

Spring 1985

RISK AND UNCERTAINTY (UTILITY, DECISION)

PAUL SNOW

University of New Hampshire, Durham

Follow this and additional works at: <https://scholars.unh.edu/dissertation>

Recommended Citation

SNOW, PAUL, "RISK AND UNCERTAINTY (UTILITY, DECISION)" (1985). *Doctoral Dissertations*. 1455.
<https://scholars.unh.edu/dissertation/1455>

This Dissertation is brought to you for free and open access by the Student Scholarship at University of New Hampshire Scholars' Repository. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of University of New Hampshire Scholars' Repository. For more information, please contact nicole.hentz@unh.edu.

INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.
2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.
3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.
4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.
5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

**University
Microfilms
International**

300 N. Zeeb Road
Ann Arbor, MI 48106



8521501

Snow, Paul

RISK AND UNCERTAINTY

University of New Hampshire

PH.D. 1985

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

Copyright 1985

by

Snow, Paul

All Rights Reserved



RISK AND UNCERTAINTY

BY

Paul Snow
A.B., Harvard College, 1973
S.M., Harvard University, 1975

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy
in
Engineering

May, 1985

ALL RIGHTS RESERVED

© 1985

Paul Snow

This dissertation has been examined and approved.

C. K. Taft

Dissertation director, Charles K. Taft
Professor of Mechanical Engineering

Robert W. Corell

Robert W. Corell
Professor of Mechanical Engineering

Eugene C. Freuder

Eugene C. Freuder
Associate Professor of Computer Science

Filson H. Glanz

Filson H. Glanz
Associate Professor of Electrical Engineering

Jeffrey E. Sohl

Jeffrey E. Sohl
Assistant Professor of Business Administration

4/8/85

Date

ACKNOWLEDGEMENTS

Thank you to William M. Golding of the Governor's Energy Office (John H. Sununu, Governor and Dennis A. Hebert, Director) for giving me work with flexible hours. Thanks also to Joyce Donohue for her patience while I wrote this.

Of course, I am grateful to the faculty of the dissertation committee for volunteering their time on a project remote from their everyday professional concerns. Special thanks to Charles K. Taft, who was generous with his time and with what material support was his to give.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iv
ABSTRACT	vi
CHAPTER	PAGE
I. THE RECEIVED THEORY OF EXPECTED UTILITY	1
The Von Neumann-Morgenstern Axioms	1
The Kelly Rule	14
The Friedman-Savage Problem	20
II. THE GAMBLER'S RUIN AND EXPECTED UTILITY	28
Exponential Utility and Ruin Constraints	28
Choice and the Probability of Ruin	34
Power Law Utility and Ruin Constraints	44
III. UTILITY-LIKE DECISION RULES	56
Power Law Buying and Selling Price Rules	56
The Buying Price and the Independence Axiom	65
Bounded Utilities and Ruin Constraints	72
IV. DECISIONS INVOLVING GOALS	82
Ruin in Two-Barrier Random Walks	82
Goals and the Allais Problem	88
Bands, Barriers and the Dividend Problem	97
V. UNCERTAINTY AND PARTIAL RISK	104
Received Theories of Decisions under Uncertainty	104
Uncertain Mixtures of Lotteries and Maximin	113
The Ellsberg Problem	123
Conclusions	135
LIST OF REFERENCES	136

ABSTRACT

RISK AND UNCERTAINTY

by

PAUL SNOW

University of New Hampshire, May, 1985

The conventional theory of decision making under risk relies on axioms that reflect assumptions about people's subjective attitudes towards wealth. The assumptions are unverifiable, and the axioms are too restrictive. They forbid some decision rules that a plausibly rational decision maker ("DM") could find useful.

A new method, at once less restrictive and less dependent upon subjective assumptions, motivates expected utility techniques by assuming that a DM wishes to place an upper limit on the probability of ruin. All bounded-above, increasing functions defined over a suitable domain can serve as utility functions.

Now, DM can evaluate lotteries according to their buying prices, useful if one plans to withdraw capital from risk. DM can rigorously distinguish "once in a lifetime" lotteries from ordinary gambles; that's helpful when facing Allais' problem. Also, DM can exploit partial knowledge of state probabilities without choosing arbitrary point estimates for all uncertain odds. That helps to resolve problems that combine elements of both risk and uncertainty, like Ellsberg's.

The new method allows DM to do everything permitted under the old axioms, and more besides, with fewer and less ambitious assumptions.

CHAPTER I

THE RECEIVED THEORY OF EXPECTED UTILITY

The von Neumann-Morgenstern Axioms

Suppose a decision maker ("DM") faces a choice among two or more gambles. A gamble or lottery is a set of outcomes and an associated set of probabilities. The outcomes are mutually exclusive and exhaustive. An outcome is either a capital transfer to (or from) DM, or another lottery, or a choice among lotteries. Note that a transfer of wealth for certain may, when convenient, be viewed as a lottery with a single outcome whose probability is one.

Assume that DM knows all of the gambles that are available when a choice must be made. Assume also that DM knows all of the possible outcomes in each available gamble. Except in Chapter V, DM will also know all of the probabilities. Although lotteries may be continuous random processes, the examples pursued here will generally be discrete lotteries, binomial or multinomial.

Von Neumann and Morgenstern [98] proposed a system of axioms that counsel DM to make one's choice in the following way. DM should choose some increasing function $U(\cdot)$. The domain of $U(\cdot)$ shall be amounts of money, either changes in wealth (sometimes called the "prizes" of the lottery) or else the total wealth DM will attain after the lottery is completed. Whether one uses total wealth or changes in wealth is a matter of convenience.

For each outcome of each gamble, DM evaluates the chosen function. Then, for each gamble, DM computes the average of the $U(\cdot)$ values

offered by the gamble. One selects for play that gamble with the highest expected value of $U()$.

The $U()$ function is called a "utility function". The advice given by the von Neumann and Morgenstern axioms is succinctly summarized as "maximize expected utility."

Several authors have offered restatements of the original von Neumann and Morgenstern work. The same theorems can be proven from many versions of the underlying axioms. Fishburn [33] reviews many proposed systems. The discussion to follow is adapted from Baumol's [6] restatement.

Some conventions are needed. The capital letters A through D will denote lotteries. The lower-case Latin and Greek letters a through d and α and ω will denote amounts of money. The lower-case letters p , q , and r will denote probabilities.

The discussion will concern lotteries with at most two outcomes, so the following notation suffices. Read " $(p: a, b)$ " as "the lottery which offers a probability p of getting a and a complementary chance, $1 - p$, of receiving b instead". The outcomes may themselves be lotteries, e.g.g. " $(p: A, B)$ " or " $(p: (q: a, b), (r: c, d))$ ". Degenerate lotteries, like " $(1: a, ?)$ " will be written simply as amounts for certain, in this case, a .

Two relationships between lotteries need to be defined. The string " $A \succ B$ " means that if DM owned B , then one would accept A in its place. More compactly, one might say that A is weakly preferred to B . The string " $A \sim B$ " means $A \succ B$ and $B \succ A$, that is, DM is indifferent between lotteries A and B .

Between money amounts, the usual symbols $>$, \geq , and $=$ have their ordinary meanings. When lotteries are compared to amounts for certain,

the lottery relational symbols, $|>$ and $|=$ will be used.

The axioms are now given. The names assigned are the ones most frequently encountered in the literature.

The transitivity axiom holds that for any three lotteries A, B and C, if $A |> B$ and $B |> C$, then $A |> C$.

The continuity axiom holds that for any three amounts a, b and c such that $a \geq b \geq c$, there exists exactly one probability p such that $(p: a, c) |= b$.

The independence axiom holds that for any four lotteries A, B, C and D such that $A |= C$ and $B |= D$, and for any probability p:
 $(p: A, B) |= (p: C, D)$.

The probability dominance axiom holds that for any two amounts a and b such that $a \geq b$, and for any two probabilities p and q, $p \geq q$ implies and is implied by $(p: a, b) |> (q: a, b)$.

Finally, the compound probability axiom holds that for any two amounts a and b and for any three probabilities p, q and r,
 $(p: (q: a, b), (r: a, b)) |= (pq + r - pr: a, b)$.

From the continuity axiom, note that if $\alpha < \omega$ are two amounts, then for every x such that $\alpha \leq x \leq \omega$, there is a function U(x) such that $(U(x): \omega, \alpha) |= x$.

If DM accepts the axioms and U() is defined as above, then we claim the following. If amounts a, b, c and d belong to the closed interval bounded by α and ω , and if p and q are probabilities, then $(p: a, b) |> (q: c, d)$ if and only if $pU(a) + (1-p)U(b) \geq qU(c) + (1-q)U(d)$.

In words, one lottery is weakly preferred to another if and only if it offers at least as great an expected utility as the alternative.

The proof is notationally tedious, but conceptually simple. Suppose that $(p: a, b) \succ (q: c, d)$. By the definition of $U(\cdot)$ and the continuity axiom,

$$a \mid = (U(a): \omega, \alpha)$$

$$b \mid = (U(b): \omega, \alpha)$$

... et cetera.

So, by the independence axiom,

$$(p: (U(a): \omega, \alpha), (U(b): \omega, \alpha)) \mid = (p: a, b)$$

$$\text{and } (q: (U(c): \omega, \alpha), (U(d): \omega, \alpha)) \mid = (q: c, d)$$

So, by transitivity,

$$(p: (U(a): \omega, \alpha), (U(b): \omega, \alpha)) \succ$$

$$(q: (U(c): \omega, \alpha), (U(d): \omega, \alpha))$$

Each side of this preference relation can be simplified by using the compound probability axiom; the sense of the preference will be preserved by transitivity. Thus, we arrive at

$$(pU(a) + (1-p)U(b): \omega, \alpha) \succ$$

$$(qU(c) + (1-q)U(d): \omega, \alpha)$$

And, by probability dominance,

$$pU(a) + (1-p)U(b) \geq qU(c) + (1-q)U(d)$$

which is the required expected utility expression.

The proof of the implication in the other direction is similar. The same steps and axioms are applied in reverse order, starting with the expected utility inequality and leading to the lottery preference statement.

The result generalizes readily to more than two lotteries and more than two outcomes. By selecting α small enough and ω large enough, all money lotteries of interest can be covered.

The axioms provide DM little guidance in selecting a particular utility function. Still, anyone who accepted the axioms would conclude that the rule by which lotteries are chosen ought to be based on maximizing the expected value of some utility function.

The axioms impress many people with their immediate plausibility. For example, a decision rule that wasn't transitive would be somewhat strange, as Tullock [94], among many others, has pointed out.

Suppose DM strictly preferred lottery A to lottery B, and strictly preferred B to C, and also strictly preferred C to A. If DM were willing to pay even a token amount to effect each of the exchanges implied by these preferences, then DM would provide a risk-free income to anyone who owned the three lotteries. The lucky owner could collect a toll as DM moved from C to B to A and then back to C, where presumably the cycle could be repeated. This situation is often called "the money pump".

The continuity axiom's substantive imposition is that there be a specific stable relationship between lotteries and amounts for certain. One could imagine that if the three amounts referred to by the axiom were close together, then the uniqueness of the probability could be defeated by "rounding errors". For example, \$ 1.97 might be indifferent in practice to (p: \$ 1.96, \$ 1.98) for any p between .5 and .6, say. Even so, this has little practical import if the reference values, the alpha and omega chosen by DM, are widely separated.

Probability dominance seems especially in line with common sense. If two lotteries have the same outcomes, but different probabilities, why wouldn't anyone want the lottery that gives more weight to the better outcome?

Compound probability is also straightforward. Two prospects that

have the same outcomes and the same probabilities are indifferent to one another, regardless of how the probabilities are described in the formulation of the decision problem. The axiom does have content. It puts outside of the theory those problems where the pleasures or pains of gambling are an issue. For example, the attractiveness of an evening's entertainment at the Salle Privée in Monte Carlo does not depend solely on the prizes and their probabilities. Gambling for sport is not the subject under discussion.

The independence axiom is different from the others, however. As Samuelson [74] notes, the axiom does not appear explicitly in von Neumann and Morgenstern's system. Malinvaud [54] shows that it is contained implicitly in their definition of a utility function. Regardless of how the axiom is introduced into the logical system, however, it plays a crucial role. As Machina [53] puts it, "It is the independence axiom which gives the theory its empirical content by imposing a restriction on the functional form of the preference function." (Emphasis is in the original.)

The axiom has not received anything like universal acceptance. Allais [1] is perhaps the best known of its critics. Manne [55] argues by analogy to various physical mixture systems that the axiom is, at best, not intuitively obvious. Wold [99] finds the axiom unsatisfactory based on his analysis of riskless preferences. McClennen [58] reviews the various arguments put forward in favor of independence and reports himself to be unswayed.

On the other hand, some authors, notably Savage [79] and Raiffa [67], view independence as an obligatory attribute of rational decision making under risk. Throughout this dissertation, there will be several

opportunities to examine the independence axiom in some depth. For now, we simply note the existence of controversy.

Von Neumann and Morgenstern developed their theory of choice under risk to support a theory of strategic play in games. An important application of the theory concerns "zero sum" games with two players. "Zero sum" means what one player wins, the other player loses, so that the sum of the amounts won and lost is zero. Strategically, it means that the players are strict competitors of one another; they will not co-operate. Chess is a classic example of a two-person zero-sum game. To avoid repetition, hereafter "game" will mean "zero-sum game" unless otherwise stated.

A two-person game can be represented as a matrix of pay-offs, as:

	1	2	3
1	\$20	\$36	\$25
2	\$35	\$32	\$30

The two players are Row and Column. In this example game, Row has two options and Column has three. If Row plays its option 1 and Column plays its option 1, then Column pays Row \$20. If Row plays 1 and Column plays 2, then Column pays Row \$36, and so on. The players make their moves without knowledge of the other's move.

This particular game has a solution in a certain sense. If Row plays its option 2, then Row receives at least \$30. The other move guarantees only \$20, although \$36 is possible. If Column plays its option 3, it need pay no more than \$30 regardless of what Row does. Column's other options expose it to possible liabilities of \$35 and \$36, although a lower-than-thirty-dollar liability is possible.

If Row plays its "maximin" move (option 2), the one that yields

the "best worst" outcome, and Column plays its "minimax" move (option 3), the one that yields the "best worst" outcome from its point of view, then \$30 changes hands.

This is a solution in the following sense. If one player plays its "best worst" move and the other player does not, "defects" as it were, then the defecting player does worse than it could have done by playing its own "best worst" option. Only if both players defect can one (and only one) defecting player do better than the guaranteed level.

Note that this solution is stable in that if one player announced its intention to play its "best worst" move, then the other player could not profit from this knowledge beyond what it would gain anyway by playing its own "best worst" move. By contrast, announcing any other strategy would allow the opponent to profit.

Although most analysts would accept that Row should play 2 and Column should play 3, at least some do not find this solution satisfactory. Ellsberg [25], in a classic paper, is put off by a strategy where a player moves solely to get out of the game with the least possible damage. In a memorable image, he compares a minimax player to someone forced into a duel against one's will. Ellsberg offers no better solution, however, and much of what he finds unsatisfactory may be attributed to the assumption of radical opposition between the players.

Not all game matrices have a solution in the sense just discussed. Consider, for example, the matrix

	1	2
1	\$80	\$20
2	\$40	\$100

Row's maximin move is to play 2. Column's minimax move is 1.

Note, however, that if Row knew that Column intended to play its minimax strategy, Row would play 1 instead of 2, to get \$80 instead of \$40.

The salient difference between this matrix and the last one is that the cell which contains the maximin value (2, 1) does not coincide with the cell containing the minimax value (1, 1). It had long been known that when the minimax and maximin cells do not coincide, then at least one player had the option to defect and do better than its own "best worst" outcome, if the other player didn't defect.

Von Neumann [97] noted that every matrix game, regardless of whether its minimax and maximin cells coincide, has a solution in the following weaker sense. For each player, there is a strategy, the "optimal mixed strategy", which assigns to each move a probability of being played, such that the other player cannot do better on average than to play its own optimal mixed strategy. If a player announces that it will pursue its optimal mixed strategy, then the other player can do no better on average than it would have done by playing its optimal mixed strategy.

For example, suppose in the game just discussed, Row assigns to each of its two moves a probability of one-half. That is, it selects its move by flipping a coin. Whichever way column plays, it effectively faces a lottery based upon Row's coin flip. The expected value of this lottery is

$$\frac{1}{2}(\$80) + \frac{1}{2}(\$40) = \frac{1}{2}(\$20) + \frac{1}{2}(\$100) = \$60$$

The situation is symmetric. If Column assigns its move 1 a probability of two-thirds and move 2 a probability of one-third, it imposes a \$60 average value lottery on Row. Neither player can impose a lottery whose average value is more favorable to its interests.

To explore this result for matrix games with an arbitrary number of options for each player, von Neumann used the theory of linear programming. The theory is applicable since each player exercises control by choosing probabilities. The figure of merit, the guaranteed average value, is linear in the probabilities chosen. Von Neumann showed that the decision problem faced by Column is the dual of the problem faced by Row. This insures that the value guarantees attainable by each player are equal in magnitude.

More recently, other authors have used techniques other than linear programming to obtain similar results, for example, Vilkas [96] and Tijds [93].

One attraction of the von Neumann approach is that it relates the game problem to a widely understood and practiced optimization technique. An interesting counterpoint to von Neumann's results is offered by Dantzig [22], who showed that every linear programming problem can, in principle, be represented as a matrix game. On a more practical plane, a succinct survey of von Neumann's and others' results in the application of linear programming to games can be found in Berge and Ghouila-Houri [8].

As satisfying as the von Neumann result was, it had one serious flaw. A player might not prefer one lottery over another simply on the basis of expected money value. If a decision rule could be devised that selected lotteries based on the average value of some function of

the pay-offs, then the original von Neumann theorem and its linear programming proof would be preserved. Expected utility is such a rule.

Thus, the axiomatic system discussed earlier provided von Neumann with what he needed to claim a larger kind of generality for his solution to the matrix game. As a theoretical advance, axiomatic utility had shown its power.

McClennen [57] notes that the introduction of utility theory does not resolve reservations about minimax strategies generally, such as those in Ellsberg's "reluctant duellist" paper [25]. McClennen finds nothing in the utility axioms that leads the player to seek such a solution. Roth [71] agrees, and explores some additional assumptions, complementary to the utility axioms, that do lead to minimax-style play in two-person games.

To apply the axioms to practical problems, DM must choose a particular utility function. The axioms themselves, however, require only that the function be increasing in its argument. That is, if the function is differentiable, then the sign of the first derivative is positive.

If the function is twice differentiable, then the likely sign of the second derivative can also be specified.

Adopt the convention that an arc of an increasing function is convex if its second derivative is positive, concave if the second derivative is negative. To picture the convention, a concave function is concave when viewed from below.

DM is said to be risk-averse if for any lottery one would prefer to receive the actuarial value of the lottery for certain rather than to undergo the lottery. The antonym of risk-averse is risk-seeking.

Anecdotal evidence suggests that most people are unwilling to play fair gambles, those whose expected money value is zero, at least not for large stakes. If so, then most people would seem to be risk-averse.

If DM were always risk-averse, and wished to choose lotteries according to the axioms, then DM's utility function would be everywhere concave. This can be shown by Jensen's inequality. If $U(\cdot)$ is any concave utility function, then for any lottery, the expected $U(\cdot)$ value of the lottery is less than the $U(\cdot)$ value of the lottery's expected pay-off. Thus, the average pay-off for sure is preferred to the lottery.

The proof of Jensen's inequality is straightforward and can be found in many textbooks, e.g. DeGroot [24, chapter 7].

The restriction on DM's choice of a utility function implied by Jensen's inequality still does not come close to choosing a particular function. The choice of a utility function might be simple if it were possible to say what the function measures about its argument, if anything.

Bernoulli [9], in a work that predates von Neumann and Morgenstern by two centuries, suggested an expected utility decision rule without the benefit of axioms. Bernoulli advised taking the logarithms of one's after-play wealths and computing the expected value of those logarithms. The lottery with the highest expected logarithmic value is chosen.

Bernoulli argues that this wealth-dependent utility did measure something about its argument. He believed that the "marginal satisfaction" of any increase in one's wealth is inversely proportional to the current size of one's fortune. That is,

$$dE \text{ is proportional to } 1/w$$

where "dE" is the infinitesimal increase in "satisfaction" (emolumentum was Bernoulli's Latin term) from an infinitesimal increase in wealth, and w is one's current wealth. The total satisfaction from a given fortune, then, is the integral of this infinitesimal satisfaction, or $\log(w)$.

Bernoulli's idea that people's affective rewards from possessing wealth is non-linear appears to have wide acceptance in the economics literature. Von Neumann and Morgenstern seem to embrace this notion, and assert that one's preferences among gambles are related to the relative strength of one's feelings for the various prizes.

It is as if a lottery offering a fifty-fifty chance of \$50 and \$100 itself conveyed a satisfaction that was, in some subjective way, midway between the satisfaction afforded by \$50 and that afforded by \$100. That is generally taken to be different from the satisfaction of \$75.

Arguments such as these rely heavily on subjective attitudes presumed, but not proven, to exist which nevertheless allow precise mathematical computations to be carried out. Blatt [10] finds such an enterprise inherently faulty.

Less sweepingly, if the foundation of decision-making were that people have complicated attitudes about money, then there would seem to be little that an engineer could contribute to the discussion.

The Kelly Rule

Kelly [45], an engineer, motivated a decision rule that is indistinguishable from Bernoulli's advice to maximize the logarithms of one's after-play wealth. The Kelly rule is not in any way based on the assumption that the logarithm measures any attitude that DM might have about wealth. Rather, Kelly shows that choosing lotteries with this decision rule maximizes one's exponential rate of capital growth, assuming indefinite play and infinite divisibility of wealth.

Neither assumption is true. DM is mortal. Transferable wealth is, in general, quantized, e.g. in dollars and cents. As a practical matter, however, DM might adopt the assumptions as reasonable approximations. If so, then that adoption and assent to the goal of maximizing the rate of capital growth are the only subjective elements in the Kelly theory.

To see how logarithmic utility accomplishes its optimization of capital growth rates, suppose that DM has the opportunity to choose an amount to hazard on the following wager. There is a probability p that the amount wagered will be lost and a complementary probability $q = 1-p$ that the same amount will be added to DM's wealth. The wager is available for endlessly repeated play with no minimum and no maximum stakes. DM can vary the stakes on each play.

Suppose further that DM wishes to maximize one's exponential capital growth rate. Let N be the number of times the gamble is played, C be DM's capital after N plays and I be DM's initial capital. The exponential growth rate of capital is defined as the limit as N goes to infinity of

$$\frac{1}{N} \log \left[\frac{C}{I} \right]$$

One is free to choose any base for the logarithm, since the policy which

attains the maximum for one base does so for all bases.

Suppose that DM bets some constant fraction f of one's current capital on each play. After N plays, let W be the number of times DM has won and L be the number of times DM has lost. Clearly, DM's wealth is

$$C = I(1 + f)^W(1 - f)^L$$

Divide through the equation by I and N , and take the logs and limit:

$$\lim \frac{1}{N} \log \left[\frac{C}{I} \right] = \lim \frac{W}{N} \log(1 + f) + \frac{L}{N} \log(1 - f)$$

In the frequentist view of probability, the limit of W over N as N approaches infinity is just the probability of winning, q . Similarly, the limit of L over N is just p .

So, we may rewrite the expression on the right as:

$$p \log(1 - f) + q \log(1 + f)$$

and we recognize the expression on the left as the exponential growth rate. Therefore, to maximize the growth rate while playing a fixed fraction of one's wealth on each play, choose f so as to maximize expected logarithmic utility.

Breiman [16, 17] extended Kelly's result, showing that logarithmic utility asymptotically optimized the capital growth rate in more general sequences of gambles. Bell and Cover [7] showed that logarithmic utility is optimal for two competitors who try to amass the greatest winnings in the course of a finite contest.

The Kelly derivation of logarithmic utility can be distinguished from Bernoulli's on grounds other than Kelly's avoidance of hedonic assumptions. Nothing in Bernoulli's analysis depended on "long run" arguments. The pleasure or pain, so to speak, afforded by a lottery did not depend on its ever being repeated. DeGroot [24] points out that

modern axiomatic utility is also assumed to hold for "one time" gambles. If a lottery is to be taken repeatedly, then the sequence ought to be viewed as a large composite gamble. Samuelson [75, 76] notes that treating a long sequence of gambles as one composite gamble will generally lead to decisions different from those advocated by Kelly. Thorp [92] makes much the same point, but takes this as something of an argument in favor of adopting logarithmic utility.

By contrast to the axiomatic theory, the Kelly approach depends on repetition. It is only in the long-run sense (indeed, an infinite-run sense) that the exponential growth rate is optimized.

The logarithmic utility function is disesteemed by some authors regardless of whether it is derived from Kelly's, Bernoulli's or anyone else's arguments. The difficulty that these critics perceive is that the logarithm is an unbounded function, both above and below. To understand why this would be thought to be a defect requires a digression on the classic St. Petersburg Paradox.

The paradox was propounded by Nicholas Bernoulli (cousin to Daniel, the author of logarithmic utility). Suppose one were offered the following gamble by a casino. One will flip a coin until it comes up heads. If it is heads on the first flip, then the casino will pay the player \$2; if on the second flip, \$4, and on the n-th flip, 2^n dollars. The expected value of this gamble is infinite. The probability that the first heads occurs on the n-th flip is 2^{-n} . The average pay-off is therefore

$$\sum_{n=1}^{\infty} (2^{-n})(2^n) = \sum_{n=1}^{\infty} 1$$

The paradox was offered as a counter-example to the view of Pascal and Fermat that lotteries should be valued according to their average

pay-offs. The value of the St. Petersburg game is not infinite, even though its expected pay-off is. That is, no DM would pay much, and especially not an arbitrarily large amount, for the privilege of playing the game. Conversely, if DM found oneself liable to offer the game to someone else, DM wouldn't pay much to be rid of the obligation.

This left open the question of how DM ought to order lotteries. Daniel Bernoulli noted that his logarithmic utility resolved the paradox.

The engineer Fry [38] pointed out in 1928 that the paradox was defective: the gamble simply couldn't be realized. After some finite number of tails, the casino would be bankrupt. The pay-off, therefore, was finite. In fact, the reader can easily calculate that even if the casino were fabulously wealthy, the expected value of the pay-off is puny. For example, a billion-dollar casino could withstand only about thirty tails or so, and thus the average pay-off achievable is about thirty dollars.

Barring commerce with an unbounded divinity (a 20th Century bar more than an 18th Century one), the paradox simply doesn't arise, ever. Surprisingly enough, one can find quite recent discussions of the paradox in the economic literature. Samuelson [77] provides a historical overview and an introduction to the recent discussions. Shapley [82, 83] and Aumann [4] engage in a lively exchange of recent (1977) vintage.

If one does not accept the Fry viewpoint, there is a difficulty with any unbounded utility, either above or below. Although Bernoulli's suggestion solves the original paradox, one can change the pay-off schedule so that it rises faster than 2^n , fast enough to make the logarithm or any unbounded utility diverge. Arrow [2] shows that this contradicts the conventional continuity axiom.

Suppose that a DM with an unbounded utility function faces three prizes. Prize "A" is the St. Petersburg problem, generalized so that DM's utility diverges. Prize "B" is \$2 and prize "C" is \$1. Clearly, the worth of prize B is somewhere between that of A and C. The continuity axiom requires, therefore, that there be some probability p such that DM believes $(p: A, C) \sim B$. There isn't: for all finite p the expected utility value of the lottery is infinite.

Arrow's point, of course, depends on the ability to realize the gamble. In a fascinating 1974 exchange, Arrow [3] rebuts Ryan's [73] assertion that it is "highly improbable" that an unbounded-utility DM would ever face any practical difficulty.

Thorp [92], in a review of Kelly's work, finds no reason to reject logarithmic utility in such arguments. One could also note that even if the gamble could be realized, Arrow's logical objection need not daunt any follower of Kelly, since Kelly did not use the continuity axiom to obtain his utility decision rule.

The Kelly derivation does not depend on any of the von Neumann-Morgenstern axioms. Of course, by choosing lotteries so as to maximize an increasing function of one's capital, DM would behave consistently with the axioms whether or not one accepted them in principle.

A generally neglected part of Kelly's work, however, strictly violates the usual utility axioms and does so in a realistic context. The violation again involves the continuity axiom, this time the implication of the axiom that the utility function be unique for each DM.

Kelly recommended the logarithmic utility rule only for sequences where DM could re-invest the winnings. Kelly considered another

situation. Suppose DM's spouse allowed the wagering of a fixed sum each week, but not reinvestment. (Kelly wrote "wife" instead of "spouse", but he was writing in a simpler time.)

Obviously, exponential growth by gambling does not arise in this situation. Under the circumstances, then, DM ought to bet the entire allowable sum on the prospect that yields the highest expected monetary value, or else refrain from gambling if no positive average-value gamble is available. This advice assumes unending play, and assumes that DM wishes to eventually overtake in total winnings anyone who divides the allowable sum differently.

Choosing the prospect with the highest expected monetary value is indistinguishable from maximizing expected linear utility.

The linear utility function is

$$U(x) = ax + b, \text{ where } a > 0$$

It is a simple matter to extend Kelly's analysis one step further. Suppose DM's spouse allowed DM to participate in two gambling sequences. One sequence would start with some fixed sum chosen by the spouse and DM would be allowed to husband (so to speak) the sum with reinvestment. The other sequence would be based on an earmarked sum periodically refreshed, but no reinvestment would be permitted.

If DM accepted Kelly's advice, then DM would select some gambles by the logarithmic utility function and others by the linear. This behavior would be inconsistent with the von Neumann-Morgenstern axioms. Even so, the Kelly DM is acting rationally in the plain language sense of the word. DM acts to further a chosen objective which is itself not facially unreasonable within the constraints placed on one's behavior by circumstances.

The Friedman-Savage Problem

Milton Friedman and Leonard Savage [36, 37] sought to infer from people's behavior in risky decisions what attitudes people held towards different levels of income or wealth.

The decision phenomenon that attracted these authors' attention was the commonplace observation that often the same individual will buy both lottery tickets and insurance policies. Participation in real-life lotteries is risk-seeking in that the price of the ticket exceeds the actuarial value of the prizes. Insurance purchases are risk-averse: the buyer of insurance pays more than the actuarial value of the risk in order to be rid of it.

Risk-seeking behavior under conventional utility is associated with a convex utility curve (second derivative positive). Risk-averse behavior is associated with a concave curve (negative second derivative). Thus, a DM who subscribed to the conventional axioms and who was risk-averse for large losses (and so bought insurance) and risk-seeking for large gains (and so bought lottery tickets) might have a curve composed of both concave and convex segments.

Suppose that the utility value of a sum measured the psychic satisfaction that the sum afforded to DM. The first derivative at that point could be interpreted as the "marginal utility" of the wealth sum. That is, it would be a measure of the increase in satisfaction that would be realized by a small increase in wealth starting at that point. If the utility curve were concave, then the marginal utility of wealth would be decreasing as wealth increased.

The avoidance of fair bets, then, could be interpreted as a kind of psychological phenomenon. A fixed increase in wealth would provide less

additional satisfaction than the decrease in satisfaction entailed by a loss of equal magnitude. A fair bet, then, would be less than fair in average satisfaction offered.

By the same logic, the purchase of lottery tickets for more than their actuarial value would not be made voluntarily unless there were increasing marginal utility of wealth for the prizes.

Friedman and Savage hoped to counter the prevailing assumption of everywhere-decreasing marginal utility of wealth by studying risk-seeking behavior.

Friedman and Savage listed five features of decision-making under risk which they felt conformed with everyday experience. These were (1) larger incomes are preferred to smaller ones, (2) low income people are willing to buy lottery tickets, (3) low income people were willing to buy insurance, (4) points 2 and 3 are true simultaneously, and (5) lotteries typically have more than one prize.

The last statement seems a little out of place. The rationale is that the lottery operator selects the prize structure of the lottery. Presumably, the operator chooses the prize structure that makes the tickets most attractive. Thus, the diversity of prizes might correspond to some widespread attribute of risk-taking among the customers. After all, except for its role in helping to sell tickets, the operator doesn't care what the prize structure is. The operator's surplus depends only on the prize total.

Friedman and Savage then construct a utility curve for income levels that is consistent with all five statements. Statement 1 requires only that the curve be increasing. Statements 2 through 4 can be achieved by a concave section for losses and a convex section for gains.

If that's all there was to the curve, then statement 5 would not be satisfied. Lottery operators would strive for the biggest prize possible to get the most value from the convex segment. The way to get the biggest prize, of course, is put all the prize money into a single prize.

Since that doesn't seem to happen, Friedman and Savage reason that there is another concave section for very high values. Their final curve, then, has concave arcs at the high and low ends with a convex transition in between.

Markowitz [56] does not find this entirely satisfactory. He notes that most people reject fair bets except for very small stakes. The Friedman-Savage curve places a convex segment at moderate levels of income. If that were right, then fair bets would be widely acceptable for non-negligible stakes.

Markowitz's remedy is a curve with not two, but three inflection points. His curve plots utility against wealth with the origin at DM's "customary" level of wealth. "Customary" is often the same thing as simple current wealth. The distinction concerns wagering episodes that involve the accumulation of many small changes in wealth. The curve does not have to be redrawn after each bet.

Markowitz finds a convex segment for small increases in wealth, allowing fair bets and even less-than-fair bets for small stakes. There is a concave segment for small losses based on psychological literature showing that people bet more conservatively when they are behind. Really large losses have a convex segment; apparently people in Markowitz's view prefer to gamble on a huge loss rather than to take a merely big loss for sure. Finally, for large gains there is a concave

segment. The curve is bounded above and below to avoid complications of the St. Petersburg kind.

In Markowitz's theory, the shape of the curve is essentially the same for the rich and the poor, although the inflection points are located farther apart for the rich.

The Markowitz curve does not explain the same phenomena as the Friedman-Savage. The diversity of prizes observation is consistent with Markowitz, at the expense of not explaining why one offers big prizes at all.

Nevertheless, Markowitz and Friedman-Savage agree that people's choices under risk are difficult to reconcile with the decreasing marginal utility hypothesis.

Several authors wrote in rebuttal to defend the notion of decreasing marginal utility. Kwang [49] argued that many large purchases were indivisible, e.g., one cannot buy half of a Mercedes automobile. Perhaps, then, one's preference for wealth consisted of several concave sections joined at discontinuities that correspond to new consumption opportunities. The slope would steepen at each discontinuity, but concavity would prevail from there to the next discontinuity. The neighborhood of a discontinuity is convex in Kwang's scheme, even though all the constituent curves are concave.

Other suggestions for introducing local convexities in the presumed-to-exist utility curve include Richardson's [68], based on the increased costs of planning that become necessary when wealth changes. Flemming [35] argues that "big ticket" consumer durables cost their buyers more than they are worth the instant after they are bought, when they become "used" goods:

Hakansson [40] introduces convex borrowing constraints which alter a concave utility function so that it resembles the Friedman-Savage shape. Kim [47] gets similar results based on assumptions about individuals' borrowing and saving interest rates.

Experimental verification of these suggestions, despite their ingenuity, seems remote. For example, Kwang's suggestion would not be falsified by a utility curve that was everywhere convex. A well informed consumer might find new buying opportunities at every wealth level.

Some economists think that people's gambling behavior does not reveal their preferences for wealth per se, but rather an amalgam of wealth preferences and superstitions about probabilities. Yaari [100] suggests that people buy both insurance and lottery tickets because they tend to overestimate the probabilities of remote events. Thus, one simultaneously insures against the unlikely hazard and strives to win the all but unattainable prize.

Rosett [70] points out that the assumption of systematically biased probability estimates in no way rules out convex wealth preferences. Indeed, there is a theory that explicitly combines both.

Kahneman and Tversky [43, 44, 95] doubt that conventional utility theories can ever be reconciled with the evidence about risky decisions obtained in psychological experiments. They put forward their own theory of decision under risk which they call "Prospect Theory". Their system includes a utility-like function to describe DM's attitudes towards prizes and "decision weights" which are similar to, but not the same as, probabilities. The decision weights differ from probabilities in the way suggested by Yaari, i.e., small probabilities have exaggerated weights.

If the Kahneman-Tversky view prevails, then the Friedman-Savage program for measuring subjective attitudes towards wealth founders. In principle, perhaps, the effects of probability biases and wealth preferences could be sorted out. In practice, however, there was such a diversity of expert opinion when only the shape of one curve was in doubt, it is unlikely that interpreting phenomena with two unknown curves will lead to consensus.

An especially interesting point is raised by Raiffa [67, chap. 4]. He thinks that wealth utility curves with convex portions are common. He recommends that DM "alter" any convex portions, as for example, by drawing a straight line that "fills in" any convexity. By this expedient, he seeks to avoid the difficulty of taking unfair gambles. If Raiffa's advice were taken, however, DM's "true" preferences for wealth could not be inferred from gambling behavior. Any choice experiment with such a DM would find a weakly concave utility curve, thus seeming to confirm a generally decreasing marginal utility for wealth. In fact, some other interpretation would be closer to the truth.

Raiffa's concerns are strictly normative; he does not espouse the conventional utility on the basis of any descriptive considerations at all. Raiffa seeks an account of how one ought to behave, not how one does behave if ignorant of the normative principles.

The advice to avoid unfair gambles is plausibly a better prescription than to "do what you feel like" if what you feel like is taking unfair gambles. It does not particularly matter to Raiffa that behaving in such a fashion might mask one's feelings, even though those feelings are for Raiffa the foundation of utility theory.

Of course, an even stronger position can be inferred from Kelly's

work. A utility decision rule simply need not depend on DM's attitudes about a particular lottery's prizes. All that matters is what the decision rule accomplishes for DM in the long run.

The logarithmic utility advocated by Kelly is everywhere concave. However, the concavity has nothing to do with any possibility of decreasing marginal subjective utility of wealth. DM may in fact be insatiably and boundlessly greedy and adopts logarithmic utility because it promises the most money in the end. Despite its concavity, logarithmic utility is the perfection of greed rather than its denial.

Of course, the particular assumptions and objectives used by Kelly do not exhaust the long-run considerations that may interest different DM's. For example, the assumption of infinite divisibility of capital turns out to "assume away" a problem that may be of great concern.

If DM chooses lotteries by logarithmic utility, one will never risk one's entire capital. That is because the logarithm of zero is minus infinity, and will outweigh any finite gain whatsoever. So, if wealth is infinitely divisible, then DM will never run out of capital or opportunities to invest it. That is, DM will never be ruined.

In real life, of course, wealth is quantized. Ordinarily, there is some smallest amount of wealth that can be transferred, and therefore a smallest amount that can be a prize in a lottery. Also, long before DM is reduced to playing for pennies, one will probably find that the variety of investment prospects available for nickels and dimes is much poorer than that available for dollars. So, even if DM places a shiny new penny in one's wallet so as to "never go broke", the assumed-to-be unending sequence of gambles could very well end for lack of other players.

The next chapter concerns decision rules that deal with the end of play by a run of adversity. It shows how DM can choose lotteries in such a way as to constrain the probability of ever losing all (or nearly all) of one's capital.

Like the Kelly results, ruin-constraining rules do not depend on the details of DM's attitudes towards the prizes. The long-term implications of the decision rule motivate its adoption. Also like the Kelly results, the rules developed in the next chapter are generally consistent with the von Neumann-Morgenstern axioms. That is, they involve maximizing the expected value of a particular utility function.

Although the curve in question is concave, that concavity neither supports nor refutes the decreasing marginal utility of wealth hypothesis. However much DM esteems a large fortune, one won't attain that fortune if one is ruined first. The adoption of a ruin constraint entails no shortage of satisfaction in great wealth, but merely a realistic attitude about what may be required to achieve it.

CHAPTER II

THE GAMBLER'S RUIN AND EXPECTED UTILITY

Exponential Utility and Ruin Constraints

Feller [29] and Epstein [28] summarize the standard results concerning the probability of ruin when playing a multinomial game repeatedly in independent trials.

Suppose a gambler with initial capital w plays the following gamble. Let (a, b, \dots) be the distinct prizes, integer amounts of money that are added to or subtracted from the player's capital. Let (p, q, \dots) be the associated probabilities of receiving each of the prizes, mutually exclusive and exhaustive. Assume that the player will keep any and all winnings "at risk" and will play indefinitely or until all of the initial capital is lost, i.e. the player is ruined.

Feller showed that the probability of ever being ruined in the repeated play of such a gamble is bounded above by

$$r^w$$

where r is non-negative and no greater than one and is a solution of the characteristic equation

$$1 = pr^a + qr^b + \dots$$

In the special case where the prizes are $(-1, +1)$ and the probabilities are $(p, 1-p)$ where p is greater than one-half, Feller shows that the probability of ruin will be exactly r^w where r is

$$r = (1 - p)/p$$

which solves the characteristic equation.

To explore the bound for other cases, call the expression that

appears on the right-hand side of the characteristic equation the "characteristic polynomial". Assume that it has no fewer than two terms and that the exponents of each term are distinct.

By inspection, the polynomial has a derivative everywhere on the "clopen" unit interval (zero exclusive to one inclusive).

If the lottery is a pure gain, then all the exponents in the characteristic polynomial are non-negative. Since they are distinct, at least one must be positive. So, the polynomial is strictly increasing for all positive \underline{r} . Since the polynomial is one when $\underline{r} = 1$, there is no solution of the characteristic equation on the open unit interval. The pure gain characteristic polynomial is less than one everywhere on the open unit interval.

A similar argument shows that if the lottery has only non-positive prizes, then the characteristic polynomial is strictly decreasing for positive \underline{r} , and so the characteristic equation can't have a solution on the open unit interval. The characteristic polynomial is greater than one on that interval.

Henceforth, we consider only those characteristic polynomials with at least one positive and one negative exponent. The negative exponent terms give the polynomial arbitrarily large positive values for \underline{r} just greater than zero. At the other end of the unit interval, the value of the polynomial is one. Notice that the slope at $\underline{r} = 1$ is equal to the expected monetary value of the gamble:

$$ap + bq + \dots$$

Thus, near $\underline{r} = 1$, if the lottery is favorable, then the polynomial approaches one from below, if unfavorable, then from above and if fair, then "head on".

In order for the derivative to change sign on the clopen unit interval, it must pass through zero since the second derivative exists everywhere on the interval. If

$$0 = apr^{a-1} + bqr^{b-1} + \dots$$

then

$$0 = apr^a + bqr^b + \dots$$

The expression on the right has the same exponents as the original characteristic polynomial. This new expression can be viewed as the sum of two parts. One part is strictly increasing and comprises the non-negative exponent terms. The other part, made up of the negative exponent terms, is strictly increasing, but negative. Clearly, these two parts can "cross" in absolute value (and so the sum of the parts is zero) at most once on the clopen unit interval.

From this, we see that there is no solution on the open unit for an unfavorable game. The characteristic polynomial would have to approach the solution from above, and then change direction at least twice to approach the solution at $\underline{r} = 1$ also from above.

A fair game uses its only zero first derivative on the clopen interval at $\underline{r} = 1$. Thus, it lacks a solution on the open interval.

Note that this behavior implies that if \underline{r} is less than one by a finite amount, the characteristic polynomial for fair and unfavorable gambles at that \underline{r} is greater than one.

The only possibility for an open unit interval solution of the characteristic equation is the favorable game. Since the favorable gamble's polynomial approaches one from above for small \underline{r} and approaches one from below at $\underline{r} = 1$, there must be exactly one solution on the open unit interval.

This implies that the characteristic polynomial is greater than one for $\underline{r} = 0$ to the solution (exclusive), and less than one for \underline{r} between the solution and one.

By an appeal to the one-dimensional random walk with an absorbing barrier corresponding to ruin, Feller [29] sorts out the physical meaning of the algebraic solutions. For the fair and unfavorable games, eventual ruin is, indeed, certain. For the favorable case, the probability of ruin is not certain, and so one ignores the formal solution at $\underline{r} = 1$ and uses that unique solution in the open unit interval, assuming that the gamble is not pure-gain, of course.

Generally, solving the characteristic equation requires numerical techniques. Fortunately, the behavior of the characteristic polynomial just discussed allows the search for a solution to proceed efficiently on a computer.

Unfavorable, fair and pure-gain gambles are trivial. For other favorable games, an estimated solution can be checked quickly for whether it is too high or too low. If the characteristic polynomial evaluated at the estimate is greater than one, then the estimate is too low; if less than one, too high.

Note that the same consideration that leads to efficient searches for numerical solutions also provides a quick check for whether a lottery has a probability of ruin greater than some chosen benchmark value.

Suppose, then, that a DM begins to play with capital \underline{w} and has chosen a maximum tolerable probability of ruin \underline{R} . To check whether a gamble exceeds that maximum level for indefinite play, one simply evaluates its characteristic polynomial at $\underline{r} = \underline{R}$. If the polynomial

value is not greater than one, then either the lottery is pure gain or the ruin solution is not greater than \underline{R} . Either way, the gamble has satisfied the chosen constraint on the ruin probability. If the polynomial value at $\underline{r} = \underline{R}$ is greater than one, then the lottery is either fair, or unfavorable, or favorable but with a ruin probability higher than the maximum tolerable value. In any event, the gamble fails to satisfy the constraint.

In actual decision problems, one contemplates playing several different gambles rather than one gamble repeatedly forever.

Consider the lottery with outcomes (a, b) and probabilities (p, q) and a second lottery with outcomes (c, d) and probabilities (s, t). Let the probabilities be such that

$$1 \geq pR^a + qR^b$$

$$\text{and } 1 \geq sR^c + tR^d$$

In words, the indefinite-play ruin probability of playing either lottery alone is within the \underline{R}^W constraint.

The compound gamble that consists of one play of the first lottery followed by an independent play of the second lottery has outcomes (a+c, a+d, b+c, b+d) and probabilities (ps, pt, qs, qt). The characteristic equation of the compound lottery is

$$1 = psr^{a+c} + ptr^{a+d} + qsr^{b+c} + qtr^{b+d}$$

$$\text{or } 1 = (pr^a + qr^b)(sr^c + tr^d)$$

When the characteristic polynomial is evaluated at \underline{R} , both quantities in the parentheses are no greater than one, so their product must be no greater than one.

The result generalizes readily to more than two lotteries and more than two outcomes per gamble. Thus, if the probability of ruin for each

independent gamble in the sequence is bounded by some $\underline{R^W}$, then the sequence as a whole is bounded by $\underline{R^W}$.

Another way to motivate this result is to view some entire sequence as the play of random samples from the set of all lotteries with an indefinite-play ruin probability less than or equal to $\underline{R^W}$. To avoid the complications of infinite sets, imagine that the actual sequence is generated from a large, but finite, "pre-sample" set, and that plays will be drawn from this set with replacement.

Let the probability that the i -th gamble is presented be $\underline{p(i)}$ and the characteristic polynomial of the i -th gamble be $\underline{P(i)}$. The characteristic polynomial for indefinite play of this compound gamble is

$$\sum_i \underline{p(i)}\underline{P(i)}$$

Since all of the $\underline{P(i)}$ are no greater than one, then the polynomial is no greater than one.

Sequences can be constructed, of course, where the overall ruin probability will satisfy some constraint even though individual gambles within the sequence exceed the limit. To construct such sequences requires information about the relative frequencies of the gambles to be encountered.

In the discussion to come, it is assumed that such frequency information is unavailable. So, to satisfy a constraint on the ruin probability in an indefinite sequence of independent gambles, DM will select those gambles that are individually tolerable for indefinite play.

Choice and the Probability of Ruin

Several authors have suggested ways in which the probability of ruin might be used by DM's in choosing among available lotteries. Both Borch [13, 14] and Roy [72] have recommended using the probability of ruin as an objective, i.e., the decision rule they proposed is to minimize the ruin probability. Ferguson [32] has explored particular circumstances where this advice has heuristic value. As a general rule, however, it has implications that are apt to disturb most DM's.

Under such a rule, any pure gain, no matter how nearly unrenumerative, would be preferred to every lottery with a finite probability of any loss, regardless of how lucrative the gamble, how small the potential loss or how remote the probability of sustaining it.

The minimization approach does have the advantage that it can be formulated as an expected utility rule. Roy presents this result by appeal to a utility defined on events rather than on amounts of money. Roy's utility function assigns a value of one to the event that ruin does not occur and a value of zero to the event that it does. Zero-one utilities do have some uses in operations research problems, but one senses that Roy offers this particular utility in a pro forma way. He does not pursue analysis with it.

Borch observes that the probability of survival, the complement of the probability of ruin, can be viewed formally as the expected value of some increasing function of the prizes offered in a gamble. Thus, a decision rule which maximized the probability of survival (and a fortiori minimized the probability of ruin) is an expected utility rule in the conventional sense.

There is a difficulty with ties. Every pure gain would have an

expected utility value of one, and every unfavorable lottery would have a utility value of zero. Borch proposes resolving the latter ties by maximizing the expected survival time, itself interpretable as an expected utility.

Assuming that DM is unwilling to prefer any pure gain to every risky gamble, the probability of ruin is ruled out as an objective. Perhaps it would be more practical to use the probability of ruin as a constraint.

Telser [91] suggests maximizing the expected monetary value of gambles subject to a ruin constraint. Baumol [5] proposes a related criterion. He explores maximizing the expected monetary value subject to a constraint on the probability that one's return falls below some selected value, not necessarily a ruin level. Pyle and Turnovsky [65] have studied the Telser and Baumol rules, along with others that they describe as "safety-first" approaches. They find that although there are circumstances in which such rules can be restated as expected utility maxims, in other circumstances the restatement is strained or impossible.

Snow [88] has shown that a decision rule indistinguishable from maximizing expected exponential utility can be derived from a constraint upon the probability of ruin. The exponential utility has the form

$$U(x) = 1 - e^{-\lambda x}$$

where λ is a positive constant and x can be either the prizes in the gamble being evaluated, or else the after-play wealths.

Suppose that DM starts with initial capital w and has chosen a maximum tolerable probability of ruin R^w which is strictly less than one. Suppose further that DM resolves to accept any favorable lottery that comes along in preference to the status quo (i.e. a prize of zero

for certain), provided that the lottery has a tolerable probability of ruin in indefinite play. This specification is a bit redundant, since any lottery that satisfies the constraint is necessarily favorable. Only favorable lotteries have an indefinite-play probability of ruin less than one.

Now suppose that DM has the opportunity to buy, for a price, the lottery with integer money prizes (a, b, \dots) and associated probabilities (p, q, \dots). The maximum buying price of this lottery is the largest number, \underline{B} , such that the lottery ($a-B, b-B, \dots$) and probabilities (p, q, \dots) is preferred by DM to a prize of zero for certain.

For our DM, this occurs when the characteristic equation

$$1 = pR^{a-B} + qR^{b-B} + \dots$$

is satisfied.

Any price greater than \underline{B} would drive the characteristic polynomial greater than one, indicating a ruin solution greater than \underline{R} , and so violating the constraint. Since the constraint is satisfied at \underline{B} , the diminished buying price lottery is acceptable in preference to zero.

A quick way to turn the buying price equation into an expected utility formula is to make the following additional assumption. Suppose that DM believes that the minimum selling price, or "certainty equivalent" as the minimum selling price is often called, of a lottery should equal its maximum buying price. In that case, the \underline{B} found from the above characteristic equation would be DM's certainty equivalent for the original (a, b, \dots) lottery.

One can then rewrite the characteristic equation by multiplying both sides by \underline{R}^B :

$$\underline{R}^B = p\underline{R}^a + q\underline{R}^b + \dots$$

If \underline{B} is the certainty equivalent for the (a, b, \dots) lottery, then the rewritten characteristic equation defines the utility function

$$U(x) = -R^x$$

The minus sign makes the function increasing in its argument. The rewritten equation says

$$U(\text{certainty equivalent}) \text{ equals the} \\ \text{expected value of } U(\text{prizes})$$

In particular, the rewritten equation defines the exponential utility function. The parameter, λ , of the usual form of that function can be found by solving

$$e^{-\lambda} = R$$

Note that if DM is transitive in exchange preferences and has the ordinary preference among amounts of money for certain (the more, the better), then DM will prefer the lottery with the highest certainty equivalent.

The logic is straightforward. DM is willing to sell a lottery (exchange the lottery for money) at the minimum selling price, the certainty equivalent. A sum of money can be viewed formally as a lottery with one prize. That said, it's easy to show that transitivity for all lotteries leads DM to order lotteries by their certainty equivalents.

The equality of buying and selling prices is a well-known necessary attribute of a conventional exponential utility decision rule. If DM accepts the usual axioms and is risk-averse, i.e., the certainty equivalent of any risky lottery is strictly less than its expected money value, then the equality of buying and selling prices is sufficient for adoption of the exponential utility. See Kemeny and Thompson [46].

Pfanzagl [63] motivates a policy of equal buying and selling prices

independently of other principles of choice. He observes that the gamble with prizes $(a+c, b+c)$ with probabilities (p, q) is in every respect identical to the lottery (a, b) with the same probabilities, combined with a simultaneous side-payment of \underline{c} .

He reasons, therefore, that the certainty equivalent of the $(a+c, b+c)$ lottery ought to be exactly \underline{c} greater than the certainty equivalent of the (a, b) lottery. Pfanzagl calls this proposition the Consistency Axiom.

If the Consistency Axiom holds, then in the case where $(a-c, b-c)$ with probabilities (p, q) is equivalent to zero, the lottery (a, b) with the same probabilities is equivalent to \underline{c} . In other words, the buying and selling prices for the (a, b) lottery are the same.

Although the Consistency Axiom has some intuitive appeal, it may be worthwhile to derive an exponential utility rule without appeal to axioms at all.

Kelly's motivation of a logarithmic utility rule, discussed in the last chapter, did not depend on any of the usual axioms. Of course, when applying logarithmic utility, one conforms to the behavior implied by the axioms.

The Kelly derivation has much to recommend itself in an engineering context, since the decision rule follows from maximizing an articulated objective. Except for assenting to that objective in the first place (and accepting some approximations), no subjective attitudes of DM need to be considered.

Suppose, then, that DM derives the buying price equation based on an acceptance of the indefinite-play ruin assumptions, but balks at adopting any of the conventional utility axioms or Pfanzagl's Consistency

Axiom. DM can still use the buying price furnished by the equation as a guide to choice among gambles.

Let the "net buying price" be the buying price B that appears in the equation, minus any costs actually incurred by DM to acquire the gamble.

If the net buying price is positive, then DM can withdraw that amount of capital from risk and still experience a probability of ruin no greater than the acceptable constraint level. The withdrawal can be made regardless of the outcome of the lottery.

The reason why this is so echoes Pfanzagl's argument for his axiom. The combination of the withdrawal (and payment of any costs) followed by a play of the lottery is in every way identical to a play of the lottery with all of its prizes diminished by the amount withdrawn (and costs).

Indefinitely repeated play of this diminished lottery has a probability of ruin equal to the chosen constraint level.

In general, DM will not be playing the same lottery indefinitely. Rather, DM will be playing a sequence of different independent gambles which, after the withdrawal, will each have a ruin probability equal to the constraint level. The sequence as a whole, therefore, will have a ruin probability of the chosen value, as was shown earlier.

Thus, a policy of choosing lotteries according to their net buying prices will maximize the spendable income that DM can surely realize from this indefinitely sustainable program of risky investment. "Indefinitely sustainable" means that there is a positive probability that DM's capital will never be wiped out, no less than the complement of the chosen ruin constraint. This is a plausible basis for choosing

among gambles.

The situation of a DM facing a forced choice among unacceptable gambles is similar. Here, choosing the lottery with the highest net buying price, i.e., the one with the smallest absolute value negative "price", minimizes the outcome-independent amount of capital that must be raised to maintain the chosen ruin probability.

The buying price equation shows the effect of withdrawing the same amount of money from risk regardless of the outcome of the lottery. This is not the only strategy for making withdrawals that satisfies the ruin probability constraint.

For example, in a pure gain lottery, one could withdraw all of the prize money gained as soon as the outcome is known. Under this scheme, there is no possibility of ruin in indefinite play.

In general, DM can choose amounts $(X(a), X(b), \dots)$ for the lottery with prizes (a, b, \dots) and probabilities (p, q, \dots) in any way that pleases DM so long as

$$U(0) \leq pU(a-X(a)) + qU(b-X(b)) + \dots$$

where $\underline{U}(\cdot)$ is the ruin-constraining utility function. That is, as long as the ruin constraint is satisfied, one can schedule withdrawals in any of several ways.

The question of what schedule yields the "best" withdrawal pattern is a complicated one. There is a temptation to view the problem as a choice among lotteries. The general withdrawal scheme of the last paragraph offers a \underline{p} chance of withdrawing $X(a)$, a \underline{q} chance of taking out $X(b)$, and so on.

Comparing this with some other pattern of withdrawal, e.g. an amount \underline{B} for certain, reminds one of a choice between lotteries.

There is a crucial difference between withdrawal plans and lotteries, however. Money that is won but not withdrawn is nevertheless still in DM's possession. The choice is not between having \underline{B} or having the outcome of the $(X(a), X(b), \dots)$ lottery. The choice concerns only which of DM's pockets will receive the money won in the real lottery.

One distinction of the buying price withdrawal plan is that it does not depend on the outcome of the lottery. Another way to characterize it is as the maximin withdrawal scheme, that is, one which maximizes the minimum amount withdrawn, consistent with the constraint.

The proof of this is easy. Suppose

$$U(0) = pU(a-B) + qU(b-B) + \dots$$

Let $(X(a), X(b), \dots)$ be some other withdrawal scheme. Suppose the minimum of the X 's is greater than \underline{B} . Then, of course, all the X 's are greater than \underline{B} . Since $U(\underline{\quad})$ is an increasing function, then

$$U(0) > pU(a-X(a)) + qU(b-X(b)) + \dots$$

since each term's argument of $U(\underline{\quad})$ is smaller the corresponding argument in the buying price equation. In words, the chosen constraint is not met.

Similarly, the buying price is the minimax withdrawal plan among those plans which just satisfy the constraint. Suppose

$$U(0) = pU(a-X(a)) + qU(b-X(b)) + \dots$$

Then \underline{B} must be no more than the greatest X . Otherwise, the buying price equation won't be satisfied. That is

$$U(0) > pU(a-B) + qU(b-B) + \dots$$

Again, the sense of the inequality is assured by the term-by-term comparison of the arguments of $U(\underline{\quad})$.

The results obtained in this section may be of both practical and

theoretical interest to those who do subscribe to the usual axioms. After all, nothing said here is a reason not to adopt those axioms if DM wants to do so.

As noted before, the conventional axioms provide little guidance for the actual selection of a specific utility function. In practice, assessing a DM's utility function is often an arduous task.

Raiffa [67, chapter 4] describes the usual method for assessing a utility function.

The assessor elicits the subject's preferences among several hypothetical gambles, plots the results and tries to fit a smooth curve to them. If no close fit is obtained, another round of questions ensues. The interview continues until a "reasonable" curve is drawn. Many rounds may be needed.

Meyer and Pratt [60] demonstrate that if the subject can provide even a few constraining features of the shape of the function, then progress can come more quickly. Naturally, progress comes even more quickly if the functional form of the subject's utility curve is known in full to the assessor.

It is possible for a DM who accepts the usual axioms to know that one's utility curve must be exponential. It suffices that DM be risk-averse and also assent to Pfanzagl's Consistency Axiom.

Interviewing about hypothetical gambles is still required to establish the parameter that determines the individual utility function. It is possible, even likely, that DM's first round of responses might not allow a close fit to any exponential curve, so repeated rounds of questions may still be required.

As shown in this section, however, whether DM knows it or not, an

exponential-utility DM will accept all favorable gambles in preference to the status quo unless the indefinitely repeated play of the gamble entails a ruin probability that exceeds some value related to the sought-after exponential parameter.

If DM knows how much capital one is willing to put at risk and can set a constraining probability of ruin, then the exponential utility parameter can be solved for analytically. Preferences among specific gambles need not be explored. The virtue of that fact in this context has nothing to do with a desire to avoid subjectivity, but rather concerns only the tedium involved in dealing with many hypothetical questions.

Note that the ruin level of capital loss has no privilege. DM could be asked to constrain the probability of ever losing any large amount of money. The mathematics leading to the buying price equation will be the same.

For the avowedly "long run" DM, the ruin level does have a particular significance: it is the amount of adversity that prevents DM from reaching the "long run". Even for the long-runner, of course, analysis could be based on the loss of any large sum. It is this flexibility that allowed a certain vagueness about the definition of capital at risk.

Power Law Utility and Ruin Constraints

Suppose that instead of playing the same gamble repeatedly, DM plays the following sequence of gambles. At each play, DM's pre-play capital, x , is multiplied by a positive factor chosen from among $(1+(a/x), 1+(b/x), \dots)$ with associated probabilities (p, q, \dots) . At each step, the cash prizes (a, b, \dots) are adjusted proportionally to DM's current capital so that the ratios $(a/x, b/x, \dots)$ are constant throughout the series.

After any number of plays, DM's capital will be the product of the initial capital w and the factors associated with each of the prizes received. The logarithm of DM's capital, therefore, will be the sum of $\log(w)$ and the logarithms of the prize factors received.

In terms of logarithms, DM's wealth is undergoing an additive random walk analogous to the random walk that arises from repeated play of a gamble with constant absolute prizes.

There are some differences between the random walks. The first is that DM's capital will never reach zero the way that the repeated proportional game has been set up. As in the Kelly analysis, there is no possibility of ruin. Nevertheless, DM can establish some minimal level of capital and designate adversity to that level as "ruin". It is computationally convenient to set that level at one unit of currency, e.g. one cent or one dollar or whatever units the prizes will be paid in. The logarithm of the ruin level is thus zero.

DM can then formulate the ruin problem as the probability that the sum of the logs of the prize factors received is $-\log(w)$, where w is the initial capital expressed as a multiple of the unit chosen as the ruin level.

The second concern is that the prizes' logarithms are generally not integers. The derivation of the characteristic equation for the additive random walk was for integers. If one proceeds by analogy to the additive case with logarithmic steps, then an element of approximation is introduced.

This is of little concern, since the logarithms can be treated as if evaluated to some fixed decimal precision. Thus, to whatever the chosen precision is, one has an "integer" representation. Since the object of the exercise is to obtain a bound, this convention should be entirely satisfactory.

That noted, DM can proceed by analogy to the additive random walk case. The probability that the logarithm of DM's wealth will ever fall to zero, starting from $\log(w)$ and playing the sequence of proportional gambles described earlier is no more than

$$r^{\log(w)}$$

where r is the solution of the characteristic equation

$$1 = pr^{\log(1+(a/x))} + qr^{\log(1+(b/x))} + \dots$$

The bound would be an exact solution for the probability of ruin if the initial wealth were an integer power of an integer base,

$$w = n^m$$

and the prize factors were n and $1/n$, with the probability of the better outcome being greater than one-half. This is just the logarithmic equivalent of the additive lottery with symmetric unit prizes. The integer base assures that the prizes could actually be exactly paid as the game unfolded.

Returning to the general case, assume that DM has chosen some constraining probability of ruin $r^{\log(w)}$, positive and less than one.

A lottery sequence has an acceptable probability of ruin in indefinite play if

$$1 \geq pR^{\log(1+(a/x))} + qR^{\log(1+(b/x))} + \dots$$

Note that since \underline{R} is a constant between zero and one (exclusive), there is some positive λ such that

$$R = e^{-\lambda}$$

so that we can rewrite the expressions in \underline{R} that appear in the characteristic polynomial as

$$\begin{aligned} R^{\log(1+(a/x))} &= e^{-\lambda \log(1+(a/x))} \\ &= (1 + (a/x))^{-\lambda} \end{aligned}$$

Thus, the characteristic polynomial test can be written as

$$1 \geq p(1 + (a/x))^{-\lambda} + q(1 + (b/x))^{-\lambda} + \dots$$

If \underline{P} is the chosen constraining probability of ruin, then it is easy to solve for λ given \underline{w} :

$$P^{\log(w)} = w^{-\lambda} = p$$

$$\text{so, } \lambda = -(\log(P)/\log(w))$$

Just as in the additive case, it is easy to show that if each proportional gamble in a series has an acceptable probability of ruin for indefinite proportional play, then so does the series as a whole.

The characteristic polynomial test can be recast as a power-law utility expression. For a prize \underline{z} , the power-law utility is

$$U(z) = -(x + z)^{-\lambda}$$

where \underline{x} is DM's capital and λ is a positive constant. The minus sign makes $\underline{U}(\underline{\cdot})$ an increasing function of its argument.

In the conventional utility theory, a lottery is preferable to the status quo if and only if its expected utility exceeds the utility of a prize of zero for certain, $\underline{U}(0)$. This condition may be written:

$$-\sum_i p(i)(x + a(i))^{-\lambda} > -x^{-\lambda}$$

where the $p(i)$'s are the gamble's probabilities, the $a(i)$'s are the cash prizes, and the summation is taken over all the prizes.

Dividing both sides of this inequality by $-x^{-\lambda}$ gives

$$\sum_i p(i)(x + a(i))^{-\lambda} / x^{-\lambda} < 1$$

which simplifies to a form of the characteristic polynomial test:

$$\sum_i p(i)(1 + (a(i)/x))^{-\lambda} < 1$$

The sum on the left is just the characteristic polynomial. Since it is less than one, the constraint is satisfied.

Thus, whenever a lottery is acceptable in the conventional expected power-law utility sense, the chosen ruin constraint is satisfied, and so the lottery is acceptable in that sense, too.

Note that maximizing this utility's expected value is not equivalent to minimizing the probability of ruin. To implement a minimizing rule would have the same limitations in proportional play that it faced under simple additive play. A small gain for certain would always be chosen over a lucrative and safe, but not perfectly safe, gamble.

Now suppose that DM wishes to select lotteries in such a way as to constrain the probability of ruin entailed by those choices. DM has two strategies, one based on an exponential relation and the other based on the power law.

Both approaches are equally good at enforcing the constraint. Thus, DM's choice between them must depend on other factors. One way in which the two strategies differ is how different lotteries become acceptable (or not) as wealth changes. To explore this difference, one can use facts

that are known about the exponential and power-law utility functions.

An important source of knowledge about utility functions generally is the theory of "risk aversion functions" developed by Arrow [2] and Pratt [64].

Let $U(\cdot)$ be any twice-differentiable increasing function. The risk aversion function for $U(\cdot)$ is defined as

$$r(x) = -U''(x)/U'(x)$$

The argument x is the total wealth or capital, rather than prizes or changes in wealth.

Arrow and Pratt showed that if the risk aversion function is decreasing in wealth, then the selling price of a given lottery approaches the lottery's actuarial value ever more closely as DM gets richer.

Behaviorally, then, favorable lotteries that are unacceptable at lower wealth tend to become acceptable as wealth increases, if the risk aversion function is decreasing. Increasing risk aversion has the opposite, and somewhat quizzical, character. The wealthier DM becomes, favorable lotteries are apt to become unacceptable even though they were acceptable when DM was poorer. Constant risk aversion means that any lottery that is acceptable at one level of wealth is acceptable at any other.

The attributes of a lottery that are held the same in the preceding discussion are the probabilities and the absolute prizes, i.e. changes in DM's wealth.

It is easy to verify that the exponential utility, $U(x) = 1 - e^{-\lambda x}$, has constant risk aversion, specifically

$$r(x) = \lambda$$

The reader can also readily verify that the wealth-domain version of the power-law utility function,

$$U(x) = x^{-\lambda} \quad \lambda > 0$$

has a risk aversion function

$$r(x) = 1 / (x(\lambda + 1))$$

which is, by inspection, decreasing in x .

A related quantity is the proportional risk aversion function.

It is defined as

$$r^*(x) = xr(x)$$

The interpretation of the proportional risk aversion function is analogous to that of the ordinary (or "absolute") risk aversion function. Suppose one varies the prizes in a gamble as DM's wealth changes proportionally to the current wealth.

Decreasing proportional risk aversion means that some gambles that were unacceptable at lower wealth become acceptable when scaled up with higher wealth. Increasing proportional risk aversion means just the opposite: some gambles that were acceptable at low wealth become unacceptable when scaled up with higher wealth. Constant proportional risk aversion means that gambles acceptable at any wealth level are acceptable at all others if the prizes are scaled proportionally.

The power-law utility function's proportional risk aversion is constant:

$$r^*(x) = xr(x) = 1 / (\lambda + 1)$$

The exponential utility function displays increasing proportional risk aversion:

$$r^*(x) = xr(x) = \lambda x$$

Arrow-Pratt risk aversion theory is a handy tool for theorem

proving and for exploring the properties of selected utility functions. Armed with the results of the Arrow-Pratt analysis, we can qualitatively characterize the difference between the power-law and the exponential ruin-constraint schemes.

The exponential method does not adapt its selections based upon changing wealth. If a lottery was unacceptable for play at the initial capital level, then it remains unacceptable even after capital has doubled or tripled. It remains unacceptable even if DM would take the lottery had the program began at the current wealth.

That is, even if the probability of accepting the gamble indefinitely from now on does not lead to an unacceptable probability of ruin, this does not rehabilitate the lottery.

Conversely, if DM has suffered adversity and capital is now considerably less than it was originally, nevertheless, all the gambles that were acceptable in the beginning remain acceptable. No account is taken that ruin has become considerably more likely.

The power-law method is somewhat more adaptive. As capital increases, a lottery whose money stakes were originally too risky may become acceptable. The rule is not totally adaptive, however. At any finite increased capital, there are going to be lotteries whose ruin probability for indefinite play from now on would be acceptable, but which will be rejected anyway.

The power-law becomes more conservative, in a sense, as risk capital increases. Even though higher stakes gambles are taken as wealth rises, the maximum tolerable probability of ruin from any higher capital is less than the initial value.

The exponential rule grows more conservative, too. Indeed, this

behavior is not confined to the exponential and power-law rules. It is a characteristic of any ruin-constraining rule that is "myopic", that allows indefinite play and that obeys the "dominance principle".

Let us say that a rule is ruin-constraining if, for all values of capital above ruin, the maximum tolerable probability of ruin is less than one. By "myopic", we mean that the rule does not make assumptions about what lotteries will be offered to the DM beyond those currently being evaluated. By allowing indefinite play, we mean that at every finite wealth level greater than ruin, the rule allows a positive probability of loss to be accepted. This, in turn, implies that one always bears some probability of ruin.

Obedience to the dominance principle imposes some common sense on the decision rule. A rule that obeys the dominance principle will always accept any lottery or series of lotteries whose worst outcome is to arrive at a level of capital no lower than the pre-play capital. Similarly, a rule that obeys the dominance principle will always reject a lottery or series of lotteries whose best outcome is to attain a capital no higher than the pre-play level.

It is easy to see that a myopic, indefinite-play, dominance-obedient decision rule's maximum tolerable probability of ruin will be non-increasing in wealth.

Let $P(x)$ be the maximum tolerable probability of ruin under such a rule at capital x . Suppose there were some capital level $y > x$ such that $P(y) > P(x)$.

Since the rule is myopic, we cannot rule out the possibility that DM will be offered the gift lottery of $y - x$ for sure at capital x . By dominance, DM will accept the gift, but then DM will bear the probabi-

lity of ruin $P(y) > P(x)$. Thus, $P(x)$ is not the maximum tolerable probability of ruin for all sequences acceptable under the rule starting from x . That is a contradiction.

Further, there must be some $y > x$ such that $P(y) < P(x)$. By the indefinite play assumption, DM is willing to bear some risk of ruin from x . By dominance, DM will only take such a risk if there is some probability of doing better than x .

Suppose for all $y > x$, $P(y) = P(x)$. Then, DM embarks from x on a sequence of gambles which yields some probability p of ruin, some probability q of doing less well than x , but not so bad as ruin, and some probability $1-p-q$ of doing better than x .

The probability of ruin at ruin is necessarily one. The tolerable ruin probability between x and ruin is no less than $P(x)$ by the non-increasing ruin probability results just shown. By hypothesis, for values greater than x , the maximum tolerable probability of ruin is $P(x)$. Assuming as always independent play, the maximum tolerable probability of ruin from x is greater than $P(x)$, since it is no less than

$$p + qP(x) + (1-p-q)P(x)$$

We recognize that as the weighted average of $P(x) < 1$, and one. Since we can always construct a sequence where p is positive, the expression is greater than $P(x)$ for some acceptable sequence. That is also a contradiction; $P(x)$ emerges as not the maximum tolerable ruin probability.

To summarize, a sensible decision rule that places a non-trivial constraint on the probability of ruin has a maximum tolerable probability of ruin that is nowhere increasing, and above every level of capital, somewhere decreasing in capital.

If DM adopts such a rule, then as risk capital increases, one will

inevitably be at risk of rejecting lotteries whose "from now on" ruin probabilities are equal to the ruin probabilities that were acceptable at the beginning of the program. Note that this has nothing to do with any subjective desire for increased caution as the loss in question escalates.

One is still left with the question of how DM might choose between two different lottery selection approaches which have the same probability of ruin but different behavior as DM's wealth changes. The instantaneous bound on the probability of ruin, just introduced as $P(x)$, provides some useful insights.

It is easy to see that this instantaneous bound is simply

$$P(x) = R^x \quad x \geq 0$$

for the exponential decision rule, and

$$P(x) = x^{-\lambda} \quad x \geq 1$$

for the power law. (Remember that the domains of these two functions differ at the low end by one unit.)

For purposes of comparison, let the initial probability of ruin be the same under each rule,

$$w^{-\lambda} = R^w$$

Both strategies have a value of one at their respective and nearly identical levels of ruin. Both strategies' bounds asymptotically approach zero as capital approaches infinity. Both bounds are strictly decreasing in capital over the entire range between ruin and infinity.

One thing that distinguishes the curves is the speed with which the instantaneous probability of ruin bound climbs in the event of adversity and trails off under the happier circumstance of prosperity. Let us discuss the contrast in the neighborhood of the initial capital.

To aid in discussing the bounds about \underline{w} , it is helpful to note the following identities:

$$R^x = P(w)^{x/w}$$

$$\text{and } x^{-\lambda} = P(w)^{\log(x)/\log(w)}$$

where $P(w)$ is the common initial probability of ruin. These follow immediately from

$$P(w) = R^w$$

$$\text{and } P(w) = e^{-\lambda \log(w)}$$

By inspection, for large \underline{w} , changes in \underline{x} will have much sharper impact on the ratio of \underline{x} and \underline{w} than on the ratio of their logarithms. Thus, moderate adversity causes a much smaller enhancement of the risk-of-ruin bound for the power-law rule than for the exponential. Conversely, moderate success mitigates the ruin bound much more for the exponential than for the power law.

This behavior ought not to be surprising. By playing the same gamble repeatedly regardless of current capital, the exponential rule commits DM to play at a modest scale compared to risen resources, and yet be a "high roller" under adversity.

In contrast, the more dynamically adaptive power-law rule continuously adjusts the scale of gambles taken to reflect current resources.

Another way to think of all this follows. Assuming that DM is free to decline unfavorable gambles, then capital will, on average, tend to increase under either strategy as play unfolds. The exponential rule tends to "front load" more of the total ruin risk of the program into the early plays.

If DM survives the first several plays, then chances are that one

has accumulated additional capital. So, thereafter, ruin is relatively remote under an exponential regime. Adversity on a handful of early plays exposes the program to great danger.

The power law is more forgiving of early adversity, but early success does not so nearly guarantee perpetual success.

These observations do not rise to the level of prescription. The overall probability of ruin is the same under either approach. One DM might prefer to get the "sweaty palms" phase of the program out of the way early. Another DM might be persuaded to endure some suspense in return for the ability to place more of one's capital at productive risk later on.

CHAPTER III

UTILITY-LIKE DECISION RULES

Power-Law Buying and Selling Price Rules

Suppose that a ruin-conscious DM chooses to constrain one's probability of ruin using an expected power-law utility decision rule. Based upon what has been developed so far, DM can distinguish between lotteries that satisfy some chosen ruin constraint and those which do not. This is only a partial ordering. It does not tell DM which of two acceptable lotteries is the "better" one.

If DM accepts the conventional utility axioms, then one could (indeed, must) adopt as a decision rule the maximization of the expected value of the chosen utility function.

On the other hand, the form of the power-law utility function has been derived in such a way that DM need not make any commitment to the usual axioms. Thus, the ruin-conscious DM is at liberty to impose any ordering one pleases on gambles, once the prospects that satisfy the chosen constraint have been identified.

For example, one could follow Telser [91] and choose among feasible lotteries according to their expected monetary values.

Another possibility rests on the observation that for at least one type of lottery there is a natural, uncontroversial ordering. The category is amounts for certain, and the ordering rule is "the more, the better".

One approach, then, to the ordering of risky gambles is to see whether such gambles can be related to amounts for certain. If so, then

perhaps risky gambles could be ordered according to their for-certain relatives.

Conventional utility theory suggests two main techniques for relating amounts for certain to risky gambles. One way is via the theoretical selling price. The selling price, \underline{S} , is the number that satisfies

$$U(S) = \sum_i p(i)U(a(i))$$

That is, the amount of money whose utility value is the same as the expected utility value of the lottery.

Selecting the gamble with the highest selling price is in every way identical to selecting the gamble with the highest expected utility value, since

$$S = U^{-1}\left(\sum_i p(i)U(a(i))\right)$$

where $U^{-1}()$ is the inverse of $U()$. Because $U()$ is an increasing function, so is its inverse. So, the highest selling price belongs to the highest expected utility gamble.

The other main relationship between risky gambles and amounts for certain is the buying price. The buying price is the amount, \underline{B} , that satisfies

$$U(0) = \sum_i p(i)U(a(i) - B)$$

That is, the amount which, when subtracted from each of the prizes, leaves a lottery which DM values the same as the status quo.

It has already been shown that the buying price has some operational meaning within the ruin-constraint framework. It is the most that a ruin-conscious DM would pay to obtain the gamble. It is also an amount available for withdrawal from risk without violating the

chosen constraint.

To simplify the discussion, one can drop the earlier section's distinction between the "net" buying price and the simple buying price. Assume that all lottery prizes are quoted net of any costs of acquisition.

If the utility function is the exponential, then the buying price and the selling price are the same. For the power law, the reader can verify that \underline{B} and \underline{S} will generally differ (except for amounts for sure).

Krantz, Luce, Suppes and Tversky [48] show that, for utilities where the prices differ, the ordering of lotteries based on the buying price will be different from that based on the selling price. The two orderings are distinct and will lead to different choices. Nevertheless, there are many noteworthy similarities between the two orderings.

The buying price and the selling price agree in sign for all utilities, so the two rules select the same lotteries in preference to the status quo.

To see that the prices agree in sign, consider any utility function $V(\cdot)$. If the buying price is zero, then

$$V(0) = \sum_i p(i)V(a(i) - 0) = \sum_i p(i)V(a(i))$$

The prices agree, therefore, at zero. It is trivial to confirm the agreement for other signs, using the agreement at zero and the increasing nature of $V(\cdot)$.

Another similarity between the buying and selling price rules is the avoidance of mixed strategies for selecting lotteries under risk. That is, the maximum buying price or selling price obtainable among a group of lotteries cannot be increased by creating a composite gamble among the alternatives.

This is simple to show for the selling price. One uses the equivalence of the maximum selling price with the maximum expected utility.

Suppose DM faces two or more alternative gambles. Rather than choose the one with the highest expected utility, DM creates a compound gamble which one hopes will have an even higher expected utility (and hence, higher selling price).

Suppose that DM creates the compound gamble by assigning to the i-th gamble the probability q(i) of being selected. The expected utility of the compound gamble is

$$\sum_i q(i) [\text{the expected utility of the } i\text{-th gamble}]$$

That is simply the weighted average of the available expected utilities, and its value cannot exceed that of its highest-valued constituent. Thus, the selling price of the compound gamble can't be higher than that of the highest-selling-price constituent.

To prove the corresponding result for the buying price requires a slightly more oblique argument.

Suppose V() is any utility function and B is the buying price assigned to some lottery by V(). Consider the function

$$F(z) = V(z-B)$$

F() is itself a utility function, being increasing in its argument. The expected value of the F-utility is V(0) for any lottery whose buying price under the V-utility is B. That is,

$$V(0) = \sum_i p(i)V(a(i) - B) = \sum_i p(i)F(a(i))$$

It is easy to see that any lottery whose buying price under V() is less than B will have an expected F-utility less than V(0). Any lottery whose

buying price under $\underline{V}(\cdot)$ is more than \underline{B} will have an expected \underline{F} -utility greater than $\underline{V}(0)$.

Maximizing \underline{F} -utility is not equivalent to choosing lotteries according to their \underline{V} -utility buying prices. The \underline{F} -utility merely establishes a threshold mark that sorts lotteries into three categories based on whether their \underline{V} -utility buying prices are bigger than, smaller than or equal to \underline{B} . The ordering of lotteries within the first two categories will generally be different under \underline{F} -utility from the ordering by their \underline{V} -utility buying prices.

Thus, \underline{F} -utility is not a surrogate for the selection of lotteries by their \underline{V} -utility buying prices, but it is a handy tool for proving theorems about buying price rules. Although it is not obligatory to do so, one can "normalize" the \underline{F} -utility by redefining it as

$$F(z) = V(z - B) - V(0)$$

The subtraction of the constant $\underline{V}(0)$ does not change the ordering of lotteries under expected \underline{F} -utility. It makes the threshold test value zero instead of $\underline{V}(0)$.

Suppose now that \underline{B} is the greatest \underline{V} -utility buying price available among the individual lotteries facing DM. If a mixed strategy could produce a composite gamble with a higher buying price, then the \underline{F} -utility would be greater than zero.

Zero, however, is the expected \underline{F} -utility of the highest buying price lottery among the alternatives. It has already been shown that a mixed strategy cannot increase the expected utility value of any choice beyond what is offered by the highest-expected-utility lottery. Since the \underline{F} -utility of any mixed strategy is at most zero, its \underline{V} -utility buying price is at most \underline{B} .

One straightforward argument in favor of the buying price ordering is that it corresponds to a familiar notion of one thing being better than another. One is usually willing to pay more for the preferred object.

If DM is of an axiomatic turn of mind, then there is Pfanzagl's [63] Consistency Axiom. Recall that this axiom holds that if all the prizes in a lottery are changed by the addition of a constant, \underline{c} , then the value of the lottery to DM changes by \underline{c} .

As explained in an earlier section, the foundation of this axiom is that a lottery with prizes (a, b, \dots) is in every way identical to the lottery with prizes $(a-c, b-c, \dots)$ combined with a simultaneous side payment of \underline{c} . Since the two contingencies are indistinguishable, Pfanzagl would assign the same value to them.

It does not necessarily follow, however, that the value of the combination of the diminished lottery and the side payment is the sum of the values attached to the diminished lottery and the payment separately.

An adherent of the conventional utility theory would argue that the side payment increases DM's wealth. If one's utility function were wealth-dependent, then the value assigned to the diminished lottery would be different depending on whether one's pre-play wealth were some value \underline{x} or $\underline{x} + \underline{c}$.

One need not appeal to psychological preferences to make this argument. Consider the ruin-conscious DM who does not make any withdrawal from one's capital. Suppose that at capital \underline{x} , the diminished lottery just satisfied the chosen constraint. Thus, the value of the diminished lottery at \underline{x} is zero.

At an increased capital like $\underline{x} + \underline{c}$, the same diminished lottery might be less ruinous and so might no longer be valued at zero. The value of the diminished lottery and the side payment, therefore, would be greater than the sum of the values of each component alone.

Although Pfanzagl's axiom is intuitively pleasing, there is no logical difficulty in violating it in the way just discussed. If DM accepts the conventional axioms, then the expected utility value of the original (undiminished) lottery evaluated at capital \underline{x} would be the same as that of the diminished lottery evaluated at $\underline{x} + \underline{c}$, since

$$U(x + a) = U(x + c + a - c)$$

for all lottery prizes \underline{a} .

In words, DM imputes the same value to the undiminished lottery as to the combination of the diminished lottery and the side payment. That value, however, simply isn't the sum of the two components.

Thus, the recognition that a lottery is the same as a diminished lottery cum side payment does not compel the acceptance of Pfanzagl's axiom. Some additional assumptions are needed, e.g., that the side payment and the diminished lottery can be valued independently of one another. Or, to use a term sometimes encountered in the economics literature, that the diminished lottery and the side payment lack "complementarity".

Acceptance of this assumption seems quite reasonable if DM intends to withdraw a sum from risk. The money withdrawn is then unavailable to the capital account to influence the evaluation of the diminished gamble.

It is easy to imagine that such a DM might make the following sequence of swaps. The undiminished lottery is exchanged for its buying price and a suitably diminished lottery. The value of the diminished

lottery to DM is zero, and DM would give it away if allowed to do so. Such a give-away would leave DM with no lottery, and only the buying price of the first lottery to show for the two transactions.

In effect, then, DM has sold the original lottery for its buying price. Thus, the real selling price of a lottery for DM is not the quantity which is conventionally dubbed the selling price, but rather the buying price. One could simply dub the buying price "the price" for such a DM.

Contrast this with DM's behavior if one subscribes to the conventional expected utility ordering. It is a standard result of the Arrow-Pratt theory that the conventional selling price is greater than the buying price for positive-valued lotteries, assuming that DM's utility function is decreasingly risk-averse.

Under a conventional utility ordering, then, DM would be unwilling to trade a gamble whose positive buying price is \underline{B} in order to receive \underline{B} for certain. DM would be unwilling to trade the lottery for any amount less than the theoretical selling price.

If DM intends to withdraw the buying price of lotteries from risk, then adopting a selling price ordering will lead to quizzical choices.

Suppose DM faced a choice between a gamble whose buying price is \underline{B} and some amount \underline{P} for certain, where \underline{P} is greater than the buying price but less than the conventional selling price of the gamble.

Under a buying price ordering, of course, DM would select \underline{P} for certain and enjoy the higher consumption that this choice allowed. With a selling price ordering, DM would choose the risky gamble, even though it offered a lower consumption value and a diminished lottery whose worth DM places at zero (and would give away if one could).

A similar contrast between the buying price and selling price orderings occurs when the DM must make a forced choice among unacceptable gambles. As noted before, for an unacceptable lottery, the buying price is the outcome-independent amount whose absolute value, if added to one's capital, maintains the chosen ruin constraint despite the gamble.

A standard result of the Arrow-Pratt theory is that for unacceptable gambles and decreasingly risk-averse utilities, the selling price is less than the buying price.

Suppose that DM faced a choice between a gamble whose negative buying price is \underline{B} and the penalty \underline{P} for sure, where \underline{P} 's absolute value is between that of the buying and selling prices.

If DM selects the gamble, then one need raise only the absolute value of \underline{B} to keep the ruin-constrained investment program "on track". A buying-price orderer would select the gamble.

Under a selling price selection rule, the penalty for certain is selected, even though more money has to be raised to repair the damage.

To sum up this section, the buying price is a computationally tractable way to relate a risky gamble to an amount of money for sure. It leads to an ordering which, although different from the conventional expected utility ordering, shares some attractive features with that conventional ordering.

Although the buying price ordering is not uniquely "rational" in the sense that it alone conforms to the objectives that a reasonable person might entertain, it is arguably at least as rational as the conventional utility ordering. This especially if DM intends to make outcome-independent withdrawals from capital using the buying price as a gauge of what is permissible within the constraint on ruin.

The Buying Price and the Independence Axiom

Recall from the first chapter the statement of the independence axiom. For any four lotteries A, B, C and D, if A is equivalent to C and B is equivalent to D, then for any probability p, the compound gamble that gives a p chance of A and a complementary chance of B is equivalent to the compound gamble with the same probabilities of C and D.

The independence axiom is highly thought of by many, but not all, decision analysts. As Slovic and Tversky [87] describe the esteem in which the axiom is held, "Many decision theorists believe that the axioms of rational choice are similar to the principles of logic in the sense that no reasonable person who understands them would wish to violate them."

A large part of this esteem is the central role played by the independence axiom in the conventional development of expected utility decision rules. If the axiom did not hold, then the logical foundation of conventional expected utility rules would buckle.

As shown in the last section, and in the earlier work of Kelly, some expected utility decision rules can be developed without reference to the independence axiom. Whether DM subscribes to the axiom or not, certain utilities will guide DM to choose gambles in such a way as to accomplish goals chosen by DM.

"Axiom-free" motivation of utility functions would be especially interesting if it allowed DM to do things with utility functions that would not be possible under the conventional axioms.

One such thing has already been discussed. DM may, for good reason, choose to order gambles according to their buying prices, prices defined in terms of a utility function.

As explained in Snow [89], unless the utility function is linear

or exponential, the buying price ordering will violate the independence axiom. Thus, use of the axiom to derive the buying price equation would lead to contradiction if DM wished to use the buying price ordering.

To see that the buying price ordering violates the axiom, let $V(\cdot)$ be some utility function other than the linear or the exponential. For definiteness, assume that $V(\cdot)$ displays decreasing risk aversion. Actually, any departure from constant risk aversion, the exclusive domain of the linear and exponential functions, would do.

Let A be a lottery, $(q: a_1, a_2)$ in our earlier notation for two-outcome lotteries. The lottery offers money prize a_1 with probability q and money prize a_2 with probability $1-q$. Let the buying price of A be some positive number c .

Let d be the buying price of the lottery $(p: c, 0)$, where p is not zero and not one, and c is the buying price of A , as just defined. Clearly, d is positive and less than c .

If the independence axiom held for the buying price ordering, then the buying price of the lottery $(p: A, 0)$ would be d . It is not. Let us assume for the sake of proof by contradiction that the buying price of the $(p: A, 0)$ lottery is d .

Writing this assumption as an equation,

$$(1) \quad V(0) = p(qV(a_1-d) + (1-q)V(a_2-d)) + (1-p)V(-d)$$

and, because d is the buying price of $(p: c, 0)$,

$$(2) \quad V(0) = pV(c-d) + (1-p)V(-d)$$

but, since c is the buying price of $(q: a_1, a_2)$,

$$(3) \quad V(0) = qV(a_1-c) + (1-q)V(a_2-c)$$

Because $V(\cdot)$ is decreasingly risk averse, if one adds $c - d$, a positive quantity, to each of the prizes in (3), one can apply the result

of Pratt [64] to derive the inequality

$$V(c-d) < qV(a_1-c+c-d) + (1-q)V(a_2-c+c-d)$$

$$\text{or } V(c-d) < qV(a_1-d) + (1-q)V(a_2-d)$$

but that contradicts the equality of the right-hand sides of (1) and (2). A similar proof could be made for an increasingly risk-averse utility function.

The buying price ordering generally violates the independence axiom, as just shown. The buying price ordering nevertheless appears to be a rational choice for at least some DM's. Thus, the independence axiom appears not to be a necessary attribute of rationality for decision making under risk.

Its violation of the independence axiom sets the buying price ordering apart from the conventional selling price ordering. Yet, we have seen that the two price-based orderings share many attributes.

To delve deeper, note first that the buying price equation, like the selling price equation, is linear in the probabilities. Historically, interest in decision rules that are linear in the probabilities goes back to Pascal and Fermat and the simplest "utility" rule: order gambles by expected monetary value.

More recently, interest in linear-in-the-probabilities rules has stemmed from the solution of two-person zero-sum games by linear programming techniques. Specifically, the root is von Neumann and Morgenstern's [98] generalization of von Neumann's [97] result that every matrix game has a solution in the "saddle point" sense discussed in the first chapter. A saddle point solution, that is, "on average".

If two players have utility functions $U(\cdot)$ and $V(\cdot)$ such that

$$U(x) = -V(-x)$$

then each player faces a linear programming problem which is the dual of the other player's problem.

This result, due to von Neumann and Morgenstern, has long been known for the conventional selling price ordering. It can also be shown that the same result holds if each player orders lotteries by their buying prices based on utility functions $\underline{U}()$ and $\underline{V}()$ that are complementary in the sense described in the last paragraph.

Assume that $\underline{U}()$ is the utility function of the maximiner and that this DM is playing Row. Row wishes to assign probabilities to its strategies to maximize the minimum buying price, \underline{B} , available among the resulting lotteries.

This problem obviously has a solution. Applying brute force, one could guess a trial solution, diminish all the prizes in the game matrix by that amount, evaluate the $\underline{U}()$ value of each cell, and solve the usual linear programming problem. If the worst column lottery's expected $\underline{U}()$ value is $\underline{U}(\underline{0})$, then the solution is found. If the worst expected utility is bigger than $\underline{U}(\underline{0})$, then guess a bigger \underline{B} ; if smaller, then guess smaller. Repeat until a solution is found.

Consider now the \underline{F} -utility function introduced earlier,

$$F(z) = U(z-B) - U(0)$$

If the prizes in the original lottery are replaced by their \underline{F} -utility values (for the correct \underline{B}), then the maximin expected \underline{F} -utility value to Row is zero.

If Column now seeks its solution of the \underline{F} -utility matrix, it finds a minimax value of zero. That is the content of the von Neumann and Morgenstern theorem.

If one minimaxes expected $\underline{F}(z)$, one maximins expected $-\underline{F}(z)$

when faced with the \underline{F} -utility matrix. We can write $\underline{-F}(z)$ as

$$-U(z-B) + U(0)$$

Since $\underline{U}()$ and $\underline{V}()$ are complementary,

$$-F(z) = V(-z+B) - V(0)$$

One can define a new utility function

$$G(z) = V(z+B) - V(0)$$

and thus write

$$-F(z) = G(-z)$$

Note that $\underline{G}()$ is the utility function that assigns an expected utility of zero to lotteries with \underline{V} -utility buying prices equal to $\underline{-B}$, and positive expected utilities to those with greater buying prices.

That the maximin value of the \underline{G} -utility evaluated at the "negative" prizes (actually, minus one times the prizes offered to Row) is zero allows the conclusion that the buying price of the maximin gamble is $\underline{-B}$.

Thus, the theoretical duality of the opponents' strategy problems in two-person games holds for buying price decision makers. Either player can guarantee oneself a buying price regardless of what the other player does. The guaranteed buying prices offered to the two players are equal in magnitude and opposite in sign. This parallels the situation under conventional utility, where the selling prices guaranteed to the opponents are also equal in magnitude and opposite in sign.

The independence axiom is not necessary, therefore, for matrix games to have dual solutions for the adversaries.

Note that the practical import of the dual analysis of games is limited. If $\underline{U}()$ is a concave function, then its complement must be convex, as can be confirmed by differentiating the defining relation

twice:

$$V(x) = -U(-x)$$

$$V''(x) = -U''(-x)$$

Since $U''()$ is negative everywhere (by the concavity of $U()$ as assumed), $V''()$ is positive everywhere, and so $V()$ is convex.

Generally, one would be surprised to find a globally risk-seeking opponent in a game. More likely, both players will have utility curves that are concave. The problems faced by the opponents, therefore, will lack the compelling symmetry afforded by duality.

This difficulty is shared by both buying price and selling price rules.

The practical point is that the buying-price DM who plays a matrix game and who is in ignorance of the opponent's utility function can employ what is widely regarded as the optimal strategy for that circumstance. That is, DM can play a maximin strategy.

Instead of maximizing the selling price, however, DM maximizes the buying price. The virtue of the strategy is the same in either case. DM can impose a floor on the worst lottery ("worst" according to DM's decision criterion) that the other player can foist on DM.

The independence axiom, therefore, is not necessary for the pursuit of game theory along very nearly conventional lines. No more than it is a necessary attribute of rationality, no more than it is necessary for motivating utility-like decision functions in the first place.

That said, the independence axiom can be recommended as a simplifying assumption in the interest of computational convenience. When the buying-price DM solves a game, one must generally guess a trial solution,

solve a diminished matrix by linear programming, adjust one's guess as needed and repeat until the desired precision is attained. The selling-price DM need solve only one matrix and quit.

Beyond games, the independence axiom also imparts convenience in the solution of complex decisions under risk. These problems are characterized by lotteries whose pay-offs are not restricted to prizes or other lotteries, but can include opportunities to choose among further lotteries.

In the conventional theory, DM solves such problems by the so-called "average and fold back" algorithm. One locates those choices which, if offered, involve deciding between amounts for certain or lotteries with money prizes only. DM calculates how one would decide if such a choice were offered.

Under conventional expected utility, one can then ignore any sequelae of a decision that aren't chosen. What had been a decision in the original formulation of the problem becomes either an amount for certain or a simple lottery.

DM can repeat the process until one faces a single decision juncture with two or more options. A full description of the algorithm is found in Raiffa [67].

As discussed in Snow [89], the method works because of the independence axiom. It doesn't work for buying-price decision makers. Such problems can still be solved. Again, one guesses a buying price for the entire complex decision, solves the problem using the appropriate F-utility, adjusts one's guess and repeats.

Abandonment of the independence axiom causes more work for DM. As a simplifying assumption, then, the axiom has considerable power.

Bounded Utilities and Ruin Constraints

The two special utility functions developed so far illustrate that expected utility decision rules need not be based upon the conventional axioms. DM's desire to place a bound on one's probability of ruin can lead to rules indistinguishable from the maximization of conventional expected utility. DM's ruin consciousness can also be used to motivate utility-based decision rules that are somewhat different from the conventional ones, namely buying price orderings.

Both functions, the exponential and the power-law, serve equally well to place a bound on the ruin probability. The choice between them turns on the strategic implications of their different shapes.

Similarly, both the buying price and selling price orderings comport with the goal of a ruin constraint. The choice between them is a matter of whether DM wishes to realize income from one's risky investment, or prefers to accumulate capital instead.

In this section, the repertoire of available shapes for ruin-constraining utility functions will be substantially broadened.

To simplify the discussion, all the utility functions in this section will be viewed as functions of DM's total capital, rather than as functions of the prizes offered by lotteries.

From the axioms presented in the first chapter, the most obvious restriction on a conventional utility function is that it be strictly increasing. Arrow [2, 3] argued that a conventional utility function must also be bounded above and below, although his prescription is not universally accepted.

The class of ruin-constraining utility functions turns out to be less rich than the set of all increasing functions, but richer than

Arrow's utility functions, since it doesn't matter whether or not the function is bounded below. Recall that neither the exponential nor the power-law functions already derived is bounded below.

To explore the class of ruin-constraining utility functions, one must take care to properly designate what level of capital constitutes ruin. A ruin level of zero, for example, turned out to be a poor choice for the power-law utility, since the function's value at zero is minus infinity. A ruin level of one capital unit yielded more sensible results.

Generalizing this experience, suppose that $V(\cdot)$ is an increasing function. Let \underline{g} be some number such that $V(\cdot)$ is defined and finite for all real numbers greater than or equal to \underline{g} . Suppose that \underline{g} is chosen as the ruin level of capital.

In order that $V(\cdot)$ constrain the probability of ruin to a value other than one when used as a utility function, it is both necessary and sufficient that $V(\cdot)$ be bounded above. Naturally, the concept of a constraint upon the ruin probability only makes sense for capital values greater than \underline{g} , the chosen ruin level of capital.

To show sufficiency, suppose $V(\cdot)$ is bounded above. Let \underline{z} be the least upper bound of $V(\cdot)$. Define the function $U(\cdot)$ as

$$U(y) = \frac{V(y) - V(\underline{g})}{\underline{z} - V(\underline{g})}$$

where y is any element of the domain of $V(\cdot)$. Clearly, $U(\cdot)$ is an increasing function with $U(\underline{g}) = 0$, bounded above by one. The function $U(\cdot)$ may be interpreted as a utility function. Since $U(\cdot)$ is obtained from $V(\cdot)$ by an increasing linear transformation, it is a standard result that $U(\cdot)$ and $V(\cdot)$ are strategically equivalent. That is, the buying and selling prices of any lottery are the same under both utilities.

Define the function $R()$ as

$$R(y) = 1 - U(y)$$

where once again, y is any element of the common domain of $U()$ and $V()$.

The function $R()$ is strictly decreasing with $R(g) = 1$, bounded below by zero.

Suppose DM embarks on an indefinite-duration program of participation in risky gambles with initial capital w , which is strictly greater than g . Suppose further that DM plays only those gambles that are acceptable under the usual expected utility conventions with $U()$, and hence $V()$, as one's utility function.

Assume that all the gambles are independent trials, although the stakes may vary with current capital if that is allowed by the utility function.

If DM makes no withdrawals from capital (nor deposits any new capital), then Snow [90] shows that the probability that DM's capital ever falls to g or lower, i.e. the probability of ruin, is not greater than $R(w)$, which is less than one.

The proof is by induction. First, it is shown that for all x greater than g , no single gamble which offers a probability greater than $R(w)$ of falling in capital from x to g in one play is acceptable.

A gamble which offers after-play levels of capital (a, b, \dots) with probabilities (p, q, \dots) is acceptable at capital x if and only if

$$U(x) \leq pU(a) + qU(b) + \dots$$

The lottery which gives a probability $r > R(x)$ of falling to g and a complementary probability of rising to some other capital $h > g$ has expected utility

$$rU(g) + (1-r)U(h)$$

which is less than $1 - \underline{R(x)}$ since $\underline{U(g)} = 0$, $\underline{U(h)} \leq 1$ and $1 - \underline{r} < 1 - \underline{R(x)}$. But $U(x) = 1 - R(x)$, so

$$U(x) > rU(g) + (1-r)U(h)$$

and the lottery is unacceptable.

Note that we need not be concerned about the possibility of after-play capital less than \underline{g} . Clearly, a lottery offering a probability of arriving at \underline{g} or less which exceeds $\underline{R(x)}$ will be unacceptable, too.

This demonstration with a two-outcome lottery generalizes readily to gambles with more outcomes.

Next, one considers sequences of acceptable lotteries. A sequence can be built in a tree-like way. The root of the tree is the first acceptable gamble that DM faces. For each of its outcomes, there is a successor lottery to be played from the new level of capital. The outcomes of these lotteries each have their own successors, and so on, as long as DM's capital is greater than \underline{g} . An outcome that leaves DM with capital of \underline{g} or lower has no successor; DM is ruined.

Let the length of a sequence of lotteries be one more than the greatest number of successors to the original lottery along any path through the "tree".

Suppose that one has such a sequence of length \underline{N} that starts from capital \underline{w} and which has terminal capital levels (a, b, \dots) including \underline{g} . The probabilities of ending up at each of the possible terminals are determined by the acceptable lotteries that make up the sequence. Let \underline{r} be the probability of finishing the sequence at \underline{g} .

Since each component lottery is acceptable, the compound probability property of expected utility requires that

$$U(w) \leq pU(a) + qU(b) + \dots + rU(g)$$

where (p, q, \dots) are the various probabilities of the non-ruinous outcomes.

One then constructs a sequence of length $\underline{N} + 1$ from this acceptable \underline{N} -sequence. To each terminal, except \underline{g} 's, append acceptable lotteries $(L(a), L(b), \dots)$. These yield new terminal capital levels (a', b', \dots) with overall probabilities from the root of the tree (p', q', \dots) . Let \underline{r}' be the new probability of ending up at \underline{g} .

Since all the additional lotteries are acceptable,

$$\begin{aligned} p'U(a') + q'U(b') + \dots + r'U(g) &\geq \\ pU(a) + qU(b) + \dots + rU(g) &\geq U(w) = 1 - R(w) \end{aligned}$$

But since $\underline{U}(\underline{\quad})$ is bounded above by one, if $\underline{r}' > \underline{R}(w)$ then all the other primed probability terms must sum to less than $1 - \underline{R}(w)$. If so, then with $\underline{U}(g) = 0$, the expected utility of the sequence of length $\underline{N} + 1$ would be less than that of the sequence of length \underline{N} . This cannot be, since the longer sequence was derived from the shorter by the addition of acceptable lotteries.

Thus, we may conclude that a bounded-above utility function $\underline{V}(\underline{\quad})$ will suitably bound the probability of ruin, since $\underline{V}(\underline{\quad})$ accepts the same lotteries as $\underline{U}(\underline{\quad})$.

Necessity is relatively simple: it is straightforward to show that an unbounded utility will not bound the probability of ruin to be less than one.

Let $\underline{T}(\underline{\quad})$ be an increasing function that is not bounded above, which is defined for all $\underline{x} \geq \underline{g}$ or for all \underline{x} between \underline{g} and whatever finite argument, if any, leads to an infinite value of $\underline{T}(\underline{\quad})$. Without loss of generality, let $\underline{T}(\underline{g}) = 0$.

For any probability \underline{p} less than one, there is some capital level \underline{h}

such that DM would accept the lottery which offers a \underline{p} probability of \underline{g} and a complementary chance of attaining \underline{h} . One can find \underline{h} by writing the expected utility equation

$$T(w) = pT(g) + (1-p)T(h)$$

$$T(h) = T(w) / (1-p)$$

$$h = T^{-1} [T(w) / (1-p)]$$

Since $T(\cdot)$ is increasing and unbounded, the inverse (or better) exists. Thus, $T(\cdot)$ offers no upper bound on the probability of ruin besides one.

Returning to bounded utilities, Snow [90] goes on to show that $R(w)$ is the least upper bound on the probability of ruin in the sense that for any positive $\underline{r} < R(w)$, there exist sequences acceptable to DM whose probability of ruin is at least \underline{r} .

It suffices to show that there is an individual lottery acceptable to DM whose one-step probability of ruin is \underline{r} . Clearly, the indefinite play probability of ruin for a sequence cannot be less than the probability of attaining \underline{g} on the first trial.

We construct an acceptable two-prize lottery which gives \underline{g} with probability \underline{r} and a complementary probability of some other capital level \underline{h} . We seek \underline{h} such that

$$U(w) = rU(g) + (1-r)U(h)$$

Since $U(g) = 0$

$$\begin{aligned} h &= U^{-1} [U(w) / (1-r)] \\ &= U^{-1} [(1-R(w)) / (1-r)] \end{aligned}$$

and since $\underline{r} < R(w)$, the last functional argument is less than one, so the inverse (or better) exists, if $U(\cdot)$ is defined as before.

If the utility function is really defined over all capital levels at or above \underline{g} , then the required \underline{h} may be an unattainably large amount.

This can be remedied by relaxing the requirement that $\underline{V}()$ be defined for all \underline{x} not less than \underline{g} . It is enough that it be defined for all \underline{x} between \underline{g} and some number \underline{h}^* (inclusive), where \underline{h}^* is the largest attainable amount of capital.

The upper bound on $\underline{V}()$ becomes $\underline{z} = \underline{V}(\underline{h}^*)$. The utility function $\underline{U}()$ can be obtained from $\underline{V}()$ by the same formula as before.

This \underline{h}^* assumption does not affect the proof that $\underline{R}(\underline{w})$ is an upper bound on the probability of ruin.

If the assumption is adopted, then all increasing functions defined and finite on the specified \underline{g} to \underline{h}^* interval become bounded.

The value $\underline{R}(\underline{w})$ remains a least upper bound on the ruin probability, but now it becomes possible to realize a one-step lottery with $\underline{r} = \underline{R}(\underline{w})$ as the probability of attaining \underline{g} , since the value of the inverse of $\underline{U}()$ at one will be \underline{h}^* . Thus, DM would accept a lottery that gives \underline{g} with probability $\underline{R}(\underline{w})$.

Of course, DM would require a complementary probability of getting \underline{h}^* . Since by assumption, \underline{h}^* is the greatest capital level attainable, this lottery will be DM's last, win or lose, at least for a while!

Turning to other concerns, note that since $\underline{R}(\underline{w})$ is an upper bound on the probability of ruin, $\underline{U}(\underline{w})$ may be interpreted as a lower bound on the probability of survival.

Note also that the initial capital is not the only capital level for which ruin and survival probabilities have meaning. For all \underline{x} at or above \underline{g} (through the highest attainable capital level), $\underline{U}(\underline{x})$ and $\underline{R}(\underline{x})$ bound the survival and ruin probabilities for a program of risky investment once the capital level of \underline{x} is attained.

Bounded utilities offer only one way to achieve an upper bound on

the probability of ruin. It is possible to construct sequences that include "unacceptable" gambles and yet which have, over all, an acceptable probability of ruin.

The usefulness of a bounded utility approach is that it bounds the ruin probability "myopically", that is, without knowledge of what gambles will be offered in the future.

The class of ruin-constraining utility functions is broad indeed. To apply the expected utility technique, DM must select a particular utility function.

If DM assents to the usual axioms and is known to prefer a particular family of well-behaved functions (exponential, power law, etc.), then the application of the ruin bounding technique is straightforward.

Otherwise, the selection of a specific utility curve requires the choice of a shape for the curve as well as a tolerable ruin constraint. A widely used guide to the behavioral implications of various shapes is the Arrow-Pratt [2, 64] risk aversion theory.

Another possible criterion is the way that the probability of ruin changes as capital varies. The qualitative differences between the power-law and the exponential curves have already been discussed in this light.

Still another way to approach the shape question is to study the probability of falling from one wealth level to another, for starting points other than the initial wealth and destinations other than ultimate ruin.

This reflects common sense. Just as DM might be concerned about the probability of ruin, so, too, one might be concerned about the chances of disastrous, but less than ruinous, adversity. Or, one might be concerned that once a particular level of wealth is attained, one is not

unduly at risk of losing a lot of it back.

Let $U(\cdot)$ be a utility function normalized as before: bounded above by one and with $U(g) = 0$. If \underline{s} and \underline{d} are capital levels such that \underline{d} is less than or equal to \underline{s} , but not less than \underline{g} , and \underline{s} is in the interval over which $U(\cdot)$ is well-defined, then the probability of falling from \underline{s} to \underline{d} once \underline{s} is attained is no more than

$$\frac{1 - U(\underline{s})}{1 - U(\underline{d})}$$

while playing acceptable gambles. The proof is the same as that for \underline{w} and \underline{g} . Of course, if $\underline{s} = \underline{w}$ and $\underline{d} = \underline{g}$, then the bound is the same as that proved earlier, $1 - U(\underline{w})$.

This generalized bound can be exploited to find utility curves that accomplish patterns of ruin constraints desired by DM. For example, suppose DM wished that the maximum probability of ever falling back from any attained wealth level by a fixed absolute amount be constant. That is,

$$\frac{1 - U(\underline{d} + \underline{a})}{1 - U(\underline{d})}$$

be constant for all attainable \underline{d} and constant \underline{a} . It is easy to confirm that the exponential utility will serve.

If DM were concerned about the probability of ever falling from any attained capital by a fixed proportion $1/\underline{a}$, where $\underline{a} > 1$, and wishes a constant bound on that prospect, then DM wants a utility where

$$\frac{1 - U(\underline{a}\underline{d})}{1 - U(\underline{d})}$$

is constant. The power law obeys that requirement.

Rather than these global patterns, DM might be interested in constraining the probability of attrition between selected pairs of \underline{s} and \underline{d} values.

Thus, the power of expected utility rules to constrain the probability of adversity not only provides a rationale for expected utility maximization outside the conventional axioms, it may also help DM to select a specific curve.

CHAPTER IV

DECISIONS INVOLVING GOALS

Ruin in Two-Barrier Random Walks

There may be some amount of money beyond which DM declines to apply the same decision rule for lotteries that served at lower levels of capital. This may be because the amount in question is a goal of the investment program: its attainment was why DM undertook to accept risks.

Maybe the amount is simply so big that DM wishes to reserve the prerogative to change investment goals, or at least pause to see whether new objectives might better fit DM's favorably altered circumstances.

Another possibility arises when the decision rule is not being applied personally by DM, but rather by an agent. The agent, who may be another human being or perhaps a computer program, acts on DM's behalf, maybe without supervision by DM. DM's chosen decision rule may serve as a convenient way for DM to give objectively verifiable guidelines about how the agent ought to handle DM's affairs.

In such a case, there might be any number of reasons why DM might prefer that the grant of agency power be limited in scope rather than plenipotentiary.

Still another possibility is that the decision rule may have a tendency to "break down", or give less satisfactory performance as capital increases. The exponential utility decision rule, for instance, plays the same gambles at great values of capital as at the original

capital. Luce and Raiffa [50] find this insensitivity to changing wealth unlikely to be desired by DM's. Krantz, Luce, Suppes and Tversky [48] note that the exponential function is bounded above, but is both unbounded and rapidly decreasing below. For gambles with substantial gains and losses, then, "unreasonably large" gains are needed to compensate for relatively small losses. Otherwise, gambles with small-to-moderate losses would be unacceptable.

This behavior may make good sense for a ruin-conscious DM with moderate means. It may make less sense after substantial gains have occurred, but the exponential will continue to do the "right thing" long after the need has passed.

Since the exponential curve is concave, it will restrict DM to favorable lotteries. Thus, barring ruin, capital will tend to increase over time. As already discussed, the "from now on" probability of ruin will fall off rapidly with increasing capital. So, if DM plays long enough, then one may be refusing gambles that are negligibly ruinous at current capital levels, however foolhardy (or intrepid) the same gambles might have been when DM began the program of risky investment.

DM does not necessarily have to do anything about this possible problem. "Indefinitely" does not have to mean forever. It can be interpreted simply as DM's desire to play a certain strategy until DM decides to play another.

There is no reason at all why, on any given day, DM could not simply end the current program and begin to play another, itself subject to interruption whenever DM subsequently pleases.

On the other hand, the theoretical machinery is available for DM to set a fixed upper wealth limit to the original program, if DM so

chooses. In the discussion to come, such an upper wealth limit will be called a "goal" or "horizon".

Up to now, DM's capital has been modeled as a random walk on a single dimension with one absorbing barrier at ruin. The model can easily be extended to add a second absorbing barrier at the goal level. With such a two-barrier model, DM will be interested in the probability of being ruined before reaching the goal.

Two barrier, one dimensional random walks have long been objects of theoretical interest. Suppose a player starts with capital w , a ruin level g which is less than w and a goal G which is bigger than w . Let $P(w,G)$ be the probability of falling from w to g before attaining G .

If DM takes only gambles that are just fair (expected money value of zero) in independent trials, then $P(w,G)$ is easy to find analytically. One can appeal to Coolidge's [21] result that any series of fair gambles is itself a fair compound gamble.

That is, the random walk is equivalent to a lottery offering the ruin capital of g with probability $P(w,G)$ and the goal capital of G with the complementary probability. Ignoring "overshoot" (the possibility that the last gamble in the sequence takes DM below g or above G), in order for the compound gamble to be fair,

$$P(w,G)g + (1 - P(w,G))G = w$$

Or,

$$P(w,G) = (G - w) / (G - g)$$

For more general lotteries, the computation of an exact ruin probability is more difficult. Feller [29] reviews many of the standard two-barrier results and techniques.

For the DM who doesn't know the actual gambles to be faced, a bound

on the probability of ruin before attaining the goal may be useful. Some care in specifying what is to be bound is needed.

The worst case probability of ruin from w before attaining G using a utility decision rule is $R(w)$, the bound without considering the goal. The reason is that perhaps no gamble will be offered that gives the capital level G as an outcome; perhaps only gambles that offer g and capital values very much larger than G will be seen.

To capture the idea of a goal with a utility rule, one must modify the utility function so that for all $x > G$, $U(x) = U(G)$. The thought is that once G is attained or exceeded, a new utility function will be adopted. The constant utility function arc will never be used for decision making once G is achieved. It serves only to assure that no greater risk of ruin is taken in pursuit of prizes bigger than G than is taken in pursuit of G itself.

The modified utility curve can be rescaled so that the new curve, $T()$, stands to the old function, $U()$, as

$$\begin{aligned} T(x) &= U(x) / U(G) \text{ for } x \leq G \\ &= 1 \quad \text{for } x > G \end{aligned}$$

Clearly, $T(G) = 1$, and if $U()$ is normalized so that $U(g) = 0$, then $T(g) = 0$, too.

By the same arguments as in the last chapter, the probability of falling to g from w before attaining G will be no more than $1 - T(w)$.

If we define

$$Q(x) = 1 - T(x)$$

then, in terms of

$$R(x) = 1 - U(x)$$

we may write

$$\begin{aligned}
 Q(x) &= 1 - T(x) \\
 &= 1 - \frac{U(x)}{U(G)} \\
 &= 1 - \frac{1 - R(x)}{1 - R(G)} \\
 \text{or } Q(x) &= \frac{R(x) - R(G)}{1 - R(G)}
 \end{aligned}$$

This same result could be motivated in a different way if it were known that \underline{G} , if it were attained at all, would be attained exactly without overshoot. If $Q(w)$ were the worst possible probability of ruin from w without attaining \underline{G} , then

$$R(w) = Q(w) + (1 - Q(w))R(G)$$

In words, the indefinite play bound must be the sum of the probability bounds of two mutually exclusive and exhaustive events. Either one is ruined before attaining \underline{G} , or else one attains \underline{G} and is ruined thereafter. Assuming, as always, independent trials, the probability of the latter compound event will be the product of its constituents.

With the above equation motivated, straightforward algebraic rearrangement yields the expected

$$Q(w) = \frac{R(w) - R(G)}{1 - R(G)}$$

Note that if there were no overshoot, it would be unnecessary to modify the utility curve in any way. Provision for overshoot results from the decision rule's myopia; overshoot is just one more attribute of the gambles to be faced about which DM is assumed to be ignorant.

In practice, if $R(G)$ is very nearly zero anyway, then there is little cause to modify the utility curve, overshoot or not. For example, in the exponential case, $R(G)$ is very much less than $R(w)$ even if \underline{G} is only moderately bigger than w . Since $U(G)$ is very nearly one when $R(G)$

is very nearly zero, the utility curve hardly needs to be rescaled. With or without the rescale, $Q(w)$ will be close to $P(w)$.

In the exponential case, the introduction of a horizon goes a long way towards resolving the objection that the decision rule doesn't adjust well to changes in wealth. Come the horizon, if it ever does, DM simply resolves to rethink the entire program. In the meantime, the existence of the goal leaves the form and the parameter of the utility curve all but unchanged, even with a rescale.

Of course, there is nothing to prevent a DM who tolerates a certain indefinite-play probability of ruin from adopting a horizon, but not recomputing one's decision rule at all. Taking such a step is not so empty of content as it might at first appear.

If the rule is being executed automatically, either by an unsupervised personal agent or by machine, then there is a foreseeable circumstance where DM might want to ensure that one is consulted for further instructions.

That circumstance concerns the availability of lotteries so favorable as to constitute "once in a lifetime" opportunities. The next section explores that possibility at greater length.

In any event, ruin-bounding utility decision rules can easily be adapted to include an upper limit on the wealth range over which they operate. This distinguishes such rules from conventional expected utility rules that lack any provision for such limits.

Goals and the Allais Problem

Allais [1] asked several "very prudent" people about their preferences in two gambling choice problems. In this retelling, the amounts of money are stated in millions of dollars; in the original the amounts were hundreds of millions of French Francs.

The first choice problem is to decide between taking one million dollars for certain (option A), or else taking a lottery that offers a ten percent chance of five million dollars, an eighty-nine percent chance of one million dollars, and a one percent chance of no change in wealth (option B).

The second problem is to choose between two lotteries. One offers an eleven percent chance of receiving one million dollars against an eighty-nine percent chance of no winnings (option C). The other offers a ten percent chance of getting five million dollars, and a complementary chance of winning nothing (option D).

Allais reports that the majority of his respondents (he does not report the number of subjects involved, but that doesn't really matter much) prefer A to B in the first problem, i.e., the million dollars for sure, and prefer D to C in the second problem, i.e., the more lucrative expected monetary value lottery.

These choices cannot be reconciled with the maximization of any conventional expected utility. If $V(\cdot)$ is a utility function, the preference for A over B would indicate

$$V(1) > .89V(1) + .10V(5) + .01V(0)$$

where the arguments of $V(\cdot)$ are given in millions. We can rearrange this inequality to read

$$.11V(1) > .10V(5) + .01V(0)$$

The preference for D over C, however, betokens

$$.10V(5) + .90V(0) > .11V(1) + .89V(0)$$

This inequality can be rearranged to read

$$.11V(1) < .10V(5) + .01V(0)$$

which contradicts the earlier inequality derived from the other expression of choice.

Allais interprets these results as evidence of a preference for certainty over chance, at least when large sums of comparable size are involved. Allais finds such a preference rationally defensible.

Tversky and Kahneman [95] report similar results with slightly more complicated test gambles and much more modest sums of money (thirty and forty-five dollars). They concur that there is a "certainty effect" of the sort that Allais describes.

Morrison [61] interprets the Allais respondents' motivation somewhat differently. He points out that someone who has the choice between A and B actually finds oneself in the same position as someone who owns one million dollars in assets and faces the choice of remaining at the status quo (option A) or running a one percent chance of losing it all in order to get a ten percent chance at an additional four million dollars (option B).

Although Morrison does not pursue the point, his argument has much the same flavor as Pfanzagl's [63] Consistency Axiom. There is some sense, indeed, in which declining an amount in order to get a lottery is the same as buying the lottery for the amount.

In any event, Morrison does not try to reconcile the conventional axioms to the Allais result. Rather, Morrison suggests that new axioms might be devised to account for asset changes that stem from having the

option to increase one's wealth for certain. Krantz, Luce, Suppes and Tversky [48] note in passing some of the difficulties such axioms might encounter in practice, in the course of their discussion of Pfanzagl's axiom.

Borch [15] takes an unusual approach to the Allais problem. He argues that the respondents are mistaken in an especially subtle sense. Very likely, the recipient of a million dollars under option A will not spend it all, but rather will invest most of the money in securities with good growth potential and little risk. In other words, they will invest in a lottery very much like the one that they turned down.

Savage [79] and Raiffa [66] both report thinking hard and long about the Allais problem, intimating that they might have chosen in the "forbidden" way prior to deeper reflection. In the end, though, both of these champions of the conventional axioms would choose according to - what else? - the conventional axioms.

In the course of their thinking, both men conclude that it is the independence axiom that is under particular attack in the Allais situation. A later argument advanced by Raiffa [67], based on an idea by Robert Schlaifer, clarifies the role of the independence axiom.

Suppose that DM is take part in a compound lottery. The lottery offers an eighty-nine percent chance of some undisclosed prize, X , and an eleven percent chance of DM receiving one's choice between two options. The options are one million dollars for certain and, in the alternative, a second lottery that offers a ten-elevenths chance of getting five million dollars against a one-eleventh chance of no change in wealth.

If the independence axiom holds, then the decision between the

million for sure and the second lottery does not depend on the value of \underline{X} . Thus, DM could state one's choice regardless of what \underline{X} happens to be.

Suppose DM chooses the million dollars. If \underline{X} turned out to be one million dollars, then the compound lottery would offer one million dollars for certain, indistinguishable from option A in the Allais problem. If \underline{X} is zero, then the compound lottery offers an eighty-nine percent chance of zero and an eleven percent chance of one million. That's Allais' option D.

By similar arguments, the choice of the second lottery causes the compound lottery to offer prizes and probabilities indistinguishable from options B and C, depending upon \underline{X} .

Thus, by choosing in the compound lottery, DM seems to be binding oneself to consistent choices in Allais' situation. Since the compound probability axiom is an inauspicious point to attack, the controversial assumption appears to be that DM can choose without knowing the value of \underline{X} . That assumption hinges on the independence axiom.

The Allais problem and other evidence gathered from psychological experiments (see, for example, MacCrimmon [51]) suggested that many people tended to behave in ways that were inconsistent with the conventional axioms. Writers on the subject came to distinguish between the prescriptive aspects of utility theory and the descriptive aspects.

That a theory fails to describe how people actually behave, many would argue, has no bearing on its status as a prescription of how people ought to behave.

The Allais problem is of no special moment for the ruin-conscious buying price DM under most utility functions. Such a DM's choices do not conform to the independence axiom anyway. In particular, that the buying

price of the three-outcome lottery of option B is less than a million dollars does not impose a restriction on the buying prices of options C and D.

An exponential-utility DM, however, does conform to the axioms, even when using a buying price ordering. As discussed earlier, however, an exponential-utility DM has an incentive to adopt a horizon. With that horizon in place, DM's choices may no longer conform to the usual axioms.

Any DM, in fact, who incorporates a horizon into one's decision rule may make the "forbidden" choices in the Allais problem on account of the discontinuity so established. The availability of the million dollars for certain may cause DM's wealth to exceed the range over which one's instructions are held to obtain.

This assumes that DM's horizon is lower than the lofty figure of one million dollars, and that DM adopts the convention that the upper horizon is exceeded whenever the worst outcome of any option is no less than the chosen figure.

That assumed, it is interesting to note that the goal seems more properly to violate the compound probability axiom than the independence axiom. In Raiffa's compound example lottery, DM may indeed select the million for certain regardless of the value of X .

What DM might deny is that a situation where one can achieve a goal for certain, given that one reaches that decision juncture, is not different from another situation where one can't. This, despite the equality of the ultimate outcomes and their probabilities.

The behavior in question presents no theoretical problem for a DM whose decision rule is not based on axioms. For the ruin-conscious

rules coupled with goals, the behavior may even illustrate a desirable "flexibility" unattainable under rigid axiomatic schemes. Even an innocuous-seeming axiom like compound probability can, in exceptional circumstances, rule out a course of action that is not obviously offensive to reason.

Even so, the intuitive appeal of the compound probability axiom is such that DM might be dubious about violating it frequently. The nature of a goal, however, provided that it is set high enough, is that its influence will be felt only infrequently.

If the upshot of goal attainment is that one's agent simply checks with DM about what to do, then compound probability may not even be violated.

In any event, the "flexibility" of the axiom-free ruin-conscious approach is limited, and it ought to be. Probably, DM almost always wants to conform to the axiom's teachings. The operative word is "almost". Still, if DM does encounter a situation where one would like to violate the axiom for whatever reason, DM can do so without eroding the logical foundations of the decision rule.

The Allais situation illustrates an interesting reason for adopting a goal. Some prospects, if they arise at all, are inherently so rare as to render arguments based upon repeated play silly. How often can DM expect to be offered the choice between a million dollars and a lottery with an even higher expected money value and no possibility of loss?

If there is such a thing as a "once in a lifetime" opportunity, then perhaps DM ought not to be bound to treat such an event as just another quotidian risky venture. If anything, the simple wealth horizon method introduced here may not go far enough in flagging exceptional offers.

Exceptions are the bane of rules. It is easy for the designer of a broadly applicable system to relegate exception-handling to the realm of the "irrational". The recognition of exceptions in evaluating lotteries, however, goes back as least as far as Bernoulli [9].

Bernoulli allowed that his tidy system of wealth preferences would break down for a prisoner who needed a certain fixed sum for ransom. Bernoulli, with no axioms to defend, was content to leave such exceptions to stand apart.

Less melodramatically, basketball coaches will spend most of any game pursuing high percentage shots. In the final four seconds of the seventh game of a championship series and down by three points, things are different. The coach whose players took the easy two-point lay-up that the other team is only too happy to concede would be fired. The long shot from three-point land is the only way to go.

Neither Bernoulli's prisoner nor the hapless coach are necessarily outside the ambit of conventional utility theory. One could assess a special utility function that placed a numerical value on personal freedom, or that was a function of more than one variable, say points, time to play and deficit to make up.

Raiffa [67] goes beyond these possibilities in suggesting a special utility for confirmed Allais "violators". Raiffa suggests that a kind of "disutility" of having tried and lost could be assessed against the zero outcome, beyond whatever value the status quo might have in its own right. This is much the same idea as Morrison's [61], discussed earlier, except that Raiffa would keep the existing axioms while Morrison would not.

Formally, Raiffa's suggestion seems like it might preserve the

axioms in the face of an Allais violation. After all, Bernoulli's prisoner and the hapless coach illustrate that, in principle, things other than money can appear as the arguments of a utility function. Nevertheless, Raiffa's suggestion fails to accomplish his intention.

If one admits the principle that the utility function can depend on other prizes and their probabilities, then Raiffa's own compound lottery argument founders. As in the case where goals are allowed, the compound probability axiom appears to be the one violated. Presumably in the second lottery, an "opportunity conscious" DM would avoid the one-eleventh chance of the "enhanced" zero regardless of the value of X . Yet the same DM will probably make inconsistent choices in the Allais problem as originally presented.

Actually, Raiffa's suggestion could easily be expanded to embrace buying price rules.

The buying price is, after all, a function of all the prizes in a lottery and their probabilities. For general utility functions, use of buying price rules leads to violation of the independence axiom.

The axioms will not tolerate utility functions that take account of the other prizes in a lottery without unravelling. Utility functions need not be restricted to the money involved in an outcome, but they can't turn on implicit qualities of the alternative outcomes, like "regret" at not having gotten one of those alternatives.

It appears, then, that Morrison's analysis is closer to the mark than Raiffa's. The necessary advice of the conventional axioms in the Allais problem is to select either A and C, or else B and D. An axiomatic system that supported the "violators" would have to be different from the conventional one.

That's interesting, because indisputably rational people have managed to disagree about the correct course of action in the Allais problem for about thirty years now. Thus, any claim that the usual axioms embody necessary attributes of rational behavior under risk lies open to question.

Such a claim seems indispensable to the position that the usual axioms ought to be adopted on prescriptive grounds by everyone. Allais' original report, being descriptive, does not refute such prescriptive claims.

The debate that his report spawned, however, does leave room for rationally founded doubt about the status of the axioms as normative imperatives.

The decision rules developed in this dissertation are neither descriptive nor prescriptive. Rather, they simply provide means whereby a DM might formulate some principles of choice in a manner suitable for algebraic implementation. A DM who wished not to take account of the probability of ruin, e.g. a follower of Kelly, violates no canon of rational behavior.

What, if anything, DM wishes to accomplish by taking risky gambles is DM's business. A mechanical procedure for selecting lotteries can be judged by whether or not it does in fact further some objective selected by DM.

It is not clear that anyone other than DM can judge the chosen objectives. Even DM's judgement about the objectives might change in the face of "once in a lifetime" opportunities, as Allais' problem suggests.

Bands, Barriers and the Dividend Problem

A practical problem, albeit one that will be heavily idealized in analysis, that involves ruin and horizons of a sort arises in the following way. Suppose a corporation is the decision maker facing risk. If the corporation prospers, then it can accumulate capital, or pay out some of its earnings as dividends.

Whatever is paid out reduces the store of capital that protects against a run of adversity. On the other hand, the stockholders, and certainly the government, are apt to insist on some pay-outs. Both are in a position to enforce their preference on management.

The class of problems subsumed under the rubric of "the dividend problem" involves striking some balance between the wish to produce income and the wish to ensure that the engine that produces the income survives.

Shubik and Thompson [84, 85] study such "games of economic survival" in the light of an assumption that the corporate decision maker seeks to maximize the discounted present value of the future pay-outs. That is, one maximizes the present value, V , of a stream of payments, $x(i)$, where the i 's run from one to T and index equally spaced periods in the future, using a per period discount rate of d , all related as:

$$V = \sum_{i=1}^T x(i)d^i$$

As the authors note, maximizing this quantity "might even involve the eventual ruin of the firm as part of the optimal policy".

It is not difficult to conjecture why that might be so. Indefinitely sustained exponential growth in the firm's earnings may be unrealistic. It might not even be a very good approximation.

It is one thing for Kelly to speak of sustained exponential growth in the limited capital of a mortal gambler over the all-too-short active lifespan one can expect. On the much larger scale of real corporations, with no pressing limits on a firm's active life, unending exponential growth is a tall order.

Absent exponential growth, the contribution of earnings far in the future is negligible. Even with exponential growth, the rate of growth cannot slip below the reciprocal of the discount factor, else the far future also becomes negligible.

In either case, one would expect the firm to sacrifice those remote earnings, i.e. suffer ruin, in the interest of boosting more nearly present and more heavily weighted earnings, if discounted present value is the objective.

Growth assumptions aside, however, the pay-out strategy studied by Shubik and Thompson renders eventual ruin asymptotically certain. Their strategy is to choose some capital level \underline{N} . At the end of each time period, if the firm's capital exceeds \underline{N} , then the surplus is paid out and the firm enters the next time period with capital equal to \underline{N} .

The best the company can hope for is to start each period with capital \underline{N} and avoid ruin until the next period. The probability of ruin doesn't fall to zero unless the firm manages to play pure gains exclusively. If the firm does survive, then it repeats its exposure to ruin in the next period, and so on.

If the probability of ruin for a single period is constant, call it \underline{R} , and assuming the gambles faced in different periods are independent trials, then the probability of surviving \underline{n} periods is

$$(1 - R)^{\underline{n}}$$

Unless $R = 0$, the limit of this expression as n approaches infinity is zero.

Eventual ruin using the Shubik and Thompson "reflecting barrier" pay-out strategy will be essentially certain, unless in each period the firm plays more conservatively than in the last. Since the firm's capital doesn't increase, conservatism must involve accepting safer and safer gambles. Unless the universe is so obliging as to indefinitely offer gambles that are both progressively safer and progressively more lucrative, the firm's growing conservatism will erode earnings.

The criterion, however, does not allow sacrifice of earnings for safety. Realistically, then, the firm is doomed.

Pessimistic or not, reflecting barrier policies are analytically tractable. It is no surprise, therefore, that they are quite popular in the literature. Gerber [39], for example, explores the similarity between the dividend problem and certain inventory control models. He analyzes several situations, arriving at solutions in numbers aided by the well-behavedness of barrier methods.

Gerber also explores another tractable pay-out scheme: the "band" strategies. In a band strategy, instead of one threshold value for pay-outs in all periods, there are several. At the end of a given period, the firm determines the highest value on its list of thresholds that has been exceeded. For each threshold, there is an associated lower capital value that is between its associate and the next lower threshold.

Once the firm determines the highest threshold that has been exceeded, it distributes capital to reduce its wealth to the associated lower capital.

In the particular cases studied by Gerber, there is some highest

threshold, a final barrier. The limiting behavior of these strategies, therefore, ultimately converges to that of the simpler barrier case: essentially certain ruin.

The present value criterion may represent rational behavior for some firms, even though it gives little weight to survival considerations. There are companies, however, that must take survival into account in order to attract business in the first place. Insurance companies have both a moral obligation and an economic incentive to avoid bankruptcy. A visibly shaky company would have difficulty selling policies, to say nothing of passing regulatory scrutiny.

De Finetti [23] has analyzed reflecting barrier strategies considering not only the present value of pay-outs but also the expected survival time of the company. The choice of pay-out strategy still renders eventual ruin inevitable, but some effort is expended to postpone the event, even at some cost in earnings. Borch [11, 12] has elaborated on and generalized de Finetti's results.

Borch proposes either maximizing the expected present value of dividends subject to a constraint on the minimum expected survival time, or combining the two expectations into a weighted objective to be maximized, e.g.

$$\alpha \log(V) + (1-\alpha) \log(D)$$

where \underline{V} is the expected present value of the dividends, \underline{D} is the expected lifetime and α is some constant between zero and one selected by the corporation or by its government regulators.

Either approach advanced by Borch is fairly simple to implement numerically. Both \underline{V} and \underline{D} can easily be tabulated for various combinations of initial capital and barrier capital, given suitable estimates

of the random processes by which income is realized and, in the case of insurance companies, claims are paid.

The tables provide DM at a glance some sense of what kinds of expected lifetimes are available. This should be helpful in choosing a realistic constraint value. In a similar fashion, the tables can be combined with some "back of the envelope" calculation to select a liveable weighting factor if the unconstrained objective approach is taken.

Segerdahl [81] contributed a discussion when Borch [11] was read before the Royal Statistical Society of the United Kingdom in 1967. Segerdahl notes with evident displeasure that in the numerical example worked by Borch, the longest expected survival time considered was about sixty years. Segerdahl then discusses a British insurance company that was founded in 1762 and was still operating two centuries later.

Segerdahl concludes that the company's managers must have operated on different principles than those presented in Borch's paper. "I do not think they are very sorry about that," writes Segerdahl, "nor are, I think, its policyholders, employees or British life insurance as a whole."

Of course, the sixty year figure was offered as an illustration, not as an exploration of the limits of the possibilities offered by the technique. Borch himself made just this point in his replies to the discussion.

Still, the general thrust of Segerdahl's criticism retains its force. For ruin analysis to be of practical use to insurance underwriters, it ought to be possible to strike a balance between dividends and solvency without accepting the inevitability of ruin.

One approach is to apply the buying price concept developed earlier. The buying price was shown to be an amount of capital that can be withdrawn from risk at will without violating a chosen ruin constraint.

Thus, the analyst can calculate a buying price for the portfolio of risks acquired during some period. This buying price is then available for immediate payment as a dividend. The calculation can be repeated in future periods and the dividend paid will reflect those periods' on-hand capital and the composition of their portfolios.

Alternatively, the analyst might prefer a stable dividend. If one had a suitable model of how the portfolio would develop over time, then one could behave as if playing a compound gamble through time. Dividend policy can be based on the assumption that one is playing this model rather than assessing each period's dividend based on the particulars of its portfolio.

Other withdrawal strategies are available, too. The buying price was shown to be a special case of the withdrawal schemes possible without violating the chosen constraint. So, the analyst might choose some other pattern of withdrawals. For example, one could pay out "windfalls" when they occur, but otherwise maintain a steady dividend.

A full analysis of the alternatives available would require a book unto itself. Conceptually, though, implementing a ruin constraint by means of a bounded utility function leads to simple solutions of dividend-type problems.

One advantage of the new approach is the ability to use directly the rich existing literature that applies utility functions to a wide variety of models of investment portfolio risks and objectives.

See, for example, Hakansson [40], Merton [59], or Pyle and Turnovsky [65].

The observation that utilities provide probability bounds in random walks allows a straightforward integration of the research that has heretofore exploited random walks (the work of Shubik and his successors) and the research that has concentrated on utility models (conventional portfolio theory).

Interestingly enough, what had appeared for some time to require a two-barrier random walk model, the band and barrier approaches to dividend policy, actually requires only a single barrier, ruin, model. Simpler models also arise because expected utility models are very tractable compared to the ordinary techniques for solving random walks.

CHAPTER V

UNCERTAINTY AND PARTIAL RISK

Received Theories of Decisions under Uncertainty

Suppose that a money pay-off to DM depends on both an act chosen by DM and an unknown state of nature. Not only does DM act in ignorance of what the state of nature is, DM does not even know the probabilities of the different states of nature. DM does know all of the possible acts, all of the possible states of nature, and the pay-off for each pair of state and act.

Such problems are often called decisions under uncertainty, as distinct from decisions under risk in which the relevant probabilities are known by DM.

The similarity between two-person games and decisions under uncertainty is apparent. One often hears uncertain decisions characterized as "games against nature". A similar matrix format is often seen in games and uncertain decisions. For decisions, the rows are devoted to the various acts and columns to "nature's moves". In fact, one popular strategy for solving decisions under uncertainty is to follow the mixed maximin strategy. That is, to force the problem into a lottery format the same way one would in a competitive game.

The mechanics of this approach is the same as outlined in the first chapter. The resulting linear programming problem can be solved within the expected utility framework.

Although the maximin approach is widely accepted as optimal in games, it is not so generally accepted in decisions under uncertainty.

The two situations are different. Nature is presumably not a rational, calculating player dedicated to the abasement of DM. Were the probabilities of the states to correspond to the minimax solution, this would be a coincidence. The symmetry of the two-person game that gives the von Neumann-Morgenstern solution its stability is irrelevant to the analysis of a game against inanimate nature.

Thus, it should come as no surprise that several authors have proposed solutions for decisions under uncertainty other than the maximin even though many of these same authors endorse maximin for use in games.

The following review is compiled from Baumol [6], Luce and Raiffa [50], and Epstein [28]. None of these sources claims to be exhaustive. Nevertheless, there is close agreement among them in their choice of proposals for discussion. What follows, then, is a survey of the mainstream of decision analysis under uncertainty.

The simplest mainstream method is the pure strategy maximin. That is, choose the act whose worst outcome is no worse than the worst outcome of any other act. It is difficult to find much to say in favor of this principle.

One knows from the game results that DM can pursue a mixed strategy that is at least as good as this pure strategy. "Good" is meant in the sense that DM would prefer (or be indifferent between) the worst lottery of the mixed strategy over the amount guaranteed by the pure strategy. Thus, any DM who saw merit in the pure strategy maximin and who ordered lotteries according to an expected utility or utility-like rule would rather pursue the mixed strategy as a general rule.

Note that one can formulate decision problems where both the pure and the mixed maximin strategies lead to the same action.

Sometimes this recommended act may be hard to live with. Luce and Raiffa give the following example. The entries are utility values.

	S1	S2
A1	0	100
A2	1	1

The maximin act, whether mixed or pure, is to choose act A2 with probability one. Even if the entry at (A1, S2) were a million units, the maximin act would still be A2.

This matrix is an excellent example in several respects. For one thing, it dramatizes how nature may differ from a competitive player. No competitor is ever going to oblige DM with a positive probability of S2. Nature, on the other hand, doesn't care, and so might offer some chance of gaining that happy state.

Another virtue of the example is that it encourages careful thought about the degree of ignorance that one is assuming. The states S1 and S2 are not things like "It will rain tomorrow outside DM's home.", matters about which DM might have some notions as to their likelihood. If they were, then DM would have much incentive to bring to the problem some estimate, however crude, of the probability of rain tomorrow.

That would make the problem too risk-like, too much like a lottery, for the current discussion. The states are things that DM knows nothing about, like the proportion of red balls in a sealed urn, and urn that might not hold any red balls (or might hold nothing else). One is deliberately extreme about DM's ignorance. The point of a problem under uncertainty is to find the "best" act without making any appeal to the state probabilities whatsoever.

That assumed, it is not obvious that choosing A2 is wrong. It is

perhaps galling that a conservative decision rule forecloses the big pay-off altogether, without any assurance that nature would not be somewhat forthcoming if given the chance.

For this reason, Savage [78] proposed the minimax "regret" criterion. Savage's proposal entails replacing each entry in the pay-off matrix with a "regret" value. The regret is the absolute value of the difference between the money entry in some cell and the greatest entry in its column. Thus, the regret values for the last example matrix would be

	S1	S2
A1	1	0
A2	0	99

One then chooses the act which has the smallest maximum entry in its row, which in this example is A1.

The thought behind the proposal is that DM seeks to protect against rueful feelings that one might experience as a result of choosing A2 if the true state of nature turned out to be S2.

Chernoff [20] criticizes Savage's proposal on three grounds. One objection turns on a technical axiomatic difficulty with imputing any significance to the absolute difference between two utility numbers. On a less obscure note, Chernoff shows that the scheme is intransitive.

An act, say A2, may be optimal among acts A1, A2 and A3, while some other act, perhaps A3, is optimal among A1, A2, A3 and A4. Thus, even though A4 is not chosen, its presence changes the ordering of the other acts.

Perhaps most interestingly, Chernoff reports the construction of examples where a small advantage in one state outweighs a large advantage

in another state. This scores a telling blow, since Savage proposed the method in order to avoid just such difficulties.

Hurwicz [41] adopts another approach to the conservatism of the maximin criterion. He proposes a "pessimism-optimism" index. Under the Hurwicz proposal, one finds the best and worst outcomes for each act. These are then combined in a weighted average, where the weights given to the two extreme outcomes are non-negative numbers that sum to one. DM selects the act with the greatest weighted average value.

Let α be the weight given to the worst outcome. The index values of the two acts in the example are

$$A1: \alpha(0) + (1-\alpha)100$$

$$A2: \alpha(1) + (1-\alpha)1 = 1$$

The higher the α , the more "pessimistic" DM is, in Hurwicz's view. In the case where $\alpha = 1$, the rule is identical to the pure strategy maximin.

It is unclear what the basis of DM's optimism or pessimism ought to be, since one has assumed away DM's use of knowledge or estimates about the true state of nature.

Luce and Raiffa offer an intuitive counterexample to the Hurwicz proposal, one with the same flavor as the one they levy against maximin.

Consider the matrix

	T1	T2	T3	...	T100
A3	0	1	1	...	1
A4	1	0	0	...	0

Both A3 and A4 have Hurwicz index values of $1-\alpha$, so DM would be indifferent between them. Yet, anyone who is likely to be disturbed by the Luce and Raiffa rejoinder to Savage may also be liable to prefer A3 over

A4. There is no binding foundation for such a feeling apart from the informal, and by assumption uninformed, guess that one of T2 through T100 is more likely to occur than T1. There are so many more possibilities.

If one is persuaded that A3 is better than A4, then this example can also be offered as a criticism of the pure maximin criterion, since pure maximin is a special case of the Hurwicz.

Both Luce and Raiffa's account and Baumol's account suggest that Hurwicz may have had his tongue in his cheek when he offered this pessimism-optimism index. Apparently, Hurwicz personally was quite content with $\alpha = 1$, i.e. the pure maximin criterion. The invention of the index was, apparently, a dryly humorous response to those who would characterize maximin as overly pessimistic.

Whatever the circumstances of its invention may have been, the index has found a place in the decision literature.

Still another proposed decision rule is an old one, suggested in somewhat different contexts by Jacob Bernoulli, Thomas Bayes and Pierre Simon de LaPlace. Suppose there are n states of nature, and that DM does not know their probabilities. If the states are mutually exclusive and exhaustive (one and only one among them will occur), then the rule says that DM ought to assign to each state a probability of $1/n$. This rule is often called the "principle of insufficient reason". When applied specifically to decisions under uncertainty, the rule is sometimes called the Bayes-LaPlace criterion.

If DM adopts the method, then one formally turns any decision under uncertainty into a decision under risk. In the recurring example, the probability of S1 would be assumed equal to that of S2, one-half for

each. DM would then view the game matrix as a choice between a lottery offering equal chances of 100 or 0, and the degenerate lottery that offers a prize of 1 for sure. However DM would choose between these lotteries if offered, so, too, would DM choose between the acts actually before one.

Note that if DM's decision rule avoids mixed strategies under risk, for example, conventional utility, then mixed strategies will be avoided under uncertainty.

One real difficulty with Bayes-LaPlace is that it is sensitive to the number of distinct states in the problem. It is not always obvious how many states there are, or ought to be.

If a decision depends crucially on whether or not it rains tomorrow, and one adopts that dichotomy as the description of the states, then the probability of rain is assessed at one-half. If one is planning a picnic, then one might equally be concerned that a windy day would be unsuitable. What are the states then?

On a "weather suitable or not" partition, there is an imputed fifty percent chance of the plans going awry. On a "sunny and not windy, rainy and not windy, windy" analysis, bad weather totes up to two thirds. Which is it?

Note the fine distinction that can arise. DM is assumed to know all of nature's possible moves in any decision under uncertainty. It does not necessarily follow from that, however, that DM knows what the "natural" partition of those moves might be. The analytic matrix with the fixed and determined number of columns is generally provided by DM, not by nature.

The ambiguities attendant to the use of Bayes-LaPlace are well

known. Nevertheless, Chernoff [20] proposed a set of axioms that lead uniquely to the adoption of Bayes-LaPlace. Although the conclusion is not widely accepted, the axioms continue to receive attention. Subsets and modifications of the Chernoff axioms lead to other decision rules.

The crucial Chernoff axiom leading to Bayes-LaPlace is based upon some unpublished work by Herman Rubin. The axiom concerns the following situation.

Suppose that DM faces two matrices. One has pay-offs that depend both on one's act and the state of nature. The other matrix's pay-offs depend only on the state of nature, i.e. each column is a constant vector. Different columns may have distinct constants. The states are the same for both matrices.

DM will "play" one matrix or the other depending on some random event that is independent of the act chosen. Rubin's axiom states that one's decision in the problem where one's act matters does not depend on the constant-column matrix.

Once the unfamiliarity of the decision terminology wears off, the reader will remember having seen something like this before. Recall Raiffa's report of Schlaiffer's analysis of the Allais problem. In that analysis, there was a compound lottery, one of whose outcomes was a prize for certain. The other outcome was a choice among lotteries. It was asserted that the decision among the lotteries ought to be independent of the choiceless prize. Rubin's axiom is clearly a parallel construction for uncertain decisions.

The next section will discuss Rubin's axiom after the development of some theoretical apparatus to pursue the analysis.

At the moment, it is safe to say that there is little consensus in

the literature about how to handle decisions under uncertainty. This can hardly be surprising. There is disagreement about how to handle decisions under risk, and uncertainty gives the analyst one less category of information to use, the probabilities of the states of nature.

A token of the current state of disagreement was invented by John Milnor, and quoted by Baumol [6].

Consider the following decision matrix.

	S	T	U	V
A	2	2	0	1
B	1	1	1	1
C	0	4	0	0
D	1	3	0	0

Act A will be chosen by Bayes-LaPlace, act B by both the pure and mixed maximin, act C by the Hurwicz index for any α less than one-half, and act D by the minimax regret criterion.

The existing techniques for dealing with decisions under uncertainty are thus shown to be incompatible and irreconcilable.

The resulting diversity of opinion has at least one salutary effect. In the absence of a single strong decision rule candidate comparable to the conventional utility theory in the domain of risk, there may be less difficulty in proposing that DM should select a decision rule based on what accomplishes most nearly the goals and aims one chooses. The claim that an uncertain decision rule is "uniquely rational" or binding on all DM's who aspire to rationality is less often heard than the corresponding claim for risk.

Uncertain Mixtures of Lotteries and Maximin

Before pursuing decisions under uncertainty, consider the following problem under risk. Suppose DM is a ruin-conscious buying price decision maker. DM faces repeated play of a compound lottery, a lottery whose prizes are also lotteries. DM knows the identity, the outcomes and the probabilities of each of the prize lotteries.

DM does not know the relative frequency of these lotteries. Assume that DM plays without memory, that is, DM cannot infer the probabilities as play unfolds. Assume that DM will make withdrawals on each play, before learning which prize lottery is offered.

Thus posed, the problem is rather simple. Each lottery has its own computable buying price. If DM withdraws the buying price of the lottery whose buying price is the smallest, then the ruin constraint will be satisfied.

This assumes, of course, that the least buying price is positive. If it isn't, then no withdrawal plan can guarantee the satisfaction of the ruin constraint without knowledge of the relative frequency with which the lotteries will be faced. DM would decline to play such a sequence if one could.

By the same reasoning, if the least buying price were positive, DM would accept the sequence, but would not withdraw more than the least price on each play. If one did withdraw more, then the satisfaction of the chosen ruin constraint could not be guaranteed.

The argument generalizes to situations where different lotteries are available on each play, rather than the same ones over and over. Suppose that instead of the repeated play of the same uncertain compound lottery whose prizes are known lotteries, DM faces a different compound

uncertain mixture of lotteries on each play. Although DM knows the lotteries offered at each play when that play takes place, DM does not know what lotteries will be available on other plays.

Without information about the sequence of compound lotteries to be faced, nor information about the probability with which the potential lotteries at each play will turn out to be the one actually played, DM can still satisfy the chosen ruin constraint. DM simply withdraws the least buying price among the lotteries potentially available.

The application of these observations to decision making under uncertainty is straightforward. Faced with a choice of acts and a knowledge of their outcomes conditioned upon nature's move, DM is in a position to create an uncertain mixture of lotteries.

Just as one does in two-person games, DM can assign probabilities to each of one's acts. That done, one faces a different lottery for each state of nature. Since DM doesn't know the probabilities of those states, but knows both the probabilities and pay-offs for each possible lottery, then DM does face an uncertain mixture of lotteries as claimed.

DM already knows how to assign probabilities to one's act so as to obtain the greatest least buying price. One uses the linear programming techniques discussed earlier in connection with games.

A policy of so behaving in all decisions under uncertainty allows DM to satisfy one's chosen ruin constraint, even though DM does not know the substance of future uncertain decisions.

One subtlety for games against nature should be noted. Suppose the lottery that yielded the least buying price happened to be a constant column. If this were a competitive game, then no problem would arise. The constant column would be the best play for the other player regard-

less of what Row did.

Against nature, however, it makes sense to ignore constant columns when selecting a strategy. Since any probability weighting of the acts doesn't change the pay-off if the constant "lottery" turns out to be the one offered, one should choose the weighting that yields the best price in case the constant column isn't the one DM has to face.

Even so, the buying price of the decision problem as a whole would be that of the constant column, if the column is the "best worst" lottery for DM.

In other words, DM should solve the game matrix in the usual way to get the buying price for the uncertain prospect. To pick a strategy, one could delete all constant columns and solve that matrix for the actual weightings. This extra step is unnecessary unless a constant column were the unique (except for other constant columns of the same value) least-buying-price lottery in the first step. Otherwise, the column won't change the solution at all.

This modification of the usual maximin approach brings the strategy into conformity with Savage's [79] independence or "sure thing" principle. As applied to decisions under uncertainty, that principle holds that the presence of a constant column ought not to affect DM's strategy. The column may affect the price under the proposed solution procedure, but Savage didn't contemplate directly such effects in his theory.

The recommendation to the ruin-conscious DM who uses a buying price rule is generally to follow the mixed maximin rule, with the modification noted for constant columns. If DM were ruin-conscious, but didn't follow a buying price rule, then this advice would not necessarily hold.

The buying price rationale gives the worst lottery some analytic

importance. The ruin-conscious DM would want to be sure that the worst lottery satisfied the constraint, but might not otherwise care about the worst lottery's value if one were not generally guided by the buying price.

Similarly, if DM adopted a utility-like buying price rule, but not for ruin constraining purposes, then the maximin advice might not apply. The worst lottery was emphasized because its satisfaction of the ruin constraint ensured that all the lotteries were satisfactory. If constraint satisfaction were not a consideration, then perhaps DM might use some other function of the available buying prices to make a selection.

Indeed, if DM were ruin-conscious and used a buying price rule, but didn't actually withdraw money from capital, the rationale given here would be inapplicable. That may be an important caveat. For example, when a constant column sets the buying price, the modified strategy chosen does not change the amount available for withdrawal. Thus, DM wouldn't be obliged to follow maximin during the "second analysis" of the matrix with its constant columns removed.

In summary, the recommendation of maximin is specific for a certain kind of DM with a certain kind of objective. Other DM's are left to decide whether the mixed maximin comports with their goals or not.

In offering maximin at all, even in a slightly modified form, one must consider some objections that have been raised against the method's use in games against nature.

An observation frequently found in the literature is that maximin is unduly pessimistic. Sometimes, this is phrased gently, as a speculation about whether one should behave as if neutral nature were indistinguishable from an intelligent competitor.

Other times, the objection is raised vituperatively with ad hominem overtones. Consider this sentence from Baumol [2, p. 581], "However, the maximiner is a fundamentally timid man who fears that his opponent (whether it be nature or another player) will always outguess him."

This is not a pretty picture. Incidentally, the "inability to outguess" aspect of the maximin strategy is often advanced as an argument in its behalf in competitive games; see, for example, Owen [62]. This attribute may really be important in some games. However, maximin has traditionally been advanced on other grounds. Its security from invidious inference exists, but is secondary in importance.

One might also point out that the opponent's desire to do DM bad is not central to maximin, either. The competition assumption is a kind of "cover story" that explains why the players don't just agree on some course of action, as players are assumed to do in so-called co-operative games.

Maximin is a response to the players' lack of knowledge of what the other side will do, given that one cannot ask and expect to get a truthful answer.

Competition is one way such a failure to communicate might arise. If asked about one's plans, a competitor would lie. In the memorable phrase of Thomas Schelling [80, p. 219], "Any message worth sending is not worth reading."

Competition is only one way communication can fail, however. Nature doesn't lie, but nature often doesn't answer our questions at all. One can easily imagine two human opponents finding themselves physically unable to communicate and so unable to co-ordinate their acts to mutual

benefit. Neither wishes the other ill, yet both might maximin.

It seems to matter little whether DM can't ask because the other player will lie, or because the "other player" simply doesn't answer. In neither case does DM receive assurance of co-operative behavior. In both cases, the best DM can do by one's own action is to establish a floor on the outcomes of the encounter.

As for timidity, the ruin-conscious DM's attitude toward risk is not all that different from a conventional utility DM who uses a concave curve. Each avoids risk in a systematic way, and there is nothing obviously irrational or over-emotional in such behavior. With maximin, a buying-price DM can gain an income and satisfy a chosen ruin constraint. Inferences about DM's emotional state seem out of place and irrelevant to the merits of DM's unexceptional behavior.

Chernoff [20] criticizes maximin from another vantage; he objects to the use of random strategies against nature on principle. Chernoff's balk reaches mixed strategies against a competitor, although without insistence in that case.

Briefly summarized, Chernoff points out that after the "coin is tossed" to select an act, one goes on to play a particular pure strategy. Either one is indifferent among the possible acts, in which case the randomization is superfluous, or else there is an element of delusion. If one has a preference among the acts, one should take the best act and be done with it. Otherwise, DM may find that the random procedure selects some other act, and so DM ends up performing that less preferred act.

In other words, DM cannot really play a mixture of acts, but must eventually play a pure strategy. Why not, Chernoff asks, play the best pure strategy in the first place?

The answer to this objection is to appreciate the difference between a preference among the acts for their own sakes, and the long-run results of a policy that guides one's selections. The ruin-conscious maximiner employs a mixed strategy to get a more preferred lottery than that offered by any pure act. Any preferences or attitudes such a DM might have about the acts are not relevant to the implementation of the chosen policy.

Other analysts encounter real confusion about maximin when they read into it the desire to make the safest play possible. McClennen [57], for example, reports great difficulty in trying to reconcile conventional utility theory with this presumed quest for safety.

Analysis from the ruin-conscious perspective clarifies the problem McClennen wrestles with. The maximiner need not be seeking the safest play; one can play maximin seeking merely an acceptably safe play. The reconciliation of that objective with utility theory is straightforward: ruin constraints can be achieved by bounded utility functions.

Finally, one can reprise an earlier objection to maximin: that an arbitrarily small advantage in one state of nature can outweigh an arbitrarily large advantage in another.

In the first place, there might not be a problem at all. If DM did know the probabilities involved, it is entirely possible that the high pay-off act would be declined.

Alternatively, the source of the problem might not be the maximin strategy. After all, the class of decision problems under uncertainty is an artificial one. It was created for the exploration and creation of analytical tools. It is intentionally extreme in its assumption about what DM doesn't know - not even a hint about the crucial probabilities.

Indeed, as Baumol [6] observes, the class is also somewhat unrealistic in what DM does know - all the possible acts and all their possible outcomes. Perhaps the Luce and Raiffa relative advantage example merely illustrates how difficult decisions can be if DM has no probability information. This possibility seems especially plausible in light of the irony that some of the methods proposed to remedy the defect themselves fall prey to similarly quizzical hypothetical problems.

Still another approach is to use under uncertainty the same flexibility that DM has in problems under risk. In the last chapter, it was shown that DM could, without abandoning claims to rationality, treat once-in-a-lifetime opportunities as exceptions. In fact, DM ought to anticipate that such exceptional circumstances might arise. This is especially so if one's risk taking instructions are to be executed without supervision by a personal agent or by a machine.

DM can simply limit the power of one's agents to act when goal-sized pay-outs are at stake, and resolve such problems oneself. The general advice to play according to maximin is unaffected, just as the general expected utility advice under risk was not scuttled by the prospect of exceptions.

In any event, it is time to take up Rubin's axiom again. The axiom arises in the following situation. There is a lottery with two outcomes. One outcome is a pay-off matrix with constant columns, i.e. DM's pay-off does not depend on one's act. The other outcome is a decision under uncertainty where one's act does matter. The ensemble of states in both matrices is the same. Rubin's axiom says that DM's choice in the "real" decision problem doesn't depend on the other matrix.

Suppose we have two matrices:

	S1	S2		S1	S2
A1	w	x	A1	a	b
A2	y	z	A2	a	b

and a lottery that yields a \underline{p} probability of facing the matrix on the left and a complementary probability of facing the matrix on the right. The interesting case is where DM must choose among the acts without knowing which matrix obtains.

Arguing from compound probability, the problem can be recast into a single matrix:

	S1	S2
A1	(\underline{p} : w, a)	(\underline{p} : x, b)
A2	(\underline{p} : y, a)	(\underline{p} : z, b)

This just brings the lottery inside the uncertain decision. Assume that goal level pay-offs are not involved, otherwise DM might not assent to the blurring of the distinction between choice junctures and chance. That is, so DM will assent to the application of compound probability.

If $\underline{a} = \underline{b}$ and if DM applies maximin to the matrix of expected utility values of the indicated lotteries, then the (a, b) matrix doesn't affect DM's choice. Each entry in the utility matrix is a linear transform of the corresponding utility for the (w, x, y, z) matrix.

Such special cases aside, it is easy to show that the choice of \underline{a} and \underline{b} will affect the probabilities assigned to the acts as DM applies linear programming to maximize the buying price of the prospect. Because numbers which differ beyond a linear transformation will appear in the programming matrix, different answers are to be expected.

This is not irrational. The constant-column matrix really does affect the biggest withdrawal that can be made on the strength of the

compound uncertain lottery.

DM can use this fact to advantage, since one's decision does not depend on preferences among parts of the decision problem faced. Once again, the viewpoint of long-run consequences leads to a different course of action than what might be expected from preferences founded on the attributes of individual outcomes.

One may, therefore, reject Rubin's axiom.

Although Chernoff's axiomatic system (which includes Rubin's) is a classic in the received theory, it is not the only attempt to justify the principle of insufficient reason axiomatically.

Recently, Sinn [86] has offered a new motivation of the principle. Sinn uses the ordinary independence axiom combined with a careful definition of uncertainty. Sinn's arguments are both elegant and ingenious.

Nevertheless, it has already been shown that a ruin-conscious DM need not accept the independence axiom. Thus, one need not adopt the principle of insufficient reason on the arguments of Sinn, either.

There is an irony in Sinn's work. His intention in linking the independence axiom and Bayes-LaPlace is evidently offered to enhance the acceptability of the latter. Given the low esteem in which Bayes-LaPlace is held, however, the linkage may provide still another argument against the independence axiom.

The Ellsberg Problem

Daniel Ellsberg [26] poses the following decision problem. There is an urn with ninety colored balls in it. Thirty of the balls are red, and each of the other sixty is either black or yellow. A single ball will be drawn from the urn at random. In the first of two decisions, DM may choose between two acts. The pay-offs are as follows:

	30	- -	60	- -
	red	black	yellow	
I	\$100	0	0	
II	0	\$100	0	

For the second decision, suppose that the urn is the same, but instead of the matrix above, DM faced

	30	- -	60	- -
	red	black	yellow	
III	\$100	0	\$100	
IV	0	\$100	\$100	

Ellsberg's problem is to specify which act DM should choose for each matrix.

These decisions are intermediate between risk and uncertainty. DM knows something about the probabilities, but not all their values.

Ellsberg put the problem to several prominent decision theorists, among them Paul Samuelson, Robert Schlaifer, Howard Raiffa and Leonard Savage. Ellsberg instructed his subjects not to apply their full and considerable analytical powers to the situation, but to give an intuitive reply.

The specific feature Ellsberg wished his subjects not to notice was that the two matrices are identical except for the "yellow" column. In

both cases, that column is constant over the acts, although with different constants in each case.

The reader may recall that under uncertainty, Savage's independence principle says that constant columns are irrelevant to DM's decision. Ellsberg shows that knowing the probability of one state would not alter the application of this axiom. Therefore, the consistent choices in the two decisions according to Savage's axiom are either I and III or else II and IV.

A brief argument will probably capture the force of the Savage position better than a bald appeal to the axiom.

One third of the balls in the urn are red. Some unknown proportion of the balls, call it \underline{p} , are black, and some unknown proportion, call it \underline{q} , are yellow. The sum of \underline{p} and \underline{q} is necessarily two-thirds. For simplicity, assume that DM decides the problems with linear utility, i.e., according to average pay-offs under risk.

The expected values of the various acts are:

I	1/3 of 100	III	$(1/3 + q)$ of 100
II	p of 100	IV	2/3 of 100

If I is better than II, then III "must be" better than IV in the following sense. There is no pair of values for \underline{p} and \underline{q} so that the expected value of I is greater than that of II and that the expected value of IV is bigger than III's. That is, if \underline{p} is less than one third, then \underline{q} must be bigger than one third, since the sum of \underline{p} and \underline{q} is exactly two thirds.

Nevertheless, a very frequent pattern of expert intuitive response was Act I over Act II and Act IV over Act III. When Ellsberg pointed out the difficulty, most, but not all, of the experts amended their responses so as to conform to Savage's axiom.

MacCrimmon [51, 52] reports similar results when a like problem was presented to business executives. Initially, the executives picked the options with the known probabilities (I and IV), but after guided discussion many, but not all, changed to Savage-compatible choices.

Among those who fell into Ellsberg's "trap" and later extricated themselves was Howard Raiffa [66]. Raiffa proposes the following analysis.

Suppose DM preferred I over II and IV over III. Raiffa then offers DM still another decision matrix. A coin will be flipped and DM has two options. DM must make a decision before the coin lands.

	Heads	Tails
Option A	Act I	Act IV
Option B	Act II	Act III

The entries mean that DM will play the urn game with the indicated strategy.

Option A dominates Option B, and therefore should be preferred to it according to Raiffa. That is, DM prefers the outcome of Option A regardless of the state of the coin.

On the other hand, the two options are identical probabilistically. Red, black or yellow, DM has an objective fifty-fifty chance of \$100 or zero under both options. Rather than preferring Option A to Option B, DM ought to be indifferent between them.

Raiffa's gamble is simply another compound lottery of the sort discussed in connection with Rubín's axiom, and before that, Schlaifer's analysis of the Allais problem. The distilled answer to all is the same: a ruin-conscious DM can have one set of preferences among prospects and another set for lotteries involving those prospects.

The crucial point is that such a DM is not bound by the indepen-

dence axiom, and is therefore free to violate it. DM would err not to notice that Raiffa's offer set up a compound lottery in which options A and B are identical. None of this binds DM's preference in other circumstances where the options faced are not identical.

The original theoretical import of Ellsberg's experiment concerned what are called "subjective probabilities". Fishburn [34] surveys the many theories of subjective probability.

The unifying theme of these theories is that DM is assumed to be able to make estimates of the probabilities of those events in a decision problem whose objective probabilities are unknown. Some theories (e.g. Jeffrey's [42]) attach probabilities to things that don't have well-specified objective probabilities. For example, one can discuss the probability that the State of Pennsylvania has an area of less than fifteen thousand square miles.

Henceforth, we restrict our attention to events that have a well-specified objective probability.

In the Savage theory [79], a popular one, these estimates are always what statisticians call "point estimates".

Suppose a decision turned on some event, and DM estimated its probability as being between sixty percent and eighty percent. Under the Savage theory, DM would settle on a single figure in that range, say seventy-five percent. Once settled upon, analysis would proceed as if the probability were known to be seventy-five percent. No account would be taken of the range of values from which DM selected this particular figure.

After the original analysis, DM could go back and perform a "sensitivity analysis", i.e. one could try different figures in the range

of interest and see whether the ultimate decision would be different with a different assumed probability. Often enough, it turns out that the optimal course of action is the same over a range of possible estimates. If so, then one says that the optimal decision is insensitive to the estimate. If not, then perhaps DM might revise the guess. After all, since it is crucial, maybe one ought to think about it some more and assess the probability carefully.

Either way, however, actions will ultimately be chosen based upon a single value for each estimated probability. There is a sense, then, in which there are no decisions under uncertainty for Savage subjective probability adherents.

One always has some estimate of the underlying probabilities, even if only that of the principle of insufficient reason. The crucial feature of the Savage theory is that one's behavior ought to be consistent with those probability estimates, regardless of any doubts or misgivings about whether they are right.

It is this feature that distinguishes subjective probability from what would otherwise be an unremarkable engineering practice: the provisional working estimate of unknown quantities. Where the engineer might take account of how crude the estimate is in various ways, including the refusal to use some estimates at all, the Savage statistician and DM would not.

If the initial majority response in Ellsberg's problem were defensible, this would be a problem for Savage's subjective probability theory. Of course, the observation that some people violate the canons of a theory does not impeach the normative quality of the theory. The Ellsberg and MacCrimmon results document only a descriptive failure,

as argued by Roberts [69] in his review of those results.

Although Ellsberg did not at first insist on the rationality of the non-conforming choices, in a reply [27] to Roberts, he did. Other authors had by then joined the fray on normative grounds.

Fellner [30, 31] argued that it is rational to "slant" downward the estimate of an uncertain probability as compared to a known probability.

This is an interesting suggestion. It introduces in a formal way the notion of a "fudge factor", an honorable heuristic engineering practice of ancient heritage. Brewer [18] endorses Fellner's basic position, but cautions against its use in situations like Raiffa's modification of Ellsberg's problem.

The difficulty that concerns Brewer is inconsistency. In the choice between acts I and II, without knowledge that III and IV are available, the majority actor fudges the estimate of \underline{p} , known to be between zero and two-thirds, so that it is less than one third. In the choice between III and IV, the same estimating procedure works against \underline{q} and puts it less than one third.

When all four acts are considered together, the sum of \underline{p} and \underline{q} as estimated by "slanting" is less than two-thirds. Since it is a given of the problem that \underline{p} and \underline{q} sum identically to two-thirds, this is a contradiction. Raiffa's modification exposes this sort of contradiction by deriving another contradiction. Of course, once one contradiction is introduced into a logical system, the possibilities for derived contradiction are endless.

In another context, Kahneman and Tversky [43, 95] explore many other situations where DMs' use of inconsistent probability estimates

leads to silly results.

In any event, Fellner and Brewer evidently resolved their differences and have written together [19] of their fundamental agreement.

One can discuss the Ellsberg problem without concern for subjective probabilities, but as a decision problem intermediate between risk and uncertainty. Inevitably, however, if a solution is suggested that cannot be reconciled with independence, then a conflict with the Savage theory will arise.

If DM's decision rule depended on the independence axiom, then there might be a logical difficulty in such a conflict. Savage's independence principle is very similar to the conventional independence axiom described in Chapter I.

Strictly speaking, one could hold to independence only under risk, and so be free to abandon it under uncertainty. Even so, DM might have a hard time justifying an ad hoc rejection under one regime and not under the other. If one does base decisions on preferences among outcomes, then it seems plausible to argue that similarities among options cannot influence the differences between them. What bearing a knowledge of the probabilities might have is hard to imagine.

Of course, the problem doesn't arise for the ruin-conscious DM, nor for the follower of Kelly. The independence axiom was not assumed under risk, and so no logical difficulty arises if it is not assumed in any form under uncertainty.

One way to avoid estimating the unknown probabilities is to apply the maximin technique. Indeed, in both of Ellsberg's problems, the majority intuitive response is maximin.

Act I's one-third chance of getting the \$100 is, by light of maxi-

min, better than the unknown, and therefore without guarantee, probability of the same amount under Act II. In the other problem, Act IV comes across better than Act III by a similar argument.

As in any other application of maximin, what the probabilities "really" are never comes up.

In Raiffa's modification, we recognize the choice between, in expected utility form:

$$\begin{aligned} \text{A: } & .5 (U(100)/3 + 2U(0)/3) \\ & +.5 (U(0)/3 + 2U(100)/3) \end{aligned}$$

$$\begin{aligned} \text{B: } & .5 (pU(100) + (q + 1/3)U(0)) \\ & +.5 ((q + 1/3)U(100) + pU(0)) \end{aligned}$$

with p plus q equal to two-thirds. Multiplying the two expressions out, we see that they are identical, yielding

$$.5U(0) + .5U(100)$$

A buying-price DM could frame a similar argument based on \underline{F} -utilities. Since Raiffa's result holds for any utility, the buying-price DM is accommodated.

It would be convenient to have a general approach to problems where DM knows something, but not everything, about the state probabilities. In the Ellsberg problem, we actually proceeded by inspection rather than by calculation.

If we set out to apply maximin in the same way as we do under pure uncertainty, we start operating on the columns. That won't do; the solution depends on recognizing the relationship between two of the columns.

Suppose DM knows the probability of one or more states, but not all (or trivially, all except one) and agrees to apply compound probability.

The easiest thing to do is to recast the problem as a compound uncertainty.

One creates new states corresponding to the original states whose probabilities are unknown. In the new decision matrix, each entry is a lottery. The acts remain unchanged.

Each lottery entry yields for the given act and the new state two sets of outcomes. First, the lottery yields the outcomes of the known-probability states with those known probabilities. Second, the lottery gives a complementary probability of the outcome belonging to the act and the corresponding original state.

The procedure is hard to say and simple to illustrate. Suppose the original matrix were the following.

	S1	S2	S3
A1	a	b	c
A2	d	e	f

Suppose further that DM knows that S1 has a probability p , but the other states' probabilities are unknown. The new matrix would be:

	"S2"	"S3"
A1	(p : a, b)	(p : a, c)
A2	(p : d, e)	(p : d, f)

It is easy to see that this is a fair restatement of the problem. If compound probability obtains, then DM would be indifferent between the original formulation and the new one. The notation "S2" is adopted to indicate that the new state corresponds to the original S2, but is not identical to it. Perhaps "S2" should be read "S1 or S2".

Although the new state descriptions are not mutually exclusive, this presents no difficulty in calculating the relevant expected

utilities. If necessary, one can invent state descriptions for the new states that are formally mutually exclusive.

In any case, the maximin solution of the recast problem is simple. One replaces the lotteries by their expected utility values (or appropriate \underline{F} -utility values) and proceeds by linear programming to find the optimal mixed strategy.

Similar techniques can be used if DM's information about state probabilities is not a point estimate. For example, suppose DM knew a lower bound for each of the state probabilities. Trivially, zero is a lower bound for any probability. So, assume also that for at least some states, DM knows a lower bound greater than zero.

Let the lower bound on the \underline{i} -th state be $\underline{l(i)}$. Define \underline{L} as one minus the sum of all the state's lower bounds. If $\underline{L}=0$, then the problem can be solved as an instance of risk; the lower bounds must be the actual probabilities. It cannot be that \underline{L} is less than zero.

If \underline{L} is greater than zero, then define a new pseudo-state for each original state. Build a new matrix with these pseudo-states and the acts, which are unchanged. The entries in the matrix are once again lotteries.

Each lottery entry has as its prizes the row of outcomes from the original matrix. The probabilities of these outcomes vary from column to column.

In the column that corresponds to the \underline{j} -th original state, the probability of the \underline{j} -th original outcome for the row in question is $\underline{l(j)} + \underline{L}$. All other probabilities are the applicable $\underline{l(i)}$.

Thus, if the original matrix (with utility value entries) were:

	S1	S2	S3
A1	a	b	c
A2	d	e	f

then the new matrix would be the same size. If $P(i) = l(i) + L$:

	T1	T2	T3
A1	$P(1)a + l(2)b + l(3)c$	$l(1)a + P(2)b + l(3)c$	$l(1)a + l(2)b + P(3)c$
A2	$P(1)d + l(2)e + l(3)f$	$l(1)d + P(2)e + l(3)f$	$l(1)d + l(2)e + P(3)f$

This second matrix can be viewed as an ordinary unconstrained game against nature. Whatever the probabilities of S1 through S3 are, the same probabilities and pay-offs can be achieved by nature assigning probabilities between zero and one which sum to one to the pseudo-states T1 through T3.

That's because any vector of probabilities for the S's that satisfies the constraints can be written

$$\begin{aligned}
 & (\text{Pr}(S1), \text{Pr}(S2), \text{Pr}(S3)) = \\
 & \quad (l(1), l(2), l(3)) \\
 & + \text{Pr}(T1) (L, 0, 0) \\
 & + \text{Pr}(T2) (0, L, 0) \\
 & + \text{Pr}(T3) (0, 0, L)
 \end{aligned}$$

Assuming, as usual, the applicability of compound probability, DM will be indifferent between the two formulations of one's decision problem. The solution of the second matrix is by ordinary maximin linear programming.

If in addition to the lower bounds, DM knows the exact probability of some states, then this additional information is easy to incorporate

into the lower bound analysis.

The exact probabilities enter the calculation of \underline{L} as if they were mere lower bounds. No pseudo-states correspond to the known-probability states. The lottery entries for the remaining columns simply acquire additional terms, in much the same way as seen in the earlier, no-bound partial knowledge example.

If DM knows both an upper and a lower bound for the state probabilities, things are not quite so neat. Nevertheless, one can easily solve such bounded linear programs using widely-available software library routines.

One can imagine even more general kinds of partial knowledge about state probabilities, to be solved by appropriately constrained programs.

In practice, this level of generality might be computationally tedious. Conceptually, though, it seems possible to unify the techniques for risk, uncertainty and partial risk. All of these can be viewed as programming problems.

In the risk case, the problem is degenerate in that all of nature's probabilities are constrained to be what they are; there is only one column. At the other extreme, uncertainty, the only constraint is that the column probabilities sum to one. In partial risk, the intermediate case in this view, the constraints are somewhere in between.

All of this suggests an important theoretical point. There is no reason at all why nature's probabilities cannot be constrained to fall within a range, rather than forcing DM to adopt a point estimate. The resulting problem will be well-formed and will admit of a solution.

Conclusions

The conventional expected utility axioms described in the first chapter, while sufficient to motivate an expected utility decision rule, are unnecessarily restrictive. Other motivations exist, for instance, Kelly's capital-growth arguments for logarithmic utility. Alternatively, DM's desire to place a constraint on the probability of ruin can be used to motivate a variety of expected utility rules, as discussed in Chapters II and III.

This new approach frees the analyst from making ambitious and unverifiable assumptions about DM's "cardinal value" for wealth. At a practical level, freedom from the axioms allows DM to pursue decision rules which are prohibited by the axioms even though they are perfectly sensible. Thus, in Chapter III, we saw that DM could evaluate gambles according to their buying prices in contemplation of withdrawing capital from risk. In Chapter IV, we saw that DM could treat "once in a lifetime" lotteries as the exceptional opportunities that they plainly are. In this chapter, we saw that DM can use partial knowledge of state probabilities without pretending that one has exact point estimates for all the uncertain probabilities. The difference between a tight estimate and a loose estimate can now receive explicit recognition in the analysis.

Risk and uncertainty are ubiquitous features of engineering systems design problems, ranging from safety and reliability concerns to the frontiers of artificial intelligence "knowledge engineering". Such problems are challenging enough without the added fetters of overly restrictive postulates that rest on speculative theories of subjective personal values.

LIST OF REFERENCES

1. M. Allais, "Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'ecole americaine", Econometrica, vol. 21, pp. 503-546, October 1953.
2. K. J. Arrow, Essays in the Theory of Risk-Bearing, Chicago: Markham Publishing, 1971.
3. K. J. Arrow, "The use of unbounded utility functions in expected utility maximization: response", Quart. J. Econ., vol. 88, pp. 136-138, February 1974.
4. R. J. Aumann, "The St. Petersburg paradox: a discussion of some recent comments", J. Econ. Theory, vol. 14, pp. 443-445, April 1977.
5. W. J. Baumol, "An expected gain confidence limit criterion for portfolio selection", Management Sci., vol. 10, pp. 174-182, October 1963.
6. W. J. Baumol, Economic Theory and Operations Analysis, Englewood Cliffs, NJ: Prentice Hall, 1972.
7. R. M. Bell and T. M. Cover, "Competitive optimality of logarithmic investment", Math. Oper. Res., vol. 5, pp. 161-166, May 1980.
8. C. Berge and A. Ghouila-Houri, Programming, Games and Transportation Networks, London: Methuen, 1965, Chapter 6.
9. D. Bernoulli, "Exposition of a new theory on the measurement of risk", Econometrica, vol. 22, pp. 23-36, January 1954; translation by L. Sommer of "Specimen theoriae novae de mensura sortis", Papers of the Imperial Academy of Science in Petersburg, vol. 5, pp. 175-192, 1738.
10. J. M. Blatt, "Expected utility theory does not apply to all rational men", in B. P. Stigum and F. Wenstøp, eds., Foundations of Utility and Risk Theory with Applications, Dordrecht, Holland: D. Reidel, 1983.
11. K. Borch, "The theory of risk", J. Royal Stat. Soc. Series B, vol. 29, no. 3, pp. 432-467, 1967.
12. K. H. Borch, The Economics of Uncertainty, Princeton: Princeton University Press, 1968, Chapter XIII.
13. K. Borch, "Decision rules depending on the probability of ruin", Oxford Econ. Papers, vol. 20, pp. 1-10, March 1968.
14. K. Borch, "Economic objectives and decision problems", IEEE Trans. Systems Sci. & Cyber., vol. SSC-4, pp. 266-270, September 1968.
15. K. Borch, "The Allais paradox: a comment", Behavioral Sci., vol. 13, pp. 488-489, November 1968.

16. L. Breiman, "Investment policies for expanding businesses optimal in a long run sense", Naval Res. Logistics Quart., vol. 7, pp. 547-651, December 1960.
17. L. Breiman, "Optimal gambling systems for favorable games", in Fourth Berkeley Symposium on Probability and Statistics, vol. 1, pp. 65-78, 1961.
18. K. R. W. Brewer, "Decisions under uncertainty: comment", Quart. J. Econ., vol. 77, pp. 159-161, February 1963.
19. K. R. W. Brewer and W. Fellner, "The slanting of subjective probabilities - agreement on some essentials", Quart. J. Econ., vol. 79, pp. 657-663, November 1965.
20. H. Chernoff, "Rational selection of decision functions", Econometrica, vol. 22, pp. 422-443, October 1954.
21. J. L. Coolidge, "The gambler's ruin", Ann. Math. Second Series, vol. 10, pp. 181-192, July 1909.
22. G. B. Dantzig, "A proof of the equivalence of the programming problem and the game problem", in T. C. Koopmans, ed., Activity Analysis of Production and Allocation, New York: Wiley, 1951.
23. B. de Finetti, "Su un' impostazione alternativa dell teoria collectiva del rischio", Trans. XV Cong. Actuaries, vol. II, pp. 433-443, 1957.
24. M. H. De Groot, Optimal Statistical Decisions, New York: McGraw-Hill, 1970, Chapter 6.
25. D. Ellsberg, "Theory of the reluctant duelist", Amer. Econ. Rev., vol. 46, pp. 909-923, December 1956.
26. D. Ellsberg, "Risk, ambiguity and the Savage axioms", Quart. J. Econ., vol. 75, pp. 690-694, November 1961.
27. D. Ellsberg, in reply to Roberts [69], Quart. J. Econ., vol. 77, pp. 336-342, May 1963.
28. R. A. Epstein, The Theory of Gambling and Statistical Logic, New York: Academic Press, 1977, Chapter 3.
29. W. Feller, An Introduction to Probability Theory and its Applications, New York: Wiley, 1957, vol. I, Chapter 14.
30. W. Fellner, "Distortion of subjective probabilities as a reaction to uncertainty", Quart. J. Econ., vol. 75, pp. 670-689, November 1961.
31. W. Fellner, "Slanted subjective probabilities and randomization: reply to Howard Raiffa and K. R. W. Brewer", Quart. J. Econ., vol. 77, pp. 676-690, November 1963.

32. T. S. Ferguson, "Betting systems which minimize the probability of ruin", J. Soc. Indust. & Appl. Math., vol. 13, pp. 795-818, September 1965.
33. P. C. Fishburn, "Utility theory", Management Sci., vol. 14, pp. 335-378, January 1968.
34. P. C. Fishburn, "Subjective expected utility: a review of normative theories", Theory and Decision, vol. 13, pp. 139-199, June 1981.
35. J. S. Flemming, "The utility of wealth and the utility of wind-falls", Rev. Econ. Studies, vol. 36, pp. 55-66, January 1969.
36. M. Friedman and L. J. Savage, "The utility analysis of choices involving risk", J. Pol. Econ., vol. 56, pp. 279-304, August 1948.
37. M. Friedman and L. J. Savage, "The expected utility hypothesis and the measurability of utility", J. Pol. Econ., vol. 60, pp. 463-474, December 1952.
38. T. C. Fry, Probability and its Engineering Uses, New York: Van Nostrand, 1928.
39. H. U. Gerber, "Games of economic survival with discrete and continuous income processes", Oper. Res., vol. 20, pp. 37-45, January-February 1972.
40. N. H. Hakansson, "Friedman-Savage utility functions consistent with risk aversion", Quart. J. Econ., vol. 84, pp. 472-487, August 1970.
41. L. Hurwicz, "Optimality criteria for decision making under ignorance", Cowles Commission Discussion Paper, Statistics, no. 370, 1951.
42. R. C. Jeffrey, The Logic of Decision, Chicago: University of Chicago Press, 1983.
43. D. Kahneman and A. Tversky, "Prospect theory: an analysis of decision under risk", Econometrica, vol. 47, pp. 263-291, March 1979.
44. D. Kahneman and A. Tversky, "The psychology of preferences", Sci. Amer., vol. 246, pp. 160-173, January 1982.
45. J. L. Kelly, Jr., "A new interpretation of the information rate", Bell Sys. Tech. J., vol. 35, pp. 917-926, July 1956.
46. J. G. Kemeny and G. L. Thompson, "The effect of psychological attitudes on the outcomes of games", in Contributions to the Theory of Games Number 3, Ann. Math. Studies, vol. 39, Princeton: Princeton University Press, 1957.

47. Y. C. Kim, "Choice in the lottery-insurance situation: augmented income approach", Quart. J. Econ., vol. 87, pp. 148-156, February 1973.
48. D. H. Krantz, R. D. Luce, P. Suppes and A. Tversky, Foundations of Measurement, New York: Academic Press, 1971, vol. I, pp. 395-397.
49. N. Y. Kwang, "Why do people buy lottery tickets? Choices involving risk and the indivisibility of expenditure", J. Pol. Econ., vol. 73, pp. 530-535, October 1965.
50. R. D. Luce and H. Raiffa, Games and Decisions: Introduction and Critical Survey, New York: Wiley, 1957.
51. K. R. MacCrimmon, "Descriptive and normative implications of the decision-theory postulates", in K. Borch and J. Mossin, eds., Risk and Uncertainty, New York: MacMillan, 1968.
52. K. R. MacCrimmon and S. Larsson, "Utility theory, axioms vs. 'paradoxes'", in M. Allais and O. Hagen, eds., Expected Utility Hypotheses and the Allais Paradox, Dordrecht, Holland: D. Reidel, 1979, pp. 333-410.
53. M. J. Machina, "'Expected utility' analysis without the independence axiom", Econometrica, vol. 50, pp. 277-323, March 1982.
54. E. Malinvaud, "Note on von Neumann-Morgenstern's strong independence axiom", Econometrica, vol. 20, p. 679, October 1952.
55. A. S. Manne, "The strong independence assumption - gasoline blends and probability mixtures", Econometrica, vol. 20, pp. 665-669, October 1952.
56. H. Markowitz, "The utility of wealth", J. Pol. Econ., vol. 60, pp. 151-158, April 1952.
57. E. F. McClennen, "Some formal problems with the von Neumann and Morgenstern theory of two-person zero-sum games, I: the direct proof", Theory and Decision, vol. 7, pp. 1-28, February-May 1976.
58. E. F. McClennen, "Sure-thing doubts", in B. P. Stigum and F. Wenstøp, eds., Foundations of Utility and Risk Theory with Applications, Dordrecht, Holland: D. Reidel, 1983.
59. R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model", J. Econ. Theory, vol. 3, pp. 373-413, December 1971.
60. R. F. Meyer and J. W. Pratt, "The consistent assessment and fairing of preference functions", IEEE Trans. Systems Sci. & Cyber., vol. SSC-4, pp. 270-278, September 1968.

61. D. G. Morrison, "On the consistency of preferences in Allais' paradox", Behavioral Sci., vol. 12, pp. 373-383, September 1967.
62. G. Owen, "A discussion of minimax", Management Sci., vol. 20, pp. 1316-1317, May 1974.
63. J. Pfanzagl, "A general theory of measurement applications to utility", Naval Res. Logistics Quart., vol. 6, pp. 283-294, December 1959.
64. J. W. Pratt, "Risk aversion in the small and the large", Econometrica, vol. 32, pp. 122-136, January 1964.
65. D. H. Pyle and S. J. Turnovsky, "Safety-first and expected utility maximization in mean-standard deviation portfolio analysis", Rev. Econ. & Stat., vol. 52, pp. 75-81, February 1970.
66. H. Raiffa, "Risk, ambiguity and the Savage axioms: comment", Quart. J. Econ., vol. 75, pp. 690-694, November 1961.
67. H. Raiffa, Decision Analysis: Introductory Lectures on Choices under Uncertainty, Reading, MA: Addison-Wesley, 1968.
68. G. B. Richardson, Information and Investment, Oxford, U.K.: Oxford Press, 1960, Chapter 1.
69. H. V. Roberts, "Risk, ambiguity and the Savage axioms: comment", Quart. J. Econ., vol. 77, pp. 327-342, May 1963.
70. R. N. Rosett, "The Friedman-Savage hypothesis and convex acceptance sets: a reconciliation", Quart. J. Econ., vol. 81, pp. 534-535, August 1967.
71. A. E. Roth, "A note on the maximin value of two-person zero-sum games", Naval Res. Logistics Quart., vol. 29, pp. 521-527, September 1982.
72. A. D. Roy, "Safety-first and the holding of assets", Econometrica, vol. 20, pp. 431-449, July 1952.
73. T. M. Ryan, "The use of unbounded utility functions in expected utility maximization: comment", Quart. J. Econ., vol. 88, pp. 133-135, February 1974.
74. P. A. Samuelson, "Probability, utility and the independence axiom", Econometrica, vol. 20, pp. 670-678, October 1952.
75. P. A. Samuelson, "Lifetime portfolio selection by dynamic stochastic programming", Rev. Econ. and Stat., vol. 51, pp. 239-246, August 1969.

76. P. A. Samuelson, "The fallacy of maximizing the geometric mean in long sequences of investing or gambling", Proc. Nat. Acad. Sci., vol. 68, pp. 2493-2496, October 1971.
77. P. A. Samuelson, "St. Petersburg paradoxes: defanged, dissected and historically described", J. Econ. Lit., vol. 15, pp. 24-55, March 1977.
78. L. J. Savage, "The theory of statistical decisions", Amer. Stat. Assoc. J., vol. 46, pp. 55-67, March 1951.
79. L. J. Savage, The Foundations of Statistics, New York: Dover, 1972.
80. T. C. Schelling, Choice and Consequence, Cambridge, MA: Harvard University Press, 1984.
81. C. O. Segerdahl, in discussion of K. Borch [11], J. Royal Stat. Soc. Series B, vol. 29, no. 3, pp. 455-457, 1967.
82. L. S. Shapley, "The St. Petersburg paradox: a con game?", J. Econ. Theory, vol. 14, pp. 439-442, April 1977.
83. L. S. Shapley, "Lotteries and menus: a comment on unbounded utilities", J. Econ. Theory, vol. 14, pp. 446-453, April 1977.
84. M. Shubik, Strategy and Market Structure, New York: Wiley, 1959, Chapter 10.
85. M. Shubik and G. L. Thompson, "Games of economic survival", Naval Res. Logistics Quart., vol. 6, pp. 111-123, June 1959.
86. H. Sinn, "A rehabilitation of the principle of insufficient reason", Quart. J. Econ., vol. 94, pp. 493-506, May 1980.
87. P. Slovic and A. Tversky, "Who accepts Savage's axiom?", Behavioral Sci., vol. 19, pp. 368-373, November 1974.
88. P. Snow, "Exponential utility and ruin constraints", IEEE Trans. Systems, Man & Cyber., vol. 14, pp. 761-764, September 1984.
89. P. Snow, "A plausible utility-like ordering for lotteries that violates the independence axiom", in Proc. IEEE Conf. Systems, Man & Cyber., pp. 7-11, 1984.
90. P. Snow, "Bounded utilities and ruin constraints", IEEE Trans. Systems, Man & Cyber. (forthcoming).
91. L. Telser, "Safety-first and hedging", Rev. Econ. Studies, vol. 23, no. 1, pp. 1-16, 1955-56.
92. E. O. Thorp, "Portfolio choice and the Kelly criterion", in W. T. Ziemba and R. G. Vickson, eds., Stochastic Optimization Models in Finance, New York: Academic Press, 1975.

93. S. H. Tijs, "A characterization of the value of zero-sum two-person games", Naval Res. Logistics Quart., vol. 28, pp. 153-156, March 1981.
94. G. Tullock, "The irrationality of intransitivity", Oxford Econ. Papers, vol. 16, pp. 401-406, November 1964.
95. A. Tversky and D. Kahneman, "The framing of decisions and the psychology of choice", Science, vol. 211, pp. 453-458, January 1981.
96. E. I. Vilkas, "Axiomatic definition of the value of a matrix game", Theory of Prob. and its Appl., vol. 8, no. 3, pp. 304-307, 1963.
97. J. von Neumann, "On the theory of games of strategy", in A. W. Tucker and R. D. Luce, eds., Contributions to the Theory of Games, Annals of Mathematics Series, 40, Princeton: Princeton University Press, 1959; English translation of "Zur theorie der gessellschaftsspiele", Math. Ann., vol. 100, pp. 295-320, 1928.
98. J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton: Princeton University Press, 1944.
99. H. Wold, "Ordinal preferences or cardinal utility?", Econometrica, vol. 20, pp. 661-664, October 1952.
100. M. Yaari, "Convexity in the theory of choice under risk", Quart. J. Econ., vol. 79, pp. 278-290, May 1965.