieved quickly, so if the idea is grasped at all, it will usually be through rote learning. The idea is then no part of the individual's repertoire of general sense-making beliefs about the world: it will merely be a context-bound formula to apply in a given situation.

Once the new concept has been successfully integrated into the individual's existing, personal cognitive structure, it becomes part of that person's repertoire of tools used to make sense of the world. At that stage, the individual readily acquires an emotional attachment to the idea: the result is a commitment to a belief. That commitment helps define the identity of the person. We can be described in terms of what we believe. (Head & Sutton, 1985, p. 95)

In summary, autonomy can be defined as both an acceptance of oneself as having the primary responsibility for one's learning of mathematics and the acceptance of oneself as the source for validating one's knowledge and solutions. Autonomy is an independence theorized to affect persistence, confidence and intellectual growth.

#### Summary and Research Assumptions

This research study is premised on several theories and perspectives. Foremost are the assumptions that beliefs and knowledge are constructs of the individual and that these constructs are formed in response to experience and self-reflection. Individuals utilize their beliefs and knowledge to create meaning from experience and subsequently to anticipate future events. Within this perspective, the theory about beliefs as articulated by Kelly (1963), Green (1971), and Rokeach (1968) coincide with constructivism, as articulated by Cobb and von Glaserfeld (1983). They share a common view of the individual as a scientist actively developing theories to explain observation and experience. Like scientists, individuals test their theories against experience.

One immediate consequence of this perspective is that all knowledge and beliefs are perceived as individualized. Yet individuals do not develop their understanding of the world in isolation from the culture and era in which they live. Toulmin (1972) suggested that the way an individual chooses to interpret experience is influenced by society. Kuhn (1962) argued that knowledge can be viewed as socially justified beliefs. That is, those individual constructs or meanings which are shared and common to a society would be considered knowledge. Under this description, discipline knowledge would represent the common constructs of the practitioners of that discipline. The discipline also would share a common criteria for validating statements. Thus, even within this individualized perspective, the construction of knowledge is seen as a flow between the individual and society. Those individual constructs that were not shared or common

to a discipline would be designated beliefs rather than knowledge.

The role of beliefs about mathematics is intrinsically tied to the nature of mathematical knowledge. In the discussion of mathematical knowledge two themes emerge. First, although all mathematical statements are scrutinized within a well-defined deductive system, the practitioners of the discipline still engage in a personal affirmation process to reestablish the meaning and validity of these statements. Secondly, the discipline of mathematics entails more than the specific content knowledge. That is, mathematics includes not only the end products--the theorems or algorithms--but the processes by which these were developed and verified. When speaking of mathematics, one also means the perspective through which an object or idea is viewed.

The processes and perspective associated with mathematics form an integral component in understanding and creating mathematics, and represent a metacognitive level in the discipline. It is one of the hypotheses of this study that students' beliefs about mathematics as a discipline and beliefs about learning and doing mathematics reveal their attempts to describe this secondary level in mathematics.

It is further hypothesized that these beliefs, rather than being extraneous, have a dynamic role in the learning and doing of mathematics. The rationale for this premise comes from multiple sources. Kilpatrick (1985), Schoenfeld (1983), and Shaughnessy and Haladyna (1984) suggested that beliefs act in a metacognitive fashion. Kilpatrick (1985) noted that: "metacognitive processes rather than being imposed on top of acquired knowledge, interact with knowledge as it is being acquired (p. 9)." Beliefs are metacognitive in the sense that they set-up expectations and anticipations which in turn delimit choices (Cobb, 1986; Schoenfeld, 1983, 1985). These expectations are theorized to affect both how knowledge is structured (Skemp, 1987) and how it is used (Buchanan, 1984; Cobb, 1986; Frank, 1985; Schoenfeld, 1985). Skemp (1987) proposed that instrumental (procedural) and relational (conceptual) knowledge structures were linked with differing expectations. Cobb (1986), Buchanan (1984), and Frank (1985) identified these expectations with differing belief systems.

What is being proposed, then, is that beliefs about mathematics and oneself as a doer of mathematics by setting up expectations impinge in the cognitive processes at vital decision making junctures. The following scenario suggests how and when this might occur. When

confronted with a situation, a student must first decide whether or not that situation is indeed a mathematical problem. Even the acknowledgement of a situation as mathematical depends on the very elements--context and clues--that an individual perceives as relevant (Kouba & MacDonald, 1987, 1991). Here then is a first juncture where students' beliefs about the nature of mathematical problems can influence their response to a problem. For example, does the student expect a problem to be identical to ones seen in the classroom? Must it contain numbers? Does the student look for the mathematical structure inherent in the problem?

Once a problem is acknowledged as legitimately a mathematical situation, the student must then decide what strategies to employ to solve the problem. It is here that Cobb (1985, 1986) and Schoenfeld (1983, 1985) propose that beliefs again enter into the deliberations. Does the student approach the problem with the expectation that its solution resides in the quick execution of a known procedure? Does the student expect to use trial and error to explore a problem before a solution process is devised?

Also, within the solution process, beliefs are theorized to affect executive-monitoring (Confrey, 1983; Confrey & Lipton, 1985; Gelman & Meek, 1986; Schoenfeld, 1985). Does the student expect the solution to make internal sense? Are the solutions expected to be consistent with other knowledge? Are solutions validated solely on the basis of a careful execution of an algorithm? Are the solutions correct only if a teacher or answer key indicates them as such?

In each of these junctures, beliefs about mathematics were conjectured to impinge upon the cognitive processes in problem solving. In addition, beliefs are hypothesized also to affect knowledge formation. They do so by establishing expectations for what is valued and attended to in an experience and how it is expected to be utilized in the future. Skemp (1987) suggested that students with a goal to understand mathematics instrumentally would focus their attention in the mathematics classroom on the procedures and rules that when executed will yield correct results. They also will disregard the rest of the instruction. Analogously, students with a goal to understand mathematics relationally would look for and attempt to develop interrelationships among the concepts.

Skemp's (1987) categorization of mathematical knowledge or understanding as either relational or instrumental encompasses qualities about the knowledge structure, as well as, expectations about autonomy/authority and about the nature of mathematics. This linkage of

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knowledge with beliefs about the discipline and about authority or autonomy also was proposed by Perry's theory on intellectual development. Confrey (1985) in her discussion on constructivism argued, as well, that autonomy was inseparable from the construction of knowledge. This linkage then forms another hypothesis for this study.

Within the constructive framework, all knowledge and beliefs are constructs of the individual. These mathematical constructs are theorized to form principally within the classroom content. In fact, several researchers have argued that the classroom experience is responsible for fostering dysfunctional beliefs which are conjectured to inhibit students' ability to engage in mathematical activities, especially problem solving (Anderson, 1984; Baroody & Ginsburg, 1986; Borasi, 1990; Buerk, 1985; Frank, 1988).

Thus, beliefs about mathematics as a discipline are theorized to affect problem solving and learning mathematics. It also is theorized that beliefs about mathematics and autonomy are linked together. Two differing belief systems, instrumental and relational, have been identified as a way to categorize students' beliefs about mathematics as a discipline and as an explanation for differing knowledge structures.

## CHAPTER III

### LITERATURE REVIEW

#### Historical Overview

Early research studies on attitudes towards mathematics focused heavily on the statistical correlation between measures of attitudes toward mathematics and achievement in mathematics. In a representative study of that period, Aiken and Dreger (1961) developed a Likert scale involving feelings of like or dislike for and feelings of confidence in mathematics which they tested on college students. Aiken and Dreger used a variety of measures to predict success a the course. They found that numerical ability and attitudes were the most significant factors in predicting course grades. They also found a correlation between the numerical ability and attitude scales which was indicative of the low to medium correlation (.19 to .54) that latter researchers would show between their own measures of attitude and achievement (Aiken, 1970a, b, 1971, 1976; Kulm, 1980; Reyes, 1984). These studies emphasized the affective subcomponent of attitude in their Likert statements.

During this period, further attitude-achievement studies were conducted on elementary, junior high, and high school, as well as, college students. While the results revealed consistent low to medium correlations, the studies were far from supporting a causal relationship. This inconclusiveness is visible in Cleveland's study of sixth graders cited in Aiken (1970a) which demonstrated "that attitude scale scores do not generally discriminate between high and low achievers in arithmetic (p. 559)." Studies across grade levels and several longitudinal projects exemplified by School Mathematics Study Group (SMSG) and National Longitudinal Study of Mathematics Ability (NLSMA) suggested that students' attitudes towards mathematics varied overtime shifting significantly during the junior high years (Aiken, 1970a). In junior high, students showed a decrease in the generally positive attitude of the elementary years with more of a decrease for females than males.

Like the work on attitudes in psychology, the research into mathematics attitudes underwent a change in emphasis in the late sixties and seventies. The research now focused

on intervening variables which might also contribute to the attitude-achievement correlation. Two variables, mathematics anxiety and sex differences, received prominence. (Aiken, 1970a, b, 1976; Reyes, 1984; Suydam & Kirschner, 1980). While mathematics anxiety was an intuitively appealing variable, the results again pointed to a consistent correlation, not necessarily causal, between anxiety and achievement. However, curriculum projects designed to reduce anxiety generally failed to improve achievement or to increase mathematics course election (Aiken, 1976; Reyes, 1984). The Mathematics Anxiety Rating Scale (MARS) (Suinn, 1972) constructed during this period, received some usage within the research literature. On this Likert-type instrument, students indicated the degree to which they were frightened by particular mathematical situations.

The results from anxiety research and the feminist movement propelled forward an investigation of sex differences in mathematics attitude and achievement. Fennema and Sherman (1978) in a representative study, revealed that while differences by sex did exist in attitudes, differences in scores on traditional achievement measures were low and could be explained by variations in course enrollment. The research by Fennema and Sherman also represented the broadening of the definition of attitude towards mathematics to include social variables in the multivariate analysis of the attitude-achievement research. New elements, such as motivation to study mathematics, perceived usefulness of mathematics, perception of mathematics as a male domain, and the influence of parents and teachers on learning mathematics was theorized to affect achievement by their influence on desire to study and to engage in mathematics. The Fennema-Sherman Mathematics Attitude Scales (F-S) (Fennema & Sherman, 1976) were developed expressly to provide data on all of these elements, again using a Likert measure which was normed for high school students. The exploration of the attitude-achievement model with sex as an intervening variable also has been investigated internationally (Hanna & Kuendiger, 1986; Mirato, 1983; Mukuni, 1987).

To this point, the research studies can be roughly understood as a matrix with the rows representing the various student populations and with columns contrasting a selected attitude measure against another psychological construct. The rows would include entries for elementary, middle, secondary, college, graduate students, as well as teachers (preservice and inservice). The attitude measures might include MARS, F-S, NLSMA, or author developed. The contrasting statistical measure might include achievement, anxiety, instructional technique, sex or



socioeconomic factors (Kulm, 1980; Reyes, 1984).

The next shift in attitude research occurred in the eighties when the focus moved to students' beliefs about mathematics. Emerging problem-solving research suggested that students' conceptions of mathematics may affect their performance, in particular, their executive monitoring of their problem solving progress. In addition to this shift from the affective subcomponent of attitude to the beliefs subcomponent, the methodology also shifted from paper-and-pencil measures to clinical interviews (Cobb, 1985, 1986; Confrey, 1981a, b, 1982, 1984; Schoenfeld, 1983).

Schoenfeld, whose research (1983) is representative of this developing area, conjectured that students' beliefs about geometry and about mathematics in general would determine the type of strategy that students would elect when approaching a problem-solving situation in geometry. The significance of Schoenfeld's research lies in its attempt to tie beliefs to actual cognitive processes.

As an area of research in mathematics education beliefs also received encouragement from the growing interest in the philosophy of constructivism which stressed the individuality of all knowledge. The evidence from this research area suggests that beliefs about mathematics may influence students' choice of heuristics in problem solving and students construction of their knowledge. Researchers also have begun to theorize that students develop dysfunctional beliefs from their classroom environment (Baroody & Ginsburg, 1986; Borasi, 1990; Buerk, 1981). Consequently, researchers also have explored teachers' beliefs about mathematics and their subsequent actions in the classroom (Cooney, 1985; Owens, 1987; Russell, 1987; Thompson, 1984; Underhill, 1988).

While the research emphasis has shifted to include beliefs, numerous studies are still being conducted on the attitude-achievement model with Likert measures (Suydam & Crocker, 1990, 1991). Mathematics anxiety and the sociological constructs remain popular topics of research.

#### Criticisms of the Attitude Research Literature

The research literature on attitudes towards mathematics has been prolific. Suydam and Kirschner (1980) reported over 450 selected citations in this area in 1980 and the number of studies has continued to grow. Shaughnessy and Haladyna (1984) conclude from their review

of the literature that: "achievement and gender differences with respect to mathematics attitude may have been overinvestigated (p. 12)." They along with other reviewers of the literature strongly suggest changes be made in the theoretical framework, methodology, and scope of future research into attitudes towards mathematics (Aiken, 1970a, 1976; Kulm, 1980; McLeod, 1987; Reyes, 1984; Shaughnessy & Haladyna, 1984; Silver, 1985; Underhill, 1988).

With regard to the theoretical framework, the criticism among the reviewers is universal-many studies have proceeded without any theory or model within which to conceptualize the research.

Research on the affective domain in mathematics education is in need of a strong theoretical basis that will be developed only through sustained, systematic efforts over time. Too much of the research in this area has had no theoretical rationale. Often researchers include an affective component in a study with little thought or planning. (Reyes, 1984, p. 572)

This deficiency is often coupled with a failure to investigate or review pertinent studies within mathematics education and in related disciplines like psychology and sociology. In addition to these difficulties, the reviewers also cited the lack of a clear definition of attitude within many studies.

Although many definitions have been proposed by psychologists, the most recent trend has been to avoid explicit definition and to settle for operational definitions implied by items of instruments measuring attitude. (Kulm, 1980, p. 356)

Finally Shaughnessy and Haladyna (1984) argue that many researchers select variables "haphazardly" without reference to any theory or hypothesis.

With regard to methodology, the consensus among the reviewers is that the research needs to move away from paper-and-pencil measures of attitude. Aiken (1970a) states that: "In general, there has been too much reliance on correlational methods and on indirect measures of behavior, such as questionnaires and other student reports (p. 588)." As in the psychological literature, the criticism here focuses on the ineffectiveness of these measures to investigate the nuances in attitude, to correlate well with action, and to distinguish individual differences in belief (Aiken, 1970a, 1976; Kulm, 1980; Shaughnessy & Haladyna, 1984; Silver, 1985). Shaughnessy and Haladyna (1984) also criticize the methodology for its tacit assumption that attitudes are static. In place of the present Likert measures, the reviewers have suggested qualitative methods, particularly clinical interviews in which students engage in mathematical situations. Again this advice coincides with the research literature in psychology which advocated the

measurement or assessment of beliefs from action or behaviors. This method was theorized to bring the salient beliefs closer to the individuals' conscious thoughts.

It is possible that a clinical methodology could allow us to witness the dynamics of attitude formation in a more natural setting than a questionnaire and could help us to explore attitudes toward different aspects of mathematics. (Shaughnessy & Haladyna, 1984, p. 7)

Attitudes spill over when students are actively engaged in a content area. . . . Thus, the monitoring of affective variables during an activity which is predominantly cognitive seems a fruitful approach for future attitudinal research. (Shaughnessy & Haladyna, 1984, p. 18)

The third area criticized within the research literature is the limited scope of the investigations. Not only are the measures "crude" but they fail to capture the multidimensional and dynamic aspect of attitude (Aiken, 1970a, 1976; Shaughnessy & Haladyna, 1984). In recognition of the complexity of attitudes, Aiken, Shaughnessy and Haladyna recommend that investigations include not only the many facets of students' attitudes (anxiety, confidence, beliefs, social influences, etc.) but also the classroom environment, especially the teachers' beliefs.

As long as we continue to investigate attitude solely by focusing on the relationship between attitude and achievement, we are not going to make strides in detecting the factors that enter into formation and change in attitudes. (Shaughnessy & Haladyna, 1984, p. 7)

Perhaps the soundest conclusion that can be drawn from the results of the studies cited in this review is that changes in attitude toward mathematics involve a complex interaction among student and teacher characteristics, course content, method of instruction, instructional materials, parental and peer support, and methods of measuring these changes (Leake, 1970). Therefore the findings of these sources of variability are severely limited in generalizability to other classroom situations. (Aiken, 1976, p. 302)

In conclusion, the reviewers recommend that the research into attitudes focus on theory development and on the investigation of attitudes with respect to problem solving and concept development (Kulm, 1980; Reyes, 1984; Shaughnessy & Haladyna, 1984; Silver, 1985). Shaughnessy and Haladyna summarize this position by stating that:

Finally, interesting effective [sic] issues have arisen in surprising places such as in research on problem solving and research in concept development. We recommend that future research on attitudes towards mathematics take a more eclectic approach, both towards its methodology and towards the conditions under which attitude is measured. (p. 18)

#### Attitude Research Literature

Although lacking in a theoretical framework, the research studies on attitudes collectively afford some insight into students' general attitudes about mathematics. The results from the National Assessments of Education Progress, NAEP, provide a consistent measure of students' attitude toward mathematics from 1977 to 1985. These results will be discussed first, followed

by corroborating evidence from other studies, especially those utilizing interviews. These interviews elaborate and often suggest qualifications of the results in the larger, more general reports. The discussion will then move to an overview of the research on mathematics anxiety, and social variables since these are theorized to affect attitudes.

Once the results on these various aspects of attitude have been outlined the discussion will focus on those research studies which address the interaction of students' beliefs and their problem solving performance. It is these studies that provide much of the motivation and direction for this investigation. Together they suggest ways in which beliefs impinge on cognitive processes. They are also relevant since they utilize qualitative methods to explore these relationships and to make inferences about beliefs from problem-solving behavior. Next, two related areas will be summarized: teachers' beliefs about mathematics and their classroom behavior and the effect of academic tasks on learning.

#### National Assessment of Education Progress

The second, third and fourth assessment reports contain a discussion of students' attitude towards mathematics. Data on students' attitudes was collected through Likert questions in four areas: mathematics in school, mathematics and oneself, mathematics and society, and mathematics as a discipline. Since Swafford and Brown (1989) reported little variation in the response percentages on items in the attitude measures from the second to the fourth assessment, only the results from the fourth assessment will be summarized here. The fourth assessment was administered to 18,033 3rd-grade students, 23,527 7th-grade students and 31,938 11th-grade students in the United States during the 1985-86 school year. Questions in all four attitude areas were given to 7th- and 11th-grade students, while only a subset of the second and third areas were given to 3rd-grade students.

#### Mathematics in School

The category, mathematics in school, dealt with students' affective response to mathematics in comparison to science, social studies, English, and physical education. Students indicated whether they liked or disliked each subject, how easy or hard the subject was for them, and how important or unimportant they thought the subject was.

For 7th-graders, science and mathematics ranked as their best-liked subject. Fifty-one percent felt that mathematics was easy while ninety-one percent believed mathematics was

important. For 11th-graders, each academic area was liked equally well. Forty percent felt mathematics was easy while eighty-seven percent believed mathematics was important.

#### Mathematics and Oneself

The category, mathematics and oneself, examined students' perception of themselves as learners of mathematics. The statements included questions about students confidence in mathematics. The results showed that students at both the 7th- and 11th-grade levels felt fairly confident in their ability with a high percentage expressing that they usually understood what was going on in mathematics. The responses from the 3rd graders indicated that they are less certain about their ability than the older students. There is also a gradual decrease in percentage of students expressing enjoyment in mathematics as age increases: 60, 55, 50 percentage, respectively. All groups expressed a willingness to work hard in order to do well in mathematics.

#### Mathematics and Society

The category, mathematics and society, dealt with students' perceptions of mathematics in their everyday life. Over three-fourths of the 7th- and 11th-graders indicated that they "thought arithmetic was important for getting a good job. The response rate dropped to about 50 percent for the 11th-grade students when 'mathematics such as algebra or geometry' was substituted for arithmetic in the previous statement. . . . Fewer than half of the students at each of the three grade levels indicated that they expected to work in an area that requires mathematics. For 3rd-grade students, the figure was only 40 percent (p. 112)." Swafford and Brown concluded that students viewed mathematics as important to society but not to them personally.

#### Mathematics as a Discipline

The last category, mathematics as a discipline, dealt with students' perceptions of the processes of mathematics and perceptions of mathematicians. The results showed that approximately 80 percent of 7th- and 11th-grade students felt that there is always a rule to follow in solving a mathematics problem and that doing mathematics requires lots of practice in following rules. The students also indicated that mathematics helps a person to think logically and that it is important to know why an answer is correct. The students, however, did not feel strongly that mathematics entailed exploring number patterns or guess and check. The survey

also revealed that 50 percent of the students perceived mathematics as made-up of unrelated topics. Less than 40 percent of the students indicated that they felt mathematicians worked with ideas or made new discoveries.

#### Corroborating and Contrasting Evidence on Students' Attitudes

While the data from the NAEP is significant for the extensiveness of the population surveyed, the results are necessarily very general. For example, the results do not distinguish between students' attitudes toward routine problems and problem solving. Since the individual response is lost in the assessment, conclusions can not be drawn on the consistency of attitudes (e.g. Are the students who view mathematics as utilizing rules also those who perceive mathematics as memorizing?; Are those who view mathematics as entailing justification also those who sense mathematics as teaching logic?) A further caution needs to be extended when interpreting the results from the NAEP or any Likert-type attitude measure since researchers have reported inconsistencies between markings on these measures and students' explanations in interviews (Bassarear, 1986; Frank, 1985; Miller, 1987).

#### Mathematics in School

Research by Haladyna and Thomas (1979) confirmed the NAEP's findings on attitudes towards mathematics showing a downward trend by age. This downward trend also appeared in reading, physical education, art, music, science and school in general. In addition Haladyna and Thomas noted differences in attitude by sex with girls favoring reading, art, music and mathematics and with boys favoring physical education and science.

Using open-ended sentences, Kiryluk (1980) queried 13-year olds in England about the aspects of mathematics they found interesting and boring. His results differentiated the qualities of mathematics that students find salient in their experience in school mathematics. The data revealed that for some students, mathematics is interesting when it is easy and routine while others like to be challenged and to discover connections. Similarly, mathematics is boring for some when it is too easy and boring for others when too hard. Additionally, Kouba and MacDonald (1987) reported that elementary students in K-6 felt that "when something becomes easy, it is no longer mathematics (p. 107)."

#### Mathematics and Oneself

The NAEP reported that students indicated that they felt fairly confident in their ability

to understand mathematics. Kloosterman and Cougan (1991) found elementary students expressed different levels of confidence depending on the mathematics topic. They also found that: "Simple beliefs like self-confidence in learning mathematics and the extent to which mathematics is enjoyable seem to form in the first two to three years of elementary school (p. 10)." Kloosterman and Cougan also reported that older elementary students believed that anyone could learn mathematics and that effort was important to achieve it. They found that younger students did not distinguish between effort and ability. For all students, they observed that to do well in mathematics meant to receive a good grade.

Through a series of interviews with college students, Martinez, Schneider and Wineburg (1986) observed that these students seemed to have developed their beliefs about their capability in mathematics in early elementary school. These students vividly recalled a situation or teacher which had influenced their self-perception. Martinez, Schneider and Wineburg contended that: "the classroom exerts a powerful influence on the formation of attitudes, yet each classroom is embedded in a broad culture that attaches its own meanings to mathematics (p. 36)."

Schoenfeld (1985b) reported similar beliefs from his interviews and surveys with high school students. In his research, the students claimed that work and not luck accounted for good grades, mathematics was an objective discipline that could be mastered.

#### Mathematics in Society

The NAEP reported that approximately 90 percent of 7th- and 11th-grade students felt that mathematics was important although many believed they would not need it themselves in their future employment. Kiryluk's research (1980) with 13-year olds in England seemed to contradict these results. He found that students held more dichotomous views on the importance of mathematics.

Miller (1987) concluded from her research that "students sense from society, in general, and parents and teachers, more specifically, that mathematics is useful, but they are not exactly sure why or how it is useful (p. 143)." She found that when pressed for examples of how they used or intended to use mathematics in the future that students were vague or cited consumer arithmetic. This lack of specificity was also noted by Hoyles (1982) in her research when 14-year olds were asked to recount instances when they felt especially good or bad when learning.

Of the 281 stories collected, one-third of the good stories and one-half of the bad stories focused on mathematics.

Hoyles also observed distinctions in the type of stories students recounted depending on the subject they were describing. She found the "tendency to focus on 'self' rather than 'work' or 'task in hand' in the mathematics stories, both good and bad (p. 360)." Her work indicated that students tended to deal primarily with feelings rather than content in mathematics, as opposed to other subjects where they discussed the content directly. Hoyles conjectured that the learning experience in mathematics produced especially strong affective reactions whether positive or negative. Hoyles described the distinction between students' stories in mathematics and other subjects by saying:

The mathematical work being undertaken is not described in any detail in the stories but merely mentioned or named. This illustrates a general trend within the stories: that is the actual mathematical content of work was rarely talked about while, in marked contrast, stories about other areas regularly included vivid and detailed descriptions of the nature of the work undertaken. It was also of interest to note that, not only did the pupils tend not to describe the actual mathematical work being undertaken in their stories, but they also did not tend to comment on its interest, relevance or future use. (p. 362)

#### Mathematics as a Discipline

The NAEP reported that students believe that mathematics entails memorization and rules while also acknowledging the importance of the rationale behind the process. Several research studies call this conclusion into question (Buerk, 1981; Frank, 1988; Martinez, Schneider & Wineburg, 1986; Schoenfeld, 1983; Yackel, 1986). These studies suggest a more dichotomous view of mathematics with a prevalence for the belief that mathematics is memorized rules. Frank (1988), Buerk (1985), and Borasi (1990) have further proposed that students hold dysfunctional beliefs that inhibit their ability to learn and solve problems. Buerk (1985) argues that:

The way most of students view the field of knowledge we call mathematics inhibits them from functioning in the ways that I believe we would like them to have them perform (p. 2). . . . In mathematics they believe that they are working with the known or absolute rather than the uncertain and variable. They believe that learning mathematics is an external, objective process in which one can only learn by reproducing the methods and the answers of others--that there are no new ideas to be discovered and no controversies. They don't know or believe that doing mathematics requires an internal, reflective process in which one uses intuition, conjectures, false starts, and testing strategies. (p. 6)

The research has shown that beliefs about mathematics are present in children in the early elementary grades and that they remain through adulthood (Kouba & MacDonald, 1987,



1991). Individuals often point to their grade school experiences as establishing their self-confidence and expectations about how mathematics operates (Kloosterman & Cougan, 1991; Martinez, Schneider & Wineburg, 1986). Often though, individuals view mathematics as narrow and focused on memorized rules. This belief also was associated with a sense of mathematics as external, incomprehensible, and transmitted leaving the student with a sense of being powerless to control mathematics (Buerk, 1985, 1981; Frank, 1988, 1985; Martinez, Schneider & Wineburg, 1986; Mau, 1991; Schoenfeld, 1985b). These beliefs are exemplified poignantly in a journal entry reported in Buerk's (1981) research on adult women's beliefs about mathematics. Like the stories reported to Hoyles, mathematics is spoken of in terms of the student's strong emotional response to it.

And on the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it into a fence forty cubits high, and infinite cubits long. . . .He said "On one side of this fence will reside those who are good at math. And on the other will remain those who are bad at math, and woe unto them, for they shall weep and gnash their teeth."

Math does make me think of a stainless wall--hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put your nose against it, it doesn't give anything back, you can't put a dent in it, it doesn't take your shape, it doesn't have any smell, all it does is make your nose cold. I like the shine of it--it does look smart, intelligent in an icy way. But I resent its cold impenetrability, its supercilious glare.

Anger, frustration, resentment, panic--I'm trying to write about what math is, and what I come up with is how it makes me feel. I resent it right now that I can't think of an intelligent definition of math--math is what you do with numbers when you're little, and what you can't do with letters when you're older--if you're me. . . .What I learned in math each year became progressively less useful and more senseless.

What strikes me as being particularly strange and yet even now completely typical of my encounters with math was my bland expectation of not understanding algebra. What I did expect was to "see" it sometimes, and other times not. . . .It was the only subject that just seemed to have certain intrinsically impenetrable aspects. I grew accustomed to hearing a sort of math-babble, words that attached themselves to nothing, numbers that seemed arbitrarily plunked into equations. I'm sure I copied it all down, too--to memorize, in case I'd need it on a test, or just on the chance that because of my very dutifulness, I would be granted sudden enlightenment. I obviously never asked questions.

So what is math? I can't get very far with an answer. I have heard math spoken of as a language. I can't imagine it. Language is personal. Language carries thought forward, discovers thought, creates thought. Whatever it is that I know as math doesn't do those things. (Buerk, 1981, pp. 158-160)

While these beliefs dominate, the studies also showed the existence of beliefs about mathematics as relating concepts, and involving discovery and personal autonomy (Kiryuk, 1980). Students also identified mathematics with the explicit presence of numbers or operations, and with classroom activities (Kouba & MacDonald, 1987, 1991).

### Intervening Variables

Much of the research in recent years on the relationships between attitudes towards mathematics and achievement in mathematics has centered on the role of intervening variables to further explain this relationship. These variables include mathematics anxiety, perceived usefulness of mathematics, perceptions of ability, influence of parents and teachers, and gender. Although the focus of this study is on students' beliefs, these affective variables have been shown to be correlated to achievement and to attitude (Meece, Parsons, Kaczala, Goff & Futterman, 1982; Reyes, 1984). They also provide important information about a student's motivation to study mathematics and, as such, are valuable tools for investigating and contextualizing students' attitudes towards mathematics. Statistical correlations have been the primary tool of investigation in this area. Many of the studies include data on all of these areas, and almost universally gender has been an underlying variable in these research studies.

The research on mathematical anxiety revealed that higher achievement levels were associated with lower levels of anxiety about mathematics and that females reported higher levels of mathematical anxiety than did males (Calvert, 1981; Kincaid, 1981; Reyes, 1984). Mathematical anxiety was also associated with beliefs about mathematics as procedural and with a sense of being powerless over mathematics (Gourgey, 1984).

Handel (1986) and Kaczala (1980) reported that gender differences appeared in students' perceptions of their ability in mathematics, in their expectancy for success in current and future mathematics courses, in their perceptions of the usefulness of mathematics, in their career goals, and in their attributions for success and failure in mathematics. Generally males showed more positive attitudes and expectations about their ability and about the utility of mathematics. Males also tended to attribute success to ability while females associated success with effort and failure with lack of ability. Handel (1986) also found that males engaged in more extra-curricular activities involving mathematics.

Parental expectation for their child's mathematical learning differed by gender. Parents felt that mathematics was more important for sons than for daughters (Parsons, Karbenick, Adler, Futterman, Heller, Kaczala & Meece, 1979). Parental positive attitudes towards their children as learners of mathematics also were correlated to their children's interest in mathematics and physical science (Elmore, Broadbooks, Pedersen & Bleyer, 1985). Teachers have also been

hypothesized to influence students attitude through possible differential interactions by gender. Peterson and Fennema (1985) reported no such connection in their research investigation of fourth grade classrooms. They did note differences in the type of activity that males and females engaged in during classtime. Males tended to work more independently while females favored cooperative situations. Parsons, Heller, Meece and Kaczala (1979) did find differences in the type of feedback the teachers provided to males and females in the 7th- and 9th-grade classes they observed. While the teachers' expectations did not differ, they found that girls received less work related criticism and less criticism on the quality of their work than did males.

#### Beliefs about Mathematics and Problem Solving Performance

While correlational studies have been the predominant mode of inquiry, several recent studies have utilized interviews to investigate the relationship between student beliefs about mathematics and their achievement, particularly, their problem solving performance (Buchanan, 1984; Cobb, 1985; Confrey, 1982, 1984; Frank, 1985; Schoenfeld, 1983, 1985). The subjects in these studies ranged from second grade to college students. In each investigation, data on students' beliefs was collected both through explicit dialogue and through inference from students' behavior while engaged in solving problems. Skemp's (1987) descriptions of mathematical knowledge as involving instrumental or relational beliefs were used to distinguish among students in the studies by Cobb, Buchanan and Frank.

Cobb (1985) examined first and second grade students' beliefs, confidence, and persistence while they solved arithmetic problems. Cobb (1983a,b) was drawn to this type of investigation from a conviction that: "Children's behavior could not always be fully accounted for solely in terms of an analysis of the children's arithmetical concepts (p. 113)."

In the report of his research, Cobb describes two second grade students, Tyrone and Scenetra, who held differing views of mathematics and differing approaches to problems. Cobb observed that for Scenetra mathematics was "an activity in which one finds unrelated rules for solving unrelated problems. The dominant theme was to get the correct answer. The mean was completely dominated by the end. The sole criterion by which she usually judged a method was whether or not it yielded a correct answer. The question of understanding why a particular method worked or broke down did not arise (p. 117-118)." In contrast, Tyrone "frequently used a known sum or difference when he attempted to find an unknown sum or difference. He did

not seem to 'notice' these relationships fortuitously or accidentally. Instead, he appeared to actively search for opportunities to use these types of methods. The dominant theme, which guided his mathematical activity was the achievement of a relational rather than an instrumental understanding (p. 117)."

The two children also differed with regard to their motivation and persistence. For Scenetra "failure did not give rise to new problems or to questions of what she could do differently in order to succeed. Instead, it led to self-doubts about her competence (p. 122)." Cobb concluded that: "In general, Scenetra did not seem to view the problems she constructed as her own; it was as if they were for her obstacles that the teacher placed in her path. Problems were threats to her self-esteem rather than challenges to her intellect (p. 122)."

For Tyrone, his "initial failure did not give rise to self-doubts about his competence. Instead, it led to a new, more demanding intellectual challenge. Unlike Scenetra, who viewed difficulties as threats to her self-esteem, Tyrone seemed to view difficulties as opportunities for fresh insights. This intrinsic desire for conceptual mastery was also manifest in his attempts to understand problems even after he had given correct answers. . . . Tyrone's confidence was such that, on several occasions, he told the teachers, 'You don't help me no more!' when the teacher attempted to give assistance (p. 123)." The students' implicit and explicit beliefs about mathematics Cobb felt formed expectations and anticipations about mathematical situations. In turn, these "anticipations delimited both what could count as a problem and what could count as an acceptable method of solution. They also seemed to constrain the sorts of implicit and explicit heuristics the children employed and could therefore be used to give at least a partial explanation of the flexibility of the children's problem solving behavior (pp. 122-123)." Cobb's analysis points again to the multifaceted nature of the attitude-achievement relationship. Beliefs about the nature of mathematics and school interact with perceptions of self, one's autonomy and confidence, to influence behavior and ultimately the ability to solve problems.

Buchanan (1984) in her research of gifted 3rd and 5th grade students reported findings similar to Cobb. Outside problem solving experts evaluated the students' problem solving performance according to task orientation, independence from teacher/experimenter, speed of solution, processes utilized, and evidence of self-monitoring behavior. On the basis of these results, Buchanan defined four basic categorizations which distinguished between the students

with regard to their source of motivation, their beliefs about mathematics, their predominant problem solving process, and their means of achieving satisfaction. Table 2 illustrates the characteristics that she observed (Buchanan, 1984, p. 67).

Buchanan noted that while students' problem solving performance did not always remain exclusively in one category students did tend to function predominantly as one of the types. For example, a student she described as Type I, was task involved; held relational beliefs about mathematics; used clarification, exploration, and evaluation; and was satisfied with a problem when it was understood. In Buchanan's classification Type II or IV students might manifest either relational or instrumental beliefs about mathematics with Type II students seeking to please authority while Type IV students sought to avoid failure. Type I students were generally relational and autonomous and Type III, instrumental and nonautonomous. Buchanan's research reaffirms that autonomy and beliefs about mathematics as instrumental and relational are interrelated with behavior although not necessarily in dichotomous fashion.

Table 2  
Model of Mathematical Problem-Solving Performance

Student Types	Source of Motivation	Belief About Mathematics	Predominant Mathematical Problem Solving Processes	Means of Achieving Satisfaction
I	Task enjoyment	Relational	Clarification	When understands the problem and relates to others - generalizes
	Intellectual challenge	Open/Flexible	Exploration	
		Makes sense	Evaluation	
II	Authority	Instrumental or Relational	Exploration	When the teacher considers the problem complete
	Appear interested			
	Be good			
III	Peer authority	Instrumental	Execution	When an answer is given
	Be first	Inflexible		
	Beat others	Manipulate numbers to get answer		
IV	Self-protect from failure or appearing dumb	Inconsistent	Efficacious Insight	Either when understands or thinks they will fail
			Relational or Not at all	

Frank (1985) investigated junior high students' beliefs about mathematics and their problem solving performance. The students in Frank's study were all participants in a summer program for mathematically and verbally gifted students. Frank analyzed classroom observation from a problem solving with computers course, student problem-solving interviews, attitude questionnaires, and a problem classification scheme in her evaluation of the students. Her data and analysis were broad in scope including information on what students' felt constituted a mathematics problem, their beliefs about mathematics as a discipline and about solving mathematics problems, the role of the teacher in learning, as well as, motivation and conceptions of their ability to do mathematics.

Frank concluded from her research that students' beliefs about mathematics delimited

their problem solving approach.

One's beliefs about when a problem is solved and what constitutes an acceptable answer depend in part on how one "sees" the problem. . . . One's mathematical beliefs seem almost to act as filter for one's view of a problem (p. 130).

Like Buchanan and Cobb, Frank observed that students would often quit a problem rather than risk further failure. She labelled this behavior "bailing out". Frank (1985) described this behavior by saying:

Bailing out is one of the most interesting and least studied problem-solving modes, and may occur at any point in the solution process. Bailing out is an attempt to gracefully get out of an uncomfortable or unprofitable situation. It can involve setting the problem aside (possibly to be worked on later); asking for help, trying to change the problem to one which can more easily be solved, or subtly setting the stage for a possible lack of success later in the solution attempt or a possible future need for more overt bail-outs. Bailing out may also be expressed by a lack of "commitment" (Confrey, 1984) to one's answer to a problem, as when students rely on the teacher or the textbook to evaluate the correctness of their answers. (p. 95)

Frank concluded from her research that bailing out was influenced by students' mathematical beliefs, needs, and motivations. Frank's research also showed the plausibility of making inferences about beliefs from problem solving behaviors and confirmed that these inferred beliefs could coincide with other reports made by the students about their beliefs.

Confrey (1982) interviewed ninth grade students in her study of students' beliefs about mathematics. She identified several themes reoccurring which appeared to profoundly influence how these students viewed and studied mathematics. The first theme denoted answer frames, expressed the belief that the primary aim of mathematics is to get answers and that the process is important only to the extent that it leads to the answer. This theme was exemplified by "students' pursuit of the problems in information gathering, in their lack of flexibility and subsequent preference for a single method, in their quick, but local generalizations and in their erroneous curtailment (pp. 25-26)." The second theme, whole number mentality, indicated the students' persistent expectation and preference for whole-number solutions. Confrey remarked that she was surprised by "the length to which they go to get a whole number answer (p. 26)." The third theme, symbolic manipulation versus representation, conveyed the belief that "mathematics seemed to be a set of symbols to be operated on within rules and if those rules failed to fit perfectly or errors were made" then the final recourse was "to perhaps try to manipulate the numbers more and hope some answer would come out even (p. 27)."

The final theme, authority, expressed Confrey's observation that students seldom

"express a view that certain problems are correct because they understand them and are certain they must be right. The authority to them is the teacher, the answer sheet or those people who wrote the book (p. 27)." Confrey argued and suggested that externalization for one's validation of mathematics was the result of previously unsuccessful mathematics performance and that it contributed to one's lack of confidence and to a sense of powerlessness over mathematics. She further contended that this lack of autonomy was potentially inhibiting students' intellectual growth.

Since Confrey felt that these themes were artifacts of the mathematics instruction in school, she proposed investigating the development and interaction of these beliefs in the classroom setting. She suggested "using clinical interviews following observations of classroom instruction to explore how students are interpreting and approaching problems in instructional settings. This type of examination, similar to the task analysis of Doyle, needs to include a strong subject-matter dimension (pp. 37-38)."

Schoenfeld's (1983, 1985) reported on problem-solving interviews with college students enrolled in a problem solving course.

Knowledge, and argumentation (proof) has nothing to do with the process of discovery or understanding. As a result, students who are perfectly capable of deriving the answers to given problems do not do so, because it does not occur to them that this kind of approach would be of value. Such students may fail to see that "proof problems" that they have already solved provide the answers to related "discovery problems" that they are now trying to solve. In fact, the answers they propose to current discovery problems may contradict the answers they have just derived in closely related proofs. Finally, it should be noted that naive empiricism is just one example of the way that people's behavior is shaped by their beliefs about the nature of mathematics." (Schoenfeld, 1985, pp. 185-186)

Like Cobb and Frank, Schoenfeld (1983) theorized that although unconsciously held, students' beliefs about mathematics established a context for mathematical situations and subsequently influenced metacognitive processes in the solution process.

### Summary

All of the studies summarized in this section demonstrate a linkage between beliefs about mathematics, autonomy, and problem-solving behavior. Cobb (1985) and Schoenfeld (1985, 1983) noted that knowing domain-specific knowledge is not sufficient to describe or predict students problem-solving behavior. Both researchers contended that students' belief about mathematics define a context for a problem and this context seemed to delimit the students'



choice of heuristics. This theme of beliefs influencing actions was reiterated by Confrey (1982) and Frank (1985) as well.

Buchanan (1984) distinguished among students with four characteristics: source of motivation, beliefs about mathematics, predominant problem-solving process and means of achieving satisfaction. She observed interrelationships between rational and instrumental beliefs about mathematics, autonomy and students' problem-solving performance. Confrey's (1982) research stressed the importance of autonomy as an influence in problem-solving behavior. Autonomy also appeared to distinguish among the students described by Buchanan (1984), Cobb (1985) and, Frank (1985).

Collectively these studies empirically support the existence of Skemp's (1987) description of instrumental and relational beliefs and a linkage of these beliefs with autonomy. In addition, these studies share a common methodology. Beliefs were inferred from dialogue and from behavior. For these studies, these two sources for information about students' beliefs appeared to coincide and make a convincing argument for the powerful influence of beliefs on mathematical activity.

Schoenfeld (1983) explicitly states and the others imply that beliefs seem to operate in a metacognitive function. That is, beliefs set up expectations about mathematical situations which, in turn, define for the students what constitutes a mathematical problem and what behaviors may or may not be appropriate or expected.

#### Related Research on Teachers' Beliefs about Mathematics and Classroom Environment

Green (1971) proposes that:

Teaching has to do, in part at least with the formation of beliefs, and that means that it has to do no simply with what we shall believe, but with how we shall believe it. Teaching is an activity which has to do, among other things, with the modification and formation of belief system. (p.48)

Several researchers have conjectured that the classroom environment may adversely affect students' learning by fostering dysfunctional beliefs about mathematics (Baroody & Ginsburg, 1986; Borasi, 1990; Buerk, 1985a; Confrey, 1982; Frank, 1988; Kouba & MacDonald, 1991). The teacher and the academic tasks comprise two vital elements in the classroom environment.

### Teachers' Beliefs about Mathematics

Teachers' beliefs about mathematics and about learning influence how the curriculum is presented and the role that students play in the classroom. Apart from mathematics anxiety, teachers' beliefs about mathematics have not been investigated in the same depth as students' beliefs.

In his Likert survey of teachers, principals, and parents, Fey (1970) collected data on the prevalent practices and expectations for mathematics instruction. He found that both elementary and secondary teachers viewed mathematics as procedures to be mastered. When asked to indicate why mathematics should be studied, the majority of the teachers responded with a variation of the mental discipline argument citing "learning to think logically, to solve problems but most of all to work hard (p. 498)." The survey also found that the majority of teachers lectured while only a small percentage utilized manipulatives or projects. The teachers also indicated that they never utilized simulations or outside speakers in their classrooms and that they found low student interest as a problem in the classroom.

The research by Cooney (1985), Owens (1987), Russell (1987) and Thompson (1984) document differing beliefs about mathematics for teachers at many grade levels. These beliefs range from a view of mathematics as predominately a collection of rules, to a view of mathematics as a logical structure, to a view of mathematics as a human creation which is intuitive and conceptual. The research of Russell and Thompson also supports the premise that teachers' beliefs about mathematics influence their instructional techniques. In particular, teachers' beliefs seem to establish the expectations they hold for their students (e.g., mathematics is comprehensible only to a few or mathematical ideas are intuitive and generally understandable). In addition, the researchers found that teachers' differed with respect to their explanations of students' difficulties: immaturity, lack of attention, lack of motivation, failure to remember previous information, inadequacies of teacher's model or explanations, or difficulties inherent in the mathematics content. Finally, the teachers revealed divergent beliefs about the best way for students to learn mathematics: by attending to the teachers' explanations and practicing examples or by appealing to students' experience and intuition, and by having students conjecture.

The teachers' beliefs in these studies suggest not only a connection with classroom

behavior but also a remarkable similarity to the students' beliefs discussed in the preceding section. In both the teachers' and the students' beliefs, several themes emerged. Mathematics was perceived as either basically procedural (instrumental) or conceptual (relational). This notion was related to their views on autonomy. Students, as well as teachers, held either the belief that mathematics was the product of geniuses and hence the rationale was knowable to only a few or the belief that mathematics could and should be internalized. In addition, many of the students and teachers interviewed, also believed that mathematical knowledge is transmitted by teachers to students and that this is best accomplished through a step-by-step presentation of the essential algorithms and procedures in mathematics.

#### Academic Tasks

Teachers also influence student behavior by the expectations inherent in the tasks they assign. Doyle (1983) found that academic tasks by their level, by the expectations associated with them, and by their content influence students' processing of information. He noted that students are very sensitive to clues about accountability and that they give serious attention to those tasks that they perceive they will be accountable for. Doyle further argued that the content is embedded in a context and that this embedding may limit the transferability of the content. This last observation concurs with Cobb's (1986) and Schoenfeld's (1983, 1985) conjectures that students' beliefs about mathematical problems seem to delimit their choice of heuristics or strategies they will employ. This contextualization of knowledge, also agrees with Kouba and MacDonald's (1991, 1987) data that students appear to limit their definition of mathematics to their classroom content and experiences. The research by Doyle, Sanford, Schmidt-French, Clements and Emmer (1985) stressed the need to evaluate academic tasks or teachers' questions in context. They found that the level of the tasks often changed in the course of classroom explanations. They speculated that teachers altered the content when they anticipated students' difficulties so as to avoid classroom disturbances. The teaching experiments by Nicholls, Cobb, Yackel, Wood and Wheatley (1990) suggest the plausibility of influencing or changing students' beliefs about mathematics by altering the type of activities the students engage in and by changing the classroom expectations. They found that those students in the classrooms that emphasized autonomy associated mathematics learning with attempts to understand for oneself. The research also found that these students scored significantly better

on comprehension and applications tests than did those students in a traditional classroom setting. Thus, academic tasks appear to influence students' construction and use of knowledge by embedding the knowledge in a context and by associating expectations with that knowledge.

#### Summary of Research Concerns from the Literature Review

The review of the literature suggests several considerations when investigating students' beliefs about mathematics. Foremost, the reviews point to the need for theory development in this area especially investigations of beliefs in relationship to problem solving and concept development (Shaughnessy & Haladyna, 1984; Kulm, 1980; Reyes, 1984; Silver, 1985; McLeod, 1987). Kulm (1980) suggested that research in this area also should be sensitive to nuances in beliefs. Citing the complex nature of attitudes, Aiken (1976) and Shaughnessy and Haladyna (1984) proposed that research should encompass information on intervening variables, as well as, information on the classroom environment. Specifically background information should be gathered on students previous experience with mathematics, their perceptions of their ability with mathematics, their level of anxiety with mathematics, the influence of parents and teachers on their views on mathematics, their perceptions of usefulness for mathematics (e.g., job and career goals) and their participation in extracurricular activities involving mathematics (e.g., puzzles, hobbies and computer usage).

In addition to this background information, several researchers stressed the importance of distinguishing among attitudes or beliefs about various topics in the mathematics curriculum especially between beliefs about problem solving and beliefs about routine problems (Kulm, 1980; Luccock, 1987; Owens, 1987). Others cited the relevance of examining both students' beliefs about mathematics as a discipline and about learning mathematics including the roles of the student and teacher (Buchanan, 1984; Frank, 1985; Buerk, 1981). The reports of varying attitudes towards school subjects in the NAEP suggests the value of collecting data on other disciplines in conjunction with an investigation of beliefs about mathematics.

These suggestions concerning the focus of research investigations were coupled with ones concerning the methodology. The reviewers universally called for investigations that utilized qualitative methods instead of the traditional paper-and-pencil measures of attitude (Aiken, 1970a, 1976; Kulm, 1980; McLeod, 1987; Shaughnessy & Haladyna, 1984). In particular, it was suggested that beliefs be inferred from interviews in which students engaged in mathematical

activities. Cobb (1985), Confrey (1981a, 1982) and Schoenfeld (1983, 1985a) pointed to moments of conflict and confusion that often arise in problem solving situations as useful for revealing students' fundamental beliefs.

The review of the literature on students' attitudes also served to affirm the theoretical assumptions underlying this study. Beliefs about mathematics as a discipline, about the roles of students and teachers in learning mathematics, and about autonomy emerged as recurrent themes in the investigations of students' beliefs and their relationship to problem solving behavior (Buchanan, 1984; Cobb, 1985; Confrey, 1982; Frank, 1985). The summary on teachers' varying beliefs about mathematics and the research on academic tasks also supported the contention that students' beliefs may be associated with their classroom environment (Doyle, 1983; Doyle, Sanford, Schmidt-French, Clements & Emmer, 1985; Nicholls, Cobb, Yackel, Wood & Wheatley, 1990; Owens, 1987; Russell, 1987; Thompson, 1984). Both the research literature and the theoretical background framed the questions, the type of data collected and the methodology employed in this research study.

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## CHAPTER IV

### METHODOLOGY

This research study investigated the effects of students' beliefs about mathematics on their learning of mathematics. The previous research highlighted several belief areas as central to understanding students' belief systems and their effect on learning: (a) students' beliefs about mathematics as conceptual or procedural, (b) students' beliefs about their own role and their teacher's role in their learning of mathematics, and (c) students' mathematical autonomy. The presentation in this chapter will begin with an overview of the study, followed by a detailed discussion of the pilot study, procedure, instruments, and analysis techniques.

#### Overview

The research study utilized a multiple case study design with analysis by and across cases. The cases represented 6 high school students enrolled in either Algebra II or Algebra II/Trigonometry. The research plan consisted of a pilot study and three data gathering phases: (a) classroom observations and assessment of the teacher's perception of her role in the learning process, (b) an assessment of student participants' beliefs about mathematics and autonomy with mathematics, and (c) an assessment of the students' newly formed mathematical constructs on functions.

Briefly the rationale behind the research plan was to develop a detailed portrait of each individual's beliefs about mathematics, to observe these individuals within the social context of their mathematics classes, and finally, to carefully examine the mathematical constructs that the individuals formed from their classroom experiences. By synthesizing the information on individual students' beliefs with the classroom observations and expectations, and comparing these results with the individual student's mathematical constructs, a description was developed for each student. These descriptions attempted to explain the type and depth of the mathematical constructs relative to the students' beliefs and to the classroom expectations. Finally, by searching the individual student descriptions for patterns and similarities, a general description for the interactions between students' beliefs and mathematical constructs interaction

was proposed.

#### Pilot Efforts

The research study was preceded by two pilot efforts conducted by the researcher whose purpose was (a) to explore and subsequently refine the interviews including the selection of mathematical problems, follow-up questions, and the interviewing technique itself, and (b), to develop a workable set of mathematics belief categorizations.

##### First Pilot

The first pilot effort during the summer of 1987 served as a trial for the collection of mathematics problems and for the interview technique. Lasting 2 to 3 hours, the tape-recorded interviews focused on two areas: (a) students' background information, including discussion of previous mathematics courses and teachers and (b) the students' solutions to mathematics problems and their rationale for their choice of procedures. Two returning adults attending the state university in New Hampshire and one local junior high student participated in the interviews.

The audio-tapes from the interviews were reviewed both by the researcher and by a mathematics education faculty member. The faculty review provided a critique of the interviewing technique through comments on the researcher and participants' interaction, (e.g. the directiveness of the questioning and the frequency of silent pauses during students' solutions to the mathematics problems.) In addition, the researcher partially transcribed the audio-tapes to provide a preliminary list of students' comments relevant to beliefs about mathematics and mathematics teaching. The audio-tapes were then reviewed a second time to ascertain the context for the student comments. Of particular importance were the questions that the researcher asked to precipitate the students' comments. As often happens in an open-format interview, many of the interesting and revealing remarks arose serendipitously. The researcher then attempted to develop additional non-directive questions, scenarios, and problems with the potential to elicit the same type of information.

##### Second Pilot

Conducted during the spring of 1988, the second pilot interviews were again used to evaluate the interview technique, problem selection, probe questions, and belief categorizations. The four participants were high school students enrolled in an Algebra II/Trigonometry class at a regional high school in southern New Hampshire. These students were representative of the

type of student who participated in the final research study.

Partial transcriptions from the audio-taped interviews were used to re-evaluate the beliefs categorizations suggested by theory and the previous pilot effort. According to categorization schemes, students' remarks often could be interpreted as indicative of multiple categories. As a result, the five originally proposed belief categorizations (structure of mathematics, form of mathematics, activity of mathematics, origin of mathematics and role of proof) proved not to be conceptually and practically disjoint. The categorization scheme was then modified to form three broader categories: beliefs about the nature of mathematics, beliefs about the students' and the teachers' role in learning mathematics, and autonomy with mathematics.

Besides facilitating the re-evaluation of the belief categorization, the partial transcriptions also provided the basis for a written summary of each student's beliefs about mathematics. These written summaries along with the audio-tapes were used by two mathematics education faculty members to again critique the interview technique and to check the conclusions reported in the summaries.

#### Procedure

##### Setting

##### School

The setting for the main study was a small high school in a university town in southern New Hampshire. Approximately 500 students were enrolled in this school which has a strong college preparatory program. Approximately 78% of the student body attends either a four-year or two-year college after high school. The mean SAT verbal and mathematics scores reported for the school were 497 and 540 respectively.

##### Classes

The student participants and classroom observations were from an Algebra II and an Algebra II/Trigonometry class both taught by Mrs. Thomas (pseudonym). The Algebra II course was one of three sections offered by the high school while the Algebra II/Trigonometry course was the sole section of its type. Demographically, the Algebra II course was composed of 8 females and 12 males with 1 sophomore, 13 juniors and 6 seniors. The Algebra II/Trigonometry class had 6 females and 8 males with 6 sophomores and 8 juniors.

The Algebra II/Trigonometry class was considered an accelerated course and enrollment



was restricted to those students who had previously taken the Accelerated Geometry course or who had received a teacher recommendation. The sophomores in this course were in the school's accelerated program beginning with Algebra I in 8th grade and Accelerated Geometry in 9th grade. The juniors in the course were evenly split between those who had enrolled in Accelerated Geometry (4) and in Geometry (4).

The composition of the Algebra II class was predominately juniors. These students had progressed through the traditional sequence of courses: Algebra I in 9th grade and Geometry in 10th grade.

#### Student Participants

The six student participants were volunteers from a pool of students from both algebra classes who met several research criteria. To form the pool of potential participants both classes were administered the Algebra I Placement Exam (College Board, 1972), and the Fennema-Sherman Mathematics Attitude Scales (1976). (See the instrumentation section in this chapter for a description of these measures.) Originally the Algebra I Placement Exam and the Fennema-Sherman Mathematics Attitude Scales were intended as a two-tiered selection criterion. First, the students were to be selected on the basis of their score on the algebra exam (75% or higher). Then from that group, students would be chosen who showed either a strong positive or strong negative attitude towards mathematics as measured by the Fennema-Sherman scales thus, assuring a variety of initial attitudes, while controlling for achievement scores. However, the first criterion was fairly limiting and when it was coupled with willingness to participate in study, the available pool did not permit selection by the Fennema-Sherman Attitude Scale criterion.

In addition to these measures, each student completed a background inventory sheet indicating previous mathematics courses with grades, age, sex, class standing, and availability during the school day. The final pool was then formed from those algebra students who were juniors and scored 75% or higher on the placement exam. Everyone in the pool received a letter inviting them to participate in the study. Those students who subsequently volunteered and received parental consent comprised the study group.

The six participants were given the pseudonyms of Ann, Tara, Tom, Keith, Steve, and Sue. Their selection criteria data and background information are summarized in Appendix G.

The placement scores of the student participants were over 75%. The score for the Algebra II/Trigonometry students, Keith, Steve, and Sue were respectively 34, 27, and 27 out of 35. As might be expected in an advanced section, the overall class scores were high. In fact, all the scores from that class were above the 75% cut-off. In relation to the other students in the Algebra II/Trigonometry, Keith's score of 34 was the highest, while Steve's and Sue's 27 were the lowest scores. For Ann, Tara, and Tom, their scores of 32, 29 and 28, respectively, represented scores in the upper 50% of the Algebra II class.

#### Data Collection

The data for this study was collected by the researcher in three phases: (a) classroom observations and teacher assessment, (b) students' beliefs assessment and (c) function assessment.

#### Phase I: Classroom Observation and Teacher Assessment

The first major phase in the data gathering was the video-taping of both classes during their respective units on functions. This phase was concurrent with the selection of the student participants. This phase (a) documented the teacher's presentation of the unit on functions, (b) observed the student-teacher interaction in the classroom, and (c) collected relevant classroom materials as examples of the teacher's expectations for the classes.

The placement of the camera on a tripod in the back corner of the room afforded a clear view of all the blackboards and the students in the class. The equipment was always readied by the researcher between classes so as to cause minimal disruption in the classroom routine. As an additional safeguard against disruption of the classroom flow, the researcher observed the classes for several days prior to the study without the equipment. In this way, the researcher and the classroom teacher felt that the students would become acclimated to the researcher's presence and the researcher would have an opportunity to observe typical classroom behavior without the camera's influence.

To further document the unit, copies of all quizzes, tests, handouts and homework assignments were collected. Right of privacy considerations and the university's human subjects restrictions prohibited the collection of the student participants' actual papers and assignments from the classes. In addition to the video-tapes and sample assignments, the researcher kept

a daily journal of impressions from each day's observations.

For the Algebra II/Trigonometry class, the function unit began about 3 weeks into the school year with the video-taping occurring over 8 consecutive days. For the Algebra II class, the unit began 5 weeks into the school year with the video-taping occurring over 11 consecutive days. These days were exclusive of the chapter tests given at the end of the units. Each class period lasted 50 minutes.

The second component of the classroom assessment entailed a series of five audio-taped interviews with the classroom teacher. These interviews (a) clarified any issues that arose from the classroom observations, (b) obtained background information on the teacher's experience and professional activities, (c) ascertained the teacher's expectations concerning the functions unit for each class, (d) recorded the teacher's philosophy of teaching and classroom policies, and (e) collected data concerning the teacher's own beliefs about mathematics. A complete list of the interview questions is given in Appendix A. Several of the instruments were used in the interviews for both the teacher and the student participants to permit a comparison of views on mathematics. These instruments are discussed in more detail in the instruments section of this chapter. Samples of these instruments can be found in the student interview protocol in Appendix B.

#### Phase II: Beliefs Assessment

Beliefs Assessment. The purpose of this second phase was to gather information on the students' background and beliefs about mathematics. Three primary belief categories were targeted: (a) students' beliefs about mathematics as conceptual or procedural, (b) students' beliefs about their own role and the teacher's role in learning mathematics, and (c) students' autonomy with mathematics.

This phase in the data collection involved a series of five interviews with each of the six participants. The audio-taped interviews occurred once a week during the students' free period and lasted approximately 45 minutes. The interviews were conducted in a small, enclosed study carrel in the library of the high school. A calculator, straight edge, pencil, and scratch paper were always available for student use.

The interview schedule including the questions and instruments is given in Appendix B. Although the sequencing of the questions was fairly consistent from student to student, which

questions constituted a given interview varied among the students. This variation arose as the students fluctuated in the length of time they spent answering a particular question. Also since the protocol format was open, the researcher's follow-up questions varied in length depending on the students' comments, thus again lengthening some interviews. The wording of the questions also varied. This variation was done consciously as the researcher attempted to mirror the terminology used by the students. For example, if a student described the process of working the problem, 6 divided by  $\frac{3}{8}$ , as "invert and multiply," then the researcher's follow-up question would be phrased "Why do you invert and multiply?" If the student said "multiply by the reciprocal," the researcher responded similarly.

Each interview began with a few minutes of informal conversation to relax the participants. The interviews were conducted in a non-judgmental manner. This disposition included a neutral stance on the validity of students' mathematical work, as well as their opinions. Through this stance, the researcher tried to avoid assuming an authoritative or expert's role, so as to empower the students' own voice.

Several techniques were employed to collect information on the belief categories. One component of this assessment entailed observing and questioning students while they solved various mathematics problems. These problems ranged from simple arithmetic algorithms to problem-solving situations. Follow-up probes explored the students' rationale for their strategies and their dependency on rules and algorithms when solving problems particularly in problem-solving situations.

To further corroborate any beliefs that might be expressed by the students or inferred from their solution to mathematics problems, the students completed several additional activities. These activities included marking and discussing a mathematics topics ranking grid and vocabulary lists, grading a sample algebra test, and responding to various scenarios on student's/teacher's roles. A detailed discussion of these activities is deferred to the instruments section.

### Phase III: Functions Assessment

In this phase, the students participated in an additional three interviews. These were again audio-taped and lasted approximately 45 minutes each. The purpose of this series of interviews was to investigate the knowledge that the six participants had constructed from their

classroom unit on functions. (See interviews six, seven, and eight in Appendix B for a list of the content questions used in this assessment.) The questions covered the concepts of slope, function (both the definition and notation), domain and range, graphing functions, intercepts, composition of functions, and word problems using function notation. With the exception of graphing straight lines, the functions unit represented new material for the students in both classes.

The content questions were developed prior to the study, but the final selection of questions was delayed until after the completion of the functions unit in each class. By delaying the selection, it was possible to include problems that were familiar to the students as well as extension and transfer problems.

### Instruments

#### Placement Instruments

Two instruments were administered in the participant selection process: the Elementary Algebra Placement Test and the Fennema-Sherman Mathematics Attitude Scale. These measures are described briefly below.

#### Elementary Algebra Placement Test

Briefly, the Elementary Algebra Placement Test is a 20 minute timed-test consisting of 35 multiple choice elementary algebra questions. The format of the questions is similar to the mathematics portion of the SAT. A sample question follows:

Column A	Column B
$3x + 5 = x - 7$	
x	6

**Responses:**

- A if the quantity in Column A is greater;
- B if the quantity in Column B is greater;
- C if the two quantities are equal;
- D if not enough information is given to decide.

This test was used with permission of the College Board. Any results or interpretations, however, are solely the researcher's.

#### Fennema-Sherman Mathematics Attitude Scales

The Fennema-Sherman Mathematics Attitude Scale consists of nine subscales: mathematics anxiety scale; confidence in learning mathematics; usefulness of mathematics;

mathematics as a male domain; attitude toward success in mathematics; effectance motivation in mathematics; and father, mother, and teacher scales measuring perceptions of attitudes toward one as a learner of mathematics. The attitude areas represented in the subscales are those theorized to be salient to students' attitudes towards mathematics. Each subscale contains 12 statements to which a respondent selects one of five possible responses: strongly disagree, disagree, undecided, agree, or strongly agree. When reporting scores, students' responses to each item within a subscale are encoded 1 to 5. The codings are given so that a 1 corresponds to the strongest negative attitude towards an item and 5 corresponds to the strongest positive attitude towards that item. Once the item's responses have been encoded, the scores from the 12 items within a subscale are averaged and reported as a mean item score.

The Fennema-Sherman Mathematics Attitude Scale was chosen for usage in this study because of its ease of administration and its common usage within the mathematics education research literature.

#### Belief Assessment Instruments

A variety of instruments were developed to assess the students' beliefs about mathematics. Foremost in consideration of the design was the advice and evidence from the psychological literature which emphasized that individual's beliefs are often unconsciously held and that they are most likely to be accurately portrayed when the individual is again in a situation similar to the one in which the beliefs were formed (Fazio & Zanna, 1981; Kleinke, 1984). For example, a student's attitudes about tests would most likely come to the surface and hence be more accurate of his or her true attitudes if the student were queried in a testing situation or immediately after a test, rather than assessing those beliefs further from the experience.

Traditional attitude measures suffer from being removed from immediate experience. The intensity of affective variables is not current in students' thoughts when they respond to these passive measures. The reliance on these passive measures has been criticized repeatedly in the mathematics education literature (Aiken, 1976; Kulm, 1980). Not only are these measures distant from the attitudes they report, but the measures are limited in their perspective. Researchers and critics have called for use of techniques that are more sensitive to nuances in beliefs (Kulm, 1980; McLeod, 1987, 1989). For example, it is commonly reported that students

believe that mathematics entails memorization and following rules (Brown, Carpenter, Kouba, Linquist, Silver & Swafford, 1988b; Carpenter, Coburn, Reyes & Wilson, 1978; Carpenter, Corbitt, Kepner, Linquist & Reyes, 1981; Crosswhite, 1972). Yet it is unclear from these reports if students think every topic in mathematics requires that they memorize rules, or if selected topics are associated with that view (Lucock, 1987).

With these cautions and criticisms in mind, the assessment of beliefs was designed to have the students actively engaged in solving problems. Students' beliefs would be inferred from their strategies used, rationale given, and from comments made during the process. In this way, the students' beliefs would be evaluated from active experience rather than from passive recall.

A second major concern when designing the belief assessment arose from qualitative issues. Qualitative research uses triangulation as a primary tool for reinforcing and clarifying data and conclusions. Triangulation, briefly, is a process of collecting information from multiple sources and comparing that information to verify its consistency and consequently to validate its representation of the situation (Goetz & LeCompte, 1984; Miles & Huberman, 1984; Yin, 1984).

Use of triangulation necessitated the development of multiple instruments to assess the same belief categories. The instruments developed fall into six categories. These are listed below in Table 8 along with the intended belief assessment category. Note that the questions in each category are not necessarily consecutive. The arrangement of the question types in the interview protocol was intentionally mixed to vary the type and intensity of questions and in this way reduce any potential anxiety or boredom in the interview process.

Table 3 summarizes the placement and distribution of questions within the interview protocol. The actual questions and instruments are located in Appendix B. The table can be read either across rows or down columns. To locate, for instance, those questions in the protocol that dealt with students' solutions to mathematics problems, begin with the Problems category under the Instrument column heading and move to the right to the Interview Questions column. To locate those instruments or questions that were intended to evaluate students' beliefs about mathematics as conceptual or procedural, begin with the column heading Beliefs - Procedural/Conceptual and move down the column selecting those instruments designated with an x.

Table 3  
Triangulation of Instruments

Instrument	Beliefs			Interview Questions
	Procedural/ Conceptual	Student/ Teacher	Autonomy	
Problems	X		X	19, 20, 21, 23 24, 25, 26, 27 28, 29, 30, 31 33, 34, 35
Sample test	X		X	32
Vocabulary list	X		X	22
Ranking grid	X			36
Student/teacher scenarios		X		37, 38, 39, 40 41, 42, 43
Background questions		X		1-18

What follows is a discussion of each category of instruments used in the beliefs assessment.

#### Problems

As indicated by Table 3, the student participants had 15 problems to solve in the course of the interview protocol. The problems varied in both mathematical content and in difficulty level. They can be grouped into two broad categories: algorithmic and problem solving. The algorithmic category included basic arithmetic and algebra problems that could be solved through the execution of a rule or algorithm. Their inclusion was to test students' rationale and the conceptual ties behind their procedural knowledge. Had the students developed a rationale for validating these procedures or had they accepted their validity on the assertion of an outside authority, (e.g. "it is a rule given by my teacher" or "it's just a rule")? The initial instruction was simply to solve the problem. The follow-up probes, however, asked the student to explain how they solved the problem. They also were asked either why the rule or algorithm worked or how they knew the answer was correct. In a few cases, the students were asked to offer an alternate procedure for solving the problem. The probe questions provided a situation to elicit students' remarks on their rationale and their source of authority.

The second category of problems was problem-solving situations. These problems



provided an unfamiliar setting in which to explore the participants' beliefs about mathematics. The psychological theory on attitudes argues that those beliefs which are most fundamental to our belief system are those to which we cling when challenged or confused. Confrey (1981) offered similar advice when she described interviewing techniques that were productive. In summarizing a quote from Toulmin she states: "[those] moments of controversy, conflict, confusion and change often reveal individuals' most fundamental, determinedly-held beliefs, and allow for inspection of the processes used in forming those beliefs." Included among the problem solving situations were several counterintuitive problems, that is, problems whose correct solution is counter to expected results. These problems afforded the opportunity again to question the students on why they believed their answers and how certain they were of their answer's validity.

#### Sample test

This researcher-developed instrument consisted of an algebra test (question #32) completed by a fictitious student named Chris. The student participants were asked to grade the test and to give their rationale for any partial credit they chose to assign. The questions in this sample test served two functions. They were intended to help illustrate a source of authority for students' knowledge. That is, whether internal or external and whether conceptual or rule bound. It was also intended to help distinguish the criteria students used when evaluating the correctness of their mathematics whether on the basis of familiarity of form, execution of a rule or on the concepts underlying the procedures. The researcher had previously utilized many of these questions in her own classroom experience to explore and expose students' common misconceptions and rationale for basic algebraic procedures. Again the intent was to place the student in a situation which would potentially challenge or conflict with the students' previous experience and thereby reveal students' fundamental beliefs.

The test incorporated four basic types of errors. The first type represented common algebraic errors, (e.g.  $(x^{13})^2 = x^{15}$  or  $6/0 = 0$ ). The second type of error involved incorrect procedures which yielded correct solutions (e.g. incorrect cancelling in  $(x^2 - 1)/(x-1) = x + 1$  or  $6/6 + 2/12 = 1/6 + 1 = 7/6$ ). The third type of "error" was represented in a problem whose solution was done correctly but which used an atypical procedure, for example:

$$\begin{aligned}
 3x-7 &= 4 \\
 3x-7+4 &= 4+4 \\
 3x-3 &= 8 \\
 3x-3+3 &= 8+3 \\
 3x &= 11 \\
 3x/3 &= 11/3 \\
 x &= 11/3.
 \end{aligned}$$

The final error involved problems with a correct solution process but an erroneous interpretation of the solution, for example:

$$\begin{aligned}
 3(x-2)+12 &= x+2(x+2)+2 \\
 3x-6+12 &= x+2x+4+2 \\
 3x+6 &= 3x+6 \\
 6 &= 6 \\
 x &= 6.
 \end{aligned}$$

As in the pilot study, students' mathematical misconceptions appeared as they graded the test. For example,  $6/0 = 0$  was often marked as a correct solution. As this provided an opportunity to probe for conceptual ties or their lack there of, the researcher questioned the student participants about their misunderstandings once they had completed grading the test. At that time, the researcher also pushed for clarification on the students' rationale when determining point value. For example, on the problem  $6/6 + 2/12 = \dots = 7/6$ , if the student had not noticed that the correct answer appeared the researcher would say, "what would you say to Chris when he or she says, 'but I got the right answer?'" Also, on the problem  $(x^{13})^2 = x^{15}$ , if the student had remarked "the answer should be  $x^{26}$ , it's wrong", the researcher would ask why  $x^{26}$  was right, again looking for conceptual ties or an adherence to memorized rules. As in the other instruments, the source of the students' authority was perceived as indicative of their autonomy with mathematics.

#### Vocabulary List

This vocabulary list was developed to provide corroboration of any beliefs stated or inferred from the students' problem solutions. (See question #22 in the student interview protocol, Appendix B.) The list consisted of 94 adjectives derived from several sources: (a) terminology used by the students in the pilot studies, (b) the adjectives used as part of attitudinal surveys in NLSMA (Crosswhite, 1972) and NAEP (Carpenter, Coburn, Reyes & Wilson, 1978; Carpenter, Corbitt, Kepner, Lindquist & Reyes, 1981; Carpenter, Lindquist, Matthews & Silver, 1983), (c) and the terms used by high school students in an essay on their likes and dislikes about mathematics written as part of an application to a gifted mathematics program at the state

university of New Hampshire.

Each student participant read the list and circled those terms that were associated with mathematics. When the student completed the list, the researcher then asked the student to review the terms circled to ascertain if any terms could be grouped together to describe a particular aspect of mathematics. For example, the terms absolute, controlled, mechanical, ordered, and structured might be grouped to describe the nature of solving problems (e.g. "each problem has a prescribed method of solution"). Following the discussion of groupings, the researcher selected particular terms and asked for the students' reasons for circling those terms.

To provide some perspective on the choice of terms selected for describing mathematics, the students repeated the task successively, describing English, history, and science. Perry's (1981) theory on intellectual development and Kolb's (1981) theory on disciplinary differences suggest that individuals may simultaneously hold widely differing perspectives on academic subjects. Is memorization an expectation for the students in all their classes or is it peculiar to mathematics? To assert that mathematics alone entails memorization, strengthens the significance of that attribute.

#### Ranking Grid

Again, this instrument provided a component in the triangulation process to validate the assessment of students' beliefs about mathematics. It also addressed a criticism from the research literature that measurements of attitudes need to differentiate between mathematical topics ( e.g. "Do students perceive all topics as boring or abstract or only selected topics?"). (See question #36 in the student interview protocol Appendix B.) The instrument in this category was a modified version of Owens' (1987) rank topic grid. Owens' topic grid is representative of a technique called repertory grid which was developed in conjunction with Kelly's (1955) theory of constructs. The purpose of this type of instrument is to illuminate the emphasis that individuals place on specific topics with regard to adjective pairs. The topics across the columns of the grid were selected to correspond to common mathematics topics studied by high school students. The adjective pairs along the rows (referred to as constructs) were modified from Owens' constructs to be more closely aligned with the terminology used by the students in the pilot.

On this instrument, students were asked to rank order the topics across the columns with

respect to the adjective pairs. For example, a student who felt that word problems were the most interesting topic would assign it a rank of 1. The most boring of the topics would be ranked 11. The remainder of the topics would be ranked accordingly between 1 and 11.

#### Student/Teacher Scenarios

This instrument consisted of a series of short scenarios and related questions to investigate the students' perceptions of the role of the teacher and themselves in the learning of mathematics. For example, the students were asked to respond to: "I would like you to imagine there is a new kid in school, an English speaking foreign student who is unfamiliar with American schools. The guidance office calls you down and asks that you show the student the ropes in your math class. What kind of advice would you give him?" As another example, students were also asked to complete the sentence: "A good math teacher is one who?"

#### Background Questions

This collection of questions provided data on the students' current and past mathematics courses, non-academic uses of mathematics, jobs, hobbies, usage, and post-secondary plans. The intent here was to draw a general portrait of the students which might offer some insight into their subsequent remarks made during the remaining interviews. The specific data collected also provided important information about motivational and affective issues: perceived usefulness of mathematics, future needs, and past experiences with mathematics.

The questions and instruments in each of the six categories were designed to open up discussion between the student and the researcher, and to provide situations likely to reveal actions indicative of a view of mathematics as conceptual or procedural as well as autonomy with mathematics, and student-teacher roles in the learning of mathematics. By using a multiplicity of instruments to explore students' beliefs, the data collected from different instruments could be compared for consistency and used to clarify the inferences and conclusions made about each student.

#### Analysis

The analysis of the data required several stages of refinement and validation before the final summaries were created. Before examining the details of that process, a brief description of the overall sequence of events will be given. The analysis process actually began in the pilot studies. There, the working definitions for conceptual/procedural and

autonomous/nonautonomous beliefs about mathematics initially were developed. Prior to any evaluation in the actual study, the audio-tapes of the student interviews were completely transcribed and then checked again for accuracy. Once completed, the transcriptions and the students' scratch work along with the tapes were coded. That is, segments in the transcriptions were marked as evidence of conceptual/procedural and autonomous/nonautonomous in the belief interviews, and coded right/wrong in the functions interviews. Next a copy of the transcriptions and scratch work, along with the tapes and interview questions were given to two independent raters who used the researcher's criteria to recode the data. The results from the raters' coding were then compared to researcher's original coding. Using both sets of codings, the researcher then wrote a report summarizing the analyses. This report was then submitted to the raters for their verification as representative of their analysis of the data. Similarly, the video-tapes of classrooms were coded and recoded, and the results summarized. The last stage in the analysis, the secondary analysis, entailed comparing the summaries for each student and suggesting hypotheses or conclusions that could be drawn from the research. The remainder of this section will describe in more detail the analysis of each of the major data collection areas: classroom assessment, beliefs assessment, and functions assessment.

#### Classroom Assessment

The purpose of this assessment was to develop a portrait of the classroom environment. Two themes were of particular interest. The first was whether the classroom environment was conducive to either a view of mathematics as conceptual or procedural and whether it was conducive to the development of students' autonomy with mathematics. The second theme was to document the unit on functions including the topics presented in the lectures and the type of questions given on the quizzes and tests.

Much of the data in this section was descriptive in nature. For example, the teacher interviews were used to record the teacher's beliefs about mathematics and her expectations for the classes. The analysis, in this case, entailed summarizing the views presented in those interviews. Similarly, the documentation of the functions units involved reporting and summarizing the topics and the evaluation measures.

However, the determination of the classes as conducive either to a conceptual or procedural view of mathematics and conducive to either autonomy or nonautonomy with

mathematics involved a careful analysis of the classroom video-tapes. The analysis of the video-tapes primarily focused on the nature of the teacher's questions in the lecture portion of the class period. These questions were coded as either asking the students to draw on conceptual or procedural knowledge. The frequency of the two types of codings, conceptual or procedural, was then summed. This frequency provided a measure of the extent to which the classroom fostered the use of conceptual or procedural knowledge. These questions formed one part of the classroom expectations concerning what was valued about mathematics. These expectations are then theorized to influence students' beliefs about mathematics.

The process of coding the video-tapes occurred in several stages. First the researcher coded one of the classroom video-tapes. This coding involved listing each question asked by the teacher during the lecture portion of the class period and then evaluating each question as indicative of either a conceptual or procedural question and the relative level of difficulty of each question. (See the coding criteria in Appendix E. Once this sample coding was completed, the researcher met with two experienced high school mathematics teachers who volunteered to serve as raters in this project. Both teachers received a small compensation for their time in evaluating the interviews. Neither teacher was associated with the school district where the study took place nor knew the teacher involved in the study. To facilitate the discussion of the coding process, the researcher had the raters view the sample video and code the teacher's questions. Once their recoding was completed, the two raters and the researchers discussed their respective markings and interpretations of the criteria. The raters and the researcher agreed that any new procedure or use of a new definition would be coded conceptual during its first day of usage but any subsequent instances would be viewed as recall and hence denoted as procedural.

The purpose of the discussion between the researcher and the raters was to define the rater's responsibility and to discuss the coding criteria. The intent, however, was not to train the raters to code the data identical to the researcher. The independence of the raters was necessary to validate any analysis or conclusions made about the classroom environment.

Each rater was given one entire set of video-tapes of the unit on functions. The raters were responsible for listing all the teacher's questions from the lectures, coding the questions, and answering the auxiliary questions about each day's lesson. (See the rater's coding sheet

in Appendix F.) The coding process required that the raters view the tapes at least twice to accomplish the tasks. Once completed, the researcher used the raters list of questions, minus the codings, to re-evaluate the videos. The match between the researcher and the raters' codings were 84% and 88%.

Although no formal criteria were given for the easy to hard scale, the practice coding session provided a limited discussion of this subjective marking. This rating was intended as a means for the raters to qualify their judgments. The coding process required the raters to take into account the context of the question (e.g. did the proceeding discussion lead to the conclusion asked for by the teacher?). Thus, a question which might be judged difficult and conceptual if evaluated in isolation could be evaluated as conceptual, led, and easy given the context for the question. The lack of a formal definition for the rating scale attributed little to the mismatches between researcher and raters. On those questions where there was agreement on the basic category, there were only 9 differences in ranking out of 113 questions. The majority of the mismatches occurred between the categories (e.g. the researcher tended to code more questions as procedural (recall) than did the raters who marked those questions as easy conceptual).

The determination of the classroom as either conducive to students' autonomy or nonautonomy with mathematics entailed consolidating data from multiple sources: classroom video-tapes, teacher interviews, and rater's auxiliary questions. The video-tapes were re-examined for instances where students acted as their own authority for the validity of their mathematics, where students generated mathematical ideas, where students worked on tasks independently of the teacher (e.g. student presentation of homework problems, use of student conjectures, use of independent projects) or instances where the teacher shared authority by redirecting a question directed to herself to other students. The teacher's comments in the interviews furnished additional evidence of the teacher's expectations about the students ability to work independently. Finally, the auxiliary questions completed by the raters supplied observations relevant to this determination.

#### Beliefs Assessment

The assessment of the student participants' beliefs about mathematics formed the cornerstone of this research study. The assessment focused on three belief categories: beliefs

about mathematics as conceptual or procedural in nature, beliefs about the student's and teacher's role in learning mathematics, and beliefs about autonomy with mathematics. Since each of the six students was treated as a separate case study the analysis proceeded on a case by case basis. This casewise analysis included both the belief and functions assessment. However, for discussion purposes, these two assessments within each case will be treated separately. As stated in the methodology section, data on these belief categories was collected through multiple instruments. For the purpose of the initial analysis, the data from the instruments was divided into two major groups: (a) data from the vocabulary list, the ranking grids, and the student/teacher scenarios and (b) data from the mathematics problems and sample test. The data from the first group was analyzed using the qualitative technique of typological analysis (Goetz & LeCompte, 1984) while the data from the later group was coded and enumerated.

The analysis of the first group of instruments proceeded in several stages. The data from these instruments served as a potential confirmation (triangulation with) of the data from the students' problem solutions and, as such, needed to be analyzed separately. The data in this first group was further divided into those questions and instruments (e.g. vocabulary lists and ranking grid) that were related to general beliefs about the nature of mathematics and those instruments relevant to student/teacher roles. The analysis in both subsections proceeded in a similar manner. In typological analysis, the data, here the transcriptions, is repeatedly and systematically reviewed with a specific focus. The transcriptions were reviewed for statements suggestive of a view of mathematics as procedural or conceptual, and statements indicative of the students' view of their own autonomy with mathematics. While these beliefs were the focus of the search through the transcriptions, any statement which expressed a clear belief about the nature of mathematics was flagged for inclusion in the report of the data. The data from the annotated transcriptions was then reported on an instrument by instrument basis. These reports included and, in fact, stressed the students' actual remarks rather than commentary on those remarks.

Once the data from the individual instruments was reported, the students' comments within each instrument were re-examined for cross instrument agreement. Any trends, themes or contradictions observed were summarized and included in a synopsis of general beliefs about



mathematics. A similar process occurred for reporting and then summarizing the individual students' beliefs about the student/teacher roles in learning mathematics.

A different process of analysis was used for the second group of data: the students' problem solutions and evaluation of the sample test. Unlike the first group, the analysis of this data entailed inferences from the students' work and subsequently necessitated corroboration for those inferences from independent sources. The transcriptions of the students' solution processes were coded according to the criteria given in Appendix C and recorded using the coding sheets shown in Appendix D. Additionally, actual passages within the transcriptions were annotated as evidence supporting the various coding categories.

The coding of the data occurred in two phases. The data was rated first for the conceptual/procedural criteria and, second, for the autonomous/nonautonomous criteria. The codings were based on a positive instance of a category rather than on its absence. For example, in order for an episode in the transcription to be designated as nonautonomous, the student would have had to make a comment or shown by her actions one of the behaviors listed in the coding criteria. She would not receive a nonautonomous coding because she failed to show any autonomous behaviors. Also the categories were not assumed to be mutually exclusive, that is, an episode could potentially be coded both conceptual and procedural if the student exhibited actions or comments exemplary of both categories. If no positive instances of either category were observed, then the episode/problem was designated undecided. Finally the coding of the data was independent of whether the student correctly solved the problem.

Once the initial coding of the data for the six participants had been completed, the researcher met with the two independent coders. The two coders were doctoral candidates in mathematics education whose own dissertations involved extensive use of qualitative methodology and whose teaching experience included the Algebra II level. The discussion within this meeting was focused on and limited to the definition and meaning of the coding categories. The researcher was wary of imposing any training on the coders which might subjugate their independence and impartiality when coding the transcriptions.

The two coders were responsible for re-evaluating the data from two of the student participants: Ann and Tara. The coders were given clean copies of the transcriptions and the scratch work along with the audio-tapes and coding sheets. The coders worked through the data

once to code it for the conceptual/procedural categories and a second time for the autonomous/nonautonomous categories.

The coders' evaluations and comments were then compared to the researcher's own coding of the data. This comparison yielded an 85% and 75% overall match in the selection of coding categories. Using both sets of codings and the comments, the researcher then wrote a summary of the evaluation of the students' problem solutions. The written summaries for Ann and Tara were then submitted to the coders for their corroboration as representative of the data they had examined and coded. The final reports given in this dissertation reflect the coders' criticisms and comments.

The final step in the analysis of the students' beliefs entailed a re-examination of the data from both of the major groups of instruments. This examination searched for any consistencies in or discrepancies between the students' beliefs as inferred from the analysis of the problem solutions and the beliefs as reported from the other instruments. The observations from this last examination were summarized and included in the final report of the data.

#### Functions Assessment

The purpose of this assessment was to evaluate the student participants' understanding of the concept of function. This evaluation included questions on the definition of function, function notation, the concept of domain and range, graphs of functions, intercepts, composition of functions, and word problems using function notation. These topics represented the core of the content common to both classes. Like the beliefs analysis, the analysis of the data in this section progressed through several stages and involved the independent coders.

The analysis began with researcher coding the students' solutions and comments on each of the functions questions. Although the primary coding was either correct or incorrect, these codings were qualified by comments and by the demarkation of C+, C and C- which indicated the relative strength of the students' correct response. That is, how complete the response was and how readily it was forthcoming (without prompting or probing by the researcher).

The data from Ann's and Tara's functions questions was recoded by the same two coders who had evaluated their beliefs data. As before, the coders were given a clean copy of the transcriptions, scratch work, audio-tapes, and coding sheets. The coders were instructed to code Ann's and Tara's responses as mathematically correct or incorrect and to express their rationale

for their coding selection. Since the interviews were interactive, the possibility existed that the researcher might have inadvertently led the student to the problem solution or curtailed the student's own response. In recognition of this possibility, the coders also were asked to comment on the interaction especially if in their perception the dialogue may have unduly influenced the student's responses.

The match on the categories selection between the coders' and the researcher's codings was fairly high: 90% and 80%. The few discrepancies that occurred were all instances where the students' comments reflected a very weak or partial understanding of the concepts implied in the question. For example, one coder rated the student's response C--, while the researcher evaluated it as W (incorrect). Since the percentages were based exclusively on category match, this example would have been tallied as a mismatch, yet both codings and the accompanying comments conveyed similar messages about the incompleteness or vagueness of the student's understanding.

Using both sets of codings and comments, the researcher then summarized the students' responses to functions questions. The summary was organized around three topics: (a) definition of function, (b) function notation, and (c) related concepts. Next the data from the beliefs assessment and the functions assessment were re-examined for any apparent consistencies or discrepancies. For example, would a student whose comments and problem solutions suggest a view of mathematics as conceptual develop connections or ties in his knowledge of functions or would his knowledge appear fragmented and rule based? The observation from this examination was reported in a conclusions section of the functions assessment.

### Secondary Analysis

For the primary level of analysis, each of the six students was treated and reported as a separate case study. For the secondary level of analysis, the data and conclusions for each student were compared. The purpose was to examine the case studies for apparent trends or contradictions. For example, were the students described as conceptual and autonomous in orientation also those who developed a well-connected understanding of functions? Was there any relationship between the designation as conceptual/procedural in orientation and autonomous/nonautonomous in orientation? Were there any students whose data appeared

**incongruent with the others? The observations from this re-examination were summarized and presented in the conclusions chapter along with a discussion of the classroom environment.**

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## CHAPTER V

### BELIEFS INTERVIEWS

This section of the data analysis presents and summarizes the student participants' beliefs about mathematics. These beliefs were gathered from the students' remarks and actions as they responded to the interviews. In this section each student is treated as a separate case study and within each case the data is organized as indicated in the following outline.

- I. Background
  - A. General information
  - B. Previous mathematics courses
  - C. Affective responses to mathematics
- II. General views about mathematics
  - A. Description of something unlike and like mathematics
  - B. Vocabulary lists for mathematics, English, science and history
  - C. Ranking grid
  - D. Synopsis of general beliefs about mathematics
- III. Student and teacher roles in learning mathematics
  - A. Description of a good mathematics student
  - B. Advice to a foreign student
  - C. Advice to a new teacher
  - D. Description of a good mathematics teacher
  - E. Summary of student and teacher roles
- IV. Conceptual and procedural views of mathematics
  - A. Discussion of conceptual problem protocols
  - B. Discussion of procedural problem protocols
  - C. Summary integrating problem protocols with general beliefs about mathematics

- V. **Autonomy and nonautonomy with mathematics**
  - A. **Discussion of autonomous problem protocols**
  - B. **Discussion on nonautonomous problem protocols**
  - C. **Summary integrating problem protocols with general beliefs about mathematics**

Keith

### Background

#### General Information

Keith, a 15 year old junior was enrolled in Algebra II/Trigonometry along with English literature, American literature, French IV, Spanish II, and U.S. History. In addition to his college preparatory course work, he was involved in French club, Granite State Challenge (a televised quick recall competition in NH), chorus, and auditions for the school musicals. He also worked as a clerk/stock person at the university book store. The job entailed minimal mathematics. The absence of the usual ready made chart occasionally required Keith to figure the mark down/up.

Keith had experience with computers, as did all the participants. He was exposed initially to computers and simple programming in elementary school. He used his home computer solely for word processing. He owned a scientific calculator but had used it only for basic computations in the Algebra II/Trigonometry class. During previous year's Chemistry class he frequently used the calculator to compute logarithms and exponents. From time to time he has read and enjoyed Games magazine but he preferred crossword puzzles.

Keith spoke enthusiastically of his love of languages, having taken French and Spanish concurrently. As a consequence, he wished to attend a college strong in languages, especially Russian and Japanese, and international studies. His ultimate career aspiration was uncertain although he was considering becoming an interpreter or working in an embassy; he even mentioned, somewhat teasingly, seeking an ambassadorship. Since Keith saw his strength as languages, "I excel in languages whereas some people do better in math and science", he freely acknowledged that he was taking mathematics merely as an entrance requirement for college. However, he stressed that he thought mathematics was important "for figuring taxes and stuff." He did not regard upper level mathematics courses as very useful except that they supposedly taught an individual logic. He described his own ability in mathematics as average,

his rationale being that since he was having difficulty in an advanced section he must be average.

#### Previous Courses in Mathematics

Keith had a varied background in mathematics. While in elementary school in Anchorage, Alaska, he participated in a gifted program from which he had vague recollections of doing mathematical brain teasers and problem solving. In 7th grade he was assigned to an advanced mathematics program with Mrs. Thomas as his teacher. Doing somewhat poorly in Algebra I in 8th grade, he retook it with Mr. Jones the following year, then progressed to Geometry with Mrs. Thomas. Finally, he moved once again to the advanced program with Algebra II/Trigonometry.

Eighth grade Algebra I. Keith had strong memories associated with his experience in eighth grade Algebra. He described being depressed and frustrated with the mathematics. He said, the year began well since it was somewhat a review from the previous year when he had been introduced to signed numbers. As the year progressed he became more and more lost, finally getting D's on his tests. It was especially difficult for him since in his other courses he was receiving A's. He attributed his lack of success to his own neglect of homework assignments. At the beginning of the year, he found the ideas mainly review, not requiring much effort. By the end of the year, he had diverted his energy into a major science project. He found it hard to recapture the momentum he had lost through his earlier neglect. He consoled himself with the thought that all was not lost or hopeless since he could retake Algebra I in the ninth grade. As a result of his experience in algebra, Keith has developed some very definite views on student/teacher roles and responsibilities. These will be discussed in a later section.

Ninth grade Algebra I. Algebra I in ninth grade proved more successful and he received A+'s. Keith attributed his success to his diligent attention to his homework and to the fact that it was a review for him. In fact, he found the course so easy, it was unnecessary to study for the tests. He said politely that he did not feel that Mr. Jones taught effectively. "He seemed to really goof off a lot in class. We didn't concentrate on math." He disliked the cartoons that appeared on their mathematics tests feeling that they were unprofessional.

Geometry. Keith described geometry as "not my thing" despite his B+ average. His grade would have been higher had he been more willing to participate in class. He

characterized himself as quiet in all his classes. In his geometry course he felt:

Some things just didn't make sense to me. . . . I had trouble applying them [formulas and theorems] to what I was given to work. I ended up understanding them but previously had really been told it's just like this equals, or this is this. I was never really told why this is this. Mrs. Thomas gave me more insight into that.

Keith drew an interesting distinction between the proofs in geometry and those in Algebra II/Trigonometry. He felt the geometry ones were easier saying:

In algebra you're just given these numbers and x's and y's and you're not really sure what they are. . . . Whereas in geometry it's part of something. . . . It was there, whereas numbers in algebra are just kinda floating out in space or just not there at all.

Algebra II/Trigonometry. At the beginning of Algebra II/Trigonometry he felt overwhelmed.

I really study for my tests . . . like for 3 hours. And I still come off with 83s. I just don't really see why because I'm doing OK on my homework. And I study my homework. I redo problems I've gotten wrong on my homework. I don't know, it's totally different.

He went on to describe his difficulties with the tests. Most of his mistakes were "dumb ones" and occasionally he had trouble with insufficient time, especially when word problems were included on the test. Keith said he had difficulty "applying things". When asked if he had similar trouble with word problems in chemistry class, he replied "no". He offered no explicit explanation for this dichotomy but referred to practice sets he had worked in chemistry class with the implication that these were comparable to the actual word problems on the chemistry tests.

### Affect

Although Keith was soft spoken and described himself as quiet in his classes, he did not seem intimidated by the interview process. He was, like the other participants anxious to be helpful.

Keith described his mathematical ability as: "I really don't consider myself really being that good in math. . . . I don't feel my ability is that great." He remarked later that he thought of himself as "average" since he was having difficulties in the advanced class. Despite his self reported difficulties, Keith exhibited a measure of self confidence while working several of the arithmetic problems in the interview protocol. He implicitly trusted his own calculations even over those of the calculator saying: "I guess I have a tendency to believe in some cases . . . my own work."



Keith confessed that:

I don't really like math. I want to do well, but it isn't as important to me as doing well in writing, something like that. . . . I can't think of anything that I really liked in math class. (laugh) . . . I mean some people like math, but I can't imagine sitting in class, I mean, I've never had a teacher that's made math fun.

Keith's response to mathematics appeared to be resigned acceptance rather than intense dislike.

#### General Views of Mathematics

A perspective on Keith's beliefs about mathematics generally was gleaned from his responses to several instruments and questions: (a) name something unlike/like mathematics, (b) vocabulary lists for mathematics, english, science, and history (see Appendix H), and (c) ranking grid of various mathematics topics (see Appendix J). Data from each instrument will be presented individually followed by a synthesis of the responses.

#### Unlike/like Mathematics

Keith when asked to describe something unlike mathematics gave the imaginative answer: "The colors blue and green". He offered the explanation that he associated colors with academic subjects : blue with French and American literature, green with science and English literature, and red with mathematics. To him, blue and green represented soft colors which illustrated the idea that you can express yourself easily since there are no definite boundaries to those colors. Red, the color of mathematics, is bold and bright and so:

You have to notice it. It's hard. Mathematics seems to leave no room for error. It is there. The answer is there. There is no difference . . . in opinion. Everywhere in the universe it has to be this.

This view of mathematics as being rigid, unchangeable, and immovable is further illustrated by Keith's response to the companion question: What is the most like mathematics? His straight forward response was "A BRICK WALL". "It is there and you just run into it. There's the answer. You can't go away from it. . . . It's there. You can't move it." He attributed part of this perception to his feeling of frustration associated with his difficulties in solving word problems.

#### Vocabulary Lists

Mathematics. Keith further reiterated and expanded on the theme of mathematics as being right or wrong and rigid in his selection of words from the vocabulary list [see Appendix H]. He chose and explained:

*rigid.* You have to get this answer or else you're wrong. You can't go away from that. The answer is always there. The formula's always the same.

*controlled*: You use a specific formula to get specific answers. It's really hard to get away from that formula.

*absolute*: There is no room for error. You're always gonna get the same answer no matter what. Other people aren't going to get different answers or else they'll be wrong. The answer is going to be known.

He continued the previously mentioned idea that numbers were not real entities by saying: "[mathematics is *abstract* since] numbers don't really have a place. I mean they're not really there. They're not objects. They're not concrete and they're just kinda anywhere."

When Keith was asked if he thought there could be indeterminate mathematics problems, he replied yes citing 9/0 as an example commenting that "We don't know how to do that yet. I guess people eventually [will] figure out how to sort of tell me what it is." The emphasis still was on the answer existing though in this case it was not yet revealed.

In his first pairings of terms Keith connected *cause and effect*, and *trial and error* saying:

You try something and by error you figure out what it is. Where trial and error cause you to try something to get an answer, so cause and effect, and trial and error. Deductive also seems to go with those too.

Again Keith focused on getting the answer although here he suggested that the process had a basis in deductive reasoning.

Keith acknowledged in some of his groupings a view of mathematics as composed of *logical and truthful facts* and *well-defined rules*. He observed that: "Logical and factual seem to go together and truthful because if it's logical it's based on a fact and therefore if it's a fact, it would supposedly be true." A mathematical fact was "any of the properties like  $a + b = b + a$  and  $a(b + c) = ab + ac$ ." Keith also connected rules and well-defined saying: "[They] seem to go together because math rules are well-defined like  $a + b = b + a$ ."

Keith grouped the terms *practical, useful, and routine* indicating "if it's useful or practical you'll use it, so it becomes routine." Another interesting combination that emerged was that he elected to circle *clever* but not *creative* as applying to mathematics. He felt "You needed to be clever to succeed in math", that is, "to use your brain" but that mathematics was not creative in the sense of using your imagination. He illustrated his point with a story.

There are a lot of people that really are good in math but in English they just can't do well. Like my father was telling me about somebody who in his high school was just a whiz kid in math but he was failing English. And he ended up . . . leaving as a sophomore in high school [to go] to MIT and did his doctorate on some formula or something there. So, I mean, I guess, you can be creative in math that way, meaning that you can explore different mathematical formulas and things such as geometry or

algebra. But I don't think you can really be imaginative and think like that.

To Keith mathematics was *individualistic* meaning mathematics was a solitary activity: "[You do] not pool your knowledge with other people". He also believed that it was *experiential*: "You better yourself in math through experience."

Keith did not circle memorize as associated with mathematics. When asked to comment on this, he remarked that although to a certain extent ".5%" of his mathematical thinking entailed memorization of formulas, he associated memorization more strongly with foreign languages especially learning vocabulary.

English. Keith's view of English offered a sharp contrast to his view of mathematics. He grouped such words as *opinionated, open ended, free, creative, thought provoking, and theoretical*.

Those go together because you can have difference of opinion(s). You can find different things in life that other people might not find and nobody's idea is right. It varies depend(ing) on the person you are.

In contrast to mathematics, English was *beautiful and elegant* as well as *expressive*, dealing with *ideas* and offering a *creative* outlet for the individual. English, for Keith, also required *analysis and insight* to understand the *symbolism and abstraction* in writing.

Science. Keith's view of science was one of a discipline filled with *controversy, uncertainty, and trial and error*. In science, theories constantly changed over time while science simultaneously demanded precision and *organization* to carry out its experiments. When asked if mathematics also was changing, he replied: "This is not true with mathematics. You are just told this is so and has been since the Greek philosophers decided it." He opted to include *valid* in his selection of terms relevant to science but not truthful. The distinction for him focused on the permanence of the ideas.

The answers that you get are valid for the time that is now. I mean, maybe a 100 years ago somebody could say, well no, this isn't right. But now since things have changed you can say, yeah, this answer is valid. Truthful [means] being right all the time. Throughout history, it's always the same. Science hasn't always been the same. It hasn't always been made up of absolute truths.

For mathematics, however, both terms applied since "The answers are always the same throughout time."

After this exchange, the researcher then posed the possibility of a non-euclidean geometry. Keith replied that his experience with measurements led him to believe that the

euclidean assumption of the angle sum being 180 degrees was true and that the other theories were not plausible. Still another contrast with mathematics was evident with his choice of *visual, beauty, discovery, and exciting* as descriptive of the excitement and sense of discovery he had felt when doing a chemistry experiment which yielded crystals. Note that none of these terms were selected as associated with mathematics.

History. For Keith, history had a sense of stability lacking in science while having a sense of *beauty* not revealed in mathematics. The stability he described with the words *fixed, factual, organized, factual, truthful, and valid* saying that "You can't change history." According to Keith, history did not have the subjectivity associated with English. Historical events were facts which were unchangeable.

Summary. In summary, Keith presented a view of four qualitatively distinct disciplines. Mathematics represented the most structured and unyielding of the disciplines, its rules and procedures being fixed and its answers being right and wrong. Mathematics was described as neither creative, beautiful, nor exciting. In science, theories were controversial. Experiments were viewed as highly organized and detailed, emphasizing the structural and rigorous nature of the changeable discipline. Science could be beautiful and exciting. Characterized by creativity and beauty as well as abstraction, analysis, and symbolism, English was seen as a discipline through which writing expressed the variability and depth of the human experience. Finally, history was perceived by Keith as a stable, factual and organized discipline whose writings could be beautiful.

#### Ranking Grid

According to the ranking grid, word problems and proofs emerged again as difficult topics for Keith (see Appendix J). Keith ranked word problems strongly in the following categories: *interesting, applied, useful, flexible, original thinking, advance, thought provoking, hard to do, difficult to learn, worst at, arbitrary, least liked, and confusing*. He stressed that: "Word problems at least they set up a situation that's interesting. . . . It gives you a situation that you can apply." Keith's responses indicated a dual impression of word problems. First, he valued their application both for its utility (*useful and applied*) and for the *interest* it incited. Second, the complex nature of word problems in turn made them *difficult to learn, hard to do, and confusing*. For Keith the real world application made word problems one of the most

interesting topics in mathematics while simultaneously being the least liked due to its difficult nature.

Geometric proofs were ranked strongly on the following categories: *original thinking, logical, advanced, thought provoking, flexible, boring, hard to solve, difficult to learn, worst at, least liked, and confusing*. Of proofs he said he saw:

No reason for proofs since you'll never use them again. I mean people say they're supposed to improve your logic, but I mean, I don't see this as improving my logic at all.

The dichotomy observed in Keith's responses to word problems was not present in his ranking of proofs. Here the complexity (*thought provoking, original thinking, and advanced*) and the apparent lack of usefulness of proofs gave rise to a strong negative response both affectively (*boring and least liked*) and cognitively (*hard to solve, confusing, and difficult to learn*).

He did, however, like and enjoy geometric constructions "Because you have a shape, you have something there, it's a picture." Keith also commented on factoring and solving equations indicating that completing them gave him "a sense of accomplishment".

Summarizing, Keith's ranking centered on word problems and proofs. In this segment of the interviews, as well as in subsequent conversations, Keith expressed his frustration and confusion associated with these two topics. Word problems represented both a source of interest in mathematics and an area of difficulty. Proofs, on the other hand, were difficult and confusing while lacking the affective motivation of interest and utility. Both were perceived as a challenge requiring original thinking to solve them yet offering multiple (*flexible*) approaches to the solution.

#### Synopsis of General Beliefs about Mathematics

Keith's description of mathematics as synonymous with a brick wall revealed a belief about mathematics as rigid in process, unchangeable over time and impermeable to human emotion or expression. Mathematics was out there, separated from the individual, forcing the individual to conform to its structure in order to get *the* answer which existed for all problems. Since mathematics was valid and truthful over all time, it was devoid of the subjectivity often associated with human opinion. In addition to stability through time, mathematics was considered as universal across cultures. The lack of individual interpretation and the need for conformity precipitated a view of mathematics as absent of beauty and creativity.

Keith viewed real world/application problems as a key motivational factor. Word problems were associated with the dual traits of utility and difficulty. However, he saw little constructive use for proofs or upper level mathematics, not accepting the argument that they taught logical reasoning. The abstraction in algebra was seen as confusing almost to the point of senselessness. For Keith, mathematics was a discipline that originated outside himself and that demanded unyielding conformity in order to reach the answer which was often irrelevant to him.

#### Student/Teacher Roles

Keith's views on mathematics learning were explored through discussions on the following: (a) his characteristics of good mathematics students and his explanation of how and why these individuals became good at mathematics, (b) his advice to an imaginary exchange student on how to succeed in a mathematics class in the United States, and finally, (c) his advice to beginning teachers and his expectations for a good mathematics teacher.

#### Good Mathematics Student

Keith's perception of a good mathematics student was one that "seems to always come off with 100's . . . They just seem always [to] be able to understand what's going on." His explanation for what makes them good was one that partially stressed developmental, experiential and affective issues.

I'm not really sure. Maybe just the development that has occurred in the last seven years or so in math. It's just that [mine] hasn't developed [his math ability] whereas since I'm writing . . . my left side of the brain is developed more. They [the specific individuals whom he saw as good at math] also seem to be good at writing. Maybe they have just learned from experience.

He continued his remarks by saying:

They seem to like math. They want to do well, whereas I don't really like math. I want to do well, but it isn't as important to me as doing well in writing.

Keith also held the belief that anyone could be good at mathematics:

If you put enough effort in, you can be as good as you want to be. . . . But if you really want to do well and if you try your hardest, I think you can probably come to [that] point.

Thus for Keith, being good at mathematics meant having the desire to do well along with a liking for mathematics and a willingness to try hard and secondly, the development of mathematical skills over the years. This ability manifested itself in 100s on tests.

### Advice to an Exchange Student

Keith offered advice to a foreign exchange student on listening in class, taking notes, doing homework and taking tests.

I would tell him to listen carefully when the teacher is speaking. . . . And take notes on everything that's on the board and write down everything in English and when you . . . get home . . . you can translate it to another language. . . . Write down everything, like I can usually tell what's important and not important. I mean, math is easily, is like world wide, . . . so you should be able [to tell]. She [the teacher] points things out, like when she starts doing work on the board. . . . It's just the basic concept. Like she does like four problems using the same concept. . . . I guess I'm telling him to pull out the basic concept from all the problems.

On homework, Keith advised:

Do all the homework, even if you don't understand it. Write down something or else you'll end [up] getting to a test and not understanding anything. . . . I did that in algebra in eighth grade. I just basically slacked off toward the end and I was like 'I'm failing anyway, why even bother.'

He continued by saying even when stuck on a problem:

Write down whatever you can from the problem and like go as far as you can and then ask that question in class or [of] other people who know how to [do it] or the teacher or whatever.

With regard to tests, Keith suggested that the exchange student:

Budget his time and if he came to a problem that seemed to be taking a lot of time skip it and go on to the next one. Just try to make sure that you leave enough time for problems that you have trouble on. Whether it might be word problems or any other kind of problem.

When asked how to prepare for a test Keith confessed:

I really never studied for [them]. That's really horrible but this year has been the first year that I've actually even attempted to try to study for a test. I guess I'd tell him to do problems on all the self tests from the book and all the chapter tests. . . . And take like a problem from each night's assignment and try to do it.

Keith explained further:

I mean, studying for a math test just seems like it's not worth it because . . . it's not like seeing a vocabulary word and knowing that this is what it's going to mean. . . . Well like maybe 15 minutes I [would] go over a concept and say OK. Or proofs, . . . I'll like study theorems, axioms and stuff, properties. But otherwise, I mean, you can come in or you can study. Like well this says  $3x + y = 7$  or whatever, and [you] go to the test and you're gonna have different problems. It just doesn't make any sense to me why you would study different problems.

### Advice to a Beginning Teacher

Keith had less advice to offer beginning teachers. He basically felt that learning was a matter of his own attitude.

That's really hard. I guess, I'm really not sure. I'm trying to think of what Mrs. Thomas

taught us since I had her in 7th, 10th, and 11th grade. For my own teachers, I don't think I could really give them any advice that I think would really help me. Since I think basically my own way I think about math. . . . Like if I go to math class with a positive attitude then I'll come out with a better grade than if I go in and say 'oh this is math' and I just don't pay any attention.

When asked what the teachers could do to help create a positive attitude in the classroom he responded:

Like math class is hard to make exciting. I can't imagine having an exciting math class. . . . Anything they can do to make it more interesting than just sitting [there] . . . watching somebody write things up on the board. . . . Half the time I'm not paying attention because I'm just so tired of thinking of all the things I have to do that night. . . . It just isn't really interesting, I mean, I'm not gonna say 'Oh wow  $3x + 2y = 7$ .'

To avoid creating a negative attitude in the classroom, Keith felt that teachers should:

If you [the student] asked for a problem and nobody else needs the problem, I think you [the teacher] should still go over it or at least she should go over it with you like on your own or something.

Recalling his experience in algebra in eighth grade, he suggested that teachers need to take the initiative with failing students.

I think that if you're really doing badly in a class. Like I was in algebra, they [the teachers] shouldn't force you but she should make you sit down and go over your tests, go over your quizzes. And say 'OK, I think this is what you're doing wrong.' . . . Try to make it as clear as possible how to do it.

#### A Good Mathematics Teacher

A good teacher, for Keith, was one who was:

Creative in her or his way of teaching, [that is] by making it interesting--by associating like one problem to life. Like something that you find in actual work or life or something like that. I mean even to describe chemical formulas would be more interesting than seeing  $2x = 7$ . [Also the teacher should] know her or his students well enough to be able to see when they're having problems and helping them on those problems so that they can succeed.

Keith, in addition, maintained that tests should be constructed by the teacher to reflect problems covered as homework:

I don't think I can deal with that [unfamiliar test questions]. I think that it would be too bazaar. . . . Too bazaar seeing something and we never did anything like this. . . . If I wasn't shown before then, there's no way I could be able to tell them. I think he [the teacher] should probably discuss those areas first before putting them on the test.

The argument was then put forth that if the student understood the underlying concept then he or she should be able to extend it's use to unfamiliar problems. Keith responded by indicating:

That's the way it should probably happen in math class. I know a lot of times I can't understand what's going [on] but I can come off with a good grade because I understand



what problems we're doing. I mean, I have the basic idea of what's going on. . . . If she had never shown us how to [do] it . . . I couldn't handle it unless she scaled the tests. . . . I would do that [the extension problems] as part of something like math club, or even just as an exercise in class, not as an actual test.

Keith was, however, more flexible with his expectations on homework. He remarked that:

My own teacher breaks down what pages we have to read. I usually don't read them because . . . she goes over them in class. But if she has assigned a homework thing [problem] that she didn't go over in class, I should be able to read them and understand them, read the description of how to do it. So no, I think that if she didn't go over it in class, you should still be able to figure it out yourself.

#### Summary of Student/Teacher Roles

Keith had an underlying belief that anyone could become good at mathematics provided they were motivated, liked the subject, and worked hard. He also expressed the opinion that experience affected one's ability. He felt strongly that students should attempt all homework assignments, asking questions of other students or the teacher when stumped. He also advised taking thorough class notes concentrating on information the teacher wrote on the board. Advice on testing, however, was somewhat contradictory. In regard to test taking, he offered the seasoned suggestion of skipping problems that were difficult although saving sufficient time to return to them later. Studying for tests represented a dilemma for Keith. He mentioned concrete actions such as attempting chapter reviews in the text and selecting sample homework problems, but he personally struggled with the utility of studying at all. Keith questioned reviewing specific problems when the actual test questions would be different.

Keith's expectations for teachers was not readily forthcoming. He did suggest that teachers need to be more creative in their teaching of mathematics and should show more applications of mathematics relevant to the students' lives. However, he did not hold out much hope since he had never had an interesting mathematics class and, in fact, could not imagine one. Keith was adamant that teachers should be aware of individual's difficulties and initiate help sessions when a student's grade was in jeopardy. He also felt strongly that no question should be turned aside because only one student had asked that particular question. Regarding tests, Keith explained that he believed they should reflect homework or examples presented by the teacher. He was more flexible with his expectation on homework assignments. He indicated if the topic or technique had not been covered in class, that it was reasonable to expect students

to be able to learn the appropriate procedure from the textbook readings.

#### Conceptual/Procedural View of Mathematics

The analysis of Keith as holding a conceptual or procedural belief about mathematics was derived from two sources: the general mathematics overview and the coding of Keith's responses to the mathematics problems in the interview protocol. (See Appendix I.) Of the 14 problem episodes that were codable for Keith, 9 were labeled conceptual and 5 procedural. Keith's ranking of 64% conceptual ordered him fifth among participants in this category. (See Appendix C for a listing of the coding criteria and Appendix I for a table summarizing the participants' coding on the conceptual/procedural and the autonomous/nonautonomous categories.) A cautionary word needs to be given about making inferences from this table. The numerical counts used within each of the coding categories represent the total number of problem episodes which were coded as indicative of that category. However, these percentages do not reflect the variation among students' responses. For example, one student may have given a very complete and rigorous justification for their process thus revealing a well connected network of conceptual ties while another student with prompting may have given only a brief rationale showing only one conceptual tie. Both would have been coded as having demonstrated conceptual ties. The researcher noted such distinctions and recorded them on the coding sheets with the demarkation of C+, C or C-. These markings reflected the completeness of the arguments the students gave and to a lesser extent whether the information was given spontaneously. It was felt that reporting these finer codings would unduly complicate the presentation of the data. However, mention will be made of these variations in responses when presenting the secondary analysis of the 6 students.

Keith's problem episodes were, as indicated, coded roughly 60/40 conceptual and procedural. Keith's episodes proved particularly difficult to code due to his reserved nature. He would rarely volunteer affective information and would in fact become quieter when confused or stumped. Of the codable episodes, the conceptual and procedural coding split along those problems that represented basic arithmetic and algebraic procedures and those that entailed problem solving and grading the sample test. That is, Keith was able to provide an intuitive justification for the basic arithmetic and algebra problems and hence received a conceptual coding. However, on the problem solving problems (#27, #35) he relied on unmonitored trial and

error and execution of formulas to attempt the problems and thereby suggested a procedural coding.

### Conceptual

Keith received a conceptual coding on many of the basic arithmetic and algebra problems by providing intuitive justification for his answers. For example on problem #21 ( $1.50 \times .25$ ) Keith was asked how he knew the answer was .375 instead of 3.75. He responded:

Because you're given four numbers after the decimal point so I moved the decimal point over four spaces. [Why?] I guess it works cause you have one quarter times one. [pause] Yeah, like a quarter of one and a half will be .375. So it will be less than 1 because you're taking a quarter of that number.

Again on problem #26 (20% of 85) Keith offered some justification for the reasonableness of his answer. After completing his work, he was asked if his computation had yielded 32 instead of 17 would he have believed his answer. He replied:

No, because 32 seems to be more close to 50% or 30% or something. Since 32 times 2 is 64, 32 times 3 is 96, so it would be closer to something like 50% or 30% somewhere around there. . . . It's too big a number to me. If I had punched [the numbers] into my calculator [and] I got the answer [32], I'd do it myself to find out if I got the same answer.

Although his estimation was fairly rough, he used it to ascertain the appropriateness of the value, in this case, rejecting it.

On problem #28 (larger of  $543 \times 29 / 32$  and  $(30 \times 543) / 28$ ) Keith demonstrated flexibility in his approach. He first evaluated the two expressions using the calculator. When asked for an alternate approach, Keith noted the similarity of factors saying:

The 543 is the same in both. . . . This is gonna be less than that since 30 times 543 is going to [be] greater. [pause] And 32 dividing or the lesser number divided by 32 will give you even a lesser than the greater number divided by 28. So, yeah, by just looking at it I probably could have done it without the calculator.

Thus on this problem, Keith could provide multiple solution techniques and one of those techniques made use of the relationship between numerators and denominators.

Keith also showed flexibility on problem #24 (Find the least radical fraction). He began here by approximating square root of 5 as 2.2 and then computing the decimal equivalent of each fraction. When asked for other techniques he used approximation and comparison of fractions similar to that described above to reaffirm his choice.

These four examples were representative of the type of problems for which Keith received a conceptual coding. In each of these, he demonstrated either a flexibility of solution

of technique or an intuitive rationale for the reasonableness of his answer.

#### Procedural

Of special interest was Keith's rationale when grading the sample test which was coded procedural for several reasons. First Keith focused his grading primarily on the familiarity of form and the correctness of the answer rather than on the validity of the mathematical process. Secondly, Keith lacked consistency in his grading policy.

Keith revealed his attention to form rather than process on several problems on the sample test. He remarked frequently that the student, Chris, "didn't do the right work" indicating that Chris was not giving the expected steps. For example, Keith on problem 11a deducted 5 points claiming:

Oh that isn't supposed to be there [referring to the adding 4 to both sides]. . . . And he got this stuff, but he screwed up right there. Well, that's right even though that isn't. [pause] I would take off 5 points since he didn't really get the right amount of work [underline added], so he did give the answer at the end. It basically came out the same. So I took off 5.

To verify that Keith was indeed evaluating on form rather than a belief that the step was illegal, he was asked if the suspicious step was in fact incorrect. Upon reflection, he reconsidered the problem and changed his marking to correct. He remarked that:

Well actually. OK, I understand. . . . We always like, [it's] how I've learned all the time. We subtract 4 from both sides, but he added 4 to both sides which will give you the same answer and which I guess is correct. . . . You could do that. . . . You [would] have an equality on both sides. So, yeah. . . . I mean, that seems correct to me now.

Keith after reconsideration agreed that the step was valid but his initial impulse was that the step was wrong because it was not the standard step that he had been taught. It was not what he was accustomed to expect.

In problem 1d Keith commented on both an initial error and the lack of "the right work".

Keith stated:

No that's wrong. . . . You can't cross off 6 and 12 because you can only do that in multiplication, so that's wrong even though this comes out correctly. You would have another step in here,  $1/6 + 6/6$  would be  $7/6$ , but he didn't even do the right work [underline added]. He didn't do it correctly. He got the right answer but didn't do it correctly.

In these two problems, Keith stressed that the problem was wrong due to steps that were either contrary to accustomed form or missing from the accustomed pattern.

A second incidence coded as indicative of a procedural view of mathematics was Keith's

reliance on the correctness of the answer as his judgmental criteria. In problem 11c, Keith checked the two given solutions, found them incorrect and then concluded: "So it's wrong. So I would have to [take off the] full . . . points [-10]." When asked what Chris had done wrong, Keith replied "[The] -3 doesn't equal 12. . . . It would have [to] be no solution then. Empty set, no solution." Keith then commented that he was not sure he would have gotten it right. He continued with

I don't know. I probably would have written down no solution. I don't know if that's right or not, but this person is absolutely not right cause I know that is not the right [answer] [underline added]. So, yeah, I'd take off the whole 10 points.

Keith based his judgement on the fact that the answer was wrong, giving no partial credit for the correct process. That Keith was capable of making such a distinction between the solution technique and the interpretation of the answer was evident by his grading on 11d. On this problem which also had a correct solution but an incorrect interpretation, Keith stated: "That's the right answer [referring to the interval]. . . . So he got that far, he just got the wrong answer, so I'd just probably take off 2 points." Since on 11b Keith failed to acknowledge the process, Keith appeared to rely solely on the answer as the basis for his judgement.

The third way in which Keith demonstrated a procedural view on the sample test was his overall inconsistency in grading. Keith's grading technique reflected an unawareness of problem similarities and resulted in grading problems as isolated entities. His grading technique provided another example of his procedural thinking because the isolation of problems suggested a lack of reflection on the evaluation process and a view of mathematics as judged on arbitrary standards.

As mentioned above problems 11b, 11c, and 11d were graded inconsistently although they represented parallel situations having correct and familiar solution techniques with incorrect interpretations. Keith also marked problems 1a, 1b, 1c, and 1d on a shifting scale. Problem 1a was graded as completely wrong (-6 points off) not only because of the incorrect answer but also because of the absence of any written work. On 1b, the initial step was seen as wrong but Keith granted 1 point for showing work. However, this policy did not extend to 1c and 1d. There the initial error negated any subsequent correct steps and so were marked completely wrong (-6 points).

Summarizing, Keith was coded as procedural on his evaluation of the sample test

because: (a) he graded on the basis of form rather than process, (b) he graded solely on the correctness of the answer, and (c) he graded inconsistently. These actions constituted a procedural view of mathematics since they stressed the right/wrong nature of answers rather than the concepts underlying the problems.

Another area in which Keith was coded as procedural was on the problem solving episodes. On problem #35 (the rope problem), Keith attempted to solve the problem by directly applying the circumference formula without aid of a diagram. This application led him in a type of circular computation that yielded again the length added to the rope. He made no attempt during the calculations to reflect on the reasonableness of these computations. When he arrived at his result of .001 miles (the length of added rope) he claimed that this represented the distance off the ground. When he recognized this value as numerically equivalent to 6 feet, he abandoned the computations and said that:

Six feet is such a small amount, I think that you would be able to fit a piece of paper, but a mouse couldn't crawl through it or a person couldn't walk under it. Is that right?

Keith's problem solving technique portrayed a blind reliance on the execution of a formula. He showed no monitoring of his solution technique to check the reasonableness of the result or the appropriateness of the computation he was executing. Keith's dialogue on this problem also showed a lack of any initial reflection on the meaning of the problem. His only question concerned the conversion units between feet and miles.

On problem #27 (the remainder problem), Keith again was coded procedural. Here the coding was given because of Keith's unmonitored trial and error process. Keith persistently checked a string of numbers: 7, 12, 13, 17, 19, 21, 22, 23, 26, 27, 28, 29, 30, 31, 32, 33, 34, ... , 52. Some of these he checked for the respective factors of 3, 4, and 5. Others he dismissed because of recognition of related factors. After 52 he jumped to 100, 102, 103, ..., 109 then 202. At this point he asked if the number was less than 1000. It was suggested that he consider 14. He complied by checking each factor. After examining this number, he still generated no hypotheses and turned again for help. This time he was asked to look at 62. He verified that the value satisfied the initial conditions but offered no reflection as to why this value worked. Keith on this problem followed a sequence of trial and errors. During the process he did not stop and attempt any generalizations or look for any patterns in the sequence of

remainders. For him, his solution technique consisted of a persistent testing of values which constituted a procedural view of mathematics because of the emphasis on the validity or invalidity of the answer as the driving force in the solution process. This persistent testing of values was done at the expense of searching for patterns and generating hypotheses. On these two examples of problem solving situations, Keith was coded procedural due to his exclusive reliance on the execution of a formula and unmonitored trial and error.

### Summary

Keith had a mixed perspective on mathematics. Many of his beliefs about mathematics and problem solutions suggested a procedural view of mathematics. Recall that Keith described his own inability to apply ideas and to solve word problems. Joining this with his views that one "uses a specific formula to get specific answers" and that one needs "to get this answer or else you're wrong" gives affirmation to what was observed in the problem solving episodes. There Keith relied extensively on the execution of a formula in his attempt to solve the rope problem. On the remainder problem, he focused his attention on the correctness of each trial number. On these problems he seemed to emphasize finding the appropriate formula for the problem and getting to the right answer.

Again in the grading of the sample test, Keith's verbal description of mathematics was reinforced by his actions. Keith marked several problems on the basis of familiarity of form which coincided with his description of mathematics as being controlled, absolute, fixed, mechanical, routine, and rigid. He also expressed the opinion that mathematics tests should be similar to homework problems. Thus, the expectation was set up that the answers and solution techniques should be parallel to what was known or familiar. It was also noted on the sample test that problems were graded right or wrong based exclusively on the answer and not on the proceeding steps. This again matched with his view that one needs "to get this answer or else you're wrong" and that mathematics was right or wrong.

Keith's verbal descriptions of mathematics as conceptual were less apparent. He had circled the terms cause and effect, deductive, logical, common sense, analyze, and rational suggesting that mathematics was logical and could be somewhat understood by the individual. He also spoke of "pulling out the basic concept" from examples in the teacher's lectures. These verbal descriptions again agreed with what was seen in the problem solutions. When asked to

justify his answers, Keith provided some intuitive rationale for algorithms and offered multiple approaches to problems. He had developed for himself a logical rationale for some of the arithmetic and algebraic algorithms. Thereby, Keith demonstrated a belief that mathematics was interconnected and based on deductive processes.

Keith's conceptual view of mathematics was significantly tempered by a pronounced belief that mathematics was driven by the need to find the correct answer. It was this need to find the correct answer that received precedence in Keith's problem solving episodes and in grading the sample test. When the going got tough, so to speak, Keith emphasized getting the answer. The more dominant belief, the one that Keith relied on in unfamiliar and new situations was the procedural view. Recall that the psychological and mathematics education literature argued that the most dominant or central beliefs are those adhered to in new or confusing situations (Cobb, 1986; Confrey, 1981, Green, 1971; Rokeach, 1968).

Although Keith held a conceptual view of mathematics, he also demonstrated a strong and central view that mathematics was procedural in nature. However, this type of apparent contradiction between beliefs is not unexpected in belief systems. As Kelly (1951) noted, all beliefs have a domain of convenience, that is, a domain in which the beliefs operates or has influence. For Keith, the domain of his conceptual beliefs was familiar arithmetic and algebraic algorithms while the procedural beliefs operated in the domain of unaccustomed problems or problem solving situations.

#### Autonomy/Nonautonomy with Mathematics

On the codable problems, Keith received a 40% autonomous rating. With this rating, he again ranked fifth among the participants. Keith willingly answered questions but he rarely initiated or volunteered information. Thus, the number of actual problems that were codable was limited. His reservation made evaluation of autonomous actions difficult since the criteria for autonomy stressed independent action. An example of this type of dilemma can be seen in the voluntary checking of answers. That Keith was capable of checking was evident in his protocol to problem #25, but the checking was initiated by request, not voluntarily. Since coding of episodes as autonomous/ nonautonomous was based on positive incidence of the category and not on lack or absence of evidence, a lack of checking was not coded as nonautonomous but rather received no coding.



### Autonomy

Evidence of Keith's autonomous coding came from problem episodes in which he (a) verbalized the expectation that problem solutions should be justifiable and (b) monitored the reasonableness of answers. On #25 (find Sandy's age), Keith quickly set up and solved the algebraic equation. When asked if he believed his result, he replied by checking his answer in the equation. Keith was then asked what would happen if the original equation was wrong. He eventually replied "If that happened then I guess you wouldn't really be able to know that it was wrong unless you went over the original equation again, just to determine it." He further indicated that:

With word problems . . . I'm not usually right. But when I look over it and by reading it and by making the equation, it makes sense to me. . . . If I go over it again, I usually come out with the same equation, and so it seems right to me.

In this dialogue, Keith expressed the idea that he expected the written statement of the problem and the algebraic equation to coincide. Although in general, he felt uncertain about his ability to solve word problems, he placed faith in his interpretations and solutions to problems when a second reading produced the same result.

In problem #30 (find the area of the triangle), Keith again offered a weaker statement of this self-reliance. Here when asked if he believed his answer he remarked: "I guess it's [the answer] all I have to believe. I mean, it's hard for me to find another answer. I mean, from what I've looked at." Keith believed his answer because he could think of no conflicting interpretation. Belief in the validity of a solution due to the absence of alternatives was not unique to Keith. This phenomenon was likewise observed in Tom's protocols.

Keith also monitored the reasonableness of his solution in problem #30. He began by selecting trial values for the various unknowns and then evaluating them against the problem criteria. Next, he moved to representing the unknowns by variables, and finally to setting up and solving an algebraic equation. Throughout he carried on a dialogue with himself about the reasonableness of his technique.

But that doesn't make sense. . . . I'm trying to think. . . . Needs height. . . . And the height of the triangle is going to be the same. . . . I don't know what the height is and I don't know what the length of the bases, I can't really. . . . But, I'm not really sure. . . . You'll get base times height equals 16 for the little one, and base times height equals 8. So it would be 16? Would that?

Concurrent with Keith's autonomous actions was a lack of confidence in his results.

This was evident in the above dialogue when Keith asked for confirmation of his answers. Keith accepted his solution but seemed unwilling to invest it with much credibility. This lack of confidence in his often correct mathematics appeared also in his protocols on the function unit, examples of which will be given in the separate discussion of those protocols.

#### Nonautonomy

Keith's nonautonomous coding resulted from several episodes in which he (a) looked to the researcher for affirmation of his answers and for hints, and (b) justified algorithms on the basis of the teachers' authority. As noted earlier, Keith relied heavily on the researcher for direction on problem #27 (the remainder problem). At one point, he even stopped to ask if a solution existed.

Another indication of nonautonomy occurred on problems #20 (6 divided by  $\frac{3}{8}$ ) and #23 (fractions greater than  $\frac{3}{4}$ ). Here Keith provided no intuitive justification for his procedures. He remarked on problem #20: "It's just that [is] the way I've been taught. . . . I really don't have a definite answer for why." When asked on problem #23 if he could have solved the problem other than computing and comparing the decimal equivalents of the fractions, his response was "I don't even think I really did [it] any [certain] way. I'm sure there is another way, since I just basically guessed." On the sample test as well, Keith relied on the appearance of learned procedures as a primary guide to grading the problems. Here again, he abdicated the validity of the mathematical process to the execution of rules and procedures he had been taught.

Ann

Background

General Information

Ann, a junior, was enrolled in Algebra II, U.S. History, American Literature, Spanish I, and Chemistry. In addition to her college preparatory program, Ann was active in cross country, competitive swimming, and track. She felt these activities, added to her school work, left little additional time for a job, other school organizations, or hobbies.

Among the resources available to Ann were a personal computer and school owned software. She used her computer primarily for word processing but also utilized programs to design greeting cards. Occasionally she availed herself of a school-owned chemistry software program to practice balancing equations. Although she owned a scientific calculator, she used it only infrequently for calculations in chemistry class. Ann enjoyed and sporadically read brain-teaser and puzzle books.

When asked to describe in what ways, if any, she used mathematics outside of her classes, Ann mentioned mentally computing running times and lap speeds in meters and in yard, logging these times and subsequently scanning the data for trends.

Although Ann planned to attend college, she was uncertain as to a specific major. She liked mathematics and had considered it as a possibility, yet when she compared herself with her two older brothers who were "mathematically oriented" she felt inadequate. Alternately, she considered a major in a foreign language since she felt adept at memorizing. Despite her uncertainty, she indicated her intent to enroll in college mathematics regardless of her major or college requirements. She added the proviso: "Unless I have a bad experience in math these next couple of years."

Among the participants, Ann was unusual in her acknowledgement of familial influences. She not only evaluated her own ability in mathematics against the abilities of her brothers, but went on to describe her family as "into mathematics."

I think math would be hard to major in. I've heard stories. One of my brother's is a senior at Princeton this year and he was going to major in math. And he was taking [pause] linear algebra. And he went through a quarter and he said he had to switch to pass. He had gotten an A in calculus here and then he went there and . . . he placed out of all the math program, except this one. He ended up having to take it pass/fail so he couldn't major in it. So now he's majoring in engineering. And then my other brother is probably going to end up majoring in math or economics. . . . I just look at both of

them and they're having trouble with math. I'd hate to see what I'd be like, cause I'm definitely the, I don't want to use the word dumbest [laughter] of my family. They're definitely both smarter than me in school.

My brothers [have] both been on the math team. They're both very mathematically oriented. My mom's a chemist, [and] my dad majored in math. He taught math. . . . Like everything I do I need to figure out mathematically [underline added]. Like I finish running a race and I go immediately and I figure out what my pace was per mile. . . . I'm always fooling around with numbers. And then with swimming, we swim in yards and then we swim in meters and I want to . . . convert them both back and forth to see how they equal out. . . . I think a lot of people are math oriented. . . . It's [not] like my parents . . . [said] "you have to do well in math and math is going to be your life." It's not like that at all. It's just that I was brought up like that. And I like math a lot. It comes easiest.

### Previous Courses in Mathematics

Algebra I. Ann felt she had a good experience with Algebra I. The content material was clear to her and she found the teaching methods of Mr. Brown compatible with her own learning style.

We thought along the same wave length, so I could understand what he was talking about. . . . I would go in there each day and I would know what was going on. And that's why when I filled out the survey [the Fennema-Sherman Mathematics Attitude Scales] [I indicated that] I liked math so much. I have had two good teachers [Algebra I and Geometry] right in a row. They've given me confidence in my math.

Geometry. Ann felt her geometry teacher, Mrs. Smith to be outstanding. In later conversations Ann offered Mrs. Smith as a model teacher. Ann was particularly cognizant of the affective atmosphere in the class. According to Ann, Mrs. Smith cared deeply about the students' understanding of mathematics and she demonstrated this caring by sharing the disappointment and responsibility when students failed, by carefully preparing for class including working all homework problems, and by offering students a voice in the learning process. Ann felt that these actions created a climate of mutual progress rather than the traditional adversarial roles between students and teacher.

Mrs. Smith would write a definition on the board that the book would give us. And then she'd go through and explain exactly what it would mean in our, in my sense. It just like clicked for me. Everything she explained clicked. And she just made it so easy. It was like Mrs. Smith really wanted us to do well. And that was her goal: for us to do well. . . . She always said that she really looked forward to us doing well. . . . It was like we were each going for the same thing. We weren't working against each other. . . . Mrs. Smith was definitely the teacher I had that really wanted to see us do well. . . . She was always organized. Always knew exactly what she was going to say. And she did our homework. . . . She had looked at every single problem she was giving us. Like the problems we had would be like 2,5,6,7,9,[and] 11. They'd be like a random set of numbers and so it showed that she had looked at each problem and she knew exactly what she wanted us to do. Mrs. Smith was just so caring and she just put so much time and effort into teaching. She was just an incredible teacher. She was always asking us for our input on how she could make the class better. It was a give and take relationship.

As the above quotation indicates, Ann deeply respected Mrs. Smith. She spoke in considerable detail about the style and activities in the class that enhanced her own learning. Since these comments coincide with her views on good teaching, further discussion will be suspended until that section.

In addition to commenting on the teaching, Ann offered extended remarks on the geometric content as well. She enjoyed geometric constructions, but found proofs difficult. For her, the difficulty did not reside in the concepts but rather in the necessity of proving connections and conclusions that were already obvious to her. Ann considered the role of proof to be contrived and artificial.

I hate proofs. The thing I hated about writing proofs is that you can sit there and say well I see how this works. You're looking at this [and] it's so obvious that this works. It's so obvious that this is true, but then you have to go and prove it. And it seems trivial that you have to prove why this works [underline added]. I just thought it was a waste of time. I was just like, they name[d] all these things. And the names they don't mean anything to math. I haven't had any dealings with them where they're important. I just need to know that they work. I don't need to know their names. It's obvious angle A equals angle A. I mean, no kidding. [laughter] Why do I have to. You know what I mean. I think that's why I really didn't like writing proofs. Because you're proving things that any person with minimal intelligence can look at and say, that's right [underline added]. And you'd have to sit there and go well, this is true because, the subtraction, you know, whatever, like. I don't even remember the name of it. But things like that, I just hated that.

When asked if she would have found writing proofs more constructive if they had been less obvious, Ann responded that she disliked the infusion of English into mathematics that occurred in proofs.

That wouldn't be fun either. Cause just the name, it's the whole idea. They're trying to tie English in with math and I don't like it when they do that. I like it when they just stick. I think writing proofs don't even have anything to do with the outside world. It's not like you're applying it. I just, didn't like proofs at all.

Algebra II. Although Ann was doing well in Algebra II, she did not find the teacher, Mrs. Thomas, as clear and as organized as her previous teacher, Mrs. Smith. In fact, Ann was quite critical of Mrs. Thomas' teaching technique both in logistical organization and affective atmosphere.

I don't think like she does, so I have a lot of trouble. . . . It's a completely different math than I've had in last 2 years. Mrs. Thomas goes about things really differently than I'm use to going about them so I'm not really enjoying it as I have in the past. It always seems like whenever Mrs. Thomas explains something she's trying to confuse me or something. You know what I mean. I just feel, that she doesn't explain things in the easiest ways there are to explain it. So I'm not having that much fun with math as I have in the past. I'm still understanding it, what's going on. I think it's just because of my background that I've had in math that I understand it. I think Mrs. Thomas is prepared

for class and she's a good teacher. I've heard she's a really good teacher. . . . I just don't feel that she'd be that upset if I didn't do well. . . . She just cares whether we get the right answer not whether we understand the concept. With Mrs. Thomas it's just like a normal class. You walk in there, and you've done your homework, and [you] wait for the bell to ring and then you leave [underline added].

### Affect

Throughout the series of interviews Ann appeared confident and at ease. She remarked after the first few interviews that she found the questions and queries challenging and that the interviewer really made her think about her mathematics. Ann felt sufficiently at ease to ask the researcher to wait, that is, to stop asking questions until she had time to reflect on her responses. Like all the students in the study, Ann was eager to answer completely the researcher's questions, asking at times if she had explained her thinking sufficiently and if the researcher had understood her remarks. Ann did not appear intimidated by the problem solving activities, in fact, she actually seemed to enjoy them. She responded straight forwardly when she did not recall an idea as happened in the function interviews. She did, however, attempt even those questions that she knew she did not understand fully.

When asked to describe her own ability to do mathematics she said: "I'm very confident in the classroom that I have. I have a grasp on what's going on in the classroom." She indicated, as well, that "Math is my strongest subject." She was also asked why she had elected to take a third year of mathematics and she responded that mathematics was a class she looked forward to and that she felt that she was good at.

I want to. I want it to be an option for me in college. It's something I've always understood. It's like the daily pick me up [underline added]. I understand what's going on in math. I don't understand what's going on in U.S. History or American Lit., but at least I understand what's going on in math. It's something I can look forward to [underline added]. Well at least I know what's going on. . . . I don't want to take away something that I'm good at [underline added].

### General View of Mathematics

#### Unlike/like Mathematics

When asked to describe something unlike mathematics, Ann suggested an essay. Her rationale was that: "It wasn't tangible. It was something that was abstract." Although Ann acknowledged that mathematics could be abstract, the variability of form and shape of an essay distinguished it from the abstraction in mathematics. She remarked that: "An essay doesn't have any particular shape or form or any set way. Any set pattern at all. They're all different

and they all vary. It's not just one."

Ann's selection for something like mathematics was a *table*. She argued that a table was representative because first, it was a geometric shape, and secondly, one used mathematics to construct a table. She continued by saying: "[It was] something that was more tangible, [and] I think [that] has more to do with math." She also indicated that "I think you find math in everything."

Ann's choice of an essay and a table as illustrative of something unlike and like mathematics foreshadowed a more intense distinction she drew between mathematics and English as subjects. The distinctions Ann drew take on even greater significance when contrasted with Tara's responses for mathematics and English. These two best friends had seemingly constructed mirrored views of the two disciplines. For Ann, mathematics was creative and flexible while English was structured by rules of grammar and organization. In contrast for Tara, mathematics was restricted and fixed while English was free and creative.

#### Vocabulary lists

Mathematics. Ann's selection of vocabulary words as descriptive of mathematics included virtually the entire list [see Appendix H]. That her choices were so inclusive was striking and telling of her perception of mathematics. Again she spoke of mathematics as permeating throughout her life.

Ann organized the vocabulary terms around two views of mathematics: "One is the math that you do in class. And the other math is the math that you find in the outside world." School mathematics was *right or wrong, boring, memorized, routine, simple, chronological, and mechanical*. In contrast, the mathematics in the outside world was *current, exciting, and even fun*.

Ann also included in her selection the following terms saying:

*common sense:* you need common sense in order to take this formula and apply it here.

*right/wrong:* Whereas math, there is usually a right or wrong answer to a question. [But] you can chose certain ways to go about your problem but in the end you're gonna come up with the same answer.

*trial and error:* that's a part of mathematics. But sometimes it's not just trial and error. There's common sense involved and how you come up with [an] answer.

*creative:* Whoever came up with the mathematical system . . . was very creative. You have to be smart to come up with the formulas. . . . I think you have to have a creative

personality. In order to apply mathematics to other things in the outside world you have to be able to understand and be creative.

*solemn*: solemn kind of goes along with boring. . . . It can get to be busy work sometimes in class. Solemn kind of fits that definition.

*language, verbal, and writing*: word problems.

*easy and hard*: some math problems are easy and some math problems are hard for me. . . . When you have the background, they're easy.

*dogmatic*: It's definite that 2 plus 4 is 6. . . . Everyone goes by that.

*free*: I'm free to do math when I want to do math. . . . I could apply free to whether you're able to do it and dogmatic to how it's done.

*humanistic*: how it relates to human. Just how you can apply it. You look in the paper and it's filled with numbers and it's filled with mathematical concepts. There doesn't have to be numbers. Like in geometry we did sentences where because you have this you can deduce this from it. . . . Like reading between the lines. . . . That's all mathematical concepts.

Ann circled, as well, the term *memorize*. However, she did not perceive mathematics as requiring much memorization. In fact, the opposite was true. Ann's memory work was restricted to vocabulary, names of theorems, and postulates in geometry. She found that mathematics became second nature to her after she completed an assignment and once she understood the rationale for the process. She went on to comment that mathematics problems (concepts) built on themselves and that understanding the foundational ideas gave insight into extension problems.

You learn how to do a problem. You learn how to add, you learn how to multiply, and then you can apply those in so many different ways. . . . Once you know how to do it, it's not like you have to keep on going and memorizing how to do it with different numbers plugged in. . . . You can build on what you know [by] put[ting] these two things together because they compliment each other. If you know that  $2 + 3$  is 5, you can plug in 4 in place of 2 and that's a new [problem], but you don't have to memorize . . . every single fact. They just compliment each other and you can build.

Like after I do one problem, I've memorized the formula and it will be with me forever. . . . Like I understood why the formula worked so I could use it and apply it.

Since Ann included most of the terms in her list as applicable to mathematics her nonselections also offer some insight into her perceptions of mathematics. She excluded the following grouping of terms: anxiety and dull; expressive, controversial, and opinionated; changing, new and theoretical; fragmented, arbitrary, and capricious; and beauty. These terms suggested to Ann an image of mathematics as unstable, controversial, and illogical. For her mathematics was agreed upon, based on common sense, and hence not subject to change or individual opinion. When asked why she had not circled beauty, Ann remarked: "No one is



going to think of a number as beautiful." Beauty was a term she reserved for literary descriptions.

Ann's responses on the vocabulary list suggested that she saw mathematics as both a well-defined process (*ordered, rational, involving cause and effect*) which could be *routine, simple, and mechanical* and as a subject that required *insight and creativity*. Her view of mathematics was broader than that of a collection of logical rules. Mathematics was concepts and those concepts were universal in scope - touching one's life in all directions. In addition Ann felt that the application of mathematics to the outside world was not only *practical* and relevant but *enjoyable* as well.

Ann's comments revealed a deep reflection of her own mathematical knowledge. She not only distinguished between school and real world mathematics but between rules and concepts. Mathematics was more than procedures and numbers to Ann. Mathematics embodied concepts. This idea was reiterated in later sections when Ann described her own learning style. Finally, implicit in all of Ann's remarks is a strong and pervasive expectation that mathematics should make sense--its rules, formulas, and applications.

English. Ann categorized the terms in the vocabulary list for English around two themes: grammar and creative writing. With grammar, she associated the terms: *memorize, objective, ordered, solemn, simple, routine, precise, mechanical, fixed, common sense, factual, rigid, and structured*. Creative writing was tied to those characteristics of writing which refer to style and freedom of expression: *expressive, individualistic, beauty, elegant, flexible, free, open-ended, opinionated, and creative*.

Ann commented on her selection of the terms *beauty* and *elegant* by saying:

[It's] because stories are very beautiful. They paint a picture in your mind of a very beautiful scene.

However, Ann saw English and mathematics as combined in word problems:

The English part, words could be beautiful and elegant whereas the math part was more of the mundane type. That's what I think [of] the numbers. More the fixed part.

Summarizing, Ann's perception of English was divided into grammar, which reflected the inflexible oriented side of the discipline, and composition, which expressed the individualistic and controversial side of writing.

Science. Under science, Ann organized her selection of terms based on her experience

in her biology and chemistry classes. For her, chemistry had a strong mathematical or logical component while biology connoted nature and ecology. Ann offered the following remarks concerning science:

*truthful*: The right or wrong aspect of it. These things are theories and they have been proved time and time again. That's why it works.

*valid*: The law of gravity is valid. The fact of it, effects our world each day.

*anxiety, boring, dull*: Last year I absolutely hated biology and the teacher was awful. That's where anxiety comes in and that's where boring comes in and that's where dull comes in. . . . I dreaded going into class each day. . . . We just didn't learn a thing all year.

*expressive*: You express how you feel about science fooling around with nature. We got into big debates last year [about] what we thought.

*insight*: You need insight in order to be able to look at [a] problem in chemistry [and] use your memorized skills to apply them to that problem. . . . You get more insight over the years, because you live more and you learn more. You see different kinds of problems being done and then you gain insight from each problem. Then you combine all that insight together and you are able to do a problem that you weren't before because you've seen problems that are close to it.

*memorize*: There's not a very big percentage that I think of as memorizing because once you memorize certain facts they can be applied throughout chemistry. . . . There's not [much] memorizing, it's just a matter of knowing how to do it.

*beauty*: Nature. What runs through my mind - a tree, grass, sky.

*capricious*: That went along with *trial and error* Like balancing equations. . . . There is an algebraic way to do it, but a lot of times the easiest way is just to plug numbers in there and see if they work. It's not completely haphazard.

*simple*: Some of it is very simple. These's concepts that are very easy to catch onto, like the symbol for Hydrogen is H.

*solemn*: That goes along with *boring*. It's not always fun and exciting when you do science.

Like mathematics, Ann's perception of science included a rational component (*deductive, cause and effect, logical, ordered, rules, and common sense*) as well as a creative component (*clever, instinctive, and insightful*). Science was, however, distinct from mathematics since it's theories and experiments often involved controversial issues such as biological engineering and environmental pollution. Science was also practical and universal. For Ann, the theories of science were valid and truthful because they had been proven repeatedly by experimentation. She did not, however, articulate the view that these theories were transient and subject to revision as knowledge revealed new data. Controversy in science was associated with ethical questions rather than validity of theories. Although Ann felt that certain rules or formulas needed

to be memorized, these procedures served as stepping stones for solving related but unfamiliar problems.

History. In this category, Ann selected terms that:

Were based on how I felt about history and [the] people that sat down and wrote the Constitution. And what was going through their heads. And about current events. . . . You have [to] be rational in order to sit down and think of laws.

Her view of history, like that of mathematics, was tempered by the school presentation of the subject. Under this perspective, history was *dull* and required *memorization* of facts that were *chronological, detailed, ordered, precise, and simple.*

Ann commented on some of her choices of terms:

*dogmatic:* because Democrats are set in their ways and [in] their beliefs. And you have the Republicans. Just look at Seabrook. They're each fighting each other. They're on different sides and they're each set on their own beliefs as to what's right.

*free and fixed:* [the] Civil War was fought to free the slaves [from] the South. And it was fought between the North and the South but everything in history is fixed. You can't change what happened in history and that's why it was fixed.

*concentration:* history takes a lot of concentration to understand exactly what each person was thinking in history when they did what they did.

*truthful:* You're putting your faith in these people to run their country or to be in charge of some part of history. They have to be truthful to you. You have to have an honest relationship between people in order to come out successful. . . . We need an open honest relationship with other countries.

*universal:* Everyone is affected. . . . And history is happening everywhere. And what happens in Morocco today might effect me somewhat latter on in my life.

These comments revealed Ann's optimistic and universal view of politics, and current and historical events. She saw history as *fixed-known facts*, as having *multiple perspectives* on interpretation and as interrelating events.

Summary. Ann's remarks on the vocabulary list showed an insightful awareness of the distinctions between how subjects are presented in school and their usage in the outside world. School subjects could be boring and laden with heavy memorization while their real world counterparts could be dynamic and enjoyable. This distinction was most pronounced when she spoke of mathematics. She described school mathematics as often focusing on the correctness of mechanized procedures while applications in the real world were pervasive and interesting. For Ann, the discipline of mathematics was not restricted merely to numbers and operations but extended to the realm of logical thinking. Ann illustrated this point by describing how deductive

thinking in geometry applied equally well to making inferences from a newspaper article.

Among the participants, Ann was unique in attributing mathematics with the qualities of creativity and cleverness when applying and developing mathematical ideas. Although Ann spoke of mathematics as including rules, procedures and formulas, she felt that once the reason behind the process was understood then the process became a natural part of one's thinking. Thus memorization was not necessary to learn mathematics. Unlike other disciplines, mathematics was devoid of controversy and individual interpretation.

Ann's perception of English was divided between two categories: grammar and writing. Grammar entailed rigid-memorized rules while writing encompassed individual expression, controversy, creativity and beauty.

When describing history, Ann separated the subject into the memorized-known facts, and into multi-perspectives and current events. Ann perceived historical events as known and fixed facts. The interpretative aspect of history arose when speculating on the individuals' perspectives both from the past and from current events. Controversy in history occurred when the individual perspectives where in conflict.

Ann's view of science and mathematics coincided in many aspects. Both subjects contained many formulas and procedures, but these were learned easily once the rationale was understood. Both disciplines were cumulative in nature with previous problems offering insight into new situations. The similarities ended when science delved into the human arena. Often science was controversial when dealing with such issues as genetic engineering and ecology. Ann, however, did not extend the controversy to scientific theories. To her, theories represented known facts whose validity has been proven by repeated experimentation.

#### Ranking Grid.

Ann's ranking grid and associated comments illustrated her preference for selected mathematics topics. (See Appendix J.) Ann found most mathematics fairly easy to learn but was mixed in her response to various topics. Ann especially enjoyed word problems, geometric constructions, and factoring; while strongly disliking proofs, absolute value and inequalities, and graphing.

Ann was emphatic in her dislike of geometric proofs. She ranked writing proofs on affective issues as the most *boring* and *least liked*, and on cognitive issues as the *most*

*theoretical, hardest to do, and least useful* of the mathematics topics. She indicated that she found it the most *thought provoking* and the most *flexible* of the topics with regard to the variety of solution techniques. She also ranked proofs as very *abstract, confusing* and requiring *original thinking*. Ann did not see proofs as applied to real world or even as mathematics. Rather proofs represented *busy work* and pointless exercises in stating the obvious.

The thing I hated about writing proofs is that you sit there and say well I see how this works. . . . It's like so obvious that this works. It's so obvious that this is true, but then you have to prove why this works. And it seems trivial that you have to prove why it works. I just thought it was a waste of time. . . . I just need to know that they [the theorems] work. I don't need to know their names [names of properties and theorems]. It's obvious, you know, [that] angle A equals angle A. I mean, no kidding. [laughter] . . . You're proving things that any person with minimal intelligence can look at and say "that's right". . . . Proofs don't even have anything to do with the outside world. It's not like you're applying it.

Ann also remarked that in writing proofs: "They're trying to do English in math and I don't like it when they do that." She made similar remarks when speaking of word problems. She referred to the written expression of the problem as the English component while the thought process and resulting equations represented the mathematics component.

For Ann, proofs appeared as both a challenge and as busy work. Her intense dislike of proofs was not motivated by an inability to construct proofs, but rather the futility of stating the obvious and the uselessness of proof to the outside world.

Ann also held graphing in low esteem, ranking it as very *theoretical* (non-applied), fairly *useless*, and highly *rigid* in the solution process. Ann was bored by graphing, finding it to be useless busy-work. Graphing straight lines entailed simply executing the point-slope formula. For Ann, the challenge and utility rested in finding the initial equation rather than graphing the results.

There's nothing to figure out. There's no challenge. I like solving the equation part of the graphing. . . . The solving equation was fun but then the graphing part is so boring. . . . [It is] like the work part of it, the drudgery [part] of it.

Ann ranked absolute values and inequalities low on *interest* and *utility*. Here the ranking focused an Ann's difficulty in tying the computation aspect of absolute values and inequalities to the number line representation. She marked absolute values and inequalities as a very *rigid-abstract* topic which was also *boring, least liked*, and among her *worst* topics.

I see why it works [finding the direction of the associated inequalities] but it's just for some reason, I just don't feel that it's that concrete . . . I don't have fun doing it.

Although difficult initially to learn, Ann felt factoring problems provided a sense of satisfaction when the task was accomplished. Factoring was like unravelling a puzzle.

Factoring was hard to learn, but once you learn it, it's a piece of cake. I really like factoring, where you get to play with numbers [until] you find something [that] works. You get all excited.

Ann also enjoyed solving equations and word problems. She found the challenge of word problems stimulating rather than intimidating.

With the exception of writing proofs and solving absolute value and inequalities, Ann found most of mathematics clear and logical as evidenced by the string of ones she marked across the logical row in her grid sheet. She perceived decimals and fractions as "virtually the same thing." Ann enjoyed geometric constructions and often used the techniques in designing patterns to decorate her notebooks and book covers.

In summary, Ann's ranking grid revealed her ease with mathematics which she perceived as logical, generally easy to learn, and useful. Proofs, absolute values and inequalities, and graphs stand out as exceptions: proofs because they were seen as useless and obvious; absolute values and inequalities because they seem to lack a connection between the computation and the number line; and graphs because they were busy work.

#### Synopsis of General Beliefs about Mathematics

Ann described mathematics as something tangible and pervasive in life. Mathematics could be simple, routine, mechanical, and boring when restricted to school mathematics. Ann, however, saw it as extending beyond school to the real world where its applications are exciting, fun, and current. She also viewed mathematics as encompassing more than numbers and operations. Mathematics was an accumulation of interrelated concepts. For Ann, these concepts were learned easily once the why behind the procedure was understood. The creation of mathematics required creativity and cleverness as well as logic and common sense. However, Ann neither described mathematics as beautiful nor controversial. For her, the concepts in mathematics represented universally agreed upon facts.

Although mathematics problems possessed right or wrong answers the process by which the problem was solved remains flexible. Often a combination of common sense, logic, and trial and error were utilized in the solution process. Insight into problems arose from experience with similar or related problems.

Ann expressed an ease and familiarity with many mathematics topics. She especially enjoyed word problems, factoring, and geometric constructions while intensely disliking proofs, graphing, and absolute values and inequalities. Ann saw all mathematics as offering challenges but it remained accessible when the learner had an adequate background. She had the expectation that mathematics should be applicable to diverse problems and that mathematical knowledge was cumulative.

#### Student/Teacher Roles

##### Good Mathematics Student

When suggesting someone who was good at mathematics, Ann chose Tim. She felt that Tim revealed his ability by his insight into problems that she was unable to solve.

He sees things in problems that sometimes I won't see things in. . . . I might get a problem right that he'll get wrong, but he'll get the problems right that I get wrong. . . . It's not like he's better or worse it's just that we can do different problems. So I admire the problems that he can do that I can't do. . . . It's just he sees things in problems that I don't see.

Ann perceived mathematics ability as a developmental process requiring good teachers and complete comprehension at each stage.

It starts back in elementary school because I think you have to have a good set of teachers. . . . Everything is built on top of each other and if you don't have the first rudimentary skills they you won't have any [background]. . . . Like I can't do what I'm doing this year if I didn't have Algebra I. If I didn't have a good Algebra I teacher, I wouldn't be able to do it. . . . You have to take all the steps and you have to understand everything in order to understand the next thing that comes.

When you have a bad teacher all of a sudden you get stopped. You have to go back. . . . I couldn't do math now even if I understood the concepts if I didn't know how to do fractions. . . . It's all like compound. . . . It's like on top of one another.

##### Advice to an Exchange Student

Ann's advice to a new exchange student was very brief. She believed it was essential to do homework. If the student had difficulty he or she should seek out the teacher, Mrs. Thomas, who was always available after school. She felt further advice was conditional on the student's ability to grasp algebra. She had no advice concerning tests since she herself found it unnecessary to study for mathematics tests.

I don't study for tests in math. A lot of people do, but like every once in a while I will, if I don't understand a concept. . . . If we [I] do the homework, the concept's there and so I don't even bother studying for it. I just know it.

### Advice to Beginning Teachers

To student teachers, Ann offered the advice that they should avoid dead time in the classroom and should fill-in any gaps with additional examples. In this way, the student teachers would be better able to control the class. When asked how they should present the mathematics, Ann suggested that:

You have to be clear. . . . It's just a matter of some teachers have the gift to be teachers and others don't. It's hard to give advice when it's more whether or not you have the knack to keep the kids interested and whether you can communicate your message to the students.

She felt the lessons would be clearer if

You apply it to terms that students will understand. It makes it easier. . . . Examples [should be] related to something that we do every day. Something [that is] an important part of our life. That we understand well. Instead of applying it to something that's abstract. Bring it towards us.

To further illustrate her point, Ann described how Mrs. Smith brought in paper models to demonstrate the derivation of the formula for the volume of a cylinder. Ann believed that drawing a diagram would not have been as effective as the actual model which the teacher unfolded to reveal the constituent parts.

People might be able to do it [with a drawing] but they might not understand it as well as to why it works. Cause you can go through the skills, you can go through the steps, but I think it's just a matter of . . . realizing why it works.

Here again, Ann stressed the necessity of the students understanding the why behind the process and by implication suggested the importance of the teachers demonstrating that rationale.

### A Good Mathematics Teacher

Ann's description of a good teacher centered on the theme of empathy with students. She emphasized that a good teacher is one who "wants the students to learn." Teachers reveal this affective quality by accepting the responsibility for student failure.

Good teachers don't feel they've done their job if you're doing poorly in class and trying. If you try and are doing poorly there's something wrong with their method of teaching instead of saying that there's something wrong with the student.

A bad teacher gives the impression that "It's more student versus teacher instead of student working with teacher."

When asked whether tests should be selected from class-type problems, Ann agreed arguing that this was necessary in order to be fair to all students. Ann remarked that she



enjoyed figuring out new problems but felt that not all students have this aptitude. She offered the compromise of using new problems as required bonus questions on the test. In this way, less adept students could try the problems without being penalized and gifted students could be challenged.

I'd like it if I could figure the problem out. . . . I don't think it should count on the test because you can't help whether or not you can figure these problems out or whether or not you have seen them before. . . . Tests should be on the material covered and that you understand so everything's fair [to] everyone that's taking that test. Unless he's [the teacher] gonna grade you differently all around. Some people are more gifted in a subject than other people. You can't help it if you're less gifted. I can see putting problems on the test but having them be extra credit and making everyone try them. . . . I couldn't see taking points off because of that.

When asked whether teachers should present examples of all homework-type problems, Ann responded by saying that she enjoyed the challenge of trying to apply concepts to new problems. In fact, these applications served as a valuable check on her understanding. If she could solve the problem then that indicated that she indeed understood the concept.

I like it when we haven't covered it in class. . . . You can't expect them [teachers] to go over every problem. . . . They'll give you the basic idea . . . and if they've explained it well, you should be able to apply it to other things. I'm not saying that everyone [every student] can because not everyone can obviously. But at least give people that . . . can apply it to other things the chance to try. If a teacher explains a method to me then I can apply that method to other things. It really teaches me that I understand this and it will help me understand it. Even though the girl sitting next to me might not get the answer right, at least I had the chance to try it and she had the chance to try it too. . . . Maybe it didn't help her but it helped me, but at least it helped one of us which is better than helping none of us.

I think part of math is being able to apply formulas and methods of doing things [to] other problems that you haven't seen before to really prove that you understand the information that you're being given [underline added]. And if a student can't do that problem I think the teacher has to go back and go over the original method and show why it applies to this problem. . . . If a student can't get a problem and it applies to the unit you're doing . . . and they don't have any idea how to do it, [then] chances are they don't understand the rudimentary facts about the problem, about the original unit being learned.

Additional information concerning Ann's views of good mathematics teaching was taken from her description of Mrs. Smith's geometry class. Ann felt that Mrs. Smith was an exemplary teacher and had personally contributed to her own confidence in mathematics. Foremost in Ann's mind was Mrs. Smith's sincerity in wishing her students to be successful. Student failure was taken by Mrs. Smith as her own failure as well. To involve students in their learning, Mrs. Smith often asked students for suggestions on improving the class.

In addition to encouragement, Ann found Mrs. Smith's lessons to be well organized and especially clear which she attributed to thorough preparation. Mrs. Smith related mathematical

definitions to students' experiences and illustrated geometric concepts with concrete models. Preparation for class also included careful selection of homework and complete solution keys. Mrs. Smith presented her solutions to the class whenever a question arose on the homework. She was often able to pinpoint students' mistakes by hearing their answers. Finally, Mrs. Smith's concern for students' learning extended to her grading policy. On test questions, she assigned point value proportionally based on the severity of the student's errors.

She's [Mrs. Smith] a conscientious teacher. She was always asking us for our input on how she could make the class better. . . . She'd have the homework done that we had done. She had it all written out exactly how she wanted us to do it. And then she [would] go over each problem and just how she did it. . . . [Then if] someone in class says "Oh I got this for an answer," she says "Oh you probably forgot this step here." Because she had gone through and she had figured out where someone could go wrong. She had put so much time and effort into it. And she always wanted to make us happy. . . . You know, she's going to put a lot of time into it so we're going to put a lot of time into it and because of that, we learned. . . . Mrs. Smith would be happy to know that we got the right answer and we know how to go about doing it. [But] Mrs. Thomas wouldn't give us any credit for the problems where Mrs. Smith would. [Mrs. Smith would say] 'Well, you did all right up to here and then you subtracted 2 instead of added 2,' so minus 2 or something like that where Mrs. Thomas would be like 'no, that's wrong.'

#### Summary of Student/Teacher Roles

Ann believed mathematical ability was a combination of having good teachers and an accumulation of knowledge over the years. She felt that doing homework was important for success in mathematics. Equally vital was understanding the rationale, the why behind the process. This understanding in turn eliminated the need to memorize mathematics and facilitated the application of the concepts to new situations. Ann felt strongly that students should be given the opportunity to attempt problems that required them to apply their knowledge. In this way, students could test their comprehension of the concepts.

Ann believed ideally that students and teachers should share responsibility for learning. Students are expected to attempt all assignments while teachers must prepare clear relevant lessons. When possible, lessons should include concrete models, applications to real life, and rationale for the procedures. Accompanying homework should be carefully selected and when questions arise the teacher should present detailed solutions. Ann argued that constructing tests parallel to homework assignments was necessary to be fair to less adept students. Challenge or applications-type problems could be included on tests if relegated to bonus questions. Grading of tests should be based on the severity of the error and on whether the overall process was understood. Finally Ann strongly believed that teachers must hold a deep desire for their

students to succeed.

#### Conceptual/Procedural View of Mathematics

The analysis of Ann as holding a conceptual or procedural belief about mathematics was derived from two sources: the general mathematics overview and the coding of Ann's responses from the transcriptions of her solutions to the mathematics problems in the interview protocols. Of the 12 problem episodes that were codable for Ann, all 12 were labeled conceptual. Ann's ranking of 100% conceptual ordered her as first among the six participants in this category. (See Appendix I for a table summarizing the participants' coding on the conceptual/procedural and autonomous/nonautonomous categories.)

Ann's clear presentation of her ideas facilitated the coding of her problem protocols. Ann was able to convey complete arguments with a minimum of interference dialogue, that is, false starts, repetitive phrases, and time filling phrases. Another distinguishing quality was Ann's ability to clearly cite mathematical examples to illustrate her points. The other participants attempted this, but their verbalizations lacked the detail and focus that were conveyed in Ann's examples.

#### Conceptual

Ann's conceptual rating on her problem protocols was due to many factors. Her protocols revealed that she could (a) justify procedures in terms of first principles or intuitive number sense, (b) find multiple approaches to problems, (c) use number sense to check the reasonableness of the solution, (d) summarized her solution process, and (e) evaluate the sample test questions on the basis of process rather than on the answer solely.

Ann was able to justify several basic procedures and formulas on the basis of first principles and intuitive number sense as seen in her responses to problems #30, #19 and #22. A simple example is evident in an auxiliary question to problem #30 (find the area of the triangle). When asked why the  $\frac{1}{2}$  was in the area formula for a triangle, Ann responded by indicating that one could visualize the triangle as half a rectangle and hence the  $\frac{1}{2}$  was necessary.

Because if it was just the area equals base times height, that would be the area of a square or rectangle. You have half base here [she points to the diagram she drew] and have half a rectangle. Because if I drew this rectangle, divide it in half, there's two triangles.

Ann was also able to justify the need for common denominators when adding fractions in problem #19 ( $1/4 + 2/3$ ). She began first by saying it was a rule she had learned in school but when asked for further details she argued that adding unlike fractions is like adding apples and oranges.

[Common denominators] makes them like the same. . . . If these were apples and these were oranges, if I add them up then I would get apple-orange juice. It would be like a mismatch of everything. . . . You need to change those each to the same thing. You need to change them each to pears so you can add them up. And then you [are] going to have pears in the end.

When asked to further associate her apple-orange-pear analogy to a picture, Ann drew the usually pie diagrams shading the given fractions. Next she showed the pie diagrams for the equivalent fractions with the common denominator. Ann's explanation to accompany her diagrams follows:

If I use a pie and this pie is divided into 4. And this one's divided into thirds. Then I have 1 of these [points to a  $1/4$  piece] and I want to add it to 2 of these [points to two,  $1/3$  pieces]. . . . I just can't add them up because I'm not dealing with the same two things. So I know I want to get something that is alike in each of them. I want to pull something out that is alike in each of them. . . . If I take this pie and I divide it into 12 pieces. And then I'd do another one. . . . Instead of being 2 [parts of the pie], it will be 8. So then I have 8 of these and there's three of these. But I know it's not 3 whole, it's 3 pieces out of the entire pie. And there are 8 pieces out the entire pie. So out of both the two pies I have 11 pieces. But since I'm adding the two together, I'm still talking about the one pies. So that's 12ths.

As the above quotation indicates, Ann had developed an internal rationale for explaining the necessity for common denominators and thus was coded as conceptual.

In her explanation for problem #22 ( $1.50 \times .25$ ), Ann justified her answer by appealing to number sense. Ann began her explanation by saying that you move the decimal point over 4 places because "I want to retain all my places." She continued by arguing that:

If I had only moved the decimal point over 3 [sic 2] places then I would have been multiplying it by either 15 times .25 or 1.5 times 2.5. So it would be a completely different problem if I didn't move the decimal point.

When asked why 3.75 could not have been a reasonable answer, Ann responded by saying:

Because I'm multiplying this number [1.50] by [a number] less than . . . 1. . . . so I should get a smaller number. . . .25 is less than 1, so I should be getting a smaller number than 1.50.

In these responses particularly the second, Ann indicated that she justifies her answer of 0.375, by stressing that the decimal value of the answer was dependent on the relative value of the constituent factors. It was reasonable for the answer to be 0.375 since it was result of

multiplication by a value less than 1. Thus, it was to be expected that the result should be less than 1.50. Although unable to rationalize the rule for moving decimal places, Ann was able to justify the numerical size of her answer by using number sense. Thus her procedural algorithm was underpinned by number sense which supported its validity.

In addition to being able to justify procedures, Ann was able to suggest multiple approaches to several problems. This ability was evident in her protocols of problems #23 (fractions greater than  $\frac{3}{4}$ ), #26 (20% of 85) and #28 (larger of  $(543 \times 29)/32$  and  $(30 \times 543)/28$ ). On problem #23, Ann first worked the problem by comparing and approximating the given fractions with other fractions that she knew were related to  $\frac{3}{4}$  or its equivalent fractions. When asked for an alternative approach, Ann suggested writing  $\frac{35}{71}$  and  $\frac{3}{4}$  as a proportion and cross multiplying. In the course of executing this procedure, Ann became confused and so she offered a third approach. This last approach entailed comparing each fraction individually with  $\frac{3}{4}$  by means of finding a common denominator and checking for the larger numerator.

Which of the fractions is more than  $\frac{3}{4}$ ? The first one is  $\frac{35}{71}$  and I know that one is close to  $\frac{1}{2}$  because  $\frac{35}{70}$  would be  $\frac{1}{2}$ . So that is not more than  $\frac{3}{4}$ . And then  $\frac{13}{20}$  is more than  $\frac{3}{4}$  because, well, 12 wait a minute.  $\frac{12}{20}$ . This one [ $\frac{13}{20}$ ] is more than  $\frac{12}{20}$  and  $\frac{12}{20}$  is  $\frac{3}{5}$ . It's ok. I was thinking this is a dead end, this is not what I want to do, so I'll take  $\frac{15}{20}$  which I know is equal to  $\frac{3}{4}$  and this is less than  $\frac{15}{20}$ , so this one is less than  $\frac{3}{4}$ . The  $\frac{71}{101}$ , well I know that  $\frac{75}{100}$  is equal to  $\frac{3}{4}$  and that  $\frac{71}{101}$  is less than that, because this [numerator, 71] is less than that [75] so you're going in the opposite direction. You're making it smaller because the fractions are further apart. The  $\frac{19}{24}$ , well  $\frac{3}{4}$  of 24 would be 18 over 24, so I know that is greater than  $\frac{3}{4}$ . And  $\frac{15}{20}$  is equal to  $\frac{3}{4}$ .

I could have put like 35 over 71 is equal to  $\frac{3}{4}$  and I could cross multiplied them and see if they equaled to each other.

I could go through and I could figure out what these numerators would have to be with that given denominator for it to equal  $\frac{3}{4}$ . . . . Then if it wasn't greater than  $\frac{3}{4}$  then I knew that the original fraction wasn't equal to  $\frac{3}{4}$ .

On problem #26 (20% of 85), Ann offered two approaches to solving it. Her first approach was to multiply 0.20 times 85. When asked if she believed that 17 was a plausible answer, Ann argued by using proportional reasoning: since 20% of 100 is 20, then 20% of 85 should be less. She also argued that the proportion  $85:100 :: 17:20$  should yield equivalent amounts when cross multiplied. In this example, Ann was able to offer multiple techniques for solving the problem and used number sense to justify the reasonableness of her result.

In problem #28 (larger of  $(543 \times 29)/32$  and  $(30 \times 543)/28$ ), Ann was able to suggest

two techniques for choosing between the fractions. Her first approach involved simply completing the computations with a calculator and comparing the resulting decimals. For her second approach, Ann began by comparing the numerators and then figuring in the effect of the denominators. The rationale behind this second approach also involved using number sense with the fraction concepts.

I could have done it in my head. . . . You're multiplying 543 by a larger number here than you are in column A. You know that you're gonna get a larger number on top. . . . Then you're going to divide this by a larger number. . . . The top in column B is greater than the number on top in column A and the number on the bottom is smaller, so you are dividing by a smaller number. And when you're dividing by a smaller number you're gonna get more parts . . . [Say you are starting with] these two quantities on top are equal, then . . . you divide this one by 5 and then this one by 4. . . . You're gonna get 4 parts here and you're gonna get 5 parts here. . . . Each part is gonna be smaller. [Ann drew two unit rectangles, dividing one into 4 parts and the other into 5 parts.] So this one is gonna be less than that one. That's exactly what's happening here [referring to the original two fractions], except this one you're even dealing with a bigger piece.

In this second approach, Ann effectively argued that the fraction in column B would yield a larger number than that in column A. In her argument Ann noted that the numerator in column B was larger and, in addition, was being divided by a smaller value than the fraction in column A. She augmented her argument with an example where the numerators were equal and the denominators were different. Finally, Ann further illustrated her point with a diagram. Ann's conceptual rating on this problem was due to her ability to present multiple solution techniques and her ability to use number sense in determining the solution.

The third way in which Ann illustrated her conceptual view of mathematics in the problem protocols was by using number sense to verify the reasonableness of her solution. The two previous examples, problems #26 and #28, demonstrated this ability. A further illustration appeared in Ann's protocol of problem #30 (rope problem). In this difficult problem solving situation, Ann progressed through a series of false starts before finally arriving at the solution with the aid of suggestions from the researcher. In this case, Ann's intuition or number sense led her to the incorrect conclusion that 1 foot could not possibly be the height of the rope above the earth. She argued that the 6 feet added to the length of the rope was insignificant in comparison to the original length: "6 feet is hardly anything to a mile . . . I don't trust that answer, I did something wrong again." She again checked her computations and concluded:

So I guess  $x$  is equal to a foot, [but] I don't believe it, because if you have 6 feet going around this table, or say like around my head . . . I'm going to get like around the globe out there [in the library] and I like spread it [the excess rope] around to like equal out.

I'm gonna end up with like maybe  $5 \frac{1}{2}$ , and then the earth is so many times greater than that globe there. There's no way that it can be 1.

The fourth category, the ability to summarize her solution process, was evident in Ann's protocols for problems #34 (If  $r$  is less than 0, which is larger  $r$  or  $r^2$ ?) and #27 (the remainder problem). In problem #34, Ann decided that either value could be larger depending on the selection of the numbers for  $r$ . She based her decision on the results of substituting for  $r$  the values of 1 and 2. When asked the follow-up question: "Is  $r^2$  always going to be greater than or equal to  $r$ ?", Ann's initial response was "Yeah". When a counterexample was suggested by the researcher, Ann reconsidered and offered a partial summary of the possibilities.

A [ $r^2$ ] is gonna be greater than B [ $r$ ] when it's an integer and B [ $r$ ] is gonna be greater than A [ $r^2$ ] when it's a fraction. When it's a fraction less than 1.

A more striking example of Ann's ability to generalize results was demonstrated in problem #27 (the remainder problem). In this problem solving situation, Ann used monitored trial and error in her search for the solution. To begin the search, she tried each of the individual integers from 2 to 13. At this point she stopped and looked for some pattern in the sequence of divisions. She then observed that 32 worked for 3 and 5. This suggested to her that perhaps the mystery number must end in 2. She tested this theory by checking which digits would give a remainder of 2 when a factor of 3, 4 or 5 was subtracted from it. For example, factors of 5 end in either 0 or 5 so she determined which digits when 0 or 5 were subtracted from it would yield a remainder of 2. She found that 2 and 7 met this condition. Having satisfied herself that ending in 2 was a reasonable criteria, Ann proceeded to check in order 32, 42, 52 and 62. When 62 was reached, she recognized the solution.

I'm just going up through [the numbers], trying to do [them] in my head. . . . I'm trying to think about whether there's another way to do it. Obviously it's not as low a number as I thought it was going to be. [Pause] I was trying to think about [whether] that number is going to end in a 2. . . . All right cause like if I have 32, that's gonna leave me with a remainder of 2, and that's gonna be 3, so it will end up being 6, remainder 2. An the number will end, so the number for 5 will either end in a 2 or a 7. And then a 4, it will be any even. It would be every other [even number], every even number that isn't divisible by 4 for every other even number. So that narrows it down to 2. Cause I know it can't be 7 now. Cause 7 would leave me with a remainder of 3 or. . . . So I know that the number has to end in 2. I can't think of any example that would work for 3. With 32 it works with 3 so that 3 would be 1 [10] with a remainder 2. So the number is gonna end in 2. So I know it's not going to be 42 cause that works out. Then 52, [pause] goes into 4 evenly. [5 into] 62 will be 12 remainder 2; and with 3, it would be 20 remainder of 2. So the number is 62.

When asked if there was another number that had the given property, she replied: "I'd

double 62." She then paused and considered saying:

Let's see if it works. No it wouldn't cause I have to stay with a 2 on the end. [pause] Well, one thing I know about 3, 4, and times 5 that would equal 60 and then you add the 2 so you have the remainder of 2. So then I suppose it could be 60 times 2, plus 2 which gives me 122. And then see if that worked for all of them, which would be . . . 40 remainder 2 . . . 30 remainder 2. And then divided by 5 would give me 24 remainder 2.

When asked for a third example she quickly used her pattern: "I'd take 60, multiply it by 3, get 180 and add 2 to it. And it would be 182 and that would work."

In the process of searching for a second example, Ann reflected on her answer and observed the generalized pattern. She then proceeded to test it against her second example. Once this example was verified, she trusted it's validity as seen by her quick generation of her third example. Ann's ability to go back and generalize from her solution earned the conceptual coding. This reflection is not evident in many of the other students' protocols for this problem. The other students had used trial and error exclusively and so their solution was generated by exhaustion and thus attempts to find a second example were relegated again to guess and check.

The final way in which Ann exhibited a conceptual coding was in her grading of the sample test. Ann graded the test questions fairly consistently and distinguished between the solution process and the answer. In her mind, both appeared to carry roughly equal weight for grading purposes. Thus she assigned half the points for process and half for the answer.

This distinction between process and answer was evident in her grading of problems IIb, IIc, and IId. On IIb, Ann noted that " $6=6$  is not equivalent to  $x=6$ , so minus 5 . . . because they at least distributed and it's only an Algebra I test." The researcher then challenged Ann's position by suggesting that she verify that 6 was not a solution. Ann then checked it, laughing at her surprise when it worked. She then checked another value, 2, into the equation.

The reason it [the 6] works is because each of the equations are equal. . . . Which means that 6 will satisfy each [side of the] equation to make them equal. But that doesn't mean  $x$  equals 6. Well what would happen if I tried 2? . . . There's an infinite number of solutions. That all numbers work, so it's not just 6.

After re-evaluating, Ann remained with her original grading, since the answer was not correct.

On question IIc, Ann first worked the problem for herself and concluded that "the answer is there is no solution." She then compared steps, conclusions, and noted:

That's not right [ $-3=6$ ] so there's no solution. And they gave 2 answers and there should be only 1 answer for  $x$ . Since they started it right, . . . minus 8.



Ann marked this problem as eight off since the student, Chris, had given an incorrect interpretation to  $-3=6$  and had indicated that there were two solutions in a non-quadratic equation.

Again on question 1ld, Ann distinguished between process and answer saying:

This is correct up to here. So they got the right answer, but this is wrong, so I'll given them half credit, minus 5.

That Ann is grading process was further demonstrated in her remarks on questions 1la and 1c. On question 1la, Ann began by working the problem for herself.

Then I'd go through and see [what] they did. All right. [pause] They added 4 to each side, so they can do that if they want. It's a waste of time but [laugh] then they combined these together. Then they added 3 to both sides. They got  $3x=11$ . There's nothing wrong with doing it that way because they're doing things to both sides of the equation [underline added]. So I wouldn't take anything off.

On question 1c, Ann argued first that the cancellation was illegal but noted that the answer was correct. Upon reflection, she remarked that there was insufficient evidence to conclude with certainty that the student was indeed using the illegal cancellation. She conjectured that possibly the student might have done the work in his/her head and written misleading marks.

Well you can't just cross them out because . . . there's something being done to it, but you can factor this, I think. . . . So that can be factored . . . then these cancel out. Which would give you  $x$  plus 1, which is the right answer. But that [pause] is complete luck that that person got it right [underline added]. So I'd take off, but then you have to think that maybe they know what they were doing in their head and they were just being weird, so I'd take off a minus [pause] 5. Give them the benefit of the doubt for that one point. [laugh] No, minus 4, be nice.

### Summary

Ann's coding as conceptual was evident in both her general beliefs interviews and in her problem solving protocols. Ann had presented a perspective of mathematics that stressed knowing the rationale behind the process. This knowledge governed how to apply concepts, how to learn mathematics, and how to solve mathematics problems. Implicit in Ann's beliefs was the deep expectation that mathematics should make sense: the rules, the formulas and the procedures. Ann saw the concepts of mathematics as integrated and interdependent. She also felt that mathematics was everywhere. For her, mathematics was more than numbers and operations on numbers; it was a way of reasoning; it was concepts; and it was applications to the real world. Ann had a keen awareness of mathematics as being both a structure and

problem solving.

Ann's problem protocols also demonstrated a conceptual view of mathematics. Ann justified rules and answers on the basis of first principles and on number sense, summarized her solution process, added insight into her choice of procedures, utilized multiple approaches to problems, and evaluated the correctness of mathematics on the validity of the process rather than on the familiarity of form. These characteristics of Ann's problem solutions tied in with her general beliefs about mathematics. Ann had stressed the importance of knowing the rationale behind the procedures and formulas. This idea was realized in her protocols where she was able to clearly articulate reasons for such fundamental procedures and formulas as using common denominators in addition of fractions, and the area formula for a triangle. She also was able to justify, through number sense, results in a decimal multiplication, again indicating that mathematical concepts should be rational.

Other of Ann's beliefs about mathematics appeared in her protocols, as well. Ann had stated that although a problem has a fixed answer, the solution process remained flexible. Ann's grading of the sample test embodied this belief. She readily accepted the unfamiliar approach to solving the linear equation. For Ann, the solution technique did not have to match the standard one to be valid. Instead, she judged it on the basis of its mathematical correctness. This basis of judgement also coincided with her view that rules have a rationale behind them and that this rationale, almost common sense, governs one's actions.

Ann also believed that solving problems sometimes entailed using trial and error guided by common sense as well as creativity, cleverness, and flexibility. In addition, she held the view that previous experience provided insight into new problems. All of these beliefs seemed to come to bear on Ann's approach to problem #27 (the remainder problem). On this problem, Ann used monitored trial and error in her initial phase of the problem, then tested conjectures formed from her experimentation and from her knowledge of elementary number theory and, finally, generalized the results. Ann approached this problem with the expectation that trying examples would yield a more efficient technique for solving the problem. It is important to note that Ann was not driven by the belief that all problems can be solved by applying a formula or writing an equation. For Ann, an appropriate solution technique was one that matched the problem conditions rather than some predetermined expectation as to the technique.

The expectation that mathematics should be applied and that answers should make sense predominated Ann's beliefs. These expectations were evident in Ann's remarks on problem #35 (the rope problem). Ann was adamant that her answer of 1 foot could not be right. Although her calculations yielded this result, her intuition told her that the value was unreasonable. Her belief that answers should make sense dominated her perspective on this problem. Even when the researcher acknowledged that 1 foot was correct, Ann only accepted the result on the basis of an authority telling her it was right. It was clear that she still remained skeptical.

Ann's view that mathematical ideas were interrelated and that solution techniques could vary was visible in her ability to find multiple approaches to mathematics problems. On the ranking grid, Ann had also stated that decimal, fractions, and percents were virtually interchangeable. Ann demonstrated her flexibility of approach on problems #23, #26 and #28. With these problems, she was able to move comfortably to and from fractional, decimal, and percent representations and thereby generate alternate techniques for solving the problems.

#### Autonomy/Nonautonomy with Mathematics

Ann's problem protocols and beliefs interviews revealed her autonomy with mathematics. Of the 12 codable problem episodes, all 12 were coded as autonomous. This ranked her first among the six participants. Ann demonstrated that she had developed an internal voice with which to judge the validity of mathematics. She was not dependent on others to provide answers or to check the reasonableness of her results. Ann was motivated by a strong belief that mathematics should make sense and it was this underlying belief that seemed to support her independence. She believed that knowledge was accessible to her and that it was vital for her understanding of mathematics. She needed to know for herself if she understood the concepts.

#### Autonomy

Within Ann's problem protocols were many instances that supported her autonomy with mathematics. She repeatedly challenged the researcher's position, voluntarily checked answers, monitored her own progress through a problem, demonstrated an expectation that results should be internally consistent, and voluntarily summarized her results.

On several occasions Ann challenged the researcher's conclusions or suggestions. Ann

did this very early in the interview protocols. On problem #22 (multiplication of decimals), when the researcher offered a statement concerning Ann's understanding of decimals, she remarked:

Right, but I needed to think about it for a while to see whether I wasn't getting tricked into saying something I didn't want to say.

As this quotation illustrates, Ann was unwilling to agree with the researcher's statement until she had satisfied herself that it was valid.

On problem #35, the rope problem, Ann again showed her reluctance to blindly accept the researcher's answer as valid. After several false starts, Ann completed the computation to yield 1 foot. The following is an abbreviated outline of the conversation between Ann (A) and the researcher (R):

A: I end up with 1. . . . I don't trust that answer. I did something wrong again. [Ann rechecks her computation.]

R: So you believe that now?

A: I have no choice. I have no other options. . . . I don't believe it.

R: Because?

A: Because you have 6 going [around it] . . . then the earth is so many times greater.

. . .

R: What if I really blow your mind and said you didn't do anything wrong. The answer is 1.

A: I'd say all right, but my perception is not that good.

R: You'd believe it because I told you?

A: I'd be baffled, but I'd believe it.

Ann was unwilling to completely accept an answer that did not coincide with her intuition. That she finally acquiesced was due to the authority of the researcher. Although not cited above, the conversation between Ann and the researcher continued and included a more detailed discussion concerning the validity of the results. It should be noted that although the researcher offered occasional hints or counter examples to students' remarks, it was rare that a complete explanation was given. This was done to avoid setting up an expectation that the researcher was a source for explanations or guidance. Ann appeared unwilling to leave the problem unresolved. Her curiosity had been piqued as well as her determination to get and understand the answer.

In both of these examples, Ann demonstrated her hesitancy to accept answers without satisfying herself about their validity. Ann often felt the need to reaffirm the validity of results whether suggested by the researcher or by her own intuition. On problems #24 and #25, Ann voluntarily checked her results. In addition, Ann also felt compelled to have not only a valid answer, but an appropriate solution technique as well. This was evident in her protocols on the function concepts. On problem #62 in that section, Ann accepted her solution, but was dissatisfied with her solution technique because it utilized trial and error to generate the solution. She felt she had been "lucky" to have stumbled across the answer. She needed to have a process not dependent on chance - an equation in this case.

Another situation in which Ann needed to verify her results was on the sample test. Here, Ann worked the problems for herself before grading Chris' work. She used her answers only as a first check on Chris' work, not expecting the intermediate steps to be the same. She always thoroughly reviewed the problem steps to ascertain the problem's validity and the severity of any errors.

Along with verifying her answers, Ann also voluntarily summarized her results in problem #27, the remainder problem. This action was seen as autonomous, first because it was done voluntarily, and secondly because the summary was not mathematically essential to solving the problem. Hence creating a summary was motivated by Ann's own desire to have closure on the problem.

#### Summary

For Ann, autonomy with mathematics coincided with a conceptual view of mathematics. She believed strongly that students should be given the opportunity to apply their knowledge to new, unfamiliar situations so that they could test for themselves how well they had understood the concept. Ann also felt that the ability to use mathematics was dependent on knowing the rationale behind the processes. She expected mathematics to make internal sense and this inner voice - common sense, insight, and logic - should guide her solution of problems.

Ann's independence was strengthened by her conceptual view of mathematics. That is, Ann was successful in her independent actions because she had an integrated and conceptual

knowledge of mathematics which she could effective use to verify her solutions. In turn, her conceptual knowledge was reinforced and driven by a deep desire to understand internally the reasons behind mathematics.

## Tara

### Background

#### General Information

Tara, a junior, was enrolled in Algebra II, U.S. History, American Literature, Spanish IV, and Chemistry. Her extracurricular activities included Spanish Club and Student Council. In addition to these school activities, she was president of her youth fellowship group and she enjoyed playing tennis, and the piano, and studying Russian and Japanese on her own. She had no interest in the mathematics team saying that she was "not a competitive person" and "situations like that make me nervous."

On weekends Tara worked as a cashier-clerk at a bookstore in the local mall. When asked if her job utilized mathematics, she replied, "No!" After a moments reflection, she recalled some usage of mathematics.

[I] recap all the figures from the day before of all of our sales . . . and you add up all the sales and all the subtractions and all the charges and everything like that. And come out with your net sales and like that. I fill in the charts. That's how we keep it [the sales information]. They are premade. It's usually the manager or the assistant manager who does that.

Tara had a personal computer available at home which she used primarily for word processing and occasionally to review with a chemistry software program. She did not read any of the popular games or puzzle magazines.

When asked to describe what ways, if any, she used mathematics outside of school, Tara indicated that she found only basic arithmetic relevant and useful to her life. She was taking advanced mathematics in high school because

My mother says you have to have [it] to get into a good college. . . . I'm still in high school and maybe I will decide to [go] into something that has math in it. . . . I don't want to limit my options.

Tara planned to attend college and eventually become a lawyer. Uncertain about a major in college, she was considering foreign relations, Asian studies or perhaps Japanese. She also indicated that she anticipated taking mathematics through calculus in college "because it's probably required." She commented further by confessing:

I'll be honest, math is not something that I'm either strong in or that I have an extreme interest in. I just find it useless. Like addition and subtraction and division are like the only things I ever use. . . . It [Algebra I and II] doesn't seem like something that's important for me, for my future life. . . . When am I ever going to use this?

### Previous Courses in Mathematics

Algebra I. Tara had a bad experience with Algebra I. She was often frustrated by her teacher's, Mr. Jones, apparent lack of professionalism and felt that she had learned only the most basic concepts. She attributed her difficulty in Algebra II to her weak background from this class.

I took Algebra I which I really despised. He [Mr. Jones] didn't teach us anything, he drew cartoons. He taught us how to meditate, totally irrelevant material. He did not teach the class. Whenever we had questions, he would do it out as though we should have already understood it and then just put it aside. Therefore all of our class [except] two people who had already taken Algebra I once before in the eighth grade [received a] grade below 70. If that says anything to you about the teacher. It was terrible. I ended up having a lot of trouble before I ended up getting a tutor. I started getting like C's and I usually don't get like C's. . . . It just shouldn't have been something that had to happen. It was really such an awful experience with algebra. I had some troubles with Algebra II because I don't know any Algebra I. Because I learned nothing in Algebra I except like  $2 + x =$ , really basic stuff that I already knew like in third grade. So I really didn't learn anything. Then I went into geometry and learned a lot.

Geometry. For geometry Tara had Mrs. Smith whom she considered to be an excellent teacher. In fact, Tara attributed her like and enjoyment of geometry to Mrs. Smith. Unlike her experience with algebra, she found geometry fairly easy to understand due to Mrs. Smith's clear and simple lectures and thorough presentation of homework problems.

She was really an exceptional teacher. She was very organized and taught very well. . . . Everyday we would first start out by correcting the homework and she personally would go over problems. I find that what Mrs. Thomas [her Algebra II teacher] does with having students do the problems very confusing. . . . When she goes through them it works out a lot better and I understand it a lot better. Mrs. Smith went through all the problems and she explained them to us. . . . She would outline notes on the board, definitions and postulates and theories or theorems or whatever that's called. And she would teach us how to do everything we needed to do in our homework in that outline form, very neat. . . . Just an exceptional class. I never had a really good math class before and it just sort of taught me that I can enjoy math. I just need to have a good structured teacher who goes through things step by step in the simplest way possible [underline added].

Algebra II. After having had Mrs. Smith, Tara described Mrs. Thomas as confusing and seemingly unconcerned about individual's progress. Tara felt that Mrs. Thomas unduly complicated the mathematical concepts and was frequently disorganized in her lectures, going off on irrelevant material. Tara also did not like Mrs. Thomas' relegated the presentation of homework problems to students in the class.

In case you haven't noticed, like almost everyday, she [Mrs. Thomas] ends up making a mistake on the board which throws me off even more, because math is something I have to concentrate a lot on. It isn't natural [to me] [underline added]. Seeing her jump around from place to place on the board, sometimes she writes up a definition and then sometimes you don't understand how, what things mean. Like right now we're doing step functions. I don't even know where that fits in, [or] what's the purpose of doing it.



Affect

At times, Tara appeared nervous during the interviews. This anxiety seemed to be situational in nature and arose whenever Tara was presented with problem solving activities or when pressed for conceptual explanations for procedures or algorithms. Tara did not, however, exhibit this anxiety when the discussion changed to teaching issues or mathematics more generally. Often during the interviews, Tara would tap loudly with her pencil or twirl her rings and bracelet, especially when uncertain how to begin a problem. Her comments were often sarcastic which seemed to be an attempt to hide her nervousness or to release some of her anxiety. It also seemed that Tara attempted to read clues from the researcher's facial expressions. When queried about incorrect responses, she immediately erased her answers and looked at the researcher asking for clues and hints.

About midpoint through the series of interviews, Tara revealed that she, Ann, and Tom occasionally talked among themselves about their experiences in the interviews. She expressed that they had at times found the interviews frustrating when they could not answer the questions and were pressed for additional information.

When asked to describe her own ability with mathematics, Tara remarked that she felt confident that she could do well, but for her, mathematics required a great deal of concentration and a very clear straight forward presentation of the content material.

My own ability to do math. I think I have the ability to do math. I think I have the ability to get an A in math. But I also think that if there's something that I don't understand and that isn't just clear in my mind then a lot of times I'm just pressured into understanding it. Like if we're having a quiz the next day and I don't understand it and I ask the teacher to explain it, but I don't understand what she said. So I go home and try to figure it out, but I'm not ready. If it hasn't just clicked in my mind, then I won't do well on the quiz. I mean, I can do well. It takes a lot [of] concentration though. A lot more effort than some of my other courses.

Tara also found mathematics interesting and fun when she understood it.

It is interesting to learn how things work out. When they click in your mind that's also something that's interesting. It's kinda fun to learn math.

General View of MathematicsUnlike/like Mathematics

When asked to describe something unlike mathematics, Tara suggested writing such as stories or novels but not poetry. She reasoned that writing was unlike mathematics because:

Nothing is correct. There is no correct answer. There's a little bit of format but there's no solid format that you have to follow. It's a lot more free. You're free to do what you want. Your mind can sort of go off on its own direction. [It does] not have to be structured by something. . . . [Whereas mathematics was structured since] everything that you do has a certain way that it's to be done. There's just a way to do it and that's how you do it.

Tara had excluded poetry on the grounds that its form, rhythm, and rhyme were often highly structured.

Not surprisingly when asked to name something like mathematics, Tara selected chemistry. She chose chemistry because it used mathematics and was highly structured with right and wrong answers.

That is like mathematics. . . . Everything that is structured. I'm thinking chemistry. The only reason I'm thinking that is cause in chemistry we're doing a lot of mathematics. It's like the two courses are kinda parallel in that we're doing a lot of mathematics and there's always a certain way to do everything in chemistry. I mean the way that we're being taught . . . there's always a correct answer [underline added]. There's not a lot of room for variation . . . when we're given a lab we're expected to come out with a result. . . . It is a little bit freer than math. Freer in that, [when] you made a mistake in chemistry and you don't come out with the right solution then you can assess why you came out with the wrong results. You can explain your error and still get a good grade. In math you can't.

Tara's selection of writing as unlike mathematics and chemistry as like mathematics presaged her dichotomous views of mathematics and English. The issues of freedom of expression, as well as flexibility of form, and acceptability of answers reappears throughout Tara's discussion of mathematics.

#### Vocabulary lists

Mathematics. Tara's selection of terms for mathematics from the vocabulary list centered on the theme of mathematics as a structured and inflexible system of rules and formulas [see Appendix H]. Tara associated the following grouping of terms with her category of *rules*: *analyze, concentration, interpretative, hard, thought provoking, and logical, detailed, precise, thorough, specific, and factual; and chronological, sequenced, organized, well-defined, structured, controlled, rigid, and fixed.* These terms invoked the image of mathematics as being highly sequenced, precise and requiring detailed analysis. Tara commented on these terms by saying:

It's a big category rules. Rules--there are certain ways that they are to be done. And it's clear and it's chronological. You have to analyze the problem as you go along in a chronological order. And you have to think about it, thought provoking. It's structured in a certain way. I think that pretty much fits structured and ordered.

This theme of mathematics as being structured and inflexible was reiterated in Tara's

remarks on the following terms:

*controlled*: There's a certain way it's to be done. . . . There's only one kind of answer you can get. It goes along with right and wrong. So math is definitely controlled.

*factual*: I figured that everything that has to do with math is based on facts. Everything that I learned is something that is concrete and that's taught to me in a certain way.

In conjunction with the terms controlled and factual, Tara was emphatic that beauty was not a term that she could ever associate with mathematics. For her, beauty had the connotation of subjective evaluation and since mathematics was based solely on facts and provoked the image of a rigid structure; the two, beauty and mathematics, were incompatible.

I don't associate beauty with math. Beauty and creativity, exciting, free are all kinds of words that express a subject to me that's more free. It is free in that it's more open to interpretation. Into individualistic, ideas.

Tara indicated, as well, that the quality of mathematics problems as "being *right or wrong* causes *anxiety*." She also stated that her mathematics was learned exclusively through *memorization*.

Everything that I ever learned in math seems to be like memorization. Like memorizing the formulas and memorizing the theorems, memorizing the postulates and the this and the that. And so it all seems to be memorization.

When queried about the percentage of time she spent engaged in memorization of mathematics, Tara offered, "40%."

Although Tara's comments stressed mathematics as structured rules which are given to students, she did offer a few remarks suggesting that mathematics utilized *common sense* as well. The examples in her remarks were in keeping with previous comments in which she indicated that only basic arithmetic was useful and relevant to her life.

*common sense*: A lot of it has to do with common sense. Like when you think about a problem when you first get it. You're thinking, well if Jill has 6 pencils and Tara has 9, how many do they have together? You think, well am I going to add that or subtract. It's just common sense, knowing whether to add or to subtract it or something like that.

*valid*: theorems, theories, working thins out are valid. Algebra and geometry. Multiplying each side by a number and you come out with the same number. It's the same number when it's reduced.

For Tara, mathematics was predominantly rules and formulas that were executed in a certain way and learned through memorization. Mathematics was hard and required concentration to arrive at the right answer. That mathematics was right or wrong caused anxiety. It also meant that mathematics was the antithesis of individualism and free expression and with

it creativity, beauty, excitement and ideas. Mathematics could touch the individual, only in that, sometimes it invoked common sense when dealing with simple arithmetic word problems.

English. Tara's choice of terms for English and her associated comments stand in sharp contrast to her views of mathematics. Mathematics was almost exclusively prescribed rules and structure, whereas English was *free, instinctive, and exciting*. Tara grouped the terms: *exciting, creative, flexible, and fun* saying that it is fun "because it is free."

She also combined *insightful, interpretative, individualistic, expressive, and universal*, suggesting that she perceived English as a human endeavor. This idea was reinforced by her discussion of the term *instinctive*:

A lot of times it's easier to write just like art—about how you feel, on the spur of the moment. It's an instinct, but then with math you kinda go back and think about it you can't just go "I'm going to do [a] word problem [underline added]."

English could also be *abstract* or *concrete* depending on the author's intention when writing.

Writing can be anything. You know, it can be abstract and you might have to analyze it or it could be very concrete and understandable like that.

Like mathematics, English also had a side that included *rules* which Tara associated with *language, verbal, and themes*. These represented the rules of grammar and the structure in essay writing. Even with this perspective, Tara did not imbibe English with the characteristics of being dogmatic, right or wrong, rigid, fixed, dull or memorized. For Tara, the rules of grammar and writing did not inhibit the individual's creative process. The language rules were seemingly more integrated with the process of writing and hence did not form a barrier to writing. In contrast, mathematics was not something that she felt was natural. It came from the outside whereas English was aligned with one's emotions and expression of self.

To Tara, English represented a discipline that was creative, flexible, exciting, and fun. Writing was a means to express feelings and was an instinctive, creative process much like art. English also included rules of grammar and writing but these received relatively little emphasis in Tara's discussion. Instead she stressed the individualistic nature of English which permitted often controversial interpretations.

Science. Tara described science as made up of three components: the first represented the established theories of science; the second, the highly theoretical and current

experimentation; and the third, those topics studied by science which were continually changing.

Basically these fall into two categories . . . what's discovered now and what's factual. Like on a periodic table, the elements on it are fact. They do exist. Then there's also the other side which is stuff that's being experimented with now. There's also another part, I guess that's three, another part that's constantly changing. Like the evolution of the world, like that. It's continually changing, and that's chronological and stuff like that.

Tara's perspective on science had some similarities to mathematics. She circled terms like *controlled*, *dogmatic*, *fixed*, *right or wrong*, *factual*, *organized*, *precise*, and *sequenced* implying that science in its established phase was *structured* and *rigid*. Tara was also *anxious* during science tests because of the *precision* and right or wrong nature of the discipline, yet not to the same extent as in mathematics.

Tests, but math is the worst [because of] the precision. You just have to be precise. Like [in] science there's always some room [for] differing opinions. In English there's always a lot of room.

The right or wrong aspect of science, Tara associated with the established part. Like mathematics, there was much that needed to be *memorized*.

In the part of it that's already established, there's always a right or wrong. There are always the theories that you have to memorize and the chronological order that things come in and stuff like that.

For Tara the two disciplines of mathematics and science diverge with respect to creativity. While mathematics was seen as rigid and inflexible, science through its experimentation was seen as alive and requiring *flexibility* and *creativity*.

On this side over here where they're experimenting [to discover] new things, you have to be creative and you have to have some flexibility. They have to be able to make up an opinion . . . a hypothesis first. They just kinda make it up and then they go and try to prove it. So in that sense, it's creative.

For science but not mathematics, Tara circled the terms: *clever*, *creative*, *flexible*, *expressive*, *free*, *insight*, and *ideas*, suggesting by these that science interacted on a more individualistic level than mathematics. It was, in some sense, a reflection of the individuals who introduced often controversial theories. Science was also labeled *uncertain* and required *trial and error*, reflecting again the experimental nature of science. It also differed from mathematics in that science was seen as *practical* as well as *fun*.

Another aspect in which science differed from mathematics was science could be *fun*.

It can be fun. What we're doing now, like formulas and stuff. And I enjoy that . . . just formulas for molecules and conversions like moles to quantities to this. The amount that is in grams.

Note that Tara's examples in the above comments were only partially formed making it difficult to ascertain what was alluded to. However, this phenomenon was not restricted to science. She exhibited a similar lack of completion in her mathematical examples.

For Tara, science had several aspects: a factual stable component, a highly creative experimental and theoretical component, and finally it dealt with changeable evolving topics. In its experimental phase, science required creativity and flexibility while in its factual state it required precision and memorization. Although its answers could be right or wrong, there was also an aspect of science where opinions were valued. Finally, for Tara, science could be fun while at times causing anxiety.

History. Tara's selection of terms and her subsequent remarks revealed a view of history as both based on fact and subject to controversy. Historical evidence supplied the factual side while the interpretations of history yielded the often prejudicial and controversial side. Tara grouped the following terms together under what she called the changeable category: *multi-dimensional, multi-perspective, interpretative, opinionated, controversial, experiential, discovery, logical, cause and effect, themes, ideas, and truthful.* She described this category by:

History is written by people who are biased about the history during that time. So it's sort of opinionated, whatever they say. Their ideas are affected by the changes that took place.

She also commented on her selection of other terms:

*truthful:* It's something that happened. A belief that somebody has.

*valid:* Everyone's opinion is valid. Everything that they wrote down is valid.

*factual:* Things that happened in history are always factual. This did happen and that did occur.

*clever:* To write a historical account you would have to be clever.

*common sense:* I was thinking about the newspaper that Thomas Payne wrote. It is also common sense when looking back on history, knowing what things are going to happen when. Things like that.

As these remarks demonstrate, Tara held as valid and truthful any opinion of history because it represented an individual's perspectives. It is interesting that cleverness again appears and was required to construct a historical account. Its absence from mathematics was even more pronounced given its presence on each of the other lists.

Summary. The vocabulary lists revealed stark distinctions between Tara's views of

mathematics, English, science, and history. These distinctions center on the themes of freedom of expression, creativity, flexibility of form, and inherent interest. For Tara, mathematics and, to a lesser extent, science were governed by rules and prescribed procedures. In contrast, English and history were not burdened with an external structure and they valued individual opinion.

For Tara, mathematics represented a collection of memorized rules given by the teacher. These rules were characterized by inflexibility and control. Each problem had a certain way it was to be done. Consequently, answers were either right or wrong which caused anxiety for Tara. The absolute nature of mathematics precluded individual expression and hence creativity. To a lesser extent, mathematics could appeal to common sense but this was usually restricted to simple applied arithmetic situations.

In contrast to mathematics, English was characterized by being free, instinctual, and exciting. English, rather writing, was a medium for expressing affective concerns, conveying opinions, and showing artistic style. Writing was beautiful, often controversial, and enveloped the creativity of its authors. Unlike mathematics, the rules in English for grammar and essay writing had a minimal impact on the writing process. They did not deter the expression of individualism. In English the opinion of the individual was highly valued. Tara found English to be exciting and fun primarily because it allowed her personal input. For Tara, English, writing in particular, flowed from inside herself.

For Tara, science shared many traits in common with both mathematics and English. Her impression of science was dominated by its dualistic nature. Science had both an established-rigorous side and an experimental side. The established side mirrored mathematics in that it was governed by memorized inflexible rules and facts which produced right or wrong answers. On the other side was experimentation which mirrored the freedom in English. The experimental process required flexibility, creativity and cleverness and often involved differences of opinion. It was this side that Tara saw as exciting and fun.

For Tara, history also had two aspects. First, it was composed of facts representing actual historical events and second, it involved multi-perspectives on these events. Controversy over historical events arose when conflicting, sometimes prejudicial perspectives were expressed by individuals. Like English and science, history, too, required cleverness and creativity in the

expression of individual opinions which again were highly valued.

#### Ranking Grid

Tara's ranking grid demonstrated her abhorrence of fractions, dislike of word problems and confidence with writing proofs [see Appendix J]. Tara ranked fractions on affective traits as the most *boring* and *least liked* and on cognitive traits as highly *confusing*, *abstract*, requiring *original thinking*, *hardest to do*, most *difficult to learn*, *worst at*, and *least useful*. That fractions ranked so low on the grid seemed incongruent with Tara's background in mathematics through Algebra II with fairly good grades and her high scores on the initial placement test. Tara commented on her situation by stating:

I hate fractions. . . . I just never learned how to do them. I really don't know how to do them, know how to do a fraction. . . . Just give me a question with a whole number, that's all I ask.

On word problems, Tara ranked them affectively as *boring* and *least liked* and cognitively as *difficult to do*, fairly *hard to learn*, *worst at*, *advanced*, *thought provoking*, *rigid*, and *confusing*. She commented only briefly on word problems: "I don't enjoy them."

In contrast to fractions and word problems, Tara ranked writing proofs high in most positive traits. On the affective traits she marked proof as *most interesting*, and *most liked* while cognitively it was seen as *easy to do*, *easiest to learn*, *best at*, *logical*, *clear*, *fairly flexible* and *applied*. Tara indicated:

I just like geometry. I don't have steady hands. I wasn't good at writing, like drawing. That wasn't my favorite project, just like writing out the proofs, cause I can't draw.

To understand Tara's ranking grid it is important to consider her previous experience with mathematics. Tara liked geometry primarily due to Mrs. Smith's outstanding teaching. Tara had stated that geometry was the first class where she felt confident in mathematics and that Mrs. Smith was extremely thorough in presenting examples of all homework problems. Tara also remarked that she liked mathematics when it clicked. Since she had been successful in geometry, her high marking for writing proofs seemed consistent. This is further corroborated by Tara's high ranking on geometric constructions, another topic from her geometry class.

Just as her success in geometry translated into a positive ranking, her failure to understand fractions translated into negative responses. Her failure was further frustrated by her repeated encounters with fractions in her classes.



Tara's dislike of word problems coincided also with her earlier comments. She voiced the belief that mathematics always had a prescribed format and this was learned through memorization. While completing the grid, Tara also remarked that "math is not something that's flexible." Since word problems do not easily fit into set patterns, Tara would find them difficult and confusing. Recall as well, that Tara described only having understood the most basic procedures from Algebra I. Here again her lack of success would be translated into her dislike of and lack of interest in word problems, and her perception of word problems as useless.

#### Synopsis of General Beliefs about Mathematics

Tara's description of writing as unlike mathematics and chemistry as like mathematics served as precursor to her overall view of mathematics. Writing was free and instinctual while chemistry was fixed and rule-oriented. Tara's view of mathematics was dominated by a belief that each problem in mathematics had a certain way it must be solved. Students learned these procedures and rules by memorizing them. The inflexibility that characterized these algorithms, forced mathematical work to be evaluated as either right or wrong. This in turn had the affective impact of divorcing mathematics from the individual. To touch an individual, Tara believed a discipline must permit freedom of expression and value that expression of opinion. Mathematics could not achieve this since its processes were predetermined and dictated from outside the individual.

Occasionally mathematics could be fun or interesting but, for Tara, this only occurred when she understood it and saw the relevancy to her life. She cited geometry as a topic which held her interest but credited this to her extraordinary geometry teacher. Similarly since fractions and word problems were little understood, they were seen as useless.

For Tara, mathematics was perceived as a collection of rules and procedures which were memorized and precisely executed. It held little intrinsic value and lacked a humanistic side. Finally, mathematics was seen as originating from outside herself.

#### Student/Teacher Roles

##### Advice to an Exchange Student

Tara's advice to a new exchange student was very succinct: always do all the homework, asking questions in class if you have difficulty; and review for tests by working selected problems from each section in the textbook. Interestingly, Tara stressed reviewing from

the textbook rather than from class notes which she felt were "more confusing."

#### Advice to Beginning Teachers

To advise beginning teachers, Tara suggested that they give very organized mathematics notes preferably in outline form as Mrs. Smith had done in geometry. During the lectures, the teachers should avoid irrelevant material and afterwards should also be available to any student who needs help. Finally, they should specify what to expect on tests which in turn should be construct so that the average student who studies can pass them.

#### A Good Mathematics Teacher

Tara began her discussion of the qualities of a good teacher by stressing that:

A good math teacher is one who understands the individual need of an individual student and can adapt their programming to every individual's needs.

She continued her discussion by indicating that it was the teacher's responsibility to initiate meetings with individuals before they were in academic difficulty.

Her expectation for the teacher also included well-prepared lectures that clearly and slowly presented a step-by-step outline of procedures needed to complete the homework assignments. These presentations were vital for Tara's own understanding of mathematics. However, she felt that the teacher was not responsible for presenting examples of all homework-type problems prior to making assignments. Her rationale was that it did not matter since homework was merely checked off as completed and had little overall impact on her grade in the class.

It's not like it matters. It would be a different story if they give it on a test. They should go over it in class. If they expect us to be able to do it, they should just tell us how to do it.

When queried further about her expectations for teachers' responsibility for tests, Tara became adamant in her insistence that teachers should model all problems that they expected their students to be able to perform. If a teacher did not hold this view, that is, if they included questions on their tests that had not been specifically covered in class, then Tara said simply, "I would quit the class." She offered the following rationale for her view:

They definitely shouldn't be, if you're getting graded on it. You should be graded on what you have learned (underline added). . . . We're in school, learning from teachers. They're getting paid to teach us and that's their job. If they don't teach us and then give us a test on it, that's not a good idea.

Even when broached with the suggestion that such test questions might represent simple

extensions to check understanding of in-class concepts, Tara reiterated her stance, "I don't think that's right." She remained adamant and inflexible in her position about tests.

From other discussions, Tara had offered Mrs. Smith as an exemplary teacher. The qualities that she stressed were Mrs. Smith's compassion for students, her thoroughness in lectures exemplified by the clear outline format she used to present the material, her inclusion of relevant examples when giving definitions, and her presentation on the board of all homework assignments in complete detail.

Tara also felt keenly Mrs. Thomas' lack of motivational remarks and application examples. Without this relevance to her life, Tara found it difficult to retain interest in the topics.

It just doesn't seem like something that's important to me. . . . When am I going to use this? . . . Maybe if we had some explanation of that, had someone come in and talk about where math is useful in different careers. I really find that a common opinion among students is "Where are we ever going to use math?" . . . I'm never going to use this, so why am I learning this? . . . When they [concepts] click in your mind that's also something that's interesting. . . . I know in engineering it's very useful and [in] computers and everything like that it's very useful. [For] my understanding, I just don't feel like--I'm sure it will be useful to me--but a lot of stuff like that as we get into some more complicated stuff, I just don't see how it would be useful. When the teacher doesn't explain it, it's hard for me to see how it's going to be used. I do like math, if I can understand it and if it's taught well [underline added]. Those are my big criteria.

Tara related an event in class that exemplified her frustration. Another student in the Algebra II class had asked Mrs. Thomas a question about the use of function to which Mrs. Thomas had replied that its relevancy would be apparent later. The researcher was present in class during this exchange and could corroborate, in this instance, the brevity of the teacher's remarks and lack of motivational examples in the introductory lessons on functions.

For Tara, good mathematics teaching was manifest in attention to individual student's needs and in clear, methodology-oriented lectures. The class lectures should also include motivational examples that were relevant to the students' lives. Tests should duplicate exactly in-class examples and homework. Finally, teachers themselves should present all homework problems on the board.

#### Summary of Student/Teacher Roles

Tara had developed very explicit roles for students and teachers. She believed the students' responsibility included doing all homework assignments, asking questions in class when having difficulty with homework, and studying for tests by solving selected problems from each section in the textbook.

Tara believed teachers had several vital responsibilities. Foremost, they should be aware of individuals' needs and try to adapt their classroom presentations to meet these needs. In concern for the individual, teachers should initiate contact with any student having difficulty before it became a serious academic liability.

Another essential responsibility that teachers held was to present clear well-organized lectures. These should be presented slowly and should stress the step-by-step process needed to complete homework assignments. Furthermore, lectures should include applications and motivational examples relevant to students' lives. When homework was discussed in class, teachers should present all of it on the board in full detail.

In evaluating the students, teachers should construct tests that parallel in-class examples and homework problems. Tara strongly believed that it was the teachers' responsibility to model and prepare students for any material they expected the students to perform. In addition, teachers should explain to students what specifically would be expected of them on the test and should construct test questions that the average-prepared student could pass.

#### Conceptual/Procedural View of Mathematics

The analysis of Tara as holding a conceptual or procedural belief about mathematics was derived from two sources: the general mathematics overview and the coding of Tara's responses from the transcription of her solutions to the mathematics problems in the interview protocols. Of the 14 problem episodes that were codable for Tara, 8 were labeled conceptual and 6 procedural. Tara's ranking of 57% conceptual ordered her as last among the six participants in this category. (See Appendix I for a table summarizing the participant's coding on the conceptual/procedural and autonomous/nonautonomous category.)

#### Conceptual

Tara's problem episodes were coded as conceptual primarily on the basis of her ability to justify procedures in the terms of first principles and to use number sense to evaluate the reasonableness of an answer. These instances, though, were relatively weak since her exposition was often confounded with extraneous information and erroneous statements.

On problem #19 ( $1/4 + 2/3$ ) Tara was able to justify the need for common denominators by associating it with the traditional pie diagrams. She first drew and shaded pie representations

for  $1/4$  and  $2/3$ . She then proceeded to draw pies divided into twelfths. Tara demonstrated the ability to use number sense to justify her answer on problem #21 ( $1.50 \times .25$ ). Tara originally worked the problem by executing the computation on the calculator. When asked how she knew that the answer of  $.375$  was correct, she replied: "This calculator is very smart. [laugh] It's an intelligent calculator. But it also makes sense." She proceeded to justify her answer by arguing:

One and one-half, well, 1 [times]  $.25$  is  $.25$ . Then half of  $.25$  will probably equal something like  $.375$ . [laugh] . . . You have  $.25$  and you're gonna multiply it by  $1 \frac{1}{2}$ . So I obviously did it with a calculator, but I would start out [pause] just to make sure it's right,  $.25$  times 1 is  $.25$ . Then times the half will be something like  $.1$  something. So if you add  $.25$  plus the  $.1$ , it's gonna come out to something like  $.3$ .

After completing this explanation, Tara confessed that on this type problem that she would "never do it by hand."

This explanation revealed that Tara could justify her answer of  $0.375$  on the basis of number sense, here, being able to mentally recast the problem into a distributive expression,  $.25(1 + 1/2) = .25(1) + .25(1/2)$  in which she could justify the resulting addends by approximation.

When the researcher continued to question Tara on the algorithm for shifting decimal position in multiplication, she reverted to the stance that the decimal position was dictated by a rule she had learned.

So if you move it, [it] just makes sense, you know what I mean? No, you don't know. Yes, you do know what I mean, you're just [being silent]. That's the way I was taught to do it.

Again on problem #24 (find the smallest radical fraction), Tara was able to support her answer by use of number sense with fractions. Tara selected between the various radical fractions by computing the decimal representations on the calculator and then comparing the results. When she finished she remarked, "I guess, I probably could have guessed that." She rationalized that observation by saying:

Because root 5 is obviously like a half a root and a half of 5 is close to 2 something, and then 1 divided by that is going to be, we 1 divided by that is going to be bigger than 1 divided by this. [points to  $1/(5 \text{ sq rt } 5)$ ] Right?

On both problem #33 (select the larger of  $(p + 2)$  or  $(2 - p)$ ) and #34 (select the larger of  $r^2$  or  $r$  when  $r > 0$ ) Tara rationalized her response by using various specific numbers to illustrate various numerical combinations. On problem #33, she quickly indicated the answer was "choice c." By this answer she was referring to the standard pattern responses on SAT-type questions where response "c" meant either none of the above or insufficient information given.

She explained her choice by saying:

Well it depends. If  $p$  is a negative number then this one's bigger. And if  $p$  is a positive number, this one's bigger. If the value, say it was 1, 1 plus 2 that would be 3, so that one's bigger. 2 minus 1, that would be 1, so that one's smaller, so this one's bigger. Then if it was 2 minus -1, if the number was negative then that would be 3. And this one would be 1 plus 2 which is also 3. Well, just a minute here, no -1 plus 2 which would be 1.

Again on problem #34, Tara relied on numerical examples to justify her response.

Is that an obvious question, or? Well let's say  $r$  is 2, then this one's bigger.  $r^2$  is 4 and that's 2 for column A. So that's greater than 0. If it were less than 0 then it would be different. No it wouldn't really be the same column A.

When asked to consider 1 as a value for  $r$ , Tara noted:

Well then they'd be equal. But you said larger, which one is larger. [laugh] So, well, unless it's 1. Or, well you can't call this 1 either. Choice c. Cause if it's a  $1/2$ , if  $r$  is a  $1/2$  then  $r^2$  is  $1/4$  which is smaller than  $r$ . So D, . . . you can't call it.

The above examples were indicative of Tara's protocols that were coded as conceptual. She utilized number sense to justify results and basic concepts to justify algorithms.

#### Procedural

Unlike her conceptual coding, Tara's procedural coding arose primarily from her responses to problem-solving situations and the sample test. In these protocols, Tara demonstrated an expectation that problem solving situations can be solved merely by executing an algorithm or formula, and an evaluation scheme based on form rather than process.

On problem #30 (find the area of the triangle) Tara showed an expectation that the problem could be solved by merely replacing the appropriate values in the area formula for a triangle. Tara began by quickly assessing that she lacked the numerical values necessary to use the area formula. She immediately asked for a hint. When none was forthcoming, she summarized her rationale for needing to know either the base or the height of triangle ABD. At this juncture, it was suggested that she visualize the diagram as two separate triangles. She surmised from this separation that triangle ABD could possibly be isosceles. Failing to convince herself that this conjecture was valid, she reiterated her need for the numerical values. Next she questioned whether a solution existed. Assured of its existence, she then commented that it would take someone of Ann's caliber to solve it. As a final attempt, Tara invented a value for the base by speculating that the base, BD, and the height,  $h$ , looked equal in length in the diagram. From this value, she then calculated the remaining variables she needed to compute

the area of triangle ABC. She offered no rationale for her guess or answer but, instead, continued to look to the researcher for confirmation.

I'm scared. BD is equal to DC. The area of [the] shaded region is 8. The area of triangle ADC is what? . . . That helps a lot [pause] Answer D. Insufficient information. The area of triangle ABC. All right. Those two are equal. [BD = DC] This obviously equal to itself. That doesn't help though. Little hint. . . . I don't know either the base or the height. . . . If I had the base then I could say equals 1/2 times h, then I could figure out what h was. Then the h is the h for the whole triangle, then I would be able to figure out the area of the triangle. But because I don't know anything [underline added]. . . I have no idea. I'm thinking about how I could figure out either the base or the height.

Even with the suggestion to visualize the triangle, ABC, as two separate triangles, Tara remained insistent that she needed the numerical values in order to calculate the area.

It looks like isosceles. Well, maybe, it is, then again, maybe it's not. Or it could be, no, that looks more isosceles which means that this would equal this. That doesn't help me, not that I know that everything is equal. I need the number [underline added]. . . . Impossible to figure out, am I correct? . . . Oh, only one person can figure it out, Ann. . . . The area of triangle ABC is? How many times can I read that problem over! Let's see. . . . Is this drawn to scale? . . . Since the bell is going to ring you can tell me. . . . I'm not sure why I think that this was equal to this [h and AD], but I have a feeling that that's wrong. . . . So then it [would] be 4 times 4. Well let's pretend that's true. How about we pretend that's true? OK? So then we pretend that the height is 4 and that this was 4 and this is 4. [BD and BC] But that can't be right, but OK. Then for the regular triangle [ABC] we have 1/2 times 8 times 4, we'll get 16, 16.

When asked if she believed her answer, Tara replied: "You'll laugh, no. [laugh] Well, I mean, I'm sure I would if I knew what the height was." To explore further her rationale for selecting 4, Tara was asked if she could pretend with other values. She took this question as a suggestion and tried other values, all the while looking for guidance from the researcher.

What times what is 16? . . . So we'll pretend that that's the 8 and that's the 2, 1/2 of I times 2, 4, 8. . . . Hey, those look the same. [laughter] Are you trying to tell me I should guess before? Yes you are. What did you get. OK. So it looks like 16 would be the answer. . . . Looks like 16. 16 is the answer of the day.

As a secondary question, Tara was asked why 1/2 was part of the area formula. Not unexpectedly, she claimed it was there because it worked. "Cause that's the formula. Good question though. I don't know. . . . Cause it works. It works, who needs a better reason."

In both her attempts to solve the problem and in her comments on the area formula, Tara revealed her dependency on the memorized area formula. Without either the explicit values for the base and height, or some visual clue as to the relationship between them, Tara was lost. She repeatedly expressed the necessity of knowing the numerical values and pushed the researcher for hints and finally, for the answer. Tara received a procedural coding on this problem due to her dogmatic reliance on the area formula with specific values for the solution

process and her inability to offer any rationale for the derivation of the area formula.

Again on problem #35 (the rope problem) Tara showed a dependence on the execution of a formula for the solution process. In this problem, Tara began by calculating the circumference. Once completed, she had no further strategy for finding the height above the ground and pleaded for a hint. The suggestion was made that she draw a diagram of the situation. She then requested the specifics she was to draw. Although the conversation continued, Tara supplied no further initiative in solving the problem. Tara's final arrival at the solution resulted from a sequence of leading questions.

Throughout the problem solution, Tara demonstrated inflexibility in applying the circumference formula. She was unable, on her own, to draw a diagram to represent the situation or to set up a variable equation using the circumference formula. That she was expecting an algorithmic process was inferred from her concluding remarks. When asked if she believed her answer, she replied that her certainty was conditional on being taught the process for solving this type problem. The following is an abbreviated excerpt from that concluding conversation:

R: On a scale of 1 to 10, how confident [do] you feel about your answer? What would you rate your answer?

T: Oh, one.

R: What would help you feel [more confident]?

T: An answer book. Or knowing how to do it. I have no idea how to do it. So if I know how to do it then I can.

R: What would help you know how to do it?

T: Learning. [laugh]

R: I'm not sure what you mean by [that]?

T: I doubt that on a test the teacher would give me this kind of question, that they never taught me how to do before. So if they taught me how to do it first, I would be able to repeat [it] [underline added].

The dependency on an algorithmic solution was evident as well on problem #27 (the remainder problem). After reading through the problem, Tara constructed a series of equations:  $x/3 =$ ,  $x/4 =$ , and  $x/5 =$ . She then immediately requested assistance. When none was forthcoming, she insisted that the researcher knew the answer and pleaded again saying she had no idea where to begin. When it was suggested that she examine some numbers, she



complied by trying the integers 1, 2, . . . , 11, 17, and 19. After that, she reverted to her usual tactic of claiming that there was no solution. Her solution protocol follows.

What is the smallest positive number which when divided by 3, 4, or 5 will leave a remainder of 2. Oh, OK. We could try to figure this out. What is the smallest positive number, which one is divided by 3, 4, or 5? [pause] divided by 3. [pause] I have no idea how to do this problem. . . . I have no clue. How [do] you do this? . . . You obviously know how to do it. . . . Well, the remainder 2 is going to be like  $2r \times$ , all right? Two  $x$ 's. So  $x$  over 3. No clues. . . . [after divisions] that doesn't work. Nope! [pause] 17, nope, 19. This is like one of those problems you skip. [laughter] One of those, all right. All right like answer, D. Choice D. Nothing. No answer. No solution. . . . This is impossible, this can't be done, so. [laugh] . . . I don't know the answer.

When asked if she thought she would eventually discover the answer by checking values, she responded by again trying numbers and failing in that attempt, again argued that there was no solution.

It doesn't seem that there's such a number, but I'm sure that there is. You probably wouldn't have the problem [without] an answer. I'm sure that there will be an answer. [Tries 20, 27, 28, 29, 30, 31, and 32]. . . . I am becoming more and more convinced that there's no such number. But is there, there is a number? And that I already passed? Oh great, we'll try like 51 or something. . . . I'm not going to stay here all day trying to figure this out.

When asked what she would do if given this problem as homework, she replied, "Skip it!" She also said that when stuck she usually looked at the answer at the back of the book and tried to work backwards. After these remarks the answer was given to Tara. She merely checked it's validity and made no attempt to rationalize why it worked. In fact, she turned to the researcher and asked for the equation to produce it.

I usually look in the back of the book and find the answer, and the work backward. Cause it's easier to understand if you know the answers. [checks validity of 62] I haven't done remainders in a long time. Yeah, looks like you're right, of course you're right. Yes it works. No wonder I couldn't find it. . . . Honestly I have no idea. I just don't have no clue. So you can tell me now. OK I mean, what's an equation that you used to get the answer [underline added]?

As the above protocol on problem #27 revealed, Tara expected the problem to be solved by an application of an equation. When none was immediately obvious, she had no options available to her to explore the problem. Again her protocols showed a dependence both on an algorithmic approach to the problem and on an authority to supply that approach when it was not readily apparent.

The above protocols for problems #30, #35, and #27 document Tara's reliance on the execution of a formula or equation to solve problem solving situations. This dogmatic expectation was coded as exemplifying a procedural view of mathematics. Tara's belief that

these processes should be modeled by the teacher if students were expected to solve this type of problem also reflected her expectation of a solution process.

Another area in which Tara received a procedural coding was on her grading of the sample test. In light of her attempts at solving unfamiliar situations, it was not surprising that she graded the sample test solely on the basis of familiarity of form and on the correctness of the answer rather than the process. Here again she revealed a dependence on the execution of memorized rules.

When evaluating question 1d, Tara began by working out the problem for herself. She then compared answers and since they matched she said simply "so that's right." No examination was given to the solution process. The answer was merely checked.

Again on question 11a, Tara evaluated Chris's work on the basis of the answer. As before, she solved the equation herself and then compared answers.

Solve for  $x$ . The answer in set notation. OK, let's calculate this part. So here he got, oh Chris you're in trouble. [she adds the points taken off so far on the test] Fifteen. . .  $3x$  minus 7 equals 4. He got it right [underline added].

When evaluating question 11d, Tara commented that the answer had a form that she recognized from her experience with absolute values and so she was inclined to believe the validity of the equation but was uncertain about the subsequent conclusion.

I don't think that makes sense when I think about it [the conclusion]. But if I remember that, we just did absolute values, a lot came out like this [underline added]. I think that

more than clues you in. . . I think that Chris got the right answer. But he did that [referring to the conclusion]. I think I'll take like four off. Four, it's important.

As Tara's comments on these three questions reveal, she was judging the test solely on the basis of the answer or the form of the solution. This trait was coded as procedural since the validity of the mathematics resided in the answer exclusively.

Tara's comments on several of the other test questions were confounded by misconceptions. For example on 1a, Tara marked the solution correct saying:

That's correct, because  $x$  to the 13th and you add the 2 so it's  $x$  to the 15th. Ok. So we'll give them six points for that.

Again on 1e, she graded it as correct: "six divided by zero is zero." On 1c, she took off three points saying:

$x$  squared minus 1 over  $x$  minus 1. OK. [pause] That's right you can cross those out, [pause] then you write  $x$  minus 1 over 1. Then you would have [a] negative. He doesn't

have a negative here. Chris. I don't think that it's wrong. I think we can give him, I think we will take three points off.

In addition to her comments on Chris's answer, Tara offered numerous remarks about the composition of the test itself, her difficulty in grading it, and the expectations that she as the teacher held for Chris's work.

He's wrong. Chris is just mistaken. . . . That's stupid [referring to his work on 1b]. He shouldn't have done that cause I taught him better than that and I told him not to do that.

It would take me a long time to correct this if I were the teacher.

This is really a short test.

I hate these, when there's no value for  $x$ . [referring to 1c]

Chris, I think you need to get a new test. [laughter] Do you think the teacher might have written a bad test? Think it was, what do you think? [addressing the researcher]

But who would write a test where two out of four, [or] a half of the problems in a section were no solution. Then again, I wrote it, so I should know.

These excerpted comments correlate with her earlier remarks on her expectations for good teaching. There, she had voiced the opinion that tests must parallel typical homework problems. These comments also reaffirm Tara's view of mathematics as rules given by the teacher which the students must memorize and repeat on typical tests.

### Summary

Tara's problem protocols were coded as both conceptual and procedural. Examining the problems in each category revealed that Tara was able to offer a rationale for her answers and processes on those problems that involved simple arithmetic algorithms. However, on the more complex problem solving situations and on the sample test she was unable to give such rationale. Instead she revealed a deep expectation that these problems should be solved by executing an algorithm or formula which had been demonstrated by the teacher.

While she showed some conceptual understanding of basic algorithms, Tara's protocols were dominated by her procedural expectations for mathematics. Tara had earlier expressed a view of mathematics as prescribed procedures which were memorized:

Everything that I learned is something that is concrete that's taught to me in a certain way. . . . Everything that I learned in math seems to like memorization.

Consequently when put into unfamiliar situations, Tara clung to her expectation that one needs to find the correct formula or equation in order to solve mathematics problems. This need for

an algorithmic approach was tied to her deep dependency on outside authorities, either the teacher or the researcher, to supply the necessary approach.

This view of mathematics as procedural in nature was also seen in her protocol on the sample test. Tara graded the test questions based solely on the match between her answers and Chris's. This reliance on answers coincided with her view of mathematics as right or wrong. To Tara, mathematics was a prescribed technique to get to the right answer and she seemed tacitly to presume that the correct answer implied the correct approach. Tara's problem protocols and beliefs about mathematics also suggested a procedural view of mathematics. Each problem had a prescribed technique and answers were subsequently right or wrong.

#### Autonomy/Nonautonomy with Mathematics

On the codable problems, Tara received a 33% autonomous rating. With this rating, she ranked last among the six participants. As with her conceptual/procedural coding, Tara's autonomous/nonautonomous coding separated into routine arithmetic problems and problem solving situations. Generally those arithmetic problems for which Tara had developed a conceptual rationale were also ones for which she demonstrated a self reliance by monitoring the reasonableness of her process and answers. Similarly in the problem solving episodes, Tara revealed a strong dependence on the researcher for answers and hints, and expressed an expectation that all solution processes should be first modeled by the teacher before students tackled them.

At first this dichotomy suggested that perhaps the original coding had been influenced by some unconscious prejudice on the part of the researcher that conceptual-autonomous and procedural-nonautonomous categories were linked. To check this possibility, the researcher consulted the original rationale for Tara's codings, looking for instances of a conceptual-nonautonomous or procedural-autonomous match. Only one example, problem #21 ( $1.50 \times .25$ ), fit the conceptual-nonautonomous category. As still another check on the results, the independent raters' coding for Tara's protocols were re-examined. The same general pattern emerged in the independent coding. Those problem protocols that were rated conceptual were also autonomous and those that were procedural were also nonautonomous. The independent coding reaffirmed the dichotomy observed between the arithmetic and problem solving episodes and between the conceptual-autonomous and procedural-nonautonomous categories.

### Autonomy

Representative of Tara's autonomous coding were problems #26 (20% of 85) and #23 (fractions greater than  $\frac{3}{4}$ ). On these problems Tara showed an expectation that the problem solution should make sense and monitored her own progress. On problem #26, Tara used the calculator to compute the percentage. She then voluntarily recast the problem as 80% of 85, recomputed the results and subtracted it from 85, yielding again 17. By the time this calculation was done and she felt satisfied that the answer was reasonable, Tara had changed the problem to model the type of situation she usually encountered at her job in the bookstore. She reinterpreted the problem as a markdown situation for which she needed to find the amount of the discount.

Find 20% of 85. . . . What we do in the store when it's a sale [is] find 25% off, find how much it costs. But we don't want that, we want 20% of 85. . . . [use calculator and gets 17]. How about saying 80%. Yes [laugh] I found 80% and subtracted it for 85.

Through this process Tara demonstrated that she expected the result to relate to her experience with percentages in the bookstore. She also checked her result in a second way,  $85 - (.80)(85)$ , to verify the reasonableness of the value. These independent actions by Tara were coded as autonomous behavior.

On problem #34 (fractions greater than  $\frac{3}{4}$ ) Tara again monitored her progress. She began by trying to select from among the various approaches to the problem. First, she considered reducing all the fractions, then finding a LCD, and finally settling on pair-wise comparisons. That she expected the problem solution to make sense was inferred from her solution technique. She drew a circle shading  $\frac{3}{4}$  of it and then as she compared each fraction in the list, she tried to visualize its proportion on the circle. As each fraction came up, she also varied her technique of comparison to fit the complexity of the numbers involved.

I suppose I would reduce them. Well, no I wouldn't. . . . I'm thinking about whether it would be better to find like a common denominator, but then I'm thinking why? You know, what's the point of doing that. [laugh] All right. I guess that I'm gonna make a little circle on my paper and divide it into fourths and shade in three-fourths so I can compare it. And then I'm gonna take the first fraction which is  $\frac{15}{20}$ , and reduce that which equals  $\frac{3}{4}$ . So that's not more than  $\frac{3}{4}$ . So we can x that out. OK, then  $\frac{19}{24}$ , OK. We'll say under,  $\frac{18}{24}$ , and I was to reduce it by 6, that would be  $\frac{3}{4}$ , but 19 is more than 18 so that must be more than  $\frac{3}{4}$ . So circle that one, cause that looks like it's more than  $\frac{3}{4}$ . [And]  $\frac{71}{101}$ . There is a problem cause I don't know what goes into 101. . . . Oh, we'll leave that one [ $\frac{71}{101}$ ] behind. Let's do  $\frac{13}{20}$ . [The]  $\frac{12}{20}$ . . . . So  $\frac{3}{5}$  is less than  $\frac{3}{4}$ . That's 13. [The]  $\frac{13}{20}$  and  $\frac{3}{5}$  is less than  $\frac{3}{4}$ , no, yeah,  $\frac{3}{5}$  is less than  $\frac{3}{4}$ . So that's not, not more than  $\frac{3}{4}$ , so I mark that one. But  $\frac{35}{71}$ ,  $\frac{35}{70}$  is  $\frac{1}{2}$  . . . a lot less than  $\frac{3}{4}$ .

Eventually, Tara returned to  $71/101$  but in the process of trying to reduce or modify it to give a fraction that she was comfortable with, she became confused. Finally, she used the calculator to compute  $71/101$  and then compared the result to  $0.75$ . Even though Tara made some subtle mistakes in her elimination process, she still expected the result to coincide with her knowledge of fractions.

Tara's autonomous codings were based on her expectation that her answers should make sense and on her monitoring of her solution process. These characteristics were evident in her protocols on basic arithmetic problems.

#### Nonautonomy

Tara's nonautonomous codings were associated with her problem-solving protocols. In these protocols, Tara demonstrated a deep dependence on the researcher for hints and answers, and a strong expectation that mathematics consisted of memorized patterns presented by the teacher. On problem #30 (find the area of the triangle) Tara looked to the researcher for direction on the solution process and verification of her answer.

Little hint. . . . Since the bell is going to ring you can tell me. . . . How about we pretend that's true? OK? . . . Are you trying to tell me I should guess before? . . . What did you get?

When questioned about the area formula for a triangle, Tara also expressed the view that the formula was valid because it worked. This rationale suggested that Tara was abdicating the validity of the formula to some outside authority: "Cause that's the formula. Cause it works! It works, who needs a better reason."

Again on problem #27 (the remainder problem) Tara relied heavily on the researcher to supply the process and in the end demanded the algorithm for generating the solution.

I have no clue. How [do] you do this? . . . You obviously know how to do it. . . . But is there a number? And that I already passed. . . . I just don't have no clue. So you can tell me now. OK. I mean, what's an equation that you used to get the answer?

A final example indicative of Tara's nonautonomous coding was problem #35 (the rope problem). Tara again requested and was dependent on hints from the researcher. She also abdicating the validity of her final result to either a textbook answer or the teacher, expressing the view that she expected the teacher to model such problems before students attempted them.

An answer book. Or knowing how to do it. I have no idea how to do it. So if I knew

how to do it then I can [be more confident] . . . . I doubt that on a test the teacher would give me this kind of question, that they never taught me how to do before. So if they taught me how to do it first, I would be able to repeat [it].

I have no idea. No clue. Hint? Time for hint. Time. A piece of paper with a hint on it.

### Summary

Tara's beliefs about mathematics coincided with her autonomous/nonautonomous codings. Tara had suggested that some practical arithmetic problems were based on common sense. This belief was manifest in her approach to several of the basic arithmetic problems where Tara demonstrated that she expected the results to make sense. She even went so far as to recast problem #26 (20% of 85) into a familiar practical word problem situation.

However, this belief that mathematics entailed common sense was not pervasive in her belief system. Tara's belief that each problem had a prescribed technique and that this technique was to be presented by the teacher were much stronger. Tara had expressed this view in her description of mathematics as being controlled, fixed and memorized and in her description of good teaching as presenting clear step-by-step outlines of problem techniques. This dominant belief that mathematics consisted of memorized techniques was repeatedly manifest in Tara's problem-solving protocols. She relied almost exclusively on the researcher for guidance and verification of her results. At times, she even openly acknowledged that her faith in her answer was dependent on having been taught that type problem. She entrusted the validity of her answer to the execution of a prescribed memorized technique.

In conclusion, Tara's problem protocols and beliefs about mathematics revealed a nonautonomous approach to mathematics. Tara perceived mathematics as originating outside of herself and consequently was dependent on the teacher to supply techniques for each problem situation and to serve as the source for verification of answers.

Tom

Background

General Information

Tom, a junior, was enrolled in Algebra II, Short Story, Spanish II and Chemistry. Outside of school his interests included playing ice hockey, racing cars, and playing guitar in a neighborhood band. Although currently jobless, Tom had worked as clerk in a car parts store and as an "executive petroleum dispenser," that is, as gas station attendant. He had no interest in the mathematics club, although he liked mathematics and found it easy.

Among the technological resources available to Tom were a home computer which he used occasionally for word processing and frequently for video games like Test Drive and King's Quest. He used his scientific calculator only in chemistry class. Since Tom did not enjoy reading he was uninterested in either puzzle books or game type magazines. Instead, he limited his magazine reading to his great passion: racing.

When asked to describe in what ways, if any, he used mathematics outside of his classes, Tom mentioned mentally figuring change at his job at the gas station. He could think of no other specific instances saying: "I don't know, just normal everyday type of things like doubling a recipe."

Tom planned to attend college possibly majoring in either mechanical engineering or a business field like accounting. As a consequence of his career choice, Tom anticipated taking many mathematics courses while in college. After college he had no specific plans remarking that:

I plan on going to college. My parents plan on me going to college. I actually plan on it too. After college, I'm really kinda up in the air. I don't really have anything to look forward to I guess. Not that exactly, I just don't have anything planned right now. It still seems so far away even though it really isn't, like just 6 years from now. Who knows what'll be going on.

Previous Courses in Mathematics

Algebra I. Although successful in Algebra I, Tom was adamant in his disapproval of Mr. Brown's teaching style. He felt that Mr. Brown was unclear in his explanations and often ridiculed students.

Mr. Brown [was] the worst math teacher I've ever had in my life. . . . I can tell you why. He didn't explain things until the night after you did your homework. He'd give the assignment. You'd go and do your homework but you didn't know what you were doing.



And then [he would] have you ask questions. If you asked too many questions he'd say, "Oh, you ask too many questions." So he'd give us more homework for the next night and we still wouldn't know what we were doing. So that wasn't really helpful. . . . He never explained things. He never made things clear. Didn't work with the students. . . . Anybody could have stood up there and did what he did. So that's basically what went on. He'd go over one problem of how to do the next night's homework and everybody would just sit there with a dazed look on their face, going "What?" cause nobody understood what was going on. I don't want to seem like stuck-up or whatever. I don't want really have that big a problem with math so I was able to keep up and figure most of the stuff out for myself. You know, but for a lot of people, if they don't have someone to show them exactly what to do, they can't do it. But I didn't really have that big of a problem. I had like all B's and A's in the class so that wasn't bad.

Geometry. Unlike his experience with algebra, Tom enjoyed geometry with Mrs. Thomas.

He spoke enthusiastically about the challenge of writing proofs, citing its emphasis on thinking processes.

I loved geometry so much. I thought it was so much fun. It was so much different from algebra, cause I don't really like algebra that much. It's not difficult for me but I just don't like it as much as I do the geometry. [In geometry] we weren't working with numbers so much as the thinking process [underline added]. Like we were doing the geometry proofs and stuff like that. We had to think [underline added] and know theorems and postulates and that kind of thing. I just found it a lot more challenging than algebra. . . . Even though it was challenging, I sorta liked it because it was challenging, cause usually math is just like do the homework [underline added], . . . do it out, like it's done, "no big deal." But geometry, I had to sit down and think, get into it more, and that was kinda fun. I just kinda liked it to be challenged [underline added].

Algebra II. Tom also had Mrs. Thomas for Algebra II. He described her as "the best math teacher I've ever had." About the class itself he said merely that he understood the material so far.

So far it seems just to be review from the Algebra I. So it's not really that hard. . . . [Functions] doesn't seem to be a big deal really. Once again, that sounds kinda stuck-up and I don't want to sound that way, but it doesn't seem that hard to me. I've always been able to do work easily in math.

With regard to doing his homework he said:

I'm unmotivated I guess. If I don't really like what I'm doing or if I already think I know how to do it, I usually just kinda blow it off. My homework grade will be very low that's usually what keeps me from getting A's in math, cause all my tests are usually up in the 90s or high 80s. I usually have like a 72 homework average so it brings me down. That's just a personal thing, just cause I'm lazy more than anything else.

When asked if he would have been more motivated in the Algebra II/Trigonometry class, Tom replied that he thought he could handle the material. He explained his confidence by describing his fifth grade class in California. In that class he reported the students were covering eighth grade level mathematics. Despite his confidence, he was unable to enroll in Algebra II/Trigonometry since he was out of sequence with the students in that program. They had

begun their mathematics track with Algebra I in eighth grade but his local junior high did not offer such advanced classes. In fact, he described his eighth grade program as being:

Geared at the lowest level in the class. So anyone who was above that, floated around. Like what am I going to do, cause they didn't offer any Algebra I class.

#### Affect

Throughout the series of interviews Tom appeared confident and at ease. Though cooperative, occasionally he became frustrated during the interviews when the researcher continued to press after he had completed a problem. He remarked: "You get me questioning myself. And when you ask questions, I stop and think why is she asking this. Is that the right answer or not." Like Ann, Tom also seemed to enjoy the problem solving situations showing great persistence and agility in his tactics.

When asked to describe his own ability to do mathematics he said:

I think it's above average. Like I said before, I never had any problems doing math. It's not something I go "Oh no, time to do math," really it's like "math, OK." Whip it off, it's not real hard. So I think I've got a little bit more ability, more than [average], above average ability in math. [I] do it real quick without even having to think about it. It's not something I have to fight with and think about every single step along the way.

He indicated, as well, that:

It's always been my easiest subject. . . . Which I don't understand cause both of my parents can't do math problems. I mean [they] can't balance a checkbook. They say "Tom come here." [laughter].

He elected to take a third year of mathematics in high school saying:

I want to. I like math. It's a class that I can do well in. That even if I don't need to take it, I'd still [take it since] it's something I enjoy. I like math, so I do it.

#### General View of Mathematics

##### Unlike/like Mathematics

Like Ann, Tom had difficulty imagining something unlike mathematics. He saw mathematics as permeating everything.

I don't think I can see anything. I mean, you use math in everything. Everything's got some math into it. You always [use math] when you build something [since] you have to know measurements and stuff and that's part of math. Making a shirt, for example, you have to know how big to make it or how small to make it or whatever. I can't think of anything right off the top of my head that doesn't use some sort of math.

When asked if that included even his rock band, Tom replied:

Well, yeah, cause you have to count and stay in time to the music. That's, I guess, a form of math. I seems to me anyway.

Tom's example of something like mathematics was particularly telling of his perceptions about mathematics. Tom described the essence of mathematics as the ability to solve problems. He stressed that mathematics was the thinking behind the procedure. He argued as well that mathematics entailed the ability to apply the knowledge to real life situations not merely numerical examples.

Similar [to math]? It's just having a problem and being able to solve it. It's just like [an] everyday thing. . . . [There is] a lot more to taking math [than] just being able to solve a number problem. Being able to do the thinking to get to those steps [underline added]. If you can do it with numbers you should be able to do it with like people for example. They should be able to come up with a solution.

In his attempt to describe something unlike and like mathematics, Tom expressed a perception of mathematics as visible in everything. Mathematics, to him, was the thought behind the procedures as well as the application of the procedures to real world situations.

#### Vocabulary Lists

Mathematics. Characteristically Tom volunteered few remarks concerning his selection of vocabulary terms [see Appendix H]. When asked about specific selections he offered the following comments.

*sequence and chronological*: Just how you do things. Like powers, parentheses, multiplication, division, addition, subtraction. It's the way that you're suppose to do them.

*individualistic*: You do [it] by yourself. Something you do by yourself.

*goals*: You're looking for the solution of a problem and that's sort of a goal. When you achieve it, when you get the right answer, you've achieved the goal you were looking for.

*universal*: Everybody uses mathematics. It's all over the place. On one of those space orbit things they sent out [Voyager II]. They have some message in math in numbers to try and communicate with people and beings or what ever. Something that everyone understands.

*writing*: You have to write them down. That's the only thing I really don't like about math, having to go through steps and writing things out [underline added]. Cause I'm lazy. I'd rather just look at something and get the answer in my head then write it out. That's why I have like problems on tests and stuff with Mrs. Thomas. Cause like I'll [think] things in my head and she can't see the work, so she thinks I cheated or something.

*cause and effect*: What you do to the problem effects your answer.

*memorized*: Cause you need to memorize just basic things. Like in multiplication. What you do first [order of operations]. [You can not learn mathematics by just memorizing] because you never get the same problem. If you went around memorizing  $2x = y$  and  $x = 3$ , that's great. But what happens when  $x = 4$  and you have to memorize that one too. And then when  $x = 5$  and all the way. . . . You could not do it. You can [however] memorize the procedure to do it.

Tom's vocabulary list suggested a multifaceted view of mathematics. He circled terms

indicative of an awareness of the structure of mathematics which entails using rules and procedures: *rules, formulas, memorized, precise, specific, short cuts, routine, organized, sequenced, chronological, and structured*. He also marked terms like *common sense* and *instinctive* indicating an intuitive component to mathematics. Still another facet might be labeled analytical to correspond with his selection of the terms: *analyze, thought provoking, concentration, and ideas*. A final facet was inferred by Tom's selection of the terms *discovery* and *trial and error*. These suggested a nonprocedural perspective to mathematics.

Tom's vocabulary list and comments reflected a view of mathematics that required both the application of rules and original thinking in the solution of problems. Tom also saw mathematics as universal, permeating all aspects of life. Finally for him personally, mathematics was easy, simple and, most importantly, fun.

English. Tom's selection of vocabulary terms associated with English suggested a dualistic view of the discipline. One aspect of English represented the format and structure of writing, while the other represented the controversial content in the writings. Under the various forms of writing, Tom grouped *abstract, creative, arbitrary, objective, opinionated, and symbolic*. In addition he circled such terms as *structured, ordered, rules, organized, and logical* suggesting that the various forms of writing were governed by structure and rules.

While the form of writing may be structured, its content was open to individual interpretation. This view was indicated by his choice of terms: *individualistic, open-ended, interpretative, opinionated, capricious, and controversial*. Since its content was *individualistic*, writing could be *free, expressive, and flexible*.

With regard to affective traits, Tom circled *anxiety, dull, boring, and hard*. As with mathematics, Tom needed to be interested in the topic before he would participate. English failed to attract his attention. In fact, he intensely disliked it. He felt the abstraction in writing was often unwarranted.

I hate English. It's boring. I don't do well in it. It doesn't really do anything for me. I don't think there is anything that can make English better. It's hard to say why [a person doesn't] really like it. It's just not as interesting as other things you could be doing. Like I just had to drop English class, American Lit. The teacher wanted us to analyze and she wanted us to say what the author was saying when he wrote down the passage. I just look at something and I read it. She was like analyzing the black hand. I read black hand and go he probably had a glove on it or something like that. These guys [classmates] are writing papers three pages long about black symbolizing death and stuff like that. I just look at things and see them or I don't. I don't like look past them as easily.

Although he disliked English, Tom believed it was necessary for an individual to learn to speak and write well. In this sense, writing was *useful, practical, and valid*.

I think that you should be able to speak well and write well. . . . So I can see why it's valid to learn. That's the basis for knowing yourself.

For him, the various forms of writing, creative, abstract or symbolic, were also structured and ordered. The content of the writing, however, could be free, flexible and expressive, reflecting the views of the individual. Through its emphasis on the individual perspectives, writing could be controversial, open-ended and opinionated.

Science. By his selections on the vocabulary list and his comments, Tom stressed a view of science as tenuous and controversial. He commented on his grouping of the terms *ideas, controversial, theoretical, and thought provoking* by saying:

It's a way of thinking in science. A lot of things are theoretical. Ideas, just because you have to have ideas. Controversial cause a lot of things clash between science and society. Science and a church or whatever.

When asked why he had not included *factual* in his list, he expressed the view that scientific knowledge was often composed of as yet unproven opinions.

I see science as we don't really know what's really going on, but we're trying to do our best to find out. So we can't say it's fact yet. And we can't prove it. We can prove some, so I guess in effect, those would be *factual* I just think that as a whole, the entire [discipline of] science is just things that people think [underline added]. They're all based on theory.

The experimental side of science was also evident in Tom's choice of the terms *trial and error* and *cause and effect*. These represented techniques employed in the scientific method.

None of Tom's comments implied a view of science that was either structured or controlled. In fact, his usage of the terms *rules* and *organized* referred to the procedures in an experiment.

I was thinking you just have to follow rules when you [do a] lab in science. . . . We really had to follow simple rules to do the experiment.

Similarly Tom associated the term *mechanical* with the apparatus or machinery in science rather than with the execution of formulas: "just like physics, how things could work."

Tom's perception of science was dominated by the experimentation inherent in the discipline. Because scientific theories were based on experiments, Tom felt they were often tenuous and controversial. Consequently, scientific knowledge was based on opinions rather

than hard facts.

History. Tom's remarks and selection of terms indicated a view of history that was centered around indisputable facts which were subject to individual interpretation and learned exclusively through memorization. The factual theme of history was represented in his grouping of the terms: *absolute, fixed, detailed, specific, sequenced, organized, and ordered*. He remarked that:

Everything is in order. Everything's already set down. You have to memorize all that happened. It's just absolute. It happened. This is the way it happened. This is the way you're going to learn it. . . . It's just set down or it happened. It absolutely happened this way. That's it.

Tom represented the interpretative theme in history in his single selection of the term *opinionated*. He argued that many historical events could be viewed from opposing positions, citing slavery and women's suffrage as two such examples.

If someone, white people, are writing about slavery then they are gonna talk about how we brought them over, how we gave [them] a good life and all they had to do was work for us. [If] some black people write about slavery, they're going to write down about how we took away their freedom. Just the opposite side of the fence. . . . It's all in peoples' minds, based on the country you're from or nationality.

When asked how he rectified the themes of history being both factual and controversial, he responded:

It's something that happened. It's just what you get from it [that is] your opinion. Like people, let's say men for example. When women started to vote, they said "Oh, this is gonna be awful. This is the worst." So when they're writing their history books down, this is the worst thing that ever happened to American politics. The women, on the other hand, are writing down [in] their books [that] this is the best thing that ever happened to American politics. It happened. It's just how it effected the people differently.

Like English, Tom saw history as *dull, boring, and causing anxiety*. He felt no need to study history. For him it was useless.

I don't like history either. I guess it's good to know but it's just unuseful. I know that you have to know what happened in the past so you won't make the same mistakes or something like that.

Tom's dislike of history was attributable to learning the subject primarily through *memorization*.

He remarked "All of history is memorized. So all you're doing is memorizing what happened in the past years." By learning history through memorizing, Tom was not actively engaged with the topic or challenged by it which for him was necessary to keep his interest in a discipline.

For Tom, history was a dull collection of memorized facts which he described as *absolute, fixed and sequenced*. Although indisputable, the facts could be interpreted differently

by individuals. Thus history was both factual in nature and controversial in interpretation.

Summary. Of the four disciplines discussed, mathematics was the only subject Tom enjoyed or found challenging. Tom openly acknowledged that he was unwilling to engage any subject that did not interest him. For Tom, mathematics utilized rules, common sense, and logical thinking to solve problems. He de-emphasized memorizing mathematics by arguing that it was impractical. Finally, he felt that mathematics was useful and universal in scope.

Unlike mathematics, Tom hated English. He was annoyed by the symbolism superimposed on written passages. He divided writing into its form and content. Within the various forms, writing was structured and ordered, while its content was free and flexible. The thoughts expressed in the content were opinionated and controversial.

Tom's view of science was focused entirely on the experimentation in the discipline. For him, scientific knowledge was based on unproven opinion and, as such, was open to controversy.

While scientific knowledge was tenuous and based opinion, historical knowledge was grounded in indisputable facts. That history had controversy was due to individual interpretation of historical events. Tom stressed that history was learned entirely through memorization. Consequently, Tom found history boring, dull and useless.

#### Ranking Grid

Tom found the ranking grid difficult to complete [see Appendix J]. For him, many of the adjective categories were indistinguishable and many of the topics intrinsically related. Tom believed that the category easy to do-hard to do was identical to the categories best at-worst at and basic-advanced. Similarly, he identified the category interesting-boring with most liked-least liked, and the applied-theoretical category with the most useful-least useful. Besides collapsing many categories, Tom also left blank the categories visual-abstract and logical-arbitrary. He said simply:

On the rest of these . . . nothing goes boom when I look at what they are. I can't really rate them.

In addition to identifying adjective categories, Tom grouped together decimals, percents, and fractions saying:

They're pretty much the same thing. . . . I can't even remember when I learned how to do like percents. I think of percents and decimals as just being the same thing, cause

all they are is just moving the decimal point 2 spots. That's just decimals. Fractions, that was like third or fourth grade.

Tom's ranking of various topics was consistent with his previous comments. He had stated that he enjoyed those topics which were the most challenging, that he found most mathematics easy and that many topics had a set solution pattern. Tom ranked proofs, exponents and word problems high in the categories of *interest*, *challenge/thought provoking*, and *original thinking*. He also marked proofs and word problems high on *flexibility* of approach, saying:

Writing proofs is the most flexible. [In] writing proofs there's so many different ways you can prove things. Sorta flexible. As to the rest, I guess, word problems cause you can set it up different ways. The rest are pretty much just clear cut [underline added].

In addition, Tom ranked word problems as the sole topic that he found *confusing*.

I'm not really confused by any of them. Word problems sometimes. It's the wording on the word problems [that] can end up being the most confusing.

Analogously Tom viewed the busy work category as synonymous with the easy to do category. He illustrated his point by citing factoring as an example.

That's what we're doing right now. It's just review. It's so easy. It just seems like busy work when she give you 50 problems of the same thing. I mean, once you know how to do it, that's it. None of them really seem confusing to me.

This theme was further illustrated in his ranking of the topics percents, decimals and fractions. Since these topics used routine thinking they were also the easiest to do and consequently, for Tom, they represented boring busy work. Interestingly, these three topics were simultaneously the most boring and the most useful to the real world.

Although Tom indicated that only proofs and word problems were flexible in their approach, he did not endow the remaining topics with the characteristic of rigidity. For Tom, the issue was not rigidity but rather anticipation of the problem approach. Most mathematics topics were simply "clear cut," requiring few decisions on the part of the student. The approach was apparent once the student had learned the process.

In summary, Tom's ranking grid reaffirmed his view that the most interesting mathematics topics--word problems, exponents, and proofs--were also the most challenging. With the exception of word problems, he described most mathematics topics a clear, easy to do, and consequently boring busy work. For Tom, proofs and word problems stand out as the only topics with a flexible solution technique. All other topics had clear cut approaches. Finally, he



described fractions, decimals, and percents as being useful and indistinguishable topics.

#### Synopsis of General Beliefs about Mathematics

From Tom's discussion of his vocabulary list, ranking grid and topics unlike and like mathematics emerged a two-sided view of mathematics. He described mathematics as both routine and thoughtprovoking. The routine aspect of mathematics was exemplified by typical arithmetic and algebraic procedures. These procedures were clear cut and easily mastered by Tom.

In contrast, word problems and proofs exemplified the side of mathematics that was thought provoking and challenging. These two particular topics required Tom to reflect on his own thinking and offered multiple solution techniques. Tom's enjoyment of mathematics was directly tied to the complexity and challenge he felt within the topics. Thus proofs and word problems held Tom's interest while decimals, percents, and fractions were seen as boring.

It is important to note that even within the procedural side of mathematics, Tom stressed the thinking process leading up to the procedure. For him, mathematics was "being able to do the thinking to get to those steps" as well as "being able to solve" the problem.

In summary, Tom's remarks suggested a conceptual view of mathematics. He emphasized that mathematics was more than the execution of procedures. It was, in fact, the thinking behind the sequence of steps. He also stressed his enjoyment in reflecting on his own thinking while working on proofs. With the exception of word problems and proofs, most mathematics problems were routine because their solution processes were clear cut, easily identified, and quickly acquired. Finally, Tom described mathematics as universal.

#### Student/Teacher Roles

##### A Good Mathematics Student

Tom's description of a student who was good at mathematics revealed much about his perception of mathematics and mathematics learning. Tom offered Linda as an example. She was not only gifted but she utilized her talent to succeed.

She's really gifted. I think we're a lot alike with only one difference. She takes her intellect or whatever to the full extent and uses it. And I kinda just use it to get by. I don't really try unless I really need to.

While attempting to describe the qualities that made Linda good at mathematics, Tom interjected himself into the discussion. He spoke of mathematics problems as solving themselves and

solutions as flowing from inside himself. He associated these characteristics with being good at mathematics.

I think of being good at math as you don't need to study. I think the mastery comes pretty easily to me. . . . I never study for a math test. . . . The problems just sort of solve themselves usually [underline added]. You sorta get that feeling that you just do it. It's like you don't have [to] sit there and fight, "OK, what do I do now?" It just flows [underline added].

Tom also described the confidence and persistence exhibited by gifted students. Despite a problem's difficulty, Tom believed that people who were good at mathematics had a firm conviction that the problem would work out and so they persisted.

It's not being afraid of the problem. Not letting it get to you. Just doing it with complete confidence that you know how to do it [underline added] [regardless of whether it is] anything that you've done before. Just do it now. [When I'm stuck on a problem] I stay on it until I can find a solution. Find the answer. There are a lot of problems that I can't answer or [that] I can't see right away. But I usually stick with it. Sometimes they work themselves out. Sometimes I have to just forget about them and go on and ask in class the next day. . . . But most of the time, I don't usually have that big a problem.

Tom continued his discussion of difficult problems by outlining his strategies for tackling them. He suggested looking for similar examples, seeking alternate approaches, and simplifying expressions to reduce the complexity of the problem.

I will go back and look at the directions or problems like it that I did. Like the last one I did and see how I solved that one. See if I can apply it the same way. Look for little things that you can do to make it easier or whatever. I just work on it. I keep throwing things up, trying different ideas, different ways to do it. It'll come out.

Tom's discussion of the qualities of someone good at mathematics also led him to consider those who failed to grasp it. Since to him mathematical understanding was based on intuition and common sense, he reasoned that an inability to do mathematics was due to a lack of common sense within the individual. According to Tom, you either had the ability or you did not. He cited his friend Paul as someone totally lacking in common sense and in mathematical ability. For Paul, mathematics was unnatural. Tom believed that mathematical ability was developed--not created--in the early grades by being challenged.

I think that people who don't do well in math don't have the understanding. Not so much understanding as certain people are good at math and certain people aren't. A lot of it has to do with like sense. I know math to me seems like common sense. It's just something that I know [underline added]. It's like what do you do when you came to a stop sign. You stop. Other people might say "Oh, well I look, I'd look around." So you look around, so keep going. There are people like that. I know people like that. Like there's one kid, Paul. I'm really good friends with him. But the kid has no common sense. And he's not good at math. I think he's really [a] smart kid. He does well in all his subjects except for math. He just can't get it. It's unnatural [underline added].

I imagine that it [mathematical ability] probably came early. I think if you're challenged as a kid early in the early grades you would [develop it].

Tom described the ability to be good at mathematics as something that was instinctual within the individual. The ability manifested itself in not needing to study for tests, in having no difficulty grasping the ideas, and in easily solving problems. For the mathematically gifted student, problems seemed to solve themselves. At times the solution process appeared to be generated by the common sense within the individual. Even when the solution was not apparent, the gifted students believed that the solution would come and so remained confident and persistent in their efforts. They were also willing to try various approaches and strategies until they finally succeeded. Tom believed this ability was developed in the early grades through challenging problems.

#### Advice to an Exchange Student

To an exchange student, Tom offered the simple and straightforward advice that he should always pay attention in class and do all the homework. When stuck on a problem Tom advised trying a different problem or checking the answer in the book and then working backwards. In either case, a student should be persistent.

A lot of times I'll just sit there and stare at it and hope something comes along. And a lot of the time [I'll] probably just blow it off and say "Oh, I just can't do it," and try another one. But maybe I'll come back to it. Look in the back at the answer and see what they got and then compare it to what I got. Things like that.

With regard to studying for tests, Tom was at a loss to give any advice. For himself, he found it unnecessary to study at all.

I don't know, I suppose some people study. I never studied for a math test before. I wouldn't know where to begin for a math test, cause I think that [either you] know how to do those type of problems or you don't [underline added]. So I usually never study for a math test. It's always worked for me before. I got a 95 on the one we took in A-2 [Algebra II] the other day. . . . I don't study for many other tests either, but especially math. It's always come easy.

Since Tom found it unnecessary to study mathematics, he felt that one either had or did not have the ability to do well in mathematics. Tom remarked that he strongly believed that not everyone had this capability nor could it be created. In fact, he felt someone should just know how to do math.

Not everyone could do math as some people. I just don't think they have the capability to understand what's going on. If they can't do it [and] if they study [then] maybe they're studying it wrong. That might be a possibility [or perhaps] doing the homework wrong. But I just think he should know how [underline added]. They don't have the capability to

understand what's going on. They can't work it out through the process or whatever.

Through his advice to an exchange student, Tom revealed his belief that mathematical ability was developed but not acquired. He felt it was important for students to pay attention in class and to complete homework assignments. The necessity for studying for tests, however, was dependent on the student's ability.

#### Advice to Beginning Teachers

Tom's advice to beginning teachers was based on both his own positive and negative experience with mathematics teachers. First of all he stressed the importance of thoroughly understanding the content to be presented. He argued that "If you don't understand, the kid won't." Next he pleaded for teachers not only to challenge their students but to separate them by ability. He dramatized this point by a discussion of his own disappointment and boredom when forced to endure classes where he already understood the material. He argued that by separating the students by ability the teacher could challenge the more able students while she assisted the weaker students.

I think it's good to challenge people. And then another thing I kinda like is separating classes. I remember in like my earlier grades we used to have like a pre-test or something. We'd take the pre-test and then correct it the next day and that way the teacher would know who already know some of it and who knows nothing and who knows all of it or whatever. And then you can give them homework accordingly. . . . [For] all those kids [that] failed it [the pre-test] then you give them the easier problems so they can build up an understanding. And then you give the people who did really well on it, like the harder problems to challenge them so that they want to do it. Cause you just wanna say, "Well I already know how to do it so why even bother." . . . I know it's easy to say that but I'm sure it's kinda hard to do that in the upper level grades. . . . I think that's important. Cause like when you have to go slow, like it hurt me a lot in seventh and eighth grade. . . . I just think that it's good to separate people by their ability.

Again reflecting on his experience with Mr. Brown, Tom suggested that teachers needed to be thorough in their presentation of the content. They should cover the basic ideas before assigning homework on it.

They should explain it thoroughly before you do the work. Cause like Mr. Brown, I had him my freshman year for Algebra I. I went from the worst teacher I ever had to the best teacher I ever had. He'd assign this problem then show us how to do it the next day in class. So we went home, did all this homework, struggled through it, going "How in the world do you do this?", reading the thing by ourselves. And then he'd explain it the next day, and then we'd go "Oh, OK." . . . I think that's important to explain it thoroughly.

When asked what teachers could do to make mathematics classes more interesting, Tom offered a unique opinion on the purpose of teaching.

Teacher's aren't hired to be clowns. . . . It's [school] never said it's a fun place to be.

And kids aren't expecting it to be one of their highlights of their life. I can think of endless other things that I'd rather be doing, that I could rather be doing right now. Trying to draw kids in for fun or something like that, that's just not what school is. If it's boring you just do your best to get through it. The way I look at it is Tom has got two days rest on the weekends so just totally forget about it.

Tom offered beginning teachers the advice that their purpose in teaching was not to entertain the students but rather to present thorough explanations of the material. They should challenge their more adept students while they helped their slower students. This assistance was most easily rendered when the class was separated by ability. Underlying all this advice was Tom's expectation that teachers should understand thoroughly the mathematics they intended to present.

#### A Good Mathematics Teacher

Since Tom offered fairly detailed advice to beginning teachers, he was fairly brief in his description of good mathematics teaching, the implication being that his advice also conveyed his connotation of good teaching as well. Tom stated simply that a good mathematics teacher was one who understood the material, explained it well, and presented basic example of concepts underlying homework assignments.

Understands what she's teaching. Who knows how to explain it. . . . They have to be able to speak well, to communicate. . . . The ones [problems] that they do on the board might be easier than the ones you have to do in our homework but the basic concept on how to do it still stays the same.

#### Summary of Student/Teacher Roles

Tom's protocols offered little explicit information about his views on the student's role in learning. He did, however, vividly explain his beliefs about mathematical ability which in turn suggested how he perceived that learning took place. Tom's fundamental belief was that mathematical insight or ability was instinctual and natural. He described it as a common sense. Problem solutions flowed from this common sense, seemingly solving themselves. Since mathematical ability was an instinct, Tom believed it was a trait that an individual either had or did not have. He felt this ability could be developed through challenging students in the early grades but that it could not be created. As a consequence of his belief that mathematical ability was synonymous with common sense, Tom believed that mathematical failure was attributable to a lack of common sense.

Tom also believed that people who were good at mathematics had great confidence and

trust in their ability to solve problems. These students believed that a solution would come to them and so they persisted when they encountered a difficult problem. To aid in the search for a solution, Tom outlined problem-solving strategies he used when stuck. He suggested looking at similar problems, working backwards from the answer, simplifying expressions, varying your approach, rereading instructions, and finally, persisting. Tom's insisted that if someone had the ability to do mathematics and had the faith in his ability, then he would eventually resolve the problem regardless of whether he had seen it before or not.

Personally, Tom described himself as someone who was "good at mathematics." Since, for him, mathematics operated at an instinctual level, he gave few guidelines on the student's role in learning. He implied that it was sufficient to pay attention in class and to attempt the homework.

Another consequence of Tom's view on learning mathematics as instinctual was that he offered few suggestions on how to teach mathematics. He felt it was important for teachers to have a strong knowledge base from which to develop clear and thorough presentations. As a result of his experience in Algebra I, Tom wanted some presentation of the ideas and examples before giving an assignment but he did not expect a complete step-by-step outline of all homework-type problems. It was sufficient to give basic examples. To clearly convey the ideas, Tom felt a teacher should be able to communicate well. Again reflecting on his experience, Tom argued for separating students in a class by ability. Through this means, a teacher could address simultaneously the needs of the weaker students for more assistance and the needs of the adept students for stimulating problems to keep their interest. Tom repeatedly stressed his wish for challenging problems. Finally, Tom expressed the view that he did not expect teachers to be "clowns", in effect entertainers. School was something you endured.

#### Conceptual/Procedural View of Mathematics

The analysis of Tom as holding a conceptual or procedural belief about mathematics was derived from two sources: the general mathematics overview and the coding of Tom's responses from the transcriptions of his solutions to the mathematics problems in the interview protocols. Of the 14 problem episodes that were codable for Tom, all 14 were labeled conceptual. Tom's ranking of 100% conceptual ordered him as tied with Ann for first among the six participants in this category. (See Appendix I for a table summarizing the participants' coding

on the conceptual/procedural and autonomous/nonautonomous categories.)

### Conceptual

Tom's conceptual rating on his problem protocols was due to many factors. His protocols revealed that he could (a) justify procedures in terms of first principles or intuitive number sense, (b) use number sense to check the reasonableness of the solution, (c) summarize his solution process, and (d) evaluate the sample test questions on the basis of process rather than on the answer solely.

Tom was able to justify the arithmetic procedure in problem #20 (6 divided by  $\frac{3}{8}$ ) on the basis of intuitive number sense. Tom began the problem by stating that: "Multiplying and dividing are opposites so when you want to divide by a fraction, you multiply by it's reciprocal." He offered no rationale for this process saying: "I don't remember any of my teachers ever telling me why. It's just that, that's how you get the solution. Chances are they probably don't even know why." However, Tom was able to justify the reasonableness of his answer by recalling that division meant dividing into pieces. Next Tom noted that 6 divided by 1 was 6. He continued his argument by saying that the divisor,  $\frac{3}{8}$ , was smaller than 1, so it should go into 6 a greater number of times. Thus his answer of 16 was to be expected.

[This answer makes sense] because fractions aren't whole numbers. Cause if you divided 6 by the smallest whole number or let's say 1, you'd get 6. But if you get a bigger answer, you know it has to be divided by a fraction. Cause a fraction is part of the whole and so there's gonna be more of it.

On problems #23 and #28 Tom showed his ability to justify his answer by using number sense. On problem #23 (find fractions greater than  $\frac{3}{4}$ ), Tom employed a mixture of reductions, approximations, and calculations in his selection of the appropriate fraction.

I'd just reduce them. Find the number that goes into each number in the numerator and denominator. If possible. You have all prime numbers. That's not fair. OK. Well then since they're all prime then . . . I guess I'd just estimate. I know  $\frac{35}{71}$  is not  $\frac{3}{4}$  cause 35 is only like one-half 70 and well it's not. [The]  $\frac{13}{20}$ , this isn't because  $\frac{3}{4}$  is  $\frac{15}{20}$ . [The]  $\frac{71}{101}$  isn't because subtract 1 from each and you would really have. Just the closest real number, 100, and  $\frac{3}{4}$  of that would be 75. And  $\frac{15}{20}$  is  $\frac{3}{4}$ .  $\frac{19}{24}$ . That one might be more, that one's close. I don't know. Do it out I suppose. Divide 19 by 24 [to] get a percentage. I don't feel like doing it. [Uses calculator] So it's 79% and that's greater than  $\frac{3}{4}$ . So that's the only one.

Again on problem #28 (larger of  $(543 \times 29)/32$  and  $(30 \times 543)/28$ ), Tom demonstrated his ability to justify his answer by number sense. Tom quickly noted that he could determine the larger value by comparing the respective numerators and denominators. He explained that

since  $542 \times 30$  was the larger numerator and it was also being divided by a smaller denominator, then it must be the larger of the two values.

Because the number you're multiplying by, you're multiplying 30 by 543 instead of 29 by 543. So that would give you a bigger number. And then you divide by a smaller number like 28, compared to 32. It will go in more times.

Tom was able to summarize his results on problems #27 and #30. On problem #27 (remainder problem), Tom began by clarifying for himself that the question asked for a single number that had a remainder of 2 when divided by 3, 4 and 5. He then considered various approaches he could try on the problem. He was hesitant to merely use trial and error since he felt it would be inefficient. He did, however, examine a few numbers to motivate his reasoning. His reasoning then moved from multiples of the numbers, to common denominators, to the answer 62. After verifying 62, he went back and satisfied himself that it was indeed the smallest possible value that met the given condition. In the process, Tom realized why 62 worked and was able to summarize that rationale and use it to find an additional number that had a remainder of 2 when divided by 3, 4 and 5.

I'm trying to think if I've ever done a problem like this before. OK, well the smallest positive number, let's put  $x$  divided by 3, will leave a remainder of 2? Will one number for each 3, 4, 5? Or one that 3, 4, 5 all three of them go into that one number? With a remainder of 2. I don't know. I'm just trying to think of a number in my head and do it out in my head, cause I don't think I could write it down. I was trying to think if 27 or something, if that would leave 2 for the 5, but 4 would go, 24 and leave 3. And 3 goes into it, so that won't work. I don't even know where to begin with this one.  $x$  divided by some number which would give you 2. [pause] Divided by 3, 4 or 5. [pause] [shakes his head] I can't do this. I don't know where. The only way I can think of to do this problem would be just to pick numbers and divide each of them into it and see what's the remainder. [hesitates] Well, because it's not a way. I mean it's not a specific [way]. It's not like a formula or something to go by. Cause if I have to pick numbers, we'd be here all day picking numbers, trying to find the one. I wanna say that I don't think there is a number. But. Cause I don't see how 3, 4 and 5, multiples of those numbers pretty much cover just about everything else they have. Skip 2, a number that 3, 4 and 5 go into that would leave a remainder of 2. Most common denominator maybe. [pause] What would that be. Try 62, 12 remainder 2. Sixty-two. Well, it worked. I don't know if that's the smallest number. I think that's the smallest common denominator. [pause] I just figured that whatever number was divided by 3, 4 or 5, 2 less than that, would have to be a whole number that they all went into, including 2. I believe it. I think it's the smallest number that all 3 go into.

When asked for another value that satisfied the condition, Tom immediately gave the value 122.

Probably it would be like 122. Yeah, that would work, 122. Just taking a number that's divisible by all three and adding 2 to it [underline added]. I don't know why I was so stumped with it. I haven't worked with remainders for a long time. They always use decimal places.

In the remainder problem, Tom demonstrated his conceptual view of mathematics by



looking for the logical connection between the unknown number and the problem conditions. He was not content merely to use trial and error. He seemed to believe that there was some logical process which would yield the solution. He also showed his conceptual ability by summarizing for himself the final result. Interestingly, once he saw the rationale for the answer, he was surprised that he had difficulty with the problem, as if he were saying the answer or the connection was quite obvious.

Again on problem #30 (find the area of the triangle), Tom summarized his observations and solution. Like most of the participants, he began by insisting that he needed the numerical value for either the height or the base in order to solve the problem. When assured that a solution was possible, he returned to the problem and observed that the two enclosed triangles must have the same area since they shared congruent bases and the same height. Hence the area was 16. Even with a solution Tom was not convinced. He remained skeptical, wishing he had a value for the height. When it was suggested that he make up a value, he experimented with a height of 4 and again arrived at 16 as the solution. He then noted that the result, 16, was independent of the value selected for the height.

Oh no, geometry. I just don't remember how to do a lot of it. . . . OK, so this is a median? Well, let's keep going. [pause] so they have the same height. Both triangles are gonna have the same height. The area shaded is 8. [pause] Is it base times height? One-half base times height or base times 1/2 height, same thing, equals area. [pause] So. [pause] How the heck can you solve that unless you know [what] the height of the triangle is? Can you? . . . You've got too many variables. The problem [is] you don't know what the base [is], wait. If you know one or the other, and since you've got the area of the triangle, you can find out the other one, but you need to know at least one. I don't see a way to solve it unless you have another number to work with. Well, I was thinking something about the base of the large triangle. BC and DC are congruent. So to find the area of this you'd have to take half of BD. To find the area of the whole triangle, you'd have to take half of BC, which would give you. So what I was thinking is the area of the whole would be 16. But. Because well half of BD times whatever the height is, gives you 8. [Writes  $(1/2 BD)( ) = 8$ ] And BD is half of BC. So if you take half of BC, you get BD, and multiply it by whatever the height is, and then it would be 16, cause it wouldn't [be] halved. If that makes sense. So I'd say 16. Yeah, 16. I'm not positive, but I don't know. That's all I can come up with, so I don't see how else I could get a number [underline added]. Well, if I had a height, then I know I could work out the problem. All right. If the height is 4 [pause] and the total area is 8. So that's 1/2 BD times 4 equals 8. [Substitutes 4 into his equation  $(1/2 BD)( ) = 8$ ] So if you divide by 4, both sides by 4, and you'd get 1/2 BD equal 2. Multiply both sides by 2 to get rid of the fraction. And you get BD equal 4. So then that would make the entire base BC would be 8. And then you've got the same height for both triangles. . . . the height is 4, BC is 8. Half of 8 is 4 so it would be 16. 4 times 4 is 16. So then it works. So you could put any height and have it work. Cause it's gonna be the same height for both triangles no matter what [underline added]. I mean I could put 1 in there and it would just change the length of BD or BC. You still come up with the same answer [underline added].

In addition to summarizing his observations, Tom demonstrated his conceptual understanding by justifying the  $\frac{1}{2}$  in the triangle area formula. He argued:

Because a triangle is like a half of a square and that's the way I always thought about it. Like if you take out a 45-45-90 triangle out of a square, you just cut the square with the diagonal. Then this triangle will be half the area of the whole square. That's the way I always thought of it. I know that triangles can be a lot of different shapes, but that's just the way I always [thought].

In both the remainder problem and the triangle problem, Tom clearly demonstrated an ability to summarize his results. It was through these summaries that Tom reassured himself of the validity of his results. Both problems were also representative of the type of protocols for which Tom received a conceptual coding for his ability to summarize and generalize results.

The last category for which Tom was coded conceptual was for his grading of the sample test. Tom utilized several evaluative schemes while grading. He appeared on several problems (1a, 1b, and 1c) to mark them right or wrong based on his remembrance of an algebraic rule. However on problems requiring multiple steps (1c and 1d), he evaluated the procedure as well as the answer. On 1c it was evident that Tom had a rationale for the immiscibility of Chris's technique. However on 1d, his protocol did not clarify whether the evaluation of the procedure was based on the incorrect execution of a rule or on some underlying conceptual rationale. Finally on problems 11a, 11b, 11c, and 11d, Tom demonstrated his ability to evaluate problems conceptually. On problem 11a he clearly reasoned through Chris' work and checked it against his own standard for valid mathematics. Thus Tom's grading suggested an evaluation of mathematics problems on the basis of both procedures and concepts. The determining factor in Tom's conceptual coding on this task was his reasoning as he evaluated problem 11a. In this discussion Tom verbalized a strong belief that the validity of mathematical procedures resided in its universal applicability. That is, a procedure was acceptable if it was not limited to a given example but could be generally applied to any similar problem. The strength of his conviction on this problem and his conceptual evaluation of several of the other problems implied that Tom had the facility to examine mathematics conceptually even if he did not utilize this strategy on all the problems.

On problem 11a, Tom was initially thrown by Chris' solution technique. He wondered aloud if this was some "old method" for working the problem. Without prompting, Tom continued to muse on the appropriateness of the procedure. He finally resolved the dilemma by visualizing

the addition of 4 and then 3 to both sides of the equation as a two-step method for adding 7 to both sides of the equation.

Once again, they got the right answer, but, how? Adding 4 to both sides.  $3x$  plus 3 equals 8 . . .  $3x$  minus 3 plus 3 equals 8 plus 3.  $3x$  equals 11. Hum. I don't know, it depends on if that way works for every time you do the problem [underline added]. I mean not just that problem, but if you could do any problem like that [underline added]. Suppose he had  $2x$  plus 8 equals 12 or a negative 12. [pause] How? I don't understand how they even. Is that like an old method or something cause I don't know if it works for every [problem], if it works like all the time. Well it should because all you're doing is, you just broke down 7 and added several parts so I guess that would be full credit. All 10 points. Cause there's nothing wrong [underline added].

On problem 1ld Tom considered both the process and the answer in his evaluation. He began by working the problem for himself by writing it as two separate inequalities:  $x + 3 < 2$  and  $x + 3 > 2$ . After he solved each piece he checked his results against the inequality presented by Chris. Since they matched, he moved on to consider the interpretation given by Chris. He argued that Chris' conclusion was incorrect because  $x$  could be negative since the expression would become positive once the absolute value was taken.

I don't know if that's how I'd do it. . . . I'd probably set that in 2 totally different problems and then subtract the 3 from both sides. . . . So he got the right answer up here but this, there [the conclusion], that is a solution because  $x$  isn't the absolute value. It's  $x$  plus 3, absolute value. And, oh,  $x$  can be any number if it's inside the absolute value because you take the positive number of it. I'd probably give him half again. Five.

As on problem 1la, Tom again demonstrated on 1d his insistence that mathematical processes be universally valid. Correct answers from incorrect work were lucky rather than valid.

They cross multiplied [cross cancelled] and that's not how you add fractions, so. He got  $7/6$  so that's totally wrong. That would be 0 points. The answer should be  $14/12$ , reduced to  $7/6$ , so it is the answer, whoa! All right, so he did get the answer right, but that's not how you do that though. So that's totally blind [luck], that's lucky then. So he did get the right answer, but that's not how you do it, so I'd still give him no [points], I wouldn't give any points for that. He got lucky when he did this out [underline added]. I think this is you, but. You know.

These four examples from the sample test question suggested that Tom was capable of conceptually evaluating mathematics. In the course of grading, Tom articulated a view of mathematics as processes that were universally applicable. He subsequently used this standard on his examination of Chris' solutions.

### Summary

Tom's coding as conceptual was evident in both his general beliefs interviews and in his problem protocols. While Tom's beliefs interviews revealed a divided view of mathematics as rules and as the rationale behind the procedures, his emphasis was on the conceptual side. He

stressed that he enjoyed and wanted the challenge of mathematics problems where he had to reflect on his own thinking. He de-emphasized memorization in mathematics implying that understanding the procedure was sufficient to retain it since mathematics was just common sense or instinct. In addition to applying rules, Tom also believed that many mathematics problems, but especially word problems, required original thinking, discovery, and trial and error. To handle difficult problems, Tom had developed explicit problem solving heuristic. These strategies reflected his expectation that not all mathematics problems could be solved by an application of a rule or procedure. Tom also expressed the view that mathematics topics could be conceptually interrelated as was the case with decimals, percents, and fractions. These conceptual beliefs about mathematics were visible in his problem protocols.

Tom's solutions to the problem solving situations reaffirmed his expressed belief that word problems required original thinking and an examination of the rationale behind the process. On problem #27 (the remainder problem) Tom considered and then rejected solving the problem by an equation. Unlike Tara, Tom was able to express the desire for a straightforward solution technique without being locked into that strategy. He did, however, insist that his solution reveal the conceptual tie between answer and problem conditions. He was looking for the thinking behind his procedure or answer. He also utilized many problem solving heuristics including rereading the problem and using sample numbers to motivate his reasoning. This flexibility of approach was evident in all his problem solving protocols confirming his view that mathematics problems often utilized original thinking and that it was vital to see the interconnection between the solution technique and the given problem.

The interrelationship that he saw between decimals, percents and fractions was evident in his problem protocols as well. On problems #23 (find fractions greater than  $\frac{3}{4}$ ) and #21 ( $1.50 \times .25$ ), Tom moved effortlessly between representing numbers as decimals, percents and fractions. His flexibility with these topics permitted him to check the reasonableness of solutions and to justify his procedure.

Tom's conceptual view of mathematics was also present in his grading of the sample test. Here Tom's consideration of both the process and solution coincided with his belief that mathematics was the rationale behind the procedures. Correct answers alone were not sufficient to justify a procedure. He believed strongly that mathematical processes should meet the test

of being universally applicable. Tom used this conceptual criteria in his grading of the test questions.

For Tom mathematics entailed the application of rules but more importantly it represented the rationale behind the procedures. He expected mathematics problems to require original thinking and mathematical procedures to be universally applicable. Tom's problem protocols reflected these beliefs through his flexibility of approach, summarization of results, and usage of number sense to justify procedures and answers.

#### Autonomy/Nonautonomy with Mathematics

Tom's problem protocols and beliefs interviews revealed his autonomy with mathematics. Of the 10 codable problem episodes, all 10 were coded as autonomous. This ranking tied him with Ann for first among the six participants.

#### Autonomy

Within Tom's protocols were many instances that supported his autonomy with mathematics. Tom voluntarily checked his answers, monitored his solution process, and summarized his solutions. Tom voluntarily checked his result on problem #24 (find the smallest radical fraction) and quickly ascertained that  $1/(5 \sqrt{5})$  was the smallest fraction by reasoning that it had the smallest numerator while simultaneously having the largest denominator. He then expressed the desire to verify his result. He believed that his selection was correct but wanted the additional support for his answer.

Which is the least of the following numbers? . . . It would be 1 divided by 5 root 5. Well, cause 1 is a small number already, and the more you divided it by, the lesser the answer is going to be. And 5 root 5 is larger [than] anything else. Did you understand what I mean? OK. Is that right? If you're not gonna tell me, then I need a calculator. Well, yes [I believe my answer] but I like to check them anyway. You know, it's a good habit to get into. [Pause, uses calculator] So I divided by 11. [sings] Well zamma, zamma, zamma. I think that was probably the smallest number.

On the problem solving situations in #27 (remainder problem) and #30 (find the area of the triangle) Tom exhibited several actions indicative of autonomy with mathematics. On problem #27 Tom monitored his solution progress and voluntarily summarized his results when finished. Tom's monitoring was evident by his initial clarification of the problem question followed by his consideration of various approaches including trial and error. Tom was constantly re-evaluating the efficiency and appropriateness of his solution techniques as they occurred to him. Tom also expressed in this protocol his expectation that arriving at a solution by trial and error was

unacceptable. Not only was trial and error inefficient but it failed to utilize fully the problem information. Tom's actions and comments suggested that he expected to find such a connection and directed his activities accordingly. When he finally arrived at the solution he continued to reflect on his answer and his reasoning until he was able to summarize a logical connection between the solution and the problem conditions.

Again on problem #30 (find the area of the triangle), Tom concluded his problem solution by voluntarily summarizing his result. After deducing that the area of the large triangle should be 16, Tom remained skeptical. He wanted additional evidence to support his deduction. In this case he wanted to verify the area by using a numerical value for the height of the triangle. Once he had experimented with a specific height, he generalized from the example that the area was independent of the given height. Tom revealed by his insistence on a second approach that he expected the problem to make sense to him.

Tom continued to show his independence or autonomy with mathematics by monitoring his progress and solutions on the problems #23, #34, and #21. In each of these examples, Tom reflected on and made decisions concerning the solution process. The decisions on the best technique were often based on efficiency, relevance to the task at hand, and familiarity or ease of usage. By monitoring his own actions, Tom demonstrated his control and awareness of the mathematical process he was using. By doing so, he also demonstrated an expectation that mathematics should make sense, that is, that his actions had a causal relationship to his solution and that he was in charge of and responsible for his progress.

Evidence of Tom's monitoring on problem #23 (fractions greater than  $\frac{3}{4}$ ) was seen in his initial dialogue. He sorted through his options: reducing the fractions, finding a least common denominator, and estimating. Tom elected estimation because the numerical values in the denominators were relatively prime, and so were not easily used to generate a common denominator.

I'd just reduce them. Find the number that goes into each number in the numerator or denominator. If possible. . . . Well then since they're all prime then . . . I guess I'd just estimate.

In addition to problem solving situations, Tom was also able to summarize his results on simpler algebraic problems like #34 (select the larger of  $r$  and  $r^2$  for  $r > 0$ ). Here Tom used a series of trials to evaluate the various possibilities before concluding that  $r$  could be either equal,

greater than, or less than  $r^2$  depending on the value substituted for  $r$ .

[ $r$  is less than] cause it's a positive integer.  $r^2$  would [be] multiplying. This is  $r$  times  $r$ . This is just  $r$ . It's more. Well 2 squared is 4, compared to 2. No, they could be equal if  $r$  was 0 or 1. They could be equal. [ $r$ ] could never be greater than [ $r^2$ ] unless you use fractions. Cause like  $1/2$  times  $1/2$  is  $1/4$  compared to  $1/2$ . So it depends on what  $r$  is. It can [be] equal, greater or less.

Tom demonstrated that he also monitored the reasonableness of his answer on problem #21 ( $1.50 \times .25$ ). Initially Tom multiplied the numbers by using a partially remembered technique learned in grade school. This technique entailed first multiplying both numbers by 100 to eliminate the need for the decimals and then reinserting the decimals by dividing by 100. When Tom executed his procedure he obtained the erroneous answer of 37.5. He immediately realized the inappropriateness of his answer by reasoning that  $1/4$  of \$1.50 was approximately 30 cents. The remainder of the discussion focused on why his algorithm failed.

Well just for my personal preference I'd multiply 1.50 by .25. I'd multiply each number by 100 to get rid of the decimal. Cause it's easier, so you'd get 150 times 25 and then just multiply it out. Actually this is a trick that my fifth grade teacher taught me that I always like to use. Whenever there's a number of zeros on the end, instead of putting it on top and always having to write zero, you just put the other number on top and thousand zero on the end. Then you can just drop the zero and then multiply through. . . . And then you can divide by 100 again, you get the decimal back so 37.5. But that's not right. I think I messed up somewhere. It would be 3.75. It wouldn't even be that. Cause a quarter of 150 is 30 cents. I'm not sure. I know I did wrong I think. . . . I know I did something wrong. Anyway I suppose I can do it out long hand. Count the decimal places four, .375. I don't know why I did that. I got the right answer. I guess I was looking for a short cut, but there wasn't really one there.

Tom had an underlying sense that his answer was wrong but was unable to sort out the error in his technique. He then returned to the traditional algorithm that involved counting the number of decimal positions. When asked why this procedure worked, Tom replied: "I was taught to do [it]. That's why I do the problems the way I do them because that's the way I was taught. Like they never explained why." Despite not having previously reflected on the multiplication algorithm for decimals, Tom clearly expected his result to match with his intuitive estimations. Only when he became confused did he resort to his memorized algorithm.

### Summary

Tom's problem-solving protocols provided many examples of his autonomy with mathematics. He demonstrated in these problems a deep belief that the problem solution should make sense and an ability to monitor his own actions and solutions. These independent actions implied that Tom expected to be his own resource for problem strategies and verification of

problem solutions. Tom also spoke of enjoying mathematics problems that required him to think about his reasoning. He emphasized the importance of perseverance and confidence in solving unfamiliar problems. He believed that a solution would arise from the problem if he kept trying strategies. These beliefs about mathematics were visible in his problem solving protocols.

Problem #27 (the remainder problem) was exemplary of his tenacity and consistent reflection on problem approaches. Without prompting, Tom repeatedly threw out for consideration possible connections between his knowledge and the problem conditions. He was adamant that the solution technique should have a logical connection to the problem information. When he finally found the solution, he was not content until he articulated for himself the logical connection. This need to make the solution reasonable in his own mind demonstrated Tom's autonomy with mathematics. Tom expected the problem to make sense to him. Also as a consequence of this expectation, he was neither dependent on others to provide verification of his solutions nor to provide direction in an unfamiliar situation.

Tom's expectation that the problem solution should make sense agreed with his description of mathematics as an internal common sense. Problem solutions seemed to him to come out of this inner voice. For Tom, this common sense appeared to represent both an intuitive or instinctual quality as well as a logical quality. Solutions seemed obvious or instinctual because the logical connection between the problem and its solution technique was self-evident to Tom. It was "clear cut." Hence he described problems as solving themselves.

Tom's belief that mathematics was common sense encouraged his autonomy with mathematics. Since he believed that mathematics was instinctual and hence knowable, he set up the self-expectation that he should be able to solve unfamiliar problems from his own resources.

Like Ann, Tom's autonomy with mathematics and conceptual view of mathematics were integrated. Those problems for which he was coded autonomous were also those for which he received a conceptual coding. Tom's conceptual view of mathematics was expressed in his belief that mathematics was the thinking behind the procedures. Tom not only valued this reasoning but enjoyed finding the logical connections in more challenging problems. Simultaneously, Tom believed that mathematics could make internal sense and that he could serve as his own resource for solutions techniques and answer verifications. He drew upon



these beliefs in his problem-solving protocols. As a consequence, Tom persevered until he could find the logical connection between the problem and its solution.

Sue

Background

General Information

Sue, a junior, was enrolled in Algebra II/Trigonometry, Chemistry, U.S. History, Short Story, Foods, and Chamber Singers. In addition to her school work, Sue was active in track, cross country, and 4H. During the school year Sue chose not to work, but during the summers and vacations she did general office work such as photocopying and collating in her mother's office. Although she had considered joining the mathematics team, Sue found her sports too time consuming to permit involvement.

She had access to a personal computer at school and through her mother's office, both of which she used solely for word processing. She initially learned about computers in elementary school where she played games, developed simple programs, and learned typing. Sue owned a scientific calculator which she used for basic computations in chemistry and mathematics class. Sue read her subscription to Games magazine primarily for the crossword puzzles. She described as "out of my league, at least right now" the magazine's puzzles that used letters in place of numbers.

When asked to describe in what ways, if any, she used mathematics outside of her classes, she mentioned calculating her daily and weekly mileage when running on 400 and 800 meter tracks. Other than using mathematics for track, she suggested basic consumer addition of purchased items.

Sue was planning to attend college possibly through a track scholarship. She indicated an interest in either elementary or secondary education. Her indecision focused on the type of relationships possible with different age groups and with her area of specialization.

I've been thinking about getting into education cause I'd like to teach. I haven't decided what I want to teach cause I like little kids but then [with] the older kids you can have a relationship. You can have a more relaxed atmosphere in the class. You can talk to the kids and joke around. And [with] little kids it's hard to get a relationship with them. But if you're going [to teach] high school then you have to major in one topic. I'm not sure I want to do that yet. I'm not sure if I want to do math. [I] like math. I'm not sure how well I'm going to do in it. And I like history. I like learning about history so that's another topic that I might do. So I'm still undecided but I think I might go into education.

She went on later to describe the distinctions she saw between teaching mathematics and history.

But another thing I've thought about was when you teach math, it's hard to get off the subject. But if you're in history you can go into other stories and I like that. I always liked history. And math you sorta have to stay on the subject. . . . You can't talk about George Washington, if you're talking about math. [laugh]

I like to talk and tell stories. Math is just sorta basic you do this and do that. So I'm not really sure yet. . . . I think about that a lot. Seeing how I could make [it] more interesting so I could do math for teaching. But then it seems impossible. You always hear about the teachers having to finish certain number of chapters in such time. And they're always complaining that they don't have enough time. I don't know if it's going to be harder to fit more fun in. Cause it always seems like we're always doing math and there's not enough time to go off. So I'm not sure. Cause like [at the] end of the year, he's always pushing us to get done with the book. . . . It's a consideration.

Due to her potential mathematics major, Sue anticipated taking mathematics in college. She indicated that regardless of her eventual decision, she enjoyed mathematics and would continue enrolling in mathematics so she could stay "well rounded".

#### Previous Courses in Mathematics

Algebra I. For Algebra I Sue had Mr. Brown. She had no comment about the teacher but stated that she enjoyed algebra better than arithmetic.

It wasn't that bad. I don't know if I tried as hard as I could of just because I didn't know what I thought [I] could do. So I was getting B's. I wasn't having difficulty. . . . I liked it more than regular math. Just division and multiplication. I liked using x's and y's. I like figuring those out.

Geometry. Sue also enjoyed geometry and liked her teacher Mrs. Smith. She found the course fairly understandable and was only confused by indirect proofs. She easily learned the statement of theorems by simply completing the homework. She stated also that she liked the concepts in geometry and found them easy to visualize.

I liked that a lot. That was fun. The theorems came pretty easy to me. I didn't have to study them once I used them in homework one night. I usually memorized the theorems. I liked drawing little pictures and figuring them out. I can remember theorems but I didn't like proving them. I hated proofs; I could never do those. I mean I could do them and I got pretty good at the end of the year. But some of them, I think it was indirect proofs when you prove the negative. . . . Those are confusing. [Also] the contrapositive. We didn't spend much time on that and then [on] all the tests and quizzes I just [guessed]. That was the only thing I really didn't like, that and proofs. . . . They weren't bad doing them but I didn't mind doing them but I didn't like them. Maybe it was I liked the teacher. I liked her a lot, Mrs. Smith. I like doing things with angles. I think it's just figuring variables out that I liked. Trying to figure out what an angle would be if this was a parallel line. I just, like the concept of it, the whole concept. Not necessary the little things that you did but the concepts I liked.

Algebra II/Trigonometry. After taking regular geometry in 10th grade, Sue elected to move into the advanced program in Algebra II/Trigonometry in the 11th grade. She was initially unnerved by the pace and style of Mrs. Thomas' teaching but as the term progressed, she

began to feel more acclimated and comfortable in the course. She continued to enjoy the x's and y's in algebra.

It started out a lot really fast. Last year Mr. Brown had advised I not take it because . . . he said I might not do very well in it. And I said, "No, sure I can do it." Then the first day of class, we took maybe two and a half pages of notes in 10 minutes. Jill Jones and I were just freaking out in that class cause it's going really fast. We thought about dropping it for a while, like going down to regular algebra. . . . I guess maybe it was just the teaching that Mrs. Thomas does. Cause she doesn't explain the notes. She just puts them on the board and you do them and then later you look at your notes when you get home. But I like it now. It's not that bad. It doesn't seem fast any more. So maybe it was just that first week jitters of the class. But I like it. I like figuring x's and y's like I said. That was fun. But we haven't done very much of it yet.

In a later interview, Sue again discussed her impressions of Mrs. Thomas. She now spoke enthusiastically about the class and Mrs. Thomas. She especially enjoyed Mrs. Thomas' willingness to joke with her students.

When I first started class the impression I got from her [Mrs. Thomas] was that everything was really strict and down to business. I thought about dropping the class. . . . During the class it just feels like nobody was asking questions so I got the impression that I couldn't ask questions because I was gonna look really stupid in front of everybody cause three quarters of the class was sophomores and I'm a junior. I got that impression but now it seems like she seems more outgoing. I just may be used to her, but I like the class. it's fun. We joke around a lot. So her impression now is that I think she really likes math and that helps, because she makes it enjoyable now. I think I was just scared at the beginning of the whole class itself. But now I like it and I think she likes it and it helps. She takes time to laugh or to make a stupid joke. She'll ask about how things are going outside of class, about games. She'll talk about her kids and say something funny about them. It just helps it when you have a class that they're laughing.

### Affect

Throughout the series of interviews, Sue appeared at ease, laughing often. She especially enjoyed sharing her opinions on teaching since she identified herself with her future vocation. As with all the participants, Sue was affected by the interview process saying: "I always wondered if I'm doing these right or wrong, making a fool of myself."

Sue generally found mathematics easy with the exception of word problems and percents. Word problems represented her nemesis while percents were her Achilles heel.

When asked to describe her own ability to do mathematics Sue said:

I think I'm pretty good at it. . . . It's never been hard cause I liked it. . . . Sometimes it's difficult, like something I do might be hard to do but it doesn't detract from the fact that I like math. And since I like the math I think I do good in it [underline added]. Cause I want to do good in it cause it's fun.

### General View of Mathematics

#### Unlike/like Mathematics

Sue described reading as a topic totally unrelated to mathematics. She argued that while engaged in reading she did not have to consciously think about the material. Instead she could just drift using her imagination to depict the scenes in the stories. In contrast mathematics necessitated conscious goal-directed thinking. Mathematical thinking was analytical and did not utilize imagination.

Opposite of math. [laugh] [pause] That's hard. . . . Like math, you're always trying to figure something out and when I read it's like I'm not worrying about anything else. So I guess reading stories. I like reading a lot and that's sort of different cause I'm not using my brain the whole time. Cause I'm using it more for imagination. It's hard to imagine in math [underline added]. I like reading. I guess that [would] be sorta opposite of math for me. In math you're using your brain to figure something out and you're on this one topic. In a story, in my mind, I picture what's happening in the story. And it's like a different place and in math you're right there doing that problem. You can't escape from it [underline added]. So like the numbers sorta sit there. It's like in reading you don't have to use your mind to figure something out. Cause it's right there.

#### Vocabulary Lists

Mathematics. Sue selected few vocabulary terms as descriptive of mathematics [see Appendix H]. She grouped together the terms *routine*, *mechanical*, *formulas*, *specific*, and *memorize*. For Sue these terms represented the automatization that occurred when procedures were repeatedly practiced. Repetition also made concepts seem so familiar that they were easily stored in memory and appeared to the individual as common sense.

I guess like an example [is] that problem [ $1.50 \times .25$ ] we just did is routine. It's like you always do it in a certain way and you always move the decimal point over and just automatically do it mechanically. Cause you don't think about it, it just happens. And memorize I guess could go in there. You memorize the routine of the mechanical things that you have to do. Formulas would go there too cause you have to memorize all those formulas and once you know them, it's mechanical, you just automatically do them.

Sue felt that these routine problems had specific solution techniques that one needed to know.

Cause there's always a specific way to do it. Not always, but there's only a couple ways you usually can do a problem and there's specific ways. You really can't make up your own way because it probably won't work [underline added]. So specific as in theorems and one way you can say 'em.

When asked how much of mathematics was memorized, Sue distinguished between the original mental act to retain the knowledge and the subsequent recall of that knowledge. She felt that initially theorems or formulas may need to be memorized but through repeated usage in problems the ideas lost the quality of being memorized. They became so integrated into one's

thought processes that the ideas now seemed to be common sense to the individual. What was once a conscious act had become an instinctual response.

Well it depends what you would call memorizing. You memorize them but then they become common sense after memorizing. . . . It's like you memorized theorems for geometry in the beginning of the year, but by the end of the year you don't have to memorize them, they're there already. At the end of the year, they're in your head. You don't really think about memorizing them any more and studying and cramming for. There's probably some memorizing, that there's always gonna be one part where you always have to memorize how to do it. Like current problems are always  $x$  minus  $y$  or  $y$  minus  $x$  and memorize that. It's really not common sense yet to me.

She also linked together *common sense* and *useful* saying that once the mathematical ideas had become common sense to you then it was easy to use them.

Common sense and useful might go together because once you know it, it becomes common sense [underline added] to yourself as when you multiply it's common sense that you always move decimal points over. And it can be useful once you know the common sense ones. The ones that just come easily.

Sue associated the terms *thought-provoking*, *trial and error*, and *current* under the heading of word problems. For Sue, the term *current* invoked the traditional word problem scenario with a boat traveling with and against the current. With regard to *trial and error*, Sue remarked:

About doing word problems or problems where I know what the answer is. And I try to figure out the problem and I don't get the right answer. I just try another way and I try another way until I get it. Think about doing.

Sue indicated as well that mathematics was *deductive*, *logical*, and *easy*. However conspicuously absent were the terms that would have suggested that mathematical rules were dogmatic like structured, rigid, or right or wrong. Sue's ranking grid later confirmed this perspective.

Sue's vocabulary list presented a view of mathematics as primarily consisting of automatized rules and formulas. Over time Sue believed that these procedures became so integrated in one's thinking processes that they seemed like common sense to the individual. In contrast, word problems epitomized that part of mathematics that could not be reduced to simple routines. Sue felt that word problems often required trial and error, and original thinking. She also indicated that mathematics was useful, deductive, and logical. Finally for herself, mathematics was easy.

English. Sue organized her vocabulary terms for English around the activities associated with writing, interpreting, and reading essays and stories. Under the writing category, Sue grouped the terms *organized*, *sequenced*, and *structured*. She felt that all writing assignments

had a prescribed format.

The way you write a story, it's always organized. You have an introduction in a story. You have your climax and all those parts of the story. I think of that when I think of writing or an essay paper. It's always organized in a certain way.

Under the interpretive category, Sue collected the terms: *interpretive, analyze, opinionated, thought-provoking, themes, controversial, open ended, integrating, and symbolic*. For Sue, examining a piece of writing entailed analyzing its structure, themes, and view point. The view point represented the individual's interpretation or opinion therefore, this facet of writing was often controversial.

Different things you can do to a story. You can interpret it, you can analyze it and look at the structure. . . . Take the whole thing apart and analyze it. And it makes you think about what you're reading and you can see if you're opinionated about it, and if there's other feelings that you have. . . . Themes and symbolic would go together in the same category cause in English we talk about the themes and symbols in it.

The reading category included the terms: *visual, exciting, creative, discovery, and expressive*. These terms suggested to Sue an activity that was solely contained within the individual's mind. When reading a story, Sue described enjoying using her imagination and creativity to visualize scenes within the writing. Because reading was very much an individual activity, stories would not be universally appealing.

I think about that [the terms in the category] when I read stories or read a book. I can picture the stories in my head. And creative and exciting, it's just I like everything in it. I like the stories that I read. They're exciting. In your head you can create the story. I circled visual, in that you create the story in your head and it makes you imaginative and creative.

Every story relates to the author. Everybody's an individual. Each story is individualistic to that person. Everything is different. What somebody likes [or] reads might not be what the other person likes or reads.

For Sue all writing had a prescribed form. These writings were analyzed for structure, themes, and interpretation. The interpretations given to the writing were controversial and arose from an individual's response from reading the material. Reading required imagination and creativity to develop a mental image of the story.

Science. Sue's discussion of the vocabulary terms for science focused on the experimental process and on science's universal appeal. Under the experimental category, she grouped: *cause and effect, visual, changing, and flexible*. Cause and effect was the technique by which science made discoveries and verified hypotheses.

[The] different experiments that we do to find out the causes of something and the effect

of it, what starts it.

Experiments also utilized *visual* observation and distinguished between chemical and physical changes.

You visually look at the snow and you take an observation. The visual observation.

I just think about chemical change and physical change that we're doing right now in chemistry.

Sue associated *flexibility* with the elasticity of various metals.

In her discussion Sue described science as investigating causes. In this regard, observation and *common sense* motivate deeper and more *thought-provoking* questions. The resulting scientific theories were built on both common sense and theory.

You think of theories you have to figure out and common sense. It's either common sense or you have to think about it. Like the snow is cold, it's common sense. And then you start thinking why is it cold and then you think of the theories that go with it.

Since questions constantly arose within individuals about their environment, Sue felt that science had a *universal* appeal.

[Science is] used all over. Everybody does it. Even if they are [not] scientific [scientists], everybody, little kids. Why is the grass green and they just think about science. I think it's universal wherever you are.

Sue found science enjoyable and *exciting* because it attempted to answer those naturally occurring questions.

I like science. It's just neat learning about different things, about why this happened and why that happened.

Sue especially liked archeology (*discovery, ancient, chronological*) since it was the overlap between science and history.

Discovery [of] the ancient civilizations. I like those kinds of things. And chronological, of course, figuring out what happened when, how people evolved. That's what I think about science. What we study about sometimes but not a lot in history.

Finally, Sue commented on her grouping of *diagrams, formulas, and theoretical* by saying that science required memorization of formulas and theories.

All the theorems that we do, and theories that we have. And the formulas we've memorized and diagrams we make. Graphs, charts, data [are] all diagrams.

Science utilized cause and effect to organize and direct its experiments. Scientific theories were based on both common sense and theoretical ideas. Since science attempted to answer naturally occurring questions from the environment, Sue found it exciting and believed that the



questions of science were naturally part of everyone. For herself, she enjoyed especially archeology because it combined science with her interest in history. Finally, school science required memorization of formulas and theorems.

History. For Sue, history was a sequence of events which could be understood in terms of causes and effects. However, within this orderly procession of facts some uncertainty existed. The uncertainty lay in the perplexing attempt to study ancient times with no written record or means of verifying ideas. Controversy often arose as historians tried to interpret and understand those ancient times.

Under the sequencing aspect of history, Sue grouped the terms: *sequenced, ordered, theoretical, and cause and effect*.

Cause history always has been a certain way. It never changes, once it happened, it happened. And it's in an order. Some of them are related in that cause and effect. In that the order that something happens is the cause and effect of it and that will make something else happen. Like we just studied in history about the British people putting all these taxes on the Americans before the war against Britain that we had. Just all the cause and effects and those happen in a certain order and they were sequenced until the effect happened.

Although much of history was factual and knowable, there still remained parts of history that were uncertain. Sue associated with this category the terms: *old, ancient, discovery, uncertain, interpretive, opinionated, and controversial*. According to Sue, historians attempted to discover what actually transpired in ancient times but they could, at best, generate theories to explain the events since historical data was uncertain and sketchy. As a result of this uncertainty, history was often controversial.

The old ancient part [was uncertain and hence controversial] because you really don't know what happened. We're just guessing--theoretical. We're making up these theories but we aren't really sure. With all these theories it's controversial. You know which is right or wrong.

Sue also circled *anxiety* but this choice reflected her immediate circumstances. She found her current teacher's demands for extensive memorization of dates difficult to meet.

Right now I'd just call it anxiety. Different parts are very awful. My teacher's . . . one of the harder teachers. . . . Just remembering dates of all the explorers. I like history a lot but it's hard to remember all those dates.

For Sue historical events could be interpreted by cause and effect. While most history was well-ordered and known, less was known about ancient times. As a result, historians' theories on these times were often controversial.

Summary. Although Sue was sparing in her selection of terms on the vocabulary lists, her remarks suggested many distinct traits among the disciplines. Universal appeal was one of the few traits common to all four disciplines. They also shared the distinction that none were rigid, right or wrong, dull or boring.

For Sue mathematics represented automatized procedures or rules. She de-emphasized the importance of memorization in mathematics. She felt that once procedures were practiced, they no longer needed reinforcement by memorization. Instead they became common sense within the individual. Like Ann, Sue saw mathematics as having set answers but with multiple solution techniques. While much of mathematics utilized these procedures word problems required a flexible and thoughtful approach including trial and error. Sue remarked as well that mathematics was logical and deductive. Finally, Sue personally found mathematics easy and useful.

In contrast to mathematics, Sue described English as emphasizing the individual. English entailed writing, reading, and interpreting, all activities which highlighted individual opinion and creativity. She perceived all writing as having a structured form while reading was unfocused allowing the individual to drift and create images from the stories. The interpretations given to essays and stories were often controversial. Finally Sue saw reading as exciting.

Sue's perception of science was of a discipline that investigated the causes of natural phenomenon. Because most people were curious about their environment, Sue felt that science was universally appealing and part of everyone's thoughts. Science used trial and error to test its theories. The resulting theories were built on common sense and scientific opinion. Although the questions of science were exciting, school science required memorization of formulas and theories.

Like science, history was also memorized. Yet history differed from science in that it was based on stable and sequenced facts. Historical events were known and documented. Any controversy in history arose from speculation about events in antiquity.

#### Ranking Grid

Sue's ranking grid revealed very strong preferences and dislikes among the mathematics topics. (See Appendix J.) She passionately hated percents while word problems followed a close second. On the positive side, she enjoyed geometric constructions and solving equations.

Sue's dislike of both percents and word problems stemmed from her confusion and inability to set up the initial equations. Sue's difficulty with percents began in ninth grade. Since her teacher had not put much emphasis on them on his unit test, she likewise ignored them.

Percents, scratch them off. I hate percents. I just never learned how to do them. It just never stuck in my head. It was just the worst unit ever. And I know we're gonna do them in A2-Trig, it's gonna be awful. [laugh] We're suppose to learn them my freshman year, but he didn't put much emphasis on them, so I just sorta blew off the unit that we did. It wasn't fun. I only had two problems on a test. I could maybe get partial credit if they were easy ones. So I didn't put much emphasis on it when I was studying.

Her dislike was so strong that she jokingly teased that the ranking grid should have a separate "hate column" for percents.

I have no idea where I should put the percents, cause it's not really boring, I just don't like them. . . . It really doesn't go in this category, it doesn't belong anywhere.

Unlike Tara's response to fractions, Sue felt the need to conquer her knowledge gap.

If I knew how to do them, it wouldn't be that hard. I think I'm gonna ask somebody to help me learn how to do them so I can do OK. Instead of going "Oh, I don't know."

Not surprisingly on the ranking grid, Sue marked percents as *hardest to do, most difficult to learn, worst at, advanced, requiring original thinking, boring, least liked, and useful.*

Word problems also received a poor ranking on Sue's grid. She marked them as *hard to do, hard to learn, requiring original thinking, and thought provoking while being flexible and applied.* Sue's reaction to word problems differed from her dislike of percents. Sue expressed more frustration than actual dislike. Sue illustrated her confusion by citing specific examples of word problems. Her difficulty and confusion focused on her inability to even get started. She described being at a loss even to define the variables initially so that she could continue on to set up the associated equation.

I can't do word problems. Like I can do most kinds of word problems. It's the ones with money. And when you have like money and sell so many tickets. An each one is worth this much and you have five students. I can't do those. I've never been able to do word problems like that. I've finally mastered current ones. The currents up and down.

It's when I don't know how to start a word problem at all is when it really eats me up. I get very aggravated [underline added]. And half the time it's like homework. I sit there and well I'll ask somebody tomorrow how to do it. Cause I hate spending a lot of time on one problem. Especially if it's not even working out from the beginning. If I get all through the end and I come up with the wrong answer it's OK. I can just go back and recheck. But if I can't even start it, I get so aggravated at the word problem. It's hard to explain what gets me upset.

It's trying to figure out how to put in the variables I'm trying to think about the one we just had. . . . We had to figure out two less than the total value of something else. But we didn't really know the other value. And we didn't know any of the other variables for the

questions or something. Or that's what it seemed like. I just got really confused how to figure out how to set up the equation just to start it. It was like basically the variables how to put them in order that got me.

Despite her confusion and frustration, Sue continued to emphasize her determination to master word problems. She was also candid in admitting that since her teachers rarely stressed word problems and only put one or two such problems on a test, that she was not highly motivated to study them.

I've just never mastered the technique. But last night some of the smart people in class couldn't do some of them, so I think they were just plain hard. I don't know how you can make a word problem easy. I guess if you just really spend a lot of time on them, cause usually you don't spend much time in math on word problems. You just do a little bit of them. The teachers assume you know how to do them. You just go really fast over the unit, so you don't spend a lot of time. . . . I've never had a test with more than three word problems, so they're never that important. And you can usually get one or two of the word problems, [so if] you [don't] answer the other one it's not that big of a deal cause it's only one problem off this whole test.

Writing proofs was another topic that Sue found difficult. She ranked them as *theoretical*, *advanced*, *confusing*, *worse at*, and *not useful*. In fact, she saw no use for proofs at all.

I guess writing proofs [has] to be the one thing that you never use in the real world, cause you don't prove things in the real world, it doesn't seem. . . . [You learned geometry] so you could do good in A2-Trig. I have no idea. [laugh] I don't know why we learned that. I mean I don't plan on making a triangular house or anything. [laugh] So, I don't know why we learned it because it's sorta like chemistry. It's not gonna help me in my day to day life getting a job. Well, I do know about the atom [but that] doesn't mean anything.

Similarly absolute values and inequalities were seen as *useless*, *theoretical*, and *abstract*. Sue said simply: "I can't relate to them."

In contrast, Sue spoke enthusiastically about geometric constructions. They were for her the most interesting topic being *easy to do* and *learn*, and a topic she *liked* and was *best at*. At the same time she found geometric constructions to use only *routine thinking* and hence were *rigid* in approach and *busy work*. Sue described covering her notebooks and book covers with geometric designs she had devised.

Yeah, I like doodle on all my papers cause like I even have them all over my cover. I just make all these abstract things all over my papers. I like three-dimensional figures. I have them all over my papers during class, make squares and boxes. Geometry did that to me. [laugh]

On the ranking grid, Sue again expressed her enjoyment in solving equations. She ranked them high in *interest*, *utility*, and *challenge*. Underlying Sue's preferences and dislike for topics were two unifying themes. She believed that all mathematics was *logical* and that with the exception of geometric constructions all topics were also *flexible* in their solution techniques.

Even though Sue found proofs useless and word problems confusing, she still believed that each had its own logical process. In fact for Sue all mathematics was logical.

I'd say that everything's logical in this [underline added]. I don't think anything's not logical. When I think of these, they're all important and they all have a certain way you have to do them. . . . When I think of a proof, I just think like once you find that side, you now that you're supposed to find that side, you know that you're supposed to find the other side. So it's sort of logical, at least in my head. So I don't know if I could rank those saying that one's not [as] logical [as] the other. It all involves logic. It's [word problems] sorta logic in that you have to find out where you are, where you want to go, what you want to do. It's just logic that you do.

The second theme that underscored Sue's rankings was a belief that most mathematic problems had multiple-solution techniques. However, geometric constructions were an exception. Sue felt that the construction techniques were rigid and fixed.

Cause you can do factoring different ways and come out with the same answer. You can do them many different way. And you can do word problems many different ways and still come out with the same answer. And exponents, they're all the same you can do different steps at different times and still come up with the same answer. . . . And writing proofs, you can put [in] all different kinds of orders and prove different ways. If I put all those [as ones], probably everything except probably geometric constructions. Cause when you make a picture of a square, there's only one way to make a square. You can make it three dimensional or you can just make a regular square. . . . So that, that would be the highest, but the rest are almost the same. Cause they're all different steps you can use to come up with the same answer.

Sue's rankings reflected her enjoyment of those topics that she found easy to do and clear: geometric constructions, solving equations, decimals and fractions. Analogously, she disliked those topics that were difficult to learn, confusing and abstract: word problems, proofs, and absolute values and equalities. Despite her clear preference, she described all mathematics as logical and most as flexible.

#### Synopsis of General Beliefs about Mathematics

Sue's beliefs interviews presented a view of mathematics as logical, deductive and flexible. Sue felt that although problems had only one answer often there were multiple-solution techniques. When practiced these procedures became common sense within the individual. Thus the procedures no longer needed to be reinforced by memorization. In contrast word problems and proofs were examples of mathematics that could not be reduced to routine procedures. They required original thinking and often trial and error to ferret out the appropriate solution.

Sue found most mathematics enjoyable and easy with the exception of percents and word problems. Generally, Sue was often aggravated at her own inability to set up the initial

equations. Equally frustrating for Sue, was the irrelevance of most of her mathematics to her daily life. She described geometric proofs as useless since no one ever proved anything in real life. Her sole rationale for studying geometry was as a preparation for Algebra II/Trigonometry. She did not dislike geometry, rather she did not see its significance.

Finally in her description of something unlike mathematics, Sue presented an image of mathematical thinking as focused and goal oriented. Mathematics problems did not allow the mind to wander or daydream but were instead intense being tied to the concrete "here and now." As a consequence, Sue did not perceive mathematics as creative or imaginative. One further implication was that mathematics was seen as divorced from the individual. Mathematics was not like reading or writing in that it did not value individual opinion or imagination. Mathematics was more conformity than originality.

#### Student/Teacher Roles

##### Good Mathematics Student

Sue offered Mary as someone who was good at mathematics. Along with her ability to do mathematics quickly, Sue described Mary as self-confident.

She's good because I think the attitude that I get from her is that she knows she's good and she knows how to do the math. She does it well enough so she can just say how to do something right off the top of her head. And it comes easy to her. . . . The attitude that she gives off is that for her it's easy and that it's no problem.

Sue felt that a person became good at mathematics primarily by being motivated and having some ability to work with numbers.

##### Advice to an Exchange Student

Sue offered straightforward advice to the exchange student. Foremost was doing all homework as a preparation for the tests. She also stressed the importance of always asking questions. She encouraged the exchange student not to be shy but to persist getting help either by asking questions in class, by asking another student or by finding the teacher after school. Sue felt strongly that asking questions was a student's right and was necessary to understand the material. Sue also felt that taking good class notes was helpful. Studying for tests, however, depended on an individual's need. She suggested reviewing notes and working sample problems from the beginning of the unit.

I'd just tell him the most important thing is to do all your homework, cause that's gonna help study for the test. You have to ask questions, cause if you don't ask them, nobody

else is probably gonna ask them. Everybody else in the class is smart so [laugh] if you don't ask, nobody else is going to. It's real important to ask the questions. No one else really cares if you ask questions in the class.

If you have a problem with it [homework], you could ask someone else to help you. If you can't do that, during class you [can] ask to put that [problem] on the board. Cause we go over homework and then you just say what problems you need help on, and you can just put them on the board and ask somebody else to do it on the board and you can learn it that way. If you really have hard trouble you can stay after and ask Mrs. Thomas. [laugh]

[Studying for tests] depends on how good of a person you are in math. Some people don't have to study at all cause they just remember it. Sometimes if I know most of the material pretty good, just tell them to look over your notes that you take. You gotta take good notes too. Look over the notes cause sometimes like I . . . do a couple problems that you probably did in the very beginning of the unit . . . [to] see if you're doing them ok. . . . Mostly reviewing notes would be the best. And if you don't understand it, to look and do a couple problems.

In a subsequent conversation, Sue outlined her philosophy and strategies for handling difficult homework problems. She began by attempting all the problems and then by referring back to class notes or to the textbook reading assignments. If those strategies failed to enlighten the problem, she gave up and asked in class. However, she insisted that if she felt that she was getting close to a solution that she would persist.

I'll write down all the problems and I'll try to attempt it. And if I can't do it, if I really don't know how to do it, I'll look in the book and I'll try to see. . . . If I read the book and [if] I still don't understand it after I've already looked at my notes from the class, I'll just say the heck with it. [laugh] I'll come to school the next day and I'll ask somebody to help me. Or I'll just put all the problems up on the board, ask them to be done. I mean I'll try, but if I can't I'll just get somebody else to help me. If I think I'm getting close or I think I should know [to do] it, I'll spend more time on something. Probably no more than 15 minutes on a problem. Cause it's sorta a long time to do a math problem [underline added]. But if I know I don't even understand the basics of it, cause sometimes you have levels in the math you have A, B, and C problems. A being real easy. If I can't even do the A problems and I'm trying really hard, I'll try hard on those cause those are supposed to be easy. Then when I get through and I get to B and C problems and I still can't do them, I won't spend as much time. Because [if] I couldn't even do the A problems, there's usually no way I can do C prob'ems. So I'll spend less time on it cause I know I probably won't get it anyway. No more than 5 or 10 minutes probably.

Sue also indicated that she always copied down the correct solutions from board and used her corrected homework to study from. In fact, she mentioned saving all her mathematics homework, tests, quizzes, and notebooks since her freshman year. She felt that, "They'll help me in college when I do math and science."

#### Advice to a Beginning Teacher

Sue's first piece of advice to a beginning teacher was to spice up their lectures by joking around. If the classroom atmosphere was friendly, then Sue felt that not only would the classes be less boring but that it would be easier to learn in that environment.

The type of teachers that I like are the ones that can joke around with the class. I like it cause sometimes when you just sit there in a class and you're [being] lectured at the whole time, it's not as fun. And it's more enjoyable when you can have a nice atmosphere and you can joke around in the class and still learn a lot. . . . And if you just add some spice to your lectures, it's better to learn. It's not as boring. It's nice to joke around once in a while.

Sue also felt it was important for teachers to be aware of the attitude they project. Often teachers gave the impression that they are rushing through the materials and did not wish to be disturbed by questions. She believed that teachers should be open to all questions.

Just to be open for questions, cause some teachers give the impression that they're too busy for questions. Maybe they don't mean to but sometimes just their attitude just says we're gonna get through this and no questions. Or they act really [like] this is wasting my time, you should know this already. I don't like [it] when teachers are like that. Just be open towards the questions from the kids.

As a follow-up question, Sue was asked how she would change the way mathematics was taught. Mirroring the other participants' thoughts, she wished that both mathematics and science could be more relevant to her life.

I don't know, I think it's fine. The only thing that comes to mind about mathematics is sometimes it just seems so unimportant [underline added]. There's not too much [need] in our daily lives when we get older to have to know the Pythagorean Theorem. Or [when are going] to have to know theorems for an octagon, how to find the volume of an octagon. I mean most people don't need that. And it just seems you put so much emphasis into something we really don't need [underline added]. It's sorta crazy, I don't mind learning it, but it just sorta seems [a] waste sometimes. You're sitting there doing your homework thinking why am I learning this it's not important.

Sue indicated that she had similar thoughts about chemistry class. Her objection extended to include the futility of learning material that would soon be obsolete.

We were talking about this in science the other day about how ten to one in a couple of years the whole atom is gonna be changed anyway. Because we really don't know what it is already and we're just making theories on it. Probably half of them are wrong anyway. So learning things that are gonna be changed seems sorta wasteless but you do it anyway [underline added].

When asked if she thought that mathematics would be changeable as well, Sue remarked that mathematics was stable with only an occasional introduction of a new short cut or theorem.

I think they'll always be the same. The only thing that will happen would be that they might find a short cut to do something or just a new theorem for somebody to make up to do during class [underline added]. . . . [Mathematics is] not changing cause most of the math things are positive and they're known [underline added].

#### A Good Mathematics Teacher

Sue's description of a good mathematics teacher began with the declaration that the teacher should love mathematics. She felt from this love would come a thorough understanding



of the discipline and a desire to teach enthusiastically.

Love math [underline added]. Cause if they like math, they're gonna understand what's happening, and they'll make it more enjoyable I think. When a teacher stops liking what they teach, I think that they become a bad teacher. Their attitude changes and it's not as fun in the class and they won't teach as well. But if they like math then I think they'll try very hard to make other people like it. It will make class more enjoyable and will be easier for the students.

Sue felt that without this interest and enthusiasm, that the class would be boring and lifeless. To illustrate her point she described her English class.

You just sit there in class and you take notes and you listen but it's so boring. They make it so boring that you don't learn very much. But when the teachers themselves are lively, and up and at it, and making jokes you learn more I think. Like my English [class that] I have next [period] is like that. I don't think she's ever said one thing to anybody about outside of class activities or joked with anybody. She's just [concerned with] class, [with] English that's all. It gets so monotonous in that class. You need something to change the program about once or twice during a period. It helps so much.

Sue felt that a teacher was not responsible for telling the students everything they needed to know to do the homework. The teachers should, however, indicate the appropriate readings in the textbook covering that topic.

If a teacher tells you that you're gonna have to read more in a chapter, that's OK. But it's when they give you homework and they don't tell you that you're gonna have to be reading on or whatever, I don't think that's fair. Just to give you homework and assume you can look it all up and find the answers. But if they tell you that you're gonna have to do that then I think that's OK. I don't think they have to tell you every little thing how to do it, or what's the homework's on that's important [underline added].

When asked how she would construct a test, Sue suggested selecting homework-type problems. She felt it was unfair to give "surprise questions", that is, questions unfamiliar to the students.

I'd give problems like the things that we've done during class on the homework. I don't think I'd give a surprise question that we hadn't done before. Like, let's say we're doing equations and the homework never ever had an equation with fractions in it. I don't [think] it's fair to give one with a fraction in it, if you haven't worked with it very much and she did say you were. I don't think that's fair to do something you've never done before on a test. That means so much to your grade. I think it should have been at least something related to it. I mean even if you had 1 question with 1 fraction on homework or during class, that's fine because if everybody reviews their notes and their homework, they'll get that problem and will remember that that was one of the things that they've done. But I don't think it's fair to just surprise and give you a word problem if there's no word problems in a unit or something. I don't think that's fair.

Sue continued with her argument by claiming that tests were suppose to measure mastery of in class material. To examine students on extension or unfamiliar material was inappropriate and unfair since students had no prior preparation.

A test is supposed to be something that to see that you're following all the material you're

doing in class [underline added]. You understand how they're doing it so you can move on and they're suppose to test you to see if you're doing well in the class and you understand the material. But if you give material on something we haven't done, that's not really related, I don't think it's fair cause that's not testing what you're doing.

Finally Sue argued that it was unfair once a student had mastered a concept to change the concept. She gave solving equations and word problems as an example of a concept and extension that would be unfair to students.

Once a person finally masters how to do something, I don't think it's fair to change it on the test. Change the concept just a tiny bit cause some times they won't be able to understand that. But they do understand the concept, it's just not fair to change it on them. . . . [For example] you might understand how to do an equation but maybe haven't mastered how to take a word problem and put that into an equation. That's sorta cheap for a teacher who put that on there, because the word problems in my mind are different than regular equations. Cause sometimes it's so much harder to figure out where to put the variables in the equation from a word problem. It's so confusing. [laugh] I don't think that's fair to put a word problem on a test if you're learning equations, unless you've done them before.

In summary, Sue believed that a good mathematics teacher loved the discipline and wanted to share that enthusiasm with her students. In addition to being knowledgeable about the subject, a good teacher needed to be aware of students' interests and use that knowledge to help students relate to the mathematics they were learning. To facilitate learning, teachers also needed to create a friendly and lively atmosphere in the classroom. Teachers were not responsible for presenting all homework-type problems in class but they should indicate the necessary reading assignments. On tests, teachers should model questions on homework and in-class problems. Otherwise, the tests would be unfair to students who were just able to master the presented concepts.

#### Summary of Student/Teacher Roles

Sue felt that students were responsible for completing all homework assignments, getting their questions answered, taking good class notes, and reviewing for tests. She repeatedly stressed the importance of clearing up any questions on the homework problems or lectures. Assistance was available from other students and from the teacher both in and out of class. By keeping corrected homework assignments, quizzes, and tests, a student could combine these with class notes to review for the tests. Occasionally reworking problems from the beginning of the unit was necessary to recall the earlier material.

Sue felt that teachers should be cognizant of the attitude they projected in the classroom. A positive atmosphere was one in which the teachers took time to joke with their students and

took time to encourage questions. A teacher also needed to be both knowledgeable and enthusiastic about mathematics. Their knowledge should include examples showing the relevancy of mathematics to students' everyday lives. When making homework assignments, teachers should stipulate if additional reading was necessary before beginning the homework. Sue felt if they did this they need not present examples of all homework problems. Finally, teachers should construct test questions parallel to homework and class examples.

#### Conceptual/Procedural View of Mathematics

The analysis of Sue as holding a conceptual or procedural belief about mathematics originated from two sources: the general mathematics overview and the coding of Sue's responses from the transcriptions of her solutions to the mathematics problems in the interview protocols. Of the 10 episodes codeable for Sue, 9 were labeled conceptual and 1 procedural. Sue's ranking of 90% conceptual ordered her third among the six participants in this category. (See Appendix I for a table summarizing the participants' coding on the conceptual/procedural and autonomous/nonautonomous categories.)

#### Conceptual

Sue's problem protocols were coded conceptual based on her ability to use number sense to justify solutions to find multiple solution approaches and to summarize results. On problem #21 ( $1.50 \times .25$ ), Sue was able to justify her answer by number sense. For Sue, the algorithm for shifting the decimal position to correspond with the total number of decimal positions was a memorized technique. It was merely the procedure one executed. Sue first attempted to create a justification for the algorithm by suggesting a need to match the hundredths places. When it was pointed out that the result had a thousandths position, she returned to her rule statement. Next, she reasoned that a multiplication product could be verified through the corresponding division. Again when queried about an explanation for the shift in decimals in division, she remarked that it made the computation easier but she did not know why it worked. Again it represented a rule. Finally, she argued that her result was appropriate because for a multiplication problem with a factor of 1.50 to yield a value of 3.75, the multiplier would have to have been 3 or 2. Since the multiplier here was only .25, 3.75 was an unacceptable value. She continued by pointing out that .25 also represented a fraction and so the resulting multiplication with 1.50 should yield a smaller rather than a larger value.

You count over how many decimal points there are, so that's 1, 2, 3. [It works] because I'm going all the way out to the hundredth's place. I'm not really sure. . . . Cause you have to get numbers out to the hundredth's position because of the hundreds. Oh the thousands place. Well. I'm not sure. I don't know, it's always been just given to do move over however decimal points there are [underline added]. You can always divide back again and check you answer. [To] take the .25 into 375 you have to move the decimal points over in both of them, cause 25 into 375, 37.5. And that's 1.5. That's right. I don't know [why you shift]. I've just always been given to always move the decimal points over in this area. I guess it's easier to divide by a whole number than it is by decimals. It's easier to keep the decimal points more organized and to know where you are. It makes it easier to find the decimal place.

[3.75 doesn't make sense] because if it was 3.75, it would at least be around 3 in front of the decimal point to make .25, to make that agree over here. . . . It should have been at least multiplied by 3 to make 1 a 3. Right, it should have been at least a 3 to make that a 3 right here, or like a 2 or something to make the whole number big. It's [.25] not a whole number itself, so it's a fraction, that it's not even equal to 1. Because it's multiplied, it's less. Will make the number smaller.

In this example, Sue relied on the execution of the algorithm to produce the correct answer.

This answer seemed reasonable to her based on her number sense. She connected the value of the answer with her knowledge of the relative sizes of different multiplications. Thus Sue was able to justify her answer to a routine algorithm by means of number sense.

Sue also used number sense to verify her answer in problem #26 (20% of 85). She wrote the proportion  $20/100 = x/85$  which she solved for  $x$ . Upon finishing her computation, Sue reflected on the reasonableness of her answer. She argued to herself that since 20% of 100 was 20, then 20% of 85 should be slightly less.

Find 20% of 85. OK, if it's 20% so what I'd do is I'd put 20%, 20 over 100. And then [x] over 85, so 85 [is] in the denominator. That's the whole number that you have, and that's saying the same as 100 so you want the fractional part. And then so I'd multiply 20 times 85. [pause] That's 1,700 divided by 100. And that's 17. [pause] I'm just thinking if that's right or not. Makes sense. [laugh] Tha's the answer I got. Cause 20% isn't very big. And 17 is not that big. The 17 is just a little bit less than 20%, less than 20. And 85 is a little bit less than 100. so proportionally that makes sense.

On problem #24 (find the smallest radical fraction) Sue was again able to use number sense, in the form of approximation, to verify her result. Sue began the problem using the calculator to figure the decimal representations of each radical fraction. Once these computations were finished, she examined each decimal to determine the smallest value. When asked for a second approach, Sue approximated the square root of 5 with 2 and then compared the resulting fractions.

I guess the first way I'd do it cause I'm not, I don't trust myself as much to go along with my roots right now. About my basic knowledge cause I, we haven't really been rooting yet, so I'd probably divide everthing out. . . . I'm just thinking about which is the least number to the right of the decimal point. You know, it would be this one right here cause it's zero.

[.089].

You could approximate because that  $1/5$  is gonna be smaller than this number right here. Cause 5 is close to 4, and the square of 4 is 2, so it's gonna be a little over 2, and this would probably be about  $1/2$ . And you could look at these two right here and this would be  $2/10$ . So this would be,  $5/10$ . And you know that this one is gonna be smaller than that one right there. And you know that this one right here is gonna be  $2/5$ , well, so that would be about  $4/10$  and this would be  $1/10$ . So you approximate, so it would still be this one would be the smallest.

In this problem, Sue was able to use multiple approaches. Her second approach utilized approximation and comparison of fractions.

Again on problem #28 (larger of  $(543 \times 29)/32$  and  $(543 \times 30)/28$ ) Sue was able to suggest multiple approaches. She began by comparing the results from computations with the calculator. For her second approach, Sue argued by comparing the relative size of the numerators against the denominators. Since  $(543 \times 29)/32$  had both the smaller numerator and the larger denominator, it must be the smaller value.

Oh, these are like the SAT's, PSAT's. All right, I guess I'll just divide them out [uses calculator]. . . . Column B. 29 is less than 30, so it would be, this over here would be 542 less than this column over here. And this is divided by a greater number and so dividing it by the greater number would make the answer smaller. So it would be column B. Cause there is more on top and less on the bottom so the lesser number would go in more times.

As Sue's opening remarks indicated she recognized this type of problem probably from the sample SAT questions in her textbook and from the challenge problems presented on the overhead in Mrs. Thomas' class. Despite her familiarity with the problem type, Sue elected to utilize the surest technique, the actual computation facilitated by the calculator. Although it was not her first choice, Sue was capable of utilizing number sense about fractions to ascertain the larger one.

Like many of the participants, Sue was able in problem #30 (find the area of the triangle) to offer a justification for  $1/2$  in the triangle area formula.

I know why, because the base times height is the area of a square or rectangle. And the  $1/2$ , since a triangle is  $1/2$  of a square. I think, I'm making sure. Yes, it would be, cause it's half a square. That's what I think of, but I don't know if it's right. For a right triangle, that would be right.

The preceding examples were representative of Sue's problem episodes that were coded conceptual. However, Sue's problem solving episodes also provided weaker examples of her conceptual ability. On problem #27 (the remainder problem), Sue was able to solve the problem with only minimal direction from the researcher. She initially set up the equation:

$x/3 + 2 = x/4 + 2$  which she solved for  $x$ . When her work yielded  $x = 0$ , she abandoned that approach to use trial and error. This approach was equally unproductive for Sue. When suggested that she consider 14, Sue immediately grasped the example's significance and extended the idea to 3 factors. She was then able to summarize for herself why 62 satisfied the initial conditions. However, when asked for a second example, Sue doubled 62 getting 124. Since upon her check, 124 failed to meet the conditions she subtracted 2 getting 122. This value worked but Sue had no rationale for why it was necessary to subtract 2. She passed it off by saying since I doubled I must subtract 2.

Why does it work, let's see. Cause when you multiply the 3, 4 and 5 together you come up with 60 and the lowest common multiple is 60, and if you want to take a remainder 2, if [you] just added 2 to that 60. When I looked at the problem, when they had the 14 as the number, two of them came out with a remainder of 2. And that was with the 3 and the 4 and when I divided them into the number 14, they both came out to 12, 14 minus 12. And they were both multiples of 3 and 4. And they were the same number, so I figured if I got the third to add the same number, then I could do it. So I multiplied them all together to find the least common multiple.

Let's see. Maybe if I multiplied 62 times 2, 3 into 124, that's 4. [pause] How about 126, no. [pause, calculating] It would be 122. I multiplied 62 by I multiplied it by 2, so I doubled the number. When I doubled the number, the remainder came out to be 4 and so I subtracted 2. I'm not sure why it works, but then I subtracted and it, it came out to 122. I divided it and add the remainder of 2. I guess you had to subtract 2 because you doubled the number, but I'm really not sure why.

Sue was able on the first example, 62, to clearly summarize her rationale. However on the second example of 122, her explanation, in essence, said 122 was correct because it worked. Thus, her work was coded as a weaker case of conceptual thinking.

A similar phenomenon was observed in Sue's other problem-solving protocols. In problem #35 (rope problem) and #30 (find the area of the triangle) Sue organized her efforts around finding equations or theorems that would satisfy the problem conditions. Unlike Tara, Sue was willing to deviate and explore other options while maintaining a focus on the equations. These two protocols were difficult to judge either conceptual or procedural since Sue clearly expected a result through the application of an equation but did not limit her thinking to that exclusively. The coding was further complicated since an algebraic solution existed for both problems. Sue's protocols differed markedly from Tara's in another aspect. Tara expected to be helped through the problems to arrive at the appropriate equations. Sue, however, demonstrated monitoring of her own effectiveness with her equations. Thus she wanted an algebraic solution but was willing to abandon that approach when it proved to be unproductive. Sue's protocols suggested a slant

towards procedural techniques but her monitoring kept it from being a dead end approach as it was for Tara. These two examples were not clearly codable as either conceptual or procedural but they were positive and strong examples of Sue's autonomy with mathematics. The specific protocols will be discussed further under the autonomy category.

In summary Sue's conceptual coding occurred primarily in those problem protocols which dealt with arithmetic situations. Her problem-solving protocols offered either weaker evidence of conceptual thinking or conflicting information. Sue was coded conceptual based primarily on her ability to find multiple approaches to problems, to utilize number sense to justify solutions, and to summarize results.

#### Procedural

Sue's procedural coding resulted from her grading of the sample test. Like Tom, Sue had a mixed strategy when grading. Her grading on problem IIa seemed conceptual but the remainder of test showed evidence of grading based on the correct application of procedures or rules, and on class expectations. Sue remarked repeatedly that her point assignment would be determined by the degree of familiarity that students were expected to have with these problems. That is, if a problem was fairly typical then it might be evaluated right or wrong irrespective of the mathematical work present.

On problem IIa, Sue observed that Chris' solution technique was unusual but that the work was essentially correct. Chris had maintained a balanced equation by performing the same operation on both sides of the equation.

It's a weird way to do it, to do number A. Cause usually you just, normally you just add 7 to both sides. But they added 4 to both sides, he didn't have to. But technically they're right, cause they did the same to both sides. Every time they did a step they did the same to each side so they still had a balanced equation. They just took the longer route. so it's still right.

On problem IIb, Sue first began to evaluate the problem on the basis of both the process and solution but then she changed her strategy to one of class expectation. On that basis, she took off 7 of the 10 points possible.

It's basically wrong, but he did the steps find until the last step, up to the 6 equals 6, it should be 0 equals 0. I don't know, I guess it depends on why they're learning in class. If they've done this for a long time, then it should be no excuse (underline added). . . . Cause sometimes if you just do one example on the board and then they have it on a test, and it's not worth as much if they haven't really gone over it. [laugh] So, I guess depending if they haven't done, been doing it a lot, I wouldn't give them a lot of credit. If the main purpose of the test was to learn, to know how to use the distributive property, how to

multiply things out, they'd be right. Cause the person did it right. But I wouldn't give them more than, I don't know four or five points. No more than half. I don't know, maybe minus 7 or something.

Sue's grading of this problem might first suggest an evaluation on the basis of a conceptual objective, understanding of the distributive property. Yet at the same time, her remarks stressed the importance of mastery, that is, the correct application of the rule. For Sue, mastery seemed tied to the degree of familiarity--how often students had practiced this rule. These comments suggested that students should be able to correctly execute any procedure solely on their recognition of the problem type.

This theme was evident in her grading of problem 11d, as well. Sue quickly noted that the conclusion was wrong. Her grading criteria again was the class expectation. On that basis she marked the problem completely wrong, -10 off.

Oh yeah, and this right here, that's wrong because it can be a negative number. If they've been doing it a lot of the time, I would just mark it all the way off wrong [underline added]. . . . Like if he had the whole unit of absolute values, I'd mark that wrong because everybody knows that's gonna be positive not negative [underline added].

On problem 11c, Sue used a different evaluative scheme. Here her sole criteria was the incorrectness of the final solution. She admitted that she did not know the correct answer but was certain that Chris' result did not check. So on that basis, the problem was completely wrong.

Both [sides of the] equations are [not] the same so it can't be any whole number. If I had been doing it I would have gone all the way out to 0 equals 15. And x's have been cancelled. I'm trying to figure out what that means. [pause] It's so much easier teachers already know the answers, know how to do the problems. I'm gonna just see if the -3 worked. [pause] I might have done something wrong, but I don't think the answer's right. Cause it can't be both of them. I don't know if the equation is right or not. I don't know if it's possible to have an the answer but I don't think that that answer is right. Well the equation looks right. Right up to the -3 equals 12 [underline added]. [pause] What they did up there looked right, but now that I think of it, cause it's just the way it's written that. [pause] . . . So I guess I'd call it wrong. I'm not sure what the real answer is or not, but it's not -3 or 12 [underline added].

On the beginning test questions, Sue used still another evaluation scheme. Here she seemed to be grading on the basis of the proper execution of learned rules. For example on 1a, Sue's remarks suggested she was searching for the rule.

X to the 13th taken twice. [pause] [I'm] trying to remember do you add them or do you multiply them [underline added]. I think you multiply these right here. And so I'd mark that wrong. I'd take off six points.

Again on problem 1c, Sue's comments indicated an evaluation by rule.



I'd mark that wrong because it's not the way to do it cause the top one can be factored. [pause] The answer's right I think. [pause] The answer's  $x + 1$  but the way that I've learned it, you can't just scratch off the. You can't just whatever it's called, reduce them. You have to be multiples. You can't reduce something that's in a subtraction or addition equation. [expression] So I guess I'd take off full points for that cause they just didn't do it right [underline added].

Interestingly, Sue moved next to problem 1d saying simply "D is correct." She failed to observe the similarity of errors with the previous problem.

Sue's protocols on the sample test showed an inconsistency in her grading criteria. Problems in part I were either right or wrong based on her understanding of the proper execution of the rules. In part II, the criteria fluctuated from problem to problem. On 11a, Sue appeared to evaluate together the answer and solution technique. On problems 11b and d, Sue evaluated Chris' work on her classroom expectations, while on problem 11c she marked the problem solely on the incorrectness of the final answer. Thus Sue's sample test protocol was coded procedural because of her inconsistency in grading and because of her criteria that problems should be evaluated solely on familiarity to the student.

### Summary

Sue's problem protocols were coded both conceptual and procedural. This dichotomy was evident in her belief protocols as well. Sue expressed a view of mathematics that was simultaneously a collection of familiar routines while being flexible and logical. These two perspectives were manifest in Sue's problem protocols.

Sue's conceptual view of mathematics was seen in her ability to suggest multiple approaches in the arithmetic protocols. In problems #26 (20% of 85), #24 (find the smallest radical fraction), and #28 (larger of  $(543 \times 29)/32$  and  $(543 \times 30)/28$ ), Sue moved easily to and from decimal and fractional representations. This coincided with her comment that she saw these two topics as similar: "Decimals and fractions, they're right next to each other." She also articulated a belief that although mathematics problems had only one answer, they often had multiple-solution techniques. Again this expectation matched with her actions in the problem protocols. At times, Sue even utilized her flexibility of approach to verify for herself the reasonableness of her answers.

In addition to finding multiple approaches, Sue demonstrated in the problem protocols that she could summarize solutions and provide intuitive justification for results from arithmetic

algorithms. These traits seemed to agree with her expectation that mathematics involved logic and common sense. For Sue, common sense meant both an easy familiarity with a procedure (routinization) and an internal acceptance of its validity. This latter perspective implied that rules and procedures should be believable, that is, they could be justified in terms of Sue's own knowledge structure. This expectation was visible in her ability to intuitively justify her result of the algorithm in problem #21 ( $1.50 \times .25$ ). Another consequence of her expectation that mathematics should be logical was her voluntary summarization of her results in problems #27 (the remainder problem) and #33 (larger of  $p + 2$  and  $2 - p$ ). By summarizing these results, Sue showed that she valued and understood the logical connections between the answer and the original problem.

In addition to her conceptual expectations about mathematics, Sue's protocols also supported a prominent procedural view of mathematics. This perspective was most evident in her grading of the sample test. Sue evaluated problems based on her estimation of how familiar students were expected to be with certain procedures. She assigned point values independent of the type of error and based explicitly on her classroom expectations. This grading policy agreed with Sue's comments on the purpose of testing. She expected mathematics tests to reflect typical homework problem and hence students should be well acquainted with the material. As a result, Sue felt comfortable assigning either full or zero credit on many of the test's questions. She assigned these values even while acknowledging and distinguishing between the process and the conclusion. Her expectation that these procedures should be familiar to and mastered by students overrode her consideration of the correct processes with incorrect conclusions. Sue also spoke of the routinization of procedures when she remarked, "There's always a specific way to do it."

For Sue mathematics could entail both logic and memorized procedures. The differences appeared to reflect a realization of how typical school mathematics operated. When in testing situations, Sue felt that the primary objective was to repeat accurately the techniques learned in class. In this situation mathematics was to be judged right or wrong depending on the student's mastery. In other situations, like when doing homework, students were exploring and trying to understand the techniques. Recall that Sue was adamant that students should push to get their questions answered. Sue wanted mathematics to make internal sense--to be clear

and logical. She also expressed the belief that teachers were not responsible for presenting examples of all homework-type problems. Thus, Sue expected to reason through and draw conclusions for herself explaining her belief that mathematics could be internally logical, like a common sense within the individual.

This dual view of mathematics was also played out in Sue's problem solving protocols. She expected to solve the problems through an application of an equation but felt that the formation of the equation might necessitate divergent thinking. She expected to use trial and error to generate the appropriate equations. Sue wanted an equation to solve the problem solving situations, but maintained her perspective that whatever equation she generated must be internally consistent with her understanding of the problem. Sue seemed to expect these problems to be like typical word problems which she hated but which were solved by finding a logical equation.

Summarizing Sue's problem protocols and beliefs interviews revealed a dualistic view of mathematics. For Sue mathematics consisted of routine procedures which could become common sense within the individual over time. She felt keenly that all mathematics was logical and flexible. These beliefs were manifest in her ability to offer multiple approaches to problem and to summarize and justify results and in her evaluation of mathematics on basis of mastery of learned procedures.

#### Autonomy/Nonautonomy with Mathematics

Sue's problem protocols and beliefs interviews revealed her autonomy with mathematics. Of the 6 codable problem episodes, all 6 were coded as autonomous. This ranked her third among the six participants.

#### Autonomy

Within Sue's protocols were many instances that suggested her autonomy with mathematics. Sue demonstrated her autonomy by monitoring her progress, by voluntarily summarizing her results, by initiating a check on her answers, and by showing an expectation that the mathematics should make sense.

Sue's monitoring of her own progress was most evident in the problem-solving protocols. In problems #27 (the remainder problem) and #35 (the rope problem), Sue verbalized her points of decision and conveyed an awareness of her actions. On problem #27 (the remainder

problem) Sue progressed through the problem by first considering trial and error. She elected to put trial and error aside in favor of writing a series of algebraic equations. When these failed she returned to reconsider the meaning of the problem. Then with the suggestion of examining 14, Sue extrapolated to the solution of 62. Throughout this process Sue verbalized her decision indicating that she was monitoring her approaches to the problem. This monitoring is evident in the following abbreviated citation from her protocol.

I suppose I could do it by trial and error. That would be one way I could do it [underline added]. [pause] There's probably some equation to do this and I'm trying to figure out which one [underline added]. [pause] All right, let's see if this works. [pause] Naugh [underline added]. [I stopped] because it's going to be a negative number when I figure out the problem [equation] that I used. [pause] I'll try a couple of trial and errors, that might work [underline added]. [pause] . . . I don't understand why my equations wouldn't work. [pause, sigh] Let's try it with the other number and see what happens. It's probably not going to work. . . . Oh, it's not working [underline added]. . . . I don't understand . . . the only way I think about having a remainder that I know of, and it's not working [pause] I don't know. I'm thoroughly confused. It is possible to figure it out isn't it. . . . I can't figure out how to put a remainder into an equation. . . . It said that the smallest positive number which was  $x$  divided by 3, 4, 5, so  $x$  divided by 3 with remainder 2 would equal the same thing as divided 4, would have a remainder of 2. That doesn't make sense though [underline added]. [pause] . . . Am I reading the problem wrong [underline added]? . . . Maybe if I multiplied 62 times 2. [pause] Three into 124, that's 4. [pause] How about 126, no. [pause] It would be 122.

These excerpts show that Sue constantly monitored her progress. When a technique failed, she returned to re-evaluate the meaning of the problem by using trial and error to explore and motivate her thinking. Despite Sue's desire and expectation that an algebraic equation should offer a solution, she was realistic in her evaluation of the equation's ability to satisfy the problem conditions. Like the other participants Sue also needed assurance that a solution did indeed exist. A quick nod was sufficient for her to continue her search, unlike Tara who insisted, as well, on being given the appropriate equation. Sue's constant monitoring indicated that she expected her solution to match the problem conditions and that she, herself, served as the judge of her effectiveness to do that. She tried various techniques dismissing them when she had determined their failure.

Similar verbalizations were evident on problem #35 (the rope problem). Sue showed monitoring of her progress by checking the reasonableness of her answers as she advanced through the problem. Sue began by recalling the circumference formula and using that to figure the length of rope. When she progressed to the second question concerning the addition of 6 feet of rope, and its height about the ground, she tried to visualize the situation. Her intuition

told her that the distance could not be very great considering the relative magnitude of addition to original length of rope. Sue had difficulty using her diagram to suggest a strategy for finding the rope's height above the earth. Through a series of questions, Sue was asked to explain the significance of the inner circle, outer circle and the distance between them. Once Sue realized that the adjoining space represented the sought after height, she easily generated appropriate equations. In the process of solving the equations, Sue added the  $2\pi$  feet into the 8000  $\pi$  miles. She did this without regard to the units, but immediately questioned her own result. It just did not sound right to her. After correcting this mistake, Sue proceeded to the actual solution. Although expressing hesitancy at actual height of 1 foot, she had implicit faith in her answer and solution process. Sue's monitoring was evident in her protocol which is presented here in abbreviated form.

All right, [I'm] trying to remember our theorems [underline added]. OK. [pause] Area, wait, circumference equals [pause]  $2\pi r$ , I think. All right let's try this formula. . . . I'm trying to figure out how to figure out how much rope you need to go around the equator and that's the circumference and that's 8000  $\pi$ . . . . I'm not sure if I should be converting this into feet [underline added] or not, or just leave it 8000  $\pi$ . So I guess it would be 8000  $\pi$  miles. [laugh] . . . Well, I'm trying to figure out, I've never had a problem like that. But I was just trying to figure out what it would look like and how I would try to figure it out [underline added]. Cause when you think about it, you only add 6 feet to the rope, that's not too much, if you're just putting out in a circle. It's hardly anything [underline added]. [pause] . . . Cause a 6 feet when it's all tightened out, wouldn't be very much around the world compared to the length the rope already is. The 6 feet doesn't add very much at all. If it was a bit snugly around it and you just added 6 feet, there would be maybe a little bit, but not very much [underline added]. So I guess a piece of paper could fit through it, but not a person or a mouse. I'm not really sure how to figure out the problem to figure out how high above the ground you would be. . . . [Series of questions by researcher] OK. Earth was 8000  $\pi$  miles. And the circumference with 6 feet was 8002  $\pi$  miles. Yeah. It doesn't sound right though [underline added]. Cause we didn't add 2 miles, only added feet. We should convert feet into miles or miles into feet. . . . Hugh numbers. . . . No! I did something wrong [underline added]. Oh no, wait. Oh, I looked farther down. [laugh] I was getting very upset [underline added]. Yeah. [laugh] OK. This is a new radius and that's the old radius. So you subtract the old one from the new one, you get 1. So that's 1 foot. All right, a paper can fit through and a mouse can fit through. I guess it's just hard to picture [underline added]. The greatness of the earth is a rope. [pause] Yeah I guess so. . . . Maybe it just doesn't seem right, but it could be possible [underline added]. . . . Yeah. I believe my work, and I did it right. I guess that makes sense [underline added]. Yeah. Is it right [underline added]?

Sue abbreviated protocol demonstrated her monitoring and her expectation that the numbers she created should make sense to her. She constantly tested these values to see if they made sense in the problem. With the exception of her initial confusion about how to begin to find the height of the rope, Sue proceeded through the problem independently. She served as her own judge of her progress and answers. Interestingly, Sue's hesitation with her answer

of 1 foot, was superceded by her faith in her ability to solve the equation. This faith coincided with Sue's remarks that she enjoyed solving equations and felt that it was one of her better topics. Only after finally committing herself to her answer did she ask for confirmation of her solution to this difficult problem. On problem #33 (larger of  $p + 2$  and  $2 - p$ ), Sue voluntarily summarized her results. Like most of the participants, Sue easily solved this problem using trial examples to motivate her answer.

I'm just trying to figure out. I guess it depends on what p is. Cause if you let p equal 5 and that would be 7 and p, 5 is this, it would be -3. But if we made this -5, that would be -3 and p was -5, that would be 7. [-7]. So it depends what p is. If it said p was greater than Q, it would be A, but if it said that p was less than Q, it would be B. P equals 0 [underline added], it would be A. No wait, no they would be equal to each other [underline added], cause 0 plus 2 is the same as 2 minus 0. That's the way I'd look at it. [laugh] So the answer to this would be D. . . . On SAT's D if you need more information.

As Sue's conclusion indicated, she was familiar with this type of problem from her practice with sample SAT questions. In addition to answering the question, Sue voluntarily summarized the conditions under which the two expressions would be equal, greater or less.

On problem #26 (20% of 85), Sue voluntarily checked her answer by using number sense. She initially calculated the answer, 17, by means of solving the proportion  $20.100 = x/85$ . She then reflected on the reasonableness of her response by imagining what type value 20% of 85 would need to be. She argued that since 85 was smaller than 100, 20% of 85 should be slightly less than 20% of 100 or less than 20. Because 17 met that criteria, she accepted the answer as appropriate.

[Solves proportion] And that's 17. [pause] I'm just thinking if that's right or not. Makes sense [underline added]. That's the answer I got. Cause 20% isn't very big. And 17 is not that big. 17 is just a little bit less than 20. The 85 is a little bit less than 100. So proportionally that makes sense [underline added].

### Summary

Sue's coding on the problem protocols revealed her autonomy with mathematics. Sue demonstrated this independence by voluntarily checking answers, by summarizing results, by monitoring her own progress, and by conveying an expectation that mathematics should make sense. Sue's monitoring of her progress was especially pronounced in her problem-solving protocols. In fact, her monitoring of the effectiveness of her strategies seemed to significantly temper Sue's expectation that these problems should be solved algebraically. This tempering added to the belief that problem-solving situations often required divergent thinking appeared to

have allowed Sue sufficient flexibility to explore and eventually solve several of the problem-solving situations. Sue was not locked in by her expectations as was Tara. Sue's belief system had also included the expectation that students should be capable of exploring and generalizing mathematics on their own.

Throughout her monitoring Sue used phrases like "That doesn't make sense, though", "I'm just thinking if that's right or not", "So proportionally that makes sense", and "It doesn't sound right, though". These utterances suggested that Sue had a deep belief that mathematics should make sense to her. She was constantly testing her work against some internal voice that warned her when the answer was wrong. This expectation seemed tied with Sue's comments that mathematics became common sense within the individual and that all mathematics was logical. Sue expected to use her knowledge to judge for herself the validity of her mathematical work. This expectation was further evident in her voluntarily checking and summarizing her results. Since these were voluntary acts, they represented an internal drive within Sue to ascertain the validity and the logic of her own ideas.

For Sue, a close connection seemed to exist between her dualistic view of mathematics and her autonomy with mathematics. Both her monitoring and her self-reliance appeared to mediate her procedural expectations concerning the problem-solving situations. When the problems did not conform to her desire to solve them algebraically, Sue was able to put aside that expectation in favor of further exploration through trial and error. Thus her conceptual approach to mathematics seemed to be supported by her autonomy with mathematics.

Steve

Background

General Information

Steve, a junior, was enrolled in Algebra II/Trigonometry, U.S. History, American Literature, Latin II, Chemistry, and T.V. Film. In addition to his college preparatory program, Steve was active in the local university marching band, 4H, and the mathematics team. Of his participation in the mathematics team Steve said: "It wasn't particularly fun. It's just one of those things. A lot of things like that I just do for my own edification." He did, however, express enthusiasm for his other interests including playing frisbee and volleyball, hiking, skiing, designing house plans, drawing, and reading science fiction, fantasy, and adventure novels. Steve also had a job in the biology lab at the local university doing odds and ends like washing test tubes. Steve thought of his job as a way to explore possible vocational interests. He went on to say:

I haven't discovered a job yet I can do mentally. It's been tough trying to find something I can do right now. I don't particularly like associating with people. I'd rather be doing something on my own. I'm considering trying to get a job at Channel 11. Because I'm quite interested in T.V., video and things like that. I rarely discovered anything that uses, that I can use my brain or anything.

Among the resources available to Steve was a home computer on which he taught himself how to program. He discussed his familiarity with DOS 3.3 and Pascal. He viewed computers as tools and developed his initial interest as means to alleviate his boredom.

For a while I tried just programming, just for the heck of it, when I was really bored. And then [now] most of the time I just use it as things come along that I need to do like for 4H, I had something I was trying to promote so my father and I were thinking well if I made a quiz out of it. So then I figured a way of writing a quiz on the computer, programming the quiz. Most of the time I just use it for homework assignments, for personal writing. . . . If I need to do something well, I just look up what I need and figure out how to make it work.

He also owned a scientific calculator which he used in chemistry class for basic arithmetic functions. Steve was unfamiliar with Games magazine but indicated that he occasionally glanced at the puzzle section in his subscription of Omni magazine.

When asked to describe in what ways, if any, he used mathematics outside of his classes, Steve initially said that he did not use it in his daily life although he played with mathematical figures in architectural drawings and doodlings.

Not really. I use it for house plans. Trying to figure out how everything works together. It's just like having to fit everything together. I've been working on a basic blueprint that would allow everything, would have a certain number of like bedrooms. Certain size of



rooms into the smallest area possible. And so you gotta try and figure out how things work together, different lengths, width, size of rooms, dimensions. I did a little bit with angles. I guess the only other time I've really ever used math is sometimes when I'm doodling. I'll end up doodling something, it looks kinda interesting. So I'll calculate and make a repeating pattern or something. Calculate the math to see how it would work out, and why it works the way it does. . . . Things like, you start with a triangle. [pause] Get that central triangle there, and then you have a thing like that. [Draws a series of nested triangles] Then this becomes the central triangle and build on that. . . . And sometimes I just calculate to see how it grows or I may suspect a relationship between how things work in the diagram. I suspect something may be an equilateral triangle or something so I calculate it out to see whether it will work out.

Although Steve planned to attend college, he was uncertain as to a specific major. He indicated that irrespective of his choice he expected to get a "PhD in something." He was considering at the moment engineering or artificial intelligence but felt that there were "a lot of things I'm kinda interested in possibly getting a PhD in." He thought MIT might be a good school for him since it had advanced courses in electronics, and computers. He remarked: "I was thinking of going just right through and getting my PhD at MIT eventually, something like that."

#### Previous Courses in Mathematics

Steve offered only sketchy information about his previous mathematics courses. What he described instead was his state of mind as he progressed through the courses. Steve had Mrs. Thomas for mathematics in 7th grade, Accelerated Geometry in 10th grade and Algebra II/Trigonometry in 11th grade. Steve had forgotten the names of his teachers in 8th and 9th grades.

General Mathematics-Seventh Grade. Steve indicated that during seventh grade he joined the mathematics team but did not stick with it, saying: "I tried for a little bit. And, I didn't get along too well then cause I didn't get along with math well then." Of Mrs. Thomas he said simply "Basically, with her it's just a matter of getting things done. She knows I can do it. So she doesn't press me hard about getting everything done."

General Mathematics-Eighth Grade. Steve described himself as not ready in eighth grade to handle the abstraction of mathematics. When asked why this happened he remarked:

I don't know. It's kinda strange. It's really complicated. I wasn't in very good mental condition yet. With everything I've been doing, I been forcing my mind to operate on several different levels of abstract of abstract stuff and then I wasn't really that good. Over the last couple of years I've really got myself in order on the right track. It takes a little while to get started.

Algebra I. Steve succinctly described his algebra class as "boring." When asked why, he commented that the pace of the class was too slow and so he easily lost interest.

I'm used to absorbing things and processing information at a fast rate. So any classes that are slow, I'm completely bored in. I generally don't do very well. It's a process of mental stimulation. So it was just like boring because the teacher's going at everybody else's speed.

Accelerated Geometry. Steve enrolled in Accelerated Geometry on the advice of his Algebra I teacher. He commented on the geometry course by saying that he found proofs useless. He much preferred to just read the statement of the theorems and to understand their meaning by applying it. For Steve, proofs were tedious.

My algebra teacher said to go into it. She could probably tell how bored I was. . . . I was feeling at that point anyway that maybe a little bit more mental stimulation. I was considering going into that. She strongly encouraged me to.

I've never really done well on proofs because I don't see the use of them. They take so long to do and it's so tedious you know. I'm used to processing, I'm used to doing the homework by taking like the theorems and things and understanding basically what they're saying, by applying it. And by applying it, I remember how it works. But theorems are like you gotta prove, like you gotta remember exactly what the theorem is. I don't know, I just don't get along with them very well.

Algebra II/Trigonometry. Steve remained somewhat bored even in the Algebra II/Trigonometry class where he felt Mrs. Thomas was moving at a quicker pace. The material was not repetitious but he simply found it easy to comprehend, at times even falling asleep in class.

Mrs. Thomas moves at the fastest possible rate. So I do fine in her class. Actually I'm a little bit bored in Mrs. Thomas's class too. It's hard to explain. It's a matter of being able to process information I've been doing. It's probably mostly because I've been so busy. I've been focusing myself to take in a lot of things at once and understand a lot of things at once and as a result I've been able to. Usually during classes, it's not very often that I actually have to look up to see what she's doing on the board to understand it. I either space out or I just look in the book to see where we're [at], to see how to do tonight's homework and go ahead and do it. Mrs. Thomas understands that. I've fallen asleep in her class before and she doesn't even bother waking me up. She knows I understand the material. That's the only reason why I'm in the math team because the only problem I have with it is making careless errors all the time.

### Affect

Steve's reaction to the interview process was difficult to ascertain. He seemed to like the attention given to his opinions, yet he presented a very stoic face throughout the interview. The researcher often chatted with the participants for a few minutes before beginning the actual interview protocols and even in those unstructured moments, Steve never showed emotion or enthusiasm. For instance, he spoke dryly of the then upcoming Thanksgiving and Christmas breaks. Whenever he spoke of himself it was from a distance and usually concerned his mental state rather than his affective reaction to situations. Steve often used computer-laced

terminology when describing his thoughts.

When asked to describe his own ability to do mathematics, Steve remarked: "I'm not sure, it's just one of those things. I never really think about it actually." He went on to say that he found mathematics easy to learn.

It's just something, one of those things I do. I take math classes, I do math. I learn how to do it and I store [it] up [in] my brain and later I just pull it out and use it whenever I need to [underline added]. It isn't very difficult. It's just a matter of getting the stuff down. Once I've got it down, it's no problem. The only reason why my math grades are not particularly high is because I make careless errors in exams. Math is just pretty easy as far as I'm concerned.

#### General View of Mathematics

##### Unlike/like Mathematics

Steve described modern abstractionist art as his example of something unlike mathematics. His rationale was that this type of art lacked a focal point and regularity.

Probably nothing that I can think of would be completely unlike mathematics. Sort of like performed art. The kind of modern art that's like a blob here and a blob there. That's about the only thing I can think of that isn't related to mathematics. Even regular art, you gotta [have] angles to your focal point and everything. . . . Simply because there's no regularity to it at all. Not as a statue, something you have to worry about balance. That's like throwing paint on a piece of paper. There's no regularity about it. It's not measured. You don't have your focal points, you lines or angles or anything else. It's just like a blob.

In contrast, Steve selected architectures or civil engineering as being something like mathematics. He asserted that both areas utilized mathematics. Upon deeper reflection Steve felt that they were more akin to applied mathematics than to pure mathematics. The distinction he described was that architecture and civil engineering were creative and hence for him like applied mathematics while drafting which was simply the direct application of lines and angle was like pure mathematics.

More towards drafting, I guess. Cause there's no creativity at all. It's all lines, angles and things like that. It's just pure math is given a bunch of things and you've to sit them down exactly right, in the right order [underline added]. And to scale and things like that.

Pure math isn't particularly creative simply because there's nothing really creative to outline and shapes and angles. It's how you put them together that's creative. Applied math, yes, would be creative. Cause it would have to be creative to apply it.

##### Vocabulary Lists

Mathematics. From his selections and comments on the vocabulary list, Steve presented a complex view of mathematics [see Appendix H]. Several themes emerged: (a) mathematics as a language, (b) mathematics as a tool of science, (c) mathematics as prescribed techniques,

(d) mathematics as logical, (e) mathematics as adaptive, and (f) mathematics as learned knowledge. Underlying Steve's remarks was an emphasis on the utility of using mathematics to describe theoretical concepts in science. It was in this capacity that Steve seemed to value mathematics.

Many of Steve's terms on the vocabulary list reflected his view of mathematics as a precise language capable of distinguishing detail and conveying abstract ideas. The following is a sample of terms and associated comments that indicated Steve's view of mathematics as a language.

*expressive:* It's a more specific way of showing something. Like you got a shape and say it's a triangle. It could be any size triangle with mathematics you can show exactly what kind of triangle it is.

*integrating:* Almost everything can be mathematically described.

*abstract:* To do theoretical work you have to think abstract. You can't actually describe something you see. You describe something you imagine. Or draw something you imagine.

*theoretical:* There's many things you don't actually have, can't actually touch or hold, so it's more theoretical. If you use like physics for example . . . like angular momentum or things like that cause they don't really know [what] it's like. What the thing is. They just state it. But the distance it's not actually sitting there, like from what they observed. Theoretically it should have the following properties and agree with the following equations.

Closely related to Steve's view of mathematics as a language was his emphasis on mathematics as a tool for science. In fact, it was mathematics' ability to quantify and precisely define theoretical ideas or complex phenomenon that made it indispensable in the sciences like physics. Steve expressed this view in his comments outlined below.

*discovery:* I guess [mathematics] could aid in discovery. You can describe something more exactly with an equation. Without math you couldn't discover a new atomic particles for instances. Cause you couldn't really describe what they were without describing their orbits and shape. How they behaved, because you can't actually describe what they look like.

*changing:* Cause they are always discovering new things and describing it with math.

*analyze:* If you analyze something you describe it.

*universal:* You can use all. You can use it for physics, architecture and it can be used around the world. And we sent out our satellite [and] everything on it was described through mathematics because it doesn't change.

Steve also saw mathematics as simultaneous, fixed, logical, and adaptive. He seemed to feel that within the logic and the structure of the rules an individual could control and adapt mathematics to new situations, again stressing the applied nature of mathematics. Under the

fixed characterization, Steve's comments expressed a view of mathematics as a collection of routine procedures which were highly structured with right and wrong ways to execute them.

The following comments from Steve's protocol fall under the fixed characterization.

*rules:* It's a lot of rules. Like the right sequences. You have rules for sequences.

*routine:* You use the same basic rules all the time.

*fixed:* Yeah, because once you get a specific equation worked out and it works out in all situations then it's fixed.

*sequenced:* Like complex equations using addition, subtraction, multiplication. If you do the addition before you do the regular multiplication you completely mess up everything, so it's gotta be sequenced.

*mechanical:* Everything works in a specific way. It's just not random. It's mechanical, it works in a specific way once you've got it.

*chronological:* Describes how things fall together in the right order. It has to be in the right order [underline added].

*factual:* Definitely, cause it's a whole bunch of different numbers and things.

Adjoining his perception of mathematics as fixed was his notion that mathematical knowledge was logical and absolute. He felt that every question in mathematics was resolvable and that mathematics was truthful because all statements were either right or wrong. Mathematics could never be subjective. He represented this view in the following comments on his selection of the vocabulary terms.

*right or wrong:* It works or it doesn't.

*logical:* Definitely logical cause it's not random. Everything has a specific way.

*organized:* It has to be. There's no real, any variables for it so it's either right or wrong. It either works or it doesn't.

*truthful:* Yeah. Because an equation doesn't lie.

*objective:* Yes because it's not really subjective. There's no way of interjecting emotional values into it.

*cause and effect:* Simply because of the fact you get an equation. . . . Everything happens in a specific way or has a different specific way of happening.

*deductive:* You can take a basic property observed and they'll combine different properties and deduce a theorem from that and that would apply to make things easier.

*common sense:* It makes common sense, so everything fits together.

*rational:* Yes cause it all makes sense if you once get into it.

Steve passed over the terms open-ended and uncertain because he believed that mathematics

represented valid and clearly defined knowledge.

*not open-ended:* Cause in the end it has an end. It may look like open-ended now, but there's a solution to it somewhere.

*not uncertain:* If it's mathematics it isn't uncertain anymore.

When asked directly if there could ever be mathematical statements that were not provable either as valid or invalid, Steve remarked: "If you had the right knowledge you could prove it one way or the other. It may take forever to work out the equation. But it's still possible."

Although highly structured, Steve felt that mathematics was a tool that could be adapted to new problems. This view was reflected in the following comments.

*control:* In mathematics you have control over everything you do.

*not rigid:* Cause you can adapt it.

Concurrent with this perspective, Steve believed that the application of mathematics might require the use of trial and error, and insight. These applications also necessitated an ability to view mathematics from many perspectives. Interestingly, Steve again stated that mathematics in itself was not creative, only in its application. The following remarks were supportive of Steve's perception that mathematics required flexibility and not mere execution of rules.

*trial and error:* Not everything is cut and dried. I mean first they start out with math, and not everything's gonna work. Like, let me see, when they discovered the thing for like pi, for instance, it was just trial and error. You just study a whole bunch of triangles, it just so happened you get the same number all the time.

*insight:* You have to have insight to figure out the theoretical problems and describe mathematically things that haven't been described before.

*multi-perspective:* Definitely because to look at something that's theoretical, you've gotta be able to see it from different angles and also to solve mathematical problems you gotta see it from different views.

*intensive:* It requires a good bit of thought sometimes. It is that way.

*thought provoking:* Cause they wouldn't have come this far if it hadn't been thought provoking. Also if you get a certain equation, well it's also a challenge to make it work out different ways, describing different thing.

*goals:* Cause you have to have a certain goal in mind for an equation. You don't just write an equation for the heck of it.

*not creative:* It can only really be creative in a sense that you can use it to create things.

One final theme that emerged from Steve's discussion was his belief that mathematics was learned and not instinctual. To test the strength of his resolve on this point, he was asked whether he thought that a deserted island society would develop mathematically. Again he

responded, "no," despite his belief that mathematics crossed cultures and was universally used. He remained insistent that mathematics was learned.

The discussion then turned to whether mathematics was learned through memorization. Steve vacillated on this issue, feeling that the rudiments like multiplication tables needed to be memorized while the application of mathematics required understanding. Part of his dilemma was that he had an easy capability to remember information. He could effortlessly memorize and recall facts and ideas. He even described his mathematical knowledge as something he stored away and retrieved when he needed it. Thus his ability to memorize helped him in mathematics, yet he also stressed the importance of understanding. Steve believed as well that the ability to use mathematics was dependent on having experience with it.

[Mathematics needs] a lot of memorization. Your ability to memorize, your ability to do anything is learned. . . . I have a tendency to memorize, to remember. I like to remember a lot of different things so I can use them later. In my case it takes very little at all to memorize something. As soon as I understand how it works, I've got it down [underline added].

You can say well OK, multiplying is, it's figurative for how many times you put a certain number together. However, you can't take that and build on it. . . . He's gotta memorize the different rules of multiplication. . . . You can't really go anywhere without memorizing.

Well anybody can memorize tons of facts. Like people can in social studies, if they memorize all the facts and regurgitate them onto a test. The only way you can use something is by understanding how it works [underline added]. If you're gonna build a car you got to understand how the car works. You can't memorize all the different parts of the car and expect to build one. And also to build one you gotta memorize it cause you get the basic things down, then you can build on it. If you know how the basic things work, then you can manipulate them.

If you got experience, after a while you begin to figure out how things fit together. The first time you try math it's rather abstract. Like if you're an engineer you've got to multiply things and it's rather abstract. It's hard to visualize. But once you get experience using it, then it's easier. Then you can go on to other things. So you can only use mathematics if you've had experience in it [underline added].

Like Ann, Steve's selection of vocabulary terms for mathematics was extensive. He omitted those terms that depicted mathematics as humanistic: individualistic, instinctive, capricious, opinionated, controversial, and beautiful. When asked why mathematics was not marked as beautiful, elegant or clever, Steve replied: "Math is objective. Math is just a tool rather than an end [underline added]." His omission of the term visual represented his view that mathematical concepts were not actual objects but rather abstractions which mathematics described.

Visual, not particularly. The only way it would be visual if you drew something on paper.

It's not really visual because you can't see it. A lot of abstract things, like atoms and things you see the equation. If you're familiar with it you see the equation and you know basically what it's talking about. But you see an equation, it's not a thing, it's not a shape. It's just an equation.

Finally, with regard to affective issues, Steve believed that mathematics was *easy* and could be *fun* "if it involves a hobby."

In summary, Steve's comments on the vocabulary terms emphasized a view of mathematics as a descriptive language for science. Mathematics precision and adaptability facilitated this application. Because mathematical rules were well ordered and logical, Steve believed that all mathematical statements were resolvable as valid or invalid. While much of mathematics merely required the accurate execution of these precise rules, its application often necessitated trial and error, and insight. To apply mathematical ideas, Steve felt that one must understand them. Only through its applications could mathematics be considered creative. Steve further believed that mathematics was not instinctual but learned. This learning occurred partially through memorization, experience, and understanding. Finally for Steve, mathematics was easy and fun when applied to his hobbies.

English. Steve's selection of vocabulary terms in this category stressed English as a human endeavor. Within this perspective were three themes: English as instinctive, expressive, and controversial. Steve's comments also represented English (writing) as structured and encompassing many forms and purposes.

As part of the categorization of English as *humanistic* Steve remarked: "It deals with human relationships and the way people get along." This view stood in sharp contrast to mathematics which Steve felt was divorced from human emotions and was never subjective. Still another distinction from mathematics was Steve's belief that English, both language and writing, were instinctual. He pointed to humankind's need to communicate and record events as exemplified in the cave paintings.

Because cavemen and people like that inscribed their woolly mammoth hunts on walls. It wasn't actually English but it was a form of scribing the best they knew how about what they've been doing, through pictures in this case.

Since English dealt with human emotions, writing was a way for the individual to express those thoughts. This view was reflected in Steve's selection of the terms *expressive*, *experiential*, and *individualistic*. With regard to his selection of the term *individualistic*, Steve



commented that: "It's very much up to the person writing about what they write about." Steve thought that individuals drew upon their experiences and emotions for inspiration in their writing.

Equally individualistic were the interpretations given to the writings of others. They were often *controversial*, *opinionated*, and subjective. English was never objective since "Even in the scientific journals, you're seeing form the point of view of the author." A further consequence of this subjectivity was that: "There's nothing right or wrong about it." In fact Steve felt that: "It's always changing." and "[It] isn't defined at all."

For Steve writing was structured by the rules of grammar, and sentence order. In addition, specific forms of writing had their own inherent order as in poetry. This structure was vital to convey clearly the meaning of the author. Steve indicated this view of writing in the following comments.

*rules:* There are certain rules to putting things together which can be broken, but generally doesn't work.

*structure:* Cause you get the sentence structure which if it isn't structured right then it doesn't mean anything. Or it means something entirely different.

*mechanical:* You've gotta have the mechanics of a sentence cause if you don't have your verb in the right place it won't make sense. It's not subject, direct object and things like that.

*ordered:* In a poem you put things together in a definite order. Put words in a definite order to create a certain mood.

*memorize:* Yeah you can memorize basic sentence structure. you won't get anywhere if you just write everything haphazardly. You gotta memorize the letters and the words and things like that.

*formula:* The way we use words in English. Or like if you're German, then in English the words go together in certain formulas. Cause other English speaking people wouldn't be able to understand your sentences. . . . It's got specific order so it's a formula for writing.

Depending on how one expressed his or her ideas, writing could be *beautiful*, *clever*, or *creative*. Since writing also helped an author create a particular mood it could be manipulated to have an effect on others as in Poe's horror tales or in propaganda. Besides creating a mood, writing could be used to convey *logical* or *deductive* arguments or *abstract* concepts.

*beauty:* Yeah, you can describe beauty or English can be beautiful.

*clever:* You can use clever with use of language like Benjamin Franklin or like Thomas Jefferson. Clever ways of putting things together that were really nice like "We the People."

*cause and effect:* It involves cause and effect cause if you write something, it can have a certain effect on somebody. And like people like Edgar Allen Poe wrote specifically for

affect.

*flexible*: You can manipulate it.

*not controlled*: Cause it can be manipulated.

Steve felt that the writing process could involve *trial and error*. "Cause if you write something it's not going to come out perfect the same time it did. You can take like 5 years to write a book cause it doesn't look right the first time." However for poetry, Steve believed that the creative process was more often *capricious*: "Yeah, cause poets don't just sit there and pour out things. You gotta be sorta capricious. You do things on a whim on inspiration, like when it hits you."

Finally Steve described English as *simple* and *easy* since "Anybody can write." English could also be *fun* and *exciting* especially when reading an engaging story.

In summary, Steve emphasized through his selection of terms and comments that English focused on the individual. This individualization was evident in the freedom to write on any topic, to express one's emotions through writing and to interpret writing through one's experience. There could be no absolutes in English since all writing was subjective. While the content of writing was subjective, its structure was ordered by grammar and form. Steve felt that this structure and conformity was necessary to communicate clearly. Depending on the style and purpose, writing could be clever, beautiful, creative, and persuasive.

Science. By his comments, Steve portrayed science as nonjudgmental and proven. To him science was based on laws which were accepted as facts. Through the use of trial and error and the scientific method, science was able to discover new phenomenon and to clarify relationships in physics and chemistry.

Steve described science as neutral. He felt that scientific knowledge was gained through impartial means and hence was set apart from controversy. In fact, he perceived it as a unifying factor among countries since science could be discussed separate from ethnic and national considerations. If controversy was associated with science, Steve believed that it arose from interpretations individuals gave to science and from uses they made of science. Science itself, however, was always objective. Steve represented this position in the following comments.

*not right or wrong*. Right or wrong is basically a conclusion. Science is sort of a neutral. Your conclusion according to what you've found could be right or wrong but science itself isn't. Science itself is facts and just basic things that have been proven. Neutral.

*not varying:* Different people will tend to come up with the same answers. Like in genes, 3 people from different sides of the globe are looking into it and they all came out with the same answer.

*language:* Well it's a universal language. Science is universal. Like forces attract. It doesn't matter where you are or what you speak.

*integrating:* It's one of the few areas where a whole bunch of people from different cultures get together, work together without really arguing over the fact that they are from different cultures.

*controversial:* What people do with science has always been controversial.

*opinionated:* Interpreting science is opinionated but science isn't itself.

*not capricious:* The way you think of it isn't going to change it all.

Science maintained neutrality by objectivity and by the validity of its laws. Steve believed that these laws were so indisputably proven that they were regarded as fixed and valid rules. Once established Steve thought that these laws were unchanging over time. He did not share the traditional view that scientific knowledge was transient. Instead he believed that experimentation extended rather than revised or replaced old knowledge. Steve's belief in the stability of scientific knowledge was evident from the following remarks.

*truthful:* True science cannot lie about anything.

*fixed:* There are certain fixed rules. Once it's been proven, it's fixed. Like exactly what a neutrino is, for instance.

*rigid:* There are certain things if violated, they're always there.

*rules:* There are certain rules like an object in motion tends to stay in motion, and things like that.

*current:* Yeah, it's always being added to.

*factual:* It's certain facts that we build on. Like an object in motion tends to stay in motion, things like that.

*formulas:* There's certain formulas. There are general rules like gravity. Gravity effects everything on earth.

*uncertain:* There are certain uncertain[ities]. Like in the certainty principles and things like that where we're not sure. [Like] exactly how light travels through space and things like that. It's sort of an idea, but we're really not certain.

*proven:* If something is proven, everyone [who] has ever studied or looked at it [at] all, always comes to the same conclusion for instance.

*not changing:* Cause it's always been there. It's just we're discovering new things.

Steve was unshakable in his position that scientific knowledge was stable. Even when challenged with counter examples concerning the terra and helio centered theories of the solar

system, Steve maintained that given proper equipment and careful observation that scientific theories were always right. He remarked:

That can always be the case. But if you get the right equipment and if you'll always observe things. Most things if you take the essential facts and you put them together then . . . you try to interpret them in a different ways. But if you look at them realistically, there's only certain ways that you [can interpret it].

The term, theoretical, for Steve, did not connote uncertainty but rather concepts and laws that were not immediately observable through the senses. He stated: "There's a lot of things we can't see in science and we describe in theoretical [terms]." To further explore Steve's position, he was asked how many times an experiment needed to be confirmed before it was accepted as valid. he replied:

It's up to whatever you're willing to accept as fact. If you're willing to stop [after] three experiments [and] say that's a fact, that's fine. . . . If you did the experiment like 1,000 times; the changes are greater that you'll hit that one mistake than if you did it say three times. That's probability.

Steve felt that all events were governed by specific laws of nature which science continually strove to discover. Scientists investigated these natural laws through strict adherence to the scientific method. Trial and error was only used as a tool for exploration when relatively little was known about a new area. To do research a scientist needed to have insight, experience, and multiple perspectives.

*mechanical:* Things have different mechanics to it, run in specific ways. We may [not] understand the ways, but everything runs in a specific way.

*cause and effect:* Science is basically cause and effect. Why things happen.

*trial and error:* Sometimes. Not as much any more cause we know so much about it. But before and still sometimes there's things where no one has ever really known about it so you gotta [use] trial and error to figure out what you're looking for. And then you could start using hypothesis and scientific method.

*insight:* You gotta see different ways to seeing problems.

*experiential:* What you figure out about science is due in part to how much experience you've had and seeing different areas, seeing different perspectives and understanding.

*multi-perspective:* You gotta see how different ways of seeing other things put together. Cause your way may not be necessarily the right way, but it may go together differently.

As a consequence of science's neutrality and strict adherence to the scientific method, Steve described the discipline of science as highly *structured, sequenced, organized, logical, and well-defined*. He remarked that: "[You] can't do much without structure in science" and "The study of science is organized." He felt that this structure also extended to the sequencing of

topics saying: "You can't get into quantum mechanics until you understand Newton's theories and things like that." In essence everything in science was logical, "Everything makes sense." Although not as pronounced as in mathematics, Steve believed that science was also "well-defined." He argued that:

That which is defined is well-defined by now [underline added]. Some of it. Whereas there are other areas that aren't well-defined like gravity is well actually it isn't well-defined then. Gravity, you could say gravity is well-defined cause you know it occurs but not one has really defined it, so gravity exists.

Unlike mathematics, Steve believed that science enveloped the individual's thoughts and experiences.

[Science is *individualistic* since] each person adds something of their personality to the study of science. . . . You put a little bit of your own experience into science.

[Science is *expressive*.] I suppose in the way. If you had a certain view and no one believed you, you could express yourself by proving it through science.

While permitting individual expression, Steve believed that science was neither instinctive nor humanistic. It always remained logical and divorced from emotion.

Steve's affective response to science was one of enthusiasm. He found it *exciting* since science was always *new* and fresh. It was constantly being *discovered*. Science was *easy* to study provided one had the appropriate background.

In science you discover things. . . . You hypothesize something and then try to invent something to solve something. So it's always new and interesting.

It's only easy if you have the background, if you know all the basic rules and everything for it. People think on different levels.

Steve also felt the learning of science required *memorization* of the natural laws. Interestingly he believed, as well, that: "You don't have to be clever, just knowing the facts and theorems [was sufficient]."

In summary, Steve's view of science was an idealization of its neutrality and methodology. He presented science as an infallible discipline based on proven laws. Science was above controversy and its theories were universally accepted as fact. Since all natural phenomenon were governed by the laws of nature, it was the aim of science to devise systematic and thorough investigations to discover those laws. Because of its continuous exploration, science was constantly new and hence was exciting and interesting.

History. Steve's selection of vocabulary terms and comments presented history as

uncertain, changing, and controversial. The uncertainty in history was associated with the gaps in historical knowledge. As new evidence was uncovered, historians were able to fill in the missing links and so historical knowledge could be perceived as changing. Controversy arose whenever historians speculated on the missing links or tried to interpret events.

For Steve uncertainty was associated with the missing links or the gaps in historical knowledge. The belief that all events could be understood in terms of causality explained historians attempts to fill in the gaps. History represented a *rational sequencing* of events by *cause and effect*. To fill in these gaps historians needed to be *creative*, to use *common sense*, and to test hypothesis by *trial and error*. The following sampling of terms and associated comments support Steve's view of history as rational but often uncertain.

*cause and effect*: Everything that happens in history has an effect.

*rational*: It's gotta make sense.

*known*: There are a few things that are known beyond doubt that happened.

*facts*: Things that have been proven as being true. Like something that's been proven [to have] taken place like [a] whole city says it takes place or we dig up an old settlement. And it's very evident that it takes place. [By] digging up Pompeii, you know that lava went through and you can see people were stuck there.

*abstract*: Kind of cause you gotta fill in "missing links."

*trial and error*: Could be. We could think well maybe this might have happened and you can go investigate it. It might be wrong.

*creative*: To fill in the missing links you've got to be creative.

*common sense*: To figure out [how to] fill in the missing links.

Because history was so uncertain, it was often controversial. This controversy pervaded in all historical interpretations. As a consequence of the uncertainty and controversy Steve believed that no interpretation could be labeled right or wrong. There could be no absolutes since all history was subject to interpretation. The controversial nature of history was represented in the following comments.

*not right or wrong*: It's an individual interpretation.

*not truthful*: Not necessarily. Because things can be hidden by someone that won't come out in history for many years, so it may not be the truth.

*not absolute*: It's just the interpretation according to one person. So it doesn't particularly absolute. There's nothing saying that what the one person interpreted has actually happened.

*theoretical*: There's a lot of things you are not really sure of cause we don't get any kind of set records.

*not controlled*: You can manipulate it and you can write it from different sides.

*culture*: Depends on the culture.

*universal*: Every culture writes histories.

*not organized*: You can write history any way you want to. It could be fictional history or you can write it from different points of view.

*opinionated*: An author gives his own opinion of it.

*varying*: Different people write history different ways.

*controversial*: You can write something that other people don't agree with.

Since new evidence was constantly being uncovered, Steve saw history as changing.

The following terms can be associated with this perspective.

*discovery*: You can have archeology as part of history, discover things.

*changing*: New facts come up change your view of history.

*flexible*: It can be changed, you need facts.

*not fixed*: It can be changed.

*new*: There are different ways of interpreting things.

Steve had presented a view of history as both controversial and based on facts. He resolved this dichotomy for himself, saying:

History is built on facts but you can have new facts that come out too. It's the structure that changes. So the way you think it took place, the way you think the order took place can always change, but the facts aren't going to change.

Although historical facts were *easily* memorized, Steve felt interpreting them could be *hard* because "You gotta think on different levels of maturity to interpret them" and "you got to think on several different levels to have several different view points." History could also be *simple*. He commented that:

Well sorta topically simple. Cause you put things together and once you get down to it the general idea is simple. So if you have the stuff right there, it could be simple just to line it up in the right order and write it down. But if you had some problem with it, it may not be. So, it could be difficult or it could be simple depending on what you run into or what you're looking for.

On a more personal level, Steve found history *dull* and *boring* saying:

Well mostly because it's one of those things that's interesting to read and figure what's happened. And internalize it, but to have to study it I think is real boring cause you have a whole bunch of facts and analyzing things. And I hate taking facts and analyzing them like doing something with them.

He argued that while science also required analysis of facts it was dynamic and changing "Whereas [in history] you just got a whole bunch of the same facts and try to interpret them in a different way." Steve regarded history as important to study so as not to repeat the errors of the past.

In summary, Steve viewed history as evolving facts surrounded by gaps or missing pieces. Historians attempted to complete this knowledge but often evidence was not available and so they developed theories to explain the causality of events. Controversy arose from those theories and from interpretations assigned to events. As a consequence, Steve believed that historical perspectives were never right or wrong. Each merely represented an individual's opinion.

Summary. Steve's vocabulary selections and comments revealed four distinct disciplines. While all four subjects were logical, they differed dramatically with regard to the certainty one could assign to statements within each discipline. They also differed with respect to the value and inclusion of individual opinion within the discipline.

For Steve mathematics represented a discipline completely apart from individual opinion and emotion. It was never subjective. Since mathematical rules and formulas were independent of human influence, they were regarded as absolute, fixed and unchangeable. Steve also believed that all issues in mathematics were resolvable as valid and invalid. Mathematics had no gray areas-no uncertainty. Pervasive in Steve's discussion was a description of mathematics as a precise and adaptable language for science. He felt that only through its applications could mathematics ever be creative. The application of mathematics required understanding of the basic concepts. Furthermore, Steve described mathematics as learned and not instinctual. Finally, Steve found mathematics personally easy and enjoyable.

Science also represented a precise and certain discipline. Steve believed that scientific laws were universally accepted as facts. Scientific statements examined apart from cultural and national interest, could be a unifying topic among nations. Unlike mathematics, science benefitted from an individual's experience and opinion. These provided insight and direction for scientific theories. Experiments were founded on the scientific method and hence were highly reliable. Steve was adamant in his faith in the results of these carefully designed experiments,



so much so, that he perceived scientific knowledge as stable over time rather than evolving. For Steve, science was dynamic and exciting. Science was clearly his favorite subject.

In contrast, English and history were disciplines based on individual expression and interpretation. Consequently statements within these disciplines were never perceived as right or wrong. They merely represented an individual's opinion. English stood apart as the sole subject which Steve believed to be instinctual. The other disciplines were all learned and only through extended experience could they become common sense within the individual.

#### Ranking Grid

Steve's ranking grid revealed strong preferences and dislikes among the mathematics topics. He enjoyed geometric constructions while intensely disliking proofs and word problems. His remarks in this section coincided with his previous comments on the vocabulary list and on his mathematics background.

Steve's attraction to geometric constructions was a result of his interest in designing houses. He remarked: "I like designing things and I think I like architecture. Architecture is a lot of geometric constructions and that kind of thing. I like drawing and designing things." He ranked geometric constructions as *interesting*, requiring *original thinking*, and *most liked*.

Proofs however were rated as *boring*, *least liked*, *hard to do*, *difficult to learn*, and *useless*. For Steve, the difficulty with proofs resided not so much with understanding the concepts but rather with their apparent uselessness. He found it hard to become enthused and motivated. He saw no purpose in proving already known facts. He remarked:

I don't like writing proofs. I figure they wouldn't accept it as true and they wouldn't have expanded on them if someone hadn't proved it before so I don't see the point of my proving it too. The only thing I could see as being valuable is being able to disprove it. If it's important enough to stick in your math book, they must have all said, pretty well certain it's true. . . . To be able to find something wrong with it, that would be the challenge.

Steve confessed that: "I can never remember all the theorems and everything. I figure out how to do it and then I don't bother memorizing exactly what it is." Steve also describes writing proofs as *routine*, *basic*, and *busy work*. For Steve, writing proofs entailed piecing together known statements or theorems. Since the theorems were given to the students, Steve felt proofs were basic and routine. He explained this position by saying:

It's all the things you've learned. It's just a matter of putting them together. So it's nothing really original about it. . . . [In] writing proofs all you basically have to do is take

all the information in you have and then put it in order so that it proves something. You just take what you've learned before and just use it, so it's not that difficult.

Steve also found word problems troublesome. He ranked them affectively as *boring* and *least liked* and cognitively as *hard to do*, *difficult to learn*, *worst at*, *arbitrary*, *confusing*, *advanced*, *challenging*, requiring *original thinking* and *flexible*. He commented on his rating by saying:

You get all kinds of strange little things you have to figure out which are kind of a pain in the neck. . . . You gotta be able to think on several different levels and from several different perspectives and be able to see several different things. . . . [Word problems] would be the most original cause you got to think in so many different ways to solve the stupid things. . . . Word problems are difficult because you've got to know everything [underline added]. It's basically an equation you have to solve. Cause you're given certain information, you got to figure out where it fits and what to use and you've got to know how to use all of those things [equations, exponents, factoring] before you can use them of course. It can be kind of twisted anyway [underline added].

Steve only commented briefly on the other topics. For graphing he remarked simply: "I hate connecting dots." With regard to solving equations he said: "You have to know a lot of things to solve equations because they can involve a lot. They can involve factoring, exponents, and a number of other things." Steve's grid showed that he found absolute values and inequalities *theoretical*, *abstract*, *arbitrary*, and *useless*. Also fractions, decimals and percents were generally rated together and were ranked *easy*, *useful*, *logical*, *clear*, and *rigid*. Overall Steve felt that most mathematics topics were easy and clear. He confided:

Most of them except for word problems I can do rather easily if I applied myself to it. Most of them I'm sick and tired of [laugh]. . . . Math the way I figure is a tool. So I just figure I had to use it and I use it all the time. I think of it as clear with a few exceptions.

Summarizing, Steve's ranking grid and comments indicated his strong interest in geometric constructions to design blueprints. Fractions, decimals, and percents were seen as easy, clear, and useful topics. However, Steve was unsympathetic to proofs. He found them pointless and routine. He argued that re-proving statements already known to be true was a futile gesture. Likewise he perceived word problems as difficult to learn and to do describing them as involving both the complexity of solving equations and the trickiness of interpreting the verbal statements. Absolute values and inequalities ranked as difficult due to their abstract and theoretical nature. With the exception of the proofs, word problems, and absolute values and inequalities, Steve felt that most mathematics was easy. Underscoring his markings was his belief that mathematics was a tool which he constantly used.

### Synopsis of General Beliefs about Mathematics

For Steve, mathematics was foremost a tool being utilized in physics and chemistry to describe and discover theoretical concepts which could only be envisioned through equations. Steve further remarked that mathematics was creative only when it was applied. The tools of mathematics were its rules and procedures which Steve felt were both fixed and flexible. While the execution of rules was prescribed, they could be adapted and applied to new situations. Since mathematics was based on proven rules, it was never subjective. In fact, Steve believed that all mathematical statements were resolvable as valid or invalid.

For Steve, mathematics was not instinctual. Rather, it was learned. He depicted his own mathematical learning as focusing on an overview of the concepts rather than the minute details. He said: " I learn how to do it and I store [it] up [in] my brain and later I just pull it out and use it whenever I need to." While stressing rules and procedures, Steve also acknowledged that the application of mathematics required understanding of the concepts as well as original thinking.

With the exception of proofs and word problems, Steve found most mathematics easy. He saw no point in proving statements already known to be true. Word problems were difficult because they could entail the interpretation of tricky statements as well as the solution of complex equations. For Steve, the enjoyment and value of mathematics resided in its applications to his more preferred areas of interest. Math was a tool Steve used in his design of houses and his exploration of science.

### Student/Teacher Roles

#### A Good Mathematics Student

When asked to describe someone who was good at mathematics, Steve used himself as an example. He believed that mathematical ability was manifest in the facility to look at a problem and instantly visualize the solution.

I don't know in my math class specifically, like people that I consider being good at math. I consider myself good at math because I can understand. I got a handle on it and everything. Kids that are really good in math are considered the ones who can look at a problem and basically visualize it and give an answer almost instantly [underline added]. We got one kid like that, that can get the right answer almost immediately. And then there's the one that was on the news, TV and the radio. A kid who was like top in the nation simply because he was able to visual[ize] the math problem, conceptualize the math problems instantly. Give out answers to math problems almost the instant after they were asked.

Steve attributed mathematics ability to a combination of concentration, practice, and acclimation toward mathematics.

I guess most of it is concentration [underline added]. You've got to have a good bit of concentration to do that kind of thing. And also practice [underline added]. There are certain types who could never be that good. Some people are more geared toward math [underline added]. What they've done is always been thinking things out in their brain, in their mind. They're used to thinking things through and thinking things through instantly. I mean it's just how you go about thinking [underline added].

Steve was unsure whether all individuals could become good at mathematics. He felt that if they were to try, they must begin at an early age to develop their minds through practice and will power.

Well I'm not so sure at this point. If you've always been encouraged to do things in your mind and you've always practicing things [underline added]. Like your parents give you lots of math when you're a kid by the time you get into high school you're going to be pretty good in math. But somebody who has never applied himself, can't all of a sudden say well I'm going to do good at math. Cause it's gonna take a good bit of willpower to actually succeed at like math [underline added].

When asked if his parents had helped him to nurture his ability with mathematics, Steve replied that he had developed his ability on his own. In his remarks he stressed his faith in his ability to conquer difficulties.

Actually no. Most of what I do now is just stuff I just developed myself. I mean I wasn't very good at math at all. But the thing is I had no reason not to be so I just sorta worked on it. . . . Just a matter of practice. I wasn't about to give up on it [underline added]. You know, like I said, just basically I know I can do this so [underline added].

When queried for specifics on how he developed his thinking, Steve responded with examples from his childhood. He described learning to entertain himself by investigating how things worked.

It's kinds strange. I've always been sorta a loner. [As] I grew up I had no brother to talk to [so] my personally was set. So I think on my own. And when I was a kid my parents were both getting on [and] so they never spent a lot of time entertaining me. We didn't have a T.V. or anything so when I was growing up I learned to entertain myself [underline added]. As a result I wanted to . . . assimilate things rather quickly and understand things. Simply because that's a lot of what I had to do when I was a kid. I'd wander around the property, figuring how things work and how they coincided [underline added]. And things like that and then also creativity. I have had problems throughout the past but just being able to work through a problem from different angles and see how to get it done. And that particular facet of creativity has take a good but. . . . Because I can simulate things cause I'm open to new ideas. It's just a matter of I say to myself well I know I can do this so after a while I can sorta [underline added]. Once I get the basics down then it just sort of follows.

Steve then returned to the original discussion about the characteristics of someone good at mathematics. He repeated his belief that to be good at mathematics required concentration.

He elaborated somewhat saying this concentration entailed the ability to focus on minute detail. He confided that he lacked this depth of concentration. He preferred instead to develop a basic overview.

The biggest problem is frequently if something doesn't stimulate me mentally, I don't particularly pay much attention to it. I absorb it and I just leave it there. So it's probably one of the reasons why I have so many problems and careless errors in math. I think I could have done a little bit further with math. If I paid more attention to it, because there was like a test in seventh grade, you got into higher math and regular math starting out. Probably the only reason why I didn't make it with the test was because of too many careless errors. That's one thing people who are good in math they generally think about concentration. Able to concentrate on a small tedious detail, which I've never done. I absorb the basic concept and understand how it works and then that's about it [underline added].

#### Conceptual/Procedural View of Mathematics

The analysis of Steve as holding a conceptual or procedural belief about mathematics was derived from two sources: the general mathematics overview and the coding of Steve's responses from the transcriptions of his solutions to the mathematics problems in the interview protocols. Of the 12 problem episodes that were codeable for Steve, 9 were labeled conceptual and 3 procedural. Steve's ranking of 75% conceptual ordered him fourth among the six participants in this category. (See Appendix I for a table summarizing the participants' coding on the conceptual/procedural and autonomous/nonautonomous categories.)

#### Conceptual

Steve's conceptual coding on the problem protocols was based on his ability to justify algorithms in terms of first principles, to generalize results, to use number sense to check the reasonableness of a result and to evaluate the sample test on the basis of process. On problem #26 (6 divided by  $\frac{3}{8}$ ) Steve demonstrated his ability to justify division by fractions on the basis of first principles. Steve began the problem by mentally inverting the fraction and then multiplying. When asked why this procedure worked, Steve initially responded that it was something he learned. He continued to reflect on the procedure and then offered a rationale based on the definition of division and multiplicative property of 1. He argued that 6 divided by  $\frac{3}{8}$  was equivalent to  $6/(\frac{3}{8})$ . Since the fraction in the denominator was inappropriate, he suggested multiplication by 1 in the form of  $(\frac{8}{3})/(\frac{8}{3})$ . This choice permitted the denominators to be cancelled, thus yielding the original algorithmic expression. Steve was less clear in his argument when responding to the query about the reasonableness of his result. Specifically, he

was asked why a value 16 was reasonable since division usually yielded smaller rather than larger results. To this question, he replied that he was now multiplying by  $2\frac{2}{3}$  and so with multiplication he should get a larger value.

Let's see, 6 divided by  $\frac{3}{8}$ . Divide and multiply by reciprocal of 6 times  $\frac{8}{3}$ . . . . 3 goes into 6 twice. It would be 2 times 8 is 16. Actually I'm not really sure [why you multiply by the reciprocal]. It's something I learned [underline added]. [pause] Oh wait. I suppose one reason would be 6 divided by  $\frac{3}{8}$ , then cause you can't have a fraction as your denominator. So you multiply to cancel out that. You'd have to multiply it by  $\frac{8}{3}$  up here too. You can only multiply by 1 so it would be  $\frac{8}{3}$ . So those cancel out and you end up multiplying by the inverse. So I guess that's the right reason for it. . . . But to divide you have to multiply if you're doing fractions. So in this case . . . by taking the inverse you have 2 and  $\frac{2}{3}$ . So you multiply those two and you get your larger number if you multiply. . . . I don't know, I can't really explain that one.

On problem #23 (find the fraction greater than  $\frac{3}{4}$ ), Steve presented evidence of both a conceptual understanding and a strict adherence to an algorithm. He began by asserting the need to find pairwise common denominators. He explained that common denominators would make the fractions have like units and as such they could then be compared. The remainder of his dialogue focused on a systematic calculation of the equivalent fractions. Steve made no attempt to use any numerical reasoning like approximations. When asked for an alternative approach he suggested drawing very accurate pie diagrams although he felt that would be difficult.

Well, let's see. In order to find whether sometime's more than  $\frac{3}{4}$ , you'd have to, well the basic thing is you're comparing. First you're comparing a  $\frac{1}{4}$  unit and you're comparing  $\frac{1}{71}$ ,  $\frac{1}{20}$ ,  $\frac{1}{101}$ ,  $\frac{1}{24}$  and  $\frac{1}{20}$ . And all of those [denominators] are bigger than  $\frac{1}{4}$ . What that indicates that for the same 360 degrees you've got 101 parts for a whole instead of 4 parts for a whole. So any of these would be, so  $\frac{1}{71}$  for instance would be smaller than  $\frac{3}{4}$ . But in order to find out if any of them is more than  $\frac{3}{4}$  you'd have to figure out whether the total number of these smaller units is enough to be more than 3 of the 4 units. So. [pause] So basically what you've got to do is, in order to find if that was true, you'd have to put them, you'd have to change the fraction so that they were both the same number of units for each of them. So like  $\frac{3}{4}$  and  $\frac{13}{20}$ , you want to know whether that  $\frac{13}{20}$  is more than  $\frac{3}{4}$ . So you say well, so  $\frac{3}{4}$  would be let's see  $\frac{15}{20}$  because for every  $\frac{1}{4}$  you've got a certain number that equals to 20. And so this number of 20, and so you just change the  $\frac{1}{4}$  to the number of 20s it can be broken down into. So in that case, it would be  $\frac{15}{20}$ s, so that would still be more than  $\frac{13}{20}$ s. So then you'd have to do that for each one of them to figure out whether one of them is more than  $\frac{3}{4}$ . you'd have to put them into the same number. Yeah, I think there's  $\frac{15}{20}$  right here. So that would be equal to  $\frac{3}{4}$ . And you'd have to do that for each one, put it into the same unit. [continues with the remaining computation]. . . . Not that I'm aware of. Unless you drew it out in a pie graph and compared the number, compared the amount of a pie graph that was shaded in. But you'd have to be very precise to do that.

Steve's initial argument indicated a conceptual understanding of the association between common denominators and the associated pie diagrams. However his reliance on the tedious

computation of the equivalent fractions based on pairwise common denominators and his inability to suggest other approaches suggested a procedural dependency.

On problem #33 (greater of  $p + 2$  and  $2 - p$ ) Steve was able to partially generalize the conditions under which two expressions would be equal or unequal. He began by substituting the values 0, 1, 2 and -2 and then from the results he voluntarily summarized his observations.

OK,  $p + 2$  and  $2 - p$ . Well I could just take a number, so that would be 4. Well let's start at basics again. Column A or Column B. So probably you could take as an example, you could take 0. Zero plus 2 equals 2 and 2 minus 0 would equal 2. So in that case, OK. P equals 0, so they're equal. So you could take 1. One plus 2 and 2 minus 1, so it would be 3 and 1. So they would be equal if it was 0, anything other than 0, then it would not be equal. Then let's see. Two plus 2, 2 minus 2, so it would be 4 and 0. A is greater. Two minus 2, 2 plus 2, so it would be 0 and 4. OK, so if it's negative, B's greater. If it's positive it [A] is greater. And if it's 0, they're equal.

On problem #26 (20% of 85) Steve was able to weakly justify his answer on the basis of number sense. With minimal written computation, Steve quickly set up the expression  $1/5 \times 85$ , which he cancelled to yield 17. When asked if this result was reasonable, Steve replied that 20% of 85 meant a portion of 85. When asked more directly if 40 would have been a reasonable answer, Steve remarked that 40 would be almost half of 85 and 20% was a considerably smaller percentage.

OK, 20% of 85. So 20% of 85 would be 20 over 100 that would be  $1/5$  times 85. And divide that and you get 17. So it would be yeah, 17. Because 20% is a part of 85. So that's a part of 85. No, 40 would be almost one half of it, and 20% is a good deal less than half.

Steve's grading of the sample test protocol was coded conceptual on the basis of his consistency in grading, his acknowledgement of the distinction between process and conclusion, and his rationalizations of procedures as he evaluated problems. Steve evaluated questions Ia, Ib, Ic and Id as completely wrong, six points off. He justified this grading in each case by specifying the error and explaining the correct procedure. While grading, Steve noted the similarity of errors in problems Ic and Id saying:

That one [Ic] would be completely wrong because you can't cancel things out unless you got two of them. . . . if it's two things multiplied on top and bottom. . . . Once again [on Id] that's entirely wrong, cause you can't cancel things out if you're adding, it's the same problem [underline added].

After having graded the initial four questions, Steve was asked for his answer to question Id. Noting that the answer was indeed correct, he changed his evaluation to three points off.

Well, let's see. I suppose that would be twelve. That would be  $12/12$  plus two. So  $14/12$ , which would be. So that's  $7/6$  all right. Well, maybe half cause the process is wrong but the end result is right.

This sequence was repeated on question 1c.

$X$  plus 1 times  $[x]$  minus 1 over  $x$  minus 1, so that's OK. So that, yeah, OK. Forgot to check it so. So the end answer is right, but the process is wrong.

Steve then acknowledged, "Well, I forgot to look at that."

On question 11a Steve accepted the alternate approach to the problem although commenting that:

It's an unusual way of doing things. Yeah, that would be right. There are different ways of doing things, whether it's unusual or not. Some people just take longer to do things.

On the remaining three questions, 11b, c and d, Steve concluded that each problem had the right process but the wrong conclusion and so he deducted half credit, -5 points. During the discussion of each problem, Steve verbalized his rationale for his answer. On problem 11d, the technique was once again unfamiliar to him but he argued to himself that it was equivalent to his strategy of rewriting the absolute value and inequality into two separate inequalities. Steve's comments on these problems are given below.

Let me see [reads steps on question 11b] that would be  $x$  and equals six. So  $x$  would equal any number in that case cause sixes would cancel out. So  $x$  could be any and the threes would cancel out so that [would] say  $x$  is equal to  $x$ . So  $x$  could be anything. It would still work. So let's see, 10 points. [pause] Right idea, wrong conclusion, right process, wrong conclusion. I don't know. [writes -5] I never want to be a teacher.

Let me see here. [examines question 11c]  $X$  would be undefined here, cause there's no number you could put in there unless, I don't know yeah. Cause if you subtract  $x$  from both,  $2x$  from both sides, you get, that cancels out. So there's no number you could put in there that would make those agree. So that would be undefined rather than -3, 12. So, [pause] it would be -5.

OK. So, the absolute value,  $x$  could equal. Answer is in an absolute value, so it could be  $x$  plus 3 or  $-x$  minus 3. So well, the whole process is wrong. Oh wait. No wait, that's right. . . . Yeah once again, right process wrong conclusion. . . . So it doesn't matter whether what is in the absolute value is positive and negative. So I guess it would be wrong conclusion again. OK. Half is the process and half is the conclusion.

#### Procedural

Steve's problem episodes were coded as procedural due to his exclusive reliance on algebraic equations in a problem-solving situation and his inability to use either number sense or first principles to justify an arithmetic algorithm. The protocol of problem #27 (the remainder problem) was indicative of Steve's dogmatic reliance on an algebraic solution. Immediately upon reading the problem statement, Steve wrote the expressions  $\# = x$  pos, and  $3x$



$+ 2 = 4x + 2 = 5x + 2$ . He then proceeded to solve the equation  $3x + 2 = 4x + 2$  for  $x$ , which yielded  $x = 0$ . He dismissed this answer and then tested the integer, 8. From this example he realized that the value for  $x$  in the original equivalency might not be the same value in each expression. As a result of his observation he rewrote the equivalency as  $3x + 2 = 4y + 2 = 5z + 2$ . From this point he attempted to solve the first two parts of the equivalence for  $y$  and then substitute that expression for  $y$  into the equation formed from the last two parts of the equivalence. When these calculations failed to yield a numerical solution, he returned to his original answer of 0. In his ensuing discussion, Steve confused the role of 0. Originally 0 had represented the value of  $x$  in the expression  $3x + 2$  but now he interpreted 0 as equal to the whole expression  $3x + 2$ . Once he had associated 0 with the unknown number, he argued that this solution was unsatisfactory because when 0 was divided by 3 it had no remainder. At this juncture Steve was encouraged to examine 14. He proceeded to dutifully verify that 14 worked for 3 and 4 but he pronounced it unsatisfactory for the third number, 5. When it was suggested that the number 14 refuted his search for a single value for  $x$ , he remarked that he was arguing theoretically and attempting to disprove the theory or question. He next considered and then dismissed the option of checking numbers one by one. He felt that this was a "stupid way." Although the discussion of the problem continued for some time, Steve remained attached to his expressions  $3x + x$ ,  $4y + 2$ , and  $5z + 2$ . He attempted no further explorations of numbers even when he acknowledged that his equations were leading him in circles. The citation that follows is an abbreviated outline from Steve's protocol the complete transcription being over five type written pages in length.

I was just trying to figure out how to do, defining all the terms first. . . . so you take the original number that would equal  $x$  so then  $3x$  plus 2 equals. OK. And you get that number and if you divide it by 4, that would. It's all gonna have to be equal, would be  $4x$  plus 2 equals  $5x$  plus 2. And so then what you'd have is, what you did is solve it. Take one half of it and then you'd solve this one and then solve that one and then solve the first two. And then you get three different answers. And if they were the same and it's possible that might be the answer. So let me see, in this case, 2s are cancelled out and it wouldn't work, cause 0 would equal 4. So that method didn't work. [pause] Ah, gee. . . . so like let's see, try 8 for instance. . . . Oh, gee. [pause] I guess the key is to figure out what the relationship with the remainder is. [Rewrites equation as  $3x + 2 = 4y + 2 = 5z + 2$ ]. . . . The other question is, since it doesn't seem to work out right is whether it's possible to have a number that would be divided by each of those and add 2. . . . I guess the only way you could do that, if you considered 0 a positive number. Then it might work, because you could prove that the original number would be 0. But if you divide 0 by 3 you get 0 and there's no remainder. So it seems as though there's no answer for that. . . . So I don't think it's possible to have a remainder of 2 for all of them. . . . [suggested that he examine 14]. Fourteen, let's see. OK. Three goes into 14, 3

goes into 14 what? Four times, I get 12, you get 2. Remainder 2. Four goes into it 3 times and a remainder 2. And 5 goes into 14 twice, so you get 10, you get a remainder of 4. So that would be 2, remainder 4 and 4 is not a remainder of 2 so. . . . Well since this it's entirely theoretical. See I was going at it from a theoretical point of view, theories can be disproven [underline added]. It seemed to me that you couldn't do that. If I really wanted to prove that you could or couldn't, I could start just grabbing numbers starting with 0 and working my way up and just checking all of them. . . . It was a stupid way of trying [to find] an actual number [underline added]. . . . In order to solve it, you have to have a different equation for each variable. You're just gonna keep on going in circles. So I'm not sure. . . . So it's more complex and all that. . . . Trying to find some other way of representing it. . . . I can't find an answer to that.

Steve's primary strategy in this problem was to set up and solve three simultaneous equations. He experimented with only three trial numbers in the course of his solution. Thus Steve's energies were directed solely at finding an algebraic equation. On the basis of his exclusive reliance on a procedural technique, Steve was coded procedural on this problem.

Steve's discussion on problem #21 ( $1.50 \times .25$ ) focused on the execution of the decimal multiplication algorithm. When asked why totaling the number of decimal positions was appropriate, Steve replied: "The number of decimal places always conserves, you know. They don't appear or disappear." When pressed for further clarification Steve remarked:

I'm not sure, it's something you memorize. That somewhere someone in the past, someone found some way of proving that. And we were told it was proven, so that's the way you do it. I'm not exactly sure. I know it works out, and if I took long enough, then I would be able to find a way of proving it.

He continued his explanation by substituting  $1/4$  for  $.25$  and calculating  $1.50 \times (1/4)$ . As he finished, he remarked that this computation yielded a smaller number.

Well  $.25$  is part of a fraction of something, it's  $1/4$ . So multiplying by a fraction is, let me see. So if you got 150 times  $1/4$ , 25 over 100,  $1/4$ , then it would be, that's the same as 150 divided by four. So you're gonna get a smaller number. And so a smaller number than 150.

In this problem Steve did not rationalize his result but rather recomputed the answer using another algorithm. His reliance on the algorithm and his relegation of the validity of the process to others suggested a procedural coding.

### Summary

Steve's problem protocols were coded both conceptual and procedural. Steve's conceptual coding was based on his ability to rationalize and generalize results on basic arithmetic and algebraic problems, and on his consistent and conceptual grading of the sample test. His procedural coding resulted from his exclusive reliance on equations and his inability to offer alternate solution techniques and justification for arithmetic algorithms. The coding on

Steve's protocols did not suggest a clear separation by problem type. He was coded both conceptual and procedural on the arithmetic algorithms and on the problem-solving protocols. What does emerge is his orientation towards the problems more generally. Steve seemed to rely almost exclusively on the execution of rules and procedures.

On problem #26 (6 divided by  $\frac{3}{8}$ ), Steve justified the division algorithm by showing the manipulation of fractions that yielded the statement equivalent to the algorithm. While his justification was valid it was procedural in nature. Like wise on problem #23 (find the fractions greater than  $\frac{3}{4}$ ), Steve's protocol focused on the step-by-step calculation of pairwise common denominators. In neither problem did Steve utilize number sense to justify his results or the algorithm. On the problem-solving protocols Steve relied almost exclusively on the execution of an equation to solve the problem and justify his results. This reliance on rules was also evident on the arithmetic algorithm problems on which he could offer no rationale or alternate solution technique. The emphasis in both the conceptual and procedural problems was on symbol manipulation, application of rules, and solving equations. Steve used these strategies to the exclusion of number sense.

This pronounced dependency on rules agrees with Steve's comments in the belief interviews. He repeatedly stressed that mathematics was a tool. The tools--the rules and procedures--had a prescribed order and required exact execution. Steve also believed that these rules were proven facts and as such need not be verified again. He expected the rules to be interrelated but he was not necessarily concerned about the rationale behind them. He wanted merely to learn what they said, store the information in his memory, and then retrieve it when he needed to apply it. This perspective seemed to be operating in Steve's problem protocols. Since rules were proven facts, he looked for some rule or procedure to solve his problems or justify his results.

#### Autonomy/Nonautonomy with Mathematics

On the codeable problems, Steve received a 75% autonomous rating. With this rating, he ranked fourth among the six participants.

#### Autonomy

Steve demonstrated his autonomy with mathematics by monitoring his progress, voluntarily checking his answers, and voluntarily summarizing his results. On problem #35 (rope

problem), Steve was coded autonomous because of his self monitoring during his solution process. He began by calculating the circumference on the earth. Proceeding to the second question, Steve hesitated uncertain whether to convert to feet or miles. He initially set up the expression  $1/(6 \text{ ft.}) \times (5,280 \text{ mi.})/(1 \text{ ft.})$  but he immediately recognized that position-wise the units were incompatible. He continued to reflect on this relationship until he resolved the conversion factors. Once converted, Steve used his diagram to motivate his equation which he solved for the diameter. Forgetting that the original question needed the radius, Steve answered 1.6 feet. In the course of explaining his result, Steve corrected himself and answered 0.8 feet. That he did not arrive at the exact solution of 1 foot was due to his use of approximations in his calculations. Steve's monitoring is seen in the abbreviated protocol which follows.

Just to be exact. You could do it logically. I suppose just by imagining it. But to do it correctly I guess you'd have to get the earth and you get the radius which equals 4,000 miles. . . . Circumference equals pi times 8,000. Yeah, miles. OK. So you add 6 feet, the length of rope. I guess you've gotta convert, 6, ah, shoot. I hate converting feet to miles. . . . You've got 6 feet and there's 5,280 feet in a mile. [pause] Well, let's see. [pause] Let's just do it like chemistry unit multiplication then. So we want miles so we have to have, so miles would have to be on top. So 5,280 miles per 1 foot times 1 over 6 feet. So no [underline added]. All right, 6 feet and then feet would cancel out so that would be, OK. Wait a minute, it doesn't make any sense though [underline added]. [laugh] [pause] Six feet the length of the rope. Six feet as a fraction. Argh! [pause] Oh, boy. [pause] What do I want to know [underline added]? How many miles is 6 feet? Brain is slow today. So that would be [pause] there's the problem, OK. One mile equals over, that's the same as 1, OK, now that makes sense [underline added]. So yeah the feet would cancel out in the end. OK. . . . No wait. . . . OK, so, so, whoops. No wait [underline added]. It's 2 pi over 5,280 so that would be pi over 2,640. So these would cancel out. OK. So you get, so it's 8,000pi plus pi over 2,640. . . . Circumference equals pi times diameter. So circumference equals whatever. So that. Add that and then divide both side by pi, cancels out and so that equals 2,640d. And so you have, so wait a minute, no, equals and then divide by 2,640 equals 8,000.0003 equals d. No, yeah. So that's miles. So how many feet, OK [underline added]. The difference is .0003. So put it back into feet again. . . . So how far above the ground would the rope be? Ah, just a little over 1. About 1.6 feet. So a piece of paper fit between the ground and the rope, yeah, sure. Mouse crawl through, yeah. Small person walk under it, no. I don't think so. OK.

So it would be, well, actually, it would be .8 feet because what you're saying is, OK [underline added]. . . . So you'd have .8 feet on each side which is the amount of difference. . . . It's not gonna make very much difference [underline added]. . . . I feel fairly sure about 8 [on a scale of 10]. You've got to be considerate of what 6 feet is, it's not too inconceivable for it to be like 1, if you spread it out all over the globe. Yeah.

Throughout the protocol, Steve asked himself questions to check the reasonableness of his results and to focus his thinking. By evaluating his own work and correcting his own mistakes, Steve demonstrated his autonomy with mathematics.

Steve also voluntarily checked his answer on problem #25 (Sandy's age). Upon solving

his equation, Steve took his answer of 3 and verified that it satisfied the verbal description of the problem.

Steve also demonstrated his autonomy by voluntarily summarizing his results on problems #33 (larger of  $p + 2$  and  $2 - p$ ) and #34 (larger of  $r$  and  $r^2$  for  $r > 0$ ). The protocol for problem #33 was discussed and cited under the conceptual coding. On problem #34, Steve began by claiming that  $r^2$  would always be greater than or equal to  $r$ . When it was suggested that he consider  $1/2$ , Steve modified his position and finally concluded for himself that:

I guess if it's  $[r]$  any number which is between 0 and 1 would be B  $[r]$ , it would be greater. And if it's 1, they're equal and if it's greater than 1, then A  $[r^2]$  would be greater.

#### Nonautonomy

Steve's nonautonomous coding was due to his reliance on authority to justify algorithms and relegation of the validity of a result to the execution of an equation. On problem #21 (1.50 x .25) Steve argued the validity of the decimal multiplication algorithm by remarking:

I'm not sure, it's something you memorize. That somewhere someone in the past, someone found some way of proving that. And we were told it was proven, so that's the way you do it.

In this instance, Steve relegated the validity of his answer to execution of a memorized algorithm.

One problem #30 (find the area of the triangle) Steve also relegated the validity of his answer. In this situation, Steve's trust in his answer was based solely on the execution of his equation. He initially tried to find a geometric theorem that would connect the problem information. Failing this, he set up and solved the equation  $1/2bh = 8$  which yielded  $bh = 16$ . While acknowledging this as an answer, Steve still wished for a geometric solution which he felt would be more convincing. When asked on a scale of 1 to 10 how certain he was of his answer Steve replied:

Let's say an 8. Cause there is a margin possible, maybe so. I don't know, it seems like it's probably certain cause you get, because there's not much that changes. This one you get the same height all right. You've got twice the base. So I don't know, it seems fairly certain, anyways. Cause you limited the amount of factors you have. It's like the more variables you're dealing with, the more uncertain it is. I suppose one way you could check it is if you did  $h$  equals  $16$  over  $b$ . That will be one half [times] 2 times  $b$ , 2 times  $16$  over  $b$ . Those cancel out.  $B$ 's cancel out and you get  $16$  again, which is, so I don't know, it's fairly certain.

Steve's argument for his answer did not detail his mathematical logic. Rather, he made a vague reference to having a limited amount of variables. This reference suggested that not only was

he uncertain and confused about his result but that his confidence in his answer resided implicitly with the execution of his equation. Unlike Tom or Sue, Steve did not intuitively verify his algebraic answer by substituting trial values into the equation for  $b$  and  $h$ . His secondary verification consisted of solving the equation  $b \cdot h = 16$  for  $h$  and substituting that result in the equation  $\frac{1}{2} \times 2 \times b \times h = 16$ . This circular argument was his only attempt to test the reasonableness of his answer. Steve's faith in his answer rested in the execution of an equation and, in that sense, he was relegating the verification of his answer to an outside authority. He looked to the procedure without any apparent internal justification.

#### Summary

Steve's coding as autonomous and nonautonomous concurred with his comments in the beliefs interviews. Two themes emerged from Steve's discussion of mathematics and mathematics learning: first Steve felt that mathematics was a tool and second he expected to be responsible for his own thinking and learning. As a tool mathematics represented a collection of valid procedures or rules which an individual applied or adapted when solving problems. These procedures were both rational and sequenced. While Steve believed that the rationale was understandable "if you once get into it," he also admitted that he preferred to learn the overview of mathematics rather than the details.

Steve spoke as well about his efforts to train his mind to accept mathematics. He described using his willpower to concentrate on mathematics. Steve's discussion of a good mathematics student also indicated he believed that mathematical ability required independent thinking and visualization of problems in one's mind. Thus by these remarks Steve suggested that he expected to solve mathematics problems by correctly applying rules and accurately solving equations, and that he expected to carry out these applications independently.

Steve's monitoring of his progress in his solutions of the problem solving protocols focused on finding and solving the correct equation. In addition, Steve tested the validity of his answers by rechecking the solution of the equations. Steve's reliance on the validity of an equation and its solution agreed with his comments about the validity of mathematics. He had remarked that mathematics was truthful because "an equation doesn't lie." Steve also accepted mathematical rules as valid truths because they were proven by some earlier mathematicians much as scientific theories were proven, accepted, and then extended. Since scientists did not

prove every theory or formula before using them, Steve felt that it was unnecessary in mathematics as well. Thus it seemed for Steve that once an equation was found that he trusted tacitly in its solution. He felt no further need to validate his result since mathematics was a collection of proven rules he had correctly executed.

Steve expected to be responsible for his actions and consequently monitoring would focus on the correct execution of the rules or solution of an equation. Thus Steve's autonomy was tied both to independent thought and to the correct execution of valid rules.

As with the other participants, Steve's autonomy with mathematics seemed interconnected with his conceptual and procedural view of mathematics. Underlying Steve's view was a perception of mathematics as a tool--a collection of valid rules and procedures. Steve's strong procedural orientation, however, was augmented by his autonomy with mathematics. For Steve the validity of mathematics was primarily built on the correct application of rules. As a result he used his autonomy, self monitoring, to check the correct execution of procedures. He also used algebraic manipulation rather than number sense in justifying basic arithmetic algorithms. Like Sue, Steve's procedural view of mathematics was mediated by his independence of thought. Yet unlike Sue, Steve did not often appeal to number sense to supplement his rationale for answers or procedures. He relied exclusively on the execution of rules.

## CHAPTER VI

### CLASSROOM ENVIRONMENT

#### Teacher

##### Background

Mrs. Thomas was the teacher in both the Algebra II (A2) and the Algebra II/Trigonometry (A2T) classes. She was an experienced high school and junior high teacher with 12 years of full-time teaching and an additional 8 years of part-time and substitute teaching. With the exception of Calculus, she had taught the full range of courses including: junior high mathematics, General Mathematics, Algebra I and II, Algebra II/Trigonometry, Geometry, Accelerated Geometry and Precalculus. She was professionally active, having directed and given workshops at local NCTM meetings. At the local high school, Mrs. Thomas was also involved in several extra curricular functions, such as, junior class advisor and coach for the mathematics team.

##### Beliefs about Mathematics

Mrs. Thomas' beliefs about mathematics were inferred from her comments as she marked the vocabulary list for mathematics, the ranking grid, and the sample test. In addition she discussed the characteristics of a good mathematics student and a good mathematics teacher. The results from these questions and instruments are summarized in the following sections.

##### Vocabulary List

Mrs. Thomas circled the following terms as applying to mathematics: *abstract, analyze, common sense, deductive reasoning, discovery, formulas, logical, sequenced, shortcuts, thought provoking* and *useful*. She commented on these selections by saying:

abstract: Because I think that mathematics helped develop abstract thinking skills. Often when you deal with mathematics you have to think abstractly.

analyze: Because when you look at a problem you have to look at what it's asking. You have to analyze what it is asking.

common sense: I think that most of the time when you're solving math problems, you have to use your common sense especially if there's [an] answer [that doesn't] makes



sense, doesn't work. More often people try to sub in (or) find a rule instead of saying "what does your common sense tell you, you should be doing."

deductive reasoning: A lot of mathematics, I don't know, you just use deductive reasoning.

diagrams: Any time you're doing problem solving, it's helpful to use some kind of diagram.

discovery: Basically more often in geometry than in other areas, but it can work in algebraic areas as well. I think discovering theorems is important, because students understand them better if they can discover them through an activity.

formulas: Because [there are] so many scientific and mathematical formulas that kids are used to working with.

fun: I think math is fun but I didn't put it down because most people don't think about that.

logical: I think mathematics helps to develop logical thinking skills because all students have to think about what they're going to do, what they need to do, what they're looking for. In any kind of a problem solving situation [you] have to come up with a logical approach. Even in just a simple solving an equation, they have to go through a logical series of steps in order to have the equation work out properly. For instance, our numbers are ordered so it helps knowing the difference between greater and less and all of that, rational numbers.

sequenced: I think of sequenced in regard to topics. In mathematics, for instance, you have to know Algebra I in order to do Algebra II. You can't do Calculus if you haven't had any other mathematics. I suppose unless you're a genius and you just start and just sort of know it all.

shortcuts: [They] are important for students to know easier ways to do things, and to understand why they work. Instead of having to write everything out all the time. I think it's good that they can learn how to take shortcuts.

thought provoking: Any kind of a problem solving situation you have to think about it, and they have to work things out and it helps them. Maybe provoking is too strong a word. . .it stimulates thoughts.

useful: Useful in day to day life. Everyday they're going to use mathematics somewhat, even though they don't think they're ever gonna use their algebra they're doing. . . .Mathematics itself is useful and they know they need it all the time.

In a subsequent discussion, Mrs. Thomas also indicated that she did not associate mathematics with memorizing saying:

I don't really like to think about them memorizing. Obviously they have to learn the slope-intercept formula and things like that so that is memorizing, somewhat. But I think that they just [learn] because when they do their homework, they're supposed to write down the rules and then show the substitution and stuff. It sorta comes automatic. If they understand it then they are going to know it. . . .You don't want to just memorize it without understanding it. They have to understand it first. But then you have some kids that say well, "I really understand it." But then they don't remember it because they haven't really learned it. So they do have to learn it, but they have to understand it first or it doesn't make any sense.

Mrs. Thomas' choice of vocabulary terms emphasized the logical and common sense

aspect of solving mathematics problems. Throughout this discussion and in later conversations, Mrs. Thomas stressed the utility of mathematics to teach one to think and reason through a situation more than the usefulness of the mathematical concepts. These themes are reflected also in her scoring of the ranking grid which is presented in the following section.

#### Ranking Grid

Mrs. Thomas marked the grid to reflect what she felt would be the students' responses to these topics. (See Appendix K for her complete responses.) On the interesting-boring line, she did reveal that she personally did not find mathematics interesting.

Interesting is a funny word. You don't really think of math things [as] being interesting. That's my feeling. I mean it's worthwhile because it's useful, but as far as being interesting. . . . I could not put anything down for some of these [topics] cause I don't think some of them are interesting at all.

On the grid Mrs. Thomas indicated that she found geometric constructions, graphing and word problems somewhat interesting. The remainder of the topics she felt uncomfortable ranking at all. Mrs. Thomas observed that students found both geometric constructions and solving equations easy to learn. She remarked that constructions were especially appealing to the lower level students. Solving equations were easy and clear because "Once they understand the procedure and the reason for it, they find it very easy."

In contrast, students found writing proofs and absolute values difficult to learn and do. Mrs. Thomas felt that proofs were hard for students to learn because they required a logical sequence, the results often seemed apparent, and students saw no relevance to their lives. She remarked:

Kids hate doing proofs. The logical sequencing of things. knowing why they need them. I guess it's more because they really feel "why do I really need to do this?". What good is it to prove something that someone could tell us is true and we believe. And if they can show us why it works, why do we need to prove it. They have such a negative attitude about why do we need to prove it. . . . Because they don't like proofs, it makes it harder for them to learn.

Mrs. Thomas explained that she tried to mediate some of the students' negative attitude towards proofs by discussing with them the importance of planning and reasoning in their lives. She commented that:

Part of it's easy, because the kids want to learn it because they want to do well. . . . The other thing we talk about is how all through their lives they are going to be working with problem solving situations. When they have to figure out a solution to something to do with scheduling or whatever in regard to their time or their homework. . . . They have to look at what the given things they have to work around and they have to come up with

a solution on how to fit these things together. . . .They're lots of ways that they have learned to plan either their time or their money or whatever. Geometry proofs help them learn how to develop logical thinking skills. And they accept that. . . .I basically say that to them. I say this to the parents at open house, that they're never gonna be asked to prove a geometry theorem again. And the usefulness is not that they can prove a geometry theorem. The usefulness of it is everyday they're gonna be faced or frequently in their lives, you're gonna be faced with problem solving situations and that you have to come up with a logical sequence of reasoning. You have to reason through things, and this is what the purpose of proofs is. Is to help people learn how to reason through things and to think through things logically.

Absolute values were difficult because students did not understand the meaning of absolute value as a distance and because they did not distinguish between "and" and "or" in the solution process. Mrs. Thomas felt that often students just learned the procedure without really understanding it.

Because they always get mixed up. They really can't understand the whole ideas of absolute value—meaning the distance like for instance. . . .They don't like it because of the fact that they have to remember all the little rules about when you have an expression. The and's and the or's, they always get [them] mixed up. . . .And they don't really understand it.

Mrs. Thomas also indicated that in general students can not apply mathematics to their world and that many will never use the mathematics they are studying.

I don't think the kids can apply it to their world. . . .Unless they go into a field that involves math, they're not really gonna need [it]. Unless they are going to go beyond high school in mathematics, they're probably not going to use a lot of this again.

Mrs. Thomas indicated that in her everyday life she used only percents, decimals, fractions, equations, word problems and geometric constructions.

Again as in the vocabulary list, Mrs. Thomas emphasized the utility of mathematics to develop logical thinking which she felt had immediate applications to everyday. Apart from its logic, Mrs. Thomas felt that many of the topics were not needed by the students unless they elected a field of study in college that required mathematics. She believed it was this apparent irrelevance that made wanting to learn proofs in geometry so difficult. She also personally believed that much of mathematics was not interesting only useful.

#### Sample Test

Mrs. Thomas' comments while grading the sample test, focused on her disagreement with the assignment of point values, the composition of the test, and the lack of explicit instructions. She did not agree with having 6 and 10 point problems when she would grade the solution or answer all right or all wrong as in the case of la, b, c, d, and e. She felt that this grading policy

would be unfair to the students. She also felt strongly that the composition of the test was not reflective of the type of problems students would typically encounter in homework and so again was unfair. She cited the two equations in 11a and 11b as unrepresentative problems. Mrs. Thomas indicated that she often suggested clues for the problem solutions. For example in Section I, rather than say simplify she would have written "factor the expressions."

She also commented that when grading she typically looks first at the answer and if correct, scanned the problems to see if it is logically set up while if it is wrong she looked for the error. If the error occurred in the initial steps, she usually marked the problem completely wrong. Since she lacked an answer key, Mrs. Thomas began grading problem 11a by checking the initial step. Upon observing that the student failed to subtract 4 from both sides of the equation, she marked the problem as incorrect, taking off all 10 points. She observed: "I don't ever teach them to do it this way because I think they tend to make more mistakes horizontally than vertically." To clarify whether Mrs. Thomas had observed that the final solution was indeed correct, the researcher queried how she would respond to a student who argued that his or her answer was correct. She responded that:

That's 'cause they made 2 mistakes on. I didn't even read the whole thing, but they obviously made 2 mistakes in order to do that. What did they do here, wait a minute. Oh, you know I didn't notice this. OK, I wasn't looking. . . . Because what I did was, I was saying oh adding 4 and I was thinking I didn't even look down there. Because I was thinking you would have zero. . . . That's right, so I wouldn't have subtracted anything, so they would have got it totally right. But I might write there a note saying if you subtracted vertically, it tends to help you see the correct operation.

As her response indicates Mrs. Thomas was initially unaware that the solution was correct. While acknowledging the correctness of the answer, she still insisted the steps were improper. She wrote on the sample test: "If you show your steps vertically you tend to minimize sign errors."

On problems 11b, c and d, Mrs. Thomas subtracted only 1 point since the student had correctly solved the equation but failed to interpret the result properly. Rather than giving the correct answer, Mrs. Thomas wrote comments that suggested to the student that he or she reconsider the interpretation. For example on 11b, she wrote: "Is  $6=6$  a true statement? If so, try to sub in a value for  $x$  - what happens?" Mrs. Thomas' remarks while grading the sample test showed a dual perspective. One problem 11a, her remarks suggest that she examined problems on the basis of familiarity of form while her comments on 11b, c and d suggest an

emphasis on guiding the student's understanding of the answer.

#### Role of the Student and Teacher

To gain insight into Mrs. Thomas' views on the role of students and teachers in learning, she was asked to describe a good mathematics student and describe her best mathematics teacher.

Good Mathematics Student. When asked to name a good mathematics student, Mrs. Thomas chose Ann (one of the participants in the study) and Melody. She summarized their traits by saying:

She [Ann] does her homework every night. She picks it up pretty easily, so she's a bright student. She's very steady and she does consistently well. . . . Tom [participant in the study] is a very bright student, but he doesn't do his homework often enough, although he's academically very good. I don't consider him a good student, because he doesn't really follow through. He doesn't accept the responsibility of working things through.

She [Melody] picks everything up very quickly the first time. . . . Melody, not only is she bright but she also is consistent. She does her work every day and she makes sure she understands it. She asks questions when she doesn't.

They really want to learn. They really are eager to do well. And they're eager to show you what they know, as well as, explain to others who are having trouble.

To follow up on Mrs. Thomas' views, she also was asked what makes a student good at mathematics. She responded that:

[For] some it's just their natural ability. [For] some it is that they enjoy doing it. Some kids enjoy being able to figure things out and get an answer. Rather than like in English, they have to discuss something or compare somethings or analyze this line of this poem. And they really don't know whether they've done it right or not. In math, they can often check their answers either in their book or they can check that. So some kids like to know that what they're doing is right. Cause it gives them a feeling of accomplishment. . . . I think kids that work systematically, that like to follow things through do well that's why when kids can learn a systematic approach, then they do better. Kids that tend to be very random in their thinking and very random in their work habits, and try to sorta jump around from thing to thing. They have a more difficult time because they're not careful with detail.

Expanding on her remark on natural ability, Mrs. Thomas stated that she believed the students in the Algebra II class had the ability to learn mathematics since they had progressed to this level. However, having the ability did not guarantee success. She attributed failure, in that case, to a mismatch between the teacher's and the students' teaching and learning style.

The kids in the Algebra II class wouldn't be there is they didn't have the ability to understand mathematics. . . . These kids have the innate ability to be able to learn, the natural intelligence to be able to learn. And if the material is presented to them in a way that they can learn, now that doesn't mean that they're always going to learn. You can have the same kid in a different class and he won't learn as well. . . . Some students and some teachers just don't mesh. That's why within a class you have to have to do

different things in teaching so you can get everyone's learning style and you can eventually pick up everyone's learning style in every unit. . . . Basically, there are some students who just don't have any of that innate ability. Their intelligence or whatever it is, is not strong enough.

Mrs. Thomas rejected the notion that a student can do well in writing but fail to comprehend mathematics. She believed that a student with the intelligence to handle writing could be successful in mathematics as well. Likewise, failure in one discipline usually signaled inability in another area and was a sign of lesser overall intelligence.

I think Karen is a very good writer. . . . I know she's at least stronger in English and stuff like that than she is in math, but she's a really good worker. She really wants to learn and she really does pretty well. But she doesn't have the confidence in her ability to do well. . . . Usually the students that can never add fractions, can also not write. It's usually because their whole level of intelligence is lower. But I think the students who can write and have trouble with math [can] usually plug away and can be successful.

In summary, for Mrs. Thomas, a student was good at mathematics if he or she had the innate ability or intelligence, consistent study habits, and learned quickly. Failure to succeed in mathematics was indicative of low intelligence, a mismatch between the student or teacher, or a lack of effort.

Best Mathematics Teacher. Mrs. Thomas cited a college course in projective geometry as her best mathematics class. She enjoyed this seminar class because the teacher created a positive atmosphere in which the students felt comfortable asking questions and discussing their ideas. The teacher emphasized understanding the concepts. In contrast, her worst mathematics class was a college abstract algebra course in which the instructor insisted that all problems be solved his way. Mrs. Thomas described: "fighting with him after every test because he only wanted things done his way. . . . [He insisted that you] do them [matrices] in exactly the same order that he did them or it was wrong."

#### Philosophy of Teaching

##### Role of High School Curriculum

Mrs. Thomas perceived the high school curriculum as serving two roles: (a) a preparation of Calculus and (b) a means of teaching logical thinking and self discipline.

The kids that don't even go into math ever. Say they are gonna finish high school and never take another math course again. I really feel that math teaches them how to think [underline added]. Math teaches them a logical theory of problem solving so that if they have to, for instance, analyze something historical they'd know how to look at all the parts and draw a conclusion. They'd know how to look at a problem and say what do I know, what do I have to figure out, what can I conclude from that. . . . They learn strategies in mathematics. The other thing they learn is consistency [underline added]. How important

it is to be consistent with your work and to do it on a regular basis instead of letting it pile up which you can't do in math. They learn just how important it is just to break things down into parts and do a little part here and there, rather than try to cram it all in at once.

While emphasizing mathematics ability to help one think and reason, Mrs. Thomas did not believe that one needed to study applications of mathematical concepts to real life. She felt it was sufficient to know such applications existed.

I disagree because at this level the kids don't see any use for it. Ninety-five percent of the time. But what's helpful is for them to see that some people use this in other places, so that it is used. The purpose of teaching an algebraic equation is not necessarily that someday they're gonna use an algebraic equation in something. . . .They're learning a lot of skills in learning mathematics. It's not the mathematics. . . .Just the skills [logic and thinking skills] they learn in doing mathematics.

#### Changes in the Curriculum

When asked what changes she would make in the current high school curriculum, Mrs. Thomas remarked: "I don't think I'd change the curriculum at all." She was concerned that any additional content would necessitate dropping topics from the current program which she felt were needed to prepare students for calculus. Thus, although she felt probability and statistics would be useful for students, she was unwilling to give up any of the present program to make room for it.

Mrs. Thomas did suggest ability grouping in the mathematics courses. Again she had reservations about her idea. First, she felt that it would be impractical in a small school like the one in which she was currently teaching. Second, she was concerned that the students in the lower ability groups would develop a false sense of their understanding of mathematics. That is, they would misinterpret their good grades in a lower ability class as equivalent to a good grade in the harder and more sophisticated higher ability class.

I suppose if we had a larger school, it probably would be nice to somehow have a little more of a breakdown for the weaker students. But on the other hand the problem with that is that then they get a false concept. . . .of what they know.

Mrs. Thomas also felt it was important to deemphasize hand computation and utilize calculators more.

While she wished to encourage calculator usage, Mrs. Thomas was unconvinced about the use of computer software and programs to clearly and effectively convey mathematical concepts. She recounted her attempts to use computers to assist in teaching graphing. She found that students failed to remember the concepts the following year and that they tended

to execute computer commands rote without understanding the underlying mathematical concepts. Mrs. Thomas also expressed her frustration with the lack of available programs.

I've made [programmed] a jeopardy game and some fairly involved ones to use, mainly games that would teach them and help them learn things, but they were games type situations. . . I've tried to use the computer in the classroom and it doesn't seem to work. When you get to the calculus level or trig. functions and you have a program that will put the function up on the board when you're plugging in, then the kids can see how it changes when you change the amplitude or change the period or something like that. But I tried to do it last year with conics so the kids could see how things were changing. There weren't any programs that I could find that had it. . . And the other experience I've had is, I've had some kids who have learned things by the computer that you have the following year and they really don't remember anything. All they remember is punching the buttons. . . They never really understand the rule. . . All that does is help them be able to visualize. I don't think it could ever take the place of having them work it out by hand, cause otherwise they don't understand it.

#### Strengths and Weaknesses

In describing her strengths as a teacher, Mrs. Thomas cited her slow and thorough explanations and her use of multiple examples. She also believed that students felt comfortable asking questions in her classrooms. While concerned that her classroom routine might be somewhat boring, she rejected the use of projects as ineffectual. Besides, she felt that students like the consistency and learned better in the structured environment.

My strength is [that] I usually like to explain things really slowly so that the student that has a difficult time can really understand it. I try really to make sure that no one [is] confused and sometimes I do two or three examples. Sometimes that bores the brighter student in the class that may not be quite bright enough to go onto a higher level, but still gets things the first time, like say Tom. He doesn't need all those examples. Although that's one of the reasons I give out the assignment sheet ahead of time. Because if they really know what's going on, they can start their homework and I don't get upset about that.

I don't do a lot of projecty things in my class. I don't know whether that's a strength or weakness. I mean, because I don't like them. Because anytime I do them, I don't think the kids gain a whole lot from them. . . The class tends to be the same all the time, so in a way, that can make it very boring. Although one year in the middle of the year, I decided to change a few things and the kids got really upset. They were used to the pattern and they liked the pattern. . . Some kids really like to have it be consistent. . . I guess I think that the kids learn better that way, they get more out of it that way. They know what's expected of them and when. In each class they know they have to take notes, they know when they can ask questions. . . So even though I consider it might be boring, I think they learn better and that's what we're after, after all is for them to learn and to grow.

#### Classroom Expectations

##### Classroom Policies

Mrs. Thomas checked daily to see if students had completed their homework assignments. She did this by walking around the classroom examining students' homework



papers to determine the degree to which the assignments were completed. She recorded in her grade book a 100 if they had done it all, a 50 if they left off the last two problems and a 0 if they had 25 percent or less done. "It's graded on whether they do it, not on how well they do it."

While Mrs. Thomas did not require class participation she considered it when determining borderline grades. At the same time, she believed that to receive an A, students must participate and volunteer to present homework problems.

For the classwork, I use that at the end of the term after I average their grades. . . . If they've really worked hard and have a borderline grade, I'll raise it up. I won't take away points from kids for not participating in class. Except that I will never give an A to someone who [hasn't] participated in class. . . . I don't think they deserve an A. An A student really has to be well rounded. Has to participate in class. . . . [However,] I'm gonna take points away from a C student because they didn't participate, because they don't feel comfortable participating. . . . If they're a weak student and they don't have the ability to participate, then you can't take away from them for not participating. Cause they're gonna get up there and make a mistake, and make a fool of themselves and they're not gonna want to try after that. . . . You can only sorta subtract points, if they have the ability to participate and do not.

In addition to homework and class participation, the students grades were determined by their test and quiz scores. The difficulty level varied with the course. That is, the Algebra II/Trigonometry class were given more challenging questions than the Algebra II class. These differences will be discussed more explicitly in the next section on class distinctions.

#### The Teacher's Expectations for the Algebra II and Algebra II/Trigonometry Classes

Mrs. Thomas viewed mathematics in the high school curriculum as serving the needs of two distinct populations of students: those who elect careers or college studies that heavily utilize mathematics and those for whom their high school courses represent their last study of mathematics. For her, the needs of the two populations help to determine the depth and manner in which the mathematics is conveyed. For the students in the latter category, she sees mathematics as teaching them to think, to reason logically, and to develop consistent work habits.

For those students that have more of a long term need for mathematics, Mrs. Thomas views the curriculum as a preparation for calculus. The distinction between these two groups are mirrored in her description of the distinctions she makes between her two courses Algebra II and Algebra II/Trigonometry.

In the Algebra II class they really just need to be able to understand it. It's nice to be

able to have them have some application but it's not essential. They have to be able to feel comfortable [since] their frustration level is very low, [while for] the Algebra II/Trigonometry kids [they may] occasionally be frustrated [but] they're gonna be able to work it out, you want to make sure they're always challenged. [You] don't [want to] let it always be easy. [For the] Algebra II kids, it's not gonna ever be easy because they don't find math that easy. They always have to work at it.

Mrs. Thomas continues to contrast the two by saying:

In the Algebra II/Trigonometry [class] you always have to make sure that most of the problems you're asking, they really have to think about. It's not just such an automatic thing that it's just 'oh we learned this' [while the Algebra II kids] need repetition in order to learn it eventually. Sometime[s] the Algebra II kids tend to learn it first then understand it later. . .where the Algebra II/Trigonometry kids, you want them to understand it right away. . . .But sometimes for the other kids, it's not possible. Some of them feel just more comfortable with learning the routine rather than understanding why it works. Some kids just don't want to even know, don't want to understand but you can't let that happen in Algebra II/Trigonometry [because] they really have to understand [since] they're gonna need to use it later on.

The classes also differ in the expectation on homework, tests and level of abstraction.

In Algebra II, I always give them at least one example. . .of every kind of problem they're going to have, so that they've at least seen the type before. I don't do that in the Algebra II/Trigonometry [since] I want them to be able to take the information I've given them and then go a little bit further and be able to use that information to solve the problems.

In the Algebra II class she explained that she sets up:

The test so that anyone who has done their homework and understands what's going on, should easily be able to get a C. . . .There's a few problems in the test that you have to really think a little harder, so that you could earn a B. . . .There's probably like one problem in the test that really requires a little bit more thinking. . .that really pulls out the A's.

In the Algebra II/Trigonometry class she does not include the very basic type problems.

"They're not necessarily patterned after what they've done, where the Algebra II they're usually patterned after the kind of things they've done." She continued to contrast the two groups by describing each respective group's ability to handle abstraction.

I want them (A2T) really to feel good about their ability. But I guess I feel that about all classes. I want them [A2T] to be able to understand. . .at a really abstract level [and] to feel comfortable with an abstract level. [Also] to feel that they really understand and appreciate the material, not just be able to manipulate [it]. But I think it's important for them to understand where some of this is going and what the usefulness of their topics are. And honestly prepare them for the next level because it almost isn't until calculus [that] they really see that math is fit together.

Mrs. Thomas also had differing expectations for the two classes regarding their use of notation.

I expect better notation in the Algebra II/Trigonometry than I do in Algebra II. [It is] more important for them to know it because they're gonna have to use it more often. I don't think it's very important at all for the other kids to know it.

Mrs. Thomas expressed fewer distinctions between the two classes with regard to the actual mathematical content in the function units.

I thought everything [in the textbooks] was important. I really want them to have a good idea what a function is. And understand the difference between linear functions and when a function is not a linear function. I think that's important for them to really feel comfortable with that and understanding that. Function, kids think it's something new and different cause they haven't heard really much about it till they get to this point. Then they realize it's something they've really seen and used. . . . I think they should be able to recognize, first of all, from a graph whether that's a function or not. They should be able to recognize for instance an absolute value graph. . . . They should be able to possibly recognize a quadratic function.

[In the Algebra II class] their's are more visual. I want them to be able to understand it visually. I think basically the expectations are the same. . . . I think just the whole idea that the pace [in Algebra II] was a whole lot slower. I don't think that the depth was any different in that topic. Or there was very little difference in depth on that topic between the two books. Most of the difference was just in pace.

In summary Mrs. Thomas distinguished between the classes in several regards. These distinctions arose from what she perceived to be the needs of the two populations served by these courses. The Algebra II/Trigonometry class served those students who were preparing for Calculus and who would potentially elect college majors requiring mathematics. For the students in the Algebra II class, the content was seen as a means to teach them to reason logically. Mrs. Thomas felt that the Algebra II students were easily frustrated and often wanted merely to learn the routine since it rarely came easy to them. While desirable for both classes, Mrs. Thomas believed that it was essential that the Algebra II/Trigonometry students understand the rationale for the procedures they were learning since they would be expected to use it later on.

In addition to understanding the rationale, Mrs. Thomas expected the Algebra II/Trigonometry students to use good notation and to handle greater abstraction than the Algebra II students. Mrs. Thomas, also differentiated between the classes with regard to tests and homework. For the Algebra II class, she constructed the tests to match the homework assignments and prepared students for their homework by including relevant examples in the class lecture. In contrast, Mrs. Thomas expected the Algebra II/Trigonometry students to be able to answer test questions that were not identical to homework and to be able to work on homework assignments without necessarily seeing examples. She felt it was important to constantly challenge the Algebra II/Trigonometry students.

However, Mrs. Thomas did not have widely differing expectations for the content. She

stressed that it was important for the students to understand the definition of function visually and to be able to differentiate between linear and non-linear functions and to recognize the graphs of straight lines, absolute values and quadratics. While seeing the content as the same in both classes, she felt the pace in the Algebra II was slower than that for the Algebra II/Trigonometry.

In the next section, the teacher's expectations and philosophy will be compared to the classroom observations and analysis.

#### Classroom Environment

This section will briefly describe the classroom environment and the functions unit in the two classes. The discussion will proceed as outlined below.

- I. Description of typical class day
- II. Description of the function units
- III. The conceptual/procedural analysis of the teacher's questions during lecture time
- IV. The analysis students' autonomy in the classroom.
- V. Summary

#### Description of Typical Class Day

The class period lasted approximately 50 minutes. As indicated in her interview, Mrs. Thomas felt that consistency in the class format was helpful for students. The analysis of the classroom video-tapes confirmed that class structure was fairly consistent. (See Appendix L.) In both the Algebra II and the Algebra II/Trigonometry class, the daily routine included discussion and presentation of the previous night's homework, lecture on the next section, and review of an SAT type question.

The homework review followed a set routine. The teacher would select a student to read the answers from the teacher's edition of the textbook and then go to the board to record the exercise numbers from those problems for which the students had difficulty. Then various class members would volunteer to write their solutions to those problems on the board. While the students were writing, the teacher circulated through the rows checking to see if the students had completed the homework assignment. If Mrs. Thomas finished before the students were done, she would place a sample SAT question on the overhead at the back of the room and encourage the students to solve it. A brief discussion followed on the SAT question. The

volunteers then returned to the board and explained their solutions to the problems. Usually this entailed a recitation of the steps written on the board. The teacher did not ask nor appeared to expect the students to give any rationale or motivation for their solutions. Many of the students in the class participated in presenting problems over the course of the units including the participants in this study. The teacher then presented any remaining problems that the students had requested but had not volunteered for. The average amount of class time spent in this endeavor was 22 minutes in the Algebra II class and 21 minutes Algebra II/Trigonometry class.

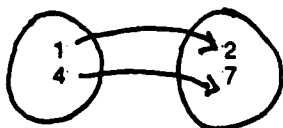
The lecture portion of the class period was primarily teacher centered. That is, Mrs. Thomas presented the relevant definitions and examples for each section in the function chapters. Confirmation of the directiveness of the teacher's lectures was provided by the two independent viewers of the video-tapes and will be discussed in more detail in a later section. It was the researcher's impression from observing the lectures and the students' presentation of the homework that the lectures generally covered examples of all the homework type problems. This observation coincided with the teacher's self-report that she felt that it was important and necessary to prepare students in the Algebra II class for their homework.

Occasionally in the Algebra II class, Mrs. Thomas assigned in-class practice problems from the textbook. The answers were then read in class and discussed. On average, the lecture time in both classes was 20 minutes. Further analysis of the lecture is included in the discussion of the teacher's questioning during that time.

As mentioned earlier, the practice SAT questions often appeared while the students were writing problem solutions on the board. At other times, these problems were used at the end of class when a few minutes remained before the bell. Overall the class format followed this routine. Occasionally the teacher would reverse the order of the lecture and the homework time.

#### Description of the Two Units on Functions

As indicated in Table 4 the topic coverage in the two classes was very similar. Both classes began their study of functions with a comparison of relations and functions. The classes compared these definitions by examining sets of ordered pairs and mapping diagrams (e.g.  $\{(1, 2), (4, 7)\}$  and



Both classes moved through a discussion of the vertical line test, domain, range, graphing linear equations, finding slopes, slope-intercept form, parallel and perpendicular lines, direct variation, function notation, and composition of functions. In the Algebra II class, the content also included one-to-one functions, step functions, and greatest integer functions. In the Algebra II/Trigonometry class, graphing inequalities was also included. While both classes were introduced to function notation,  $f(x)$ , it occurred at different points in the units. The Algebra II students encountered the notation just prior to their discussion of composite functions at the end of the unit. In contrast, the Algebra II/Trigonometry students saw function notation at the beginning of the unit. However, this notation was not used or emphasized in the remaining

Table 4  
Summary of Function Units

	<u>Algebra II</u>	<u>Algebra II/Trigonometry</u>
DATES:	Oct. 4-19 (11 class days)	Sept. 21-29 (7 class days)
TEXT:	<u>Algebra 2 with Trigonometry</u> Dilley, Meiring, Tarr & Taylor  D. C. Heath Lexington, MA 1987	<u>Algebra 2 and Trigonometry</u> Dolciani, Wooton, Beckenbach & Sharron Houghton Mifflin Boston, MA 1983
CHAPTER:	3 (pp. 87-130)	3 (pp. 67-96)
TOPICS:	(Covered in lecture and text) graphs and relations absolute value functions definition of function one-to-one* vertical line test domain range slope of line linear functions equation of a line slope-intercept form perpendicular lines parallel lines direct variation step functions* greatest integer function*	functions and relations definition of function vertical line test domain range function notation composition of functions absolute value functions graphing linear equations graphing linear inequalities** slope of a line equation of a line slope-intercept form perpendicular lines parallel lines direct variations

\*Additional topics covered in Algebra II

\*\*Additional topic covered in Algebra II/Trigonometry

topics e.g. linear equations and absolute value functions were described in terms of  $x$  and  $y$ .

As noted earlier, the function unit in the Algebra II class was 4 days longer than in the Algebra II/Trigonometry class. That extra time was used to present the additional topics, to review, and to work on in-class exercises sets. In both classes, the teacher followed the textbook's ordering of topics and notation.

Each class had a quiz and test during and at the end of the unit. (See Appendix M for a copy of these instruments.) In both classes, the questions on the quizzes and tests were representative of the type of problems encountered in homework or in class. Since the teacher usually assigned the even problems for homework, she often would select the odd problems as examination questions.

On all the examination questions the researcher was able to identify homework questions of the same difficulty and with the same instructions. This was equally true of the Algebra II/Trigonometry class as with the Algebra II class. This matching of the examination questions with homework contradicts somewhat Mrs. Thomas' report on her testing policy. She had indicated that for the Algebra II/Trigonometry students she generally asked questions which required extension of the concepts not merely repetition of the homework. While this failed to happen on both the quiz and test, the test was substantially longer and more involved than that given to the Algebra II class. For example, the test for Algebra II/Trigonometry (Algebra II) class contained 4 (3) problems on direct variation, 4(3) problems on writing linear equations, 6(4) problems which necessitated graphing, 4(5) problems on function notation and 2(0) problems asking for explicit definitions. The graphing problems were especially telling. The Algebra II/Trigonometry test asked the students to graph  $y = 1/2x + 2$ ,  $x = y^2$ ,  $y = |x| - 3$ ,  $x + y \leq 2$ ,  $|2| = |y|$ , and the intersection of  $y \leq 3/4 x + 3$  with  $y \geq -1/3 + 1$  while the students in the Algebra II class graphed  $y = |x|$ ,  $x + y = 5$ ,  $x = y^2 + 3$  and  $y = [x] + 3$ . Mrs. Thomas also remarked in her interview that she had intentionally made the Algebra II test easier than she usually would, since the previous test in that class had been hard and she wanted to even out the students' grades.

#### Analysis of the Teacher's Questions during Lecture Time

As part of the examination of the classroom environment, the lectures were analyzed for the type of question-conceptual or procedural-that the teacher asked. This measure provided

a means to examine the implicit classroom expectations. Since the teacher's questions compose only one part of the lecture activity, this discussion will begin with a few general comments on the teacher's lecture style and on the class atmosphere.

Mrs. Thomas presented well-organized and tightly composed discussions on the mathematics topics in the functions unit. She was generally clear and articulate in her explanations and continuously wrote her verbalizations on the board. She took time to draw very neat and accurate graphs of the various functions studied in the units. She maintained a quick pace and the class was rarely off task.

Mrs. Thomas' lectures generally stressed the procedural aspects of the topics. Only two of the lessons in either class used real-life applications as motivation for studying the topics. She utilized student's summer and part-time jobs as an example of direct variation and parking garage rates as an example of a step function.

While the class was tightly structured, the students appeared comfortable and at ease with Mrs. Thomas. She usually began or ended the class period with a few minutes of informal conversation about school events or occasionally teased a student. The students felt free during class to quietly get up and sharpen their pencils or retrieve a piece of graph paper from the back of the room. They quietly chatted among themselves while the homework solutions were being put on the board and during in-class exercises. They also appeared comfortable asking and answering questions in class.

#### Analysis of Questions

The analysis of the teacher's questions entailed coding each question as basically procedural or conceptual. A procedural coding indicated that the teacher's question called for the execution of an algorithm or computation, or recall of a basic fact or step in a known algorithm or procedure. A conceptual coding indicated that the question called for (a) an application of a definition, (b) synthesis of concepts, (c) the anticipation of a step in an unknown algorithm or procedure, (d) a conclusion, (e) a rationale for a procedure, (f) an example to illustrate a concept or (g) the selection of an appropriate procedure. These basic categories were augmented with a rating scale of 1 to 3 to indicate the relative difficulty of the question and with a designation of "led" or "not led" to indicate the teacher's preparation in the lecture for the accompanying question. Thus the teacher's questions were evaluated in their context.



The lectures and more specifically the questions were examined and evaluated by the researcher and by two independent raters. These raters were both experienced high school mathematics teachers. On those questions where the researcher and the raters were in agreement on the basic categorization, 78% of all the teacher's questions during the lecture were coded procedural in the Algebra II class while only 30% of the questions received that coding in the Algebra II/Trigonometry class. A further examination of the codings revealed that approximately 60% of the procedural codings in the Algebra II class were designated as led. In the Algebra II/Trigonometry class, 75 % of the conceptual codings were also denoted as led. In addition, nearly all the questions whether conceptual or procedural, were rated 1, or easy. Sample questions from the lectures are given in Table 5.

Table 5

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Sample Questions from the Lectures

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Algebra II

Procedural

Given  $y = 3x + 2$ . If  $x = 0, 1, 2$ , then  $y = ?$

If Kathy earns \$5 per hour and works 25 hours, then what's her salary going to be?

Conceptual

Are all lines functions?

What do you know about lines that are perpendicular?

Algebra II/Trigonometry

Procedural

If  $f(x) = 2x + 3$ , then  $f(5) = ?$

The y-intercept is the place where the line does what? It crosses ?

Conceptual

Is every function a relation?

What are some examples of equation of lines?

In a mapping diagram, if the range has more values than the domain is it a function?

---

Summary

The analysis of the lectures revealed that in the Algebra II class the predominate mode for the teacher's questions was procedural, easy and led. While in the Algebra II/Trigonometry class, the questions were predominately conceptual, easy and led. These apparent differences in the questioning modes coincided with Mrs. Thomas' expectations for the two classes. She held higher expectations for the Algebra II/Trigonometry class and felt it was important to challenge that group. For the Algebra II class, she held the expectation that it was sufficient for

these students to be able to do the problems since they were easily frustrated and generally found mathematics difficult.

Analysis of Students' Autonomy in the Classroom

To ascertain the degree to which the students were afforded opportunities to work independently of the teacher or to serve as the authority for their knowledge or solutions, the independent raters answered several questions based on their observations of the classroom video-tapes. In particular, the raters summarized their observations on students' participation during the lecture time and on the teacher's use of problem solving, conjectures or discovery in the class. For both classes, the raters observations stressed the teacher-centered orientation in the lecture. In addition, the raters cited no instances of conjecturing, problem solving, or discovery learning in their examination of video-tapes.

Describing the lessons in the Algebra II class, the rater commented that:

During the lecture period, the teacher was the only person talking except when students answered questions.

[Lectures were] teacher centered.

[Student participation was] minimal. Teacher asked some questions during the lecture, but students mainly watched, took notes.

Similar comments were reported by the second rater in his observations of the Algebra II/Trigonometry class.

Mainly lecture format [Student participation was] minimal, very quiet (subdued). There was appropriate response when a question was asked.

Lecture interspersed with questions designed to get students' attention and seek their input. Students were fairly attentive despite interruptions [from outside the class]. They took notes and asked questions at pertinent points. They also answered specific question but at least 1/2 were "passive" participants.

Very fast paced lecture format. . . .Lecture format woven into periods of questions and answers.

Both raters noted that the students' participation in the lecture focused on and seemed to be limited to responding to the teacher's questions. Their energies seemed instead to be directed towards taking class notes. However, the raters noted more student involvement and interest during the lesson on direct variation since Mrs. Thomas utilized information from their lives to set up the initial examples.

While Mrs. Thomas' lecture style generated little active student involvement apart from

answering questions, she did open up the class to student participation during the homework component of the class period. During this portion of the class, students suggested the homework problems they wished to discuss, and then volunteered and presented their solutions to the class. Except for points of clarification or correction of serious errors, Mrs. Thomas remained silent while the students discussed their solutions. The observation of Mrs. Thomas' class suggested that it was primarily through the presentation of the homework solutions that the students were able to serve as their own source of authority for their knowledge. During the lecture, however, the student's assumed a more passive role with Mrs. Thomas directing the presentation and serving as the sole authority.

#### Summary

The class format and tests, as well as, the teacher's lecture style and questions seemed to suggest a focus on learning the procedures in this unit on functions. The examination questions in both classes matched with the homework problems and the in-class examples. In addition to matching the homework, these questions stressed the execution of practiced procedures: finding the equations of straight lines, setting-up a proportion to solve a direct variation, evaluating a function at a given  $x$  value, computing slope, and graphing several familiar equations.

The teacher's lecture style also reinforced the perception of mathematics as entailing the execution of procedures. Mrs. Thomas presented the content through a strictly lecture format. She did not utilize any problem solving or discovery, and rarely utilized real life examples. As a result, her presentations were teacher-centered and accomplished through her demonstration of the relevant procedures. Her questioning of students during the lecture was primarily at a procedural or easy conceptual level and she generally prepared or led the students to the answer or conclusion she wished them to draw. Thus by her questioning level, she implicitly emphasized the procedural aspect of the mathematics being studied.

Finally by being teacher-centered, the lectures stressed the teacher's role as the authority for and source of the knowledge and de-emphasized the students' role in creating for themselves the concepts and connections. This teacher-orientation, however, was mitigated during the homework portion of the class period. It was then that the students served as their own authority for their solutions.

The questions during the lectures and on the tests and quizzes suggest an emphasis on the procedural aspects of studying functions. In addition, the lecture format employed by Mrs. Thomas stressed the teacher's role as the source and authority for mathematical knowledge. Based on these observations it was concluded that the classroom environment implicitly focused on a procedural view of mathematics.

## CHAPTER VII

### FUNCTION INTERVIEWS

In this section of the data analysis, the students' conceptualization of functions are presented and summarized. Students' remarks and actions as they responded to the interview questions form the basis of this discussion. (See Appendix B, interviews 6, 7, and 8.) In this section each student is treated as a separate case study and within each case the data is organized as indicated in the following outline.

- I. Definition of function
  - A. Purpose for studying functions
  - B. Functions and relations
    1. Definition
    2. Examples
    3. Domain and range
- II. Function notation
  - A. Interpretation of  $f(x)$
  - B. Composition of functions
- III. Related Concepts
  - A. Linear functions, slopes, and intercepts
  - B. Slope
  - C. Intercepts
  - D. Word problem
- IV. Summary

#### Keith

In the function interviews, Keith was coded correct on 17 problems and incorrect on 3. This coding of 85% correct ranked Keith second among the six participants on this category. Keith's comments during the interviews revealed several strengths and weaknesses with function concepts. Keith had a strong grasp of function notation including composition of functions. Keith also showed a conceptual understanding of linear equations which included graphing, slopes,

and the slope-intercept equation. While facile with function notation, he had difficulty recognizing graphs of functions. Keith seemed to implicitly associate continuity and curvature of a graph with the definition of a function. He was uncertain about the domain and range of graphs and function equations, as well as the definition. These strengths and weaknesses will be discussed in more detail in the following sections.

### Definition of Function

#### Purpose for Studying Functions

When asked what purpose he saw for studying functions, Keith readily acknowledged that he saw none and was confused by the new emphasis on the  $f(x)$  notation. He remarked:

Why do you study functions? What good are they? I don't really know someplace. I don't really understand functions. I just don't understand why you can't say that  $y$  equals like  $2x$  plus 3 instead of like  $f$  of  $x$  equals. I just don't understand what is  $f$ . I guess I know what  $x$  is, but what is  $f$ ? And why is it  $f$  of  $x$ . I guess I understand that  $f$  is the total equation, and  $x$  is the variable.

He continued to discuss the relationship between  $x$  and  $f(x)$  and suggested that this notation allowed one to write a statement "complete in itself." He explained this remark by saying:

If you're given a variable like  $x$  equals 3, and then  $f$  equals  $2x$  plus 3. Then you can just say  $f$  of  $x$  equals 9. And it's complete in itself.

Returning to the original question about the rationale for studying functions, Keith commented that:

What good are they? [pause] I'm trying to think of ways that they might help you in daily life, but I can't ever remember using or saying to myself well  $f$  of  $x$  equals. [laugh]. . . They only thing that I can really think of that it would help you be able to solve problems better. I don't know. [laugh]

#### Functions and Relations

Definition. Keith's initial description of function focused on the notation. He seemed to associate a function with an operation. That is with evaluating the function at a particular  $x$  value.

Explain what is meant by a function? We were asked this on a quiz once too. Let  $f$  of  $x$ , would be like  $x$  squared plus 5. A function [pause] allows one to use a variable to solve an equation. That's how I see it anyway. If you had  $f$  of  $x$  and you can substitute  $x$ , whatever, like you have  $f$  or 2 and use 2 in for  $x$ . I don't know. . . I don't remember how to graph functions. . . I feel like I need a  $y$  in there too, though. [pause] No, maybe not. Well then if you substituted 2 for  $x$  then you'd have a 9. I think that's how you do it, I can't remember. If you substitute 1 for  $x$ , then you'll end up with a 6. So choose a 3 for  $x$  you'll have a 15. [Pause, plots points on the graph] How's that? I don't know what I did.

When examining the graphs in problem #45, Keith's first response was: "Can I ask you

what the difference [is], because I don't think it's ever been made clear to me what the difference is. I don't even think I've actually been told what a function and relation can be." When the vertical line test was mentioned as a reminder, Keith proceeded to examine each graph. Like many of the participants, Keith was confused by the piecewise function in A. In particular, he was not comfortable with the use of the open and closed circles. After the researcher explained their meaning, Keith remarked: "I guess it's a function, I don't know. [laugh] [pause] . . . I remember something about what functions do. . . .It goes through more than 1 times." Keith applied the test correctly to graphs C and D, but vacillated on graphs B and E. On graph B, he was thrown by the symmetry and felt:

It was a relation. And because it may not if it only goes through once, maybe it [is] a function. But it seems to be a mirror image, which would seem to me to be a relation. [pause] I'll say a relation.

When pressed further to explain his rationale, Keith commented:

I really don't know an exact reason why it's not a function. But it looks to me like it's a mirror image, sorta. So that in relation basically means that. I mean they're interrelated with each.

To test his adherence to symmetry as a discriminating factor, the researcher then sketched a graph of  $y = x^2$ . Keith's subsequent comments suggested that he had been associating the term, relation, with its everyday usage. He was looking for a relationship or pattern to the graph rather than strictly applying the vertical line test.

I remember this. . . .I remember these hyperboles or whatever they are. [Parabola] Yeah, that's right. Using my English. I think hyperboles mean something in English. . . .That's [ $y = x^2$ ] a function. . . .Well, the vertical line only went through once. But also, I mean, just from memory. I remember doing functions. . . .So yes, cause I remember having these at the same time.

Keith's comments also suggested that he was relying on familiarity with the graphs to decide whether it was a function. Keith rejected graph E as function saying:

I don't know, it just looked weird to me. [laugh]. A function I take as being curved. . . .That isn't really curved.

As a follow-up the researcher sketched the line  $y = x$  to which Keith indicated: "I would say that would be a function. Because it can only be traversed through one point." When asked later if all straight lines are functions, Keith quickly replied that a vertical line was not because "when you're doing that vertical line thing it will pass through it in all points." To ascertain how confident he felt in his response, Keith was asked to rate his certainty. He replied "one"

indicating his insecurity with his memory of the definition.

In summary, Keith's remarks while examining the graphs implied that he associated the idea of function with continuity or with familiarity. He did not consistently apply the vertical line test when checking each graph. Instead, he relied on his familiarity with the graphs to help determine its classification.

Examples. In later interviews Keith was questioned again about his understanding of function. On problem #61, Keith quickly recognized that if  $a$  and  $b$  were distinct values that the ordered pairs would contradict the vertical line test. When pressed for the implication for  $a$  and  $b$ , Keith insisted that it was impossible. He suggested instead that one could change the original  $x$ -coordinates to  $(2, a)$  and  $(-3, b)$  and thereby guarantee a function.

Keith also applied the vertical line test in problem #63. He began the problem by describing for himself the elements that constituted the domain and the range. Using a coordinate axes, he visualized these sets as forming points on a horizontal line. On the basis of this depiction, he concluded that the original set of ordered pairs would be a function.

. . . Like Brown is the range in each case. And like Bill is part of the domain. Jane is part of the domain. Sarah is part of the domain and Tom is part of the domain. And then the Jones' family will be part of the range. And then Allen, Carol, Dave, George and Patty will be part of the domain.

Next Keith was asked whether the ordered pairs would remain a function if the coordinates were reversed. Here again Keith used an axes to confirm his thinking. This time he made the association with the axes more concrete by identifying the names with actual numerical values. Since his graph formed a vertical line, he rejected it as a function.

Well, now the range is constant. [Referring to the original problem]. I don't know, I don't see why not. I guess it could be. I just don't have as good of a reason for why it should be as before. Because like the range is constantly changing. Now the domain, it's saying the same. . . . If you had the domain being  $x$ , the range is  $y$ . Now the range is changing so, if you had  $(1, 0)$ ,  $(1, 1)$ ,  $(1, 2)$  and you have something like  $(1, 1)$ ,  $(1, 2)$  or maybe the other one. No, I don't think. That couldn't, because it would be like having a vertical line, which can't be a function. Because the  $x$ , the domain stays the same with Brown. It's gonna be Brown, Jane; Brown, Bill. . . . And the range is going to be changing with like Jane, Carol, Bill, Sarah, Tom. So, you'll have like everything will be in a vertical line because this  $x$  coordinate will stay the same where the  $y$  is changing. So it would be a vertical line and you can't do the whole vertical line thing. [laugh]

While Keith had difficulty determining whether noncontinuous graphs were functions, he appeared comfortable examining sets of ordered pairs. His explanation on problem #63 showed that he had formed a connection between the concepts of domain and range as coordinates of



individual points, graphs, and the vertical line test.

**Domain and Range.** As discussed in the previous section, Keith identified the domain with the x-coordinates and the range with the y-coordinates. When asked for an example to illustrate these concepts, Keith selected the function  $f(x) = 2x + 3$ . He proceeded to plug in values for x (0 and 1) and calculate the f(x) or y value (3 and 5). He then concluded that:

So the domain would be 0 and the range would be 3, and if this [x] assumed 1, it would be 5. Domain would be 1 and range would be 5. And it would continue on like that.

Keith's remarks seemed to suggest that he viewed these concepts as tied to specific ordered pairs rather than the set of all x-values and y-values that satisfy the equation. When queried about what would constitute the whole domain, Keith indicated: "I think the domain could equal all real numbers." He reasoned that:

Because anything. That doesn't make sense to me though. Because well, yeah, maybe it does. Because you can substitute in anything for domain, but then again you can substitute anything for the range too. . . . Yeah, it's like if I can substitute in a lot of things, I mean every number in for x and always get a number for y. But it won't be all real numbers. Whereas I can substitute in every number for y and not get every value for x.

To clarify Keith's comments, the researcher proposed the function:  $f(x) = x^2 + 5$ , and asked Keith to plug in values for both x and y and then to identify the domain and range. Keith selected several y values all of which would yield perfect squares when 5 was subtracted. For example for  $y = 30$ , he calculated that  $x = 5$ . The researcher then suggested that he try a variety of values like a negative number or zero. After checking several more values, Keith noted that "there are only a few numbers that can be plugged into the domain and only a few numbers that can be plugged into the range." Keith continued to try values at the researcher's urging before deciding that the domain could be "any value" and the range would be "5 or bigger."

A similar commentary occurred when Keith discussed the domain and range of the function  $f(x) = \text{square root of } x - 4$ . Again he tried specific values for x and recorded his results in a table of values. He initially concluded that "x should be 5 or more." When pressed, he tried  $4\frac{1}{2}$  and 4 as values for x although he was uncertain about taking the square root of 0. He remarked and reasoned:

I'm just trying to think if I could plug in 4, but I don't know if you could determine the square root of 0. I don't. [A square root] is a number times a number. Like the same number times a number equals number like 2 times 2 equals 4, so the square root of 4 is 2. But 0 times 0 would equal 0, so I don't see why you can't have a square [root] of

0 equals 0. But 0 is just a bizarre number because there's so many inconsistencies, [laugh] with 0 that are different than with positive numbers.

Through prompting, Keith was able to determine the complete range of the function.

As these examples demonstrate, Keith understood that the domain and range of a function were connected to the  $x$  and  $y$ -coordinates. However, naming the domain and range for a specific function proved more problematic. He seemed to rely on his computation of  $x$  and  $y$  pairs to determine his answer.

### Function Notation

#### Interpretation of $f(x)$

Keith was clear in his use of function notation. He was able to identify  $f(x)$  with  $y$  and easily graphed  $f(x) = 2x + 3$  by computing a table of values. Although Keith was facile with evaluating expressions, he remained confused about the rationale for  $f(x)$ . He saw no apparent reason for this notation instead of  $y$ .

#### Composition of Functions

Keith's ease with function notation extended also to the composition of functions. Not only was he able to evaluate composite functions at a particular value but he demonstrated in problem #55 that he could write the general expression for  $h(x) = f(g(x))$  when  $f(x) = 1/x$  and  $g(x) = x - 3$ . When asked to describe the domain of  $h(x)$ , Keith reasoned for himself that the innermost  $x$  would represent the domain and the final value or  $f(g(x))$  and hence  $h(x)$  would be the range. With this interpretation he stated that:

$x$  can be equal to anything. I'm pretty sure, yeah. . . The one that I'm really fussy about would be like 3, because that would give you 0 and I've never been able to. That's like a non-existent number. It's undefined or something. I actually don't see any reason why you can't plug in 3, except that it is undefined. And I don't know if that could actually be calculated to be a specific number. Maybe somebody else knows how to calculate it, but I don't know how to calculate it.

In addition Keith correctly argued that in general  $f(g(x))$  does not equal  $g(f(x))$ . He supported his argument by suggesting and examining two functions,  $g(x) = 2x - 2$  and  $f(x) = x + 1$  and the two compositions  $f \circ g$  and  $g \circ f$ , whose respective composites yielded different values when he evaluated them at  $x = 2$ .

### Related Concepts

#### Linear Functions

Keith's understanding of linear functions was fairly complete. He recognized that non-

vertical linear equations were functions. In addition, Keith was articulate and confident in his description of the components of the equation  $y = mx + b$ . He explained that:

The  $y$  equals  $mx + b$ .  $M$  is the slope,  $b$  is the  $y$ -intercept where the line hits the  $y$ -axis.  $X$  will be the  $x$ -coordinate and  $y$  will be a  $y$ -coordinate. So you can plug in numbers for  $x$  and  $y$ . Or you can plug in a number for  $x$  and get a number for  $y$ . . . .

When asked specifically about the purpose of  $x$  and  $y$  in the equation, Keith noted that the  $x$  and  $y$  were not assigned a fixed value "because  $x$  and  $y$  can vary, but the  $y$ -intercept will always be the same and so will the slope."

Keith's familiarity with linear equations also included graphing. He was able to graph  $y = 2x + 3$  both by constructing a table of values and by using the slope and  $y$ -intercept. However, Keith was initially uncertain about the connection between the graphs of  $f(x) = 2x + 3$  and  $y = 2x + 3$ . He did not immediately recognize that they would give the same graph. He reasoned that:

So it would be  $y$  equals  $mx$  plus  $b$  which would mean that  $y$  equals  $2x$  plus  $3$ . Let me graph this one first all right? [pause] If I plugged in, hum [pause] maybe it is the same. Slope would be  $2$  over  $1$  which says rise  $2$  and run  $1$ . And  $b$  as a  $y$ -intercept and the  $y$ -intercepts  $3$ . Yeah, OK. They are the same then. I thought it would end up being opposite. . . .After looking at it, seeing that  $y$  equals  $mx$  plus  $b$  and  $f$  of  $x$  equals  $2x$  plus  $3$ . Which would be the same as  $2x$  plus  $3$ , yeah. [laugh]

Keith's understanding of the linear equations also included slope and to a lesser extent intercepts. His comments on these topics will be discussed in the next two sections.

### Slope

When asked to explain the concept of slope, Keith readily volunteered the slope formula and its usage in the slope-intercept equation. He also added that the slope of a line represented the "rise over the run" or the "slant of the line." In problem #50, he easily calculated the slope from two of the given points and realized that "these points are on the [same] line, so they [also] would have the same slope."

### Intercepts

Keith's discussion of problem #53 revealed both an understanding and misconception about intercepts. Keith recalled that the  $x$ - and  $y$ -intercepts represented the points where "the line hits the  $x$ -axis and where it hits the  $y$ -axis, right." He noted that if  $x$  is  $0$ , then  $y$  was  $24$ . Similarly, if  $0$  was substituted for  $y$  he observed that:

You'll end up with  $x$  minus  $1$ , times  $x$  minus  $2$ , times  $x$  minus  $3$ , times  $x$  minus  $4$ . So that means that  $x$  can equal  $1$ ,  $2$ ,  $3$  or  $4$ . So the  $y$ -intercept will be  $24$  and the  $x$ -intercept will

be 1, 2, 3, 4.

When the researcher followed up Keith's discussion by asking for a sketch of the graph, a misconception was revealed. Keith plotted their respective intercepts on the axes and then proceeded to connect the y-intercept with each successive x-intercept. This resulted in a series of 4 lines which shared a common y-intercept. Keith stated that:

It would be sort of like that, with one point and four lines coming diagonally off of it. . . .I wasn't thinking of it as one equation. I was thinking of it as separate. Like y equal x minus 1, and y equals x minus 2. . . .I just wasn't thinking of it as one. If you substitute in 0 for y or whatever then I was thinking like at 0 equals x minus 1.

One possible explanation for Keith's comments could be that he was extending the technique for solving an equation by setting each factor equal to zero back to the original function and thereby breaking it apart into its factors. Yet it is unclear from the discussion whether Keith ever entertained the possibility that a single graph could have multiple intercepts or whether he associated intercepts exclusively with straight lines. Even when questioned further about his graph, Keith maintained that his sketch was appropriate for the given equation:

$$y = (x-1)(x-2)(x-3)(x-4).$$

#### Word Problem

Keith had no difficulty with the function notation in word problem #59. He quickly substituted in part a, 10 for t and calculated the population and in part b, 275 for P(t). While he readily solved the problems, Keith was surprised that the solution in part b was not a whole number. He remarked:

That's weird. Because it's not, well, I mean it isn't weird. . . .It's just, it's gonna come out to be a fraction. OK, or a decimal. So that would be 11.4 days.

#### Summary

Keith's interview revealed several strengths and weaknesses in his knowledge of functions. He demonstrated a facility with function notation including evaluating functions at a given x, composing functions, and associating f(x) with y. He was also able to solve a word problem involving function notation. In addition, he explained clearly the components of a linear equation and could use that information to graph an equation. He was equally comfortable with the concept of the slope.

In contrast, Keith was confused about the definition of function and seemed to implicitly associate continuity and curvature of a graph with being a function. He also had difficulty

describing the domain and range of a function. Keith seemed to relate these concepts to single coordinate pairs rather than the entire graph.

In conclusion, Keith's discussion of functions showed several areas of consistent understanding, as well as, areas of misconceptions. Although Keith did not perceive any use for functions or function notation, he appeared facile and comfortable with most of the concepts in the unit on functions.

## Ann

In the function interviews, Ann was coded correct on 17 problems and incorrect on 3. This coding of 85% correct ranked Ann first among the six participants on this category. Ann's conceptual understanding of functions had several strengths and weaknesses. She seemed to have a strong grasp of the basic definition of function but almost no facility with function notation. With regard to linear equations, Ann had developed an extensive and well connected conceptualization. Despite her weakness with function notation, Ann was able to utilize number sense to solve a word problem involving function notation.

Definition of Function

Purpose for Studying Functions

When asked the reason for studying functions, Ann cited step functions as a model of a parking lot fee scale.

Why did we study functions? What good are they? Ha, ha. [pause] Well we look at step functions. They were an example of when you go to like a parking garage or something. They'll like say it's like \$2.00 for your first hour and a dollar for every one after that. So it's 2, and then you pay 1 no matter where you are in there. So it ends up looking like this, whatever. [draws rough sketch of step function] They don't overlap. That's an example of a use. You can see how the price increases for every hour. But for an hour and a half it's still the same as it was for 2 hours. That's the way you look at it. It's usually like when you're 1 minute over then.

She also indicated that: "We can graph lines with functions." When asked what she could do now that she could not do before with the  $x$  and  $y$  notation, she acknowledged that she saw no advantage.

I have no idea why, what the importance of it is in there. [laugh] Beyond me. That's a good point, I don't know. I have trouble with that. I solve these equations right here [Problem #59] by just ignoring the  $t$  [in  $P(t)$ ]. [laughter] It just works out fine. I don't know why the  $t$  is in there. It just like says what you're solving for. Like  $p$ , it's the people like relative to the time is equal to. . . . The equation should tell that already because you're obviously, you've got your two variables. I don't know. [laugh]

Functions and Relations

Definition. Initially Ann was vague in her definition of function saying: "A function is like  $f$  of  $x$ ." After examining the graphs in problem #45, she selected D as the only example of a function indicating that she had done the vertical line test. Like many of the participants, Ann also was confused by the noncontinuous examples. She misunderstood them as representing two separate graphs depicted on the same set of coordinate axes. In addition, graph C was interpreted as vertical at it's ends rather than steeply sloping. These misinterpretations were

discussed and Ann was told that her usage of the vertical line test was incorrect. She reflected again on the graphs indicating this time that D was the only example of a non-function. At this point her thoughts seemed to crystalize and she recapitulated her definition.

Yes I remember now, I just all of a sudden, it was like, then I remember writing not a function a lot. All right. A function is when there's only one answer for  $x$ . There's only one solution for  $x$  that works in a given equation. Like if it said  $y = x + 3$ . Every answer for  $x$ , there would only be one answer for  $y$ . So that would be a function because there was only one solution for  $x$  for one solution of  $y$ . I mean that's an example of a one-to-one function.

In addition to voluntarily distinguishing between a one-to-one correspondence and the definition of function, Ann illustrated her definition of function by drawing the straight line  $x = 1$ . She commented on this graph by saying:

Say  $x$  is equal to 1. That means  $x$  are equal [on] all this line. But in a function [there] would only have one point,  $y$  would only be equal to one point on this line. It couldn't pass back over this line. So  $y$  could be equal to 2 or 3 or 4 or 5 or whatever all the way up. It could [be] any integers, negative numbers, but it can only be equal to one of them. It cannot be equal to more than one of them.

A one to one function would be [pause] when like right here this is a graph. [Sketches a roughly v-shaped graph centered about the  $y$ -axis]. This is an example of something that is not a one to one function because my horizontal line check. Because it's where  $y$  doesn't work. It's the exact opposite of what a function is.

Examples. After her initial confusion, Ann succinctly stated and illustrated her definition of function and one-to-one correspondence. Her conceptualization of function also extended to the more abstract and non-numerical situations suggested by questions #61 and #63. On question #61, Ann saw immediately that if  $a$  and  $b$  represented distinct values that the graph could not possibly be a function.

They [the  $y$ -coordinates] can't be equal. . . . Well, this line [point] you go over 2 and say you go up 1. And this one, you go over 2 and say this  $b$  is equal to 2. This line right here so it can't be a function because [of] the vertical [line] test.

When asked what she could change to make both points fall on the graph of a function, Ann remarked: "These points [ $x$ -coordinates] would have to be different here." She was then asked for a second way to change the points. This time, she suggested making  $a$  and  $b$  equal.

Ann also extended the concept of function to a collection of a non-numerical ordered pairs in problem #63. She used a graphical representation to help her interpret the ordered pairs. She began her discussion with a joke about a marriage between the collection of ordered pairs.

No, because what if they get married. Cause if they get married that's Carol Brown, not Carol Jones and a function doesn't move. [laughter]. All right now. [pause] I don't think it does because I can't see. Maybe I'm being too rigid with this. Your first name you go

2 over [laugh] your last name you go 3 up. . . . I mean, the only way I could see it is that all the Brown's live on the same [horizontal] line. And all the Jones' will live on the same line. And they'll live in different places on the line because you don't live the exact same spot. So it's like each line is a different family. Like each house is a different family. That's the connection I can make. [laugh].

When asked whether the collection of ordered pairs would remain a function if the ordering was reversed, Ann again referred to her graph saying:

That's not a function because they're all the same. They're all going this way [vertical] and that's not a function. They can't all go that way because you can't see that person because you have to look up. You have to be side by side.

Domain and Range. Ann's conception of domain and range was explicitly tied to the particular notation used to describe the function. She defined the terms by saying: "The domain is the value of the  $x$ . And the range are the possible values for the  $y$ ." She illustrated her definition by sketching a line segment beginning at  $(-5, 7)$  and ending at  $(7, 1)$ . She remarked:

If I had a line like this. And there were points on the line and say it would be all of these  $x$  values, any of them on the line. And say it, we were told the points ended here. It would be all the points between here  $[-5]$  and there  $[7]$  would be the domain. And the same for all the values of  $y$  would be the range.

Ann also indicated that if the line was extended the domain would include all real numbers. However, Ann's understanding of domain and range seemed limited to examples involving  $x$  and  $y$  notation. She did not immediately connect the concepts to equations written in function notation,  $f(x)$ . For Ann, function notation depicted only one variable,  $x$ .

I have no idea how to do that, because the range is equal to the  $y$  value, and the domain is equal to the  $x$  values. And I don't know how I could pull the  $x$  or the  $y$  values out of there.

When asked to consider  $y = 2x + 3$  and  $f(x) = 2x + 3$ , Ann hesitantly suggested that they were the same. She then returned to the original example [problem #52] and rationalized that the domain would need to be greater than or equal to 4 for the square root to be defined and as a result the range would be greater than or equal to 0.

#### Function Notation

##### Interpretation of $f(x)$

Ann's difficulty with function notation was pervasive. As illustrated above, the notation caused confusion when describing domains and ranges. It also seemed to interfere with her understanding of the word problem in question #59. Ann consistently interpreted  $f(x)$  to mean  $f$  times  $x$ .



### Composition of Functions

This misconception of  $f(x)$  as meaning  $f$  times  $x$  was pronounced when she was asked to interpret composition notation,  $f(g(x))$ . She indicated that:

Here's the number for  $x$  and you plug that in. Say like  $x$  is equal to 3, and then  $g$  is equal to 5 and then that whole thing and  $f$  is equal to 7, and then you solve for this by multiplying it out. I think that's the way you do it. And then I don't know what you're gonna get here, and I don't know what I am solving for. I have absolutely no idea what it's connection is to everything else or what it means. And that's the way it turns out. [laugh].

Even when presented with the specific examples for  $f(x)$  and  $g(x)$  from question #55 and reminded that previously  $f(x)$  had been interpreted as  $y$ , Ann remained insistent that the parentheses meant multiplication. Ann also indicated that she was often confused about whether problems written in function notation required her to replace  $x$  with a value or to solve for  $x$ .

### Related Concepts

#### Linear Functions

Ann's conceptualization of straight lines seemed complete. She easily used both graphs and equations to augment her explanations of domains and ranges, and functions. She readily acknowledged that straight lines could be functions. "If it is not a vertical line."

#### Slope

She also was able to give both a mathematical description of slope as well as a intuitive interpretation.

The change in  $y$  over the change in  $x$ . . . . A slope is just like the slope of a hill. Every hill has a different slope considering how much rise you have, how far over you go. Rise over run.

On problem #50, Ann immediately recognized that using the different ordered pairs would yield the same slope.

#### Intercepts

Ann's understanding of intercepts was explored in problem #53. Ann began this problem by multiplying out the factors with the intention of solving for  $x$  and  $y$  as she had done for equations of the form  $y = mx + b$ . Halfway through the process she stopped and remarked:

Now I have the feeling I am doing all this for nothing. Cause I still don't know after I get this all done. [pause] Well in order to find the  $x$  and  $y$  intercept for each of these, I want to solve for  $x$  and I want to solve for  $y$ .

Ann then changed strategies and constructed a table of values for  $x$  and  $y$  listing the ordered

pairs: (0, 1), (24, 0), (0, 2), (0, 3), and (0, 4). She next graphed these five points. Since she had been expecting a straight line she was confused by this configuration. She continued to explore and plot additional values of  $x$  and  $y$ . Ann was trying to determine: "If it [the graph] hits it [x-axis] on other spaces." When asked what was true about an  $x$ -intercept when it was substituted back into the equation, Ann responded: "Your  $y$  value is always going to be equal to 0." Having spoken that relationship, Ann saw the connection with her search and summarized:

Oh I see. So your  $x$ -axis when you plug your  $x$  is gonna be equal to 0. And 0 is the only num[ber], 24 is the only number you can get for  $y$  when  $x$  is equal to 0. So 24 has to be your only answer. And these numbers are the only numbers you can get for having  $y$  equal to 0, so those have to be the only numbers.

#### Word problems

In addition to a facility with straight lines, Ann demonstrated a flexibility with functions in problem #59. Ann assumed initially that  $P(t)$  meant  $P$  times  $t$  and so set up the equation  $P(10) = 250 - 20$  which she solved for  $P$ . As she proceeded to the next question she realized that her equation  $275t = 25t - 20$  would yield a negative value for  $t$ . She argued that since her negative value for  $t$  was not in the restricted interval that the original equation was not applicable in this case. When Ann was asked explicitly about her interpretation of  $P(t)$ , she asked if it should have been  $P$  of  $t$ . Once this was confirmed she quickly restated the problem as  $275 = 25t - 20$  which she solved for  $t$ . She claimed that her answer made sense since the values (10, 230) and (11  $\frac{4}{5}$ , 275) seemed proportional. Although the proportion would not check, she did verify her answers by substituting them in the original equation.

The total number of people,  $P$ , who have caught the flu after  $t$  days is closely approximated by the formula,  $p$  times  $t$  equals  $25t$  minus 20. . . . Well, I just want to plug in 10 to the formula because 10 fits in here. So, because they're asking how many days, and  $t$  is days. So it's  $p$  times 10 equals 250 minus 20. . . . So 23 people have caught the flu after 10 days. All right. [pause] Well, this is how many people, so  $275t$  equals  $25t$  minus 20. [pause] [pause] I'm gonna get a negative number here. So my guess is that, so  $t$  doesn't fit into this. I'm trying to solve for  $t$  right now and in this formula with 275,  $t$  doesn't fit into the inequality so  $t$  can't, so this problem doesn't work for this formula.

$P$  times  $t$ . It's not supposed to be? It's  $p$  of  $t$ ? Oh, all right, so if I went back it would be  $t$  equals 250 minus 20, so  $t$  would equal, or  $p$  of  $t$ , I should say,  $p$  of  $t$  equals 230, so it's 230. And then if I did this again I'd end up with  $275$  equals  $25t$  minus 20. . . . That's almost 11 and  $\frac{1}{5}$  days.

It makes sense with formula really right here. Because you could like, well I could do a ratio of 10 to 230 equal 11.2 over 275 [pause]. . . I thought that they'd be equal because proportionality they should be equal. But they aren't. I don't know why it wouldn't be equal.

Although Ann was again confused by function notation she was able to reason through the problem. Her answers needed to be reasonable to her before she accepted it. She attempted a proportional argument to justify her answer not realizing that the underlying relationship was not a simple direct proportion.

#### Summary

Ann's knowledge of functions reflected a deep understanding of the relationship between equations and their graphs. She demonstrated this ability with her ease in constructing examples of graphs to illustrate the terms function, one-to-one, domain, range, and intercepts. Her interview also revealed an expectation that her solutions should be consistent with her knowledge. For example while associating intercepts with straight lines, Ann continued to work on the intercept problem (#53) until she resolved the dilemma between her association and her graph. Ann showed this same tenacity on the word problem. There she utilized a proportion to try to confirm her solutions from the equations. While Ann's understanding of functions was well-connected to graphs, she was baffled by function notation. She repeatedly voiced the misconception that  $f(x)$  meant  $f$  times  $x$ . With the exception of her confusion with function notation, Ann showed a conceptual understanding of functions including the definition, terminology, and graphs.

## Tara

In the function interviews, Tara was coded as correct on 13 problems and incorrect on 7. This coding of 65% correct ranked Tara last among the six participants on this category. Tara's understanding of functions appeared tied to isolated procedures. She did not move easily between equations and their graphs and seemed dependent on set forms like  $y = mx + b$  to construct the graph. In addition, her use of function notation was limited to plugging in values for  $x$ . She did not acknowledge that  $f(x)$  was equivalent to, or associated with,  $y$ . This lack of association caused confusion not only with the ideas of domain and range but with graphing as well. A more detailed discussion of her understanding follows.

Definition of FunctionPurpose for Studying Functions

Tara stated that functions were used to "determine the relationship between a variable and the equation that it stands for." She continued her explanation by saying:

So that when we have a number to plug into an equation, you can do that. Then you can plug that number back into another equation. You can get one answer. From one function and then do that function of another one. Like  $f$  of  $x$ ,  $g$  of  $h$ , like that. And then you can just keep going on the line and find out the answer.

When asked why she had studied functions, Tara replied:

I wondered that about most math. Well that's a good question. Yeah, it is. I think it does in the long run have to do with the relationship between the variable and the equation. I think that was the idea of it, to understand that.

She concluded by saying: "I liked it. It was pretty easy."

Functions and Relations

Definition. Tara's definition of function was: "A formula into which a number is put to get an answer." As an example of a function she wrote the equation " $f = x + 2$ ." She indicated the  $f(2)$  would be evaluated as  $2 + 2 = 4$ . As a non-example Tara offered the expression  $3x + y + z$  with  $x = 3$ ,  $y = 3$ , and  $z = 4$ , claiming that: " $f$  isn't in the formula. This is just an equation and they're numbers that you substitute in. But the  $f$ , you don't have a formula which is a  $f$  and you don't have a number that you're plugging in."

In contrast to her verbal description of the functions as equations with  $f$ 's, Tara was able to quickly recognize graphs of functions by an application the "function test". Pointing to graph D, she said: "If you draw this [vertical] line here it intercepts [it] in three points." She went on

to comment that for functions, "There are no vertical lines. There are no two x's that are the same."

**Examples.** In subsequent interviews Tara's understanding of functions was further tested in questions #61 and #63. In these questions Tara was asked to respond to more abstract situations involving the definition of function as a set of ordered pairs without repetition in the first coordinate. On problem #61, Tara was unable to recognize any relationship between a and b which would allow the graph to be a function. In fact, she was uncertain how a graph and a function were related.

If (2,a) and (2,b) are points on the graph of a function, what can you conclude about a and b? I've no clue. [underline added] [laugh] What on earth? (2,a) [pause] Well, they're on the same line. [pause] Well, consider you have 2, over 2 and up well a, it's gonna be there and if you've gonna go over 2 and up b, and it's gonna be there. And they're gonna be on the same line, [underline added] whose slope is something, 1 or something. They're [the points] parallel [underline added]. OK, that's a good thing to say. Next [underline added]. [pause] I have no idea what's a function of a graph.

To clarify the language, Tara was told that a graph was just a picture of the function and she was shown a sketch of a sine function. Tara responded that it was not a function "because you can draw a [horizontal] line through it. It's not a function. It hits in more than 1 place." At this time, Tara was reminded that a horizontal line was indicative of a one-to-one function rather than a definition of a function. She then remarked.

Right, that's what I meant. OK. Right I see. [pause] Let's say it's on the same line parallel to the y-axis. I don't know quite what you want. It's not a one-to-one function. I have no clue. Well that's true. [laugh].

Later in the same interview, Tara was asked in question #63 about the feasibility of a set of ordered pairs forming a function. Like the other participants, Tara visualized the ordered pairs as points on a graph with the points forming two parallel horizontal lines. She rationalized that:

If you have all of these, then it's gonna be one line. If you have all these then it's gonna be all these. All the y's in both, and this is the same. It's gonna be this kind of a line. And all the y's in this are gonna be the same. It's gonna be this kind of line. [pause] So that's a function.

When asked if the collection of ordered pairs would remain a function if the ordering within each pair reversed, Tara was uncertain. She remarked that: "I don't think so, but I'm not sure because I don't remember. There's one that doesn't work. I think it might be this one."

**Domain and Range.** Tara's conception of domain and range appeared tied to the x and y notation. She defined the terms by saying: "It has to do with these pairs here. [(x, y)] And

the x terms are the domain and the y terms are the range." As an example of domain and range, Tara drew a line passing through (0, 8) and (8, 0) and designated two points on this segment, (8, 0) and (7, 1). She concluded that the domain was {7, 8} and the range, {0, 1}. When asked for the domain and the range of the entire line, Tara restricted the graph to the first quadrant and indicated that: "The domain and range would be all real numbers greater than 0." Tara was then asked to indicate a point on the graph where x was 9. She then realized her error and wrote  $\{x: 0 \leq x \leq 8\}$  and  $\{y: 0 \leq y \leq 8\}$ . To clarify Tara's understanding of domain and range, she was asked to examine the graph of the piecewise function  $y = 4$  if  $x \geq 1$  or  $y = 3$  if  $x < 1$ . Tara interpreted the graph as "they're two different lines." She gave the domain as "x is a little less than 1" and then clarified it to include 1. She gave the range as "y less than 3." Only with directed questioning did Tara realize that y was only equal to 3. Tara continued her analysis by remarking that the "second one [has] x greater than or equal to 1, right? And y is equal to 4." Tara maintained that the two pieces of the function be treated as distinct graphs and as such the domains and ranges were assigned for each piece.

In a subsequent question, #52, Tara was asked for the domain and range of  $f(x) = \text{square root of } x - 4$ . Tara acknowledged that the function had a domain but "I don't see any range in here." She later remarked that: "I always thought of a range as a y value, but it doesn't appear to be one." Even when pointedly asked about the  $f(x)$  symbol, she maintained that "there's only one variable. You know what I mean? If there's only one variable, how could you have a range?" For the domain, Tara began by indicating that "x could be any number. Cause [if] it was -2 -4, you'd have -6 and you could always take the square root of that." When asked for the value of the square root of -6, Tara corrected herself saying: "Oh no, you can't have a negative number. Because you have to have a positive times a negative to equal a negative." She went on to conclude that x "has to be greater than 4." She excluded the possibility of x equaling 4 by arguing that it would yield the square root of 0 and "well you can't do the square root of 0." Her rationale for this conclusion was that: "0 is not a number. That's my opinion." When queried further about this statement she remarked:

I don't know. [laugh] I'm just trying to figure out an explanation why you can't. If you have a 0, square root of 0. What times what equals 0, anything. Anything times 0 equals 0. [And] 0 times 0 equals 0. So the square root of 0 would be 0. I don't like 0's. . . . I just never learned to work with 0's. That could be it. . . . Oh, I hate 0's. Zeros and fractions. That should have been one of those categories on the 1-11 thing [ranking grid].

Oh, most difficult, 0's.

As a result of her confusion with the square root of 0, Tara indicated that the domain,  $x$ , was "anything greater than 4." When it was suggested that she replace  $f(x)$  with  $y$ , she reasoned that the range "has to be above 0." The protocol for problem #52 confirmed that Tara's understanding of domain and range was strongly associated with the symbols  $x$  and  $y$ . She did not acknowledge  $f(x)$  as representing a second variable and as a consequence the function had no range. Her uncertainty about the existence of the square root of 0 further attributed to her difficulty with the problem.

### Function Notation

#### Interpretation of $f(x)$

Tara's facility with the  $f(x)$  notation was mixed. She expressed in problem #46 a perception of  $f(x)$  as a "do something" symbol, associating it with evaluating for  $x$ .

The function of  $x$  equals  $2x$  plus 3. So that's the formula. Right?  $F$  of  $x$  means the formula and so then after that you would say like  $f$  of 3, so it means you're using  $x$  instead of a number in the formula. So if you do  $f$  of 3 then you can plug the 3 in where the  $x$  is. . . . You can plug a number in to get an answer.

In problem #47, Tara was asked to graph  $f(x) = 2x + 3$ . She proceeded by choosing to substitute 3 and 4 for  $x$ . As a result of her computation, she wrote the expressions  $f(3) x = 9$  and  $f(4) x = 11$ . She graphed her first result as two points on the  $x$ -axis,  $(3,0)$  and  $(9,0)$ . Connecting these points, she concluded that the graph was a horizontal line coinciding with the  $x$ -axis. When asked to graph the equation  $y = 2x + 3$ , Tara utilized the point-slope method and correctly graphed the equation. When queried further, she was able to generate points on the graph by "plugging" in numbers for  $x$ :  $(2, 7)$  and  $(3, 9)$ . Next Tara was asked to discuss the similarities and differences between the two equations and their resulting graphs. She argued that they should be distinct.

Well this one. In this one the  $x$  just stands for [pause] a certain number and when you plug in that number you get, I guess you're, I don't know. Hum, I don't remember. [pause] All right, let's see. [pause] Well this one only has one variable. This one has two. [taps pencil].

When confronted with her simultaneous descriptions of  $x = 3$  and  $x = 9$ , Tara eventually recognized her error and saw the connection between the two graphs.

No, no, no, no, no, that's not what I'm telling you. The function of 3, all right, is 9. Right, so when you plug 3 into this equation, it comes out to be 9. So then the function of 3 is 9. . . . So wait, wait.  $f$  of  $x$ , 3 equals 9, so then  $2x + 3$  would have to equal 9. That's

what you're trying to tell me. . . . Oh that was stupid.  $x$  equals 3, so then it would be like 3, 9. I see. So it will come out the same as this.

### Composition of Functions

Tara illustrated her explanation of composition with the functions  $g(x) = x + 2$  and  $f(x) = x + 3$ . She then proceeded to compute  $f(g(x))$  when  $x = 2$ . She wrote  $g(x) = 2 + 2 = 4$ ,  $4 + 3 = 7$ . Her verbal explanation to accompany her example follows:

Well I know this.  $f$  of  $g$  of  $x$ . Well, first you figure out a function of, what of  $x$  and then  $g$  of  $x$ . So I mean if, you have to be given something to be able to do an example. Let's say  $f(x) = 2 + 2$ .  $f$  of, no  $g$  of  $x$ . I don't know, that's not right. Right, right, right. [writes  $g(x) = x + 2$ ]. And  $f(x) = x + 3$ . OK. So then you make, you do and then it says  $g$  of  $x$  equals 2. So you go  $g(x) = 2 + 2$  which equals 4. And then you plug that back in and do  $f(x)$ , so you go  $4 + 3$  which equals 7.

Again on problem #55, Tara successfully computed the composition of the given functions  $f(x) = 1/x$  and  $g(x) = x - 3$ . She was able to give the generalized equation for  $f(g(x))$ . In her discussion of the notation for both single functions and compositions, Tara emphasized the computational aspect of the symbols. Her examples consisted of evaluating linear functions at integer values. However she had not connected the function notation and its evaluation with graphing.

### Related Concepts

#### Linear Functions

Tara's understanding of linear functions appeared triggered by the  $y = mx + b$  notation. When presented with the  $f(x) = 2x + 3$  in problem #47, Tara was unable to graph the function. To her, the function notation had only one variable,  $x$ , and as a consequence any computed values were graphed on the  $x$ -axis. However, when asked to graph  $y = 2x + 3$ , Tara succinctly used the slope-intercept technique. She also could construct the associated table of values.

When asked if straight lines were functions, Tara remarked: "If it's a horizontal straight line, it is. But if it's a vertical one, it's not." When asked if these were the only possible examples, Tara re-evaluated her response and replied: "So all straight lines, I would say, with the exception of the vertical lines."

#### Slope

As part of her explanation for slope, Tara drew a straight line on a coordinate axes which she marked with several dots. She commented that: "It [slope] is the distance from this point to whatever the next point is. This distance right here and it's the same all the way down.



$y = mx + b$ . This is the slope. [She circled the  $m$ ] When asked for clarification of her term, distance, she replied: "No this is the distance, like of, just 1 unit. That's just the distance of 1 unit and how much the line goes down or up in that 1 unit, is the slope." Although Tara recognized that the slope would remain the same irrespective of the points on the line chosen, she incorrectly computed the value of slope in problem #50 as the differences in the  $x$ 's over the differences in  $y$ 's.

### Intercepts

Tara's conception of the  $x$  and  $y$ -intercept of a graph or equation seemed tied directly to the  $y = mx + b$  representation of a line. On problem #53, she began by suggesting that she needed to expand the expression and combine like terms. Tara noted immediately that: "If  $y$  equals 0, then  $x$  equals 1, 2, 3, 4." However, she made no connection between her observation and her search for the  $x$  and  $y$ -intercepts. In fact, when asked directly if what she found related to the intercepts she replied:

Oh, no. I could be. Well it could answer both if you wanted it to. I mean, you have to have a  $y$  value. I mean, you have to plug something in.

To further explore Tara's remarks, she was asked to use a graph to illustrate the  $x$  and  $y$ -intercepts. In response, she drew a pair of axes and placed a dot on each axis. She pointed to these dots and indicated that they were the intercepts. She added that: "like in  $y = mx + b$ ,  $b$  is the  $y$ -intercept." Next Tara was asked what type of coordinates intercepts had, to which she answered: "I see what you mean. Hum. [pause] This is a stumper. [Clicks pen, pause] No clue. [pause]." To make the subsequent probes more focused and concrete, Tara was asked to determine the  $x$  and  $y$ -intercepts of the equation  $2x + 3y = 6$ . As part of her solution, Tara drew a linear equation that passed through  $(0,6)$  and  $(4,0)$ . She generated these points by using an erroneous application of the slope-intercept equation, that is, she wrote  $3y = 2x + 6$  and determined  $b = 6$  and  $m = -2$ . Her graph indicated that she started at 6 on the  $y$ -axis and from there was counting down 2 units and over 1 unit. She misread the  $x$ -axis and so marked the  $x$ -intercept  $(4,0)$  instead of  $(3,0)$  which her techniques should have determined. At this point in the discussion, Tara was asked to verify the points by substituting them into the equation. She noted that the values  $(0,6)$  did not work but could offer no rationale for their inconsistency.

It won't work. Cause of the 0 here, that shouldn't matter. [She wrote  $3(6) = 2(0) + 6$ ] Well probably because we're not writing in some of the points, and you're not, the  $x$  and

y are the points, not the y-intercept, and the x, they're points on the line. . . . I have no idea, so I'm making it up as I go along. So that's probably why you're not following it, I mean. I understand sorta, I mean, I understand this equation and I understand how to graph it.

When it was suggested to Tara that there was a mismatch between her equation and the form  $y = mx + b$ , she proceeded to divide the coefficient on x by 3, but failed to divide the 6. Thus when she again checked (0,6) into  $y = -2/3x + 6$  she encountered an error. She remarked: "Well, that doesn't work. Comes [to] the same thing as it did before. I have no idea. You can show me how to do it now [underline added]."

Tara's protocol for problem #53 suggested that her understanding of intercept was keyed to a recognition of an equation in the form  $y = mx + b$ . However, this form was not distinguished from other equations of the form  $ay = cx + d$ . Tara also did not appear to have connected the equation,  $y = mx + b$ , to the coordinates of the intercepts and in turn, to the satisfaction of those points in the equation and to their depiction on the graph. She seemed to have only one way to generate the y-intercept, that is, by pulling it from the b position of the slope-intercept form of the equation. As a result, Tara's knowledge of intercepts and graphing linear equations appeared fragmented and mechanical.

#### Word Problems

In problem #59, Tara initially interpreted the function notation,  $P(t)$ , as P times t saying: "People times time equals. Like the people per day. People per amount of time." Consistent with her interpretation, Tara replaced t by 10 on both sides of the equation in part a, getting  $10P = 250 - 20$ . Solving this equation for P yielded 23. Again in part b, Tara set up the equation with the assumption that  $P(t)$  meant P times t. She quickly noted that her equation  $275t = 25t - 20$  would result in a negative value for t. She then modified the problem to read  $275 = 25t - 20$  and proceeded to solve for t yielding approximately 12.

Well, if I subtract the 25t, I'm gonna get 250t equals negative 20. [pause] I see where I made [my mistake]. 275 equals 25t minus 20. Divide by 25. [Calculating] Approximately 12. People. [People?] Ah, days.

When queried about her change of equations, Tara indicated that she was uncertain why it worked.

Cause 275 with p as the number of people. So 275 people, then p times t, right, so that's in the formula equals the 25t minus 20. Right, so I just plug the 275 in, but it didn't work,

so I mean that's what I think it should be. I think it should be the second equation. But I'm not really sure how to use it. [underline added]

When asked explicitly about the notation  $p(t)$ , Tara remarked: "Well see that's what I was thinking. People times time equals. Like the people per day. People per amount of time." The researcher then proposed the problem  $f(x) = 2x + 3$  and asked if  $f(x)$  meant "f times x". Returning to the problem, she re-evaluated her two solutions and commented that her answers now made sense.

Oh I see, you're using t not as time, but as a function. Are you saying the function of P equals. Is that what the problem says. Well, that's a little different then. . . Well, yeah, it makes this one make more sense. . . Right, so then that makes this better. That makes that 12 days.

#### Summary

Tara's understanding of functions appeared to be based on the recognition and application of known procedures. These procedures seemed to lack a conceptual underpinning. For example, her attempts to find the x- and y-intercepts of a linear equation were centered on the recognition of the b position in the slope-intercept form. She did not seem to connect the intercepts with the ordered pairs  $(x, 0)$  and  $(0, y)$ . In addition, Tara did not readily associate ordered pairs with satisfying the equation.

Tara also had a weak and mechanical conceptualization of function notation. Her usage suggested that she associated the symbols with an operation to perform. She was able to compute expressions like  $f(2)$  and  $f(g(2))$ . However, she did not perceive  $f(x)$  as equivalent to  $y$  and as a result she felt that expressions involving  $f(x)$  lacked a range or  $y$  values and hence could not be graphed. In a similar fashion, Tara's definition of function was based on seeing an  $f$  in an equation or visually applying the vertical line test.

Overall, Tara's understanding of functions and graphing seemed to be built on the execution of procedures. She did not easily shift between equations and graphs and so her understanding appeared segregated. She also seemed dependent on her recognition of symbols or forms to motivate her actions. When problems failed to conform to standard notation or when she made an error, she was at a loss. This dependence on form also extended to her understanding of domain and range which she associated exclusively with  $x$  and  $y$  notation. In conclusion, Tara's understanding of functions appeared to be based on unconnected and memorized procedures.

Tom

In the function interviews, Tom was coded correct on 15 problems and incorrect on 5. This coding of 75% correct ranked Tom fourth among the six participants on this category. Tom's comments during these interviews suggested that he held a clear understanding of the concept of function while at the same time he was confused by function notation. Tom easily graphed linear equations and determined slopes, domains and ranges. In addition to function notation, Tom had difficulty with composition of functions and with determining the intercepts of a nonlinear equation. These strengths and weaknesses will be discussed in detail in the following sections.

#### Definition of Function

##### Purpose for Studying Functions

When asked about the purpose for studying functions, Tom responded honestly that he saw none except perhaps for some future use in calculus. He remarked:

Why did you study functions? [pause] What good are they? I really couldn't tell you. I don't know, I imagine it will probably be useful some time down the road in like Calculus or Pre-Cal or something like that. Other than that, I can't really think why.

When asked explicitly about the rationale for function notation, Tom replied:

I don't know. . . . Do you know why? I mean if it's helpful I should know, I guess, what I would be using it for. . . . I was gonna say, I was thinking that it showed that there's a limit. . . . For certain things there's a limit. Just general, like one-to-one functions together, like there's one  $x$  for every one  $y$ . That's the only thing I can think of.

Tom's later comment suggested that he might see functions as a particular type of relationship.

##### Function and Relations

Definition. Tom's explanations and comments were often predicated on the statement, "I don't remember." This was the case when Tom was asked what the term function denoted.

He replied:

I don't even remember what it is. That's with graphing right? . . . So the function of a line is just the equation of that line. . . . I don't remember what specific things are, I just remember how to do those type problems.

While his initial explanation was vague, Tom easily picked out the graphs of the functions in problem #45. Seeing the graphs appeared to trigger Tom's recall of the function concept.

He responded:

Oh, all right, all right. That's. OK, functions, they have to pass a line test, I remember that. . . . It's [graph D] got more than one point on the  $y$ -axis. And this one [graph E] is

a function. It's only got one point. For every 1 point on the x-axis, there's one point on the y.

When Tom returned to examine graph A, he was uncertain as to the meaning of the opened and closed circles on the graph. Once this was explained, Tom concluded that:

I guess it is a function then. Cause for every x there's one y. [pause] For every y there's one x. That's what I was thinking. One. Yeah, for every y there's one x. So. [pause] Yeah that's a function.

From this discussion it was unclear to the researcher whether Tom associated the vertical line test with the relationship of function or one-to-one. To clarify the issue, the researcher drew the graph  $y^2 = x$  and asked Tom if this graph was a function. He replied:

That's not a function. No, because this. [pause] . . . I don't remember, I know it's not a function. That's all I. [laugh] Cause I know it's got two points on the x-axis. . . . Well, two points for x. Each one.

Again the researcher followed up with the graph  $y = x^2$  to which Tom indicated that it would be a function. When asked why, Tom replied with some frustration: "I can't remember rules and stuff." He later remarked that:

Cause its only got one point on the x-axis. I mean it's got different. Because when you draw a vertical line down it, there's only one x point through that vertical line. [x point?] It makes complete sense to me, I can't explain it. I know there's only one point. There's points down here. And they go up and when you draw a vertical line through it, it's gonna go through one point. If you had a line like that [sketches  $x = -y^2$ ], it would go through two and that's not a function. That's how I remember.

While Tom's discussion at times was confusing and his terminology nonstandard, his selection of graphs and his drawings confirmed his assertion that he understood the mechanics of the vertical line test and its connection to functions.

Examples. Subsequent interview questions further affirmed Tom's understanding of the concept of function. On problem #61, Tom readily concluded: "Then 2 and any number all right so, that wouldn't be a function though. . . . Then it would be a one point graph though." Tom's diagram which accompanied his explanation showed that he correctly perceived that a and b would need to be the same for the graph to be a function.

Again on problem #63, Tom was able to quickly ascertain that the set of name pairs would yield a function. Tom utilized a coordinate axes to help visualize the situation. He reasoned:

Yeah, it would [be a function]. Cause it only has one y. One y value, so it would be one straight line all the way across. OK. And Bill might be here, Jane would be here, Dave would be there or whatever.

Tom's graph showed a series of points forming a horizontal line. In a similar fashion he argued with his graph that if the coordinates were reversed that the graph would form two vertical lines which would fail to be a function.

Domain and Range. Tom was fairly facile and articulate in describing the domain and range. He stated: "Domain is the  $x$ , all the  $x$  values, or what  $x$  is and the range is what  $y$  equals, or could equal." He illustrated his definition with the graph  $y = 2$  and concluded that: "so the range would equal 2, and the domain would equal all real numbers."

When considering  $f(x) = \text{square root of } x - 4$ , Tom rewrote the problem as  $y = \text{square root of } x - 4$  and succinctly concluded that:

Well, the domain has to be all real numbers greater than or equal to 4. Cause you can't have the square root of a negative number. Well, you can but it's not a real number. Something like that, I don't know. [Writes  $D = R \geq 4$ ] And the range would equal [pause] all, it could be all positive number. All real numbers greater than or equal to 0. [Writes  $R = R \geq 0$ ]

As these examples illustrate, Tom felt comfortable with the concept of domain and range.

### Function Notation

#### Interpretation of $f(x)$

While Tom worked easily the domain and range, he was uncertain about function notation. He hesitantly suggested that  $f(x)$  was the same as  $y$  saying: "I couldn't [graph it], cause I don't have  $y$  values. I don't know, maybe it's just that  $f$  means a  $y$ ." Once he made this association, he easily graphed  $f(x) = 2x + 3$  by writing  $y = 2x + 3$  and making a table of values for  $x$  and  $y$ . However, in a later discussion on composition of functions Tom failed to recognize expressions like  $g(2)$ . He interpreted  $g(2)$  to mean "g times 2 equals x minus 3." When pressed he responded hesitantly that:

It would be 5 minus 3, is that? Substituting, I don't know. I don't have the faintest idea how. I don't know how to do these. I've never done these before.

Tom's comments were referring both to the specific function expression and to the composition of functions more generally.

#### Composition of Functions

As the preceding comments indicate, Tom did not recall the meaning of composition notation. In fact, Tom remarked that he was absent the day the class discussed composition. Tom also confessed that he had not attempted to read the section on his own and had instead

skipped over it. In the following citation Tom describes his unsuccessful strategy and his confusion with the composition questions on the class quiz. Tom's comments refer to the functions in problem #55:  $f(x) = 1/x$  and  $g(x) = x - 3$ .

This is that one day, remember I told you about the function quiz I thought I was gonna do really well on it. It was the last page that I wasn't there for that day. Like on the test, I was trying to get  $x$  alone. I was like trying to solve it. Like it was  $f$  times  $g$  times 2. So I was like say[ing], "OK, we'll find  $g$ ." I was dividing by  $x$ . Everytime I saw a  $g$  in there, I'd put  $x$  minus 3 over  $x$  in the problem. Then  $f$ , I'd put you know multiply by  $x$ . It would be  $f$ , all right, divide by  $x$ , multiply by a reciprocal. And they'd cancel out, so  $f$  equaled 1. That's what I was trying to do. I don't know what those things mean.

Roughly, Tom's explanation described his misconception of  $f(x)$  to mean  $f$  times  $x$  and his subsequent attempt to solve for  $f$ . His process can be understood through the steps illustrated below:

$$\begin{array}{lcl} f(x) = \frac{1}{x} & \text{and} & g(x) = x - 3 \\ \\ \frac{f(x)}{x} = \frac{1}{\frac{x}{x}} & & \frac{g(x)}{x} = \frac{x - 3}{x} \\ \\ f = \frac{1}{x} \cdot \frac{x}{1} & & g = \frac{x - 3}{x} \\ \\ f = 1 & & \end{array}$$

#### Related Concepts

##### Linear Functions

As discussed earlier, Tom showed some initial hesitation in equating  $f(x)$  with  $y$ . However, once he had made this association, he had no difficulty graphing  $f(x) = 2x + 3$  or  $y = 2x + 3$  by constructing a table of values and by using the slope-intercept method. As an extension question, Tom was asked if either method was better. He responded:

It doesn't matter. Whichever one, I don't know. . . [pause] Well I suppose when you have the  $2x + 3$ , you can do a line without even having to know what  $y$  is.

His comment was taken to mean that through the slope-intercept method one did not need explicit  $x$  and  $y$  coordinates.

When asked about the role of  $x$  and  $y$  in the equation  $y = mx + b$ , Tom indicated that:

$X$  and  $y$  represent the  $x$  coordinate and  $y$  coordinate. . . Cause you can substitute any number in for  $x$  and  $y$ . Cause if you substitute a number in for  $x$ . Say you put 20 in and then it will give you the  $y$  coordinate, whatever it is.

In addition to recognizing the components of a linear equation, Tom acknowledged that:

"not all straight lines" are functions. He cited  $x = 2$  as example where it failed.

### Slopes

Tom also was adept with computing and discussing the slope of a line. He remarked that:

It's a change in  $y$  over a change in  $x$ . So if you have two sets of points (2, 1) and (4, 5), I don't know. Then you take the  $y$  sub 2 minus  $y$  sub 1 over  $x$  sub 2 [minus  $x$  sub 1]. . . It doesn't matter which one you put in. So you can [have] 5 minus 1 over -4 minus 2 equals 4 over -6, or  $-2/3$ . So that gives you the slope.

Tom geometrically described slope as the "angle off of the horizontal." In addition to recognizing the irrelevancy of order when computing slopes, Tom also observed that: "If the points [are] on the line, then any points on that line when you use the slope [or] rise and run method, then you'll get the same slope."

### Intercepts

Tom's discussion of the intercepts in problem #53, revealed that he associated intercepts exclusively with graphs of straight lines. Upon reading the problem  $y = (x - 1)(x - 2)(x - 3)(x - 4)$ , Tom quickly observed that if  $x$  was 1, 2, 3 or 4 that  $y$  would be zero. After making this observation, Tom plotted these 4 points on the  $x$ -axis and connected them with a horizontal line. Next he used these points to compute the slope as 0. He then returned to the original equation saying: "I'm trying to think how I can get it into  $y$  equals  $mx$  plus  $b$ ." Failing to find a way he remarked: "I don't know how to do this." Reflecting further on the problem, Tom concluded that his graph suggested "so  $x$ -intercept is any number, for all real numbers." From his graph, Tom conjectured that the  $y$ -intercept would be (0, 0) since that ordered pair also lay on the same horizontal line. The researcher tried to challenge that assumption by suggesting to Tom that he substitute (0, 0) into his equation. This suggestion revealed a misconception. Tom argued that:

No, you don't use the  $x$  value in that cause you're only finding the  $y$ -intercept. The  $y$ -intercept is where it crosses the  $y$ -axis, right? OK, so no matter what that number is, if it's -85, you don't put (-85, 0) back into your equation. You only put the 85 in. [pause]

The researcher then tried to redirect Tom's attention to the plausibility of the  $x$ -intercept as being all real numbers. The researcher suggested explicitly that Tom try  $x = 1/2$  in the equation. Upon completing the computation, Tom replied, "You can't have half numbers. You can only have 1, 2, 3 and 4. Because if you have half values, then one of these [factors] are gonna end up being 0, so when you multiply by the rest you get 0."

Since Tom was still confused, the researcher then injected the possibility that Tom's graph



was not a straight line. Tom's response was "You can't have x- and y-intercepts for a line that's not straight. . . It wouldn't be a function that you could put in slope-intercept form." Reflecting on his statement, he then remarked: "Of course you could have x- and y-intercepts. If you had a line that was curved." Tom drew an arc from the positive y-axis to the positive x-axis to illustrate his point.

While the conversation continued, Tom never resolved the question about the coordinates of y-intercept or the shape of the graph itself. Tom's thinking appeared to be guided by the assumption that since the question asked for intercepts then the equation and the associated graph must represent a straight line. When this assumption was challenged, Tom acknowledged the plausibility of a nonlinear graph with intercepts. Tom's insistence that the graph was a straight line also seemed to direct his thinking with regard to the y-intercept. He was apparently expecting the y-intercept to be visible in the original equation as it is in the slope-intercept form. The discussion also suggested that Tom was unfamiliar with substituting in ordered pairs to see if they satisfy the equation. He seemed to rely on replacing one variable and solving for the second as his sole technique for generating and verifying ordered pairs. While Tom recognized the forms  $(x, 0)$  and  $(0, y)$  as intercepts, he was not able to utilize this information to generate the intercepts here. Thus, Tom's remarks suggest that his understanding of intercepts was tied very closely to his matching an equation to slope-intercept form and reading the b position.

#### Word Problem

When Tom attempted the word problem in question #59, he read it "people times days equals 25 days minus 20." As a result of this interpretation of the equation, Tom obtained an answer of 23 people in part a. At this juncture, the researcher reminded Tom that  $P(t)$  did not mean P times t. Tom proceeded then to solve part b and remarked:

Can it give me a decimal number? [pause, uses calculator]. So 11.8 people. It said 11, yeah. So t equals 11.8. Wait, that's wrong. That doesn't make much sense really. Oh that's how many day. OK, 11.8 days. All right, that makes sense. I was thinking 11.8 people.

Since the researcher injected a comment on the notation before Tom had tried part b, it is unclear whether he would have realized his own misconception as he solved part b and obtained a negative result.

### Summary

Tom's interviews suggested several strengths and weaknesses in his understanding of functions. Although he could not cite any applications of functions, Tom thought they would be useful in calculus or pre-calculus. Tom's responses seemed to indicate that he had a clear definition of function although his language was confusing at times. In addition to recognizing which graphs were functions, Tom also was able to describe the domain and range of several functions. Tom demonstrated that he could graph linear functions by constructing tables and by the slope-intercept method. He also was facile at computing and describing the slope of a line.

In contrast, Tom was uncertain about the meaning of function notation. This confusion extended to evaluating expressions as well as to composing functions. When discussing intercepts, Tom's comments suggested that he did not believe it was possible or useful to substitute both coordinates of a point into an equation to check the point's validity. As with many of the participants, Tom's remarks revealed areas in which he appeared to develop a strong conceptual understanding and areas in which he seemed to rely on procedures alone.

### Sue

In the function interviews, Sue was coded correct on 16 problems and incorrect on 4. This coding of 80% correct ranked Sue third among the six participants in this category. Sue's interviews showed that she had several areas of strength and weakness with regard to her understanding of functions. While she could apply the definition of function when examining graphs and sets of ordered pairs, she inconsistently interpreted function notation as multiplication and as equivalent to  $y$ . This confusion with the notation proved problematic when she tried to interpret and evaluate composite functions. Sue also seemed to limit her definition of domain and range to the specific  $x$  and  $y$  coordinates she constructed for an equation. Sue seemed to derive insight from her graphs and expected them to be consistent with the equations. These strengths and weaknesses will be discussed in detail in the following sections.

#### Definition of Function

##### Purpose for Studying Functions

Sue frankly admitted that she did not know why the class had studied functions and in fact, she felt that they were irrelevant to her life. She laughingly reported that:

I don't know, so we could do graphs and we could make the teacher happy. I don't know. . . .When are we going to be using functions later on in life? We can use them if we're some kind of chemist or if we're some kind of engineers or something like that. But, I'm not going to go in the grocery store and use functions to buy my food. I mean I'm gonna use multiplication and division and addition. I know they're suppose to be helpful, but I'm not sure why. I don't know.

Sue went on to compare the study of mathematics to the study of chemistry. Both helped one to be intellectually well rounded but often lacked relevancy.

You're always learning. Like [in] chemistry we're learning things that probably in 10 years are gonna be totally wrong anyway. Even if you get into college, they might be different by the time we're seniors or something. I don't think it's very useful. It's good to be well rounded and know all these things, but why? . . .Math is never gonna change though. They'll just find new theorems to make us learn. More theorems to test us on. To me some of the stuff we learn is totally irrelevant towards life. I mean who sits down and goes well I'll do my daily quota of functions today. We do crossword puzzles, not functions. [laugh]

When pressed further Sue admitted that:

I don't know. I guess I just never really understood exactly what a function was. I mean I did them and I memorized basically how to do them, but I never knew when we use it. I just used it. [laugh]

The researcher then asked Sue if she was bothered by not knowing why she had studied functions. Sue was equally forthright in her response. She replied:

Only when people ask me. It doesn't bother me if they never ask. [laugh] No, it doesn't bother me that much. I mean if it were something more important, I would have probably asked.

Sue cited division as an example of a topic which was important to understand the rationale for.

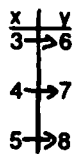
### Functions and Relations

Definition. When asked for a definition of function, Sue remarked:

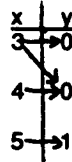
Oh, no, that question. All right. We had a question, what a function was. The test helped because she asked questions you know, and figure this relation and then by looking at the problem you could figure out "Oh, that's not a function." [laugh] I remember that. . . [pause] There was only one number in the domain that corresponded to only one number in the range. [pause] It's either the function or relation so when you had an x and a y. Each one goes with one like that.

Sue illustrated her discussion with the following two examples.

function



relation



Sue's diagram correctly identifies a function, but her example of the relation is confused. Her intent had been to associate an x with two y values yet these were not distinct and so the example is incorrect. When asked for clarification, Sue restated her verbal definition: "There is one number in the domain that corresponds to only one number in the range. I think that's what a function is. Watch it be what a relation is."

Sue showed a similar lack of confidence in her recall of the vertical line test. She was uncertain whether the test signified a function or relation. She commented:

God. One of them you can prove by using the vertical line test. To prove if it was a function or a relation, but I don't know which one. . . You take a point and you make vertical lines like that. And it passes through 2 points, then it's not a function or relation. I think it's not a function. It's like probably a relation because it hits that twice.

After the researcher affirmed Sue's supposition, she quickly indicated that graphs C and E were functions and graph D was a relation. Sue was uncertain how to interpret the piecewise function of graph A. When the open and closed circle were explained, Sue labeled graph A a function. The asymptotic function in graph B proved problematic for Sue. She believed that graph would eventually cross the y-axis both above and below and hence would fail to be a function. She argued that:

This one right here because it's getting closer to the y [axis] and will eventually cross it. And this one will eventually cross it. So if you made a vertical line 2 would go through so I guess that would be a relation. . . . I remember when they said you never made them cross the line when they made pictures of it. But in theory since it continues, it's gotta cross it eventually just because of the rule. . . . If you have a straight line, you have another line that curves even slightly towards that other line, it's gonna hit it eventually and intercept it.

It was the researcher's supposition that Sue was attempting to recall a theorem from geometry about two lines in a plane either being concurrent, parallel, or intersecting. To challenge her conclusion the researcher labeled the graph  $f(x) = 1/x$  and suggested that if the graph crossed the y-axis that it must have a y-intercept. This suggestion, however, was problematic in itself. While Sue noted that the expression would become  $1/0$ , she claimed "1 over 0 is the same thing as 0. Cause you can't divide a number by 0." The ensuing discussion eventually resolved this misconception and Sue concluded that the graph would not intersect the y-axis and therefore the graph would be a function. Although Sue reached this correct conclusion, the discussion with the researcher was very directive in nature.

Examples. Sue's discussion of the definition and graphs were essentially correct but her hesitancy suggested an uncertainty in her knowledge of the function definition. Sue's analysis of problems #61 and #63 further supported the assumption that she understood and could use the definition of function. Like other participants, Sue quickly asserted that the two points lie on the same vertical line and thus could not represent a function. When queried how she might guarantee that the points would permit a function, she suggested changing the initial coordinates to 2 and -2. When the researcher disallowed changing the x coordinate, Sue reflected a moment and then said: "a and b could be the same number."

On problem #63, Sue recognized that the set of name pairs would form a function. She visualized the situation by assigning 3 to the name Brown and 1, 2, 4, and 5 to the names Bill, Jane, Sarah and Tom. Sue was equally confident about her response when asked if the reverse order also would yield a function. She remarked:

No. Because if you have a number, then you can't have the same number in the domain go [to] more than 1 number. This is a family name right here and those are the kids. You're gonna have the same name go to all of them. You can't have that. Only one. [pause] Only one number in the range can go to one number in the domain.

The researcher followed up these two questions by asking whether the original ordering would work if both families had a child named George. Sue noted:

They [both Georges] have the same number. . . .(10, 6) [6 - Jones], the kids came first I assume. Then that would be a (10, 3). [pause] So that wouldn't work. If there was only a George. It wouldn't work, because it would be the same number in the domain and you can't have that. As long as all the kids are different numbers, it's fine.

While Sue's initial comments appeared inconclusive, her responses to question #61 and 63, suggested that she had connected the vertical line test to the definition of function and to the relationship among the ordered pairs.

Domain and Range. Sue's conceptualization of domain and range seemed to be tied to the coordinates of explicit ordered pairs. When asked to define these terms, Sue stated: "The domain is the x-coordinates and the range is the y-coordinates." Sue illustrated her definition by examining two ordered pairs (2, 4) and (6, 10) from which she concluded that the domain was 2 and 6 and the range was 4 and 10. The researcher then asked Sue to draw a graph and give its associated domain and range. Sue selected and drew the line  $y = x$  on which she labeled 4 ordered pairs on the graphs (1, 1), (2, 2), (3, 3) and (4, 4). She remarked that: "The domain would be, you have to keep going, so it would be, it's basically all the numbers." While Sue's verbal remarks indicated all numbers, she wrote the set  $\{ . . . , 1, 2, 3, 4, . . . \}$ . When queried about her notation, she explained: "The dots mean it's all numbers before it and numbers after it." Further discussion revealed that Sue was thinking of "just whole numbers." The researcher then asked Sue if she could name a point on the line between her dots. After this question, Sue realized that the domain would need to include fractions and decimals as well as whole numbers. She indicated this set by a script R. When asked to define that notation, Sue responded:

Just every number there is, I think. I don't remember ever defining it in class. We did it in the very beginning and I remember I was confused about that. So whenever I think about it, I just think about whole numbers though.

As a follow-up, Sue was asked to consider graph E from question #45. The researcher marked the x- and y-axes with specific numbers. Sue labeled several ordered pairs on the graph and indicated that they constituted the domain. When pressed for the domain of the whole graph, Sue remarked: "I have no idea. . . .I don't remember doing pictures like this. But I don't remember how we got them or how we figured them out." Sue had a similar response when asked to consider the graph  $y = x^2$ .

Sue's initial comment when examining the equation  $f(x) = \text{square root of } x - 4$  was that:

"We haven't done the domain and range with roots, whatever." She proceeded to square both sides to eliminate the square root but indicated:

I have no idea what I'm doing. We haven't done anything with square roots in finding the domain and range of those yet. I don't remember. All I can do is just make up a theorem to just do it, that's all I can do."

When Sue returned to the question a week later, her response was correct and succinct. She also revealed that during the previous week her mathematics class had discussed square roots. While she was able to ascertain the domain and range of the original function, she was confused by the fact that  $y^2 = x - 4$  did not yield the same ordered pairs and hence the same domain and range.

### Function Notation

#### Interpretation of $f(x)$

Throughout the interviews Sue's understanding of function notation vacillated. Sue began the discussion by describing  $f(x) = 2x + 3$  as an operation. She stated:

Well basically, it's asking what the  $x$  equals. Shouldn't there be more information? What  $x$  equals. . . . If it was asking like the function of  $x$  and then you find out what the equation equals out [to] or whatever.

As the citation suggests, Sue was expecting to be given a specific value for  $x$  for which she could substitute and get an answer. However when asked to graph the function, Sue first solved for  $x$  getting the equation  $x = 3/2$ . She indicated that:

I'm not sure exactly, but. [pause] This is a big guess. [pause, student taps pencil]. The only way I could think about would be like that. I got  $x$  by itself, so  $x$  I had equals 3 over 2. The rise over the run or whatever. It's the only thing I could think about doing to figure that out. So I went over 3 and up 2, over 3 and up 2. [Starting] from the origin right there.

When the researcher suggested that she check one of her points from the graph into the equation, Sue remarked: "I don't understand how you can have the equation not equal anything." In the ensuing discussion, Sue eventually graphed correctly  $y = 2x + 3$ . When explicitly asked what relationship existed between  $f(x) = 2x + 3$  and  $y = 2x + 3$ , Sue responded:

[pause] I don't know. All I could think of would be to find out what the domain and range is. I'm not really sure how. I don't understand why we do it. It was one of those things that you sorta learned and you just did it. [underline added]

Sue then observed that:

Like all the numbers in this line are gonna work in the equation. [Sue indicates both equations]. . . I always thought the  $f$  or  $x$  was basically the same as saying  $y$ . So it would work. . . . It makes sense. If you say to yourself that the  $y$  is the same thing as  $f$  of  $x$ . So you just substituted  $y$  for  $f$  or  $x$ , it's gonna be the same. . . . I guess what I was first thinking

was  $2x$  plus 3 equals 9 or something. You have to figure out what the  $x$  is equal to. And so that's what I was thinking about equaling to something. But if you're doing graphs and anything then you don't need to have a number that's equal to, cause you can just make a domain and a range and figure it out from that way.

Sue's comments suggested that she perceived function notation as indicative of a memorized procedure. However, Sue was vague in her description of how that procedure was executed. While finally acknowledging that  $f(x)$  was equivalent to  $y$ , it was unclear the extent to which she believed and could utilize this equivalence. Her responses to the questions on composite functions in the next sections confirmed that Sue had only a vague recall of how to evaluate the notation and how  $f(x)$  was related to  $y$ .

#### Composition of Functions

When asked about the meaning of the notation  $f(g(x))$ , Sue explained:

When you have this it means that, what they usually do is give you a given number for each variable here. . . I'm trying to remember if you just put them in and multiply them out or not. [pause]. . . The product of these two. I can't remember.

Sue's evaluation of the composite functions in problem #56, showed that she did interpret the parentheses as multiplication. Specifically, Sue evaluated  $f(g(2))$  by distributing the 2 to both functions. That is,  $f(g(2)) = f(2)$  times  $g(2) = (1/2)(1) = 1/2$ . While Sue remembered having seen these problems and that they had been easy for her, she was uncertain of her technique. When asked to consider  $g(2)$  separately, Sue computed the value as  $2 - 3 = 1$  which she then substituted into  $f$  getting  $1/1 = 1$ . Having recalled the procedure, somewhat, Sue went on to evaluate the other parts of the question. However, she indicated on part c that  $f(g(3)) = f(0) = 1/0 = 0$ .

Although Sue successfully described  $h(x)$  as  $1/(x - 3)$ , she was confused about how to define the domain. When it was suggested that she "plug in 4 for  $x$ ," Sue remarked:

I guess you could. But there's no  $y$ . Yeah, there is, sorry. I don't know. [pause] I always thought when it said  $f$  of  $x$ , it was the same thing as saying  $y$ . Just that it was more of a technical language.

Sue then proceeded to evaluate  $y = 1/(x - 3)$  at  $x = 4, 3, 5, 6, -4$  and  $-3$ . From her results Sue concluded: "it  $[x]$  could be any number." As before, Sue equated  $1/0$  with 0 and so saw no difficulty in the domain.

Sue's discussion of the meaning of  $f(g(x))$  initially suggested an interpretation of multiplication. While she eventually evaluated the composites correctly, her misconception



proved resilient. When asked a week later if  $f(g(x)) = g(f(x))$ , Sue returned to her first idea of multiplication of functions. She illustrated her view by examining the two composites with  $f(x) = 2 - x$  and  $g(x) = 5x + 1$ . She explained:

Usually they have a number. If you don't know what the number is, you can't say they're equal. . . . Can't remember how to do these. . . . If they're [the functions] different and if you multiply them in different orders. Oh wait. [pause] I guess it would be the same, actually. Because like it doesn't matter which way you're multiplying the orders. If I'm thinking of the right way. . . . I don't know what it means. I never understood exactly what they meant. I could do it, but I never understood. I never questioned why.

Sue wrote  $f(g(x))$  as  $(5x + 1)(2 - x)$  and evaluated  $f(g(1))$  as  $(5 + 1)(2 - 1) = (6)(1) = 6$ .

This return to the idea of multiplication demonstrated the tenuousness of Sue's understanding of function notation. Sue, herself, also admitted that the notation was unclear to her and that she had managed in class by merely applying procedures. Sue's confusion about function notation also proved problematic in word problem #59. Details of her solution to that problem will be discussed in a later section.

#### Related Concepts

##### Linear Functions

As discussed in the previous section, Sue had difficulty interpreting and using function notation. This difficulty carried over to her graphing of linear functions. Sue solved  $f(x) = 2x + 3$  for  $x$  getting  $x = 3/2$ . She began at the origin plotting points by going over 3 and up 2. When it was suggested that she check an ordered pair in the equation, Sue confessed that she had no idea how to place the numbers into her equation,  $x = 3/2$ .

In contrast, Sue successfully graphed  $y = 2x + 3$ . Initially she utilized a slope of  $2/3$  and started again at the origin. She then changed her mind and used a slope of  $3/2$ . After following through on the suggestion to make a table, Sue realized that her graph of  $y = 2x + 3$  was incorrect and that the slope was 2 and y-intercept 3. She commented:

I remember the definition. Right there. [writes  $y = mx + b$ ] So that the  $m$  was the slope and that 2 equals the slope. And  $b$  is the y-intercept. That makes more sense. It's coming back. Cause that's where you start from and then you go up 2 and over 1. I get it.

When asked specifically how the graph of  $f(x) = 2x + 3$  and  $y = 2x + 3$  were related, Sue's immediate response was: "I have no idea." Eventually she suggested that: "f of x was basically the same as saying y." She also reported that the points in the table "will fit into the equations." While Sue held this expectation for the points, she was unable to carry through the

idea in problem #53 in order to get the intercepts.

While Sue could identify the slope and the y-intercept from the equation  $y = mx + b$ , she was uncertain about the role of  $x$  and  $y$ . She remarked:

I have no idea. . . .How the  $x$ 's and  $y$ 's are used for making a line for the coordinates. Why is the  $b$  there. . . .I don't know, to me the question [is] sorta irrelevant. . . .I can't think of why you even need them. . . .It's like why do we breathe? [laugh]. . . .I guess in my mind, it's just always been given to have them there.

Sue's remarks convey not only uncertainty but also suggest a mechanical view and use of the slope-intercept equation.

### Slopes

Sue offered the analogy of a ski slope to describe the meaning of slope. She described:

When you think of a ski slope, it's something that has [pause] steepness sort of. Going uphill or downhill, depending which way you look at it. It's hard to explain. . . .Something with a grade.

When asked to determine the slope from the points  $(0, 1)$ ,  $(2, 4)$  and  $(6, 10)$ , Sue immediately graphed the points. She then determined the slope by counting over the number of  $x$  units and the number of  $y$  units between two of the points. She explained:

I went from one point to the other point. I did the rise which is 1, 2, 3, then you go over 2 for the run.

Sue also realized that the slope would remain unchanged if she had computed it from  $(0, 1)$  and  $(6, 10)$ . She observed that:

You would have gotten a fraction that could be reduced to  $3/2$ . Cause it's a straight line, so the slope wouldn't change.

When asked if she knew another technique for determining slope, Sue recalled the slope formula. She explained the formula saying:

If you subtract the first one [y-coordinate] from the second one, you're gonna come out how high you went. . . .And that would get your rise. . . .The  $x$ -coordinate is right here, would find out what your run would be. You subtract those. . . .And it would give you your rise and your run.

While Sue's understanding of the equation  $y = mx + b$  appeared perfunctory, her understanding of slope seemed well connected. She quickly associated the action of rise over run with the coordinates of the points and with the slope formula.

### Intercepts

Sue had difficulty determining the intercepts of the non-linear equation in problem #53.

This difficulty seemed to arise from her tacit assumption that the graph was linear and from her inability to substitute ordered pairs or a single  $y$ -coordinate into the equation. Her discussion also revealed several other points of confusion. Sue asked several questions before she began. She wanted to know what the capital  $y$  in the equation designated and if the factors represented ordered pairs. When satisfied that this was a typical equation, Sue proceeded to expand the factors. She noted as she completed the multiplication that:

I have no idea if I'm doing this right or not oh. [pause, continuing to expand] Oh. [sigh, pause] God, it doesn't look right. I don't know, it's just too big. . . . I have no idea what to do with this.

While Sue could describe the type of ordered pairs associated with the  $x$ - and  $y$ -intercepts, she could not capitalize on that information. She did not know where in the equation to substitute the zero. This confusion also extended to the equation  $y = 3x + 2$ . She recognized the  $y$ -intercept as 2 from the  $b$  position in the slope intercept equation but could not determine the  $x$ -intercept. She acknowledged:

[laugh] Basically. [pause] I don't remember how to do these things, God. I mean I know that it's 0, but I don't know where I should put the 0. Cause it doesn't seem logical. I don't know why. I mean all I can think about would be putting 0 equals 3x plus 2. But I don't think it's right. I don't ever remember doing it before. I don't remember doing 0 equals 3x plus 2 to the problem.

When asked explicitly if it was legal to put in a number for  $y$  and figure out  $x$ , Sue replied: "Yeah. [pause] It would be harder, but you can still do it." Sue then constructed a table by substituting values for  $x$  into her expanded equation. Her selection of  $x$  as 1, 0 and 2 revealed two points with  $y$  values of 0. Sue pondered this situation and said:

It doesn't look right, something's wrong. There's two 0's in the range. . . . I'm just gonna check it and see what it looks like. I mean, I'm just freaking out over nothing. [laugh]

Sue used a graph as her check. She commented:

Oh great. [pause] I went the wrong way. [pause, laugh] Something's not wrong. I'm expecting a line.

The researcher then asked Sue how she could know if the equation was linear. Apart from plotting points she said "I have no idea." She then returned to calculating  $x$  and  $y$  coordinates. When these additional points failed to confirm a straight line, she purposed that the graph might be a parabola since it was possible for that graph to have multiple intercepts.

Sue's discussion of linear equations, slopes and intercepts seemed to be centered around her recall of formulas and procedures and her construction of graphs. Her use of the slope-

intercept equation seemed fairly rigid. Sue could recognize the slope and y-intercept, and substitute in x values but was baffled when asked to substitute in a y value or an ordered pair. This type of usage suggested a procedural understanding. In contrast, her usage of slope showed a connection between the graph and the formula. Much of Sue's motivation for and analysis of situations seemed tied to her graphs. Although Sue had very limited techniques for constructing tables and subsequently graphs, she appeared to have the expectation that her graphs should be consistent with the equations and standard patterns. Sue also continually monitored her process checking them against her recall of familiar procedures.

#### Word Problems

Sue's confusion with function notation proved problematic in her solution to the word problem in question #59. She interpreted  $P(t)$  as P times t--"people times days." Sue quickly solved part a using this interpretation but stopped abruptly when she realized that part b would yield a negative number of days. While Sue acknowledged that people times days had no real world significance, she did not abandon her interpretation.

Sue tried to reconcile this situation by using a proportional argument. Two hundred thirty people became ill in ten days, so two hundred-seventy people should necessitate an additional day or two. She approximated the solution to be 11 days.

When the researcher insisted on an exact answer, Sue rechecked her calculations for both parts of the question. It was only when the researcher explicitly suggested that  $P(t)$  represented function notation that Sue acknowledged that possibility. She then replaced  $P(t)$  with y and quickly solved both questions. Sue noted that her approximation of 11 days was indeed close to the actual solution. Sue also checked her solution in part a by replacing y with 230 and solving for t which yielded her original value of 10 days.

#### Summary

Sue freely acknowledged that she saw no use for function notation or functions in general. She also claimed that this lack of relevance did not disturb her. She accepted functions as just another topic in Algebra II/Trigonometry. Sue could verbalize a definition for function and apply the vertical line test. She was also able to extend the concept of function to the abstract situations in problems #61 and 63. In addition Sue was able to graph linear functions and easily determine slope both from the graph and from the formula.

In contrast, Sue was inconsistent in her use of function notation. She interpreted  $f(x)$  to mean  $f$  times  $x$  and to be equivalent to  $y$ . This confusion hindered her conceptualization of composite functions. When specially reminded that  $f(x)$  meant  $y$ , Sue was able to evaluate the composites at given a value of  $x$ . Another pervasive problem for Sue was her inability to check an ordered pair in an equation or to solve for  $x$  in an equation by first substituting in for  $y$ . This inability made it difficult for Sue to analyze her own mistakes. While Sue was hampered by this inability, she still attempted to verify her answers by using a graph. Sue's concept of domain and range seemed tied to specific ordered pairs rather than to the whole graph. Intercepts also seemed linked to the  $b$  position in  $y = mx + b$ . Sue was unable to utilize her knowledge of the coordinates of the intercepts to help her generate the intercepts in problem #53 and in  $y = 3x + 2$ .

### Steve

In the function interviews, Steve was coded correct on 14 problems and incorrect on 6. This coding of 70% correct ranked Steve fifth among the six participants on this category. He seemed to have a strong grasp of function notation but no recall of the definition of function. He also tacitly associated functions with any equation involving two variables or  $f(x)$ . Graphically he connected function with continuity. With regard to linear equations, Steve had developed an extensive and well-connected conceptualization. Details are provided in the following sections.

#### Definition of Function

##### Purpose for Studying Functions

Steve described the purpose of a function in terms of defining a relationship between two numbers. This relationship then facilitated graphing. He remarked:

Well, a function defines something. . . .In order to graph anything, you've got to have a function. Because you can't graph anything with one number. So to figure out the other number you've got to find a relationship between the two numbers which is the function.

When asked what function notation could do that the  $y$  notation could not, Steve suggested that:

If you get into something bigger. If you're graphing in 3-D rather than just the  $x$  and the  $y$ , then I suppose it might be a little bit confusing, unless you had a set function. Cause if you had to find something, say you had to find  $z$  through  $y$  then the function wouldn't necessarily be  $y$  or  $z$ . It would be a little less complicated to do it that way, I suppose. I can't think of any other reason.

##### Function and Relations

Definition. Steve's definition of function was vague. He suggested that it applied to a relationship some between  $x$  and  $y$ , that it was related to graphs, and that it was written as " $f(x)$  equals some expression involving  $x$ ." His comments never specified the particular relationship between  $x$  and  $y$ . He stated:

I've been trying to figure out the actual word. Figuring out what it means and then how the meaning applies to math. A function would be a use or something like that. The function of  $x$  equals  $y$  so that kind of blows away that theory. . . .The most general way I've seen it used is the function of  $x$  equals  $y$ , something of that form. . . .Something you do to  $x$  so that it equals  $y$ . I guess there's a relational type thing. Like  $y$  is related in this way to  $x$ . . . .Graphing can mostly be expressed as a function. If you're graphing a polynomial, then you got a function of  $x$  equals.

When asked explicitly for an example of a function and nonfunction, Steve selected  $P(x) = ax^2 + bx + c$  as a function and  $x = 6$  as a nonfunction. He claimed the later was not a function because "that's just a statement." As Steve tried to explain his rationale, he retracted

his claim saying:

I suppose anything could be a function. if you stop to think about it. That 6 could be 2 times 3. . .  $f(x) = 2x$ . I don't know. . . It's a function of y equals x and x equals 6. . . So, I can't think of anything that wouldn't be a function. . . Depending on how you look at it, anything can be a function.

Steve's comments suggested that any relationship that could be written as  $f(x)$  equals or any equation in  $x$  and  $y$  would define a function.

Steve also failed to recall the definition when he examined the graphs in question #45.

Here he seemed to base his judgement on the connectedness or the continuity of the graph.

He labeled graphs A, B, and E relations, D a function and C undecided. Steve reasoned:

I was thinking of what is not a function. . . A function is a relation, a relation is not a function, something like that. A relation of two things are related. . . I know that one is a function. . . For one thing, it's connected, it's all one thing. . . These are relations cause they're related. . . But if it's a function, it's all one thing. . . Let's see, I know the circle is a function because the equation for a circle is a function. The function of  $x$  equals, I can't remember exactly what it is now, but I know it's a function. I'm just trying to think. That's the problem with math. Cause math is very quantitative and it's tough trying to stick qualitative things to it. Trying to describe it in something other than just numbers. [underline added]

Again when asked if all straight lines are functions, Steve reiterated his position that any relationship that could be described mathematically with an equation was a function. He commented that:

Any straight line would be a function. . . Horizontal line or vertical line or diagonal are OK. . . Once you stuck a coordinated plane you could define it with a function. That's analytical geometry. Anything that can be defined with math.

At this juncture the researcher mentioned the phrase vertical line test. Steve acknowledged only a vague recollection of it saying:

It's something about passing between one or two points of the vertical line. I can't remember exactly what it was. I know it's back there in my mind somewhere. I knew it at one time. . . But things like that I just kind of store until I need them and if I come to a situation where I need that, I would look it up, then I would use that. And if I used it often enough, I would remember it. And then as soon as I stopped using it, it would just go back into storage. Right now, it's in storage.

Examples. Steve's discussion of problems #61 and #63 did not vary from his earlier thought on the meaning of function. On problem #61, Steve insisted that for (2, a) and (2, b) to be on a graph of a function that a and b needed opposite signs. The remainder of his explanation focused on trying to ascertain a specific equation that for  $x$  equal 2 would yield two  $y$  values which differed only in sign. The researcher attempted to challenge Steve's conclusion that a and b were opposites by drawing the graph of  $(y - 2)^2 = x$ . Despite this graph, Steve

remained insistent that his interpretation was the correct one.

On problem #63, Steve's first reaction was that this description could not yield a function because it could not be explicitly written as an equation. When the researcher suggested that a defining rule could be to assign to each first name a person's last name, Steve reconsidered. He then argued that: "I can't think of anything that isn't a function."

Steve's comments throughout these questions suggested that he viewed functions as synonymous with continuous graphs and with any mathematical equation. He also remarked that he did not clearly remember the distinction between function and relation and in fact by his definition all equations would be functions.

Domain and Range. Initially Steve was uncertain about the meaning of the terms. He reported:

lck! [pause] Domain and range. I know what it is but I can't quite remember. [pause]. . .It's gotta be something with relations and functions. . . .I've got a definition in my head. . . .Every point on y corresponds with the unique point on x or one of them for something. And I know it's got a good bit to do with domain and range, but I can't remember exactly what the relation is.

A week later when Steve returned to the question he replied:

In some cases one of them is x and one of them is y. I think x is the domain and y is the range. For some reason I think x is the domain and y is the range. For some reason I think that it isn't true in all cases. For graphing purposes the domain is x and the range is y. [pause] I think.

Steve was able to apply somewhat these meanings to the example  $f(x) = \text{square root of } x - 4$  in question #52. Steve quickly noted that x needed to be greater than 4. He reasoned:

If you were gonna graph it, it would be easiest to [with] numbers greater than 4. Cause it's negative numbers then you'd end up with imaginary numbers which are rather hard to graph. I'd imagine that would be pretty difficult, cause they don't really exist. They're more theoretical.

Steve also noted that if  $x = 4$  then the range was 0. However, he was uncertain about how to find the range of the function generally. When shown a sketch of  $y = x^2$ , he observed that in that position that the parabola had no negative y values. Although he made this observation, it was unclear to the researcher if Steve realized the connection between the graph and the range of the function. To further check Steve's understanding, the researcher sketched another graph. This time a line with a hole at (3, 4). He reported that:

The domain would be x, the range would be [y]. . . .X and y are numbers greater than 3, 4 and less than 3, 4.



Steve confirmed in this last example that he recognized the domain and range from the graph:  $3 < x < 3$  and  $4 < y < 4$ . In the discussion, however, he did not return to the original question about the range of  $f(x) = \text{square root of } x - 4$ .

### Function Notation

#### Interpretation of $f(x)$

As noted in the previous sections, Steve readily identified  $f(x)$  with  $y$ . When discussing  $f(x) = 2x + 3$ , Steve wrote underneath the equation  $f(x) = y$  and constructed an  $x - y$  table. Steve also evaluated expressions of the form  $f(2)$  and associated the answer with  $y$ . Steve usage of the notation extended as well to the composition of functions.

#### Composition of Functions

Steve explained the composite  $f(g(x))$  as:

So you've got  $g$  of  $x$  equals  $y$ , then  $f$  of  $g$  of  $x$  would equal  $z$ . . . I suppose that's a means of defining three different numbers with just one number. That's one variable. [pause]

Steve illustrated this composition process by selecting  $f(x) = x + 4$  and  $g(x) = x^2 + 2$  which he evaluated at  $x = 1$ . Although his explanation of how he evaluated the composition was clear and correct, his written work showed the equation  $f(g(x)) = g(x) + 4$ .

Steve also was facile at evaluating the compositions in problem #55. He quickly moved through the questions, computing the values mentally. On  $f(g(3))$ , he marked the composition undefined saying:

One over zero. It is undefined. [pause] Cause 0 can't be split into fractions. Zero is not a fraction of anything.

Steve was uncertain about the domain and range of  $h(x) = f(g(x))$ . His confusion arose from what he saw as two domains and two ranges. The function  $g(x)$  had a domain of  $x$  and a range of  $y$  or  $g(x)$ , but then  $g(x)$  or  $y$  then became the domain of  $f$ . Steve tried to reason through his dilemma by visualizing which variables would be on the coordinate axes. he remarked:

I'm not sure what the domain is. [pause] It could be either  $x$  or it could be function  $g$  of  $x$ . I'm not sure which. Mostly because I'm really not sure what the domain is. Cause I know how to use [it], I just use [it] in graphing. [pause]. . . Let's see, what would you graph it against. . . I guess  $x$  would be the domain. I'm not really sure. [pause]

Steve did not resolve his question about the range. Asked to evaluate  $h(5)$ , Steve executed a series of equations which he solved for  $x$ . He began by setting  $5 = f(x)$

(e.g.  $5 = 1/x$ ) getting  $x = 1/5$ . Next he wrote  $1/5 = g(x)$  (e.g.  $1/5 = x - 3$ ) getting  $x = 16/5$ . To clarify Steve's thinking, the researcher asked if  $h(5) = ?$  was the same as  $h(?) = 5$ . Steve indicated that  $h(5)$  was claiming that 5 was in the range and that the statement was asking for the domain. He made an analogous claim for  $h(?) = 5$ . The researcher then asked Steve to reconsider his earlier work when he evaluated  $f(g(2))$ . Steve indicated that he was unsure whether his answer of -1 represented the domain or the range. He commented:

I'm not sure. I think I got out of that an answer for the range. [pause] I'm not really sure. Cause I'm not really sure what domain and range really is.

While Steve remained unresolved about the domain and range, he easily computed the general expressions for  $f(g(x))$  and  $g(f(x))$  for the functions from problem #55:  $f(x) = 1/x$  and  $g(x) = x - 3$ . He then algebraically checked to see if the two expressions were equivalent. His work and rationale follow.

$$\frac{1}{x-3} = \frac{1-3x}{x}$$

$$x \neq -3x^2 + 10x - 3$$

According to that, no it wouldn't [be equal]. Because that's just  $x$  and that's  $x$  times a few different things. So I guess it wouldn't.

Steve's comments and discussion demonstrated that he identified  $f(x)$  with  $y$  and that he was adept at evaluating functions at a given  $x$  value. This facility with the notation also extended to the composition of functions. With compositions though, he was uncertain as to which variable represented the domain and range.

### Related Concepts

#### Linear Functions

Steve appeared at ease with linear functions. He easily constructed a table of values for  $f(x) = 2x + 3$  and recognized the equation as linear. Steve also succinctly described the components of  $y = mx + b$ . He stated:

Let's see.  $m$  equals slope,  $b$  equals  $y$ -intercept. And, I guess, then  $x$  and  $y$  would be any 2 points on the line defined by  $y = mx + b$ .

#### Slope

Steve was also succinct in his description of slope. He explained:

I remember this one cause I use it a lot. Slope is rise over run, mathematically. . . .The number of units something goes up for every number of units that goes forward.

Steve was able to tie this verbal description to his recall of the slope formula and use it to calculate the slope between two points in problem #50. Steve also recognized that the slope "should be the same cause if it's on the same line, unless the line curves, it's gonna have the same slope."

### Intercepts

Steve applied his knowledge of the coordinates of the intercepts to help him locate the intercepts. He noted immediately that if  $x = 0$  then  $y = 24$ . To find the x-intercepts he replaced  $y$  with zero and expanded the factors. Once expanded Steve attempted to use synthetic division to locate the roots. He indicated that he had recently learned that technique in class. However, Steve made several arithmetic errors when expanding the factors. These errors were coupled with the misconception that the roots came from all the coefficients of the polynomial. Thus his attempts failed to reveal the roots. When the researcher suggested to Steve that he return the original equation, he quickly noted the x-intercepts would be  $(1, 0)$ ,  $(2, 0)$ ,  $(3, 0)$  and  $(4, 0)$ .

The remaining discussion focused on the graph of the function. Steve proposed that the graph would be a parabola turned on its side with a vertex at the y-intercept. When asked about the multiple x-intercepts, Steve argued that the graph would coincide with the x-axis from  $x = 1$  to 4. When asked to compute the function's value at  $x = 2 \frac{1}{2}$ , Steve modified his claim and suggested it might "look like a bunch of waves."

### Word Problem

Steve's ease with function notation facilitated his solution of the word problem in question #59. He mentally substituted the values for  $t$  and  $P(t)$  in the respective places and solve the equation to answer both questions.

### Summary

Steve's interviews revealed areas of understanding and confusion. One area in which Steve showed an ease was with function notation. He not only associated  $f(x)$  with  $y$  but could evaluate functions at a given  $x$ . His understanding also extended to composition of function. He clearly articulated the relationship between the components in that notation. That is, in  $f(g(x))$ ,  $g(x)$  was both equivalent to  $y$  and to the new independent variable of  $f$ . In addition to function notation, Steve easily discussed the components of  $y = mx + b$ , graphed a linear equation, and computed a slope. He also recognized that slope is constant and independent

of the particular points used to compute it.

In contrast to his knowledge of the notation, Steve had no recall of the concept of function. He implicitly associated a function with connectedness or continuity and with any equation involving  $x$  and  $y$  or  $f(x)$ .

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## CHAPTER VIII

### DISCUSSION AND CONCLUSIONS

#### Discussion

The preceding chapters have presented the theoretical framework, the data on the participants' beliefs about mathematics, and their knowledge of functions, and the analysis of the classroom environment. This chapter will tie these separate components together. Recall that beliefs have been theorized to serve three basic and general functions: (a) they assign meaning and interpretation to experience, (b) they provide a basis for predicting future events, and (c) they organize thoughts into conceptually related clusters.

Research has suggested that beliefs influence both how mathematical knowledge is acquired and how heuristics are employed in problem solving (Cobb, 1985; Kouba & McDonald, 1991; Schoenfeld, 1985). Beliefs are theorized to influence problem solving by establishing expectations about what constitutes a problem and about what content and context clues are relevant. In addition, beliefs are theorized to influence knowledge acquisition by establishing expectations for what is valued and attended to in an experience, and for how that knowledge is to be utilized in the future.

Skemp's writings (1987) suggest that what students value in a mathematics lesson reflects differing belief systems which in turn give rise to differing knowledge structures. A student with an instrumental view of mathematics expects mathematics instruction to focus on rules and their execution, with results being evaluated right or wrong. This student also expects to learn mathematics through the memorization of the appropriate method for each type of problem. In contrast, a student with a relational view expects mathematics to be rational and knowable, and expects to actively reflect on that knowledge.

The knowledge structures associated with these dichotomous views of mathematics differ markedly. Instrumental (or procedural) knowledge emphasizes exclusively the execution of rules. Symbols in this type of knowledge are viewed as objects to be manipulated. Procedures are not integrated and do not generalize to other situations. In relational (or integrated conceptual)

knowledge, procedures are supported by the underlying rationales. Symbols also develop meaning and procedures are more easily remembered and recalled.

In addition to the proposed relationship between beliefs about mathematics and knowledge structures, Skemp (1987) believed that the assimilation of new knowledge involved the acceptance of that knowledge by the student. This acceptance can occur in two ways. The student can accept the validity of the new information on the basis of the teacher's authority or accept its validity because the new information is in congruence with his or her previous knowledge.

These various theories suggest connections among beliefs about mathematics, autonomy, and knowledge structures. In particular they suggest that a conceptual view of mathematics is associated with autonomy and with a relational knowledge structure. Analogously, a procedural view of mathematics is associated with an external source of validity and with an instrumental knowledge structure. The results summarized in the Student Ranking Table in Appendix I support the plausibility of these connections. The table reveals that generally those participants with a higher percentage in the conceptual coding category also were those coded high on autonomy and who subsequently were coded correct on a higher percentage of the function questions.

These results, however, cannot establish causality between the factors. In a similar way the specific comments made in the belief interviews cannot be causally linked to responses in the function interviews. Yet, the interviews might suggest plausible inferences between the two data sets. The following discussion summarizes and compares each participants' beliefs about mathematics and autonomy with their knowledge of functions.

#### Keith

Keith's beliefs assessment conveyed a view of mathematics as procedural. Keith described mathematics as "a brick wall" since it allowed no room for error. On the vocabulary list Keith selected the terms *rigid*, *controlled*, and *absolute* to reflect the right-wrong aspect of mathematics. "You use a specific formula to get specific answers" and "you have to get this answer or else you're wrong." Keith saw mathematics as useful but not beautiful or exciting. However, he did feel that the solution of word problems and proofs required original thinking. He also felt that one needed to be clever to succeed at mathematics. Keith saw real-world

applications as important for motivation. He indicated that he expected tests to be like homework or class problems, although he did not expect the teacher to prepare him for each problem type.

The problem protocols in Keith's beliefs assessment also pointed towards a procedural view of mathematics. He was coded conceptual on several problems because he was able to justify intuitively arithmetic and algebraic procedures, and because he showed flexibility in his solution techniques. The procedural coding resulted from his use of unmonitored trial and error and dependence on the execution of formulas in the problem-solving situations. Keith also graded the sample test on the basis of familiarity of form rather than process. Keith's autonomy during the problem protocols was limited. He showed some monitoring of his processes and an expectation that his solutions should be justifiable.

Keith's functions interviews stand in contrast to his beliefs assessment. The conceptual and autonomous codings from the belief's assessment showed that 62% of the problem episodes received a conceptual rating while 40% received an autonomous rating. These relatively low percentages along with Keith's description of mathematics as prescribed rules suggested that Keith viewed mathematics as primarily procedural in nature. Based on this conclusion, Keith's functions assessment was anticipated to show a reliance on procedures. Instead his protocols implied an integrated understanding of and ease with many of the topics. Keith was able to use function notation in evaluating expressions like  $f(x)$  and  $f(g(x))$  and in solving word problems. Keith also associated  $f(x)$  with  $y$  and quickly graphed  $f(x) = 2x + 3$  both by a table of values and by using the slope-intercept form. He articulately described the components of  $y = mx + b$  including the roles of  $x$  and  $y$  in the equation. Keith did, however, have difficulty applying the vertical line test to graphs of function. He tacitly assumed that continuity was implied in the definition. He also limited his definition of domain and range to the explicit  $x$  and  $y$  coordinates he computed. Finally Keith insisted that the graph of  $y = (x - 1)(x - 2)(x - 3)(x - 4)$  would represent a series of 4 lines each with the same  $y$ -intercept of 24.

The researcher hypothesized that this incongruity between the beliefs assessment and the function assessment was partially attributable to Keith's reserved nature. As noted in the analysis of his interviews, Keith rarely volunteered information. This silence made the autonomous coding especially difficult to determine since the coding was premised on voluntary

actions or comments. That is, a student only received an autonomous or nonautonomous coding for a problem episode if he or she showed a voluntary action or comment that indicated either coding category criteria. Hence the actual number of problem episodes that could be evaluated for Keith was fairly small. For instance, Ann's comments permitted coding on 12 problem episodes, while only 5 problem episodes could be coded for Keith. Thus Keith's evaluation was based on fewer problem episodes. In a similar fashion, Keith's quiet demeanor may have also unduly influenced his conceptual/procedural coding. He may have been less willing to volunteer his thoughts or conjectures. The researcher observed that Keith tended to become quieter when he was confused or uncertain.

#### Ann

Ann's interviews revealed a view of mathematics as conceptual. Ann repeatedly asserted that it was important to understand the rationale in mathematics. This rationale for procedures not only made the ideas reasonable but it also helped to extend the procedures to new situations. Ann felt that it was unnecessary to memorize mathematics if one understood it. She enjoyed the challenge of applying mathematics to new situations and to real-life applications. In fact, she saw mathematics everywhere in the world around her. Ann also explained that in mathematics, especially in word problems, it was often necessary to use trial and error to gain insight into the problem. To her the creation of mathematics required creativity and cleverness. While Ann saw mathematics problems as having only one right answer, she felt that there could be multiple-solutions techniques. Finally she saw mathematical knowledge as integrated concepts and cumulative in nature.

Ann's verbal reports were consistent with her actions and comments during the problem phase of the beliefs assessment. Ann demonstrated that she could use multiple approaches, justify procedures, summarize her solutions, and use number sense to check an answer. In addition to these actions, Ann evaluated the sample test on the basis of process rather than answers alone. Throughout the problem protocols, Ann revealed her autonomy with mathematics. She constantly monitored her progress checking not only that the procedures were executed correctly, but that the solutions made sense to her. She even challenged the researcher's questions and suggestions. These actions showed Ann's pervasive expectation that mathematics should make sense. That the rationales behind the procedures were knowable and



vital.

Ann's autonomy and her view of mathematics as conceptual appeared consistent with the type of knowledge she constructed from the functions unit. Ann illustrated her definition of function and one-to-one with examples and graphs. She also was able to extend her use of the definition of function to the abstract situations in questions #61 and #63. (See Appendix B for a statement of the problem.) While confused by function notation, Ann still was able to describe the domain and range of several functions. Ann also demonstrated that she had integrated graphs with equations and the coordinates of points. She moved easily between these three representations and used them in conjunction to validate her work. Ann utilized this facility when she attempted to locate the intercepts in problem #53. Like the other participants, she had originally expected this problem to be linear. However, she persisted until she was satisfied that the graph accurately reflected the multiple intercepts. This tenacity and need to find closure was also evident in her solution to the word problem in question #59. Here Ann utilized number sense to compensate for her confusion with function notation. She expected her answer to make sense and actively sought alternate strategies to verify her solutions.

Overall Ann demonstrated a clear and integrated knowledge of functions and linear equations. While uncertain of function notation, she showed a meaningful understanding of the symbols in the equation  $y = mx + b$ . Ann was also able to transfer her understanding of functions and intercepts to several non-routine problems. Of the six participants Ann alone cited a specific application, the parking garage fee scale, in her rationale for studying functions. Ann's interviews suggested that she had developed a conceptual (relational) knowledge structure.

Ann's autonomy and conceptual view of mathematics appeared consistent with her knowledge structure of functions. She had emphasized the importance of understanding of the rationale and the expectation that this rationale behind procedures would facilitate her thinking in unfamiliar situations. Ann's functions assessment demonstrated that had she developed this rationale and indeed could apply it. Ann also had indicated her interest in the applications of mathematics. Again, her interviews showed that she had remembered one of the few examples of real life applications given in class. As in her beliefs assessment Ann repeatedly modeled her expectation that her solutions should make sense and be consistent with her knowledge. Thus, Ann's autonomy and conceptual view of mathematics were matched with her conceptual

understanding of functions.

### Tara

Tara's comments during the beliefs interviews presented a view of mathematics as procedural. Tara's selection of vocabulary terms emphasized mathematics as prescribed rules that were evaluated right or wrong. She saw mathematics as controlled since "There's a way it's to be done. . . . There's only one kind of answer you can get. It goes along with being right or wrong." She also explained that: "Everything that I ever learned in math seems to be like memorization." Tara also believed strongly that teachers were responsible for presenting step-by-step instructions for all problems that their students were expected to solve on tests.

These beliefs coincided with Tara's actions as she discussed and solved the problems in this second phase of the beliefs' analysis. During her work on the problems, especially the problem-solving questions, it was apparent that Tara expected to apply known algorithms or equations to solve the problems. She rarely utilized exploration or trial and error as a motivational tool and she readily expressed her expectation that the researcher should supply hints or answers to these nonstandard problems. Also when grading the sample test, Tara based her assessment on the familiarity of the form rather than on the validity of the process. Thus in the beliefs' assessment, Tara revealed a procedural view of mathematics and a dependence on an outside authority to validate her solutions and her knowledge. This dependency also extended to the expectation that all problem-solution techniques should be presented by the teacher.

Her explicit beliefs and her actions while solving problems appeared consistent with the type of knowledge she constructed from the functions unit. Tara's definition of function stressed the procedural aspect of evaluating functions: "A formula into [which] a number is put to get an answer." Tara could readily evaluate function notation, but failed to associate it with the  $y$  variable. She insisted, when asked about domain and range of an equation in function notation, that the equation had no range. While she recognized the graphs of functions by the "function test" (vertical line test), her usage of the test seemed mechanical rather than based on the definition of function. This hypothesis was confirmed by her failure to extend the idea of function to the more abstract situations of problems #61 and #63. (See Appendix B.) Tara could not extend the concept of intercepts to the nonlinear equation in problem #53. Her discussion

showed that she could only generate the y-intercept by recognizing an equation in the form  $y = mx + b$ . She also did not appear to realize that each ordered pair from the graph must satisfy the equation. Overall Tara's knowledge of functions seemed to be based on procedures using symbols. Her knowledge of the various procedures appeared segregated. When confused or faced with contradictory evidence, Tara again turned to the researcher with the expectation that the dilemma would be explained and the correct procedure illustrated.

Tara's knowledge of functions appeared consistent with the assessment of her beliefs about mathematics. In her discussion, she had emphasized mathematics as memorized rules and that problems should conform to class procedures and examples. Her function assessment demonstrated that she had constructed a collection of procedures that could be applied to routine problems. These procedures, however, were not flexible, did not readily extend to new situations, and appeared disjointed. She also seemed dependent on the recognition of a familiar form to trigger an appropriate response. Thus, Tara's procedural view of mathematics and her lack of autonomy seemed congruent with her procedural (instrumental) knowledge of functions.

#### Tom

In his interviews, Tom conveyed a view of mathematics as utilizing rules, common sense, and logical thinking. While rules were a significant part, he felt that mathematics was "being able to do the thinking to get to those steps." Tom found that many problems just solved themselves, but when a solution was not immediate he suggested looking for similar examples, trying alternate approaches or simplifying expressions. Like Ann, Tom felt that memorization was not necessary if one understood the concept or procedure. Not only was mathematics useful, but Tom found it challenging and interesting, as well. In fact, mathematics was the only school subject that actively engaged his interest and efforts.

Tom's actions and comments as he solved the problems in the beliefs assessment were consistent with his views of mathematics. Tom demonstrated that he could justify procedures, summarize solutions, and apply number sense to check the reasonableness of answers. Tom also evaluated the sample test on the basis of process rather than familiarity of form. Tom revealed his autonomy with mathematics by voluntarily checking his answers and by monitoring his solution processes.

These expectations and actions from the beliefs assessment appeared consistent with

Tom's responses during the function interviews. Tom demonstrated his understanding of functions through his recall of the definition, recognition of graphs of functions, and his application of the function concept to the abstract situations in problems #61 and #63. (See Appendix B.) Tom was equally adept at determining the domain and range of a function, graphing linear equations, and computing the slope of a line. While Tom's responses showed a clear and an integrated understanding of functions and graphing, he was uncertain and confused by function notation. This confusion included a failure to recognize  $f(x)$  as  $y$ , evaluate expressions like  $f(2)$ , and use composition notation. With the exception of function notation, Tom's responses generally showed a conceptual understanding of functions and related topics.

As the above discussion indicates Tom valued the rationale behind procedures. His function interviews revealed that he had put that belief into action by constructing an integrated knowledge of functions and graphing. He easily moved between graphs and equations and could cite specific examples to explain his ideas. However, Tom's codings in the Student Ranking table in Appendix I would suggest a higher percentage of correct responses in the functions category. This discrepancy is tenable. Throughout the interviews, Tom indicated that he was often inattentive to his schoolwork. Although Tom enjoyed mathematics, he often found the homework boring and tedious and so would skip it. While not blaming his teacher, Tom also confessed that he had not read or completed the homework on the composition section even though he had been absent during the class discussion. Thus, Tom's uncertainty with the composition notation is understandable given his confession and attitude toward schoolwork.

#### Sue

In her beliefs assessment Sue indicated that mathematics required analysis and logic but did not utilize imagination or creativity. While Sue felt that every problem had a correct answer, most problems had multiple-solution techniques. In addition word problems often necessitated exploration before an appropriate equation could be written. Sue felt that mathematical knowledge was initially learned by memorization, but through repeated usage that knowledge became integrated into one's thinking. Her remarks suggested that she saw mathematics as automatized rules and formulas. Sue also expressed her frustration at the irrelevance of much of the mathematics she studied in school. She felt her schoolwork in mathematics would help her be well-rounded academically, but that it had no tie to her real life.

Sue's comments presented a view of mathematics as both conceptual and procedural. This mixture of views was also manifest in her discussion of and solutions to the problems in the beliefs assessment. Sue demonstrated in these interviews that she could justify solutions by using number sense, apply multiple approaches to a problem, and summarize results. At the same time, Sue also graded the sample test primarily on the basis of familiarity of form rather than process. In addition, Sue began the problem-solving protocols with the expectation that an equation or formula could be applied or that one could be written. While Sue held this expectation, she also consciously monitored her progress and abandoned that view when it failed to produce any tangible or immediate results. This monitoring along with her expectation that her results should make sense appeared to override her expectation for an equation or formula and hence facilitated her problem solving.

Sue's belief's assessment revealed that she had beliefs and actions indicative of both a procedural and conceptual view of mathematics. Along with this mixed perspective, Sue also demonstrated a consistent autonomy with mathematics by monitoring her actions and by expecting her answers and solution techniques to make sense. This same mixture was evident in her interviews on functions. Sue's interviews demonstrated that she could give the definition for function, apply it to determine whether a graph was a function, and extend the concept to the abstract situations in questions #61 and #63. (See Appendix B.) She also was able to graph linear equations and determine the slope from points. In contrast, her discussion of the intercept problem suggested that her use of the slope-intercept equation was more procedural than conceptual. She seemed only to be able to find the y-intercept by locating the b position of the equation. She was also uncertain how to check an ordered pair in an equation and even what the role of x and y were in the equation  $y = mx + b$ . While these difficulties suggested a more procedural understanding of the unit, she constantly checked her answers against her graphs. By doing so, she gave the impression that she was looking for and expecting the various procedures to be consistent and intuitively correct. Sue's autonomy again aided her solution process even when her factual and conceptual knowledge was inadequate.

Sue's beliefs assessment presented a view of mathematics as a mixture of both memorized procedures and logic. Her problem solutions in this assessment also revealed a pervasive sense of autonomy with mathematics. Sue's functions assessment mirrored this same

mixture of dependence on known procedures with the expectation that the solutions should be consistent with her knowledge. It was, in fact, Sue's autonomy that moderated her procedural view of mathematics and facilitated her problem solving.

#### Steve

Like Sue, Steve presented a mixed view of mathematics. He perceived mathematics as a language and as a collection of prescribed rules or techniques. For Steve, the rules of mathematics represented established results, so he felt it was unnecessary to re-verify them. Foremost though, Steve saw mathematics as a tool of science. Scientist used mathematical equations to describe theoretical ideas or complex phenomenon. While Steve saw the rules as fixed, he also believed they could be adapted to fit new situations or applications. Steve felt that to apply mathematics often necessitated trial and error and insight. It was the applications of mathematics that Steve found interesting and creative. Steve also believed that mathematics was not instinctual but learned and that this learning required memorization, experience, and understanding. Using a computer analogy, Steve described his own understanding of mathematics as: "I learn how it to do it and I store [it] up [in] my brain and later I just pull it out and use it whenever I need to." Overall, Steve's discussion of mathematics stressed its procedures which required strict adherence to the rules and its applications which required insight to adapt or apply the rules.

Steve's problem protocols in the beliefs assessment also demonstrated a mixed view of mathematics. Steve received his conceptual codings because he justified algorithms, generalized results, utilized number sense, and evaluated the sample test on the basis of process. In contrast, Steve received his procedural coding because on the problem-solving questions he relied almost exclusively on equations and formulas to solve the problems. It was noted in the discussion of Steve's protocols that even in the problems coded conceptual, Steve based his arguments and solutions on symbol manipulation, application of rules, and solving equations. Steve's autonomous and nonautonomous codings followed a similar pattern. Steve demonstrated his autonomy by monitoring his results, by voluntarily checking his answers, and by summarizing his results. Steve's also received nonautonomous codings because of his reliance on authority to justify algorithms and because of his relegation of the validity of results to the execution of an equation. Thus, in both Steve's comments and problem solutions a common theme emerged.

This theme stressed the accurate execution of rules and equations as a tool primarily for solving and verifying results.

Steve's function interviews showed a clear understanding of function notation. In particular, Steve grasped the relationship between  $f(x)$  and  $y$  and conveyed an understanding of the roles of the independent and dependent variables. He also easily discussed the components of  $y = mx + b$ , graphed a linear equation, and computed a slope. While Steve was comfortable with these topics, he was confused on the definition of function. Steve associated function with any equation that defined a relationship between two variables. Steve also expressed some frustration with the concept of function because it could not be represented by numbers or an equation: "That's the problem with math. Cause math is very quantitative and it's tough trying to stick qualitative things to it. Trying to describe it in something other than numbers." Two strategies were prominent in Steve's protocol. Whenever possible, Steve looked for some algebraic manipulation to solve the problem and whenever he graphed points, he would attempt to write an equation to describe that set of points. While at times these strategies were productive, at other times his search for an equation overshadowed the original question or concept. He gave the impression that he needed an equation in order to provide concrete evidence for his thoughts. Steve's solution process was aided by his monitoring of his execution of procedures and his expectation that his solutions should be consistent with his equations and knowledge.

Steve's beliefs assessment and function assessment seemed to coincide. In the beliefs assessment Steve expressed in many ways that mathematics entailed the execution and application of procedures. Steve's comments and actions suggested a reliance on finding and solving equations. He described mathematics as truthful because "equations don't lie." Overall, he saw "math as just a tool rather than an end." He also believed that any mathematical statement could be proved valid or invalid because "If you had the right knowledge you could prove it one way or the other. It may take forever to work out the equation. But it's still possible." These remarks support the contention that to Steve, equations and their solutions represented a very vital element in mathematics. The problem protocols in Steve's beliefs assessment also affirmed this supposition. This emphasis on equations and manipulation appeared again in his function interviews. Steve demonstrated mastery of those aspects that

involved the use or description of equations. In contrast though, Steve had difficulty with the definition of function since it was a "qualitative thing." In both the beliefs assessment and in the function interviews, Steve's autonomy facilitated his use of these procedures.

#### Cross-case Discussion

As the preceding summaries illustrate, the six participants differed in their views of mathematics, in their autonomy, in their approach to problems, and in their knowledge of functions. The following discussion will highlight those areas which most clearly distinguish among the participants.

#### Beliefs about Mathematics

The students' beliefs about mathematics can be distinguished by their views on the nature of mathematics, the intellectual characteristics needed for mathematics, the utility of mathematics, the way to learn mathematics, and the teacher's role.

#### Nature of Mathematics

Ann's and Tara's views of mathematics provide the most striking contrast. Ann perceived mathematics as a way of thinking and stressed the importance of knowing the rationales underlying procedures. Tara stressed memorizing and executing procedures and rules. Tara also saw mathematics as rigid and inaccessible to individual choices. While acknowledging that mathematics problems have one right answer, Ann felt that the solution techniques could vary depending on the individual's insight and preference.

Keith, Tom, Sue, and Steve held views of mathematics that were a mix of these two perspectives. Like Ann, Tom valued the reasoning behind problem solutions and Sue shared Ann's belief that problems often had multiple solutions. Keith, however, was like Tara in that he saw mathematics as rigid and as a collection of prescribed rules. While Steve also stressed this perspective, he added that mathematics was a language and a tool of science.

#### Intellectual Characteristics

Ann was unique in her perspective on the intellectual characteristics needed for mathematics. She felt that to do mathematics one must be creative, clever, insightful, and logical. Tara again provided a stark contrast to this view. She emphasized that she did mathematics by copying and repeating strategies demonstrated by the teacher. She, unlike Ann, did not hold the expectation that one should be able to modify rules or procedures to apply them



new situations. On word problems, or in problem-solving situations if Tara was unable to apply a known formula or write an equation, she tried trial and error. However, for Tara this process entailed more guessing than deduction and conjectures.

Again, the other participants take positions between these two extremes. Tom, Sue, Steve, and Keith all shared Ann's view that word problems and proofs required insight, logic, originality, and trial and error. In addition Steve felt strongly that to apply mathematics required cleverness. Although generally concurring with Ann's view, the others did not agree that one needed to be creative in mathematics. For example, Sue saw creativity as a trait unrelated to mathematics, since for her it implied freedom of thought and self-expression. For Sue, mathematics was too rigid and rule-oriented to permit the individual a voice or choice.

#### Utility of Mathematics

Among the participants, Ann and Steve saw mathematics as relevant or useful in their lives. Ann perceived mathematics to be everywhere in life. She even saw mathematics in the deductions and inferences that one made while reading a newspaper. For Steve, mathematics was useful for its applications in the sciences and in architecture. Apart from consumer applications Tara, Sue, Keith, and Tom saw no real need or use for their mathematics. They also shared the belief that their mathematics coursework should to be useful in advanced courses like calculus and useful in learning reasoning, but this view was expressed vaguely and skeptically.

#### Learning Mathematics

The participants were divided on the issue of whether or not mathematics is learned primarily through memorization. While acknowledging that some basic facts needed to be memorized or made routine, Ann and Tom felt they did not need to memorize. They both stated that once they understood a concept that it naturally became part of their thinking. Tom also believed that mathematical ability was partially innate. Sue and Keith felt that they learned mathematics through repetition and some conscious memorization. In contrast, Tara openly admitted that all of her mathematical learning was achieved through memorization.

#### Teacher's Role

Among the participants, Tara alone held the expectation that the teacher should present step-by-step explanations of each of homework problem. While the others did not share Tara's

view, Ann went further to say that she wished the teacher would not demonstrate everything. She enjoyed challenges and used the occasions when the teacher did not explain everything to test her own understanding of the concepts. There was a general consensus among the participants that tests should mirror homework exercises and class examples. Tara was adamant in this view, saying she would quit any mathematics class that did not conform. While Ann agreed to this view of tests, she did so because she felt non-routine problems would be unfair to the less mathematically adept students. She felt that more difficult problems could be given as optional bonus problems. There was also universal agreement among the participants that the mathematics instruction should contain examples of real-life applications. They felt such applications were essential for motivation and for maintaining interest.

#### Autonomy

Each of the participants exhibited some autonomy either by voluntarily monitoring their work, checking their answers, or summarizing their results. Each held the expectation that mathematical processes and solutions should make sense, although they on the differed on the degree to which they were able to capitalize on that expectation. For example, Sue and Tara reached a point in several solutions where they realized their answers were inappropriate. Because they lacked some domain-specific knowledge, or held inflexible procedures, they were not able to modify their responses to be more appropriate. Of all the participants, Ann exhibited the most consistent and ongoing use of self-monitoring and self-reflection. She rarely put a problem aside until she was satisfied with her response. Like Ann, Tom, Steve, Sue, and Keith frequently used number sense and alternate techniques to verify solutions. For Steve and Sue, their autonomy also seemed to mediate their procedural expectations in the problem-solving protocols and in the functions assessment.

#### Approach to Problems

The problem-solving situations and the evaluation of the sample test revealed distinctions among the participants. When confronted with a problem-solving situation, Tara expected to apply a formula or write an equation. Failing this, she used unmonitored trial and error. She was easily frustrated and insisted that the researcher should supply hints, clues, or the necessary formula or equation. In contrast, Ann would begin by exploring the situation, either by thinking of a simpler case or using trial values. She used these strategies effectively to refine her

approach and to suggest reasonable conjectures. Like Ann, Tom also demonstrated an ease with problem-solving heuristics. He stated that he tried to look for logical connections between the problem and its solution. Like Tara; Keith, Sue, and Steve approached the problem-solving situation with the expectations that they should be solved by applying a formula or writing an equation. Unlike Tara, they were willing to abandon this strategy when their self-monitoring showed it to be ineffective.

The students' evaluation of the sample test also revealed distinctions. Ann, Tom, and Steve graded the test on the basis of the validity of the processes used while Keith, Sue, and Tara evaluated the problem solutions by their match to standard or familiar procedures.

#### Knowledge of Functions

In addition to the differences in beliefs about mathematics, autonomy, and approach to problems, the students demonstrated differing understandings of the function concepts. One problem in particular, question #53, highlighted those differences especially well. (See Appendix B.) In this question the students were asked to determine the  $x$ - and  $y$ -intercepts of the nonlinear equation  $y = (x-1)(x-2)(x-3)(x-4)(x-5)$ . The students differed not only in how they approached the problem, but on the flexibility of their content knowledge. Tara and Sue had only one strategy for solving for the  $y$ -intercept. They tried to put the equation in  $y = mx + b$  form and read the  $b$  term. While Sue, Keith, and Tom each were able to recognize the form of the intercepts as  $(x, 0)$  and  $(0, y)$ , they could not utilize this information readily. Sue found it confusing to substitute for  $y$  instead of  $x$ . All three were baffled when the researcher suggested that they substitute simultaneously both coordinates of a point into the equation. Not only was this action new to them, they saw no reason for it.

Only Ann had integrated into her understanding the relationship between the ordered pairs, the original equation, and the graph. She moved quickly among these three representations or perspectives and drew motivation as well as confirmation from them. With the exception of Ann, the other participants were not especially willing to consider the possibility that the graph was nonlinear. In fact, Keith forced the graph to be linear by dividing the original equation into five separate linear equations:  $y = x - 1$ ;  $y = x - 2$ ;  $y = x - 3$ ;  $y = x - 4$ ; and  $y = x - 5$ . Sue admitted that she did not know how to determine when an equation might be linear. While initially confused, Tom and Steve were able to eventually ascertain the intercepts

and suggest that the graph should be nonlinear. These observations, however, were not made without considerable prompting and questioning by the researcher. Here again Ann's autonomy and need for the concepts to make sense seemed to compel her to explore the question until she understood what type of graph could have multiple intercepts.

#### Additional Comments

Ann's interviews revealed not only a conceptual view of mathematics, but an facility with content as well. Ann was able to give spontaneous examples of graphs and equations to illustrate her definitions and explanations. These explanations were clear and succinct. Her conversations were not broken by false starts and ramblings. Thus, Ann's interviews demonstrated a qualitative difference beyond the substance of the discussion.

Just as Ann's interviews were characterized by their clarity, Tara's were often confused and vague. During the interviews Tara repeatedly asked for assistance and confirmation for her answers. When none was forthcoming, she would attempt to read the researcher's facial expressions for clues. While mathematical discussions were not easy for Tara, she was very articulate and confident in discussions of English literature and personal matters.

The discussion of the other participants also revealed individual characteristics. Like Ann, Tom also was able to give examples, although his discussion was marked by a terseness. Keith's discussions were characterized by their brevity, since he rarely volunteered information. Sue's discussions were punctuated by a jovial forthrightness. She would quickly tell you when she did not understand a question or concept and then laugh at her own confusion. Steve, like Keith, was somewhat reserved. His remarks usually were tied to references to the sciences or to computers. He often described his mind and thought processes in terms of a computer model.

These differences are not always reflected in the preceding discussions of the students' beliefs, autonomy, and knowledge of functions. They do, however, provide additional insight into the distinctions among the participants. For example, Ann, Keith, and Tom were close in the number of function questions they answered correctly but this proximity does not illustrate the distinctions in clarity, succinctness, and spontaneity exhibited in their remarks.

### Summary of Results

The results suggested that generally the students' actions and knowledge in the function protocols were consistent with their beliefs about mathematics and their autonomy. Within this consistency though, each student conveyed a unique set of beliefs about mathematics and demonstrated different degrees of autonomy. The Student Ranking Table in Appendix I summarized the differences among the students.

The differences observed among the students were in keeping with the results found by Buchanan (1984) in her investigation of beliefs and problem solving. Buchanan noted that students whose primary beliefs about mathematics were relational or instrumental showed both autonomous and non-autonomous actions in their problem-solving approach. They also had differing sources for their motivation. Collectively these results suggested a continuum of beliefs from conceptual to procedural rather than the dichotomous views of mathematics proposed by Skemp (1987). In this study, Ann and Tara represented views of mathematics at the extremes of the continuum. The remaining students would be located between Ann and Tara.

Skemp (1987) also conjectured that students' autonomy was related to their beliefs about mathematics. Briefly, he proposed that relational (conceptual) views of mathematics would coincide with autonomy, while instrumental (procedural) views would be associated with a lack of autonomy. This simple dichotomous view was not reflective of the data reported here. While Ann and Tara seemed to fit Skemp's dualistic model, the other participants did not. In fact, the protocols of Steve and Sue demonstrated that autonomy can enhance problem solving and hence mediate an otherwise procedural expectation that all problems should be solved by applying a formula or solving an equation.

The third hypothesis suggested by Skemp (1987) was that different beliefs and autonomy generate divergent knowledge structures. The students' beliefs and autonomy did appear consistent with their knowledge of functions, although the results did not confirm the two divergent structures proposed by Skemp. Just as the students' beliefs were a mixture of conceptual and procedural views, so were their understandings of functions a mixture of memorized procedures and integrated concepts.

While not validating the dichotomous views that were originally hypothesized, the results did confirm interrelationships among the factors investigated: beliefs about mathematics,

autonomy, and knowledge structure. What appeared was a complex and subtle interdependency. For example, Steve's and Sue's autonomy seemed to mediate their procedural expectation in the problem-solving protocols and in the function assessment. For Tara, her attempts to validate her answers or explore her ideas often were frustrated by her apparent lack of domain-specific knowledge or her inflexibility with procedures.

In addition to the results reported above, several other observations arose from the data. Like Frank (1985), the researcher noted that the participants often prefaced their comments with phrases such as: "I don't remember," or "we were never told this," or "I'm not sure this is right." Frank designated this as 'bailing out' and attributed it to "an attempt to gracefully get out of an uncomfortable or unprofitable situation (p. 95)." The researcher noted a similar implication in the use of these phrases in the participants' problem protocols. The results also seemed to confirm the conclusions drawn by Cobb (1985), Frank (1985), and Schoenfeld (1985) that students' problem-solving heuristics were in keeping with their global beliefs about mathematics.

One final issue needs to be discussed concerning these results. Doyle (1983) noted that what students attend to in class often reflected their perception of the class' requirements. In both the Algebra II and the Algebra II/Trigonometry classes, the teacher's expectation on the function unit seemed to reflect the procedural aspects of the topic. Mrs. Thomas also held the belief that for the Algebra II class it was important to present samples of all homework problems and that tests should be fairly consistent with these problems. The tests and quizzes given in both classes conformed to this expectation and also tended to include procedural and recall type questions. During the lecture portion of the class period, Mrs Thomas reinforced this expectation by her continual use of procedural questions. However, the class format provided some opportunity for autonomy with the students presenting their homework solutions at the board.

The participants' understanding of functions needs to be considered against the background of this classroom environment. With the exception of Ann, the other participants strongly believed that mathematics test questions should match homework assignments and class examples and hence agreed with their instructor. Primarily the classroom analysis showed that the students' procedural beliefs about mathematics in general and their procedural expectations for the content were not challenged by the teacher's actions. Thus, Tara's view of mathematics as a disjoint collection of memorized rules seemed to be further reinforced in her class. Ann's

strong belief that mathematics involved ideas and creativity stood in stark contrast to the teacher's own beliefs that mathematics was useful, but not interesting. Also the teacher's belief that the applications of the mathematics were not important, was in contrast with the students' desire for real-life examples to provide motivation for studying functions or mathematics in general.

#### Implications for Future Research and for Teaching

This research study investigated the relationship between students' beliefs, autonomy, and knowledge of functions against the background of the classroom environment. While the environment was observed and analyzed, it was not an integral component of the investigation. Thus the effects of that classroom on beliefs and knowledge were not explicitly studied. This aspect, then, is one possible area that should be studied further. An additional area of research suggested by the results is students' autonomy. The apparent power of these expectations to influence students' actions, especially in problem-solving situations, had not been anticipated by the researcher. Their potential warrants further study.

The participants' discussion of their past experiences, the teacher's role in learning mathematics, and their own learning strategies suggests that beliefs about mathematics are connected with classroom experiences and classroom expectations. This further suggests that the classroom environment, which includes the teacher's own beliefs about mathematics and the teacher's presentation of and expectations for the mathematical content, may convey unspoken messages to the students about the nature and processes of mathematics. If subsequent research affirms the influence of beliefs and autonomy on learning, then the classroom teacher needs to be cognizant of these unspoken messages and perhaps modify classroom activities to foster a more conceptual view of mathematics. This view might be encouraged through student-centered activities, especially problem-solving situations, and the establishment of classroom expectations which include explanations, explorations, and autonomy.

#### Conclusions

The results from this research investigation suggest three hypotheses concerning students' beliefs about mathematics, autonomy, and mathematical knowledge. First, students' beliefs about mathematics rather than being dichotomous form a continuum from strongly conceptual in outlook to strongly procedural. Second, students' autonomy augments their beliefs about mathematics

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and often mediates them. Third, students' beliefs and autonomy appear to concur with their problem-solving strategies and with their knowledge of mathematics. Collectively these hypotheses suggest that students' beliefs and autonomy are an integral component of students' conception of mathematics and influence both how problems are approached and how mathematics is learned. Further study needs to be done on how and when these beliefs are formed and under what conditions these beliefs are modified or changed. Finally, the interplay among beliefs, autonomy, and learning need to be investigated in the actual classroom context.



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**APPENDIX A**

**TEACHER INTERVIEW PROTOCOL**

Teacher Interview Protocol

Interview #1 (Background Questions)

I would like to interview you so that I have your perspective on what I observed in your class. First, I would like to ask you about your background.

- 1) Where did you go to college?
- 2) What degree did you get?
- 3) How many years have you been teaching?
- 4) What courses have you taught?
- 5) What professional organizations do you belong to?
- 6) Do you attend any mathematics education conferences?
- 7) Have you ever talked at any of the conferences?
- 8) What other school activities are you involved with?
- 9) Have you ever used a computer?
- 10) What hobbies or outside interests do you have?
- 11) What do you see as your overall goals and objectives for the Algebra II and Algebra II/Trigonometry classes for the year?

Interview #2 (Classroom Expectations)

- 12) How does your expectations for the Algebra II/Trigonometry class differ from those for the Algebra II class?
- 13) In the Algebra II class do you try to present all of the types of problems that those students will see in their homework?
- 14) What do you see as the overall purpose of mathematics in the high school curriculum?
- 15) If you could change anything in the curriculum, what would you change?
- 16) In the functions unit do you have any specific goals in mind for your classes?
- 17) Do you think it is important for the students to be able recognize the graphs of certain functions?
- 18) Were there differences in your expectations for the two courses?

- 19) What is your grading policy in both classes?
- 20) How does board work, quizzes and homework fit into your grading scheme?
- 21) How important is mathematical notation in understanding mathematics?
- \* 22) How much time should a student spend memorizing mathematics?
- 23) I read a book by Morris Kline entitled Why The Professor Can't Teach Math. His premise was that to teach mathematics apart from it's appreciation is useless mathematics. Do you agree or disagree with his view?

**Interview #3 (Beliefs About Mathematics Ability)**

- 24) What do you perceive as your strengths and weaknesses as a teacher?
- \* 25) Describe one of your students whom you think is good at mathematics?
- \* 26) What makes him or her good at mathematics?
- \* 27) How did she or he become good at mathematics?
- 28) Describe your best mathematics teacher.

**Interview #4 (Beliefs About Mathematics)**

- \* 29) Grade sample test. (see student interview)
- \* 30) Vocabulary list. (see student interview)

**Interview #5 (Beliefs About Mathematics)**

- \* 31) Ranking chart. (see student interview)

\* Indicates questions or activities asked in the student interviews as well.

**APPENDIX B**

**STUDENT INTERVIEW PROTOCOLS**

Student Interview ProtocolsInterview #1 (Background Questions)

- 1) How old are you?
- 2) Are you a junior?
- 3) What courses are you taking now?
- 4) Do you belong to any clubs or groups?
- 5) Do you belong to the mathematics team?
- 6) Have you ever used a computer?
- 7) Do you own a calculator?
- 8) Do you ever read any game or puzzle books?
- 9) Do you have a job?  
Do you use mathematics in your job?
- 10) What other mathematics classes have you taken?  
Could you tell me about those classes?
- 11) What do you plan to do after high school?
- 12) Do you anticipate taking mathematics in college?
- 13) In New Hampshire you are required to take only 2 years of mathematics, so why are you taking a third year?
- 14) How do you use mathematics in your everyday life?
- 15) Now use your imagination. Name something that is the most unlike mathematics that you can think of. What makes it unlike math? Name something that is the most like mathematics. What makes it like math?
- 16) How would you describe your own ability to do mathematics?
- 17) Would you describe someone in your class that is good at mathematics? What makes them good?
- 18) How does someone get to be good at mathematics?

Problems. General instructions on all mathematics problems was: Read the problem out loud and tell me what you're thinking as you do the problem.

- 19)  $1/4 + 2/3$

Follow up:

Why do you need common denominators?

20) 6 divided by  $\frac{3}{8}$

Follow Up:

Why do you invert the divisor?

Does your answer make sense? Why? When you do divide, your answer gets smaller. Does your answer make sense?

**Interview #2 (Problem Protocols and Belief Questions)**

21)  $1.50 \times .25$

Follow Up:

How do you know where to place the decimal point?

Does your answer make sense? Why could it not have been 37.5 or 3.75?

22) Vocabulary list. (see following list)

Instructions: Circle the words that you associate with mathematics (English, history or science). Read them out loud as you go. Are there any words that you think go together? Why?

## VOCABULARY LIST

absolute	abstract	analyze	ancient
anxiety	arbitrary	beauty	boring
capricious	cause & effect	changing	chronological
classical	clever	common sense	concentration
controlled	controversial	creative	cultural
current	deductive	depth	detailed
diagrams	discovery	dogmatic	dull
easy	elegant	exciting	experiential
expressive	factual	fixed	flexible
formulas	fragmented	free	fun
general	goals	hard	humanistic
ideas	individualistic	insight	instinctive
integrating	intensive	interpretative	known
language	logical	mechanical	memorize
multi-dimensional	multi-perspective	new	objective
old	open ended	opinionated	ordered
organized	practical	precise	rational
right/wrong	rigid	routine	rules
sequenced	short cuts	simple	solemn
specific	structured	symbolic	truthful
theoretical	thorough	thought provoking	themes
trial and error	universal	uncertain	useful
valid	varying	verbal	visual
well-defined	writing		



Interview #3 (Problem Protocols and Belief Questions)

- 23) Which of the following fractions is more than  $3/4$ ?  $35/71$ ,  $13/20$ ,  $71/101$ ,  $19/24$ ,  $15/20$ .

Follow Up:

Could you have done it without a calculator?

- 24) Which is the least of the following numbers?

$1/5$ ,  $\sqrt{5}$ ,  $1/\sqrt{5}$ ,  $\sqrt{5/5}$ ,  $1/5 \sqrt{5}$ .

Follow Up:

How else could you do the problem?

- 25) Jimmy was trying a number trick on Sandy. He told her to pick a number, add 5 to it, multiply the sum by 3 then subtract 10 and double the result. Sandy's final answer was 28. What number did she start with?

Follow Up:

Do you believe your answer?

- 26) Find 20% of 85.

Follow Up:

Do you believe that number? Why does it seem reasonable?

- 27) What is the smallest positive number which when it is divided by 3, 4 or 5 will leave a remainder of 2? Note: Two is the smallest integer that satisfies this relationship. However, all the student participants tacitly assumed that divided by meant a factor greater than zero (e. g.  $3k + 2$ ,  $k > 0$ ).

- 28) Which is larger the value in column A or in column B?

A	B
$\frac{543 \times 29}{32}$	$\frac{30 \times 543}{28}$

Follow-up:

Could you have answered the question without multiplying it out (or using the calculator)?

29) Which is larger the value in column A or in column B?

$$P < O$$

A

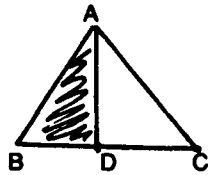
B

$P^3$

O

Follow-up:

If students gave an incorrect response, the researcher asks the students to try various numbers, including a counter-example to the students statement.



30) In the diagram above, if  $BD = DC$  and the area of the shaded region is 8, the area of the triangle ABC is.

Follow-up:

If stuck, the researcher asked the student to tell what they felt they needed to know in order to solve the problem. Since this usually involved the length of BD or BC, the researchers suggested the student make up a number and try it out.

Interview #4 (Problem Protocols and Belief Questions)

31) Smith gave a hotel clerk \$15 for his cleaning bill. The clerk discovered he had overcharged and sent a bellboy to Smith's room with five \$1.00 bills. The dishonest bellboy gave three to Smith, keeping two for himself. Smith has now paid \$12.00. The bellboy has acquired \$2.00. This accounts for \$14.00. Where is the missing dollar?

32) Sample test. (see following Interview #4)

Instructions:

Pretend now that you are the teacher. I want you to grade this test. The first five problems are worth six points each, and the last four are worth ten points each. Grade it, and tell me why you are taking off the points that you are.

Follow Up:

After grading the test, the researcher queries the student about any misconceptions they may have allowed to stand. Also on question II (a) the researcher usually attempted to clarify if the students believed that the problem was done incorrectly, inefficiently or the student marked it based on not matching the standard procedure.

33) Which is larger, the value in column A or in column B?

A	B
$P + 2$	$2 - P$

Follow-up:

Again, counterexamples were suggested if students offered incorrect solutions.

34) Which is larger, the value in column A or in column B?

$R > O$	
A	B
$R^2$	R

Follow-up:

Counterexamples were suggested if students offered incorrect solutions.

35) The radius of the earth is approximately 4,000 miles. What length of rope would be

needed to "fit" around the equator? Now suppose we add about 6 feet to the length of the rope (i.e. 2 feet), how far above the ground would the rope be? Would a piece of paper fit between the ground and the rope? Could a mouse crawl through? Could a person walk under it?

**Follow-up**

Since the purpose of the question included the students' reaction to the feasibility of the answer, the researcher interacted with the students, assisting if necessary, the students' understanding of the question and monitoring the appropriate usage of units. How certain do you feel about your answer? On a scale of 1 to 10 how confident are you?

## ALGEBRA REVIEW TEST

Chris

name

\*\* Show all work \*\*  
 \*\* Write neatly \*\*  
 \*\* Circle your answers \*\*

I. Simplify the following (6 points each):

(a)  $(x^{13})^2 = x^{15}$

(b)  $\sqrt{16+9} = \sqrt{16} + \sqrt{9}$   
 $= 4 + 3$   
 $= 7$

(c)  $\frac{x^2}{x} = \frac{1}{1} = x + 1$

(d)  $\frac{1}{6} + \frac{2}{2} = \frac{1}{6} + \frac{2}{2}$   
 $= \frac{1}{6} + 1$   
 $= \frac{7}{6}$

(e)  $\frac{0}{0} = 0$

II. Solve for X. Write answers in set notation. (10 points each)

(a)  $3X - 7 = 4$

$-3x - 7 + 4 = 4 + 4$

$3x - 3 = 8$

$3x - 3 + 3 = 8 + 3$

$3x = 11$

$\left\{ \frac{11}{3} \right\}$

$\frac{3x}{3} = \frac{11}{3}$

$x = 11/3$

(b)  $3(X - 2) + 12 = X + 2(X + 2) + 2$

$3x - 6 + 12 = x + 2x + 4 + 2$

$3x + 6 = 3x + 6$

$6 = 6$

$\{ x = 6 \}$

$$(c) \frac{1}{2}(4x - 8) = 5x - 3(x - 4)$$

$$2x - 3 = 5x - 3x + 12$$

$$2x - 3 = 2x + 12$$

$$-3 = 12$$

$$\{-3, 12\}$$

$$(d) |x + 3| < 2$$

$$-2 < x + 3 < 2$$

$$-2 - 3 < x + 3 - 3 < 2 - 3$$

$$-5 < x < -1$$

No solution,  
absolute value  
must be a  
positive number.

$$\emptyset$$

Interview #5 (Problem Protocols, Belief Questions, and Student/ Teacher Roles Questions)

- 36) Ranking chart. (see following chart)

Instructions:

What I would like you to do is rank order the topics across the top from one to eleven. For example, if you find decimals the most interesting, you would give it a one, the most boring topic an eleven. Let me know what you're thinking as you fill out the chart.

- 37) I would like you to imagine there is a new kid in school. This student is an English speaking foreign student who is unfamiliar with American school. The guidance office calls you down and asks that you show him the ropes in your mathematics class. What kind of advice would you give him?

Follow-up questions if necessary:

What would you tell him about homework? Tests? Lectures? Your teacher? What should he do if he gets stuck on his homework?

- 38) In a few years I will be teaching teachers how to teach mathematics. Do you have any advice to pass on to them?

- 39) If you could change anything about mathematics, or the way it is taught, what would you change?

- 40) How would you fill in the blank "a good math teacher is someone who"?

- 41) I have a friend who likes to make mathematics tests what he calls a learning experience. He puts problems on the test that the students have never seen before but are related to the ideas they have had in class. Do you agree or disagree with my friend's philosophy?

Follow-up, if necessary:

How would you convince him that this is not right?

- 42) I have had students say to me "You didn't go over that in class, but you gave us homework on it anyway. I don't think that's fair." Do you agree or disagree with those students?

- 43) I have also had students say to me "You waste too much time in class going over things that you don't ask us on the test." Do you agree or disagree with their view?

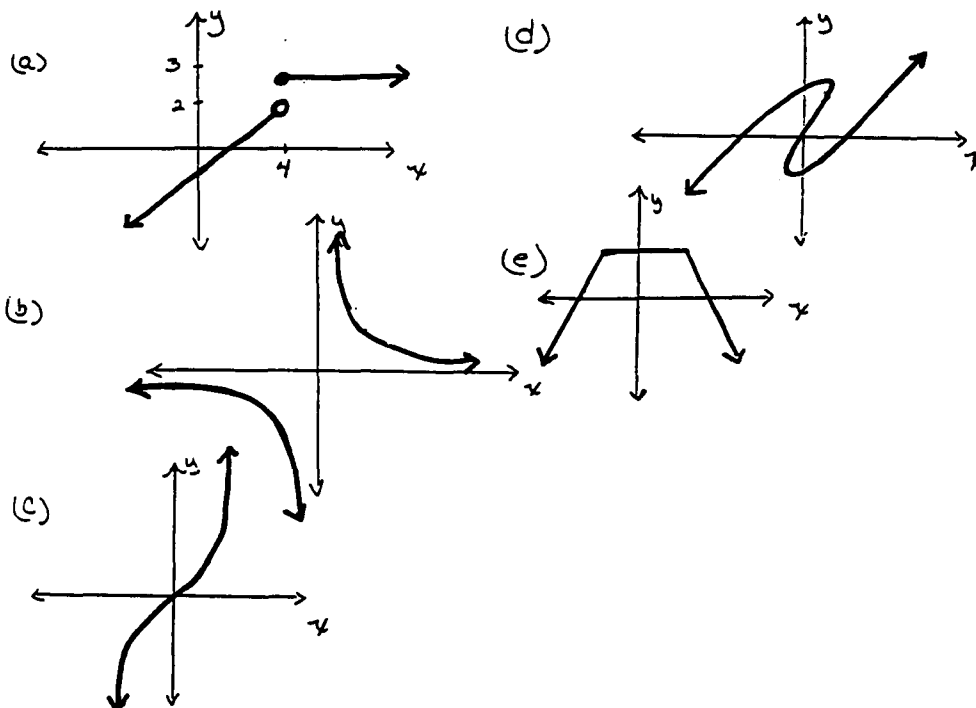




Interview #6 (Function Problem Protocols)

44) Describe what is meant by a function. Give an example of something that is not a function.

45) Which are functions? Which are relations?



46) What does  $f(x) = 2x + 3$  mean?

47) Graph  $f(x) = 2x + 3$ . Are straight lines functions?

48) What the  $x$ 's and  $y$ 's in the formula:  $y = mx + b$ ?

49) What is slope?

50) Points  $(0, 1)$ ,  $(2, 4)$ , and  $(6, 10)$  lie on the same line. Compute the slope of the line?

Follow-up:

If you used a different pair of points what would you get? Why?

51) What is meant by the domain and range?

Follow-up:

Referring back to the graphs in #45, ask the student to give the domain and range of the graphs. Or ask the student to write an equation of a function and give its domain and range.

52) What is the domain and range of  $f(x) = \sqrt{x - 4}$ ?

Interview #7 (Function Problem Protocols)

53) Find the  $x$  and  $y$  intercepts for the graph of

$$y = (x-1)(x-2)(x-3)(x-4)(x-5).$$

54) What does the notation  $f(g(x))$  mean?

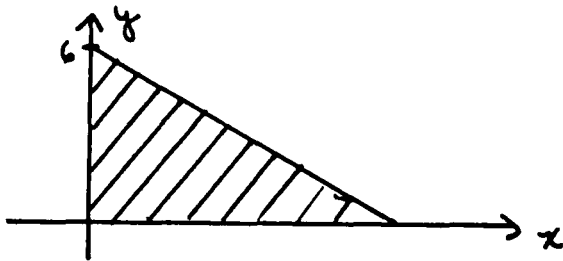
55)  $f(x) = 1/x$  and  $g(x) = x - 3$ .

(a) Find  $f(g(2))$ ,  $f(g(0))$ , and  $f(g(3))$ .

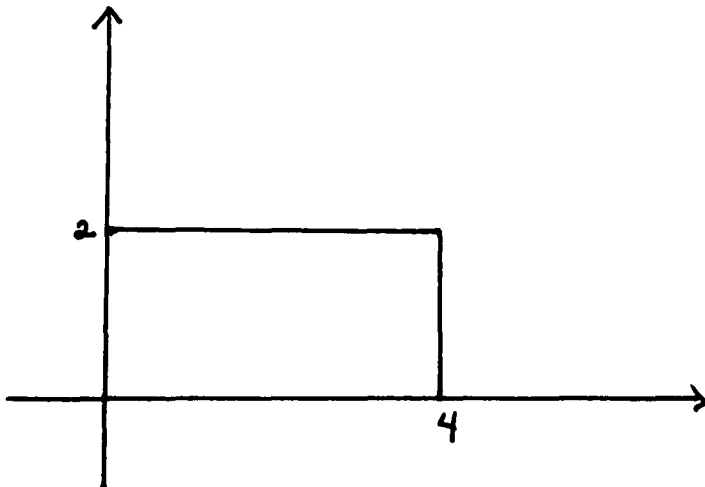
(b) What is the domain of  $h(x) = f(g(x))$ ?

56) Does  $f(g(x)) = g(f(x))$ ?

57) The  $y$ -intercept of the line in the figure is 6. Find the slope of the line if the area of the shaded triangle is 72 square units.



58) Prove that the line segment joining the midpoints of the successive sides of a rectangle form a rhombus.



Interview #8 (Function Problem Protocols)

- 59) During a flu epidemic in a small town, a public health official finds that the total number of people  $P$  who have caught the flu after  $t$  days is closely approximated by the formula:
- $$P(t) = 25t - 20 \quad (1 < t < 29).$$
- (a) How many have caught the flu after 10 days?
- (b) After approximately how many days will 275 have caught the flu?
- 60) Why did you study functions? What good are they?
- 61) If  $(2, a)$  and  $(2, b)$  are points on the graph of function, what can you conclude about  $a$  and  $b$ ?
- 62) It costs a recording artist \$2100 to make a master tape and \$1500 for each 1000 tapes produced. The tapes sell for \$5 each. How many tapes must be sold before a profit is made?
- 63) In the Brown family there are these people: Bill, Jane, Sarah, and Tom. In the Jones family there are Allen, Carol, Dave, George, and Patty. Now if I write these peoples names as ordered pairs, that is as  $(\text{Bill, Brown})$ ,  $(\text{Jane, Brown})$ ,  $(\text{Sarah, Brown})$  and so on for the Brown family. Also do the same for the Jones family. Does this collection of ordered pairs describe a function? If the order is reversed will it be a function?

**APPENDIX C**  
**CODING CRITERIA**

Coding CriteriaConceptual View of Mathematics

Summarizing, a conceptual view of mathematics holds that mathematics is composed of integrated concepts. Students perceive the concepts which underpin procedures as rational, knowable, and vital for their understanding. This view of mathematics is evident by a student's non-reliance on rules and known procedures in problem solving situations, by self-reflection on the selection of procedures and their execution and by the development of a rationale for basic procedures.

Students with a conceptual view:

- 1) could justify procedures on the basis of first principles or on intuitive number sense,
- 2) have integrated procedures as opposed to having isolated applications or have multiple ways to approach a problem,
- 3) use number sense to facilitate approximation or check reasonableness of answers,
- 4) have the ability to summarize or generalize a process used to solve a problem (not mere repetition of steps--must add some interpretation or perspective to summary), or
- 5) graded sample test on the basis of validity of process not just answers or familiarity of form.

**Procedural View Of Mathematics**

Summarizing, a procedural view of mathematics is primarily one that views mathematics as an isolated collection of procedures rules to be memorized. The importance lies in the execution of these procedures and not in the rationale behind them. This view would manifest itself in problem solving situations as an exclusive reliance on formulas or equations to solve these problems. In addition, a procedural view of mathematics would be evident in the student's inability to offer any rationale for basic arithmetic and algebraic procedures.

Students with a procedural view would:

- 1) execute algorithms without evidencing any ties to other idea or ability to justify procedures in terms of first principles,
- 2) have the expectation that mathematics can be solved by merely applying a given algorithm or by solving an algebraic equation,
- 3) used unmonitored trial and error, or
- 4) graded the sample test on the basis of: answers, not process, familiarity of form and marked problems incorrect on the basis of form.

**Demarkations:**

strong evidence

sufficient evidence

weak evidence

Uncertain - unable to code problem dialogue

**Autonomy**

Autonomy is defined here as more than an independence of action. Autonomy is associated with an expectation that mathematics should make internal sense to the individual. By this, it is meant that the student believes she is the primary source of justification for her mathematics. She expects her answers or solutions to internally consistent with her knowledge. A student lacking such an expectation would require an outside authority - teacher or text or answer key to judge the soundness of her solutions. Mathematics for a student who lacked autonomy would represent an external knowledge.

An autonomous student would:

- 1) show independence from the researcher by (a) challenging the researcher's suggestions or (b) delay or put off responding to researcher's ideas or questions until they have had an opportunity to check a computation or idea for themselves,
- 2) check answers voluntarily,
- 3) monitor the reasonableness of answers as problem solution progresses,
- 4) show an expectation that the problems should make sense,
- 5) rephrase problems in her own words, or
- 6) conclude a problem by summarizing idea for herself.

A nonautonomous student would:

- 1) relay on the researcher to supply answers, hints or clues,
- 2) express the view that mathematics is memorized rules given by the teacher, or
- 3) expect others to judge the validity of answers.



**APPENDIX D**  
**SAMPLE CODING SHEETS**

Sample Coding Sheets**Conceptual/Procedural**

Page	Problem	CPT	PCD	Uncertain	Evidence

**Autonomous/Nonautonomous**

Page	Problem	AUT	NAUT	Uncertain	Evidence

**APPENDIX E**

**CODING CRITERIA FOR TEACHERS' QUESTIONS**

Coding Criteria for Teachers' Questions

Procedural

A teacher's question should be coded procedural if:

- (a) it calls for an execution of an algorithm or rule,
- (b) it calls for recall of a basic fact or algorithm,
- (c) it calls for a computation, or
- (d) it calls for the next step in a known algorithm or procedure.

Conceptual

A teacher's question should be coded conceptual if:

- (a) it calls for the application of a definition,
- (b) it calls for conclusions to be drawn,
- (c) it calls for synthesis of separate concepts,
- (d) it calls for anticipation of the next step in an unknown algorithm or procedure,
- (e) it calls for an example to illustrate a given concept,
- (f) it calls for selection among procedures, or
- (g) it calls for a rationale for a procedure.

Rhetorical

A teacher's question should be coded rhetorical if insufficient time or no time is allowed for students to respond and the teacher immediately answers the question.

Led

A teacher's question should be coded led if the teacher's preceding examples or comments draws students' attention to the salient details to answer the question.

Numerical Rating

1-easy; 2-medium; 3-hard.

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**APPENDIX F**

**SAMPLE CLASSROOM CODING SHEET**

Sample Classroom Coding Sheet

Date of Video \_\_\_\_\_

Video tape Counter Reading	Teacher's Question	LED	CPT	PCD	Rating
_____	_____	_____	_____	_____	_____
_____	_____	_____	_____	_____	_____
_____	_____	_____	_____	_____	_____
_____	_____	_____	_____	_____	_____

1. Topics covered in the lesson.
2. Any evidence of problem solving in the lesson period?
3. Any evidence of conjecturing or testing hypothesis? Discovery?
4. Did the teacher motivate the topics by applications? Or motivate by relating to previous topics?
5. How would you describe the teaching style that you observed on this videotape?
6. How would you describe the student involvement during the lesson time?

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**APPENDIX G**

**G-1 COURSE ENROLLMENT**

**G-2 FENNEMA-SHERMAN SUBSCALES**

Table G-1. Course Enrollment

	Participants					
	ANN	TARA	TOM	KEITH	STEVE	SUE
Current Course	A2	A2	A2	AT	AT	AT
Algebra I	9th	9th	9th	8th/9th	9th	9th
Geometry Course	G	G	G	G	AG	G
Placement Score	32 (91%)	29 (83%)	28 (80%)	34 (97%)	27 (77%)	27 (77%)

A2 Algebra II  
 AT Algebra II/Trigonometry  
 G Regular Geometry  
 AG Accelerated Geometry



Table G-2. Fennema-Sherman Subscales: mean item scores\*

Subscale	Participants					
	ANN	TARA	TOM	KEITH	STEVE	SUE
Math anxiety	4.33	2.58	4.92	4.17	4.67	4.42
Confidence	4.58	2.75	5.00	4.25	4.67	4.08
Usefulness	4.17	3.33	4.75	4.42	5.00	3.92
Attitude toward success in math	4.83	4.75	3.42	4.50	4.00	4.42
Effectance motivation	4.33	3.25	3.75	3.83	4.00	3.42
Teacher	4.25	4.25	4.17	4.33	4.08	4.42
Mother	4.83	4.17	4.58	4.00	4.58	3.75
Father	4.83	4.67	4.92	4.17	4.42	3.75
Math as a male Domain	5.00	5.00	4.00	5.00	4.58	5.00

\*The scores represent the mean response to questions within each category. All questions were based on a 5-point Likert scale with 5 representing the most favorable response.

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**APPENDIX H**

**H-1 VOCABULARY LIST FOR KEITH**

**H-2 VOCABULARY LIST FOR ANN**

**H-3 VOCABULARY LIST FOR TARA**

**H-4 VOCABULARY LIST FOR TOM**

**H-5 VOCABULARY LIST FOR SUE**

**H-6 VOCABULARY LIST FOR STEVE**

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Table H-1. Vocabulary List for Keith

	M	E	S	H		M	E	S	H
absolute	X		X	X	humanistic		X		
abstract	X	X			ideas		X	X	
analyze	X	X	X		individualistic	X	X		
ancient			X	X	insight	X	X	X	
anxiety					instinctive				
arbitrary	X				integrating	X			
beauty		X	X	X	intensive	X		X	
boring					interpretative	X		X	
capricious					known				X
cause & effect	X				language		X		
changing		X	X		logical	X			
chronological		X		X	mechanical	X			
classical		X		X	memorize				
clever	X	X	X		multi-dimensional	X	X		
common sense	X				multi-perspective		X		
concentration	X		X		new		X		X
controlled	X		X		objective	X			
controversial		X	X	X	old		X		X
creative		X			open ended		X		
cultural		X		X	opinionated		X		X
current	X	X	X	X	ordered	X			X
deductive	X		X		organized	X		X	X
depth		X	X		practical	X	X		
detailed	X	X	X		precise	X		X	
diagrams	X		X		rational	X			
discovery		X	X		right/wrong	X			
dogmatic					rigid	X			
dull					routine	X		X	
easy		X			rules	X			
elegant		X			sequenced	X			X
exciting		X	X		short cuts	X			
experiential	X		X		simple				
expressive		X			solemn				
factual	X		X	X	specific	X			
fixed	X			X	structured	X			
flexible		X			symbolic		X		X
formulas	X		X		themes		X		
fragmented					theoretical		X	X	
free		X			thorough	X		X	
fun		X			thought provoking	X	X	X	
general	X			X	trial & error	X		X	
goals					truthful	X			
hard					uncertain			X	
universal	X		X		verbal		X		
useful	X		X		visual		X	X	X
valid	X	X	X	X	well-defined	X			
varying			X		writing		X		X

Note. M = math; E = English; S = science; H = history.

Table H-2. Vocabulary List for Ann

	M	E	S	H		M	E	S	H
absolute	X	X		X	humanistic	X	X	X	X
abstract	X	X			ideas	X	X	X	X
analyze	X	X	X	X	individualistic	X	X	X	X
ancient	X		X	X	insight	X	X	X	X
anxiety			X		instinctive	X	X	X	X
arbitrary		X			integrating	X	X	X	X
beauty		X	X	X	intensive	X	X	X	X
boring	X	X	X	X	interpretative	X	X	X	X
capricious		X	X		known	X	X	X	X
cause & effect	X	X	X	X	language	X	X		X
changing		X	X	X	logical	X	X	X	X
chronological	X	X	X	X	mechanical	X	X	X	X
classical	X	X	X	X	memorize	X	X	X	X
clever	X	X	X	X	multi-dimensional	X	X	X	X
common sense	X	X	X	X	multi-perspective	X	X	X	X
concentration	X	X	X	X	new		X	X	X
controlled	X	X	X	X	objective	X	X	X	X
controversial			X	X	old	X	X	X	X
creative	X	X		X	open ended	X	X	X	X
cultural	X	X	X	X	opinionated		X	X	X
current	X	X	X	X	ordered	X	X	X	X
deductive	X	X	X	X	organized	X	X	X	X
depth	X	X	X	X	practical	X	X	X	X
detailed	X	X	X	X	precise	X	X	X	X
diagrams	X	X	X	X	rational	X	X	X	X
discovery	X	X	X	X	right/wrong	X	X	X	X
dogmatic	X	X	X	X	rigid	X	X	X	
dull		X	X	X	routine	X	X	X	
easy	X	X	X	X	rules	X	X	X	X
elegant		X		X	sequenced	X	X	X	X
exciting	X	X	X	X	short cuts	X		X	
experiential	X	X	X		simple	X	X	X	X
expressive		X	X	X	solemn	X	X	X	
factual	X	X	X	X	specific	X	X	X	X
fixed	X	X	X	X	structured	X	X	X	X
flexible	X	X	X	X	symbolic	X			X
formulas	X	X	X	X	themes	X	X		X
fragmented		X	X		theoretical		X	X	
free	X	X		X	thorough	X	X	X	X
fun	X	X	X	X	thought provoking	X	X	X	X
general	X	X	X	X	trial & error	X		X	
goals	X	X	X	X	truthful	X	X	X	X
hard	X	X	X	X	uncertain	X	X	X	X
universal	X		X	X	verbal	X	X		X
useful	X	X	X	X	visual	X	X	X	X
valid	X	X	X	X	well-defined	X	X	X	X
varying	X	X	X	X	writing	X	X	X	X

Note. M = math; E = English; S = science; H = history.

Table H-3. Vocabulary List for Tara

	M	E	S	H		M	E	S	H
absolute	X			X	humanistic		X		X
abstract		X	X		ideas	X	X	X	
analyze	X	X	X	X	individualistic	X	X		
ancient		X		X	insight	X	X	X	
anxiety	X		X		instinctive	X			
arbitrary					integrating	X		X	
beauty		X			intensive	X		X	
boring					interpretative	X	X	X	X
capricious					known			X	X
cause & effect	X	X	X	X	language		X		
changing		X	X	X	logical	X		X	X
chronological	X	X	X	X	mechanical		X	X	
classical		X		X	memorize	X		X	
clever			X	X	multi-dimensional	X		X	X
common sense	X		X	X	multi-perspective	X	X	X	X
concentration	X	X	X		new		X	X	
controlled	X		X		objective	X		X	
controversial		X	X	X	old		X	X	X
creative		X	X		open ended		X	X	
cultural		X		X	opinionated		X	X	X
current	X	X	X		ordered	X		X	X
deductive	X		X		organized	X		X	X
depth	X		X	X	practical			X	
detailed	X		X		precise	X		X	
diagrams	X		X		rational	X	X	X	X
discovery			X	X	right/wrong	X		X	
dogmatic	X		X		rigid	X		X	
dull					routine				
easy					rules	X	X	X	
elegant		X			sequenced	X	X	X	X
exciting		X			short cuts			X	
experiential			X	X	simple				
expressive		X	X		solemn		X		
factual	X		X	X	specific	X	X	X	
fixed	X		X		structured	X		X	
flexible		X	X		symbolic	X	X	X	
formulas	X		X		themes		X	X	X
fragmented				X	theoretical	X		X	X
free		X	X		thorough	X		X	X
fun		X	X		thought provoking	X	X	X	X
general					trial & error			X	
goals	X	X	X		truthful		X		X
hard	X				uncertain			X	X
universal	X	X	X	X	verbal		X		
useful	X		X		visual		X	X	
valid	X		X	X	well-defined	X	X	X	X
varying		X	X	X	writing		X	X	X

Note. M = math; E = English; S = science; H = history.

Table H-4. Vocabulary List for Tom

	M	E	S	H		M	E	S	H
absolute	X			X	humanistic				
abstract		X			ideas	X	X	X	
analyze	X	X	X		individualistic	X	X		
ancient				X	insight	X			
anxiety		X		X	instinctive	X			
arbitrary		X			integrating	X	X		
beauty					intensive	X			
boring		X		X	interpretative	X	X		
capricious		X			known				
cause & effect	X		X	X	language		X		X
changing					logical	X	X		
chronological	X			X	mechanical			X	
classical		X			memorize	X			X
clever					multi-dimensional	X		X	
common sense	X				multi-perspective	X			
concentration	X	X	X		new				
controlled					objective		X		
controversial		X	X		old				X
creative		X			open ended		X	X	
cultural				X	opinionated		X		X
current			X		ordered		X		X
deductive					organized	X	X	X	
depth					practical	X	X		
detailed		X		X	precise	X			
diagrams	X		X		rational				
discovery	X				right/wrong	X			
dogmatic					rigid				
dull		X		X	routine	X			
easy	X				rules	X	X	X	X
elegant					sequenced	X			
exciting					short cuts	X			
experiential			X		simple	X			
expressive		X			solemn				
factual		X		X	specific	X	X		X
fixed				X	structured	X	X		
flexible		X			symbolic		X		
formulas	X		X		themes		X		
fragmented					theoretical		X	X	
free		X			thorough		X	X	X
fun	X				thought provoking	X	X	X	
general					trial & error	X		X	
goals	X				truthful		X		
hard		X			uncertain				
universal	X	X	X	X	verbal		X		
useful	X	X	X		visual				
valid	X	X		X	well-defined		X		
varying		X			writing	X	X		

Note. M = math; E = English; S = science; H = history.

Table H-5. Vocabulary List for Sue

	M	E	S	H		M	E	S	H
absolute					humanistic				
abstract					ideas	X			
analyze	X				individualistic	X			
ancient			X	X	insight				
anxiety				X	instinctive				
arbitrary					integrating	X			
beauty					intensive				
boring					interpretative	X		X	
capricious					known				
cause & effect			X	X	language		X		
changing			X		logical	X			
chronological			X		mechanical	X			
classical					memorize	X			
clever			X		multi-dimensional	X			
common sense	X		X		multi-perspective		X		
concentration					new				
controlled					objective				
controversial		X		X	old				X
creative		X			open ended		X		
cultural				X	opinionated		X		X
current	X				ordered				X
deductive	X				organized		X		
depth					practical				
detailed					precise				
diagrams	X		X		rational				
discovery		X	X	X	right/wrong				
dogmatic					rigid				
dull					routine	X			
easy	X				rules				
elegant					sequenced		X		X
exciting		X	X		short cuts				
experiential					simple				
expressive		X			solemn				
factual					specific	X			
fixed					structured		X		
flexible					symbolic		X		
formulas	X		X		themes		X		
fragmented			X		theoretical			X	X
free					thorough				
fun					thought provoking	X	X	X	
general					trial & error	X			
goals					truthful				
hard					uncertain				X
universal	X	X	X	X	verbal				
useful	X				visual		X	X	
valid					well-defined				
varying					writing		X		

Note. M = math; E = English; S = science; H = history.

Table H-6. Vocabulary List for Steve

	M	E	S	H		M	E	S	H
absolute	X		X		humanistic		X		X
abstract	X	X	X	X	ideas		X		
analyze	X	X	X	X	individualistic		X		
ancient	X	X	X	X	insight	X			
anxiety					instinctive				
arbitrary		X			integrating	X	X		
beauty		X			intensive	X			
boring					interpretative	X	X		X
capricious		X		X	known				
cause & effect	X	X	X	X	language		X		
changing	X	X		X	logical	X			
chronological	X	X		X	mechanical	X			
classical		X	X	X	memorize	X			
clever		X			multi-dimensional	X			
common sense	X	X	X	X	multi-perspective	X	X		
concentration	X	X	X		new	X			
controlled	X		X		objective	X			
controversial		X	X	X	old	X			X
creative	X	X	X	X	open ended	X	X		
cultural		X		X	opinionated	X	X		X
current	X	X	X	X	ordered	X			X
deductive	X	X	X	X	organized	X	X		
depth	X	X	X	X	practical	X			
detailed	X	X	X	X	precise	X			
diagrams	X	X	X	X	rational	X			
discovery	X	X	X	X	right/wrong	X			
dogmatic					rigid				
dull					routine	X			
easy	X	X		X	rules	X			
elegant		X			sequenced	X	X		X
exciting	X	X	X	X	short cuts	X			
experiential	X	X	X	X	simple				
expressive	X	X	X		solemn				
factual	X	X	X	X	specific	X			
fixed	X		X		structured	X	X		
flexible	X	X		X	symbolic		X		
formulas	X	X	X		themes		X		
fragmented				X	theoretical	X		X	X
free	X	X	X	X	thorough	X			
fun	X	X	X	X	thought provoking	X	X	X	
general		X	X	X	trial & error	X			
goals	X	X	X	X	truthful	X			
hard					uncertain				X
universal	X	X	X	X	verbal				
useful	X				visual		X	X	
valid	X				well-defined	X			
varying	X				writing		X		

Note. M = math; E = English; S = science; H = history.



**APPENDIX I**

**STUDENTS' OVERALL RANKING ON CONCEPTUAL/PROCEDURAL  
AND AUTONOMOUS/NONAUTONOMOUS CODINGS**

Student Rankings

Name	Beliefs				Functions	
	C	P	A	NA	CT	IC
Ann	100%*	0%	100%	0%	85%	15%
Tom	100%	0%	100%	0%	75%	25%
Sue	90%	10%	100%	0%	80%	20%
Steve	75%	25%	75%	25%	70%	30%
Keith	62%	38%	40%	60%	85%	15%
Tara	57%	43%	33%	67%	70%	30%

**Note.** C = conceptual; P = procedural; A = autonomous;  
 NA = nonautonomous; CT = correct; IC = incorrect.

\*Proportion of problems that were codable as conceptual to the total number of problems codable as conceptual or procedural.

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**APPENDIX J**

**J-1 RANKING GRID FOR KEITH**

**J-2 RANKING GRID FOR ANN**

**J-3 RANKING GRID FOR TARA**

**J-4 RANKING GRID FOR TOM**

**J-5 RANKING GRID FOR SUE**

**J-6 RANKING GRID FOR STEVE**

Table J-1. Ranking Grid for Keith

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	4	10	11	5	6	1	7	3	9	2	8	boring
easy to do	4	3	2	1	5	11	6	10	9	8	7	hard to do
applied	3	4	5	7	8	1	10	9	6	2	11	theoretical
easy to learn	1	2	3	5	6	10	4	9	11	8	7	difficult to learn
most useful	2	3	6	7	9	1	8	10	5	4	11	least useful
best at	1	2	3	4	5	11	6	9	10	8	7	worst at
routine thinking	8	7	6	1	2	11	3	4	10	9	5	original thinking
visual	6	7	8	9	10	3	11	2	4	1	5	abstract
logical	7	8	9	2	3	11	4	5	1	10	6	arbitrary
most liked	2	3	4	5	6	10	7	9	11	1	8	least liked
basic	1	3	4	2	5	10	6	9	11	8	7	advanced
busy work	1	2	3	4	5	11	7	8	10	9	6	thought provoking
clear	3	4	5	1	2	11	6	9	10	8	7	confusing
flexible	7	8	4	3	6	2	5	10	1	11	9	rigid

Table J-2. Ranking Grid for Ann

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	6	7	5	4	2	1	10	8	11	3	9	boring
easy to do	8	1	2	4	5	9	6	3	11	10	7	hard to do
applied	2	5	4	6	9	1	8	10	11	3	7	theoretical
easy to learn	Lines 2 and 4 combined										difficult to learn	
most useful	3	4	2	1	7	5	6	10	11	9	8	least useful
best at	5	7	3	2	1	9	8	4	11	6	10	worst at
routine thinking	1	3	4	6	8	10	5	2	9	11	7	original thinking
visual	4	5	6	7	10	2	8	3	9	1	11	abstract
logical	1	1	1	1	1	1	3	1	5	1	7	arbitrary
most liked	5	6	4	2	1	7	8	9	11	3	10	least liked
basic	3	1	2	4	11	10	6	5	7	9	8	advanced
busy work	5	3	6	8	10	11	7	2	1	9	4	thought provoking
clear	1	1	1	1	1	5	1	1	9	2	9	confusing
flexible	9	8	7	5	6	4	11	2	1	3	10	rigid

Table J-3. Ranking Grid for Tara

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	6	9	11	2	8	5	10	4	1	3	7	boring
easy to do	6	5	10	7	4	11	9	2	1	3	8	hard to do
applied	3	4	5	6	8	1	10	7	11	2	9	theoretical
easy to learn	2	6	11	7	3	10	8	4	1	5	9	difficult to learn
most useful	1	2	9	5	7	6	8	3	10	4	11	least useful
best at	5	6	11	2	8	10	7	4	1	3	9	worst at
routine thinking	6	7	10	1	2	8	9	5	3	4	11	original thinking
visual	9	10	11	5	6	3	8	2	4	1	7	abstract
logical	2	3	7	9	10	5	11	8	1	6	9	arbitrary
most liked	6	5	11	4	7	10	8	3	1	2	9	least liked
basic	1	2	3	4	5	11	6	9	3	10	7	advanced
busy work	3	4	6	8	2	11	7	1	10	9	6	thought provoking
clear	4	5	11	6	8	10	9	3	1	2	7	confusing
flexible	6	7	8	5	9	4	10	2	3	4	11	rigid

Table J-4. Ranking Grid for Tom

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	11	10	9	5	4	3	2	6	1	7	8	boring
easy to do	1	2	3	4	5	11	10	6	9	7	8	hard to do
applied	1	2	3	4	7	6	10	11	5	9	8	theoretical
easy to learn	3	2	1	6	4	11	10	5	8	7	9	difficult to learn
most useful	1	2	3	4	7	6	10	11	5	9	8	least useful
best at	1	2	3	4	5	11	10	6	9	7	8	worst at
routine thinking	2	1	3	4	5	9	10	6	11	8	7	original thinking
visual												abstract
logical												arbitrary
most liked	same as line 1											least liked
basic	same as line 2											advanced
busy work	1	2	3	4	5	11	9	6	10	7	8	thought provoking
clear						11						confusing
flexible						2			1			rigid

Table J-5. Ranking Grid for Sue

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	10	11	8	2	4	5	9	3	7	1	6	boring
easy to do	11	3	4	5	8	10	9	2	6	1	7	hard to do
applied	2	5	3	4	8	1	9	7	11	6	10	theoretical
easy to learn	11	3	5	4	8	9	10	2	7	1	6	difficult to learn
most useful	4	2	3	1	9	5	6	7	10	8	11	least useful
best at	11	2	3	5	9	8	6	4	10	1	7	worst at
routine thinking	11	2	3	5	8	10	9	4	7	1	6	original thinking
visual	7	8	9	5	6	3	10	2	4	1	11	abstract
logical	1	1	1	1	1	1	1	1	1	1	1	arbitrary
most liked	11	8	10	3	5	6	7	2	4	1	9	least liked
basic	11	1	2	5	7	8	6	4	10	3	9	advanced
busy work	7	2	3	10	8	11	5	4	9	1	6	thought provoking
clear	5	1	2	8	9	11	6	4	10	3	7	confusing
flexible	7	6	4	3	5	2	10	9	1	11	8	rigid



Table J-6. Ranking Grid for Steve

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting	6	5	4	3	7	11	8	2	10	1	9	boring
easy to do	3	2	1	9	8	11	6	4	10	5	7	hard to do
applied	1	2	3	6	9	5	10	4	8	7	11	theoretical
easy to learn	4	3	1	7	6	11	5	2	10	8	9	difficult to learn
most useful	1	2	3	4	7	6	10	5	9	8	11	least useful
best at	4	3	2	10	9	11	8	1	6	5	7	worst at
routine thinking	4	3	2	9	7	11	6	5	1	10	8	original thinking
visual	4	6	3	10	8	5	9	1	7	2	11	abstract
logical	3	2	1	9	8	11	4	6	5	7	10	arbitrary
most liked	5	4	3	9	8	10	6	2	11	1	7	least liked
basic	5	4	2	10	9	11	7	3	1	6	8	advanced
busy work	5	4	3	10	9	11	6	2	1	8	7	thought provoking
clear	2	3	1	9	10	11	6	5	4	7	8	confusing
flexible	11	10	9	2	3	1	6	5	8	7	4	rigid

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**APPENDIX K**

**K-1 VOCABULARY LIST FOR TEACHERS**

**K-2 RANKING GRID FOR TEACHERS**

Table K-1. Vocabulary List for Teacher

M		M	
absolute		humanistic	
abstract	X	ideas	
analyze	X	individualistic	
ancient		insight	
anxiety		instinctive	
arbitrary		integrating	
beauty		intensive	
boring		interpretative	
capricious		known	
cause & effect		language	
changing		logical	X
chronological		mechanical	
classical		memorize	
clever		multi-dimensional	
common sense	X	multi-perspective	
concentration		new	
controlled		objective	
controversial		old	
creative		open ended	
cultural		opinionated	
current		ordered	X
deductive	X	organized	
depth		practical	
detailed		precise	
diagrams	X	rational	X
discovery	X	right/wrong	
dogmatic		rigid	
dull		routine	
easy		rules	
elegant		sequenced	X
exciting		short cuts	X
experiential		simple	
expressive		solemn	
factual		specific	
fixed		structured	
flexible		symbolic	
formulas	X	themes	
fragmented		theoretical	
free		thorough	
fun		thought provoking	X
general		trial & error	
goals		truthful	
hard		uncertain	
universal		verbal	
useful	X	visual	
valid		well-defined	
varying		writing	

Note. M = math

Table K-2. Ranking Grid for Teacher

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric const.	absolute values	
interesting						3		2		1		boring
easy to do	3	1	4	5	8	10	6	7	11	2	9	hard to do
applied	1	2	3	5	11	4	11	1	11	6	11	theoretical
easy to learn	4	1	3	5	8	9	7	6	11	2	10	difficult to learn
most useful	1	2	3	6	9	5	10	8	7	4	11	least useful
best at	6	1	5	2	7	10	8	4	11	3	9	worst at
routine thinking	7	1	5	2	6	9	4	3	11	8	10	original thinking
visual	3	1	2	4	9	7	8	6	11	5	10	abstract
logical	4	1	2	3	10	8	9	5	7	6	11	arbitrary
most liked	6	1	5	3	7	9	8	4	11	2	10	least liked
basic	5	1	4	2	7	10	8	6	11	3	9	advanced
busy work	5	1	5	2	5	10	5	5	11	3	9	thought provoking
clear	7	1	6	2	3	10	8	5	11	4	9	confusing
flexible	6	11	5	4	2	1	9	7	3	2	10	rigid

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**APPENDIX L**

**CLASSROOM TIME ANALYSIS**

Algebra II Classroom Time Analysis

- 10/4 Video missing due to equipment failure
- 10/5 LECTURE - 30 min. (18 min. lecture, 12 min. drill)  
HOMEWORK - 14 min. (8 min. student's presentation, 6 min. teacher's presentation)
- 10/6 HOMEWORK - 26 min. (24 min. students, 2 min. teacher)  
SAT QUESTIONS - 2 min.  
LECTURE - 18 min. (3 min. drill)
- 10/7 MISCELLANEOUS - 3 min.  
LECTURE - 24 min.  
HOMEWORK - 17 min. (16 min. students, 1 min. teacher)  
SAT QUESTIONS - 2 mins.
- 10/10 HOMEWORK - 24 min. (22 students, 2 min. teacher)  
LECTURE - 18 min.  
NEW HOMEWORK - 4 min.
- 10/11 QUIZ - 23 min.  
HOMEWORK - 11 min.  
LECTURE - 7 min. (5 min. drill)
- 10/12\* MISCELLANEOUS - 3 min.  
LECTURE - 27 min.  
REVIEW QUIZ - 5 min.  
\*Shortened class period
- 10/13 MISCELLANEOUS - 3 min.  
HOMEWORK - 30 min. (18 min. students, 12 min. teacher)  
REVIEW QUIZ - 16 min.
- 10/17 HOMEWORK - 37 min. (21 min. students, 16 min. teacher)  
LECTURE - 10 min.  
MISCELLANEOUS - 1 min.
- 10/18 SAT QUESTIONS - 9 min.  
LECTURE - 9 min.  
HOMEWORK - 17 min. (15 min. students, 2 min. teacher)  
CORRECTED WORKSHEETS - 19 min.
- 10/19 HOMEWORK - 22 min. (18 min. students, 4 min. teacher)  
DRILL (REVIEW EXERCISES) - 17 min.  
ANSWERS READ - 2 min.  
PROBLEMS WORKED - 6 min. (by teacher)
- 10/20 UNIT TEST
- AVERAGE TIME SPENT ON HOMEWORK - 22 min.  
AVERAGE TIME SPENT IN LECTURE - 20 min.

Algebra II/Trigonometry Classroom Time Analysis

9/21 and 9/22 audio-taped

9/23 MISCELLANEOUS - 1 min.  
HOMEWORK - 20 min.  
LECTURE - 21 min.  
REVIEWED PREVIOUS TEST - 6 min.

9/26 MISCELLANEOUS - 1 min.  
HOMEWORK - 30 min. (21 min. students, 9 min. teacher)  
LECTURE - 17 min.  
MISCELLANEOUS - 1 min.

9/27 MISCELLANEOUS - 2 min.  
QUIZ - 23 min.  
LECTURE - 21 min.

9/28 MISCELLANEOUS - 3 min.  
HOMEWORK - 23 min. (21 min. students, 2 min. teacher)  
SAT QUESTIONS - 2 min.  
HOMEWORK - 19 min. (17 min. students, 2 min. teacher)  
MISCELLANEOUS - 1 min.

9/29 MISCELLANEOUS - 7 min.  
HOMEWORK - 14 min.  
SAT QUESTIONS - 1 min.  
REVIEW FROM CHAPTER - 8 min.  
REVIEW - 9 min.

9/30 UNIT TEST

AVERAGE TIME SPENT ON HOMEWORK - 21 min.  
AVERAGE TIME SPENT IN LECTURE - 20 min.

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**APPENDIX M**

**CLASSROOM QUIZZES AND TESTS**



Algebra II Quiz  
3.1 - 3.4 (50 pts)

State the domain and range of each relation

(1)  $\{(2, 3), (4, 7), (2, 8)\}$

(2)  $\{(4, -1), (8, -1), (15, -1)\}$

The domain of the relation is  $\mathbb{R}$ . Graph.

(3)  $\{(x, y): y = -3x\}$

Graph each relation. State its domain and range.

(4)  $\{(x, y): y = |x| + 2\}$  (5)  $\{(x, y): y = x^2 + 1\}$

Domain =    Domain =

Range =    Range =

Give the domain and range of each relation. State whether the relation is a function.

(6)  $\{(5, 2), (7, 4), (9, -3), (8, 4)\}$

Graph each relation. Determine if it is (a) a function, (b) a 1 - 1 function.

(7)  $\{(x, y): y = 4x - 1\}$  (8)  $\{(x, y): x = y^2\}$

Find the slope and y-intercept.

(9)  $2(-x) = 2(y - 4)$

Write the equation of the line through points A, B.

(10) given: A(-2, 5), B(6, -1)

Algebra II Test  
Chapter 3

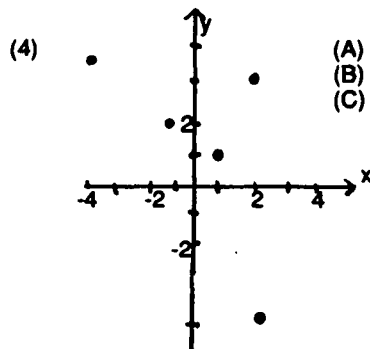
State the domain and range of each relation.

(1)  $\{(3, 5) (2, 7) (-3, 7)\}$  (2)  $\{(x, y): y = |x|\}$

Graph the relation and state its domain and range.

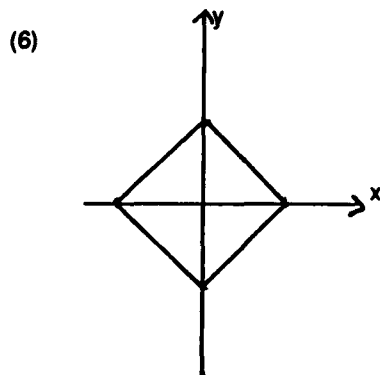
(3)  $x + y = 5$

- (A) List the ordered pairs in the relation.  
(B) State the domain and range.  
(C) Is it a function?



Determine whether each relation is a function? A 1 - 1 function?

(5)  $\{(x, y): x = y^2 + 3\}$



Find the slope and y - intercept.

(7)  $2(x - y) = 3(x + y)$

Write the equation of the line in slope-intercept form:

(8) Line through  $(-1, 4)$  and parallel to  $y = 6$ .

(9) Line through  $(8, -5)$  and perpendicular to  $y = -2x + 7$ .

Given that  $x$  and  $y$  vary directly, find the following:

(10) If  $y = 2$  when  $x = 8$ , what is the value of  $x$  when  $y = 3$ ?

Solve:

(11) Suppose  $y$  varies directly as the square of  $x$ . If  $y = 24$  when  $x = 2$ , what is the value of  $y$  when  $x = 3$ ?

(12) Given that  $T$  varies directly as the square of  $p$ , find the constant of proportionality if  $T = 18$  when  $p = 4$ .

Simplify: (13) [6.8]      (14) [-3.2] [5.1]

Graph the equation for  $-2 \leq x \leq 2$ . (15)  $y = [x] + 3$

Let  $f(x) = 2x + 3$  and  $g(x) = x^2 - 1$  and  $h(x) = [x]$ . Simplify each expression.

(16)  $g(-3)$       (17)  $f(x + 4)$       (18)  $g(f(-4))$

(19)  $f(g(a))$       (20)  $g(h(f(-1/4)))$

Algebra II/Trigonometry Quiz  
3.1 - 3.5 (50 pts)

- (1) If  $g: x \rightarrow 2x^2 + 3x - 5$ , find  $g(-3)$ .
- (2) If  $f(x) = 2x + 4$  and  $g(x) = x^2 - 2x + 1$ , find  $f(g(1/4))$ .
- (3) Determine whether each of the following is a function.
  - (a)  $\{(2, 3) (2, -1) (3, 4) (5, 4)\}$
  - (b)  $\{(x, y) : |y| = x + 1\}$
- (4) Graph the equation:  $y = |x + 3|$
- (5) Graph the inequality on a coordinate plane.  $x + 6 \leq 3y$
- (6) Graph both inequalities on the same plane, and shade the intersection.  
 $2x < y + 3$  and  $y < -x$
- (7) Given  $A(3, -7)$   $B(-2, -5)$  find the slope of  $AB$ .

Algebra II/Trigonometry Test  
Chapter 3

Define: (1) a relation (2) a function

State the domain and range of each function and give a rule for the pairing.

(3)  $\{(2, 5), (7, 20), (50, 149), (1, 2)\}$

Let  $f(x) = 3x$  and  $g(x) = x/3$ , then

(4)  $g(f(6)) =$

Let  $f(x) = x/(x - 1)$  and  $g(x) = 1/x$ , then

(5)  $f(g(4)) =$

Let  $f(x) = x^2$ ,  $g(x) = 3x$  and  $h(x) = x - 3$ , then

(6)  $f(g(h(2))) =$

Determine whether each relation is a function.

(7)  $\{(0, 0), (2, 4), (2, -4)\}$  (8)  $\{(x, y) : xy = 1\}$

Graph each relation. State the domain and the range of each relation. State whether the relation is a function.

(9)  $\{(x, y) : y = 1/2x + 2\}$  (10)  $\{(x, y) : x = y^3\}$

(11)  $\{(x, y) : y = |x| - 3\}$ .

Graph each inequality on the coordinate plane.

(12)  $x + y \leq 2$

Graph. Show intersection of solutions.

(13)  $y \leq 3/4 x + 3$        $y \geq -1/3 x + 1$

Find the slope of the line determined by the given points.

(14)  $(-1, -3)$   $(3, 4)$  (15)  $(-3, 3)$   $(2, 3)$

Write the equation of the line given the following conditions:

(16) points  $(-1, -2)$   $(4, -3)$ .

(17) line through  $(4, 6)$  perpendicular to the line with equation:  
 $y = 2/3x - 7$ .

(18) line through  $(-12, 8)$  and parallel to line with equation:  
 $y = -3/2x + 20$ .

(19) line with x-intercept 5 and y-intercept -3.

Given a direct variation, find the following.

(20) If  $d = 220$  when  $t = 4$ , find the value of  $d$  when  $t = 3$ .

(21) Suppose  $y$  varies directly as the square of  $x$ . If  $y = 24$  when  $x = 2$ , find the value of  $y$  when  $x = 3$ .

(22) The surface area ( $A$ ) of a cube varies directly as the square of the length ( $e$ ) of the edge of the cube. If the surface area is  $37.5 \text{ cm}^2$  when the edge is  $2.5 \text{ cm}$ , what is the surface area when the edge is  $3 \text{ cm}$ ?

- (23) Power ( $w$ ) in watts is proportional to the square of the current ( $a$ ) in amperes. If the power of a radio station is 1000 watts when the current is 5 amperes, what must the current be in order for the power to be 36,000 watts?
- (24) If  $f(x) = 1 - x^2$  and  $g(x) = |x|$ , find  $f(g(3))$ .
- (25) Graph the relation  $\{(x, y) : |2x| = |y|\}$ .

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