Spring 1988

Solitary waves and nonlinear Klein-Gordon Equations

Panagiotis Andreas Papadopoulos
University of New Hampshire, Durham

Order Number 8816698

Solitary waves and nonlinear Klein-Gordon Equations

Papadopoulos, Panagiotis Andreas, Ph.D.

University of New Hampshire, 1988

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

SOLITARY WAVES AND NONLINEAR KLEIN-GORDON EQUATIONS

BY

PANAGIOTIS ANDREAS PAPADOPOULOS
B.S. , University of Ioannina, 1980
M.S. , Adelphi University, 1984

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

in

Physics

May , 1988

This dissertation has been examined and approved.

Harvey Shepard
Dissertation director, Harvey Shepard
Professor of Physics

Ludwig C. Balling
Ludwig C. Balling, Professor of Physics

John Dawson
John Dawson, Professor of Physics

Robert H. Lambert
Robert H. Lambert, Professor of Physics

Dawn C. Meredith
Dawn Meredith, Asst. Professor of Physics

April 22, 1988
Date

ACKNOWLEDGEMENTS

I would like to thank Professor Harvey Shepard for introducing me to the subject of nonlinear dynamics and for his help throughout this work.

I thank David Campbell and North-Holland Publishing Company for permission to use several figures and tables from reference 1.

To all friends who through the years of agony supported and believed in me I dedicate this work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

SOLITARY WAVES AND NONLINEAR KLEIN GORDON EQUATIONS

by

Panagiotis A. Papadopoulos
University of New Hampshire, May, 1988

We analytically study kink-antikink (K-K) collisions in the classical one spatial dimension and time phi-fourth field theory as an example of inelastic collisions between solitary waves. We use the linear eigenvalue collective coordinate approach to describe the system in terms of the separation distance between the kink and the antikink and the amplitude of shape vibrations generated on each kink as a result of the collision. By calculating the energy given to the shape vibrations as a function of the incoming velocity, we find the critical value of the initial velocity above which the two colliding kinks always separate after the collision. A model previously proposed to explain the two-bounce collisions in terms of a resonant energy exchange between the orbital frequency of the bound K-K pair and the frequency of shape vibrations is modified using our analytical results. We derive a (data-free) formula that predicts the values of the initial velocities for which resonance occurs. A generalized version of this modified model is shown to give good results when it is applied to K-K collisions in other similar field theories. In the Appendices Nonlinear Klein

Gordon equations with solitary (travelling) wave solutions are reviewed and solved for particular cases. The solutions are related to soliton solutions of the sine-Gordon equation. Also the phi-fourth equation perturbed with a constant force and dissipation is solved, and finally, we present new kink-bearing integro-differential and nonlinear differential equations.

CHAPTER I

INTRODUCTION

Solitary Waves and Solitons

When a solution of a nonlinear field equation represents a localized wave of permanent form that travels with uniform velocity and no distortion in shape, we call it a solitary wave. If these solitary waves have in addition the property of retaining their shapes and velocities (asymptotically in time) even after collisions, we call them solitons. By the term " localized " we mean that the energy density of these solutions at any finite time t is finite in some finite region of space and falls to zero at spatial infinity fast enough to be integrable. Since there is no distortion in shape as the wave propagates, the energy density should also move undistorted with constant velocity.

Despite the complexity introduced by the presence of the nonlinear terms in the field equations, there are a large number of such nonlinear equations or coupled field equations (when several fields are involved), which have solitary wave solutions and even solitons[22].

Understanding some or all solutions of a set of classical relativistic nonlinear field equations could help to elevate the knowledge to quantum level and enable one to study the vacuum, the one-particle states, and other features of the corresponding quantum field theory. Solitary waves, as spatially localized nonlinear excitations, are found in a variety of natural systems from organic polymers to biological structures. Yet, despite the clear importance of these solitary waves, their interactions, which describe the dynamical properties of the system, are not well understood.

In this work we examine the interactions of solitary waves called kinks (K) and their spatial reflections called antikinks ($\bar{K}$) in the non-integrable two (one space and time) dimensional classical theory. Since the major difference between solitons and solitary waves is that the latter can collide inelastically, we investigate K-$\bar{K}$ collision processes as an example of the inelastic scattering of solitary waves.

Previous Studies of Kink Collisions

Kink interactions in ϕ^4 via collisions have been studied by several previous authors [1,2], numerically and, to some extent, also theoretically [2]. The results of numerical simulations [1] showed that when two kinks collide the interaction is inelastic and either the kinks exit the interaction region with some final velocity less than the initial velocity or they form a "bound state" and never separate . That demonstrates the fact that a kink is a solitary wave and not a soliton. Whether we have trapping or liberation is determined by the initial velocity. In fact above some value of the initial velocity Uc (the critical velocity), we always have inelastic scattering and below that value, trapping. The negligible presence of " radiation " in the whole process brings up the question of where did the energy go. It is then clear that fluctuation modes trapped in the kink were excited during the collision. An early theoretical model was given by Sugiyama [2] where a collective coordinate is introduced into the K$\bar{K}$ system, and an Ansatz solution, where kinks and their fluctuation modes are linearly superposed, is tested. He derives and approximately solves the equations of motion and estimates the energy given to the internal modes in terms of the probability to excite these modes from the ground state to higher states as result of the collision .

Even if we accept his initial setup of the problem, we find his calculations in disagreement with ours. In his effort to solve the complicated equations of motion he makes approximations that neglect terms essential to the physics of the problem. Also it is not clear how the energy of the fluctuation modes can be extracted from the final integral expression he derives. Nevertheless, in reference [1] the numerical and theoretical results were extended. The authors, looking below the critical velocity in narrow velocity ranges, find that it is possible for the kinks to even escape from the bound state and become liberated. That was an unexpected result. In fact, the kinks bounce back and forth before they are scattered away. As the initial velocity is increased (but still less than Uc), trapping and liberation alternates, up to the point where the initial velocity reaches the value Uc, above which the kinks always scatter inelastically. The contribution of radiation is still negligible.

In their attempt to explain the results they observe, they attribute the phenomenon to an energy exchange mechanism between the moving kinks and their fluctuation modes. A phenomenological model is proposed that predicts the observed results with reasonably good agreement. Yet, there is no complete theoretical method that leads to the numerical results.

It is the purpose of this thesis to examine in detail kink-antikink interactions via collisions and offer a theoretical approach that makes the comparison of theory to the (numerical) experiment clearer.

Organization of the Dissertation

We start in Chapter II by introducing theory and the associated field equation. We are rescaling fields and coordinates, and the resulting field equation, free of parameters, is solved to give the kink and antikink solutions. Looking for hidden degrees of freedom we perturb the static kink solution around its equilibrium by adding a small,time-and space-dependent quantity $y(x,t)$, to the unperturbed static kink. Putting this perturbed form of the kink solution in the field equation and linearizing in y we solve the resultant field equation for y and discuss the physical nature of each solution found (fluctuation modes).

In Chapter III we introduce the collective coordinate approach as a possible way of studying kink collisions [3,4,5]. According to this approach you assign a coordinate to each degree of freedom known from the fluctuation modes and collectively describe the system by means of these collective coordinates. This is an approximate method since our system actually possesses infinite degrees of freedom, and the linear perturbation analysis can reveal only a few. Trying to decide how to insert these "collective" coordinates in the system we discuss two possible ways: The parametrical collective coordinate (PCC) and the linear eigenfunction collective coordinate (LECC). In Chapter IV we adopt the latter one and extend the approach to include both kink and antikink. This form of the solution (with kinks and fluctuation modes linearly included), is then used in Chapter IV to describe kink collisions. Upon using the lagrangian and the (assumed) Ansatz solution, the equations of motion for the collective coordinates and the total energy of the system are derived. Calculating the energy in the shape modes as a function of the incoming kink velocity we determine the critical value of the initial velocity Uc above which a kink collision always ends up with reflection.

Chapter V is devoted to "linking" theory with the results of numerical simulations of kink collisions carried out by Campbell et.al.[1]. We combine the results found in Chapter IV with a modified version of the model the authors first proposed to explain their own numerical results. This combination bridges the gap between theory and experiment and "liberates" the old model from the parameters Campbell had introduced in the model (determined from the data) and offers a better explanation of quantities not well understood before. As a final check we generalize our modified model and apply it to the Double Sine-Gordon and Modified Sine-Gordon field equations with success. We conclude in Chapter VI with some remarks on the approximations made and procedure followed, and we end with some suggestions and directions for future work.

In the Appendices we list all the calculations made and the results of the integrations used.The general nonlinear Klein Gordon equation is studied for solitary (travelling) wave solutions which we relate through a transformation to the sine Gordon soliton solutions.We also give a parametric potential reducible to ϕ^2 , ϕ^2, ϕ^6 potentials for special values of the parameter and the perturbed ϕ^4 equation with dissipation and constant force term is solved. Finally we present new types of integrodifferential equations and differential equations which have kink type solitary solutions.

CHAPTER II

THE CLASSICAL PHI-FOURTH MODEL

Definition of the Model

In what follows we will be working with the two-dimensional field theory described by the lagrangian density

$$(2.1) \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 - \frac{1}{4}\lambda \phi^4 - \frac{1}{4}\frac{m^4}{\lambda}$$

m^2 , λ being positive parameters.

The resulting field equation from (2.1) is given by

$$(2.2) \qquad \phi_{tt} - \phi_{xx} = m^2\phi - \lambda\phi^3$$

Rescaling fields and coordinates (see Appendix A) we can eliminate the parameters and the field equation takes the final form

$$(2.3) \qquad \phi_{tt} - \phi_{xx} = \phi - \phi^3$$

with conserved total energy

$$(2.4) \qquad E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\phi_t)^2 + \frac{1}{2}(\phi_x)^2 + \frac{1}{4}(\phi^2 - 1)^2\right]$$

Higher dimensional versions of this theory play a central role in the models of the spontaneous symmetry breakdown required in the unified theories of the electromagnetic and weak interactions [40-41]. The field equation (2.3) has static solution

(2.5) $$\phi = \pm \tanh \frac{x - X_0}{\sqrt{2}}$$

Xo=constant

the solution with the positive sign plotted in fig.1 will be called the kink (K) and the one with the negative sign the antikink ($\bar{K}$).

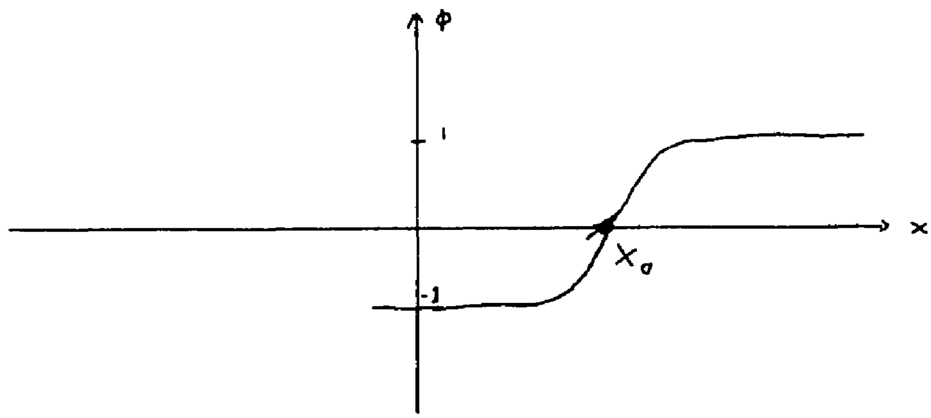


Fig.1

Note that a change in Xo merely shifts the solution in space. This is just a reflection of the translational invariance of the field equation. The other symmetries of the lagrangian under $X \longleftrightarrow -X$ and separately under $\phi \longleftrightarrow -\phi$ are reflected in the relations which take on a particularly simple form when Xo is chosen equal to zero.

(2.6) $$\phi_K(x) = -\phi_{\bar{K}}(x) = \phi_{\bar{K}}(-x)$$

The solitary wave properties of the solution can be seen by plotting the energy density (fig.2)

(2.7) $$\varepsilon(x) = \frac{1}{2}(\phi_x)^2 + \frac{1}{4}(\phi^2 - 1)^2 = \frac{1}{2}\operatorname{sech}^4 \frac{x - X_0}{\sqrt{2}}$$

is clearly localized near Xo, and it resembles a " lump " of matter in the sense that is a static self-supporting localized packet of energy.

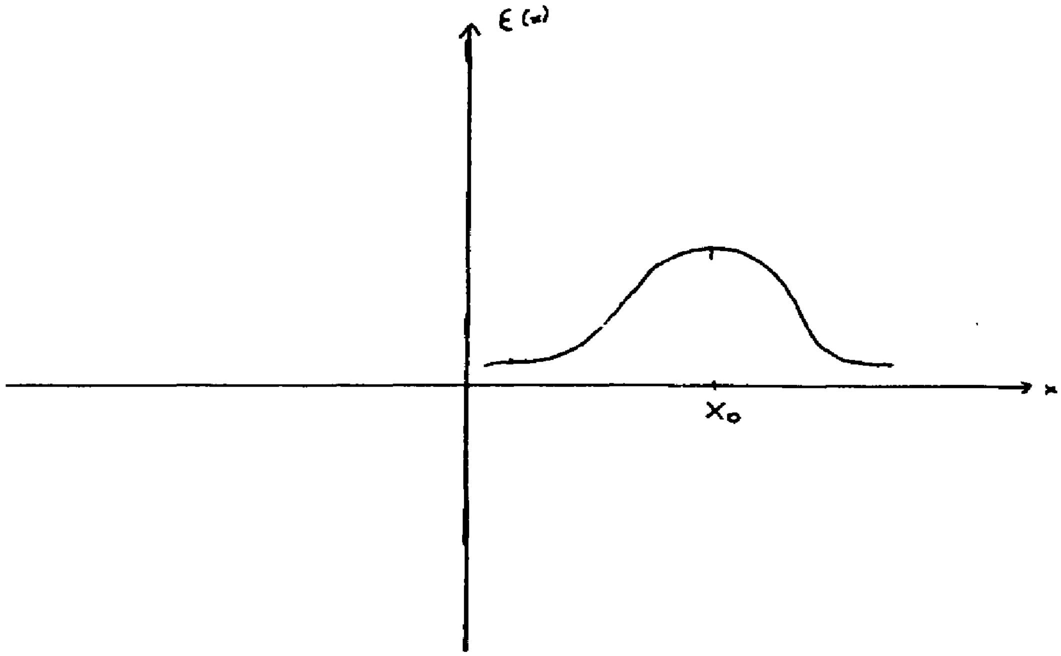


fig. 2

Given the static solution : since the system is Lorentz invariant and is a scalar field one needs only to transform the coordinate variables to obtain a moving kink solution. This gives

$$\phi(x,t) = \tanh \frac{x - X_0 - vt}{\sqrt{2(1-v^2)}} \tag{2.8}$$

(where $|v| << 1$)

and corresponding energy density

$$\varepsilon(x,t) = 1/2(1-v^2) \operatorname{sech}^4 \frac{x - X_0 - vt}{\sqrt{2(1-v^2)}} \tag{2.9}$$

Notice that as a result of the motion the " lump " of matter has been squeezed in width; consequently, its amplitude has been increased. We can also see the particle-like behavior of the kink from the total energy expression

$$(2.10) \qquad E = \int_{-\infty}^{\infty} dx \left[\frac{\phi_t^2}{2} + \frac{\phi^2}{2} + \frac{1}{4}(\phi^2 - 1)^2 \right] = \frac{2\sqrt{2}}{3} \frac{1}{\sqrt{1-v^2}}$$

If we define the "mass" M of the kink to be the total static energy of the kink

$$(2.11) \qquad M = E_{\text{static}} = \int_{-\infty}^{\infty} dx \left[\phi_x^2/2 + (\phi^2-1)^2/4 \right]$$

then $$E = M/\sqrt{1-v^2}$$

This is the same as the Einstein mass-energy equation for a particle. Therefore it is clear why in the quantum version of this theory the kink solution behaves like a particle state [22].

Fluctuation Modes

In order to study how the kinks interact with each other in a collision it is important to know how a free static kink will react to a small perturbation in space and time. The kink, for instance, could absorb the disturbance and translate by a small quantity or could vibrate or emit "phonons". In order to study this, we place a static kink at the origin and add a small perturbation quantity y(x,t). We assume that the kink is able to absorb y(x,t) so that

$$\phi(x,t) = \phi_0(x) + y(x,t) \tag{2.12}$$

(ϕ_0 the static kink solution)
is still a solution of ϕ^4 equation. Substituting this in the full equation

$$\phi_{tt} - \phi_{xx} = \phi - \phi^3$$

we find that y(x,t) must satisfy the equation

$$\begin{aligned} y_{tt} - y_{xx} &= y - 3\tanh^2(x/\sqrt{2})\,y - y^3 \\ &\quad - 3y^2\tanh(x/\sqrt{2}) \end{aligned} \tag{2.13}$$

Since y(x,t) is assumed to be small, y(x,t) $\ll$ 1, the nonlinear terms in y can be neglected. This simplifies the equation to

(2.14) $$y_{tt} - y_{xx} = y\left(1 - 3\tanh^2\frac{x}{\sqrt{2}}\right)$$

Looking for y in the form

(2.15) $$y(x,t) = e^{i\omega t}\, y(x)$$

we finally end up with the eigenvalue problem

(2.16) $$y_{xx}(x) + \left[\omega^2 + 1 - 3\tanh^2\frac{x}{\sqrt{2}}\right] y = 0$$

For $|X| \to \infty$, y describes plane waves $y \propto e^{iKX}$

providing $\omega^2 \geq \omega_0^2 = 2$ and $\omega^2 = K^2 + 2$

or $\underset{x \to \mp\infty}{y(x) \propto e^{\pm KX}}$ providing $\omega^2 + K^2 = \omega_0^2$, $\omega^2 < 2$

We are seeking solutions $y(x)$ in the form

(2.17) $$y(x) = e^{-KX} f(x)$$

Expand $f(x)$ in terms of $\phi_0(x)$ and its first derivative;

$$f(x) = a_1 \tanh\frac{x}{\sqrt{2}} + a_2 \operatorname{sech}^2\frac{x}{\sqrt{2}} + a_0$$

$$= a_1 \tanh\frac{x}{\sqrt{2}} + a_2' \tanh^2\frac{x}{\sqrt{2}} + a_0'$$

$$= N\left(c_1 \tanh\frac{x}{\sqrt{2}} + \tanh^2\frac{x}{\sqrt{2}} + c_2 \right)$$

We try the solution

$$y(x) = N e^{-Kx}\left[c_1 \tanh\frac{x}{\sqrt{2}} + \tanh^2\frac{x}{\sqrt{2}} + c_2 \right] \tag{2.18}$$

(Where N is normalization constant.)

We put $y(x)$ into (2.16), equate terms, and we get the relations

$$c_1 = \sqrt{2}\,K \qquad \omega^2 + K^2 = \omega_0^2 = 2$$

$$c_2 = \frac{(2K^2 - 1)}{3} ,$$

The solution then reads

$$y(x) = e^{-Kx} N\left[\tanh^2\frac{x}{\sqrt{2}} + \sqrt{2}\,K \tanh\frac{x}{\sqrt{2}} + \frac{2K^2-1}{3} \right] \tag{2.19}$$

This solution describes bound states

with $K^2 + \omega^2 = 2$ and $\omega^2 < 2$

For $|X| \to +\infty$ the solution is finite.

for $X \to -\infty$ if the solution is finite, we must have

$$\left[\tanh^2 \frac{x}{\sqrt{2}} + \sqrt{2}\, K \tanh \frac{x}{\sqrt{2}} + \frac{2K^2 - 1}{3} \right] \longrightarrow 0$$

Then

$$\left[1 - \sqrt{2}\, K + \frac{2K^2 - 1}{3} \right] \longrightarrow 0$$

$$3 - 1 - 3\sqrt{2}\, K + 2K^2 = 0$$

$$K = \begin{cases} \sqrt{2} \\ 1/\sqrt{2} \end{cases}$$

Therefore, there are two discrete bound states

(2.20) $$y(x) = e^{-\sqrt{2}x} N \left[\tanh^2 \frac{x}{\sqrt{2}} + 2 \tanh \frac{x}{\sqrt{2}} + 1 \right]$$

with eigenvalue $\omega = 0$ and $N = \sqrt{3/4\sqrt{2}}$

(2.21) and $$y(x) = N e^{-x/\sqrt{2}} \left(\tanh^2 x/\sqrt{2} + \tanh x/\sqrt{2} \right)$$

with eigenvalue $\omega = (3/2)^{1/2}$ and $N = \sqrt{3/2\sqrt{2}}$.

$y(x)$ is normalized as $\int_{-\infty}^{\infty} y^2(x)\, dx = 1$

For the radiation modes ($\omega^2 > \omega_0^2 = 2$), K is imaginary and the solution gives

$$y(x) = \frac{N}{3} e^{iKx} \left[3 \tanh^2 \frac{x}{\sqrt{2}} - 2K^2 - 1 - 3i\sqrt{2}\, K \tanh \frac{x}{\sqrt{2}} \right] \qquad (2.22)$$

with $\omega^2 = K^2 + \omega_0^2$ $\qquad \omega_0 = \sqrt{2}$

Normalizing as

$$\int_{-\infty}^{\infty} dx\, y_K^*(x)\, y_{K'}(x) = \delta(K - K')$$

$$\left(\frac{N}{3} \right)^{-2} = 4\pi \left[2 (K^2 + 1)^2 + K^2 \right]$$

(for a different approach see Appendix B)

Comments on the Fluctuation Modes

1. With $\omega = 0$ and

$$y(x) = \sqrt{\frac{3}{4\sqrt{2}}}\; e^{-\sqrt{2}\,x} \left(\tanh^2 \frac{x}{\sqrt{2}} + 1 + \tanh \frac{x}{\sqrt{2}} \right)$$

can be written after some simple algebra, as

$$y(x) = \sqrt{\frac{3}{4\sqrt{2}}}\; \operatorname{sech}^2 \frac{x}{\sqrt{2}} = \sqrt{\frac{3}{2\sqrt{2}}}\; \phi_{0x}(x)$$

where $\phi_0(x)$ is the static kink solution

In terms of the initial assumed solution

$$\begin{aligned} \phi(x,t) &= \phi_0(x) + y(x,t) \\ &= \phi_0(x) + \sqrt{\frac{3}{2\sqrt{2}}}\; \phi_{0x}(x) \\ &\cong \phi_0(x - \mathcal{X}) \end{aligned}$$

$$\mathcal{X} = -\left(3/2\sqrt{2}\right)^{1/2}$$

Thus the first type of small oscillation y(x) expresses the possibility that the kink will absorb a disturbance by translating its center of mass by a small amount $\mathcal{X} = -\left(3/2\sqrt{2}\right)^{1/2}$

without expending energy ($\omega = 0$), since the energy of the kink as we have seen depends only on its velocity and not on its position. This result is expected since ϕ^4 is translationally invariant in space and an expansion around the kink gives a mode with $\omega = 0$, (known as the "Goldstone mode" in solid state and particle Physics), as a result of breaking the translational symmetry of the equation by placing the kink at a definite location. In other words, having

$$\phi(x) = \tanh (x - X_0)/\sqrt{2}$$

is equivalent to having

$$\phi(x) = \tanh \frac{x}{\sqrt{2}} + y(x)$$

2. The second type of disturbance, with $\omega = (3/2)^{1/2}$

$$y(x,t) = e^{i\sqrt{3/2}\, t} e^{-x/\sqrt{2}} \left(\tanh^2 \frac{x}{\sqrt{2}} + \tanh \frac{x}{\sqrt{2}} \right) \sqrt{\frac{3}{2\sqrt{2}}} :$$

Again we can rewrite $y(x)$ as

$$y(x,t) = e^{i\sqrt{3/2}\, t} \sqrt{\frac{3}{2\sqrt{2}}} \tanh \frac{x}{\sqrt{2}} \operatorname{sech} \frac{x}{\sqrt{2}}$$

This corresponds to a small disturbance of the kink and since it travels with the kink (recall that corresponds to a bound state), it can be thought of as a shape oscillation.

3. The last type of fluctuation corresponds to the case where the kink decays by emitting radiation. This kind of disturbance involves higher energies ($\omega^2 = k^2 + 2 > \frac{3}{2}$) than the shape oscillations.

Thus, concluding, we have seen in our linear approximation that there are at least three possible ways for the kink to react to a disturbance. It either translates or changes its shape slightly, or radiates, or possibly any combination of those three ways.

CHAPTER III

COLLECTIVE COORDINATE DESCRIPTION OF A PERTURBED KINK

Introduction

If we knew an exact time dependent solution that involves two (or more than two) kinks, it would be easier to reveal the solitary wave properties of the kink. That is, we would learn how they interact when they come close, why they are unable to maintain their shapes after they collide, how their internal modes are coupled, and many other questions. We don't have such an analytic multikink solution. Thus, any attempt to study kink interactions must face the problem of approximating a solution.

The analysis of the previous section serves the purpose of extracting as much information as possible and then uses this as a guide to construct the solution. A well known method is the collective coordinate approach [4,5]. Here we isolate as many degrees of freedom as possible (in our case, among infinitely many) and through them collectively describe the system. As has been indicated, such degrees of freedom could be the rigid translation, shape oscillation and radiation modes of the kink.

The translational mode enters the solution by allowing the location of the kink, X_o , to become time dependent coordinate . Thus at first we write

$$\phi(x,t) = \tanh\left(x - X_0(t)\right)/\sqrt{2} \tag{3.1}$$

Any attempt to linearly superpose the full translational mode (2.20) would lead to "secular" terms (a perturbation whose effect increases in time), since (2.20) is related to the (translational) symmetry of the system [1]. We ignore the radiation from the trial solution for the following reasons: It takes more energy to excite radiation compared to the other modes (remember $\omega_k^2 = k^2 + 2$); thus, it will take a strong collision (high initial kink velocities) to excite the radiation and that would demand a relativistic treatment which would make the problem much more complicated. On the other hand, numerical simulations [1,2,7,8,16] and theoretical estimates [2,14] indicate that radiation is little excited in a perturbed kink. By definition a solitary wave is almost stable in external disturbances, and as one goes from solitary waves to soliton the stability is guaranteed. The presence of radiation with the dissipative character would weaken the stability of the kink too fast to explain its solitary behavior. For purposes which will become obvious as we proceed, we need to include mostly internal modes in which to store energy and not lose much energy, so that the total energy of the system remains constant. Certainly, it is idealistic to totally exclude radiation; we expect a small amount of energy to be lost if one recalls the inelastic character of the kink collisions. We shall account for this collectively later in our analysis.

Parametric Collective Coordinate Approach

We write the trial solution as

$$\phi_p \left(x;\ X_o(t), A(t) \right) = \tanh \frac{x - X_o(t)}{\sqrt{2}\, A(t)} \tag{3.2}$$

where Xo(t) describes at any time the position of the kink relative to the origin and A(t) the variation of the kink width. Inserting (3.2) in the lagrangian density

$$\mathcal{L} = \frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - \frac{1}{4} \left(\phi^2 - 1 \right)^2$$

and integrating over all space to obtain

$$L = \int_{-\infty}^{\infty} \mathcal{L} \, dx$$

$$= \frac{1}{2} \frac{M \dot{X}_o^2}{A} + \frac{1}{2} M \frac{\dot{A}^2}{A} b - \frac{1}{2} M \left(A + \frac{1}{A} \right) \tag{3.3}$$

where $M = \frac{2\sqrt{2}}{3}$ is the mass of the kink and $b = \left(\frac{\pi^2}{6} - 1 \right)$

From the lagrangian L, we use the Euler-Lagrange equations and obtain the equations of motion for Xo(t) and A(t)

$$\dot{X}_o = \alpha A$$

$$b \left(\frac{\ddot{A}}{A} - \frac{1}{2} \frac{\dot{A}^2}{A^2} \right) = -\frac{1}{2} + \frac{1}{2A^2} - \frac{\dot{X}_o^2}{2A^2} \tag{3.4}$$

α =constant

A trivial solution is

$$\dot{X}_0 = U$$

$$A = (1 - U^2)^{1/2}$$

in terms of the full (trial) solution, this gives

$$\phi_p(x,t) = \tanh \frac{x - C - Ut}{\sqrt{2(1-U^2)}} \tag{3.5}$$

which is an EXACT solution of the ϕ^4 corresponding to a rigidly translating and Lorentz - contracted kink.

In Appendix C a general solution of the coupled system is found to be

$$\dot{X}_0(t) = \frac{c_1}{\omega} \sin(\gamma + \omega t) + c_2$$

(3.6)

$$A(t) = \frac{c_1}{\omega\alpha} \sin(\gamma + \omega t) + \frac{c_2}{\alpha}$$

where $\omega^2 = \alpha^2 + 1/b$

$$b = \frac{\pi^2}{6} - 1$$

$$-b c_1^2 = \alpha^2 - (1+\alpha^2)/c_2^2$$

c_1 , γ, c_2 constants .

Inserting (3.6) in (3.2) the kink solution reads

$$\phi_p(x,t) = \tanh\left[\frac{x + \frac{c_1}{\omega^2}\cos(\gamma+\omega t) - c_2 t + c_3}{(c_1 \sin(\gamma+\omega t) + \omega c_2)(\alpha\omega)^{-1}}\right] \tag{3.7}$$

This solution represents a wobbling and translating isolated kink and it is not an exact solution of the field equation. Particular solutions have been found for Xo(t) and A(t) in references [1],[9] which are special cases of our general solution. The question is, however, how close we approximate the perturbed kink with the parametric approach. Any trial solution for a single kink, at least in first order, should give a static kink plus the shape (static) mode .

That is ,

$$\phi(x,t) = \tanh\frac{x-X_0}{\sqrt{2}} + \varepsilon \tanh\frac{x-X_0}{\sqrt{2}} \operatorname{sech}\frac{x-X_0}{\sqrt{2}} \tag{3.8}$$

(for static Xo).

if we let the parameter A in

$$\phi_p = \tanh\frac{x - X_0(t)}{A\sqrt{2}}$$

be close to its value, A=1 , for which ϕ_P is a kink (static) solution i.e $A^{-1} = 1 + \varepsilon$, $\varepsilon << 1$

$$\phi_P = \tanh \frac{x - X_0}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} (x - X_0) \operatorname{sech}^2 \frac{x - X_0}{\sqrt{2}} + O(\varepsilon^2) \tag{3.9}$$

then $\phi_P \rightarrow \phi$ (3.8) only when $x \cong X_0$

This shows that A(t) does not adequately describe shape oscillations, since the term TANH(X-Xo/$\sqrt{2}$) SECH(X-Xo/$\sqrt{2}$) is not present when ϕ_P is expanded in powers of ε . In fact, A(t) here can describe the coupling of the continuum with the translational modes leading to a "wobbling" kink, since Xo(t) takes care of the translational modes and what is left are modes from the continuum. It is interesting to observe that the frequency of the wobbling kink

$$\omega^2 = \frac{\alpha^2 + 1}{\frac{\pi^2}{6} - 1} = (1 + \alpha^2)\ 1.55 \simeq (1 + \alpha^2)\,\omega_S^2$$

where $\omega_S^2 = \frac{3}{2}$, (is the frequency of the shape modes, found in the previous section.)

From the equations of motion for Xo(t) and A(t) (3.4), when $\alpha = 0$ we get the solution

$$X_o = \text{constant} = C$$

$$A(t) = \pm\sqrt{1-c_2^2}\ \sin(\gamma + \omega t) + c_2$$

$$\omega_s^2 = (3/2) \qquad \gamma,\ c_2 = \text{constant.}$$

and $$\phi_p = \tanh\left[\frac{(x-c_1)/\sqrt{2}}{\sqrt{1-c_2^2}\ \sin(\gamma+\omega t) + c_2}\right]$$

This is a static wobbling kink (breather) [12,13,] and shows the possiblity that a kink could absorb an external disturbance without moving in space but rather changing its shape harmonically in time. It may be an accidental similarity that the frequency ω_s appears in this special case, since we have seen that A(t) intrduced in the kink's waveform doesn't account for the shape eigenfunction at least to $\mathcal{O}(\varepsilon')$. If the shape mode is excluded, since translational modes are absent (Xo= constant), we are forced to attribute the picture observed to the continuum. But the continuum starts from $\omega = \sqrt{2}$? Nevertheless the solutions we have found describing an approximate wobbling (moving or static) kink don't necessary indicate the existence of an exact wobbling kink solution. Rather the results express the sensitivity of the kink distributing its energy among its degrees of freedom and yet retaining its identity as a whole. For reasons that will be obvious as we proceed we need an approach that expresses the trial solution in terms of a kink and its discrete modes.

Linear Eigenfunction Collective Coordinate Approach

Here we choose again Xo(t) (conected with the translational modes), to describe the position of the kink at any time and linearly add the shape oscillations in the solution. The trial solution then reads

(3.11)
$$\phi_L(x; A(t), X_o(t)) = \tanh \frac{x - X_o(t)}{\sqrt{2}} + A(t)\sqrt{\frac{3}{2\sqrt{2}}}\tanh\frac{x - X_o(t)}{\sqrt{2}}\,\mathrm{sech}\,\frac{x - X_o(t)}{\sqrt{2}}$$

A(t) now becomes the amplitude of the shape oscillation.
Then, a priori, the solution can answer where the kink is and how its shape is excited. Before we let two kinks collide we examine again the case of single kink with its shape modes excited.

Inserting (3.11) into the lagrangian density we get

(3.12)
$$L = \frac{\dot{A}^2}{2} + \frac{1}{2}M\dot{X}_o^2 + \frac{\dot{X}_o^2}{2}(AC_1 + C_2A^2) - M - \frac{\omega_s^2 A^2}{2} - C_3A^3 - C_4A^4$$

$$C_1 = \left(\sqrt{3/\sqrt{2}}\right)\pi/4 \qquad C_2 = 7/10$$

$$C_3 = \omega_s^3\,\pi/16\sqrt{\frac{1}{\sqrt{2}}} \qquad C_4 = \omega_s^4\sqrt{2}/70$$

1. If we keep $\sigma(\dot{X}_o, A)$

$$L = \frac{\dot{A}^2}{2} + \frac{1}{2}M\dot{X}_o^2 - M - \frac{\omega_s^2}{2}A$$

which gives

$$\left.\begin{array}{l}\ddot{A} + \omega_s^2 A = 0 \\ M \; \dot{X}_o = \text{const.}\end{array}\right\} \Rightarrow \begin{array}{l} A(t) = A_o \cos(\gamma + \omega_s t) \\ \dot{X}_o = U_o \end{array} \qquad (3.13)$$

2. If we keep all the nonlinear terms

$$L = \frac{1}{2} \dot{X}_o^2 \left(M_K + M_S(A) \right) - M - \frac{1}{2} \omega_s^2 A^2 - C_3 A^3 - C_4 A^4$$

where M_K = mass of the Kink = $\frac{2\sqrt{2}}{3}$

$$M_S(A) = C_1 A + C_2 A^2$$

The resulting equations of motion are: for Xo

$$\dot{X}_o \left(M_K + M_S(A) \right) = \text{constant} = \alpha \qquad (3.15)$$

for A(t) $\ddot{A} + \omega_s^2 A = -3C_3 A^2 - 4C_4 A^3 + \frac{1}{2} \dot{X}_o^2 \frac{\partial M_S}{\partial A}$

$$\ddot{A} + \omega_s^2 A = -3C_3 A^2 - 4C_4 A^3 + \left[\frac{\alpha^2}{2} \frac{C_1 + 2C_2 A}{(M_K + M_S)^2} \right] \qquad (3.16)$$

since $M_K \gg M_S$, $(A \ll M_K)$ the above equation becomes

$$\ddot{A} + \omega_s^2 A = 3 C_3 A^2 - 4 C_4 A^3$$

(3.17)

$$+ \frac{\alpha^2}{2 M_K^2} \left[C_1 + 2 C_2 A \right] \left[1 - 2 M_s / M_K + 3 M_s^2 / M_K^2 \right]$$

or $$\ddot{A} + \omega_0^2 A = C_1' + C_2' A^2 + C_3' A^3$$

where

$$\omega_0^2 = \omega_s^2 + \left[C_2 - C_1^2 / M_K \right] \frac{\alpha^2}{M_K^2}$$

$$C_2' = \left(\frac{-3 C_1 C_2}{M_K} + \frac{3}{2} \frac{C_1^3}{M_K^2} \right) \frac{\alpha^2}{M_K^2} - 3 C_3$$

$$C_1' = C_1 \alpha^2 / 2 M_K^2$$

$$C_3' = \left(\frac{6 C_1^2 C_2}{M_K^2} - \frac{2 C_2^2}{M_K} \right) \frac{\alpha^2}{M_K^2} - 4 C_4$$

We shall seek the solution as series of successive approximations. Following Landau [23] (see Appendix D) the solution reads

$$\dot{X}_0 = \alpha / (M_K + C_2 A^2 + C_1 A)$$

(3.18)

$$A(t) = K \cos \omega t + (2 C_1' + K^2 C_2') / 2 \omega_0^2 - \frac{C_2 K^2 \cos 2 \omega t}{6 \omega_0^2}$$

$$- 1/8 \omega_0^2 \left(C_2'^2 K^3 / 3 \omega_0^2 + C_3' K^3 / 4 \right) \cos 3 \omega t$$

where

$$\omega = \omega_0 - \left[c_1' c_2' K / \omega_0^2 \right] - \left[5 c_2'^2 K^2 / 12\, \omega_0^3 \right] - 3 c_3' K^2 / 8\omega_0$$

Again we see the "wobbling" behaviour of the kink and the presence of sinusoidal shape disturbances, as expected i.e for A small, from (3.18)

(3.19) $$X_0(t) = \frac{\alpha}{M_K} - \frac{\alpha c_1}{M_K} A(t) + \mathcal{O}(A(t))$$

where $\mathcal{O}(A)$ are higher orders of A(t). If radiation modes were included, the trial solution would become

(3.20) $$\phi(x,t) = \tanh \frac{x - X_0(t)}{\sqrt{2}}$$

$$+ \ A(t) \tanh \frac{x - X_0(t)}{\sqrt{2}} \operatorname{sech} \frac{x - X_0(t)}{\sqrt{2}}$$

$$+ \int dk\ A_k(t)\, e^{i\omega_k t}\, y_c(k, x)$$

CHAPTER IV

KINK-ANTIKINK COLLISIONS

Introduction

In this Chapter, using the linear eigenfunction collective coordinate approach (LECC) to describe two colliding and perturbed kinks ($K-\bar{K}$), we derive the equations of motion and the total energy of the system which in turn we use to examine theoretically the results of numerical simulations of kink collisions carried out by Campbell and his coauthors[1].

First we examine the trapping of a kink and antikink in a bound state. We find that as a result of the impact the shape modes are turned on, absorbing an amount of energy Esh at the expense of the kinetic energy of the kinks. As the kinks are scattered away, having less kinetic energy than it takes to overcome their mutual attraction, they reach some maximum separation distance Xo(t) and return to collide again unable to completely separate. We approximate Xo as a function of time and kink's initial velocity and we plot Xo vs time for three different incoming velocities. Estimating Esh as a function of the initial velocity we determine the critical value of the initial velocity Uc above which kinks can "afford" the shape disturbance and separate. After the trapping is explained and Uc is determined we comment on previous similar calculations[2] and point out similarities and differences. We unfold the details as we proceed.

Collective Coordinate Equations of Motion

Let us consider the case where a kink collides with an antikink .In the center-of-mass frame, we assume a trial solution of the form [2,3]

$$\phi(x,t) = \tanh\frac{x + X_0(t)}{\sqrt{2}} - \tanh\frac{x - X_0(t)}{\sqrt{2}} - 1$$

$$(4.1) \qquad + A(t)\, y_s(x + X_0(t)) - B(t)\, y_s(x - X_0(t))$$

$$+ \int dk \left[a_k e^{-i\omega_k t} y_c(x,k) + a_k^* e^{i\omega_k t} y_c^*(x,k) \right]$$

where y_s , y_c are the eigenfunctions of the shape and radiation modes respectively, eqs. (2.21),(2.22). This solution describes two kinks (i.e a kink and antikink) moving in opposite directions with velocities $\mp\dot{X}_0(t)$ separated by a distance $2X_0(t)$. The fluctuation modes are present and are controlled by the amplitudes A, B, a_k .

The constant -1 is added to satisfy the kink boundary conditions

$$(4.2) \qquad \begin{array}{l} \phi(x,t) \to \pm 1 \\ \phi_x(x,t) \to 0 \end{array} \quad \text{as} \quad \begin{array}{l} x \to \pm\infty \\ a_k \to 0 \end{array}$$

At first we shall neglect radiation completely; we will discuss how radiation affects the collision process later. Observing that kinks and antikinks are spatial images of each other, we see that it is not necessary to introduce a separate notation for the amplitudes A(t) and B(t) since the effect of an impact should be equally shared by both kink and antikink. Therefore we shall assume A=B and the final form of the

Ansatz reads .

$$\phi(x,t) = \tanh\frac{x+X_0(t)}{\sqrt{2}} - \tanh\frac{x-X_0(t)}{\sqrt{2}}$$

(4.3)
$$-1 + A(t)\sqrt{\frac{3}{2\sqrt{2}}}\,\tanh\frac{x+X_0(t)}{\sqrt{2}}\,\mathrm{sech}\,\frac{x+X_0(t)}{\sqrt{2}}$$

$$-A(t)\sqrt{\frac{3}{2\sqrt{2}}}\,\tanh\frac{x-X_0(t)}{\sqrt{2}}\,\mathrm{sech}\,\frac{x-X_0(t)}{\sqrt{2}}$$

Upon substitution of $\phi(x,t)$ into the lagrangian density, eq.(2.1) we obtain the lagrangian of the system

$$L = \left[M + I(X_0)\right]\dot{X}_0^2 - U(X_0) + \dot{A}^2(t) - \omega_s^2 A^2$$

(4.4)
$$- 2W(X_0)A^2(t) + 2F(X_0)A(t)$$

$$+ Q(X_0)\dot{A}^2(t) + K(X_0)A^2 + 2C(X_0)\dot{A}(t)\dot{X}_0$$

where we are keeping terms up to second order in $A(t), \dot{A}(t), \dot{X}o(t)$

and $$C(X_0) = \int_{-\infty}^{\infty} dx\, f_0'(x-X_0)\, f_3(x+X_0)$$

$$M = 2\sqrt{2}/3$$, the kink mass

$$I(X_0(t)) = \int_{-\infty}^{\infty} dx\, f_0'(x+X_0)\, f_0'(x-X_0(t))$$

$$(4.5) \qquad W(X_0(t)) = \int_{-\infty}^{\infty} \frac{dx}{2} \Big[f_1''\big[f_0(x+X_0) - f_0(x-X_0) - 1\big]$$

$$- f_1''\big[f_0(x+X_0)\big]\Big] f_3^2(x+X_0)$$

$$F(X_0(t)) = \int_{-\infty}^{\infty} dx \Big[f_1'\big[f_0(x+X_0)\big] - f_1'\big[f_0(x-X_0)\big]$$

$$- f_1'\big[f_0(x+X_0)\big] - f_0(x-X_0) - 1\Big] f_3(x+X_0)$$

$$K(X_0(t)) = -\int_{-\infty}^{\infty} dx \Big[f_3'(x+X_0)\, f_3'(x-X_0)$$

$$+ f_1''\big[-1 - f_0(x-X_0) + f_0(x+X_0)\big]\Big] f_3(x+X_0) f_3(x-X_0)$$

$$U(X_0(t)) = \int_{-\infty}^{\infty} dx \Big[\tfrac{1}{2}\big[f_0'(x+X_0) - f'(x-X_0)\big]^2$$

$$+ f_1\big[f_0(x+X_0) - f_0(x-X_0) - 1\big]\Big]$$

$$Q(X_0(t)) = -\int_{-\infty}^{\infty} dx\; f_3(x+X_0)\, f_3(x-X_0)$$

$$f_0(z) = \tanh z/\sqrt{2}$$

$$f_3(z) = \sqrt{\frac{3}{2\sqrt{2}}}\ \tanh(z/\sqrt{2})\,\mathrm{sech}(z/\sqrt{2})$$

$$f_1(z) = (z^2-1)^2/4$$

and primes denote differentiation with respect to Xo. The explicit expressions for all integrals are given in the Appendix E. The functions F(Xo), C(Xo), Q(Xo) are in disagreement with similar calculations made by Suguyiama [2]. All of the functions have been checked numerically, and their plots are shown in figures (4-10). From the lagrangian the total energy of the system is given by

$$E = \left[(M + I(X_o))\dot{X}_o^2 + U(X_o)\right] + \left[Q(X_o) + 1\right]\dot{A}^2$$

$$+ A^2(t)\left[\omega_s^2 + 2W(X_o) - K(X_o)\right] - 2F(X_o)A(t) \tag{4.6}$$

$$+ 2C(X_o)\dot{A}(t)\dot{X}_o(t)$$

and the resulting equations of motion from

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_{oi}} - \frac{\partial L}{\partial X_{oi}} = 0 \qquad X_{oi} = X_o(t), A(t) \tag{4.7}$$

for $\quad X_{oi} = X_o(t)$

$$\ddot{X}_o(t)\left[2M + 2I(X_o)\right] + \dot{X}_o I(X_o) + U'(X_o) + A^2(2W + \omega_s^2 - K)' \tag{4.8}$$

$$-2A(t)F'(X_o) - 2\dot{A}^2 Q'(X_o) + 2\ddot{A}(t)C(X_o) = 0$$

for $\quad X_{oi} = A(t)$

$$\ddot{A}(t)\left[1 + Q(X_o)\right] + \dot{A}(t)Q'(X_o)\dot{X}_o(t) - F(X_o) + C(X_o)\ddot{X}_o$$

$$+ \; C'(X_o)\,\dot{X}_o^2 \; + A(t)\left[\omega_s^2 + 2W(X_o) - K(X_o)\right] = 0 \qquad (4.9)$$

Far from the interaction area (large Xo(t)), the functions K, W, F, C, Q, I exponentially go to zero (see fig. 4-10), and the equations of motion reduce to

$$\left.\begin{aligned} \ddot{X}_o &= 0 \\ \ddot{A} + \omega_s^2 A &= 0 \end{aligned}\right\} \Rightarrow \qquad \begin{aligned} &\dot{X}_o = \text{constant} \\ &A(t) = A_o \cos(\gamma + \omega t) \end{aligned} \qquad (4.10)$$

As expected, we obtain two free kinks moving in opposite directions with constant velocities and harmonically oscillating shapes . For small Xo as the kinks collide with each other, equations of motion are not correct to give an analytic description of what is happening. The kinks interact in a strong nonlinear manner and start to recover their identity as Xo(t) increases. For intermediate separation distances, it is interesting to observe that the equation of motion for Xo(t) ignoring A(t) terms reads

$$\ddot{X}_o \left(M + I(x_o)\right) = -\frac{1}{2}\frac{\partial U(X_o)}{\partial x_o} \qquad (4.11)$$

This describes a Newtonian-like particle of variable mass (M+I) in a potential U(Xo) (see fig.8). Thus, as the kink approaches the antikink it speeds up under the influence of the attractive potential and similarly it slows down after the collision until it reaches a constant velocity as $U(x_o) \to 2M$ and $U'(x_o) \to 0$ $I(x_o) \to 0$

To examine this further we expand U(Xo) for large Xo and obtain

(see Appendix E)

(4.12)
$$U(X_0(t)) = 2M - 12Me^{-2\sqrt{2}X_0(t)}$$

which gives an "effective potential"

$$U_{eff} = U(X_0) - U(\infty) = -12Me^{-2\sqrt{2}X_0(t)} \qquad (4.13)$$

In terms of the separation distance 2Xo this is a Yukawa-like potential with range $1/\sqrt{2}$ which is the inverse of the lowest frequency of the continuum. We can gain some insight into this if we crudely think of kinks as the source of mesons (radiation) trapped in the reflectionless potential U= 2-3sech(X/ $\sqrt{2}$) eq. (2.16) as shown below

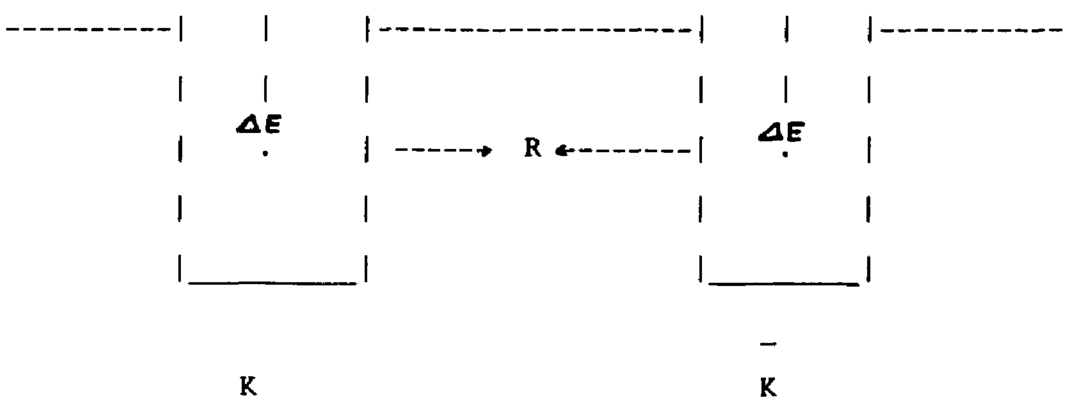


Figure 3

Then the kinks interact via meson exchange through the range R, resulting in the attractive forces we have seen before. R estimated from the uncertainty principle, is given by

$$\begin{aligned} \Delta E \Delta t &= \hbar = 1 \\ R &= \Delta t \end{aligned} \quad \Rightarrow \quad R \propto 1/m$$

where m is the mass of the continuum starting from $m=\sqrt{2}$. For weak binding, since the kinks spend most of their time far from each other, only the lowest frequency is necessary to account for the attractive forces, and that is what we see in the attractive effective potential $-12M \exp(-2X_o\sqrt{2})$.

Graph of INT.Q(Xo) vs. Xo

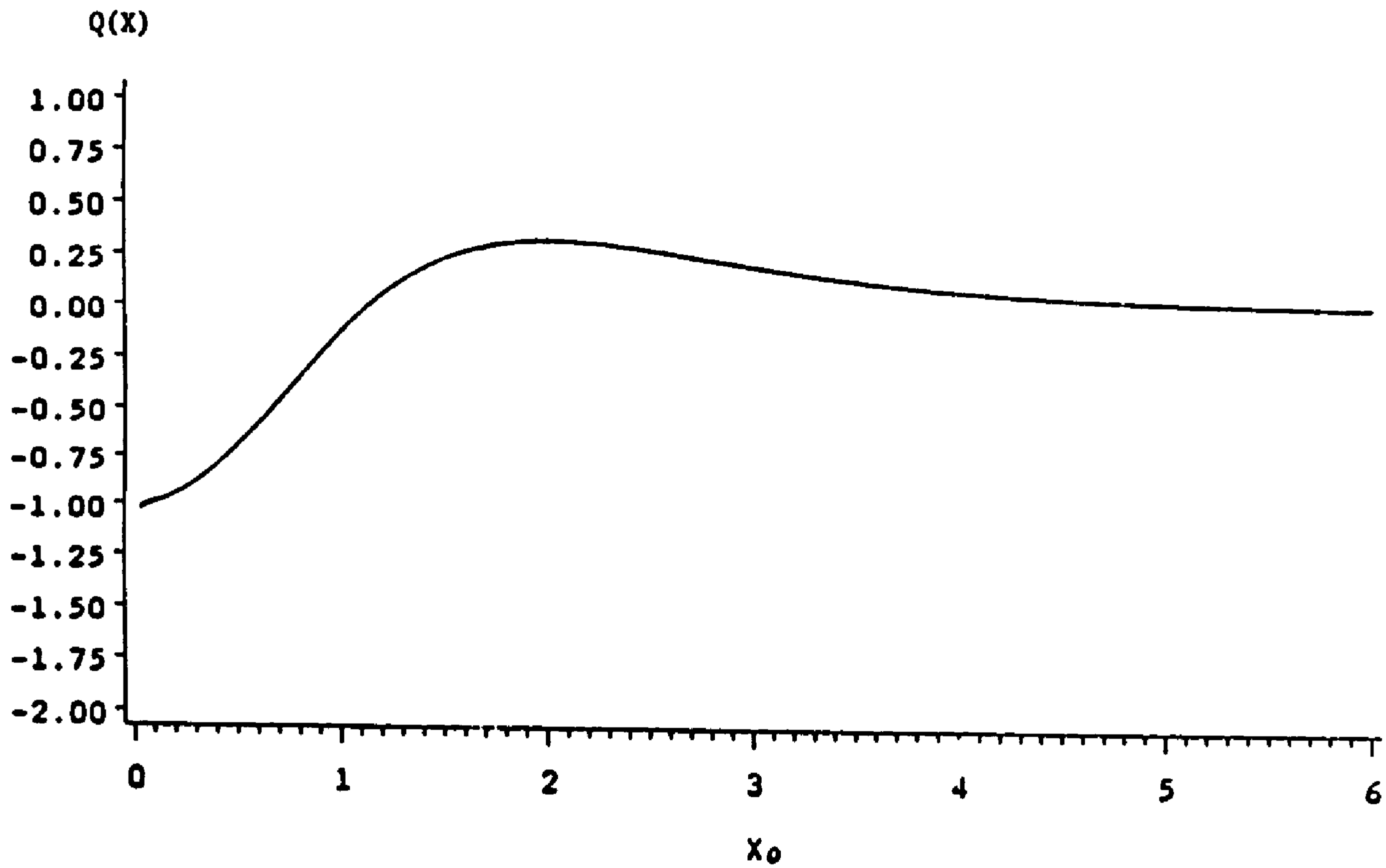


FIGURE 4

Graph of INT.I(Xo) vs. Xo

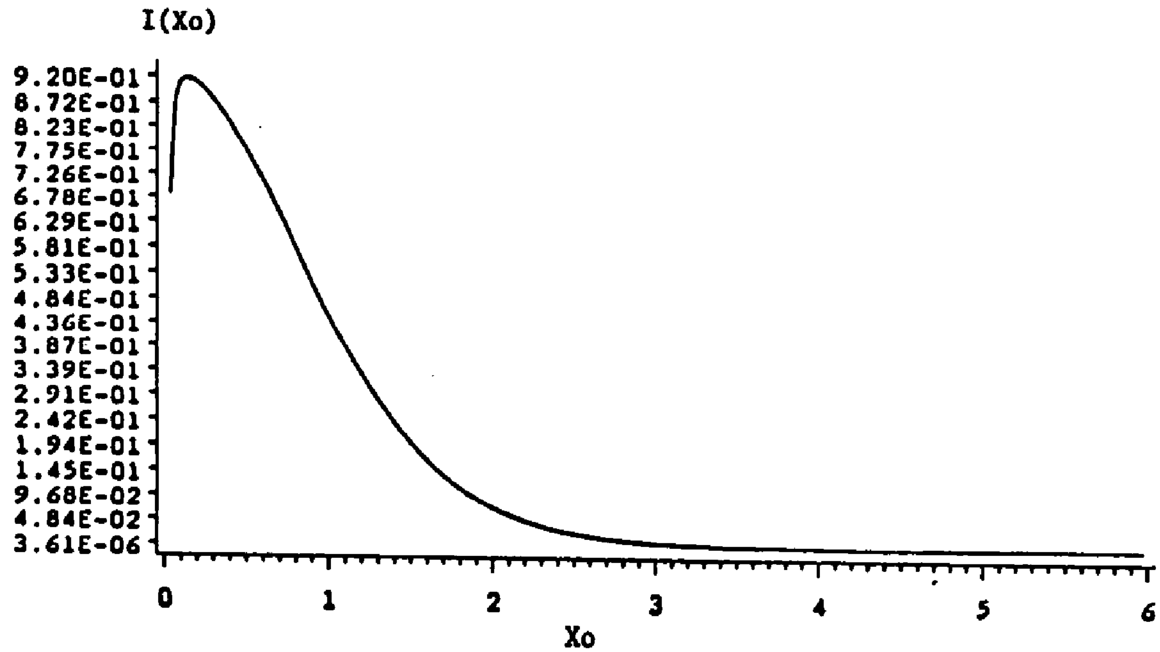


FIGURE 5

Graph of INT.C(Xo) vs. Xo

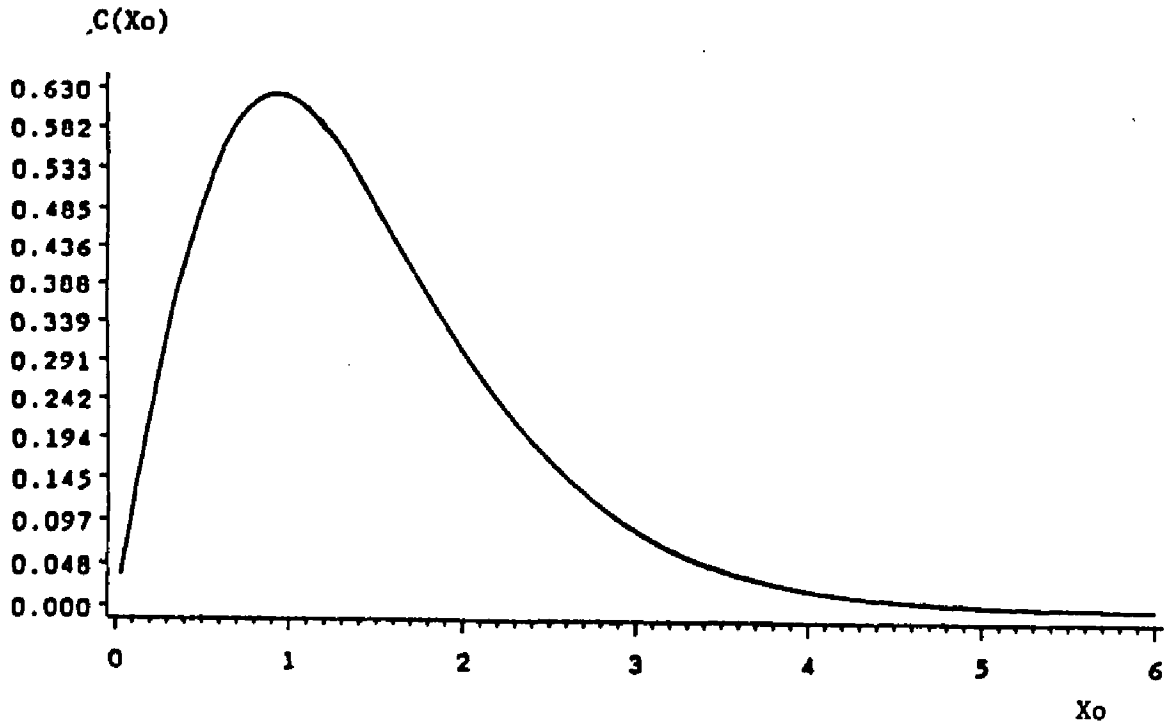


FIGURE 6

Graph of INT.F(Xo) vs. Xo

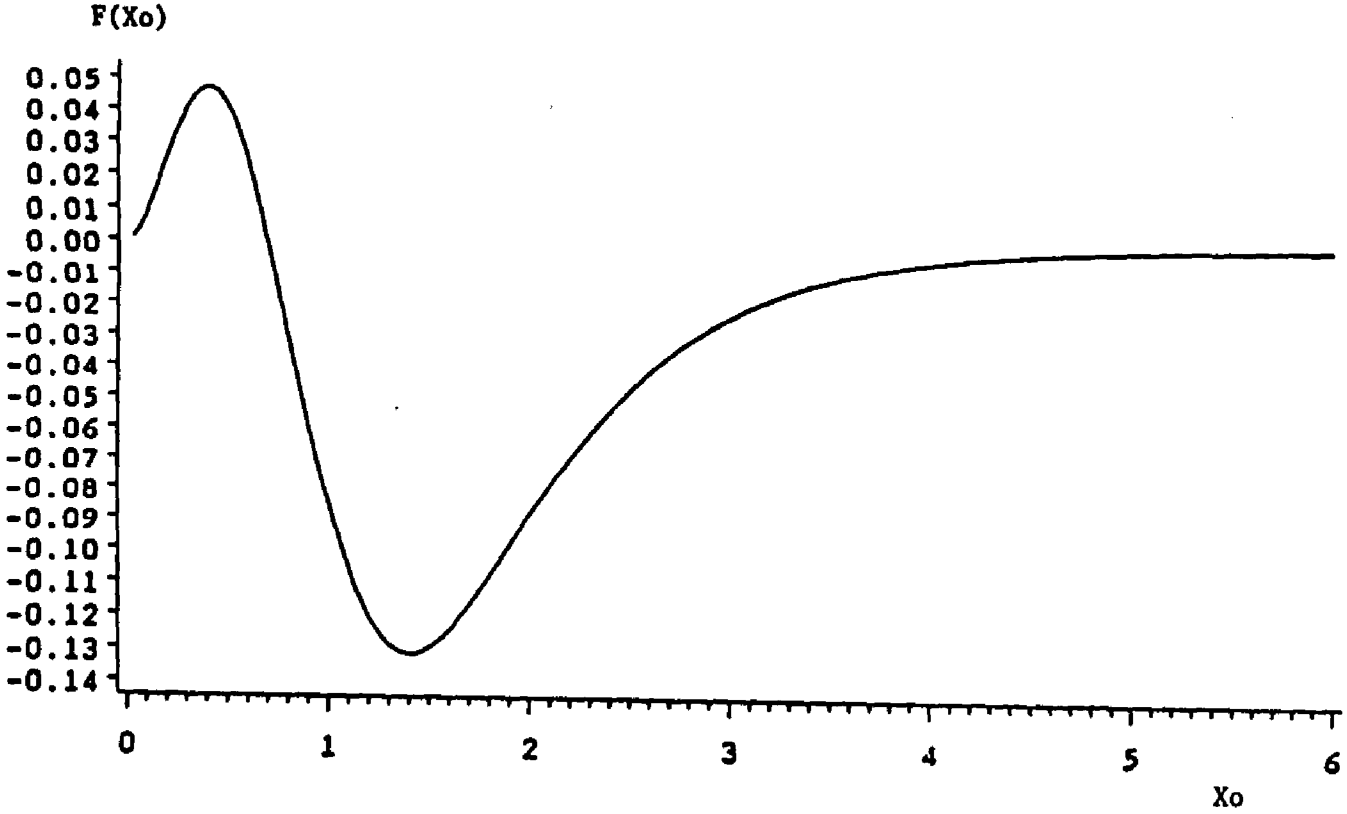


FIGURE 7

GRAPH OF INTEG. U VS XO

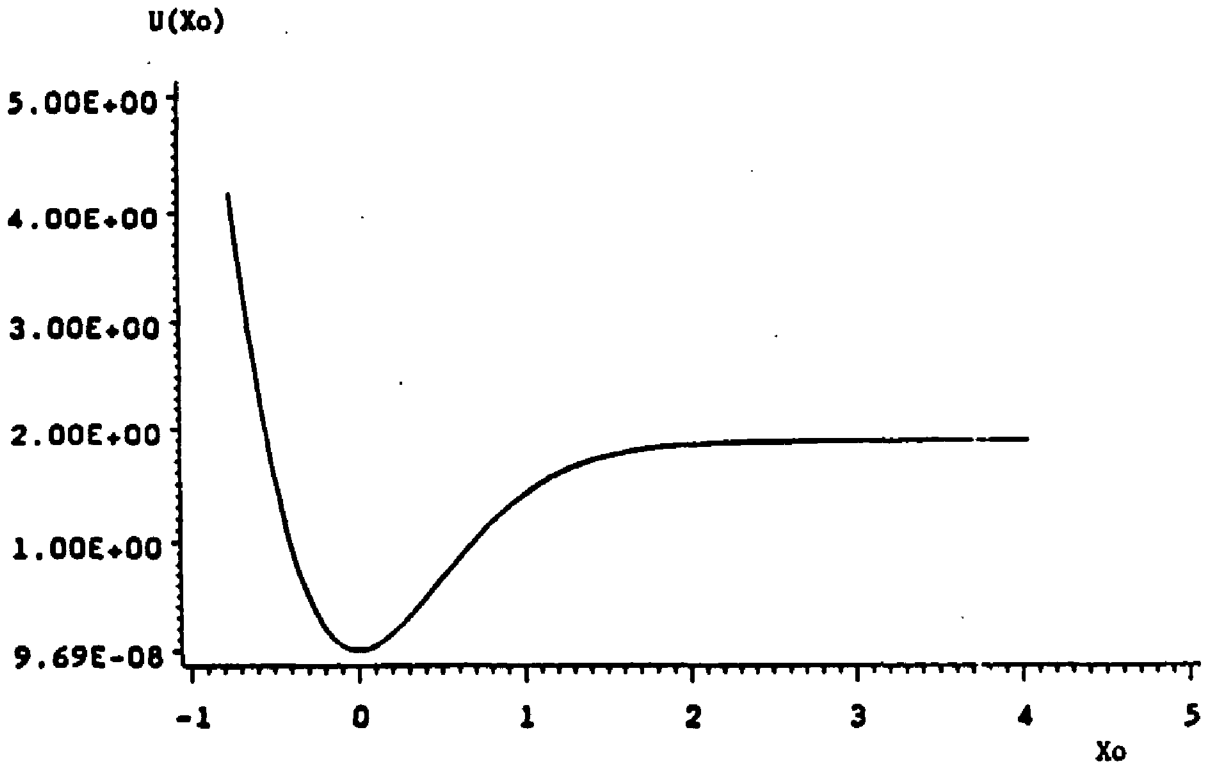


FIGURE 8

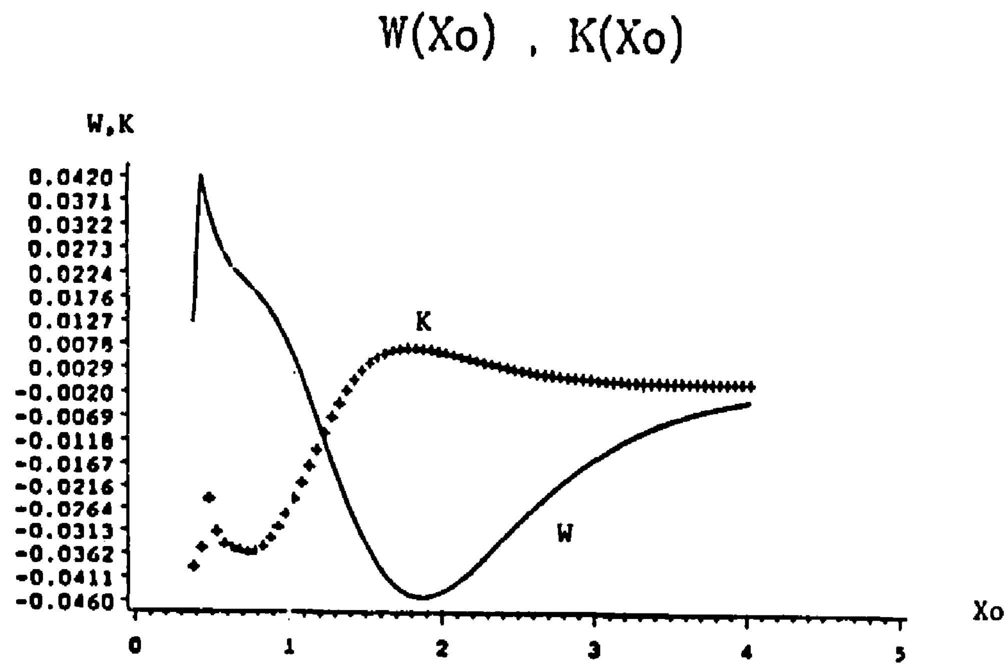

W(Xo) , K(Xo)
W,K
0.0420
0.0371
0.0322
0.0273
0.0224
0.0176
0.0127
0.0078
0.0029
-0.0020
-0.0069
-0.0118
-0.0167
-0.0216
-0.0264
-0.0313
-0.0362
-0.0411
-0.0460
K
W
0
1
2
3
4
5
Xo

FIGURE 9

GRAPH OF 2W-k+1.5 VS XO

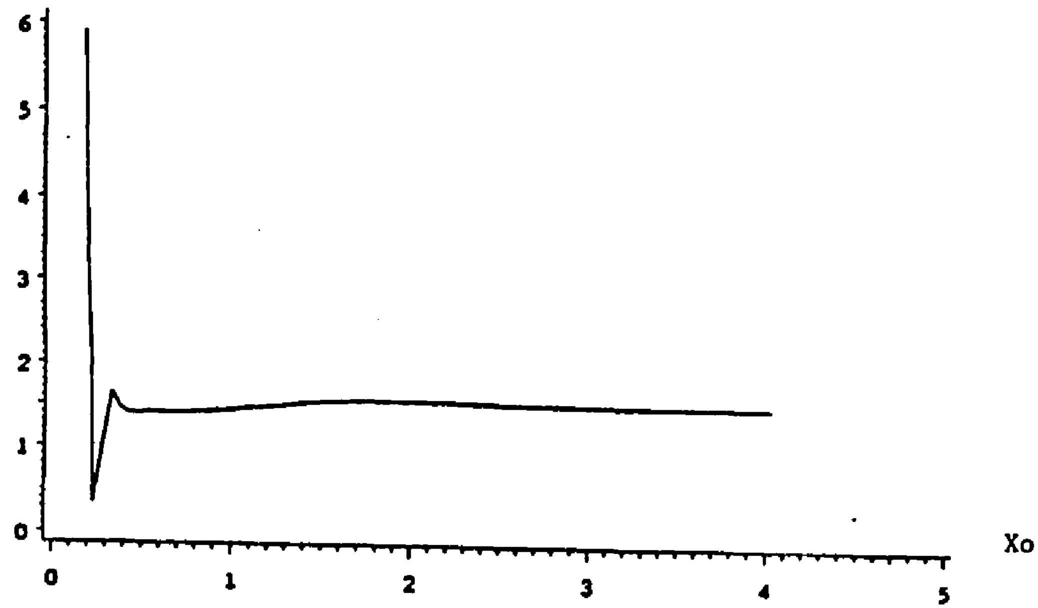


FIGURE 10

Energy in Shape Oscillations.

If we think of kinks as extended deformable, particles with only two degrees of freedom (namely, the rigid translation and shape oscillation) moving in an attractive potential, the simplest reasoning to account for a capture is an energy transfer from the kinetic energy of the moving kinks to the shape modes. As a result of this energy "loss" kinks do not have enough energy to escape from the potential and are trapped. On the other hand, if before the collision the kinks have enough kinetic energy to excite the shape modes, (at the time of the collision), and still have enough energy remaining to move back through the potential, we have reflection. To support this hypothesis we calculate Esh as a function of Ui and then equate Esh with the initial kinetic energy (of kink and antikink) MUi^2. The value of Ui satisfying this equation determines the maximum velocity Uc below which we have capture and above which we have reflection.

The total energy of the system, eq. (4.6)

$$
\begin{aligned}
E = {} & [M + I(X_0)]\,\dot{X}_0^2 + U(X_0) \\
& + [Q(X_0) + 1]\,\dot{A}^2(t) + A^2(t)\left[\omega_s^2 + 2W(X_0) - K(X_0)\right] \\
& - 2F(X_0)A(t) + 2C(X_0)\dot{A}(t)\dot{X}_0(t) \\
\equiv {} & E_K + E_{SH}
\end{aligned}
$$

$$E_K = [M + I(X_0)]\,\dot{X}_0^2 + U(X_0)$$

$$E_{SH} = \left[Q(X_o) + 1 \right] \dot{A}^2(t) + A^2(t) \left[\omega_s^2 + 2W - K \right] - 2FA + 2C\dot{A}\dot{X}_o$$

Where we identify E_K ,the energy of the kinks (including interaction and free of shape oscillations A=0) and Esh the energy taken to excite the "shape modes" of the kink. We assume $E_{SH} \ll E_K$. Before we find Esh we want to know how A(t) behaves as a function of time. From our previous analysis (see eq.3.15) we expect A(t) to have sinusoidal behavior. To prove this we examine the equation of motion for A(t):

$$\ddot{A}\,(1 + Q) + \dot{A}\,Q'\,\dot{X}_o + C\,\ddot{X}_o + C'\,\dot{X}_o^2$$

(4.14)

$$- F + A(t) \left[\omega_s^2 + 2\,W - K \right] = 0$$

We observe that all the functions above are functions of Xo which in turn is a function of time. In order to solve this equation we need an analytic expression for Xo(t). As a first approximation, we set

(4.15)

$$E_{tot} = E_K + E_{SH}$$

$$= E_{SH} + \left[M + I(X_o) \right] \dot{X}_o^2 + U(X_o)$$

with $$E_{tot} = E_{init} = 2M + MU_i^2$$

The above equation gives

(4.16) $$\dot{X}_o(t) = \left[\frac{2M - U(X_o) - M\varepsilon}{M + I(X_o)} \right]^{1/2}$$

where $M\epsilon = Esh - MUi^2$ represents the binding energy for $Esh > MUi^2$ and the final kinetic energy for $Esh < MUi^2$. We are interested in the first case. Expanding U(Xo) and I(Xo) for large Xo and integrating the above equation (see Appendix E), we get

$$X_o(t) = \frac{1}{\sqrt{2}} \ln\left[\sqrt{\frac{12}{\epsilon}} \sin\sqrt{2\epsilon}\,(t+\delta)\right] \tag{4.17}$$

$\delta = 0.2$

Where we have considered ϵ as constant with respect to time since ($\epsilon \ll 1$, weak binding), and kinks spend most of their time far from the scattering region where we have seen A(t) obeys the equation of a harmonic oscillator. On the other hand, any time variation will hardly affect the leading term in (4.16), the constant 2M. Equation (4.16) numerically integrated for Xo(t) is found to be in good agreement with (4.17) as can be seen in Fig. 12. Inserting (4.17) in the equation of motion for A(t), eq.(4.14) and numerically solving it, we find that indeed A(t) has sinusoidal behavior as can be seen in Figure 13.

Next we need to calculate Esh. For this we need A(t) and Xo(t) which means that we have to solve the equations of motion eq.(4.8),(4.9). We will show that there is an easy way to calculate Esh without solving the complicated coupled system of equations of motion. The equations of motion read

$$\ddot{A}\, 2C(X_o) - \dot{A}^2 Q'(X_o) - 2AF'(X_o) = -\Big[+2\ddot{X}_o\,(M + I(X_o)) + U'(X_o)\Big] - \dot{X}_o^2\, I(X_o) - A^2(2W - K + \omega_s^2)' \tag{4.8}$$

$$\ddot{A}(1+Q) + \dot{A}Q'\dot{X}_o + A(\omega_s^2 + 2W - K) + c\ddot{X}_o + c'\dot{X}_o^2 - F = 0$$

(4.9)

The term ($2W - K + \omega_s^2$) plotted in Fig. 10, is approximately a constant with respect to Xo therefore ($[2W - K + \omega_s^2]' \to 0$), and the above equations become

$$\ddot{A}\,2c - \dot{A}^2 Q' - 2AF = -\left[2\ddot{X}_o(M+I) + \dot{X}_o^2 I + U'\right]$$

(4.18)

$$\ddot{A}(1+Q) + \dot{A}Q'\dot{X}_o + A(\omega_s^2 + 2W - K) = -\left[c\ddot{X}_o + c'\dot{X}_o^2 - F\right]$$

upon substitution of (4.16) in (4.18) all the functions are expressed in terms of Xo(t) and ε then (4.18) becomes

(4.19) $$\ddot{A}(2c) - \dot{A}^2 Q' - 2AF = \xi_1(X_o)$$

$$\ddot{A}(1+Q) + \dot{A}(Q'\dot{X}_o) + A(\omega_s^2 + 2W - K) = \xi_2(X_o)$$

where

$$-\left[2\ddot{X}_o(M+I) + U' + \dot{X}_o^2 I\right] = \xi_1(X_o)$$

$$-\left[c\ddot{X}_o + c'\dot{X}_o^2 - F\right] = \xi_2(X_o)$$

At this point we use the information about A(t). The sinusoidial behavior of A(t) eq.(3.15) suggests that there are time(s) t and coresponding Xo(t) for which $\dot{A}(t) \to 0$ as A(t) approaches its maximum

A_{max}. For this Xo eq.(4.19) gives

(4.20)
$$\ddot{A}\,2C - 2AF = \xi_1(X_o)$$
$$\ddot{A}\,(1+Q) + A(\omega_s^2 + 2W - K) = \xi_2(X_o)$$

Solving (4.20) algebraically for A and A in terms of Xo the energy is found from eq(4.6) with $\dot{A}$=0

(4.21)
$$E_{SH} = A^2\left[\omega_s^2 + 2W(X_o) - K(X_o)\right] - 2F(X_o)A$$

as a function of Xo and .

Fixing ε and plotting Esh vs Xo we find that Esh has maximum (for A=0) at Xo=1.03 for various values of ε . Since

(4.22)
$$\varepsilon = \frac{E_{SH}(X_o = 1.03)}{M} - U_i^2$$
or

$$U_i^2 = E_{SH}/M - \varepsilon$$

for each pair (Esh[Xo, ε], ε) there corresponds one Ui. For =0 the critical velocity is determined by U_c^2 = [Esh/M]. The results are plotted in Fig. 14. Using eq.(4.21),(4.20) we find Uc= 0.22 a 15% deviation from the numerical results of Campbell et.al. (Uc= 0.2598) and Uc =0.30 (Sugiyama). Considering the nonlinearity of the problem and the collective form of the solution, this represents excellent agreement.

Comments

To the best of our knowledge the first theoretical approach which determines the critical velocity was given by Sugiyama [2]. He uses a linear eigenfunction collective coordinate approach (LECC) with different shape amplitudes A(t) and B(t) for the Kink and the antikink which later in his calculations come out to be identical. Estimating radiation to be very small he neglects radiation terms in his Ansatz and derives the same form of lagrangian (4.4) as ours when A=B. We have checked all the functions involved in the lagrangian both analytically and numerically and we have found F,Q,C listed in his Appendix(the only ones he makes available) in disagreement with ours (see Appendix E). As he proceeds in his analysis he approximates eq.(4.16) by

$$\dot{X}_o^2(t) = \frac{E - U(X_o)}{M + I(X_o)} \tag{4.23}$$

where he actually sets $\varepsilon = 0$. Even if ε is small compared to the total energy E, setting $\varepsilon = 0$ eq.(4.23) approximates Xo for free kinks and one can't use Xo to describe trapped kinks as he later does. On the other hand the presence of ε in the expression for Xo determines the dependence of the separation distance Xo from the initial velocity Ui which as we shall see in the next section controls the most important feature of kink collisions: the bouncing of the kinks.

With $\varepsilon=0$, Xo becomes independent of Ui and since A=A(Xo,t), Esh=f(A) the energy in the shape modes becomes independent of Ui which is not the case. As he proceeds in his analysis he divides the spatial region of Xo in two parts according to, if Xo >1 or Xo<1. He does not explain why he is choosing this particular value of Xo for which, as we have shown, the shape amplitude takes its maximum value. Then he approximates the equations of motion for the shape amplitudes by

$$\frac{d}{dt}(\dot{A} + Q\dot{B} + C\dot{X}_o) = F \qquad X_o < 1$$

(4.24)
$$\ddot{A} + \omega_s^2 A = 0 \qquad X_o \geq 1$$

Where he neglects terms proportional to A and B as too small to be considered. We disagree with this approximation since continuity at Xo=1 , where according to his assumptions Q,C,F vanish, the above equations of motion fail to bridge the two regions. With Xo=1 one gets

$$\ddot{A} = 0 \qquad X_o < 1$$

(4.25)
$$\ddot{A} + \omega_s^2 A = 0 \qquad X_o \geq 1$$

After he solves the above equations of motion he relates the amplitudes A and B before and after each collision (at Xo=1) in terms of creation and annihilation operators and derives an integral expression for the total energy in the shape modes Esh in terms of the transition probability of finding the kinks from the initial ground state into a final excited state. He plots Esh vs initial velocity Ui (for which we fail to "see" how Ui was introduced in his calculations) and from Esh=MUi he determines Uc=0.25 in agreement with his numerical simulations where he finds Uc=0.3. From his plot of Esh vs Ui (see Fig. 11) we observe Esh to be dangerously increasing function of Ui which disagrees with both our analytical results and Campbell's simulations(see Fig. 15) and possibly indicates the error involved in the assumptions argued before.

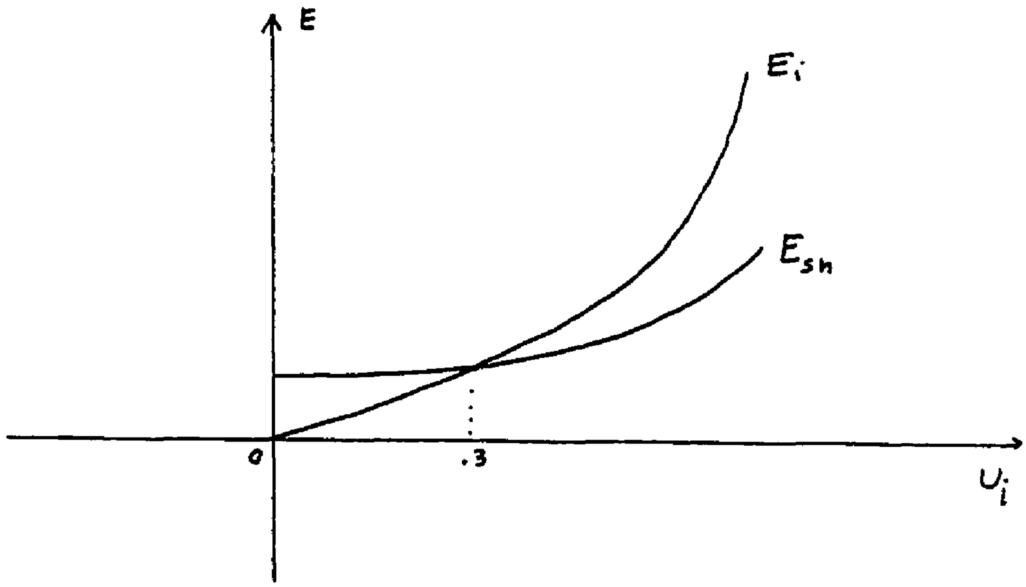


Figure 11

Regardless of the validity of Sugiyama's analysis, our goal in this section was to estimate Uc using a much simpler approach with care taken to avoid regions in Xo(t) for which the Ansatz is not believed to work accurately. Our results indicate that we have succeeded although an exact expression for A(t) would approximate Esh closer to its real value. Campbell et.al.'s work, on which we will refer to frequently in the next section, is only able to estimate Uc numerically.

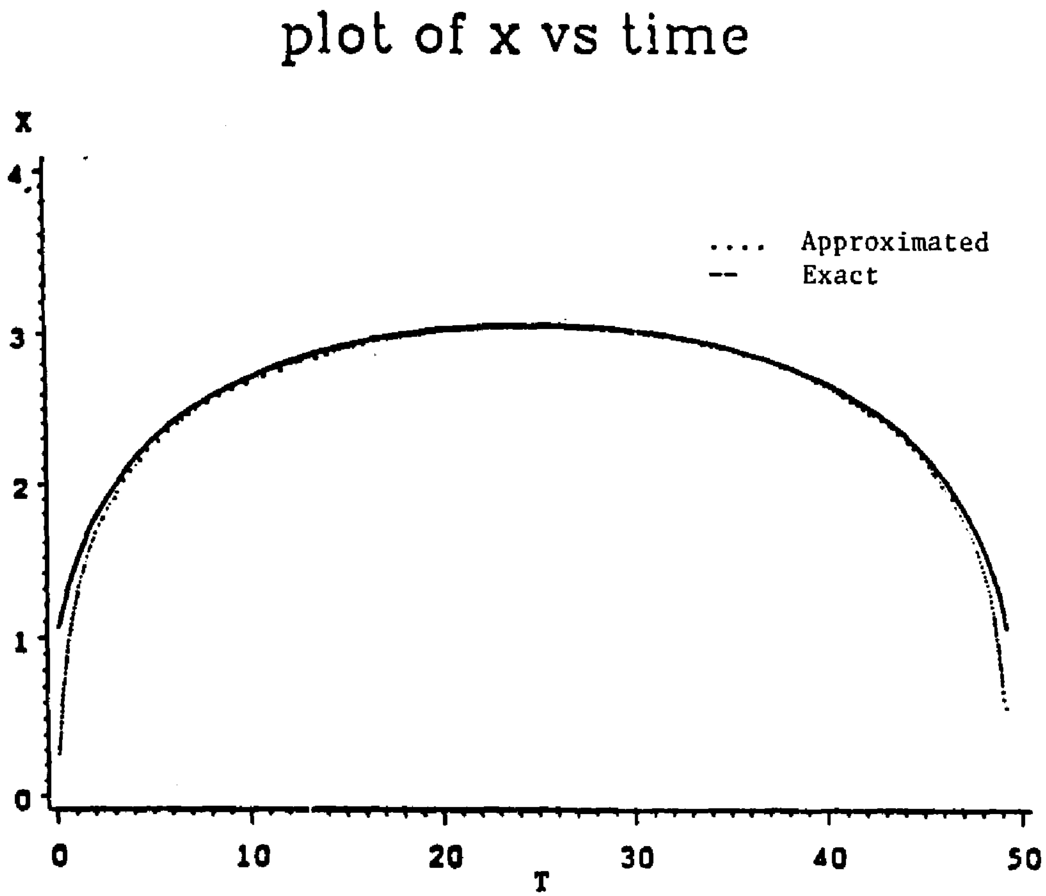


FIGURE 12

plot A and At

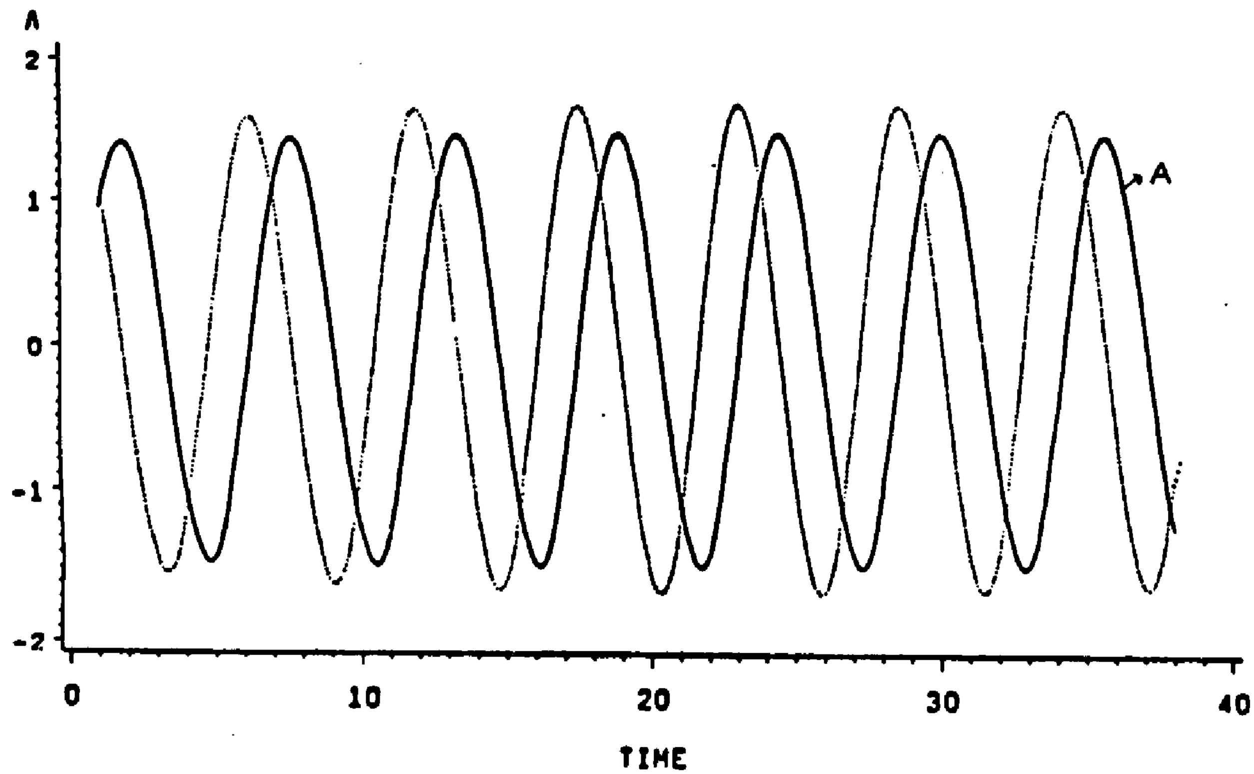


FIGURE 13

plot of shape energy vs initial kin. energy

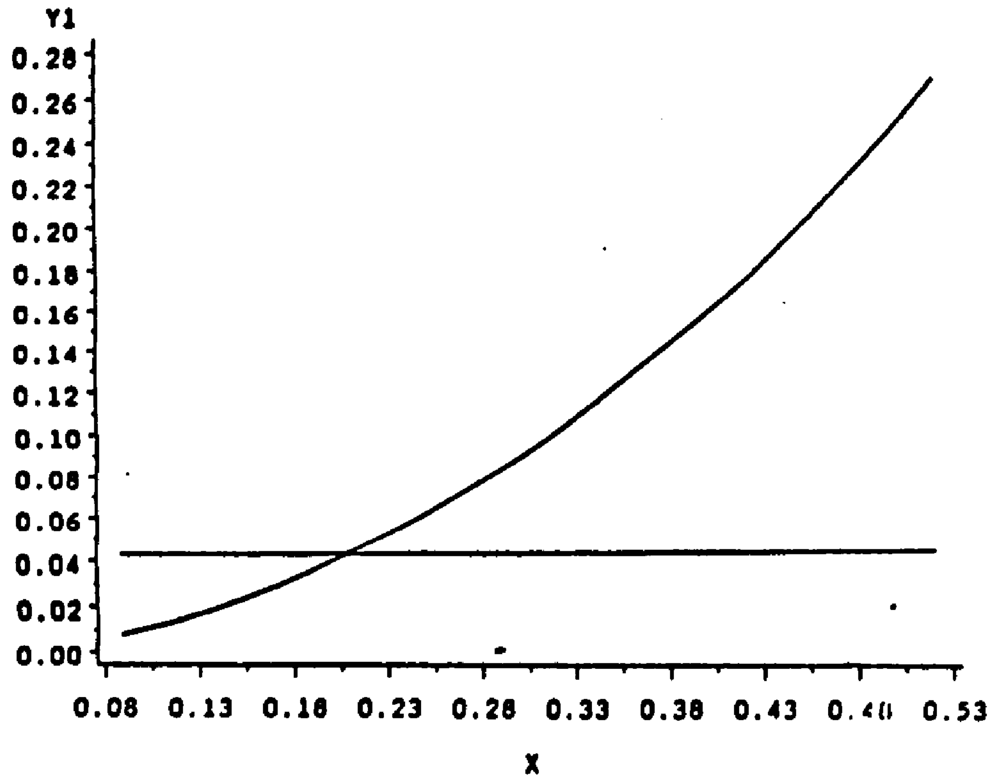


Figure 14

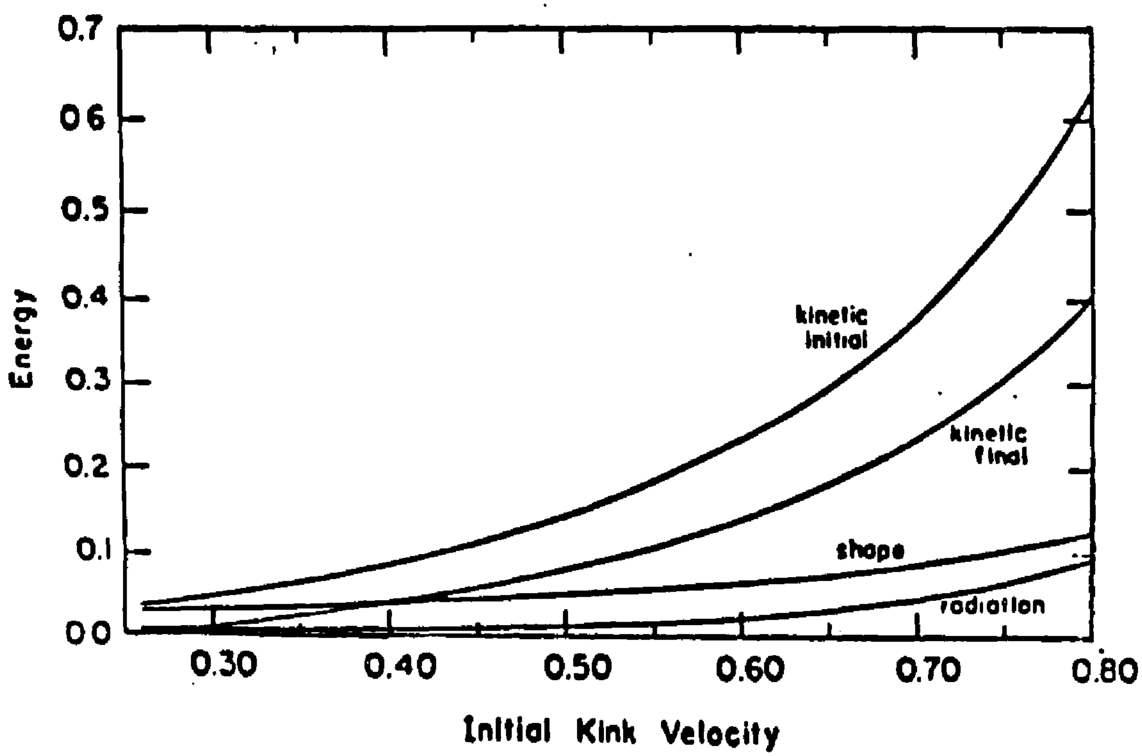


Fig. 15 The partitioning of the initial $K\bar{K}$ kinetic energy into final kinetic energy, shape mode energy, and radiation energy, plotted as a function of the initial $K\bar{K}$ speed for $v > v_c$. (Taken from ref. 1)

CHAPTER V

KINK BOUNCING

Introduction

Even if one expects for incoming kink velocities less than Uc a kink collision always to end up with trapping numerical simulations[1] have shown this is not the case. In fact for certain initial velocities less than Uc the kinks bounce twice before they finally separate. In reference [1] the bouncing of the kinks was explained via resonance energy exchange between shape and translational modes. It was proposed that the first impact turns on the shape modes which removes enough kinetic energy from the moving kinks, resulting in the trapping of the kinks in their mutual attractive potential. Separating to a maximum distance Xo they turn back to collide again for a second time. If the time it takes for the kinks to cover this maximum distance and come back satisfies a certain time condition for which the second impact turns off the shape vibrations, the translational modes can reabsorb the energy stored in the shape modes giving enough energy to the kinks to escape. Heuristic arguments combined with numerical data enable the authors of reference [1] to derive an equation which relates the time between collisions and initial velocities and predict the values of the initial velocities for which "resonance" occurs.

In what follows we provide theoretical arguments that support the model and liberate the final equation from the data dependent parameters. First we calculate the time between collisions and then derive the resonance time condition. Equating times we get an expression which predicts the observed initial velocities leading to a resonance. Free of unknown parameters the model is generalized and applied with success to modified sine Gordon and double sine Gordon field equations as a final check of its validity.

Time Between Collisions

Since the energy stored in the shape modes is small, the binding of the kinks in the attractive potential is very weak. Therefore we expect the kinks to spend most of their time far from the interaction area. Then we can use the expression derived previously for large Xo(t) (see Fig. 16-18)

$$X_o(t) = \frac{1}{\sqrt{2}} \ln\left[\sqrt{\frac{12}{\varepsilon}} \sin \sqrt{2\varepsilon}\,(t+\delta_o)\right] \tag{4.26}$$

where δ_o is determined from Xo(t=0).

The initial velocity contained in the constant ε =(Esh/M-Ui^2) controls how far the kinks separate before they turn back to collide again. The more initial kinetic energy available the farther they separate (for the same time). Xo(t) has maximum when

$$\sqrt{2\varepsilon}\,(t + \delta_o) = \pi/2 \tag{4.27}$$

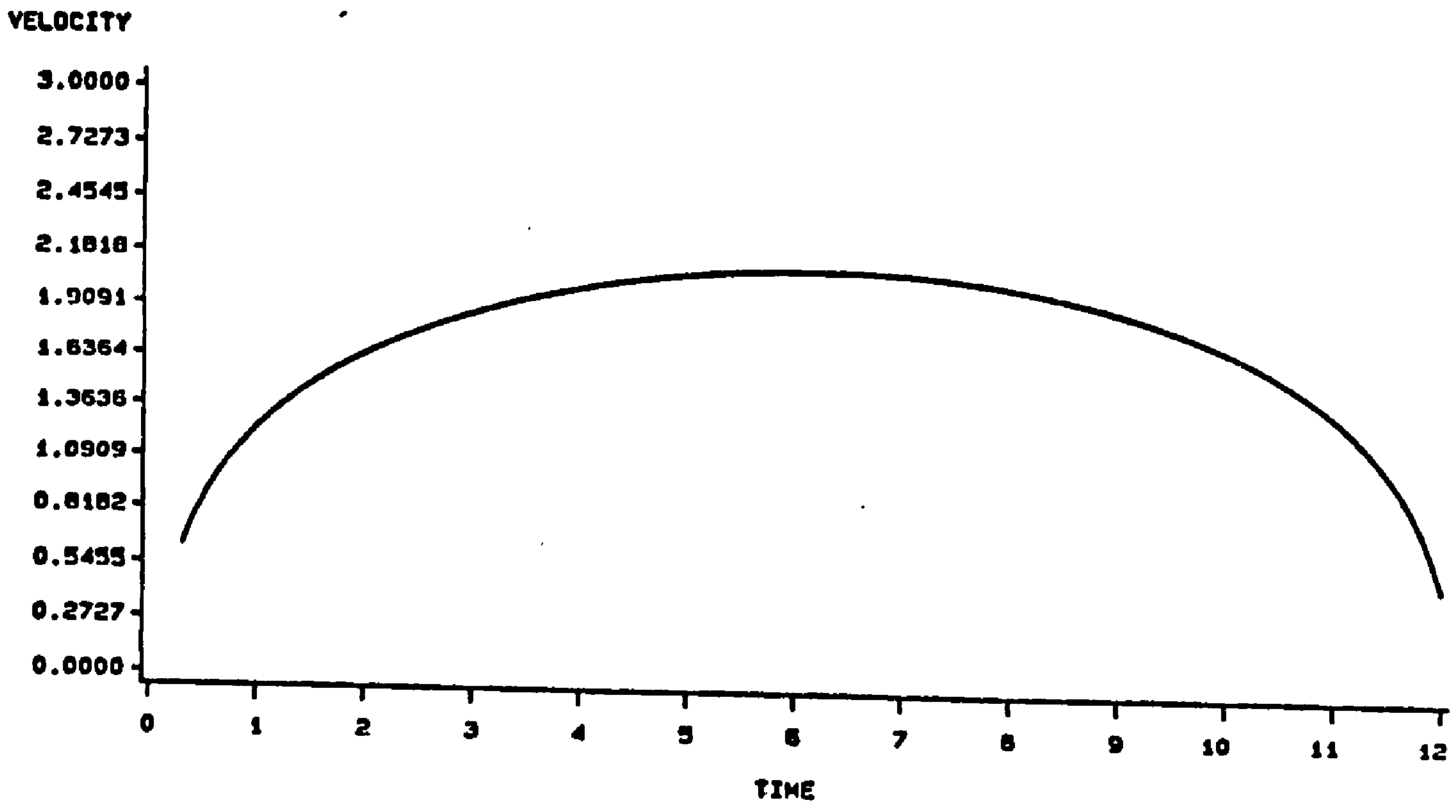
therefore the total time between the first and second collisions is given by

$$T = \frac{\pi}{\sqrt{2\varepsilon}} - 2\delta_o \tag{4.28}$$

setting $X_o(t \to 0) = 0$ gives $\delta_o = \frac{1}{\sqrt{24}} = 0.2$

(4.29) thus $$T = \frac{\pi}{\sqrt{2}} \frac{1}{\left[E_s(U_i)/M - U_i^2\right]^{1/2}} - 0.4$$

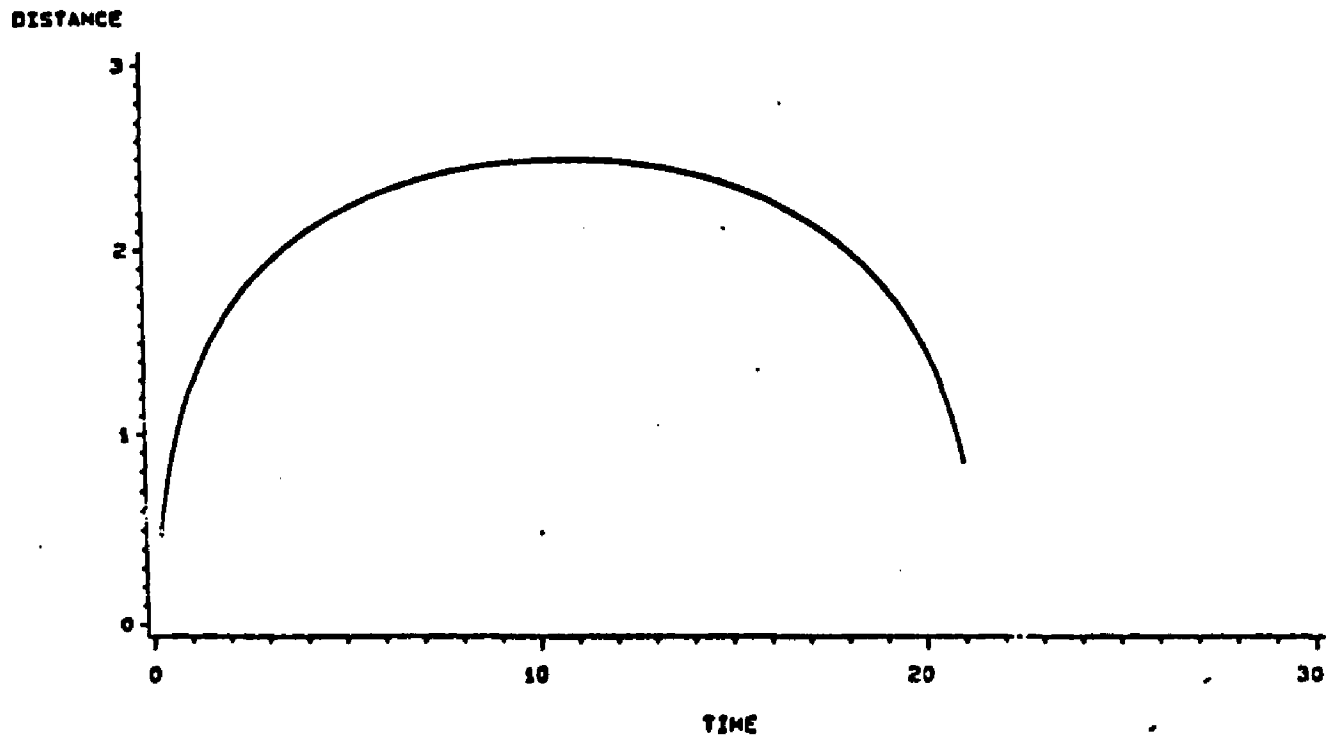
GRAPH OF Xo(T) VS T



INITIAL VELOCITY V1=0.1

FIGURE 16

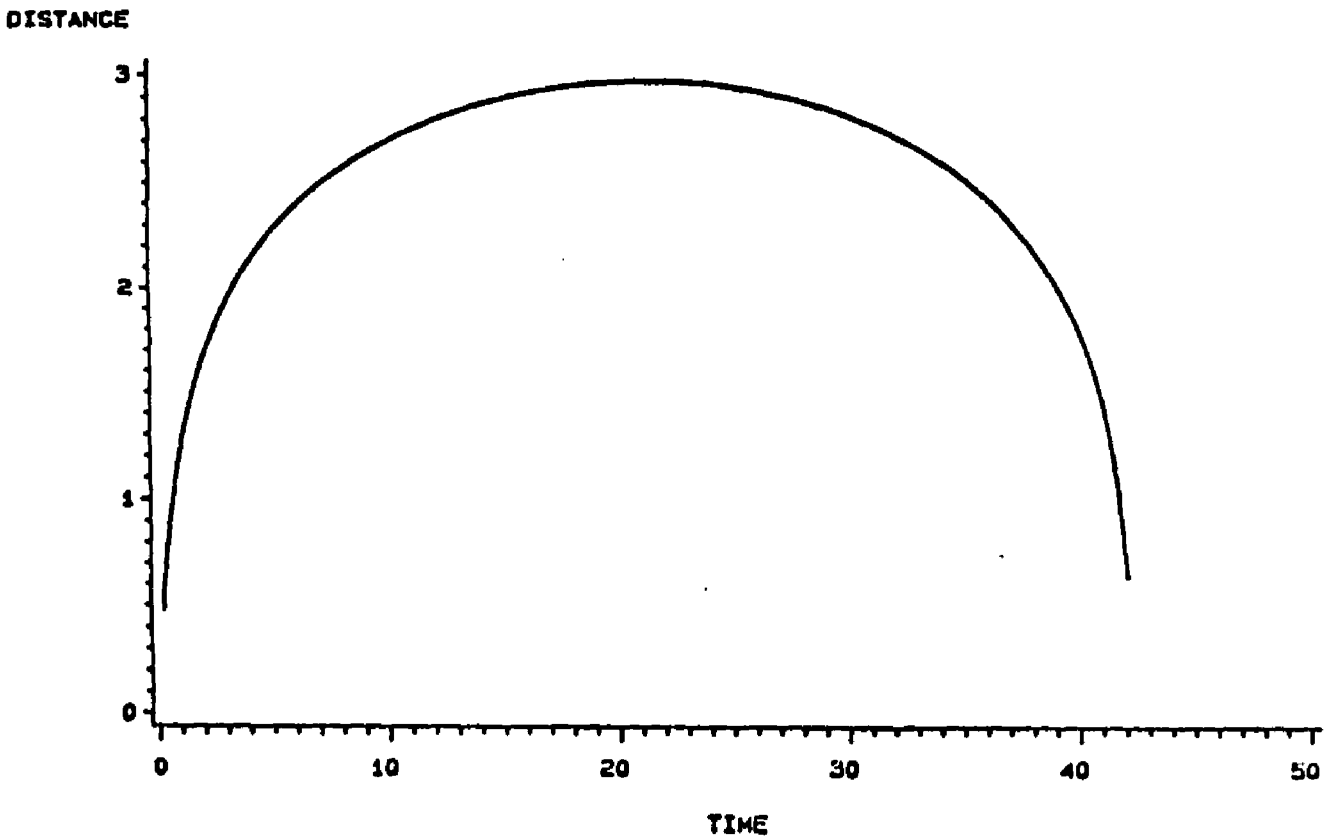
SEPERATION DISTANCE Xo(T) VS TIME



Initial velocity Ui=0.22

FIGURE 17

SEPERATION DISTANCE Xo(T) VS TIME



Initial velocity Ui=0.25

Resonance Condition

We have seen that (at least outside the interaction area) the amplitude A(t) of the shape modes is harmonically changing in time eq.(3.15). Thus with

(4.30) $$y_s = A(t)\sqrt{\frac{3}{2\sqrt{2}}}\tanh\frac{x-X_0}{\sqrt{2}}\operatorname{sech}\frac{x-X_0}{\sqrt{2}}$$

and $\ddot{A} + \omega_s^2 A = 0$ writing

(4.31) $$A(t) = S e^{i\omega t} + S^* e^{-i\omega t} = S(t) + S^*(t)$$

(where S=S(t=0) is a complex constant and A(t) is real)

Esh is given by

(4.32) $$\begin{aligned} E_{SH} &= \frac{1}{2}\left(\dot{A}^2 + \omega_s^2 A^2\right) \\ &= \frac{1}{2}\left\{\left[i\omega_s S e^{i\omega_s t} - i\omega_s S^* e^{-i\omega_s t}\right]^2 + \omega_s^2\left[S e^{i\omega_s t} + S^* e^{-i\omega_s t}\right]^2\right\} \\ &= 2\omega_s^2 S^*(t)\, S(t) = 2\omega_s^2 |S|^2 \end{aligned}$$

and the total shape energy for both kinks

(4.33) $$E_{SH} = 4\omega_s^2 |S(t)|^2$$

We make the assumption that this energy is given shortly after the first impact and fast enough that we can write

$$(4.34) \qquad S_{\alpha f+}(t) = \alpha\, S_{\beta ef}(t) + \rho$$

with ρ , α complex constants.

The above equation relates the constant amplitude (S) just before and just after each collision. In reality ρ is a function of time and Xo, and expresses the weak coupling of the internal modes among themeslves. We actually assume that ρ can be written as

$$(4.35) \qquad \rho(t) = \rho\,[\,1 + f(x_o,t)\,]$$

with $f(x_o,t)$ representing all the extra terms that we take to vanish fast enough in time that we can approximate

$$\rho(t) \sim \rho \quad \text{=constant}$$

Adopting primes (′) to denote the constant amplitude S(t) right after each collision we write

$$(4.36) \qquad S'(T) = \alpha\, S(T) + \rho$$

where T is the moment of the collision.

If the first collision takes place at t= T_1

$$S'(T_1) = \alpha S(T_1) + \rho$$

thus

$$A'(t) = (\alpha S(T_1) + \rho) e^{i\omega_s(t-T_1)} + (\alpha S(T_1) + \rho)^* e^{-i\omega_s(t-T_1)}$$

Before the first collision the shape oscillation is not excited, that is,

for $t < T_1$ $S(t) = 0$, then

$$A(t) = S e^{i\omega_s t} + S^* e^{-i\omega_s t} = 0 \quad \Rightarrow \quad S = 0$$

for all $t \leq T_1$

Continuity at T_1 demands

$$\left.\begin{array}{l} A'(t = T_1) = A(t = T_1) = 0 \\ S(T_1) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \rho = -\rho^* \\ \rho = \pm i|\rho| \end{array}$$

ρ is pure imaginary

and $S'(T_1) = \rho$ at $t = T_1$

and for later times t $S'(t) = \rho e^{i\omega_s(t-T_1)}$

At $t = T_2$ kinks come back to collide again :

$$S''(T_2) = \alpha\, S'(T_2) + \rho$$

is the constant amplitude right after the second impact. Since the field equation possesses time-reversal invariance, if we go backwards in time S''^{*} interchange with S' . Thus if the initial transformation is to work

$$\left(S''^{*}(T_2) \right)' = S'^{*}(T_2) \qquad \text{or}$$

$$\alpha \left[\alpha^{*} S'^{*}(T_2) + \rho^{*} \right] + \rho = S'^{*}(T_2) \qquad \text{or}$$

$$\left.\begin{array}{l} |\alpha|^2 = 1 \\ \alpha \rho^{*} + \rho = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} |\alpha|^2 = 1 \\ -\alpha\rho + \rho = 0 \end{array}\right\} \Rightarrow \alpha = 1$$

Thus

$$S''(T_2) = S'(T_2) + \rho = \rho + \rho\, e^{i\omega_s (T_2 - T_1)}$$

Similarly at $t = T_3$ if a third collision occurs

$$S'''(T_3) = \left\{ \rho \left[e^{i\omega_s (T_2 - T_1)} + 1 \right] e^{i\omega_s (T_3 - T_2)} + \rho \right\}$$

and so on for the n^{th} collision

$$S^{n\prime} = \rho \left[1 + \left[e^{i\omega_s (T_n - T_{n-1})} \left[1 + e^{i\omega_s (T_{n-1} - T_{n-2})} \left[\cdots \right.\right.\right.\right.$$

(4.37)

$$+ \quad . \quad . \quad . \quad \left[1 + e^{i\omega_s(T_2 - T_1)} \right] \Big]$$

For the ϕ^4 kinks (experimentally) we have only two collisions. If the energy Esh= $4\omega_s^2 |S''|^2$ is to be zero

$$S''(T_2) = \rho \left[1 + e^{i\omega_s(T_2 - T_1)} \right]$$

must vanish. Then

$$1 + e^{i\omega_s(T_2 - T_1)} = 0$$

or the time between collisions must be

$$(T_2 - T_1) = (2n + 1)\pi/\omega_s$$

But this time was found, from eq.(4.28), to be

$$\pi/\sqrt{2\varepsilon} - 0.4$$

therefore,

$$(2n+1)\pi/\omega_s + 0.4 = \pi/\sqrt{2\varepsilon}$$

(4.38) $$\varepsilon = E_{SH}(U_i)/M - U_i^2$$

$$(2n + \delta)\pi/\omega_s = \pi/\sqrt{2\varepsilon}$$

where δ =1.15

So far we have neglected radiation for two reasons: First, because the numerical simulations [1] indicate that the energy carried by the radiation modes is not very important in the collision process (see Figure 15 from reference [1]) and second any attempt to include radiation modes y_c ,eq. (2.22), in the trial solution will make the resulting equations of motion very difficult to solve and the crucial assumption we have made namely, conservation of total energy, wouldn't go along with the dispersive character of the continuum if it was to be included. However, since the sum of translating and "wobbling" kinks is not an exact solution of the ϕ^4 equation (at least in our approximation), the shape modes will decay in second order in the expansion $\phi(x,t) = \phi_0(x) + y(x,t)$ around the K or $\bar{K}$. This energy (if we accept the energy exchange mechanism) will be given back to the translational modes of the K and $\bar{K}$ further weakening the binding. We collectively account for this retransfering of the energy from the internal modes to the translational mode (while they are still trapped), by rewriting the binding energy as

$$\varepsilon M \rightarrow M \varepsilon a$$

where a is a constant, and $(1-a)M\varepsilon$ collectively describes the effect of the decay radiation. If radiation created at the time of the first impact (prompt radiation) is present, a describes both prompt and decay radiation and doesn't have to be less than 1. For

Φ^4 we neglect prompt radiation and α is expected $\alpha < 1$, then from eq. (4.38) we have

$$(2n + 1.15)\frac{\pi}{\omega_s} = \frac{\pi}{\sqrt{2\alpha}\ \sqrt{\varepsilon(U_i)}}$$

(4.39)

$$(2n + 1.15)\frac{\pi}{\omega_s} = \frac{\pi}{\sqrt{2\alpha}} \frac{1}{\sqrt{E_{SH}(U_i)/M - U_i^2}}$$

For Ui=0

$$(2n + 1.15)\frac{\pi}{\omega_s} = \frac{\pi}{\sqrt{2\alpha}} \frac{\sqrt{M}}{\sqrt{E_s(o)}}$$

since $\sqrt{\frac{E_s(U_i=0)}{M}} = 0.216$ α is given from

(4.40) $$0.216\,(2n + 1.15)\frac{\sqrt{2}}{\omega_s} = \frac{1}{\sqrt{\alpha}} > 1$$

the smallest integer n which satisfies this equation is n=2 and α=.599 With α determined, the initial incoming velocities for which bouncing occurs are given by (4.41)

(4.41) $$U_i^2 = \frac{E_{SH}(U_i)}{M} - \frac{1.25}{(2n+1.15)^2}$$

Resonance Velocity Windows

The data reveals that there exists a small velocity range centered at each U_i for which we still have "resonance", Campbell et. al. call these velocity ranges " windows ". In order to understand these windows better we go back to $S''(T_2)$ and let

$$\omega_s (T_2 - T_1) = (2n\pi + \pi) \pm \partial/2$$
$$= (2n\pi + \pi) \pm \omega_s \Delta T/2$$

We want to know what ∂ must be so that (see eq. (4.37))

$$\left| S''(T_2) \right|^2 = |\rho|^2 \left| 1 + e^{i\omega_s (T_2 - T_1)} \right|^2 = |\rho|^2$$

which means $E_{sH} = 4\omega_s^2 |\rho|^2$

wasn't transfered back to the translational modes, and the second impact left the energy in the shape modes unaffected. Inserting ∂ in $S''(T_2)$ we find

$$1 = \left| 1 + e^{i(2n\pi + 1) \pm \partial/2} \right|^2$$

or

$$\sin^2 \partial/4 = 1 \Rightarrow \partial = 2$$

then $\omega_s \Delta T = \partial$ with

$$\Delta T = \frac{\partial T}{\partial U} \Delta U$$

$$\Delta U_i = 2 / \omega_s \frac{\partial T}{\partial U_i}$$

$$T = \frac{\pi}{\sqrt{2}\sqrt{0.599}\ \sqrt{E_s(U_i)/M - U_i^2}} - 0.4$$

which gives

$$\Delta U_i = \frac{1}{\omega_s U_i (2n + 1.15)^3} \tag{4.42}$$

Comments

Strictly speaking in order to understand the energy exchange between the shape and translational mode one needs an exact time dependent expression for the shape amplitude A(t) which will describe the shape disturbance at any time and particularly at the times when kinks collide. Unable to have at our disposal such an expression for A, in this section we have adopted the basic idea of Campbell's model which assumes that the effect of the impact on the kinks can be described by the relation [1]

$$S' = \alpha_1 S + \alpha_2 S^* + \rho$$

(α_1 , α_2 , ρ complex constants)

and S', S the complex amplitude just before and just after each collision. Campbell et.al. in their effort to determine the constants use information from their numerical simulations. Wishing to avoid the use of the data we have modified the above relation by writing S' as

$$S' = \alpha S + \rho$$

and demanding in addition to time reversal invariance, continuity of the shape amplitude at the time of each collision. Following our analysis we have derived eq. 4.39

$$\frac{\pi(2n\pi + \delta)}{\omega_s} = \frac{\pi^2}{\sqrt{2\alpha}\,\left(E_s(U_i)/M - U_i^2\right)^{1/2}}$$

$\delta = 3.6$

which is equivalent to Campbell's

$$\frac{(2n+1)\pi}{\omega_s} = \frac{\pi}{\sqrt{2a}\,(U_c^2 - U_i^2)^{1/2}}$$

and to the equation which best fits the data results .

$$\frac{(2n+\delta/\pi)\pi}{\omega_s} = \frac{\pi}{\sqrt{2}}\;\frac{1}{0.74\,(U_c^2-U_i^2)^{1/2}} \qquad \delta = 3.3$$

Campbell determines a from the data by first plotting the final kinetic energy versus the initial velocity U_i assuming

$$MU_f^2 = (MU_i^2 - E_{SH})\,a_1$$

For U_i close to U_c he sets $E_{SH} = MU_c^2$ and from the data he finds that

$$U_f^2 = (U_i^2 - U_c^2)\,a_1$$

is well described with a_1=0.84

Assuming smooth continuation between the binding energy ($U_i < U_c$) and the outgoing kink velocity for $U_i > U_c$ he lets the binding energy to be

$$M\varepsilon = (MU_c^2 - MU_i^2)\,a$$

or

$$\varepsilon = (U_c^2 - U_i^2)\,a$$

and approximates $a = a_1$ =0.84 therefore $\sqrt{a}$ = 0.916

Our method gives $\sqrt{\alpha}$ =0.773 which is in an excellent agreement with the numerical one $\sqrt{\alpha}$ = .74 . In our analysis we were able to understand the phase shift δ as $\delta = \pi + \delta_0 \omega_s$ where δ_0 is determined by

$$X_0(t) = \frac{1}{\sqrt{2}} \ln\left[\sqrt{\frac{12}{\varepsilon}} \sin\sqrt{2\varepsilon}\,(t + \delta_0)\right] \tag{4.43}$$

at t= 0

We can't take the value of δ_0 seriously since (4.43) is valid for large Xo and any short range approximation is not to be trusted. However, δ_0 was surprisingly found close to what it should be and offers an explanation of the origin of which the old model fails to explain.

We also, through eq.(4.40) explain the minimum number of full oscillations, n>3 for which resonance was observed. In Fig. 20 (taken from reference [1]) the two large spikes in each picture correspond to the two K-$\bar{K}$ reflections, while the bumps between the spikes correspond to the sum of the tails of the shape waveforms centered on the kink and antikink. The number of bumps as one moves from one picture to another is increasing by one and is given by N=2-n where n is the number of complete shape oscillations with n>2 exactly as was predicted by (4-40). Above the critical velocity the big bump to the right disappears as the kinks reflect to infinity and never come back to collide again.

The crucial assumption of conservative enrergy exchange between the internal modes is confirmed in Figure 19 where the ratio of the (time-averaged) kink speed after a K-$\bar{K}$ collision to the initial velocity is plotted as a function of the initial velocity. From the plot (taken from reference [1]) one can see the (relatively) elastic nature of the reflections below the critical velocity Uc.

The new information we get from our analysis is the presence of Esh(Ui)/M in formula (4.41) instead of Uc^2. For Ui close to Uc and from the plot of Esh vs Ui (Fig. 14) we see that Esh/M is well approximated by Uc^2.

In table 1. the predicted (by the model) values of the resonance velocity windows Ui and their center (Ui) are compared with those found numerically. The table is taken from Campbell et. al. where he uses eq. (4.41) and (4.42) to predict Ui and ΔUi respectively, with the term Esh/M approximated with Uc^2. The agreement, especially for initial velocities close to Uc, is remarkable.

However a more accurate description of E_{SH} which in turn means an exact expresion for A(t), would lead to higher precision in predicting Ui directly from

$$(2n + \delta/\pi)\frac{\pi}{\omega_s} = \frac{\pi}{\omega_0}\,\frac{1}{\left[E_{SH}(U_i)/M - U_i^2\right)^{1/2}\sqrt{a}}$$

Nevertheless our results are "data free" analytical ones and our goal was to show that the model can be analytically formulated to predict the experiment. Its validity is checked in the next section for more kink-bearing field equations where kink collisions lead to resonance.

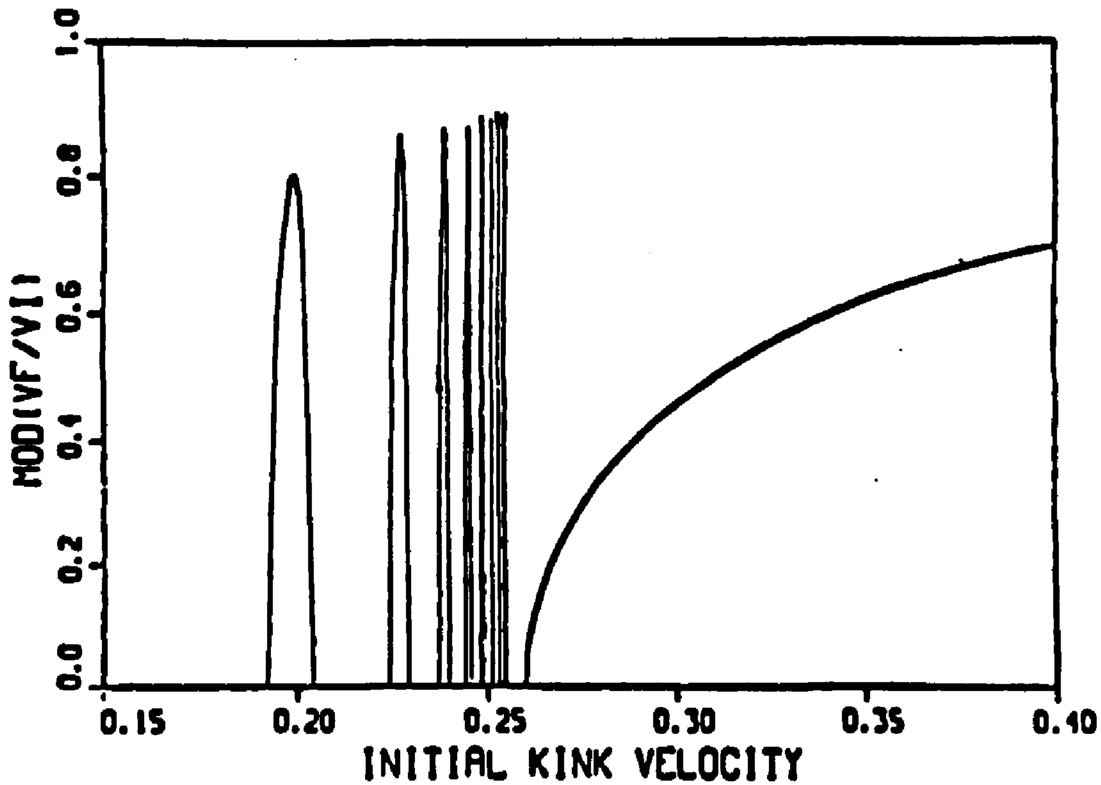
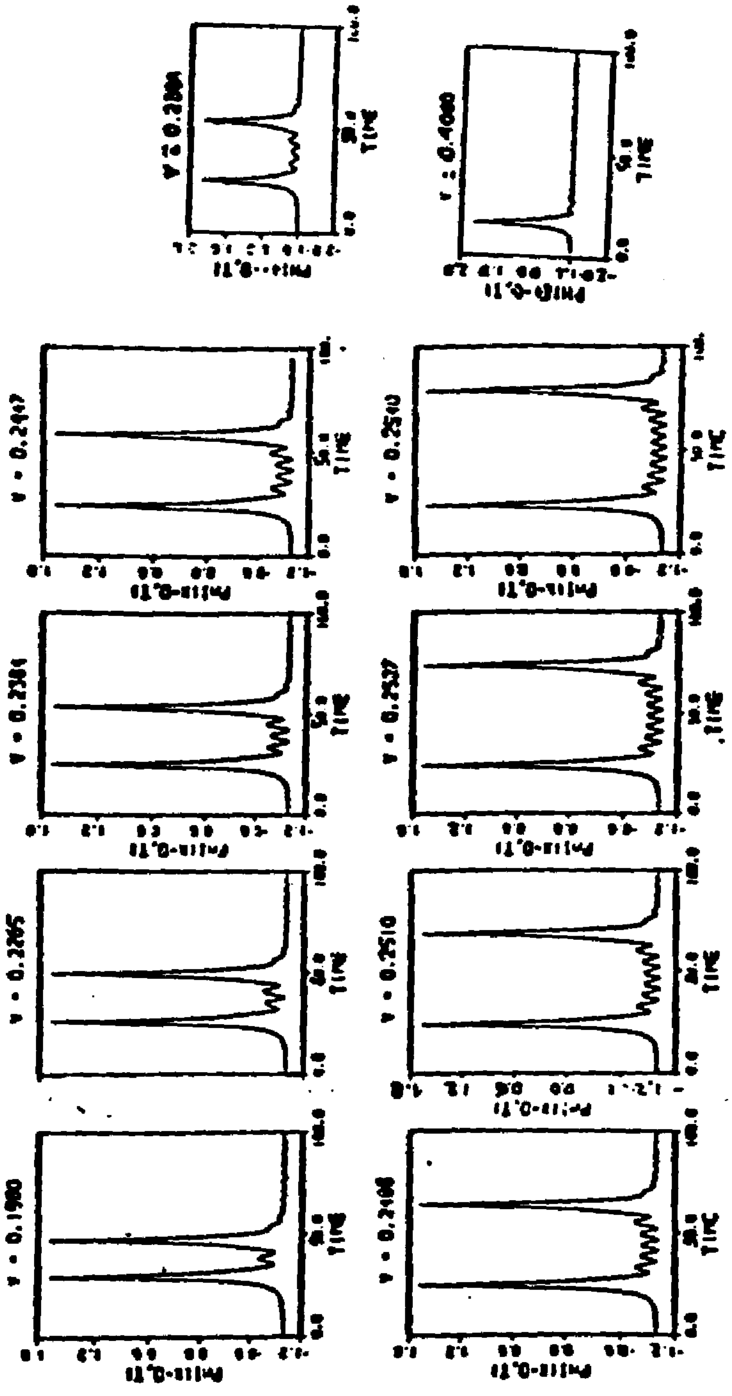


Fig. 19 The ratio of the (time-averaged – see section 4) kink speed after a $K\bar{K}$ collision to the initial speed, as a function of the initial velocity. Note the relatively elastic nature of the reflections below v_c. (From reference 1)



$\phi(x = 0, t)$ versus t for incoming speeds at the centers of the first eight two-bounce windows.

FIGURE 20

Table I

A tabulation of the range, center, and width in initial velocity of the reflection windows observed in ϕ^4 $K\bar{K}$ collisions. The last three columns give respectively, the theoretically predicted value of the window center and two quantities which our theory predicts to be constant. In the last two columns v = center of window and Δv = width (Taken from reference 1)

$N_\omega = n - 2$	Range of v	Center	Width	Predicted Center	$(v_c^2 - v^2)(2n+1)^2$	$\Delta v(2n+1)^3$
1	0.1926 – 0.2034	0.1980	0.0109	0.1990	1.39	3.74
2	0.2241 – 0.2288	0.2265	0.0048	0.2250	1.31	3.50
3	0.2372 – 0.2396	0.2384	0.0025	0.2370	1.29	3.33
4	0.2440 – 0.2454	0.2447	0.0015	0.2437	1.29	3.30
5	0.2481 – 0.2490	0.24855	0.0010	0.2478	1.29	3.38
6	0.2507 – 0.2513	0.2510	0.0007	0.2505	1.30	3.44
7	0.2525 – 0.2529	0.2527	0.0005	0.2524	1.31	3.4
8	0.2538 – 0.2541	0.25395	0.0004	0.2538	1.33	3.7
9	0.2548 – 0.2550	0.2549	0.0003	0.2548	1.33	4.3

Generalized Model.

Going back to the formulas we have derived, eq. (4.12) , (4.15) we see that for separation distances far from the interaction region conservation of energy reads

$$M \dot{X}_o^2(t) - A e^{-2\omega_o X_o(t)} = 2M - \varepsilon M \tag{4.44}$$

Where A is the amplitude of the effective potential

$$V_{eff} = A e^{-2\omega_o X_o(t)}$$

and ω_o the lowest frequency of the continuum.

Solving (4.44) for the general case with A, ω_o unknown we get

$$X_o(t) = \frac{1}{\omega_o} \ln\left[\sqrt{\frac{A}{\varepsilon}} \sin\sqrt{\varepsilon}\,\omega_o(t+\delta_o)\right]_{\delta_o = 1/\omega_o\sqrt{\varepsilon}} \tag{4.45}$$

In terms of ω_o, A the expression which relates times between collisions (4.38) and small oscillations reads

$$\frac{2n\pi + \pi + \delta_o\omega_s}{\omega_s} = \frac{\pi}{\omega_o\sqrt{\alpha}\sqrt{E_{sh}(U_i)/M - U_i^2}} \tag{4.46}$$

The L.H.S was derived for the general case independent of the details of the original field equation. The R.H.S depends on the structure of the kinks. That is Esh, Uc, ω_o, ω_s , δ are different for different types of kinks. We try the generalized model for the cases of the parametrically modified-sine-Gordon (MSG) and double-sine-Gordon(DSG) theories, where double resonances were (numerically) observed in kink collisions[17,26]. For these theories the scalar field satisfies an equation of the form

$$\Box \phi + \frac{2}{1+|4\eta|}\left(-\sin\phi/2 + 2\eta\sin\phi\right) \qquad -\infty < \eta < \infty \qquad \text{DSG}$$

$$\Box \phi + \frac{\partial U(\phi)}{\partial \phi} = 0 \quad , \quad U(\phi) = (1-r)^2(1-\cos\phi)(1+r^2+2r\cos\phi) \qquad \text{MSD}$$

$$-1 < r < 1$$

We check the models using the formula which predicts the center of the resonance velocities, eq.(4.38)

$$\frac{2n\pi + \delta}{\omega_s} = \frac{\pi}{\omega_0\sqrt{a}} \frac{1}{\left[E_{SH}(U_i)/M - U_i^2\right]^{1/2}} \tag{4.38}$$

Since Esh, to be determined, requires solving the equations of motion for each theory, we appproximate Esh with MUc^2= Esh since the dependence of Esh(Ui) on Ui is very weak and the variation of Esh from Ui is small (see plot Esh vs Ui)

then

$$\frac{2n\pi + \delta}{\omega_s} = \frac{\pi}{\omega_0\sqrt{a}} \frac{1}{\sqrt{U_c^2 - U_i^2}}$$

Knowing $\omega_s, \omega_0, U_c, \delta$ we estimate $\sqrt{a}$ as follows:

At first we neglect radiation (thus $\alpha = 1$). For Ui=0 no resonance n must be the integer before n+1 for which resonance occurs.

$$\frac{2n\pi + \delta}{\omega_s} = \frac{\pi}{\omega_o U_c}$$

$$(2n\pi + \delta)\, \omega_o U_c / \omega_s \pi = 1$$

The value of n which comes closest to satisfying this equation is chosen. Then to account for radiation we insert α

$$\frac{2 n_{min} \pi + \delta}{\omega_s} = \frac{\pi}{\omega_o U_c \sqrt{\alpha}}$$

solving for α

$$\alpha = \left[\frac{(2n + \delta/\pi)\, \omega_o U_c}{\omega_s} \right]^{-2}_{n \equiv n_{min}}$$

If $\alpha < 1$ we have more decay than prompt radiation

If $\alpha > 1$ (which means prompt radiation is stronger) the kinks are bound stronger.

The assumption that the effective potential has the assumed form of eq. (4.44) has been checked in references[17,26]. Using the values they have found for Uc, ω_o , ω_c , δ and applying eq. (4.46) we summarise the results in Table 2.

	δ	Uc	ω_o	ω_s	R (parameter)	$\sqrt{a}$(p)	$\sqrt{a}$(d)	n
DSG	3.99	.359	.866	.8409	-1.0	0.82	0.86	n>1
DSG	3.40	.2305	1.0	.69204	1.2	0.95	1.0	n>1
MSG	1.60	.175	1.222	1.1205	-	1.38	1.0	n>2
MSG	1.70	.2925	1.66	1.1675	-	0.94	1.34	n>1

TABLE 2. Application of the generalized model to DSG and MSG equations.

where $\sqrt{a}$ (p) is the predicted value of the parameter a using our model and $\sqrt{a}$ (d) is the parameter a found from the data. The results would be more accurate if we knew E(Ui). Using E(Ui=0) instead of WUc^2 in (4-38), we would be able to describe the parameter a more accurately. We would also like to point out that for the last predicted value in Table 2 the old model[1] fails completely to approximate the parameter a.

CHAPTER VI

CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK.

In this thesis we have studied kink interactions via collisions. A kink (K) from the "left" collides with the spatial reflection of itself, the antikink($\bar{K}$), coming from the "right". We have seen that as long as K and $\bar{K}$ are far enough apart and the interaction is very weak, the particle-like nature of the kinks dominate and the picture is described by two particles of mass M moving in opposite directions with constant velocities. As the kinks come closer, their internal structure starts to unfold. The kinks interact via their internal modes resulting in an attractive effective force accelerating the particle-like kinks towards each other. It is almost impossible (at least in our approach) to account analytically for what is happening when the kinks actually meet. Perhaps one needs to include more terms in the solution (radiation modes) and keep higher order coupling terms in the lagrangian. Somehow (for certain initial velocities) all the energy released on the impact is given back to reform the kink-pair (that's the beauty of the solitary essence of the kinks) and far from the interaction region we see again two particles scattered inelastically with their wave-like properties present by means of shape disturbances riding on the particle. Even if the energy taken to create these shape disturbances is very small compared to the initial total energy of the kinks (2M), it is enough to trap the pair for initial velocities less than Uc as our calculations have shown. That was one of our goals, to calculate the energy in the shape modes and show why the kinks are trapped for certain velocities below Uc. (Campbell's calculations require the critical velocity be estimated from the data).

The second goal was to explain the bouncing observed, namely why the kinks even if they couldn't escape after the first impact come back to collide again and then escape. To explain this phenomenon we needed an analytical picture of what is happening at the time of the first impact. With our set up of the trial solution (4.2), even if we were able to solve exactly the equations of motion (4.8),(4.9), we wouldn't be able to describe correctly the picture since the trial solution was built up to work outside the scattering area. In the absence of knowledge of what is happening for small separation distances, we use a modified version of Campbell's model to explain the transferring of energy from the translational modes to the shape modes. We idealize the effect of the impact on the kinks by assuming that each impact relates the amplitude of the small shape oscillations just before and right after a collision by a linear relationship of the form (4.36). This assumption enables us to relate the energy in the shape modes right after each collision in terms of what this energy was just before the collision. This is the mechanism of the energy exchange between the internal modes at the time of each impact. Knowing the energy in the shape modes from the equations of motion and the initial velocity of the kinks when they are far apart, we are able to find the binding energy of the kinks which in turn determines the time it takes for the kinks to collide twice. Relating this time with the time the model requires for the shape modes to retransfer their energy to the translational modes, we were able to predict the initial velocities for which the bouncing takes place. This mechanism namely that the kinks have the ability to exchange energy between translational modes and internal modes is not restricted only for the ϕ^4 theory or for that purpose only to kink interactions. In nuclear physics, in time-dependent Hartree-Fock calculations the collisions of the individual nuclei viewed as "solitary waves" shows similar resonant energy exchange between their two internal modes [42]. The DSG, and MSG equations also show the same behavior. The interesting thing is that all these systems possess (at least) two internal modes which is consistent with the assumptions of the proposed model[1].

Nevertheless, after all the details are stripped away the question is how well we still understand the collisions. We have studied kink interactions for system, for which we don't even have an analytic multi-kink solutions (which may not exist). We have described linearly and collectively the system in addition to introducing a mechanism (of energy transfering) which is not directly related to the structure of the corresponding field equation. The model succeeds in predicting the ϕ^4 numerical experiments, and when is generalized predicts the results of other theories too. The main reason for the effectiveness of the proccedure followed is that we work with small velocities, thus small binding energies which allow the kinks to spend most of their time far from the scattering region for which we know very little. This allows one to neglect higher nonlinear terms which are important only for small separation distances and concentrate on how most accurately to describe the solution for large separation distances where the coupling of the internal modes has already been washed out. The calculations were carefully chosen to be carried out in the region where the trial solution is trusted, and the "collective coordinates" (directed from the physics of the problem), were chosen to dominate in this region. The model which actually "links " the two regions doesn't analytically provide information on how these two regions are connected but rather collectively describes how the energy exchange takes place. Again the model works because of the solitary wave properties of the kink which allows it to retain its identity against small external disturbances. This allows the disturbance due to the collision to be absorbed fast enough that the actual time of interaction is very small compared to the time kinks spend far from the scattering region. The crucial assumption that the total energy is conserved, and only a small fraction of it is exchanged between the internal modes at each collision, is supported by the fact that radiation (thus, energy dissipation) is little excited in the collision process (see Fig.5). If one wishes to understand better what is happening when the kinks actually "meet", he has to revise the Ansatz solution. As we have seen from the Einstein-like energy expression (2.10) for the kinks, as the velocity increases the kink "reacts" by increasing its amplitude and decreasing its width. On the other hand due to the attractive effective potential, as the kinks come closer they start to speed up. We expect then the kinks' shape to suffer a big change for small separation distances, and it is therefore meaningless to include the shape eigenfunctions in the solution as too small to add any information about the kink which itself is a big local disturbance.

Intuitively, we suggest that one should replace the kinks and the shape modes in the trial solution (4.3) by

$$A(t,-Xo)\ TANH(\ X-Xo/B(t))\ \ -\ A(t,Xo)\ TANH[X+Xo/B(t)]$$

where the collective coordinates A and B now describe changes in amplitude and width respectively. Adding radiation and keeping all the terms resulting from substituting the trial solution in the lagrangian one should get from the equations of motion A,B as functions of Xo and time. The coupling among A,B and radiation perhaps would explain the energy transferring. Even if radiation is not explicitly involved in the collision process for large Xo we believe it to be responsible for the interaction between the kinks. The excitation of the shape modes is a result of this interaction. For instance for the case of soliton collisions a similar study shows that an effective potential [10] similar to the kink type potential exists there too. Since the solitons (of sine-Gordon) do not have internal "storage" for energy (no bound states) the energy exchange takes place between radiation and translation mode resulting for large Xo elastic scattering. For kinks which are not solitons, in our opinion the initial collision always excites both radiation and shape modes. The exchange of radiation provides the interaction between the kinks and the shape oscillations and explains the inability of the kinks to scatter elastically. The radiation energy is exchanged, not lost, this is why our results which did not include radiation explicitly were accurate. However , since a kink is a solitary wave and not a soliton some decay energy is present and we have to account for this by the constant a in eq. (4.39).

It would be interesting for purposes of physical implications to have an experiment with more kinks involved. For instance one could first let a kink and antikink collide and as the pair is scattered one could send another pair with a kink from the "left" and antikink from the "right" to collide with the first, already, scattered pair. If the shape mode-translation coupling works, then one would have four kinks with their shape modes excited or if the initial velocities of each pair were properly chosen one could have possibly, resonances among the kinks of the two pairs. In solid state that could be used to control special properties of the kink, for instance increase the kinks' temperature by exciting its shape modes which in turn would affect other properties (i.e, conductivity) of the material as a whole. Also it would be interesting to perturb a pure soliton-like system by a kink source. For instance a soliton of the s-G equation collides with an impurity which we take to be a kink. Will the soliton retain its identity after the collision in addition to exciting the kinks' shape? Or will there be capture ? If both retain their identity then it is possible that K-S is a solution of a very interesting nonlinear equation.

Nevertheless the problem of understanding the solitary properties of a kink is still open . The property of a kink's internal structure that makes the kink able to generally maintain its shape and act as a particle-like object is still not understood. We have just begun to scratch the surface.

APPENDIX A

DERIVATION OF THE ϕ^4 FIELD EQUATION AND KINK SOLUTION

Applying the variational action principle

$$\delta \left[\int dt \int_{-\infty}^{\infty} dx \, \mathcal{L}(x,t) \right] = 0$$

to the lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \partial^\mu \phi + m^2 \phi^2 - \frac{\lambda}{4} \phi^4 - \frac{m^4}{4\lambda^4}$$

we get

$$\sum_{\mu=1}^{2} \frac{d}{dx_\mu} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi}{\partial x_\mu} \right)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi_x} + \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \phi_t} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{d}{dx}(-\phi_x) + \frac{d}{dt} \phi_t - \left[m^2 \phi - \lambda \phi^3 \right] = 0 \Rightarrow$$

$$-\phi_{xx} + \phi_{tt} = m^2 \phi - \lambda \phi^3$$

For static solution ($\phi_t = 0$), this becomes

$$-\phi_{xx} = m^2\phi - \lambda\phi^3$$

or

$$-\frac{\phi_x^2}{2} = \frac{m^2\phi^2}{2} - \frac{\lambda}{4}\phi^4 \qquad + \text{ constant}$$

choose the constant C so that

$$\phi_x \underset{(x\to\infty)}{\longrightarrow} 0$$

and $\phi(x\to\infty)$ minimizes the total energy

then $\quad C = -\dfrac{m^4}{\lambda^4}$

$$-\frac{\phi_x^2}{2} = \frac{m^2\phi^2}{2} - \frac{\lambda}{4}\phi^4 - \frac{m^4}{4\lambda}$$

$$-\frac{\phi_x^2}{2} = -\frac{m^4}{4\lambda}\left[\left(\frac{\sqrt{\lambda}}{m}\phi\right)^2 - 1\right]^2$$

$$\left[\left(\frac{\sqrt{\lambda}}{m}\phi\right)_x\right]^2 = \frac{m^2}{2}\left[\left(\frac{\sqrt{\lambda}}{m}\phi\right)^2 - 1\right]^2$$

let
$$\sqrt{\lambda}/m \;\phi \to \phi'$$
$$m\,x \to x'$$

then $$\phi'^{\,2}_{x'} = \frac{(\phi^2 - 1)^2}{2}$$

$$\pm \int_{\phi(X_0)}^{\phi(x)} \frac{d\phi'}{(\phi'^2 - 1)} = \frac{x - X_0}{\sqrt{2}}$$

which gives

$$\phi' = \pm \tanh \frac{x' - X_0'}{\sqrt{2}}$$

and

$$\phi(x) = \pm \tanh \frac{m}{\sqrt{2}} (x - X_0)$$

with corresponding moving (Lorentz boosted) solutions

$$\phi'(x', t) = \pm \tanh \left\{ \frac{(x' - X_0') \mp Ut'}{[2(1 - v^2)]^{1/2}} \right\}$$

$$\phi(x, t) = \pm \tanh \left\{ \frac{m}{\sqrt{2}} \frac{(x - X_0) \mp Ut}{[2(1 - v^2)]^{1/2}} \right\}$$

In terms of the rescaled field ϕ', the field equation becomes

$$\phi_{tt} - \phi_{xx} = \phi - \phi^3$$

APPENDIX B

FLUCTUATION MODES

We are solving equation (2-16) using a different approach. The eigenvalue problem reads

$$y_{0xx} + \left[\omega^2 + 1 - 3 \tanh^2 \frac{x}{\sqrt{2}} \right] y_0 = 0$$

or

$$y_{0xx} + \left[\omega^2 - 2 + 3/\cosh^2 x/\sqrt{2} \right] y_0 = 0$$

This type of problem is worked out by Landau [25]. Keeping the same notation, we identify

$$E = \omega^2 - 2$$

$$U_0 = 3 \quad , \quad \alpha = 1/\sqrt{2}$$

and with the substitutions

$$y_0 = \frac{w}{\cosh^s \alpha x} \qquad s = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{4U_0}{\alpha^2}} \right]$$

the equation reads

$$w_{xx} - \left[2s\alpha \tanh \alpha x \right] w_x + (\alpha^2 s^2 + E)\, w = 0$$

$$\xi = \sinh^2 \alpha x$$

letting

$$\varepsilon^{1/2} = -E/\alpha^2$$

we transform the above equation to a hypergeometric one

$$\xi(1+\xi)w'' + \left[(1-s)\xi + 1/2\right]w' + \frac{1}{4}(s^2-\varepsilon^2)w = 0$$

which has solutions

$$w_1 = F\left(-\frac{1}{2}s+\varepsilon,\ -\frac{s}{2}-\frac{\varepsilon}{2},\ \frac{1}{2},\ -\xi\right)$$

$$w_2 = \sqrt{\xi}\ F\left(-\frac{s}{2}+\frac{\varepsilon}{2}+\frac{1}{2},\ -\frac{s}{2}-\frac{\varepsilon}{2}+\frac{1}{2},\ \frac{3}{2},\ -\xi\right)$$

then

$$y = (1+\xi)^{-s/2}\ w_1$$

$$y = (1+\xi)^{-s/2}\ w_2$$

are the even and odd solutions of y

In order that $y \to 0$ as $\xi \to \infty$, the parameters

$\frac{\varepsilon}{2} - \frac{s}{2}$ must be negative integer or zero

$-\frac{S}{2} + \frac{\varepsilon}{2} + \frac{1}{2}$ must be negative integer

respectively for each solution.

Thus for bound states

$$S - \varepsilon = -\frac{1}{2} + \frac{1}{2}\sqrt{1+24} - \sqrt{-2(\omega^2-2)} = n$$

$$\omega^2 - 2 = -\frac{1}{8}\left[-(1+2n) + \sqrt{1+24}\right]^2$$

$$= -1/8\ [4 - 2n]^2$$

where n takes positive integral values starting from zero. There are a finite number of levels determined by the condition $\varepsilon > 0$ i.e

$$2n < \sqrt{(1+24)} - 1$$

$$n = 0, 1$$

for $n = 0$

$$\omega_0^2 = 0$$

$$y_{00} = \sqrt{\frac{3}{4\sqrt{2}}}\ \operatorname{sech}^2 \frac{x}{\sqrt{2}}$$

for $n = 1$

$$\omega_1^2 = 3/2$$

$$y_{01} = \sqrt{\frac{3}{2\sqrt{2}}}\ \tanh\frac{x}{\sqrt{2}}\ \operatorname{sech}\frac{x}{\sqrt{2}}$$

APPENDIX C

EQUATIONS OF MOTION FOR PCC

We solve the coupled equations of motion, for the case of the parametric collective coordinate approach (PCC) for the collective coordinates A(t) and Xo(t). The equations are given by

$$\dot{X}_o = \alpha A$$

$$b\left(\frac{\ddot{A}}{A} - \frac{1}{2}\frac{\dot{A}^2}{A^2}\right) = -\frac{1}{2} + \frac{1}{2A^2} - \frac{\dot{X}_o^2}{2A^2}$$

Decoupling the equations, we get

$$b\left[\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{2A^2}\right] = \frac{-(1+\alpha^2)}{2} + \frac{1}{2A^2}$$

or

$$b\left[\frac{\dddot{X}_o}{\dot{X}_o} - \frac{1}{2}\frac{\ddot{X}_o}{\dot{X}_o^2}\right] = \frac{-(1+\alpha^2)}{2} + \frac{\alpha^2}{2\dot{X}_o^2}$$

In order to eliminate the constant $(1+\alpha^2)/2$ take the time derivative of both sides , then

$$b\left[\ddddot{X}_o\,\dot{X}_o + \dddot{X}_o\,\ddot{X}_o - \ddot{X}_o\,\dddot{X}_o\right] = \dot{X}_o\,\dddot{X}_o\,(1+\alpha^2)$$

or

$$\ddddot{X}_o + \left[(1+\alpha^2)/b \right] \ddot{X}_o = 0$$

which can be integrated to give

$$\ddot{X}_o = c_1 \cos(\gamma + \omega t)$$

where C, γ are arbitrary constants

and $$\omega^2 = \frac{1+\alpha^2}{b}$$

Integrating again we obtain

$$\dot{X}_o(t) = \frac{c_1}{\omega} \sin(\gamma + \omega t) + C_2$$

Inserting the solution back to the equation for Xo we have

$$b \dddot{X}_o \dot{X}_o - \frac{b}{2} \ddot{X}_o^2 = \frac{-\omega^2 b \dot{X}_o^2}{2} + \frac{\alpha^2}{2}$$

$$-b\left[\frac{c_1}{\omega} \sin(\gamma+\omega t) c_1 \omega \sin(\gamma+\omega t)\right] - \frac{b c_1^2}{2} \cos^2(\gamma+\omega t) =$$

$$\frac{\alpha^2}{2} - \frac{\omega^2 b}{2}\left(\frac{c_1^2}{\omega^2} \sin^2(\gamma+\omega t) + c_2^2 + \frac{2 c_1 c_2}{\omega} \sin(\gamma+\omega t)\right)$$

$$\Rightarrow \quad -b c_1^2 = \alpha^2 - (1+\alpha^2) c_2^2$$

APPENDIX D

SHAPE AMPLITUDE SOLUTIONS FOR LECC

In order to solve

$$\ddot{A} + \omega_o^2 A = c_1 + c_2 A^2 + c_3 A^3$$

we try a solution in the form (see Landau [23])

$$A = A^{(1)} + A^{(2)} + A^{(3)} + \cdots$$

and

$$\omega = \omega_o + \omega^{(1)} + \omega^{(2)} + \cdots$$

where $A^{(1)}$ is the solution of the unperturbed equation

$$\ddot{A} + \omega_o^2 A = 0$$

$$A^{(1)} = \kappa \cos \omega t \qquad\qquad \kappa = \text{constant}$$

Insert $A = A^{(1)} + A^{(2)} = \kappa \cos\omega t + A^{(2)}$

in the initial equation and keep $\mathcal{O}[A^{(2)2}]$

$$\ddot{A}^{(2)} + \omega_o^2 A^{(2)} = 2\omega_o \omega^{(1)} \kappa \cos\omega t + c_1 + c_2 \kappa^2 \cos\omega t$$

$$= 2\omega_o \omega^{(1)} \kappa \cos\omega t + c_1$$

$$+ c_2 \kappa^2/2 + \cos 2\omega t \; (c_2 \kappa^2/2)$$

Set $\omega^{(1)} = 0$ to get rid of the resonance term

$$\ddot{A}^{(2)} + \omega_0^2 A^{(2)} = \frac{2c_1 + c_2 K^2}{2} + \frac{c_2 K^2}{2} \cos 2\omega t$$

with solution

$$A^{(2)}(t) = \frac{2c_1 + c_2 K^2}{2\omega_0^2} - \frac{c_2 K^2 \cos 2\omega t}{6\omega_0^2}$$

Going to $A^{(3)}$

$$\ddot{A}^{(3)} + \omega_0^2 A^{(3)} = 2c_2 A^{(1)} A^{(2)} + c_3 A^{(1)3} + 2\omega_0 \omega^{(1)} A^{(1)}$$

$$= c_3 K^3 \cos^3 \omega t + 2\omega_0 \omega^{(2)} K \cos \omega t$$

$$+ 2c_2 K \cos \omega t \left[\frac{2c_1 + c_2 K^2}{2\omega_0^2} - \frac{c_2 K^2 \cos \omega t}{6\omega_0^2} \right]$$

or $$\ddot{A}^{(3)} + \omega_0^2 A^{(3)} = \left[\frac{-c_2^2 K^3}{6\omega_0^2} + \frac{c_2 K (2c_1 + c_2 K^2)}{\omega_0^2} \right.$$

$$\left. + 2\omega_0 \omega^{(2)} K + 3c_3 K^3/4 \right] \cos \omega t$$

$$+ \left(-c_2^2 K^3 / 6\omega_0^2 + c_3 K^3/4 \right) \cos 3\omega t$$

Looking for a solution in the form

$$A^{(3)} = \Gamma_1 \cos 3\omega t + \Gamma_2$$

then

$$A^{(3)} = -\frac{1}{\omega_0^2}\left(\frac{c_2^2 K^3}{3\omega_0^2} + \frac{c_3 K^3}{4}\right)\cos 3\omega t$$

and

$$-\omega^{(2)} = \left[\frac{c_1 c_2 K}{\omega_0^2} + \frac{5 c_2^2 K^2}{12\omega_0^3} + \frac{3 c_3 K^2 \omega_0^2}{8\omega_0^3}\right]$$

Collecting terms we have

$$A(t) = K\cos\omega t + \frac{2c_1 + c_2 K^2}{2\omega_0^2} - \frac{c_2 K^2 \cos 2\omega t}{6\omega_0^2}$$

$$-\frac{1}{8\omega_0^2}\left(\frac{c_2^2 K^3}{3\omega_0^2} + \frac{c_3 K^3}{4}\right)\cos 3\omega t$$

APPENDIX E

INTEGRAL CALCULATIONS - EVALUATION OF Xo(t)

By direct integration the explicit functions W[Xo(t)], K[Xo(t)], Q[Xo(t)], C[Xo(t)],I[Xo(t)],F[(Xo(t)],U[Xo(t)] are as follows:

$$K(X_o) = \frac{792}{sh^2\sqrt{2}X_o} + \frac{1170\, ch\sqrt{2}X_o}{sh^6\sqrt{2}X_o} + \frac{4124\, ch\sqrt{2}X_o}{sh^4\sqrt{2}X_o}$$

$$+ \frac{404\, ch\sqrt{2}X_o}{35\, sh^2\sqrt{2}X_o} - \frac{756}{sh^7\sqrt{2}X_o} - \frac{1134}{sh^5\sqrt{2}X_o} - \frac{4092}{10\, sh^3\sqrt{2}X_o}$$

$$- \frac{12}{sh\sqrt{2}X_o} + \sqrt{2}X_o\, ch\sqrt{2}X_o \left(\frac{882}{sh^6\sqrt{2}X_o} + \frac{756}{sh^2\sqrt{2}X_o} \right.$$

$$\left. + \frac{216}{sh^4\sqrt{2}X_o} \right) - \sqrt{2}X_o \left(\frac{792}{sh^9\sqrt{2}X_o} + \frac{1698}{sh^7\sqrt{2}X_o} \right.$$

$$+ \frac{1122}{sh^5\sqrt{2}X_o} + \frac{216}{sh^3\sqrt{2}X_o} + 9\sqrt{2} \left(\frac{.35}{sh^8\sqrt{2}X_o} \right.$$

$$\left. \left. + \frac{295}{5\, sh^6\sqrt{2}X_o} + \frac{199}{12\, sh^4\sqrt{2}X_o} + \frac{5}{6\, sh^2\sqrt{2}X_o} \right) \right)$$

$$- \sqrt{2}\, x_o \,\mathrm{ch}\sqrt{2}\,x_o \left(\frac{35}{\mathrm{sh}^9\sqrt{2}\,x_o} + \frac{75}{2\,\mathrm{sh}^7\sqrt{2}\,x_o} + \frac{35}{4\,\mathrm{sh}^5\sqrt{2}\,x_o} + \frac{1}{4\,\mathrm{sh}^3\sqrt{2}\,x_o} \right) \Bigg].$$

$$W(x_o) = \Bigg[-\frac{6}{35} + \frac{657}{7\,\mathrm{sh}^2\sqrt{2}\,x_o} + \frac{766}{\mathrm{sh}^4\sqrt{2}\,x_o} + \frac{1490}{\mathrm{sh}^6\sqrt{2}\,x_o} + \frac{825}{\mathrm{sh}^8\sqrt{2}\,x_o} + \frac{6}{35}\,\frac{\mathrm{ch}\sqrt{2}\,x_o}{\mathrm{sh}\sqrt{2}\,x_o}$$

$$- \frac{93\,\mathrm{ch}\sqrt{2}\,x_o}{\mathrm{sh}^3\sqrt{2}\,x_o} - \frac{612\,\mathrm{ch}\sqrt{2}\,x_o}{\mathrm{sh}^5\sqrt{2}\,x_o} - \frac{675\,\mathrm{ch}\sqrt{2}\,x_o}{\mathrm{sh}^7\sqrt{2}\,x_o}$$

$$- \sqrt{2}\,x_o\,\mathrm{ch}\sqrt{2}\,x_o \left(\frac{36}{\mathrm{sh}^3\sqrt{2}\,x_o} + \frac{471}{\mathrm{sh}^5\sqrt{2}\,x_o} + \frac{1215}{\mathrm{sh}^7\sqrt{2}\,x_o} + \frac{825}{\mathrm{sh}^9\sqrt{2}\,x_o} \right) + \sqrt{2}\,x_o \left(\frac{36}{\mathrm{sh}^2\sqrt{2}\,x_o} + \frac{441}{\mathrm{sh}^4\sqrt{2}\,x_o} + \frac{1062}{\mathrm{sh}^6\sqrt{2}\,x_o} + \frac{675}{\mathrm{sh}^8\sqrt{2}\,x_o} \right) \Bigg]$$

$$Q(X_0) = \left[\frac{-6\cosh\sqrt{2}X_0}{sh^2\sqrt{2}X_0} + \frac{6X_0(1+ch^2\sqrt{2}X_0)}{\sqrt{2}\ sh^3\sqrt{2}X_0} \right].$$

$$I(X_0) = \left[\frac{-2\sqrt{2}}{sh^2\sqrt{2}X_0} + \frac{4X_0\, ch\sqrt{2}X_0}{sh^3\sqrt{2}X_0} \right].$$

$$C(X_0) = \left[\frac{\pi}{2}\sqrt{\frac{3}{2\sqrt{2}}}\ \tanh\frac{X_0}{\sqrt{2}}\ \mathrm{sech}^2\frac{X_0}{\sqrt{2}} \right].$$

$$F(X_0) = \pi\sqrt{\frac{3}{2\sqrt{2}}}\left[\frac{21}{32}\tanh^2\frac{X_0}{\sqrt{2}} + \frac{21}{8}th^5\frac{X_0}{\sqrt{2}} \right.$$

$$\left. + \frac{1}{16}th^4\frac{X_0}{\sqrt{2}} - \frac{47}{32}th^6\frac{X_0}{\sqrt{2}} - \frac{15}{8}th^3\frac{X_0}{\sqrt{2}} \right]$$

$$U(X_0) = 4\sqrt{2}\left[-\frac{2}{3} + \sqrt{2}X_0 + \frac{3}{th\sqrt{2}X_0} \right.$$

$$\left. \frac{-2+3\sqrt{2}X_0}{\tanh^2\sqrt{2}X_0} + \frac{2\sqrt{2}X_0}{\tanh^3\sqrt{2}X_0} \right].$$

All the above integrals have been checked numerically. The integrals Q,C,F are in disagreement with Sugiyama's corresponding integral calculations[2]. His calculations give

$$Q(X_0) = \sqrt{2}X_0\left(3\,\mathrm{cosech}\sqrt{2}X_0 + 4\,\mathrm{cosech}^3\sqrt{2}X_0\right) - \left(6\,\mathrm{cotanh}\sqrt{2}X_0\,\mathrm{cosech}\sqrt{2}X_0\right)$$

$$C(X_0) = \frac{\pi}{2}\sqrt{\frac{3}{2\sqrt{2}}}\,\tanh\sqrt{2}X_0\,\mathrm{sech}^2\sqrt{2}X_0$$

$$F(X_0) = \pi\sqrt{\frac{3}{2\sqrt{2}}}\left[-3 + 3\tanh^2\sqrt{2}X_0 - \frac{3}{2}\mathrm{sech}^4\sqrt{2}X_0 + \frac{9}{2}\mathrm{sech}^2\sqrt{2}X_0\right].$$

The function U(Xo) physically represents the "potential" between the kink and the antikink, and I(Xo) describes the interaction of the kinks viewed as deformable particles. Q(Xo) represents the interaction of the shape modes between the individual kinks. The other functions describe the coupling of the different modes among kink and antikink.

Evaluation of Xo(t):

From equation (4.16) and for large separation distances Xo(t) one can approximate the motion of each kink in terms of the collective coordinate Xo(t), by expanding U(Xo) and I(Xo) and solving the resulting equation. The contribution of I(Xo) for large Xo is very small compared to the leading coefficient of Xo(t), the constant M, therefore I(Xo) can be ignored simplifying equation (4.16) to

E-1 $$M \dot{X}_o^2 + U(X_o) = 2M - \varepsilon M$$

Expanding U(Xo) for large Xo

E-2 $$U(X_o) = 4\sqrt{2}\Big[-2/3 + \sqrt{2}X_o + 3 + 6e^{-2\sqrt{2}X_o} + 2\sqrt{2}X_o(1 + 6e^{-2\sqrt{2}X_o}) - (2 + 3\sqrt{2}X_o)(1 + 4e^{-2\sqrt{2}X_o})\Big]$$
$$= 2M - 12Me^{-2\sqrt{2}X_o}$$

and inserting U(Xo) in E-1 we get

E-3 $$M\dot{X}_o^2 - 12Me^{-2\sqrt{2}X_o} = -\varepsilon M$$

Solving for Xo(t) we find

E-4 $$X_o(t) = \frac{1}{\sqrt{2}} \ln\left[\sqrt{\frac{12}{\varepsilon}} \sin\sqrt{2\varepsilon}(t + \delta_o)\right]$$

where δ_o is determined by the condition

$$X_o(t=0) = 0 \quad \Rightarrow \quad \delta_o \simeq 0.2$$

APPENDIX F

SOLITARY WAVES AND NONLINEAR KLEIN GORDON EQUATIONS

The ϕ^4 equation is one among a large number of nonlinear equations which possess solitary wave solutions. In this Appendix we consider the family of the nonlinear Klein-Gordon equations (NLKG) which ϕ^4 belongs to, and we derive the solitary wave solutions of some member equations of this category. We use special transformations and Euler's substitutions to solve the NLKG equations which in general have solutions expressed in terms of elliptic integrals. It is interesting that all the solitary solutions found are expressed in terms of the hyperbolic tangent kink solution [TANH(x)] of the equation (the simplest one) which in turn as we show in Appendix G can be expressed in terms of the sine-Gordon soliton solution. We consider the nonlinear Klein Gordon equation (NLKGE) given in the general case by

$$\Box\phi + \frac{\partial U(\phi)}{\partial \phi} = 0$$

(where $U(\phi)$ is a positive function of ϕ).

We are interested in travelling wave solutions $\phi(\xi)$ where

$$\xi = \frac{x \pm vt}{(1-v^2)^{1/2}}$$

Then in terms of the K-G equation can be integrated once to give

$$\phi_\xi^2/2 = U(\phi) + \text{constant}$$

Going further and demanding that the energy be finite and the solution localized in space, we set the constant equal to zero so that if ϕ_i are the roots of $U(\phi) = 0$, then our solutions (see Appendix A) should satisfy the boundary conditions

$$\phi_{x \to \pm\infty} \to \phi_i \quad , \quad \phi_{\xi\,(x \to \pm\infty)} \longrightarrow 0$$

Thus what we are left to examine is the first order ordinary differential equation

$$\frac{\phi_\xi^2}{2} = U(\phi)$$

We let $U(\phi)$ be a polynomial of n^{th} degree in ϕ

$$U(\phi) = a_n \phi^n + a_{n-1}\phi^{n-1} + a_{n-2}\phi^{n-2} + \ldots a_0$$

and study some particular combinations of α_n , ϕ^n which generate solitary waves.

1. If $U(\phi)$ is a second degree polynomial in ϕ we get the inhomogeneous K-G equation

$$\Box\phi + c_1\phi = \text{constant}$$

which has general solution in the form

$$\phi = \text{constant} + \text{plane waves}$$

For U a third degree polynomial in ϕ

$$U(\phi) = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3$$

$$\int \frac{d\phi}{\left[a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3\right]^{1/2}} = \pm \xi\sqrt{2}$$

If $\omega_1, \omega_2, \omega_3$, are the roots of the polynomial $U(\phi)$, then

$$\int \frac{d\phi}{\left[(\phi-\omega_1)(\phi-\omega_2)(\phi-\omega_3)\right]^{1/2}} = \pm \xi\sqrt{2}$$

The transformation [36]

$$z^2 = \frac{\omega_1 - \omega_2}{\phi - \omega_3} \qquad K^2 = \frac{\omega_2 - \omega_3}{\omega_1 - \omega_3}$$

gives

$$\int \frac{dz}{\left[(1-z^2)(1-K^2z^2)\right]^{1/2}} = \pm\frac{\xi}{\sqrt{2}}\sqrt{\omega_1-\omega_3}$$

$$\omega_1 \neq \omega_3$$
$$\omega_1 > \omega_3$$

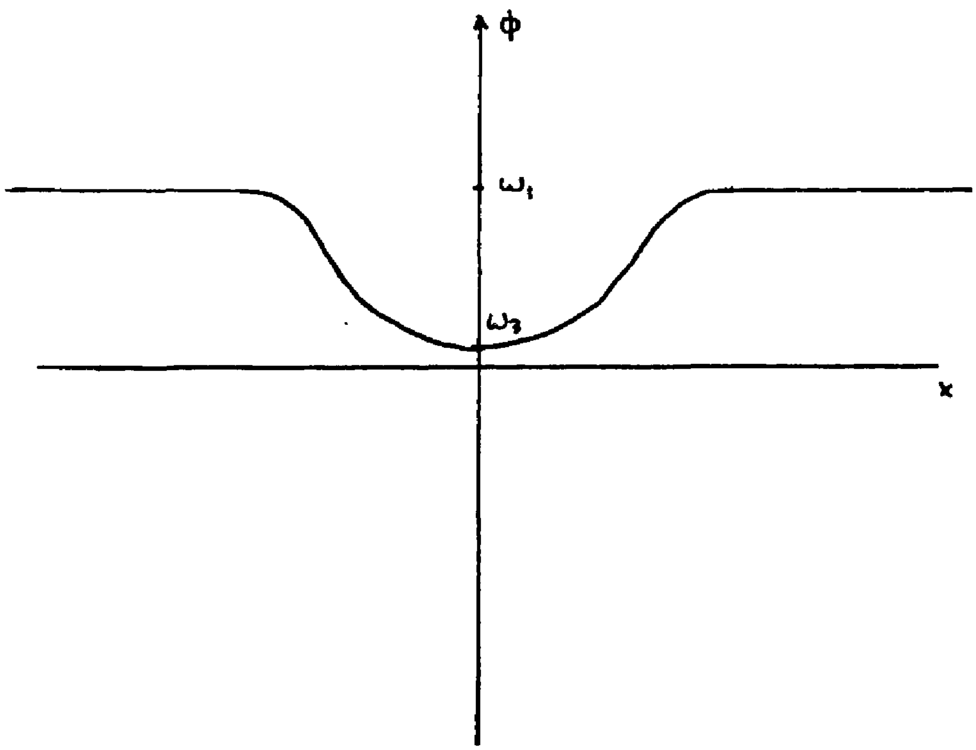
In the case where $\omega_2 = \omega_1 \Rightarrow$

$$\kappa^2 = 1$$

$$\int \frac{dz}{\sqrt{(1-z^2)^2}} = \pm \frac{\xi}{\sqrt{2}} \sqrt{\omega_1 - \omega_3}$$

$$\phi = (\omega_1 + \omega_3) - \omega_3 \tanh^2\left[(\xi/\sqrt{2})\sqrt{\omega_1 - \omega_3}\right]$$

This is a solitary wave as can be seen in the figure below.



If one , searching for internal modes, uses the same analysis as in Chapter II he finds that the system doesn't possess bound states.

2. $U(\phi)$ is a fourth degree polynomial in ϕ : The solution is given by

$$\int \frac{d\phi}{\sqrt{(\phi-\omega_1)(\phi-\omega_2)(\phi-\omega_3)(\phi-\omega_4)}} = \pm \xi \sqrt{2}$$

(Again ω =1, 4 are the roots of $U(\phi) = 0$)

By means of the transformation

$$z^2 = \left(\frac{\omega_2-\omega_4}{\omega_1-\omega_4}\right)\left(\frac{\phi-\omega_1}{\phi-\omega_2}\right) \qquad , \qquad K^2 = \frac{(\omega_2-\omega_3)(\omega_1-\omega_4)}{(\omega_1-\omega_3)(\omega_2-\omega_4)}$$

the above integral becomes

$$\int \frac{dz}{\sqrt{(1-z^2)(1-K^2z^2)}} = \pm \xi \sqrt{\frac{(\omega_2-\omega_3)(\omega_1-\omega_3)}{2}} = \pm\xi'$$

if : $\omega_3 = \omega_4$, then $K^2 = 1$

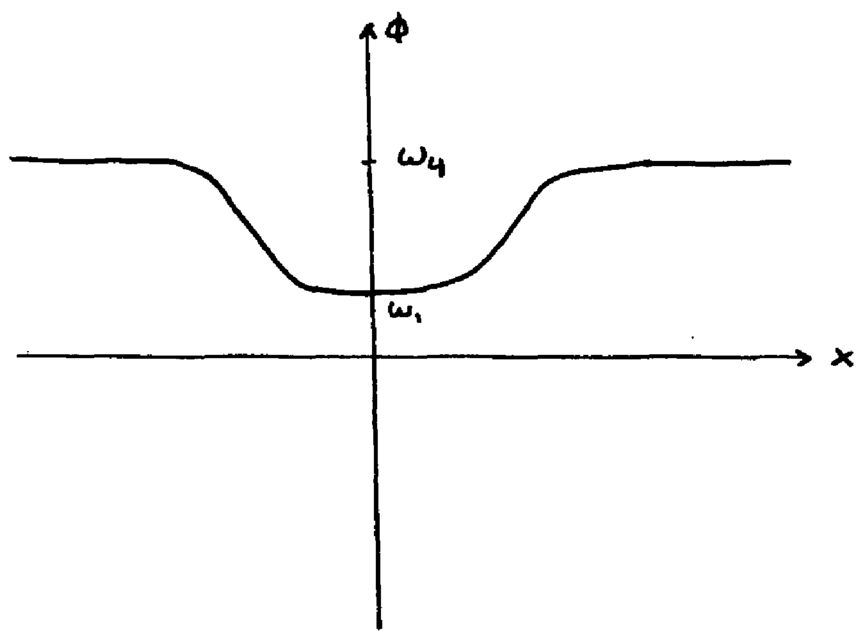
$$z = \coth \xi'$$
$$z = \tanh \xi'$$

(Notice that in the above integral for z, if z is a solution then $1/z$ is a solution too).

In terms of z the solutions read

a) $$\phi(x,t) = \frac{\omega_1\left[\frac{\omega_2-\omega_4}{\omega_1-\omega_4}\right] - \omega_2 \tanh^2\xi'}{\left[\frac{\omega_2-\omega_4}{\omega_1-\omega_4}\right] - \tanh^2\xi'}$$

b) $$\phi(x,t) = \frac{\omega_1\left(\frac{\omega_2-\omega_4}{\omega_1-\omega_4}\right) - \omega_2 \coth^2\xi'}{\left[\frac{\omega_2-\omega_4}{\omega_1-\omega_4}\right] - \coth^2\xi'}$$



For the case when U has two double roots:We write the potential as

$$U(\phi) = (\alpha\phi^2 + \beta\phi + \gamma)^2$$

or to make it more general ,

$$U(\phi) = \frac{\varepsilon^2}{4}(\alpha\phi^2 + \beta\phi + \gamma)^2$$

Providing $\alpha\gamma < 0$ the solution is given by

$$\phi(x,t) = -\frac{\beta}{2\alpha} \pm \frac{\sqrt{\beta^2 - 4\alpha\gamma}\ \tanh \xi\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

where $$\xi = \epsilon \frac{(x \mp vt - X_0)}{[2(1-v^2)]^{1/2}}$$

This is a shifted kink. The ϕ^4 kink solution is a special case with

$$\beta = 0$$

$$\alpha = 1$$

$$\gamma = -1$$

$$\varepsilon = 1$$

As we expect there is only one bound state corresponding to the shape modes and its value depends on the constants α , β , γ

If two of the roots are zero:

$$U(\phi) = \phi^2 (\phi - \omega_1)(\phi - \omega_2)$$

and

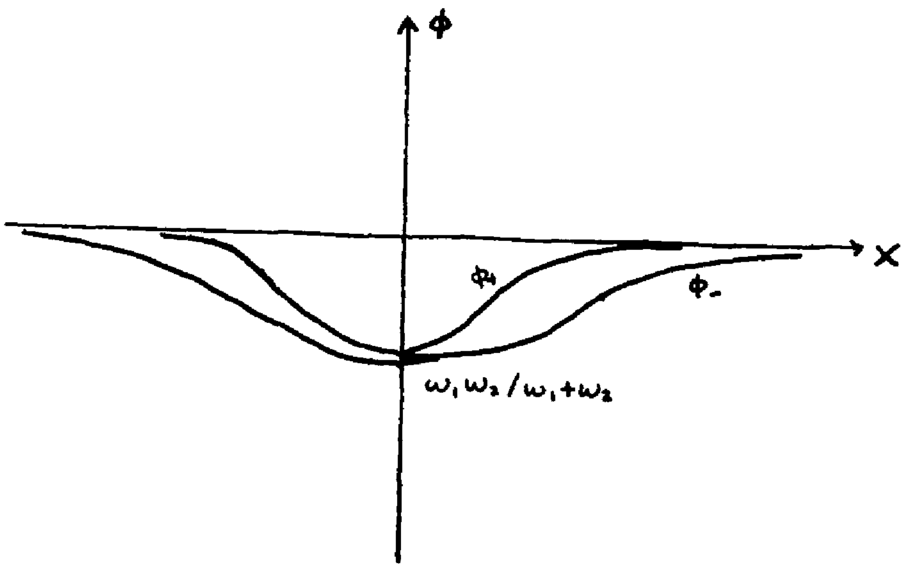
$$\int \frac{d\phi}{\phi\sqrt{(\phi-\omega_1)(\phi-\omega_2)}} = \pm \xi \sqrt{2}$$

The Euler substitution [36]

$$(\phi-\omega_1)(\phi-\omega_2) = t \mp \phi$$

will give the solution

$$\phi(x,t) = \frac{\omega_1\omega_2 (1 - \tanh^2 \xi\sqrt{2})}{(\omega_1 + \omega_2 \mp 2\sqrt{\omega_1\omega_2} \tanh \xi\sqrt{2})}$$



when $\omega_1 \rightarrow \omega_2$ then $\phi = \frac{1}{2}\omega_2 (1 \pm \tanh \xi\sqrt{2})$

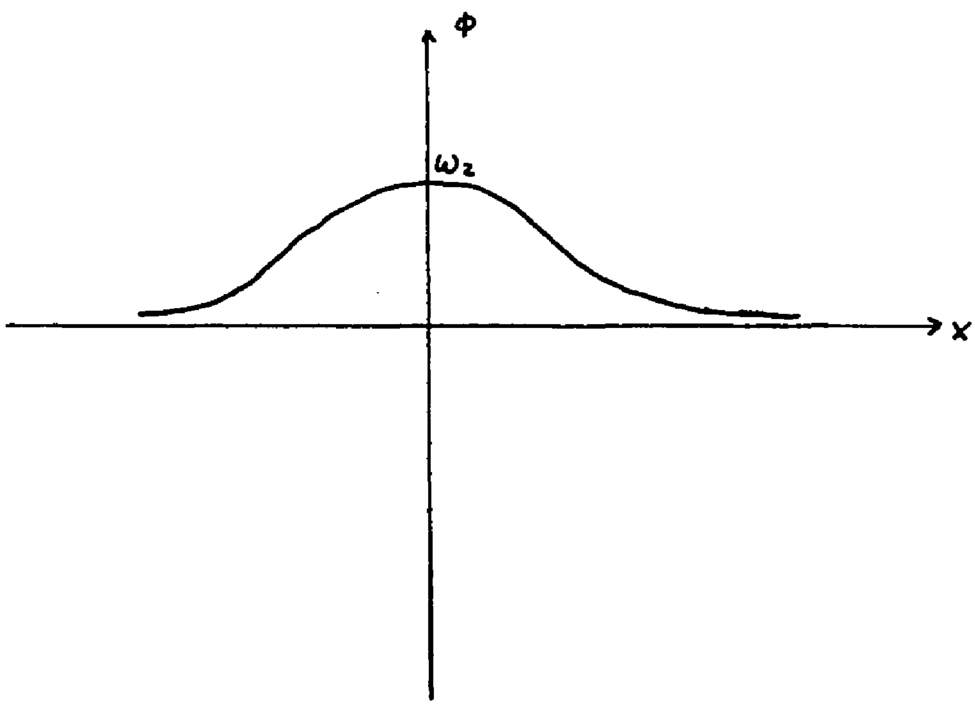
and again the shifted kink solution is recovered.

The second Euler substitution [36]

$$(\phi - \omega_1)(\phi - \omega_2) = t^2 (\phi - \omega_1)^2$$

gives the solution

$$\phi = \frac{\omega_1 \omega_2 (1 - \tanh^2 \xi\sqrt{2})}{(\omega_1 - \omega_2 \tanh^2 \xi\sqrt{2})}$$



when $\omega_1 \to \omega_2$ $\phi = \omega_2$ =constant

The third Euler substitution

$$(\phi-\omega_1)(\phi-\omega_2) = (\phi t - \sqrt{\omega_1\omega_2})^2 \qquad \omega_1\omega_2 > 0$$

gives

$$\phi = \frac{e^{\pm \xi\sqrt{2\omega_1\omega_2}}}{1 - \left[\dfrac{\omega_1+\omega_2 - e^{\mp\xi\sqrt{2\omega_1\omega_2}}}{2\sqrt{\omega_1\omega_2}}\right]^2}$$

when $\omega_1 \to \omega_2$ $\phi = -\omega_1 + h\xi \pm \omega_1$ shifted kink and antikink.

For a physical application of this case see reference [37] where the authors studying a classical nonlinear scalar field coupled by the Yukawa interaction to a fermion field end up with a field equation described by a fourth degree polynomial which gives the three solutions found before.

3. U(ϕ) is a sixth degree polynomial in ϕ :

case one:

$$U(\phi) = \frac{\phi^2}{2}(\alpha^2 - \beta^2\phi^2)^2$$

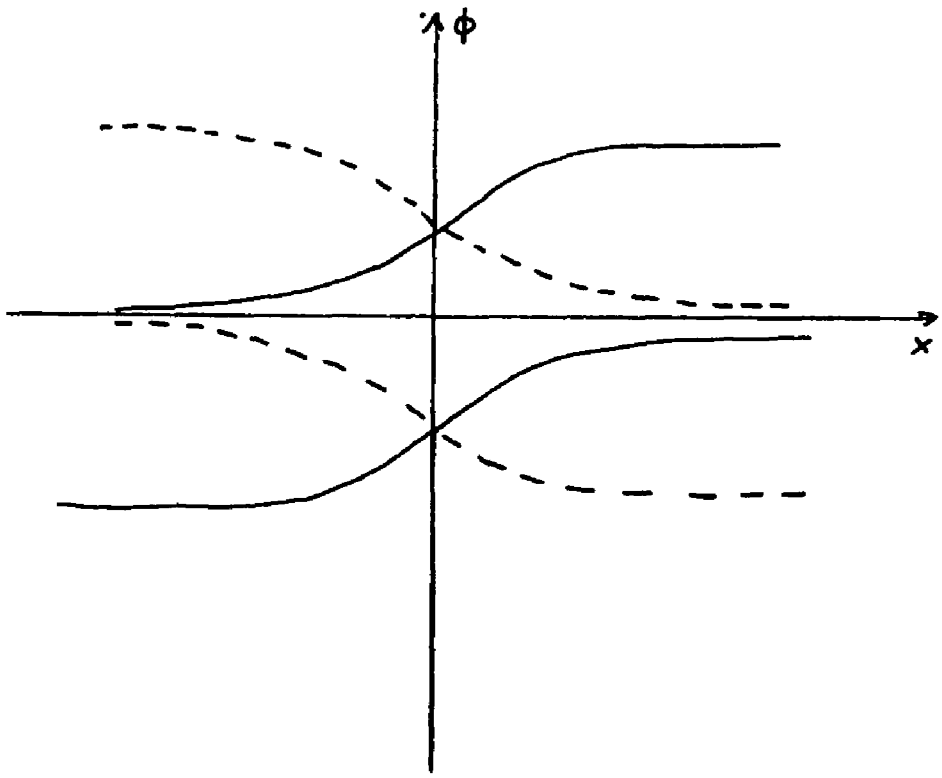
that is the case where we have two roots zero and two double roots.

The solution from

$$\int \frac{d\phi}{\sqrt{U(\phi)}} = \pm \xi \sqrt{2}$$

is given by

$$\phi(\xi) = \begin{cases} \dfrac{\alpha}{\beta}\, e^{\pm \xi \alpha^2} \Big/ \cosh\left\{ \sinh^{-1} e^{\pm \xi \alpha^2} \right\} \\[2em] -\dfrac{\alpha}{\beta}\, e^{\pm \xi \alpha^2} \Big/ \cosh\left\{ \sinh^{-1} e^{\pm \xi \alpha^2} \right\} \end{cases}$$



Second case : two roots are zero

$$U(\phi) = \frac{1}{2} \phi^2 (\alpha + \beta\phi^2 + \gamma\phi^4)$$

Letting $\phi^2 \rightarrow \phi'$ and performing again Euler's substitutions,

providing $\omega_1\omega_2 > 0$ we find

$$\phi_1 = \pm\left[\frac{\omega_1\omega_2\ (1 - \tanh^2\xi\sqrt{\omega_1\omega_2}\ )}{\omega_1 + \omega_2 \mp 2\sqrt{\omega_1\omega_2}\ \mathrm{th}\,\xi\sqrt{\omega_1\omega_2}}\right]^{1/2}$$

$$\phi_2 = \pm\left[\frac{\omega_1\omega_2\,(1 - \tanh^2(\xi\sqrt{\omega_1\omega_2}))}{\omega_1 - \omega_2\tanh^2(\xi\sqrt{\omega_1\omega_2})}\right]^{1/2}$$

and the trivial constant solutions

$$\phi = 0 \; , \; \omega_1 \; , \; \omega_2$$

for $\omega_1 \rightarrow \omega_2$

$$\phi_1 \rightarrow \pm \sqrt{\frac{\omega_{1,2}}{2}} \left(1 \pm \tanh(\xi\, \omega_{1,2}) \right)^{1/2}$$

$$\phi_2 \longrightarrow \pm \, \omega_{1,2}$$

For physical applications (solid state,quantum field theory) of the ϕ^6 theory see references [32,33,34,35,36].

4. Parametric sixth degree in ϕ potential.

We introduce the parametric potential

$$U(\phi) = \frac{1}{4} \left[(1 + c)^2 \phi^2 - 1 \right]^2 \left[(1-c) \mp c\,\phi \right]^2$$

As c goes from -1 to 1, U describes ϕ^2 , ϕ^4 , ϕ^6 , potentials respectively

$$U(\phi) = \frac{1}{4}\left(2 \pm \phi\right)^2 \qquad c = -1 \qquad \phi^2$$

$$U(\phi) = \frac{1}{4}\left(\phi^2 - 1\right)^2 \qquad c = 0 \qquad \phi^4$$

$$U(\phi) = \frac{1}{4}\left(4\phi^2 - 1\right)^2 \phi^2 \qquad c = +1 \qquad \phi^6$$

(The constant 1/4 could be replaced in general by some constant C which would change the argument of the solution from ξ to ξ 2C)

If one considers ϕ as constant, as ϕ goes to zero U describes the sine-Gordon equation.

The parametric potential U can also be written as

$$U(\phi) = \frac{1}{4}\left[2\cos^2 c/2 \quad \phi^2 - 1\right]^2 \left[\sqrt{2}\sin c/2 \mp \phi \cos c\right]^2$$

Solving the corresponding field equation

$$\Box\phi + \frac{\partial U(\phi)}{\partial \phi} = 0$$

for travelling wave solutions $\phi(\xi)$ we get

$$\frac{\cos c}{4(1-2\cos^2 c)} \ln \frac{\left[2\sin^2 c/2 + \cos c \;\; \phi\right]^4}{\left[2\cos^2 c/2 \;\; \phi^2 - 1\right]^2}$$

$$-\frac{\sin^2 c}{\left[\sqrt{2}\,(1-2\cos^2 c)\cos c/2\right]} \, th^{-1}\left(\sqrt{2}\cos\frac{c}{2}\;\;\phi\right) = \pm\frac{\xi}{\sqrt{2}}$$

5. $U = \dfrac{a}{p+1} \;\; \phi^{p+1}$

the solution is given by

$$\phi_p(x,t) = \left[\pm\frac{(1-p)}{2}\sqrt{\frac{2a}{(1-v^2)(1+p)}}\,(x \pm vt)\right]^{2/1-p}$$

$$p \neq \pm 1$$

and is not a solitary wave. It is interesting to observe that the same equation has n-1 solutions for each value of p (with $p \neq 1$), given by

$$\phi_{p,n} = \left[\frac{(2-1/n)^2\; 4n^2}{(1-p)^2\, a}\right]^{\frac{1}{p-1}} \left[(x-t)(x+t)\right]^{\frac{2n-1}{1-p}}$$

6. For $U = \frac{1}{2}\phi^2(\phi^n + \alpha^n)$ the solitary solution is given by

$$\phi(x,t) = a\left[\tanh^2(\xi n\sqrt{\alpha^n}) - 1\right]^{1/n}$$

7. Similarly for $U = \frac{1}{2}\phi^2(\phi^n + \alpha^n)^2$

$$\phi(x,t) = \left(\frac{-\alpha^n}{2}\right)^{1/n}\left[1 \pm \tanh\left(\frac{n\alpha^n \xi}{2}\right)\right]^{1/n}$$

Concluding we have seen that the solitary wave solutions of NLKGE considered in this Appendix are all expressed in terms of TANH (ξ) which happens to be the solution of the ϕ^4 equation. One might ask: Are these solutions related (except the fact that they belong to the same family NLKGE) in some way that all can be derived from a general solution which is expressed in terms of the ϕ^4 solution?

In Appendix G we try to relate the ϕ^4 kink with the sine-Gordon soliton in order to understand better the relationship between solitary waves and their cousins, solitons.

APPENDIX G

ARE KINKS OF ϕ^4 AND SOLITONS OF SINE GORDON EQUATION RELATED ?

We have seen (Chapter III) that for certain incoming velocities the energy exchange mechanism allows two colliding kinks to behave (almost) as solitons since they are scattered with (almost) no loss of energy as a result of the resonance energy exchange between the shape and translational modes. We examine the possibility of expressing the kink solution of ϕ^4 equation in terms of the soliton solution of the sine Gordon equation.

We consider the general case

G-1 $$\Box\phi = \alpha\phi - \beta\phi^3$$

Looking for travelling wave solutions in the form

$$\phi \equiv \phi\left(\frac{x \pm vt}{\sqrt{1-v^2}}\right) \equiv \phi(\xi)$$

the above equation can be written as

G-2 $$-\phi_{\xi\xi} = \alpha\phi - \beta\phi^3$$

Let

G-3 $$\phi = m \sin\psi/2$$

where ψ satisfies the field equation

(G-4) $$\Box\psi = \pm\lambda\sin\psi$$

(G-5) or $\psi_{\xi\xi} = \mp \lambda \sin\psi$ and

G-6 $$\frac{\psi_\xi^2}{2} = (\pm\lambda\cos\psi + c) \qquad \lambda, c = \text{constants}$$

From eq. (G-2) upon using (G-3) and (G-5)

$$-\left[\frac{m}{2}\psi_{\xi\xi}\cos\frac{\psi}{2} - \sin\frac{\psi}{2}\, m\,\frac{\psi_\xi^2}{4}\right] = \alpha m \sin\frac{\psi}{2} - \beta m^3 \sin^3\frac{\psi}{2}$$

$$\pm m\lambda \mp m\lambda \sin^2\psi/2 \pm m\lambda\left(\frac{1}{2} - \sin^2\frac{\psi}{2}\right) + \frac{mc}{2} - \alpha m$$
$$+ \beta m^3 \sin^2\psi/2 = 0$$

Thus

$$\beta = \pm\, 2\lambda/m^2$$

G-7

$$\alpha = \pm\, 3\lambda/2 + c/2$$

therefore

$$\phi = m \sin\psi/2$$

and ψ is the solution of

$$\psi_\xi^2/2 = (\pm\lambda\cos\psi + c)$$

(with $(\pm\lambda\cos\psi + c)$ positive for real solutions)

For ϕ^4 equation $\alpha = 1$, $\beta = 1$ then from G-7

$$\lambda = \pm \frac{m^2}{2}$$

G-8

$$c = 2 - 3m^2/2$$

and the solution becomes

$$\phi = m \sin \psi/2$$

G-9

$$\pm \lambda \cos\psi + c = \psi_\xi^2/2$$

The solution ψ for any m is given in general, in terms of the elliptic integrals. For the special case $\lambda = \pm c$ one obtains

$$\frac{m^2}{2} = (2 - 3\frac{m^2}{2}) \qquad \text{and} \qquad m = \pm 1 \Rightarrow \lambda = c = 1/2$$

$$\frac{m^2}{2} = -(2 - \frac{3}{2}m^2) \qquad \text{and} \qquad m = \pm\sqrt{2} \Rightarrow \lambda = 1 = -1$$

for $m = \pm 1$

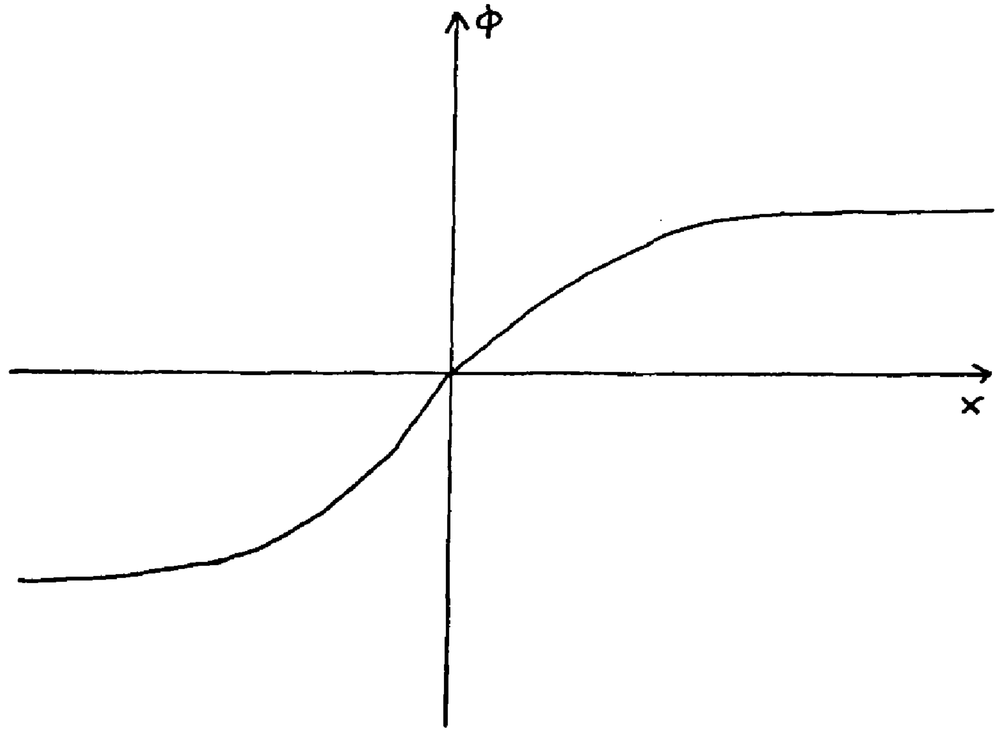
$$\psi = 4\tan^{-1} e^{\pm\xi/\sqrt{2}} - \pi$$

and

$$\phi = \pm \sin\left[2\tan^{-1} e^{\pm\xi/\sqrt{2}} - \pi/2\right]$$

G-10

which is the kink-antikink solution of ϕ^4 eq. as can be seen below.



For $m = \pm\sqrt{2}$ $\lambda = 1$, $c = -1$

G-11 $$\phi(x,t) = \pm\sqrt{2}\,\sin 2\tan^{-1} e^{i\xi}$$

which is an imaginary solution of the ϕ^4 equation.

Thus summarizing the results :

For the general ϕ^4 equation $\Box\phi = \alpha\phi - \beta\phi^3$

there exists a transformation

$$\phi = m \sin \psi/2$$

where

$$\Box \psi = \pm \lambda \sin \psi$$

or λ , c =constants

$$\psi_\xi^2 = \pm 2 \lambda \cos \psi + 2c \qquad \xi = \frac{x \pm vt}{(1-v^2)^{1/2}}$$

which gives the travelling wave T-W solutions of the general equation, in terms of the known T-W solutions of the sine-Gordon eq. G-6. For the special case α =1, β =1, $\lambda = \pm c$ one finds the kink solution of the (pure) ϕ^4 equation (G-10) and an imaginary solution (G-11).

Thus we conclude that it may be useful to view a kink as a perturbed soliton and the solitary properties of a kink as due to soliton properties left after the perturbation. It starts off as a soliton

$$\phi = \sin \psi/2 \approx \psi/2 \qquad \text{(for small } \psi \text{)}$$

and develops as a solitary wave. The result is not surprising since for small ψ the s-G equation reads

$$\Box \psi = \pm \lambda (\psi - \psi^3/3! + \cdot \cdot \cdot \cdot)$$

and the powers of ϕ^4 equation are recovered

$$\Box \phi = \alpha \phi - \beta \phi^3$$

Therefore one might think of ϕ^4 equation a s-G one which has been disturbed by external force F, such that

$$\Box \psi = \pm \lambda \sin \psi + F(x,t)$$

and the effect of F(x,t) is to neutralize all higher terms above Similarly if one perturbs the ϕ^4 by some external force , it might be possible to find solitons. Extending the logic to more solitons we might say that N interacting solitons may form a kink or solitary wave For instance the coupled equations below

$$\Box \phi - \phi + \phi^3 = -\left[(n^2+1)\sin^n \psi/2 - (n^2+n)\sin^{n+2} \psi/2 - \sin^{3n} \frac{\psi}{2} \right]$$

$$\Box \psi + \sin \psi = 0$$

$$n = 0, 1, 2, \ldots .$$

have running solutions

$$\phi(x,t) = \sin^n(\psi/2)$$

$$\psi(x,t) = 4 \tan^{-1} e^{\xi}$$

One can see, that for small ψ, $\phi \simeq \prod_i^n \psi/2$

the solitary solution ϕ is described by N interacting solitons.

The external field can also be written in terms of ϕ , and be such

that the transformation $\phi = \sin(\psi/2)$ makes it vanish. For instance the coupled equations

$$\phi_{tt} - \phi_{xx} = \alpha\phi - \beta\phi^3$$

$$+ \phi\underbrace{(\phi_x^2 - \phi_t^2)^n - (A\phi^2 + B)^n}_{F(x,t)}(m^2 - \phi^2)\phi^n$$

$$\psi_{tt} - \psi_{xx} = -\lambda \sin\psi$$

have solutions

$$\phi = m \sin\psi/2$$

$$\psi_\xi^2/2 = 2\left[c - \lambda\cos\psi\right]$$

with $A = \dfrac{\lambda}{m^2}$, $B = \dfrac{c-\lambda}{2}$,

$$\xi = \frac{x \pm Ut}{(1-U^2)^{1/2}}$$

If one use the same transformation (G-3)

$$\phi = m \sin(\psi/2)$$

to solve

$$\Box\phi = \alpha\phi - \beta\phi^3$$

$$\Box\psi = -\lambda\sin\psi$$

and not restricting the solution to running waves , one finds

$$\frac{1}{2}(\Box\psi)\cos\psi/2 - \frac{1}{4}\sin\psi/2\,(\psi_t^2-\psi_x^2) = \alpha\sin\psi/2 - \beta m^2 \sin^3\frac{\psi}{2}$$

$$\left[-\lambda\cos^2\psi/2 - \frac{(\psi_t^2-\psi_x^2)}{4} - \alpha + m^2\beta\sin^2\psi/2\right] m\sin\frac{\psi}{2} = 0$$

The condition for the transformation G-3 to work is

$$\psi_x^2 = \psi_t^2$$

which for non travelling wave solutions will relate the two independent variables x,t for any ψ solution. Wishing, still to relate the two equations for the general case we are forced to introduce another term to the R.H.S of ϕ^4 which will cancel the undesired term. Then

$$\Box\phi = \alpha\phi - \beta\phi^3 - \frac{1}{4}(\psi_t^2-\psi_x^2)\, m\sin\psi/2$$

$$\Box\psi = -\lambda\sin\psi$$

or in terms of the ϕ field

$$\Box\phi = \alpha\phi - \beta\phi^3 - \phi\,(\phi_t^2-\phi_x^2)/m^2-\phi^2$$

$$\Box\psi = -\lambda\sin\psi$$

For the pure ϕ^4 equation $\alpha = \beta = 1$

this gives $m = \pm 1$ $\lambda = -1$

and $\phi = \pm \cos \psi/2$

where ψ is any solution of the s-G equation.

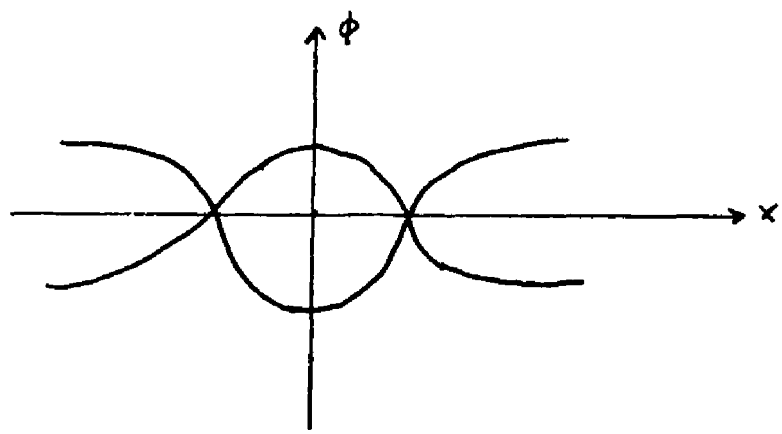
Thus if we choose the soliton-antisoliton scattering solution

$$\psi(x,t) = 4\tan^{-1}\left(\frac{\sinh vt\gamma}{v\cosh x\gamma}\right), \quad \gamma = (1-v^2)^{-1/2}$$

$$\phi(x,t) = \pm \cos\left[2\tan^{-1}\left(\frac{\mathrm{sh}\, vt\gamma}{v\,\mathrm{ch}\, x\gamma}\right)\right]$$

notice that as $t \to \infty$ $\psi = s + \bar{s}$ =soliton + antisoliton

then $\phi_{(t \to \pm\infty)} = \pm\left[\cos(s)\cos(\bar{s}) - \sin(s)\sin(\bar{s})\right]$



Even if the transformation fails to connect ϕ^4 and s-G for exact time dependent solutions, it does not necessarily mean there is no such transformation, or for that matter exact time-dependent solutions of ϕ^4 . Here we were hoping that expressing the solution in terms of the knowing solutions of the sine-Gordon equation we would be able to solve ϕ^4 for the general case as we did for the case of running wave solutions. Nevertheless we have seen that a kink and a s-G soliton are related.

The need for introducing the extra term into R.H.S of (G-12), relating the two field equations through this transformation, perhaps indicates that the "secrets" of the solitary properties of the ϕ^4 kink could be understood better by perturbing the ϕ^4 eq. and expressing the solution in terms of s-G solutions. The ϕ^4 hyperbolic type kink solution is not the only one which has the soliton solution of s-G built in. If one takes for instance the ϕ^6 field equation (see Appendix F) where the potential is given by

$$U = \frac{1}{2} \phi^2 \left(a^2 - \beta^2 \phi^2 \right)^2$$

he finds that the solution can be written in terms of ψ as

$$\phi = \frac{a}{\beta} \sin \psi/4$$

$$\psi = 4 \tan^{-1} e^{\xi}$$

Similarly the NLKG equation with potential (see A-F)

$$U = \frac{1}{2} \phi^2 (\phi^n - a^n)$$

has solution

$$\phi^n = \alpha^n \sin^2 \left[\tan^{-1} e^{\alpha^n n \xi / 4} \right]$$

expressed also in terms of ψ , and so they do all the NKG equations

which have their solution expressed in terms of ϕ^4 kink.

APPENDIX H

ϕ^4 EQUATION WITH DISSIPATION

We have shown that the ϕ^4 equation has a general travelling wave solution given by elliptic functions and the kink solitary solution is a particular case of the elliptic parameters (see Appendix G). It is interesting to see if the same method would allow one to solve the ϕ^4 problem if one adds a term proportional to the first derivative of time. Then the field equation reads

H-1 $$\Box\phi - \phi + \phi^3 + \beta\phi_t = 0$$

(where β is a real parameter)

Looking again for running solutions (G-1), and s-Gordon equations transformed to ODE

H-2 $$-\phi_{\xi\xi} \pm v\gamma\beta\phi_\xi - \phi + \phi^3 = 0$$

H-3 $$-\psi_{\xi\xi} = \lambda \sin\psi$$

where again $$\xi = \frac{(x \pm vt)\delta}{(1-v^2)^{1/2}} = (x \pm vt)\gamma\delta$$

Integrating H-3 and chosing the integration constant to be λ/δ^2 one finds

H-4 $$\frac{\psi_\xi^2}{2} = \frac{2\lambda}{\delta}\cos^2\psi/2$$

Relating ϕ and ψ by the transformation

H-5 $$\phi = m \sin \psi/2 + m$$

and upon using H-4 equation H-2 gives

$$2m\lambda \sin\psi/2 - 2m\lambda \sin^3\psi/2 \pm \upsilon\gamma\beta\delta m\sqrt{\lambda}/\delta$$
$$\mp \upsilon\gamma\beta\frac{\delta}{\delta} m\sqrt{\lambda} \sin^2\psi/2 - m - m\sin\psi/2$$
$$+ m^3 \sin^3\psi/2 + m^3(1 + 3\sin^2\psi/2 + 3\sin\psi/2) = 0$$

Collecting terms one gets the system of equations

$$m^2 = 2\lambda \quad , \quad \pm\upsilon\gamma\beta = \frac{1-m^2}{\sqrt{\lambda}}$$

H-6 $$\pm\upsilon\gamma\beta = 3m^2/\sqrt{\lambda}$$

$$2\lambda - 1 + 3m^2 = 0$$

which has solution

$$m = \pm 1/2$$

H-7 $$\lambda = 1/8$$

$$\pm\upsilon\beta/(1-\upsilon^2)^{1/2} = 3/\sqrt{2}$$

Solving for ψ from H-4 and substituting in H-5 the solution to

H-1 reads

H-8 $$\phi = \pm\frac{1}{2}\left[\sin\left\{2\tan^{-1} e^{\frac{x \pm \upsilon(\beta)t + x_0}{\sqrt{2(1-\upsilon^2)}}} - \frac{\pi}{2}\right\} + 1\right]$$

(where x_o is constant determined by $\phi(x=t=0)$)

and describes a shifted kink(antikink) solution.

From H-7 one finds that the velocity of the kink is function of the parameter β and is given by

H-9 $$U(\beta) = sgn(\beta)\, 3/(2\beta^2+9)^{1/2}$$

For negative β (dissipation) the solution reads

$$\underset{\beta<0}{\phi(x,t)} = \pm\frac{1}{2}\left[\sin\left[2\tan^{-1} e^{\frac{(x-|U(\beta)|t+x_o)}{\sqrt{2(1-U^2)}}} - \frac{\pi}{2}\right]+1\right]$$

and for positive β

$$\underset{\beta>0}{\phi(x,t)} = \pm\frac{1}{2}\left[\sin\left[2\tan^{-1} e^{\frac{(x+X_o+|U(\beta)|t)}{\sqrt{2(1-U^2)}}} - \frac{\pi}{2}\right]+1\right]$$

The kink solutions are indeed a surprise. Perturbing the ϕ^4 equation, by introducing the extra term in G-1, one does not expect to find kink solutions. From the linear picture once dissipation is introduced in the system the solution gets distorted and dies out as a result of the energy loss. The nonlinear picture differs remarkably from that and the damping term only shifts the (kink) solution, restricting the kink to move with certain velocities depending on the amplitude β of the dissipation introduced in the system. Thus in a physical system where energy loss is present (water waves) only kinks which satisfy (H-8) are expected to be observed. The question is how kinks supply this energy loss (β <0) or what happens to the energy added (β >0) if this solution has any meanning at all. If one perturbs the system further by adding a constant force term to (H-1)

$$\phi_{tt} - \phi_{xx} + \beta\phi_t - \phi + \phi^3 = \beta_0$$

using the transformation

H-10 $$\phi = m \sin\psi/2 + \kappa$$

and working the same way as before ,he finds the system of equations

H-11
$$m^2 = 2\lambda$$
$$2\lambda - 1 + 3\kappa^2 = 0$$
$$3\kappa m^2 = \kappa(1-\kappa^2) - \beta_0$$
$$\pm v\gamma\beta m\sqrt{\lambda} = 3\kappa m^2$$

If $\beta_0 = 0$ $\Rightarrow$
$$\beta_0 = 0$$
$$\lambda = 1/8$$
$$m = \kappa = \pm 1/2$$

the previous solution is recovered. For $\beta_0 \neq 0$ from H-11

H-12
$$m^2 = 2\lambda = 1 - 3\kappa^2$$
$$2\kappa - 8\kappa^3 + \beta_0 = 0$$

Solving H-12 for k (k < $\left|\sqrt{\frac{1}{3}}\right|$ for real m) in terms of β_0 one can find m and λ from

$$\lambda = \frac{1 - 3K^2}{2} = f(\beta_0) \qquad m = \sqrt{2\lambda} = f_1(\beta_0)$$

where the solutions of k are given by

$$K_1 = S + T$$

$$K_2 = -\frac{1}{2}(S+T) + \frac{i}{2}\sqrt{3}\,(S-T)$$

$$K_3 = -\frac{1}{2}(S+T) - \frac{i}{2}\sqrt{3}\,(S-T)$$

with

$$S = \sqrt[3]{\frac{\beta_0}{16} - \sqrt{-\frac{1}{12^3} + \frac{\beta_0^2}{16}}}$$

$$T = \sqrt[3]{\frac{\beta_0}{16} - \sqrt{-\frac{1}{12^3} + \frac{\beta_0^2}{16^2}}}$$

The general wave solution of H-10 is then given by (H-9) where ψ is the solution of s-G. Thus the method accounts for finding a solution of the ϕ^4 equation with both time derivative term and constant force.

APPENDIX I

NEW EQUATIONS WITH KINK SOLUTIONS

It is interesting to mention that in our study of solitary waves we have found new nonlinear equations which do not belong to the family of the nonlinear Klein-Gordon equations since they can not be written in the form

$$\Box\phi + \frac{\partial U(\phi)}{\partial\phi} = 0$$

1. The first one is given by the equation

I-1 $$\phi_{tt} - \phi_{xx} = \phi - \phi^3 + \delta\phi\phi_x + \varepsilon\phi\phi_t$$

where ε, δ are constants ,and has solution

I-2 $$\phi = \tanh\left[\frac{2(x \pm vt)}{-(\delta \pm v\varepsilon) \pm \sqrt{(\delta \mp \varepsilon v)^2 + 8(1-v^2)}}\right]$$

If one attempts to find the bound states for this equation by letting $\phi = \phi_0 + y$ as in Chapter II , he finds that the number of bound states depends on the values of the constants ε , δ .We don't know if I-1 can describe any physical system. If it does, then in addition to the kink solution one has the possibility of fixing the number of bound states by choosing the amplitudes ε , δ of the disturbances. This number reduces to two bound states when d,e are both zero (ϕ^4). Again one could check numerically if the resonance energy exchange between the bound modes works here too.

2. We also have found that the integrodifferential equations

I-3 $$\Box \phi = \phi - \phi^3 + \delta \phi_t \int \phi_x dt + \varepsilon \phi_x \int \phi_t dx$$

I-4 $$\Box \phi = \phi - \phi^3 \pm \varepsilon v^2 \phi^m \int (\phi^n)_x dt \mp \varepsilon \phi^m \int (\phi^n)_t dx$$

have kink-like solutions given by

$$\phi = \tanh\left[\frac{2(x \pm vt)}{-(\delta \pm v\varepsilon) \pm \sqrt{(\delta \pm v\varepsilon)^2 + 8(1-v^2)}}\right]$$ for H-3

$$\phi = \tanh \frac{x \pm vt}{\sqrt{2(1-v^2)}}$$ for H-4

Again we don't know what physical system can be modeled by the above equations. In this notice we just list the new equations and emphasize that they have kink solutions.

REFERENCES

[1] D.K.Campbell, J.F.Schonfeld and C.A.Wingate, Physica 9D (1983) 1-32.

[2] T.Sugiyama, Prog. Theor. Phys. 61 (1979) 1550.

[3] S.Jeyadev and J.R.Schrieffer, Synthetic Metals, 9(1984) 451.

[4] J.L Gervais and B.Sakita , Phys. Rev. D11 (1975) 2943.

[5] E.Tomboulis, Phys. Rev. D12 (1975) 1678.

[6] G.Eilenberg, Z.Physik B27 (1977) 199-203.

[7] C.Wingate, Siam J.Appl. Math, 43 (1983)120.

[8] M.Moshir, Nucl. Phys. B 185 (1981) 318-332.

[9] M.J.Rice, Phys. Rev. D15 (1977) 2866-2874.

[10] R.Rajaraman, Phys. Rev. D 15 (1977) 2866-2874.

[11] M.Ablowitz, M.D.Kruskal and J.F.Ladik, Siam J.Appl. Math. 36(1979) 428.

[12] H.Segur, J.Math.Phys. 24 6 (1983) 1439.

[13] H.Segur, Phys. Rev. Letters 58 (8) 1987, 747.

[14] R.J.Flesch and S.E.Trullinger, to be published.

[15] J.K Perring and T.H.R. Skyrme, Nucl. Phys. 31(1962) 550-555.

[16] D.K.Campbell and M.Peyrard, Physica 1 (1986) 47.

[17] M.Peyrard and D.K.Campbell, Physica 9D (1983) 33-51.

[18] A.R.Bishop, J.A. Krumhansl and S.E.Trullinger, Physica 1D (1980) 1-44.

[19] R.K.Dodd, J.C.Eilbeck, J.D.Gibbon and H.C.Morris, Solitons and Nonlinear Wave Equations, (Academic Press, 1982).

[20] Lecture notes in Physics 189, Nonlinear Phenomena, (Proceedings, Oaxtepec, Mexico 1982).

[21] Novikov, S.V.Manakov, L.P.Pitaevskii and V.E.Zakharov, Theory of Solitons, The inverse Scattering method, (Consultants Bureau, New York 1984).

[22] R.Rajaraman, Solitons and Instantons, (North-Holland, 1984).

[23] L.D.Landau and E.M.Lifshitz, Mechanics (Pergamon Press, 1982).

[24] F.G. Bass, Yu.S.Kivshar, V.V.Konotop and Yu.A.Sinitsyu
Phys. Reports 157(1988).

[25] L.D.Landau, Quantum Mechanics, (Pergamon Press, 1964).

[26] D.K.Campbell, M.Peyrard and P.Sodano,
Physica 19D(1986) 165-205.

[27] M.Peyrard and M.Remoissenet, Phys. Rev B26(1982).

[28] C.A.Condat, R.A.Guyer and M.D.Miller,
Phys. Rev. B27(1983) 474.

[29] R.Klein, W.Hasefratz, N.Theodorakopoulos and W.Wunderlich,
Ferroelectrics 26(1980) 721.

[30] J.L.Gervais and A.Jevicki, Nucl. Phys. B110 , 93 (1976).

[31] M.J.Rice and E.J.Mele, Solid St. Commun. 35, 487(1980).

[32] T.Barnes and G.J.Daniell 1984 Phys. Lett. 142 B 188.

[33] S.N.Behera and A.Khare 1980 Pramara 14 327.

[34] P.M.Stevenson 1984 Phys. Rev. D30 1712.

[35] S.Coleman 1975 Phys. Rev. D11 2088.

[36] I.S.Gradshteyn and I.M.Ryzhik,
Table of integrals,Series,and Products,
(Academic press,1980).

[37] J.Kuczynski, R.Manka and S.Ladkowski,
J.Phys.A:Math.Gen. 19(1986) L91-L95.

[38] Ru-Keng Su and Tao Chen,
J.Phys.A:Math.Gen. 20(1987) 5939.

[39] P.Lal, Phys.Lett. 111A(1985) 389.

[40] S.Weiberg, Phys. Rev. Lett. 19(1967)1264.
A.Salam, Elementary Particle Theory
Relativistic Groups and Analyticity
(Nobel Symposium No. 8)N.Svartholm, ed.
(Almquist and Wiksell, Stockholm,1968).

[41] H.Fritzsch and P.Minkowski, Phys. Repts. 73(1981)67.