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EFFECTS OF ROTARY INERTIA AND SHEAR DEFORMATION ON EXTENSIONAL VIBRATIONS OF CONTINUOUS CIRCULAR CURVED BEAMS

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BY

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To my parents, my wife,

and my son, Omar

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NOMENCLATURE

A	cross-sectional area
γ	mass per unit volume
R	radius of circular arc
a_n	n^{th} constant of integration
E	modulus of elasticity
G	modulus of rigidity
I	in-plane moment of inertia
K	shape factor
P	dynamic concentrated load
$\bar{F}$	shear force as a function of θ and t
$\bar{T}$	axial force as a function of θ and t
$\bar{M}$	moment as a function of θ and t
$\bar{P}$	dynamic distributed load as a function of t
u	radial displacement as a function of θ and t
w	tangential displacement as a function of θ and t
ψ	bending slope as a function of θ and t
M	normal function of moment
p	normal function of the distributed load
F	normal function of shear
T	normal function of axial force
U	normal function of radial displacement
W	normal function of tangential displacement
Ψ	normal function of bending slope

α	central angle of the arch
θ	angular coordinate
t	time
β	angular deformation due to shear
ϕ	total angle between the deformed and undeformed center lines
b	frequency parameter
s	shear effect parameter
r	rotary inertia parameter
e	base of Naperian logarithms
i	$\sqrt{-1}$
Ω	natural frequency
h	horizontal displacement
v	vertical displacement
ξ	roots of characteristic equation
M^F	fixed-end moment
V^F	fixed-end vertical reaction
H^F	fixed-end horizontal reaction
[D]	displacement vector
[X]	constant of integration vector
[F]	force vector
[S]	dynamic stiffness matrix
[TT]	equilibrium force vector
[A], [B], $[A_1]$, $[A_2]$, $[B_1]$	coefficient matrices described by Appendices

$[B_2]$
$[\bar{B}_1]$
$[\bar{B}_2]$
$[\bar{T}\bar{T}]$

coefficient matrices described by Appendices

$[A_{11}]$
$[A_{12}]$
$[A_{21}]$
$[A_{22}]$
$[\bar{A}_{11}]$
$[\bar{A}_{12}]$
$[\bar{A}_{21}]$
$[\bar{A}_{22}]$
$[S_I]$
$[S_{II}]$
$[S_{III}]$
$[\bar{S}_I]$
$[\bar{S}_{II}]$
$[\bar{S}_{III}]$
$[Z]$

coefficient matrices

$[D_1]$
$[F_1]$
$[X_{AC}]$
$[X_{BC}]$
$[F_2]$
$[X_I]$

Vectors

$[X_{II}]$, $[D_I]$, $[D_{II}]$, $[\bar{X}_I]$, $[\bar{X}_{II}]$	Vectors
$[c\ \xi]$, $[c\ \xi\ e^{\xi\theta}]$, $[m]$, $[d]$, $[m\ e^{\xi\theta}]$, $[d\ e^{\xi\theta}]$	Coefficient row matrices described by Appendices
$[\ \ ]^{-1}$	inverse of enclosed matrix
$\lVert\ \ \rVert$	determinate of enclosed matrix

ABSTRACT

EFFECTS OF ROTARY INERTIA AND SHEAR DEFORMATION ON EXTENSIONAL VIBRATIONS OF CONTINUOUS CIRCULAR CURVED BEAMS

by

MOHAMED EL-SAID SAID ISSA

University of New Hampshire, May, 1983

This dissertation is devoted to the dynamic analysis of continuous circular curved beams. The dynamic stiffness matrix is derived for the determination of natural frequencies of continuous curved beams undergoing in-plane vibrations. The formulation of stiffness matrix may be widely applied to problems with various consideration of Bernoulli-Euler Theory, Rayleigh Theory and Timoshenko Theory. Using this formulation for dynamic loading, the fixed-end moment, shear and thrusts for concentrated and distributed loads have been derived. Two continuous circular curved beams subjected to free and forced vibrations are given to illustrate the application of the proposed method and to show the effects of rotary inertia, shear deformation, axial deformation, frequency of the applied load and the central angle of the arc on the beams.

CHAPTER I

INTRODUCTION

Dynamic problems, which are of interest to the structural engineer, play an important role in the fields of civil, mechanical, and aerospace engineering. The problem of curved beams vibrating in the plane of initial curvature of the arc has been discussed by many researchers. In 1928 Den Hartog [1] introduced the Rayleigh-Ritz energy method for finding the fundamental natural frequencies of circular arcs with hinged and fixed ends. His work was extended by Volterra and Morell [2, 3] for vibrations of elastic arcs with hinged and fixed ends having the center lines in the form of cycloids, catenaries, and parabolas. Both in-plane and out-of-plane vibrations were considered. Waltking [4] investigated the effect of extension of the central line on the flexural motion of a pinned circular ring segment. Morley [5] presented an exact solution for the thin ring along with the first ten modes for symmetrical and anti-symmetrical vibrations. Using the basic equations of motion given by Love [6], Archer [7] made a mathematical study of the in-plane inextensional vibrations of circular ring segments of small cross section with an additional term to represent damping effects. Nelson [8], combining the Rayleigh-Ritz technique and Lagrangian multipliers, obtained natural frequency equations in the form of infinite series for in-plane vibrations of a simply supported ring segment. Suzuki, Takahashi and Ishiyama [9, 10] have obtained the natural frequencies and the mode shapes for curved beams with variable curvatures, and a generalized method for the analysis of free and forced in-plane vibrations of a multispan circular curved frame was developed by Wang and Lee [11, 12].

The elementary Bernoulli-Euler equation of motion of beams used by the work mentioned above is derived on the assumption that the deflections of beams are due to flexure only and that both rotary inertia and transverse shear effects are neglected, it is considered adequate for the usual engineering problems. However, for beams having large cross sectional dimensions in comparison to their lengths, and for beams in which higher modes are required, the Timoshenko Theory [13], which takes into account these two effects, provides a better approximate solution to the actual beam behavior [29].

Considerable research has been devoted to study the effects of rotary inertia and shear on straight beam vibrations. Cheng [14] derived the dynamic stiffness formulation in closed form for analyzing continuous beams and frameworks. Cheng and Tseng [15] presented a dynamic stiffness matrix formulation and computational procedures for dynamic response of general plane beam-column system. In case of curved beams, Philipson [16] took into account the rotary inertia and shear effects on thin rings and established equations for the radial and tangential displacements. Seidal and Erdelyi [17] based on beam theory to develop a method for studying the bending vibrations of non-thin complete circular rings. The in-plane vibrations of a circular ring including rotary inertia and shear effects was investigated by Rao and Sundararajan [18]. Recently, Wang and Guilbert [19] expanded Wang and Lee's [11] generalized method for continuous circular curved beams by including the effect of rotary inertia and transverse shear deformation.

To the author's knowledge, no investigations have been made for the free and forced extensional vibrations of multispan circular curved beams including shear and rotary inertia effects. In this dissertation, the dynamic stiffness matrix is derived in terms of rotary inertia, radial shear deformation, and bending deformation. The individual parameter may be dropped when the appropriate deformation is not considered. Therefore, the formulation of stiffness matrix may be widely applied to various cases of Bernoulli-Euler Theory, Rayleigh Theory, and Timoshenko Theory. Using this formulation for dynamic loading, the fixed-end moments, shear and thrusts for concentrated loads and distributed loads have been derived. Numerical results of a three-span circular curved beam are presented to show the effects of central angle, axial deformation, transverse shear deformation, and rotary inertia of the circular curved beam upon the natural frequencies of the curved beam. Also, the effects of the frequency of the applied load, axial deformation, transverse shear deformation, rotary inertia, and the central angle of the arc on the joint moments of the circular curved beam are studied.

CHAPTER II

GENERAL DERIVATION

1. Assumptions

The following assumptions are presented into two parts:

A. Basic assumptions

a. Displacements are limited to vibrations within the original plane of curvature of the circular curved member.

b. The centroidal axis of the circular curved member is considered to be extensional.

c. Linear stress-strain relations assumed.

B. Simplifing assumptions

a. Plane cross sections remain plane after deformation.

b. The density and cross section of the member is constant.

c. The effect of damping upon the circular curved member is neglected.

d. The vibrations of the circular curved member are considered small. As a result, the effect of high order differentials are neglected.

2. Basic Equations of Motion

Consider a circular curved beam element undergoing in-plane vibration as shown in Fig. 1. By taking the equilibrium of the forces acting on that element in radial and tangential directions and the moment about C, give (the complete derivation is given in Appendix A)

$$\left.\begin{aligned} \frac{\partial \bar{F}}{\partial \theta} + \bar{T} &= f_{I1} R \\ \frac{\partial \bar{T}}{\partial \theta} - \bar{F} &= f_{I2} R \\ -\frac{\partial \bar{M}}{\partial \theta} + \bar{F} R &= m_I R \end{aligned}\right\} \qquad (1)$$

where f_{I1} is the inertia force in the radial direction, f_{I2} the inertia force in the tangential direction, m_I the rotary inertia, $\bar{F}$ the shearing force, $\bar{T}$ the axial force, $\bar{M}$ the moment, and θ the angular coordinate.

The expressions for the inertia force and rotary inertia are given as follows [31]:

$$\left.\begin{aligned} f_{I1} &= \gamma A \frac{\partial^2 u}{\partial t^2} \\ f_{I2} &= \gamma A \frac{\partial^2 w}{\partial t^2} \\ m_I &= \gamma I \frac{\partial^2 \psi}{\partial t^2} \end{aligned}\right\} \qquad (2)$$

Substituting equations (2) into equations (1) we have

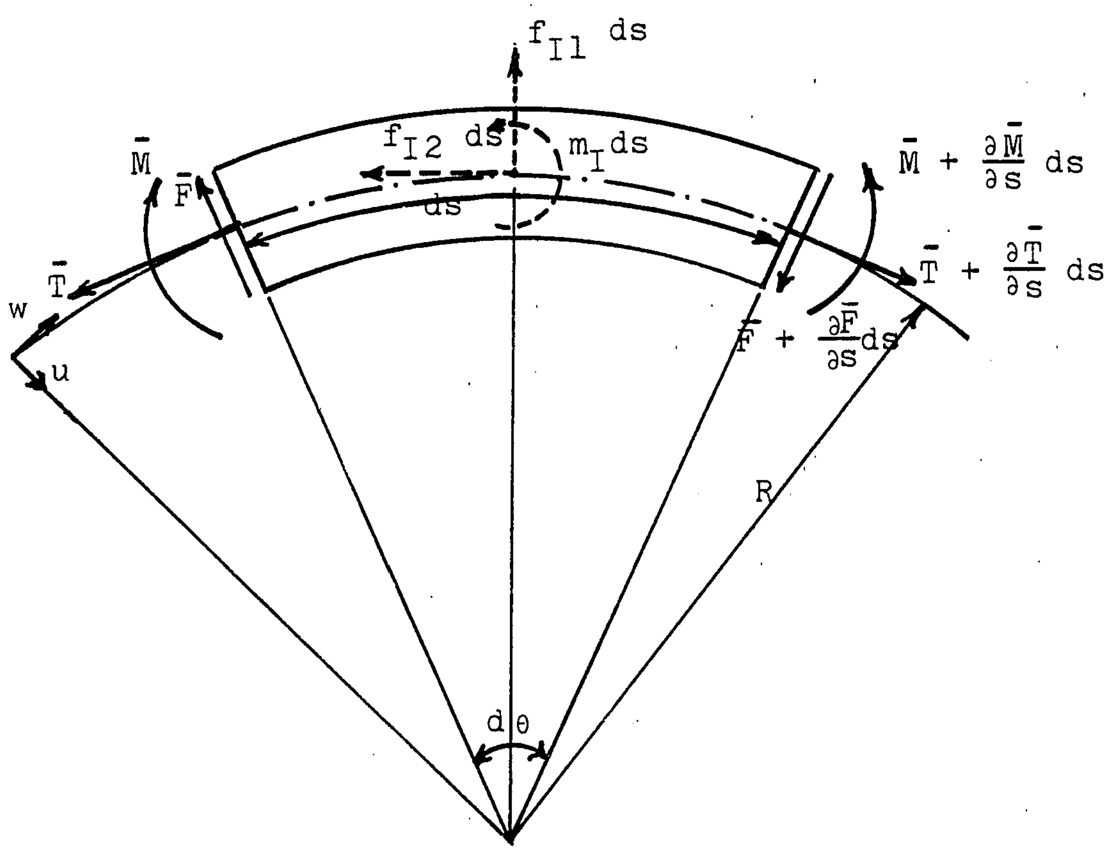

Figure 1. Curved element subjected to positive forces and moments

$$\frac{\partial \bar{F}}{\partial \theta} + \bar{T} = \gamma AR \frac{\partial^2 u}{\partial t^2} \qquad (3)$$

$$\frac{\partial \bar{T}}{\partial \theta} - \bar{F} = \gamma AR \frac{\partial^2 w}{\partial t^2} \qquad (4)$$

$$\frac{\partial \bar{M}}{\partial \theta} + \bar{F}R = \gamma IR \frac{\partial^2 \psi}{\partial t^2} \qquad (5)$$

where u is the inward radial displacement, w the tangential displacement in the sense of increasing θ, R the mean radius of circular arc, t the time, A the cross sectional area, γ the mass per unit volume, I the moment of inertia of cross section, and ψ slope due to bending.

Figure 2a shows the flexural vibration of an element for the Bernoulli-Euler beam, it is seen that the x-axis is perpendicular to any beam cross section (abcd). Figure 2b represents the effect of shear deformation on beam vibration by assuming that the beam cross section remains plane after deformation, we notice that the cross sections are no longer perpendicular to the x-axis (i.e. Timoshenko beam). The shear deformation β can be expressed [33] by

$$\beta = \phi - \psi \qquad (6)$$

where ϕ is the total angle of the deflection curve of the beam.

Referring to Fig. 2c the rotation due to tangential displacement and radial displacement are given, respec-

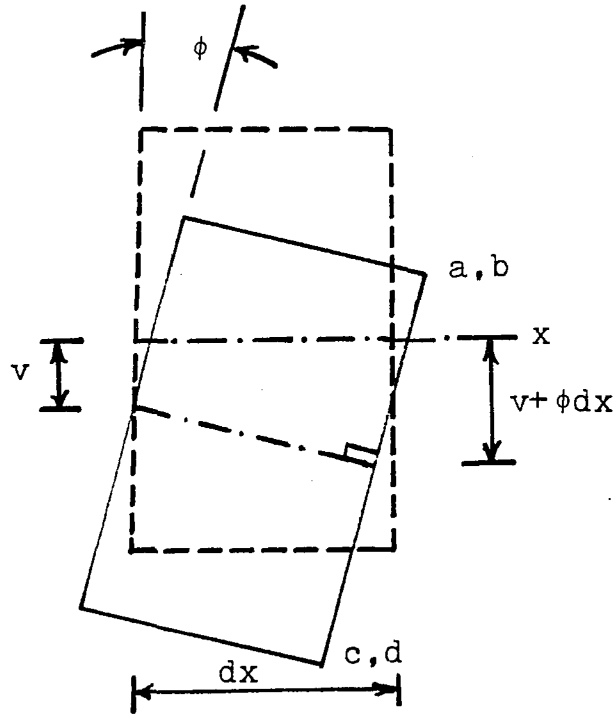

Figure 2a. Deformation of Bernoulli-Euler beam element showing x-axis perpendicular to cross sections.

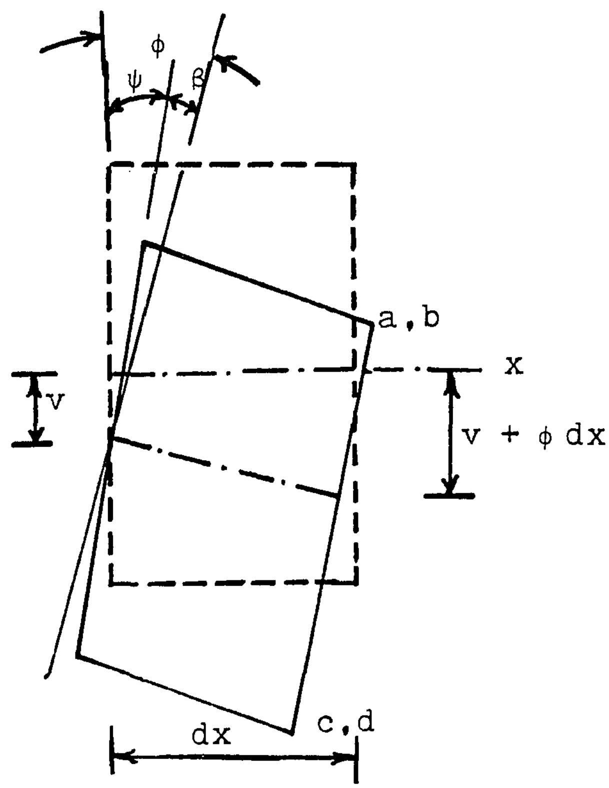

Figure 2b. Deformation of Timoshenko beam element showing shear deformation, x-axis are no longer perpendicular to cross sections.

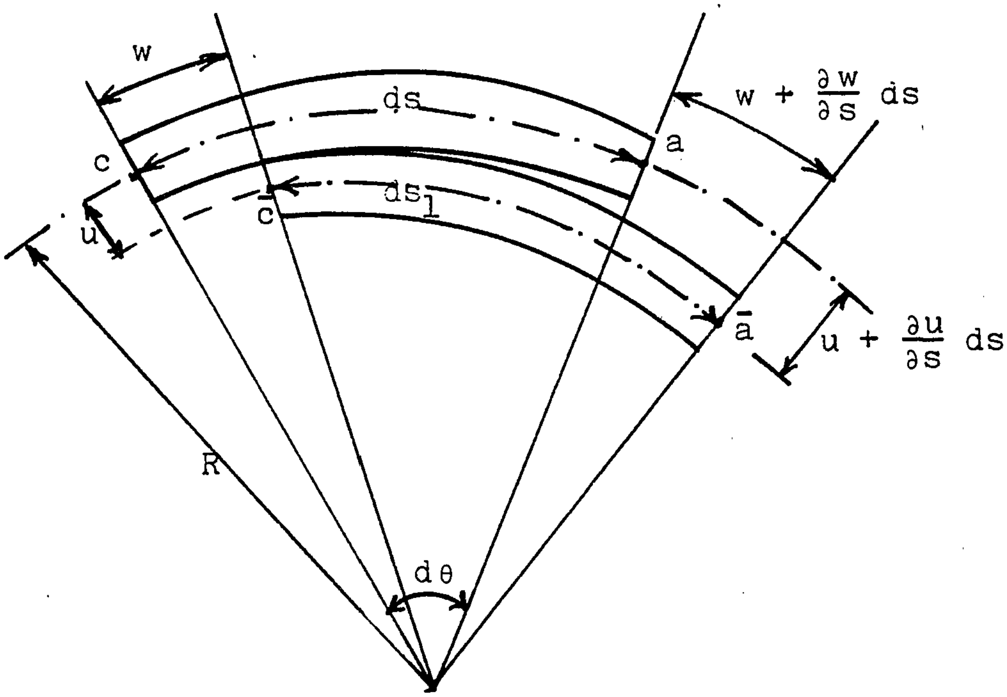

Figure 2c. Deformation of curved element.

tively, as

$$\phi_1 = \frac{w}{R} \tag{7}$$

$$\phi_2 = \frac{(u + \frac{\partial u}{\partial s} ds) - u}{ds} = \frac{\partial u}{\partial s} \tag{8}$$

The total angle ϕ between the deformed and undeformed center line can be obtained from equations (7) and (8) as

$$\phi = \phi_1 + \phi_2 = \frac{1}{R} (w + \frac{\partial u}{\partial \theta}) \tag{9}$$

From equations (6) and (9) one obtains

$$\phi = \psi + \beta = \frac{1}{R} (w + \frac{\partial u}{\partial \theta}) \tag{10}$$

Equation (10) may be rewritten in terms of the shear angle as

$$\beta = \frac{1}{R} (w + \frac{\partial u}{\partial \theta} - R\psi) \tag{11}$$

Referring again to Fig. 2c, the extension of the element can be expressed as

$$de = ds_1 - ds = [(R - u)d\theta + (w + \frac{\partial w}{\partial s} ds) - w] - Rd\theta$$

$$= (\frac{\partial w}{\partial \theta} - u) \, d\theta \tag{12}$$

The tangential strain and tangential force may be written, respectively, as

$$\varepsilon_s = \frac{de}{ds} = \frac{1}{R} (\frac{\partial w}{\partial \theta} - u) \tag{13}$$

$$\bar{T} = \sigma_s \cdot A = EA\,\varepsilon_s \qquad (14)$$

where σ_s is the tangential stress.

From the elementary theory of beams, the bending moments and shear force are given, respectively, as follows:

$$\bar{M} = -\frac{EI}{R}\frac{\partial\psi}{\partial\theta} \qquad (15)$$

$$\bar{F} = kAG\beta \qquad (16)$$

where E is the modulus of elasticity, k the shape factor of the curved member, G the shear modulus, and ε_s the tangential strain.

From equations (11) and (16), the following equation can be obtained:

$$\bar{F} = \frac{kAG}{R}\left(\frac{\partial u}{\partial\theta} + w - R\psi\right) \qquad (17)$$

Combining equations (14) and (13) one obtains

$$\bar{T} = \frac{EA}{R}\left(\frac{\partial w}{\partial\theta} - u\right) \qquad (18)$$

Introducing equations (17) and (18) into equation (3) yields

$$\frac{kAG}{R}\left(\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial w}{\partial\theta} - R\frac{\partial\psi}{\partial\theta}\right) + \frac{EA}{R}\left(\frac{\partial w}{\partial\theta} - u\right) = \gamma AR\frac{\partial^2 u}{\partial t^2} \qquad (19)$$

Substituting now equations (17) and (18) into equation (4) we have

$$\frac{EA}{R}\left(\frac{\partial^2 w}{\partial\theta^2} - \frac{\partial u}{\partial\theta}\right) - \frac{kAG}{R}\left(\frac{\partial u}{\partial\theta} + w - R\psi\right) = \gamma AR\frac{\partial^2 w}{\partial t^2} \qquad (20)$$

Finally, the substitution of equations (15) and (17) into equation (5) gives

$$\frac{EI}{R}\frac{\partial^2\psi}{\partial\theta^2} + kAG\left(\frac{\partial u}{\partial\theta} + w - R\psi\right) = \gamma IR\frac{\partial^2\psi}{\partial t^2} \qquad (21)$$

Equations (19), (20) and (21) constitute the equations of motion of extensional circular curved member undergoing free vibrations.

3. General Solution of the Equations of Motion

Assuming the beam is excited harmonically with a natural frequency Ω and letting

$$\left.\begin{aligned} w(\theta,t) &= W(\theta)\, e^{i\Omega t} \\ u(\theta,t) &= U(\theta)\, e^{i\Omega t} \\ \psi(\theta,t) &= \Psi(\theta)\, e^{i\Omega t} \end{aligned}\right\} \qquad (22)$$

where $i = \sqrt{-1}$ and $W(\theta)$, $U(\theta)$ and $\Psi(\theta)$ are the normal function of w, u and ψ, respectively. Substituting equation (22) into equations (19), (20) and (21) and omitting the common term $e^{i\Omega t}$, one obtains

$$-\frac{d^2U}{d\theta^2} - \left(1 + \frac{E}{kG}\right)\frac{dW}{d\theta} + R\frac{d\Psi}{d\theta} + \frac{E}{kG}U - \frac{\gamma R^2\Omega^2}{kG}U = 0 \qquad (23)$$

$$\frac{E}{kG}\frac{d^2W}{d\theta^2} - \left(1 + \frac{E}{kG}\right)\frac{dU}{d\theta} - W + R\Psi + \frac{\gamma R^2\Omega^2}{kG}W = 0 \qquad (24)$$

$$\frac{EI}{kAG} \frac{d^2\Psi}{d\theta^2} + R \frac{dU}{d\theta} + RW - R^2\Psi + \frac{\gamma IR^2\Omega^2}{kAG} \Psi = 0 \qquad (25)$$

The foregoing equations may be written, respectively, as

$$-\frac{d^2U}{d\theta^2} - (1 + \frac{s^2}{r^2}) \frac{dW}{d\theta} + R \frac{d\Psi}{d\theta} + \frac{s^2}{r^2} U - b^2 \cdot s^2 U = 0 \qquad (26)$$

$$\frac{s^2}{r^2} \frac{d^2W}{d\theta^2} - (1 + \frac{s^2}{r^2}) \frac{dU}{d\theta} - W + R\Psi + b^2 \cdot s^2 W = 0 \qquad (27)$$

$$R \cdot s^2 \frac{d^2\Psi}{d\theta^2} + \frac{dU}{d\theta} + W - R\Psi + R \cdot s^2 \cdot r^2 \cdot b^2 \Psi = 0 \qquad (28)$$

where b, s and r are dimensionless parameters and are corresponding to the effect of bending deformation, shear deformation and rotary inertia, respectively. The expressions for these non-dimensional parameters are given as follows:

$$b^2 = \frac{\gamma AR^4\Omega^2}{EI} \qquad (29)$$

$$s^2 = \frac{EI}{kAGR^2} \qquad (30)$$

$$r^2 = \frac{I}{AR^2} \qquad (31)$$

Equation (26), (27) and (28) represent three linear differential equations with constant coefficients and can be solved by the use of symbolic operator method.

Let the differential operator D be $\frac{d}{d\theta}$, then equations (26), (27) and (28) become

$$(-D^2 + \frac{s^2}{r^2} - b^2 \cdot s^2)U - (D + \frac{s^2}{r^2} D)W + (RD)\Psi = 0 \qquad (32)$$

$$-(D + \frac{s^2}{r^2} D)U + (\frac{s^2}{r^2} D^2 + b^2 \cdot s^2 - 1)W + R\Psi = 0 \qquad (33)$$

$$DU + W + R(s^2D^2 - 1 + r^2 \cdot s^2 \cdot b^2)\Psi = 0 \qquad (34)$$

Equation (33) may be rewritten as

$$R\Psi = (D + \frac{s^2}{r^2} D)U - (\frac{s^2}{r^2} D^2 + b^2 \cdot s^2 - 1)W \qquad (35)$$

Substituting equation (35) into equations (32) and (34) yields, respectively

$$(\frac{s^2}{r^2} D^2 + \frac{s^2}{r^2} - b^2 \cdot s^2)U - (\frac{s^2}{r^2} D + \frac{s^2}{r^2} D^3 + b^2 \cdot s^2 D)W = 0 \qquad (36)$$

$$(s^2D^3 + \frac{s^2}{r^2} D^3 + r^2 \cdot s^2 \cdot b^2 D - \frac{s^2}{r^2} D + s^4 \cdot b^2 D)U - (\frac{s^2}{r^2} D^4$$

$$- \frac{s^2}{r^2} D^2 + s^4 \cdot b^2 D^2 + s^4 \cdot b^2 D^2 - b^2 \cdot s^2 + r^2 \cdot s^4 \cdot b^4$$

$$- s^2D^2 - r^2 \cdot s^2 \cdot b^2)W = 0 \qquad (37)$$

Solving equations (36) and (37) simultaneously yields (See Appendix B for the complete derivation)

$$W^{VI} + k_1 W^{IV} + k_2 W'' + k_3 W = 0 \qquad (38)$$

$$U^{VI} + k_1 U^{IV} + k_2 U'' + k_3 U = 0 \qquad (39)$$

where

$$\left.\begin{aligned} k_1 &= 2 + 2b^2 \cdot r^2 + b^2 \cdot s^2 \\ k_2 &= 1 - b^2 + 2b^4 \cdot r^2 \cdot s^2 + b^2 \cdot r^2 - b^2 \cdot s^2 + b^4 \cdot r^4 \\ k_3 &= b^2 - b^4 \cdot r^2 \cdot s^2 + b^2 \cdot r^2 - b^4 \cdot r^2 + b^6 \cdot r^4 \cdot s^2 - b^4 \cdot r^4 \end{aligned}\right\} \quad (40)$$

The solution of equation (38) may be expressed as

$$W(\theta) = \sum_{n=1}^{6} a_n e^{\xi_n \theta} \qquad (41)$$

where a_n ($n = 1,2,. . . 6$) are constants to be determined from the boundary conditions, and ξ_n ($n = 1,2,. . . 6$) are the roots of the characteristic equation of equation (38) and may be written as

$$\xi^6 + k_1 \xi^4 + k_2 \xi^2 + k_3 = 0 \qquad (42)$$

The solution of equation (39) can be written as

$$U(\theta) = \sum_{n=1}^{6} z_n e^{\xi_n \theta} \qquad (43)$$

Substituting equations (41) and (43) into equation (36) yields the relation between the constants a's and z's as follows:

$$z_n = \lambda_n a_n \tag{44}$$

where

$$\lambda_n = \frac{\xi_n + \xi_n^3 + b^2 \cdot r^2 \xi_n}{\xi_n^2 - b^2 \cdot r^2 + 1} \tag{45}$$

From equations (43) and (44) one obtains

$$U(\theta) = \sum_{n=1}^{6} \lambda_n a_n e^{\xi_n \theta} \tag{46}$$

Introducing equations (41) and (46) into equation (35) yields

$$R\psi(\theta) = \sum_{n=1}^{6} a_n c_n e^{\xi_n \theta} \tag{47}$$

where

$$c_n = \lambda_n \xi_n + \frac{s^2}{r^2} \lambda_n \xi_n - \frac{s^2}{r^2} \xi_n^2 - b^2 \cdot s^2 + 1 \tag{48}$$

CHAPTER III

DERIVATION OF DYNAMIC STIFFNESS OF CIRCULAR CURVED BEAM

Consider the in-plane vibration of a curved member having constant cross section subjected to translational and rotational displacements at the two ends A and B as shown in Fig. 3.

For harmonic vibrations, let

$$\bar{M}(\theta,t) = M(\theta)\ e^{i\Omega t} \qquad (49)$$

$$\bar{F}(\theta,t) = F(\theta)\ e^{i\Omega t} \qquad (50)$$

$$\bar{T}(\theta,t) = T(\theta)\ e^{i\Omega t} \qquad (51)$$

where M, F and T are normal function of $\bar{M}$, $\bar{F}$ and $\bar{T}$, respectively.

Substituting equations (22), (49), (50) and (51) into equations (15), (17) and (3) and omitting the common term $e^{i\Omega t}$ yields

$$M(\theta) = -\frac{EI}{R}\,\Psi'(\theta) \qquad (52)$$

$$F(\theta) = \frac{kAG}{R}\,[U'(\theta) + W(\theta) - R\Psi(\theta)] \qquad (53)$$

$$T(\theta) = -F'(\theta) - \gamma AR\Omega^2\,U(\theta) \qquad (54)$$

Introducing equations (41), (46) and (47) into equations (52), (53) and (54) give

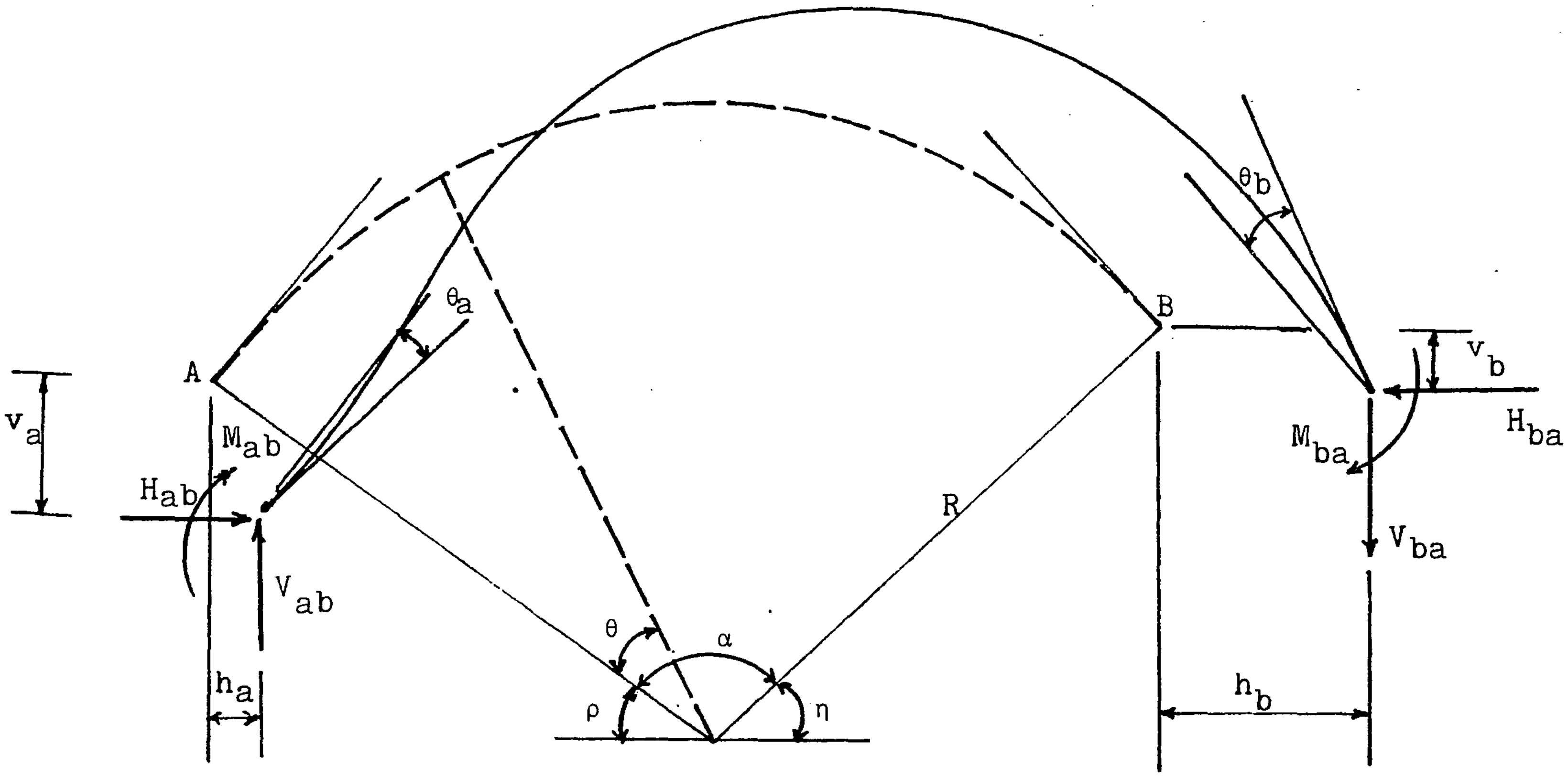

Figure 3. Positive displacements, forces and moments with common factor $e^{i\Omega t}$ omitted

$$M(\theta) = -\frac{EI}{R^2} \sum_{n=1}^{6} c_n \, a_n \, \xi_n \, e^{\xi_n \theta} \tag{55}$$

$$F(\theta) = \frac{EI}{R^3} \sum_{n=1}^{6} m_n \, a_n \, e^{\xi_n \theta} \tag{56}$$

$$T(\theta) = -\frac{EI}{R^3} \sum_{n=1}^{6} d_n \, a_n \, e^{\xi_n \theta} \tag{57}$$

where

$$m_n = \frac{\lambda_n \xi_n + 1 - c_n}{s^2} \tag{58}$$

$$d_n = m_n \, \xi_n + b^2 \lambda_n \tag{59}$$

and ξ_n, c_n, s and b are defined previously.

Referring again to Fig. 3, the boundary conditions are

$$\left.\begin{aligned}
\theta_a &= \Psi(\theta = 0) \\
\theta_b &= \Psi(\theta = \alpha) \\
v_a &= U(0) \sin \rho - W(0) \cos \rho \\
v_b &= U(\alpha) \sin \eta + W(\alpha) \cos \eta \\
h_a &= U(0) \cos \rho + W(0) \sin \rho \\
h_b &= -U(\alpha) \cos \eta + W(\alpha) \sin \eta
\end{aligned}\right\} \tag{60}$$

The bending moments and thrusts at both ends may be expressed as

$$\left.\begin{aligned}
M_{ab} &= M(\theta = 0) \\
M_{ba} &= -M(\theta = \alpha) \\
V_{ab} &= F(0)\sin\rho - T(0)\cos\rho \\
V_{ba} &= F(\alpha)\sin\eta + T(\alpha)\cos\eta \\
H_{ab} &= -F(0)\cos\rho - T(0)\sin\rho \\
H_{ba} &= F(\alpha)\cos\eta - T(\alpha)\sin\eta
\end{aligned}\right\} \tag{61}$$

Substituting equations (41), (46) and (47) into equation (60) yields

$$\left.\begin{aligned}
R\theta_a &= \sum_{n=1}^{6} a_n c_n \\
R\theta_b &= \sum_{n=1}^{6} a_n c_n e^{\xi_n \alpha} \\
v_a &= \sin\rho \sum_{n=1}^{6} \lambda_n a_n - \cos\rho \sum_{n=1}^{6} a_n \\
v_b &= \sin\eta \sum_{n=1}^{6} a_n \lambda_n e^{\xi_n \alpha} + \cos\eta \sum_{n=1}^{6} a_n e^{\xi_n \alpha} \\
h_a &= \cos\rho \sum_{n=1}^{6} a_n \lambda_n + \sin\rho \sum_{n=1}^{6} a_n \\
h_b &= -\cos\eta \sum_{n=1}^{6} a_n \lambda_n e^{\xi_n \alpha} + \sin\eta \sum_{n=1}^{6} a_n e^{\xi_n \alpha}
\end{aligned}\right\} \tag{62}$$

Equations (62) may be rewritten in the following forms:

$$[D] = [A]\,[X] \tag{63}$$

where

$$[D] = \begin{bmatrix} R\,\theta_a \\ R\,\theta_b \\ v_a \\ v_b \\ h_a \\ h_b \end{bmatrix} , \quad [X] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \tag{64}$$

and [A] is given in Appendix C.

Substituting equations (55), (56) and (57) into equation (61) one obtains

$$\left.\begin{aligned}
\frac{M_{ab}}{R} &= - \frac{EI}{R^3} \sum_{n=1}^{6} c_n\, a_n\, \xi_n \\
\frac{M_{ba}}{R} &= \frac{EI}{R^3} \sum_{n=1}^{6} c_n\, a_n\, \xi_n\, e^{\xi_n \alpha} \\
V_{ab} &= \frac{EI}{R^3} \sin\rho \sum_{n=1}^{6} a_n\, m_n + \frac{EI}{R^3} \cos\rho \sum_{n=1}^{6} a_n\, d_n \\
V_{ba} &= \frac{EI}{R^3} \sin\eta \sum_{n=1}^{6} a_n\, m_n\, e^{\xi_n \alpha} - \frac{EI}{R^3} \cos\eta \sum_{n=1}^{6} a_n\, d_n\, e^{\xi_n \alpha} \\
H_{ab} &= - \frac{EI}{R^3} \cos\rho \sum_{n=1}^{6} a_n\, m_n + \frac{EI}{R^3} \sin\rho \sum_{n=1}^{6} a_n\, d_n \\
H_{ba} &= \frac{EI}{R^3} \cos\eta \sum a_n\, m_n\, e^{\xi_n \alpha} + \frac{EI}{R^3} \sin\eta \sum_{n=1}^{6} a_n\, d_n\, e^{\xi_n \alpha}
\end{aligned}\right\} \tag{65}$$

Equation (65) may be rewritten in the matrix form as

$$[F] = \frac{EI}{R^3} [B] [X] \qquad (66)$$

where

$$[F] = \begin{bmatrix} M_{ab/R} \\ M_{ba/R} \\ V_{ab} \\ V_{ba} \\ H_{ab} \\ H_{ba} \end{bmatrix} \qquad (67)$$

and [B] is given in Appendix C and [X] is defined already.

Premultiplying equation (63) by $[A]^{-1}$, one obtains

$$[X] = [A]^{-1} [D] \qquad (68)$$

Substituting equation (68) into equation (66) gives

$$[F] = \frac{EI}{R^3} [B] [A]^{-1} [D] \qquad (69)$$

Equation (69) may be rewritten as

$$[F] = [S] [D] \qquad (70)$$

where [S], the dynamic stiffness matrix for a curved member, directly relates the end moments and thrusts to end rotations and deflections. [S] is given by

$$[S] = \frac{EI}{R^3} [B] [A]^{-1} \qquad (71)$$

CHAPTER IV

A SINGLE CIRCULAR CURVED BEAM FIXED AT ENDS

I. Dynamic Concentrated Load

The circular curved member shown in Fig. 4 is subjected to a dynamic concentrated load at any point C. The two segments, AC and CB will be considered as free bodies to determine the constants, a_n, given in equations (41), (46) and (47).

Consider first arc AC as a free body. The use of Fig. 5 and equations (41), (46) and (47) will give the following matrix equations, where displacements at the fixed-end A equals zero.

$$[D_1] = [A_1]\,[X_{AC}] \qquad (72)$$

where $[A_1]$ is given in Appendix D, and

$$[D_1] = \begin{bmatrix} R\theta_c \\ U_c \\ W_c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [X_{AC}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \qquad (73)$$

Also, the use of Fig. 5 and equations (55), (56) and (57) will give the following matrix equation:

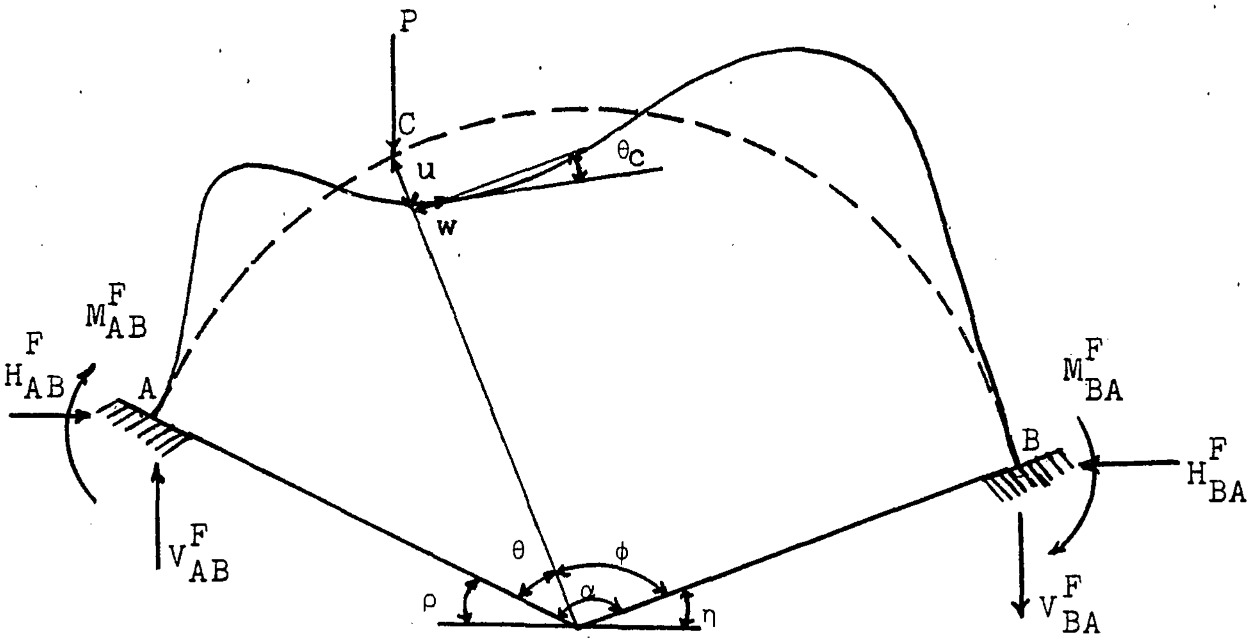

Figure 4. Fixed-end curved member under the effect of a dynamic concentrated load with the common factor $e^{i\Omega t}$ omitted

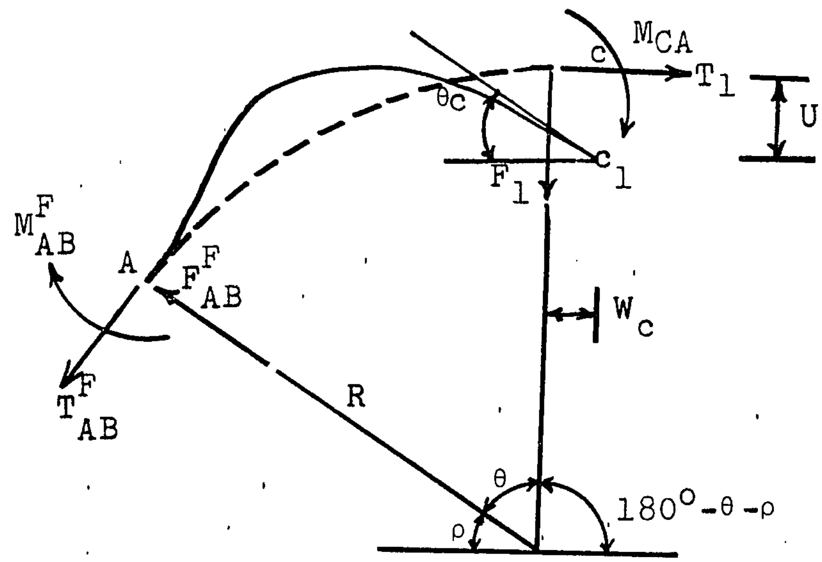

Figure 5. Displacements, forces and moments of arc AC

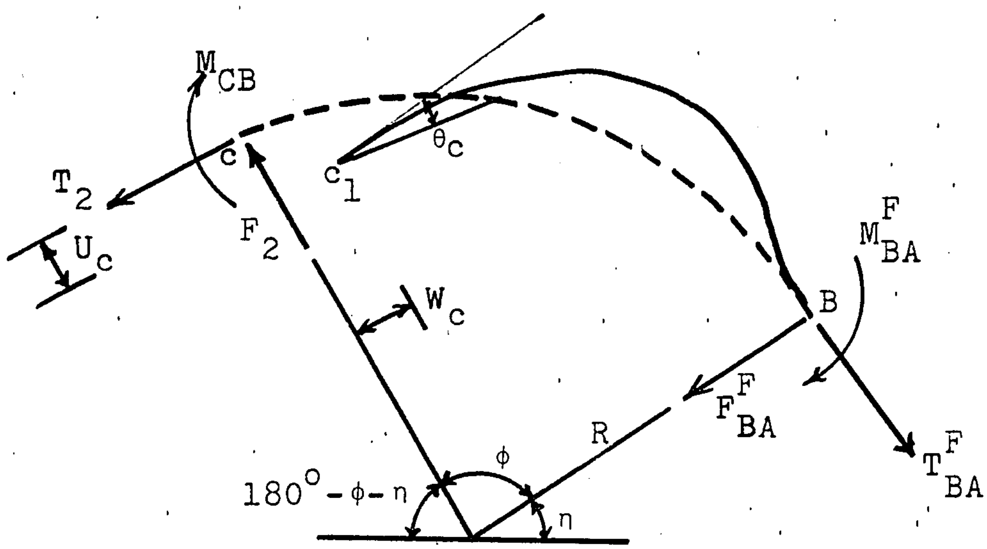

Figure 6. Displacements, forces and moments of arc BC

$$[F_1] = \frac{EI}{R^3} [B_1] [X_{AC}] \qquad (74)$$

where $[B_1]$ is given in Appendix D, and

$$[F_1] = \begin{bmatrix} M_{CA/R} \\ -T_1 \\ F_1 \\ M^F_{AB/R} \\ T^F_{AB} \\ F^F_{AB} \end{bmatrix} \qquad (75)$$

Consider next arc BC as a free body. The use of Fig. 6 and equations (41), (46) and (47) will give the following matrix equations, where displacements at the fixed-end B equal zero.

$$[D_1] = [A_2] [X_{BC}] \qquad (76)$$

where $[A_2]$ is given in Appendix D, and

$$[X_{BC}] = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \\ \bar{a}_5 \\ \bar{a}_6 \end{bmatrix} \qquad (77)$$

$[D_1]$ is defined previously.

Using Fig. 6 and equations (55), (56) and (57), the following matrix equation is obtained

$$[F_2] = \frac{EI}{R^3} [B_2] [X_{BC}] \tag{78}$$

where $[B_2]$ is given in Appendix D, and

$$[F_2] = \begin{bmatrix} M_{CB/R} \\ T_2 \\ -F_2 \\ M^F_{BA/R} \\ -T^F_{BA} \\ F^F_{BA} \end{bmatrix} \tag{79}$$

Using static condensation method, equation (72) may be rewritten as

$$\begin{bmatrix} D_I \\ --- \\ D_{II} \end{bmatrix} = \left[\begin{array}{c|c} [A_{11}] & [A_{12}] \\ \hline [A_{21}] & [A_{22}] \end{array}\right] \begin{bmatrix} X_{ACI} \\ ---- \\ X_{ACII} \end{bmatrix} \tag{80}$$

From equations (80) we have

$$[D_I] = [A_{11}]\,[X_{ACI}] + [A_{12}]\,[X_{ACII}] \qquad (81)$$

$$[D_{II}] = [A_{21}]\,[X_{ACI}] + [A_{22}]\,[X_{ACII}] \qquad (82)$$

Solving equation (82) for $[X_{ACII}]$ gives

$$[X_{II}] = [S_{II}]\,[X_I] \qquad (83)$$

where

$$[S_{II}] = -\,[A_{22}]^{-1}\,[A_{21}] \qquad (84)$$

Substituting equation (83) into equation (81) yields

$$[X_{ACI}] = [S_I]^{-1}\,[D_I] \qquad (85)$$

where

$$[S_I] = [A_{11}] + [A_{12}]\,[S_{II}] \qquad (86)$$

From equations (83) and (85), the following equation can be obtained:

$$[X_{ACII}] = [S_{III}]\,[D_I] \qquad (87)$$

where

$$[S_{III}] = [S_{II}]\,[S_I]^{-1} \qquad (88)$$

and

$$\left.\begin{aligned}
A_{11} &= \begin{bmatrix} c_1\, e^{\xi_1\theta} & c_2\, e^{\xi_2\theta} & c_3\, e^{\xi_3\theta} \\ \lambda_1\, e^{\xi_1\theta} & \lambda_2\, e^{\xi_2\theta} & \lambda_3\, e^{\xi_3\theta} \\ e^{\xi_1\theta} & e^{\xi_2\theta} & e^{\xi_3\theta} \end{bmatrix} \\
A_{12} &= \begin{bmatrix} c_4\, e^{\xi_4\theta} & c_5\, e^{\xi_5\theta} & c_6\, e^{\xi_6\theta} \\ \lambda_4\, e^{\xi_4\theta} & \lambda_5\, e^{\xi_5\theta} & \lambda_6\, e^{\xi_6\theta} \\ e^{\xi_4\theta} & e^{\xi_5\theta} & e^{\xi_6\theta} \end{bmatrix} \\
A_{21} &= \begin{bmatrix} c_1 & c_2 & c_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix} , \quad A_{22} = \begin{bmatrix} c_4 & c_5 & c_6 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ 1 & 1 & 1 \end{bmatrix} \\
X_{ACI} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} , \quad X_{ACII} = \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} , \quad D_I = \begin{bmatrix} R\,\theta_c \\ U_c \\ W_c \end{bmatrix} , \quad D_{II} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}\right\} \quad (89)$$

Applying the same manipulation to equation (76) gives

$$\begin{bmatrix} D_I \\ --- \\ D_{II} \end{bmatrix} = \left[\begin{array}{c|c} [\bar{A}_{11}] & [\bar{A}_{12}] \\ \hline [\bar{A}_{21}] & [\bar{A}_{22}] \end{array}\right] \begin{bmatrix} X_{BCI} \\ ---- \\ X_{BCII} \end{bmatrix} \tag{90}$$

From equation (90) one has

$$[X_{BCII}] = [\bar{S}_{II}]\,[X_{BCI}] \tag{91}$$

$$[X_{BCI}] = [\bar{S}_I]^{-1}\,[\bar{D}_I] \tag{92}$$

where

$$[\bar{S}_{II}] = -[\bar{A}_{22}]^{-1}\,[\bar{A}_{21}] \tag{93}$$

$$[\bar{S}_I] = [\bar{A}_{11}] + [\bar{A}_{12}]\,[\bar{S}_{II}] \tag{94}$$

From equations (91) and (92), the following equation can be obtained:

$$[X_{BCII}] = [\bar{S}_{III}]\,[D_I] \tag{95}$$

where

$$[\bar{S}_{III}] = [\bar{S}_{II}]\,[\bar{S}_I]^{-1} \tag{96}$$

and

$$\left.\begin{aligned}
&\bar{A}_{11} = \begin{bmatrix} c_1 & c_2 & c_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} c_4 & c_5 & c_6 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ 1 & 1 & 1 \end{bmatrix} \\
&\bar{A}_{21} = \begin{bmatrix} c_1\, e^{\xi_1\phi} & c_2\, e^{\xi_2\phi} & c_3\, e^{\xi_3\phi} \\ \lambda_1\, e^{\xi_1\phi} & \lambda_2\, e^{\xi_2\phi} & \lambda_3\, e^{\xi_3\phi} \\ e^{\xi_1\phi} & e^{\xi_2\phi} & e^{\xi_3\phi} \end{bmatrix} \\
&\bar{A}_{22} = \begin{bmatrix} c_4\, e^{\xi_4\phi} & c_5\, e^{\xi_5\phi} & c_6\, e^{\xi_6\phi} \\ \lambda_4\, e^{\xi_4\phi} & \lambda_5\, e^{\xi_5\phi} & \lambda_6\, e^{\xi_6\phi} \\ e^{\xi_4\phi} & e^{\xi_5\phi} & e^{\xi_6\phi} \end{bmatrix} \\
&X_{BCI} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix}, \quad X_{BCII} = \begin{bmatrix} \bar{a}_4 \\ \bar{a}_5 \\ \bar{a}_6 \end{bmatrix}
\end{aligned}\right\} \tag{97}$$

and D_I and D_{II} are defined already.

The dynamic equilibrium at point C from Fig. 4 gives

$$\left.\begin{aligned} &\frac{M_{CA}}{R} + \frac{M_{CB}}{R} = 0 \\ &-T_1 + T_2 = P\cos(\rho + \theta) \\ &F_1 - F_2 = P\sin(\rho + \theta) \end{aligned}\right\} \qquad (98)$$

Introducing equations (74) and (78) into equation (98) yields

$$[\bar{B}_1]\,[X_{AC}] + [\bar{B}_2]\,[X_{BC}] = [TT] \qquad (99)$$

where

$$[TT] = P \begin{bmatrix} 0 \\ \cos(\rho + \theta) \\ \sin(\rho + \theta) \end{bmatrix} \qquad (100)$$

and $[\bar{B}_1]$ and $[\bar{B}_2]$ are given in Appendix E, and $[X_{AC}]$ and $[X_{BC}]$ are defined previously.

Combining equations (85) and (87), one obtains

$$[X_{AC}] = \begin{bmatrix} [X_{ACI}] \\ \\ [X_{ACII}] \end{bmatrix} = \begin{bmatrix} [S_I]^{-1} \\ \\ [S_{III}] \end{bmatrix} [D_I] \qquad (101)$$

Similarly, from equations (92) and (95) the following equation can be obtained

$$[X_{BC}] = \begin{bmatrix} [X_{BCI}] \\ \\ [X_{BCII}] \end{bmatrix} = \begin{bmatrix} [\bar{S}_I]^{-1} \\ \\ [\bar{S}_{III}] \end{bmatrix} [D_I] \qquad (102)$$

Substituting equations (101) and (102) into equation (99) yields

$$[D_I] = [Z]^{-1} [TT] \qquad (103)$$

where

$$[Z] = [\bar{B}_1] \begin{bmatrix} [S_I]^{-1} \\ \\ [S_{III}] \end{bmatrix} + [\bar{B}_2] \begin{bmatrix} [\bar{S}_I]^{-1} \\ \\ [\bar{S}_{III}] \end{bmatrix} \qquad (104)$$

and $[D_I]$, $[T]$ are defined already.

Therefore,

$$[D_1] = \begin{bmatrix} [D_I] \\ \cdots \\ [D_{II}] \end{bmatrix} = \begin{bmatrix} [Z]^{-1} [TT] \\ \\ \text{-------} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (105)$$

Finally, $[X_{AC}]$ and $[X_{BC}]$ can be obtained by multiplying equation (72) and (76) by $[A_1]^{-1}$ and $[A_2]^{-1}$, respectively. The results are:

$$[X_{AC}] = [A_1]^{-1} [D_1] \qquad (106)$$

$$[X_{BC}] = [A_2]^{-1}\,[D_1] \qquad (107)$$

Referring again to Figures 5 and 6, the fixed-end reaction may be determined by using equations (78) and (82), and the results are

$$\left.\begin{aligned} M^F_{AB} &= -\frac{EI}{R^2}\sum_{n=1}^{6} c_n\, a_n\, \xi_n \\ M^F_{BA} &= \frac{EI}{R^2}\sum_{n=1}^{6} c_n\, a_n\, \xi_n\, e^{\xi_n \phi} \end{aligned}\right\} \qquad (108)$$

$$\left.\begin{aligned} V^F_{AB} &= F^F_{AB}\sin\rho - T^F_{AB}\cos\rho = \frac{EI}{R^3}\sum_{n=1}^{6} a_n \cdot (m_n \sin\rho + d_n \cos\rho) \\ V^F_{BA} &= F^F_{BA}\sin\eta + T^F_{BA}\cos\eta = \frac{EI}{R^3}\Sigma\, \bar{a}_n\, e^{\xi_n\phi} \cdot (m_n \sin\eta - d_n \cos\eta) \\ H^F_{AB} &= -F^F_{AB}\cos\rho - T^F_{AB}\sin\rho = \frac{EI}{R^3}\sum_{n=1}^{6} a_n \cdot (-m_n \cos\rho + d_n \sin\rho) \\ H^F_{BA} &= F^F_{BA}\cos\eta - T^F_{BA}\sin\eta = \frac{EI}{R^3}\sum_{n=1}^{6} \bar{a}_n\, e^{\xi_n\phi} \cdot (m_n \cos\eta + d_n \sin\eta) \end{aligned}\right\} \qquad (109)$$

II. Dynamic Distributed Load

(a) Equations of Motion

Consider the in-plane vibration of a curved element as shown in Fig. 7. Following the same derivation as given

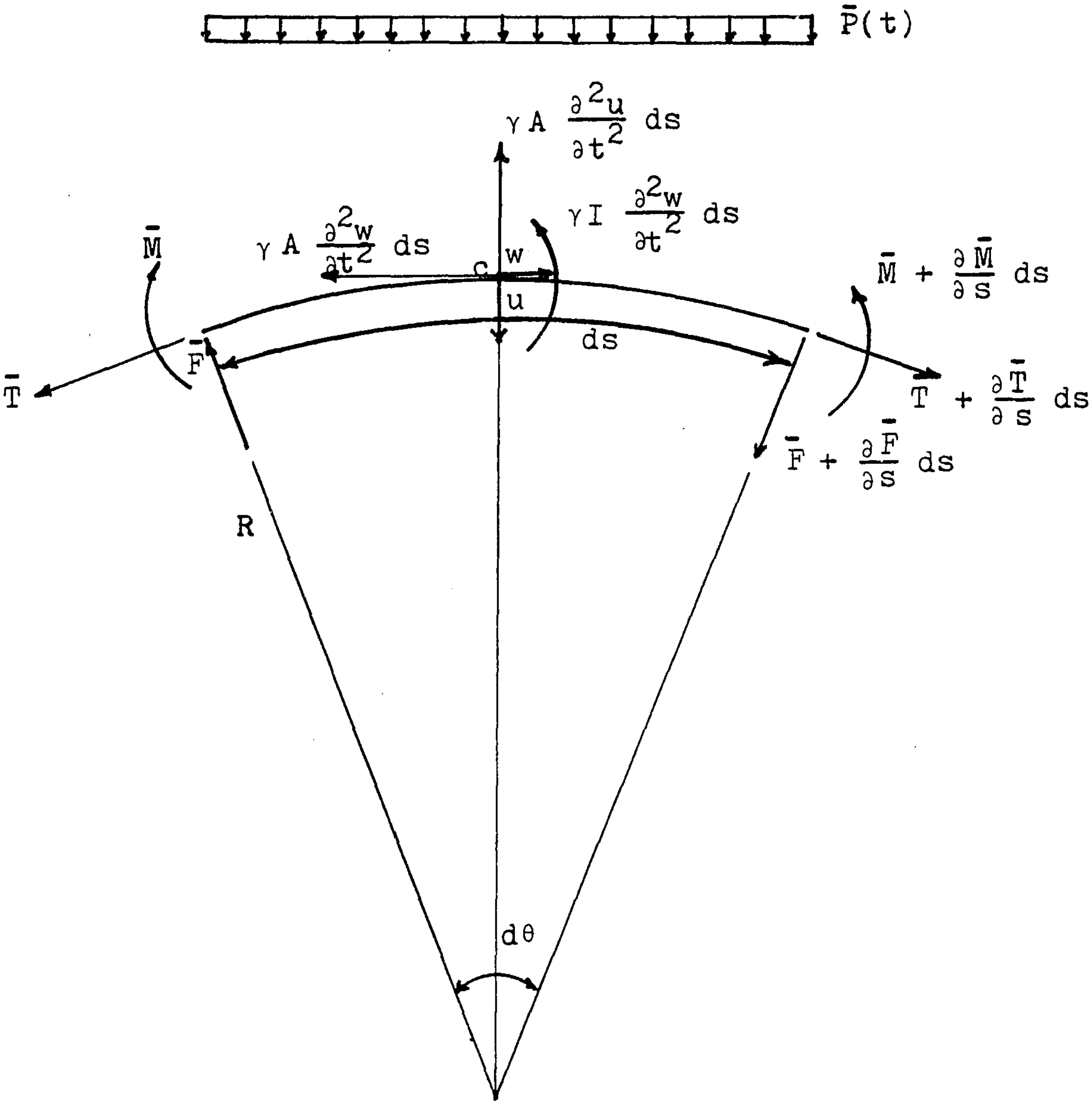

Figure 7. Element of a curved member subjected to forces, moments and load

in Appendix A, the equations of motion are

$$\frac{\partial \bar{F}}{\partial \theta} + \bar{T} + \bar{P}R = \gamma AR \frac{\partial^2 u}{\partial t^2} \tag{110}$$

$$\frac{\partial \bar{T}}{\partial \theta} - \bar{F} = \gamma AR \frac{\partial^2 w}{\partial t^2} \tag{111}$$

$$-\frac{\partial \bar{M}}{\partial \theta} + \bar{F}R = \gamma IR \frac{\partial^2 \psi}{\partial t^2} \tag{112}$$

Introducing equations (17) and (18) into equation (110) yields

$$\frac{kAG}{R}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{R}(kAG + EA)\frac{\partial w}{\partial \theta} - kAG\frac{\partial \psi}{\partial \theta} - \frac{EA}{R}u - \gamma AR\frac{\partial^2 u}{\partial t^2} - \bar{P}R = 0 \tag{113}$$

Substituting equations (17) and (18) into equation (111) one has

$$\frac{EA}{R}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{R}(EA + kAG)\frac{\partial u}{\partial \theta} - \frac{kAG}{R}w + kAG\psi - \gamma AR\frac{\partial^2 w}{\partial t^2} = 0 \tag{114}$$

Substituting of equations (15) and (17) into equation (112), the following equation is obtained

$$\frac{EI}{R}\frac{\partial^2 \psi}{\partial \theta^2} + kAG\frac{\partial u}{\partial \theta} + kAG\, w - kAGR\psi - \gamma IR\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{115}$$

(b) <u>General Solution of the Equation of Motion</u>

For the harmonic vibration assume:

$$\left.\begin{aligned} w(\theta,t) &= W(\theta)\ e^{i\Omega t} \\ u(\theta,t) &= U(\theta)\ e^{i\Omega t} \\ \psi(\theta,t) &= \Psi(\theta)\ e^{i\Omega t} \\ \bar{P}(t) &= p\ e^{i\Omega t} \end{aligned}\right\} \qquad (116)$$

where Ω is the natural circular frequency of n^{th} mode and W, U, Ψ, p are normal function of w, u, ψ and $\bar{P}$, respectively.

Substituting equation (116) into equations (113), (114) and (115), and using symbolic operator method, one obtains

$$(-D^2 + \frac{s^2}{r^2} - b^2 \cdot s^2)U - (D + \frac{s^2}{r^2} D)W + (RD)\Psi - s^2 \cdot \frac{PR^4}{EI} = 0 \qquad (117)$$

$$-(D + \frac{s^2}{r^2} D)U + (\frac{s^2}{r^2} D + b^2 \cdot s^2 - 1)W + R\Psi = 0 \qquad (118)$$

$$DU + W + R(s^2 D^2 - 1 + r^2 \cdot s^2 \cdot b^2)\Psi = 0 \qquad (119)$$

where s, r and b are defined previously.

Solving equations (117), (118) and (119) simultaneously

$$W^{VI} + k_1 W^{IV} + k_2 W'' + k_3 W = 0 \qquad (120)$$

$$U^{VI} + k_1 U^{IV} + k_2 U'' + k_3 U = k_4 \frac{pR^4}{EI} \qquad (121)$$

where

$$k_4 = (b^2 \cdot r^2 - r^4 \cdot b^4 \cdot s^2 + r^4 \cdot b^2) \qquad (122)$$

and k_1, k_2, and k_3 are defined already.

The solution of equation (120) is again given by equation (41).

Equation (121) is a non-homogeneous sixth-order linear differential equation with constant coefficients, its general solution takes the form of

$$U(\theta) = U_c(\theta) + Up \qquad (123)$$

where Up is a particular solution of equation (121) and $U_c(\theta)$ is the complementary solution which is the solution of the homogeneous part and is given by equation (46).

By inspection of equation (121), its particular solution is

$$Up = \left(\frac{k_4}{k_3}\right) \frac{pR^4}{EI} \qquad (124)$$

Thus, the general solution of equation (121) can be written as

$$U(\theta) = \left(\sum_{n=1}^{6} \lambda_n a_n e^{\xi_n \theta} + \frac{k_4}{k_3} \cdot \frac{pR^4}{EI}\right) \qquad (125)$$

Substituting equations (41) and (125) into equation (118) leads to

$$R\Psi(\theta) = \sum_{n=1}^{6} a_n c_n e^{\xi_n \theta} \qquad (126)$$

where c_n are given by equation (48).

(c) Fixed-end Moments and Thrusts

Consider the circular curved member subjected to a harmonic uniformly distributed load as shown in Fig. 8.

Introducing equations (41), (125) and (126) into equations (52), (53) and (54) give

$$M(\theta) = -\frac{EI}{R^2}\sum_{n=1}^{6} c_n\, a_n\, \xi_n\, e^{\xi_n \theta} \tag{127}$$

$$F(\theta) = \frac{EI}{R^3}\sum_{n=1}^{6} m_n\, a_n\, e^{\xi_n \theta} \tag{128}$$

$$T(\theta) = -\frac{EI}{R^3}\sum_{n=1}^{6} d_n\, a_n\, e^{\xi_n \theta} - pR(1 + b^2 \cdot \frac{k_4}{k_3}) \tag{129}$$

The boundary conditions for both ends "A" and "B" being fixed are

$$\left.\begin{array}{ll} U(\theta = 0) = 0, & U(\theta = \alpha) = 0 \\ W(\theta = 0) = 0, & W(\theta = \alpha) = 0 \\ \Psi(\theta = 0) = 0, & \Psi(\theta = \alpha) = 0 \end{array}\right\} \tag{130}$$

The system of equations resulting from the introduction of equations (41), (125) and (126) into equations (130) can be expressed as

$$[\bar{\bar{T}}]\,[X] = -\left(\frac{k_4}{k_3}\right)\frac{pR^4}{EI}\,[F] \tag{131}$$

where

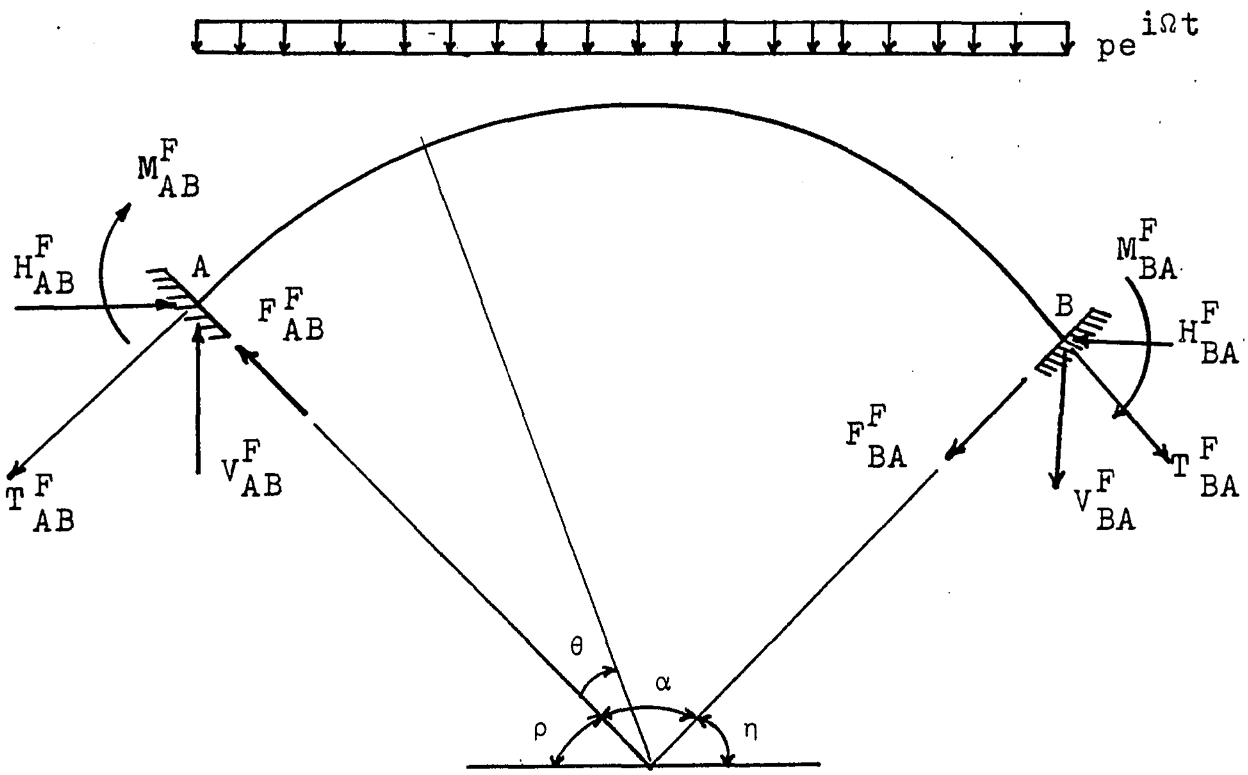

Figure 8. Fixed-end curved member under the effect of uniformly distributed load

$$[F] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (132)$$

and $[\bar{\bar{T}}\bar{\bar{T}}]$ is given in Appendix F and $[X]$ is defined previously.

Premultiplying equation (131) by $[\bar{\bar{T}}\bar{\bar{T}}]^{-1}$ will yield

$$[X] = -\left(\frac{k_4}{k_3}\right) \frac{pR^4}{EI} [\bar{\bar{T}}\bar{\bar{T}}]^{-1} [F] \qquad (133)$$

Introducing equation (133) into equations (127), (128) and (129) yields the fixed-end bending moments and thrusts at ends "A" and "B" due to a harmonic uniformly distributed load as follows:

$$M^F_{AB} = \sigma_A \, pR^2 \qquad (134)$$

$$M^F_{BA} = \sigma_B \, pR^2 \qquad (135)$$

$$V^F_{AB} = F^F_{AB} \sin\rho - T^F_{AB} \cos\rho = \tau_A \, pR \qquad (136)$$

$$V^F_{BA} = F^F_{BA} \sin\eta + T^F_{BA} \cos\eta = \tau_B \, pR \qquad (137)$$

$$H^F_{AB} = -F^F_{AB} \cos\rho - T^F_{AB} \sin\rho = \varepsilon_A \, pR \qquad (138)$$

$$H^F_{BA} = F^F_{BA} \cos\eta - T^F_{BA} \sin\eta = \varepsilon_B \, pR \qquad (139)$$

where

$$\sigma_A = \left(\frac{k_4}{k_3}\right) [c\ \xi] [\bar{T}\bar{T}]^{-1} [F] \qquad (140)$$

$$\sigma_B = \left(\frac{k_4}{k_3}\right) [c\ \xi\ e^{\xi\theta}] [\bar{T}\bar{T}]^{-1} [F] \qquad (141)$$

$$\tau_A = -\left(\frac{k_4}{k_3}\right) \sin\rho\ [m] [\bar{T}\bar{T}]^{-1} [F] - \left(\frac{k_4}{k_3}\right) \cos\rho\ [d] [\bar{T}\bar{T}]^{-1} [F] + \cos\rho \left(1 + b^2 \cdot \frac{k_4}{k_3}\right) \qquad (142)$$

$$\tau_B = -\left(\frac{k_4}{k_3}\right) \sin\eta\ [m\ e^{\xi\theta}] [\bar{T}\bar{T}]^{-1} [F] = \left(\frac{k_4}{k_3}\right) \cos\eta\ [d\ e^{\xi\theta}] [\bar{T}\bar{T}]^{-1} [F] - \cos\eta \left(1 + b^2 \cdot \frac{k_4}{k_3}\right) \qquad (143)$$

$$\varepsilon_A = \left(\frac{k_4}{k_3}\right) \cos\rho\ [m] [\bar{T}\bar{T}]^{-1} [F] - \left(\frac{k_4}{k_3}\right) \sin\rho\ [d] [\bar{T}\bar{T}]^{-1} [F] + \sin\rho \left(1 + b^2 \cdot \frac{k_4}{k_3}\right) \qquad (144)$$

$$\varepsilon_B = -\left(\frac{k_4}{k_3}\right) \cos\eta\ [m\ e^{\xi\theta}] [\bar{T}\bar{T}]^{-1} [F] - \left(\frac{k_4}{k_3}\right) \sin\eta\ [d\ e^{\xi\theta}] [\bar{T}\bar{T}]^{-1} [F] + \sin\eta \left(1 + b^2 \cdot \frac{k_4}{k_3}\right) \qquad (145)$$

and $[c\ \xi]$, $[c\ \xi\ e^{\xi\theta}]$, $[m]$, $[d]$, $[m\ e^{\xi\theta}]$, $[d\ e^{\xi\theta}]$ and $[\bar{T}\bar{T}]$ are given in Appendix F.

CHAPTER V

INEXTENSIONAL VIBRATIONS

1. Derivation of the Equations of Motion

Assuming the centroidal axis of the curved member to be inextensional, equation (13) becomes

$$u = \frac{\partial w}{\partial \theta} \qquad (146)$$

Substitution of equation (146) into equation (17) yields

$$\bar{F} = \frac{kAG}{R} \left(\frac{\partial^2 w}{\partial \theta^2} + w - R\psi\right) \qquad (147)$$

Eliminating $\bar{T}$ from equations (3) and (4) and employing equation (145) gives

$$\frac{\partial^2 \bar{F}}{\partial \theta^2} + F = \gamma AR \left(\frac{\partial^4 w}{\partial \theta^2 \, \partial t^2} - \frac{\partial^2 w}{\partial t^2}\right) \qquad (148)$$

Substituting equation (147) into equation (148) yields

$$\frac{kAG}{R} \left(\frac{\partial^4 w}{\partial \theta^4} + 2\,\frac{\partial^2 w}{\partial \theta^2} + w - R\,\frac{\partial^2 \psi}{\partial \theta^2} - R\psi\right) =$$

$$\gamma AR \left(\frac{\partial^4 w}{\partial \theta^2 \, \partial t^2} - \frac{\partial^2 w}{\partial t^2}\right) \qquad (149)$$

Combining equations (15) and (147) with equation (5) gives

$$\frac{EI}{R}\frac{\partial^2\psi}{\partial\theta^2} + kAG\left(\frac{\partial^2 w}{\partial\theta^2} + w - R\psi\right) = \gamma IR\frac{\partial^2\psi}{\partial t^2} \qquad (150)$$

Equations (149) and (150) constitute the equations of motion of circular curved member undergoing inextensional free vibrations.

2. General Solution of the Equations of Motion

Assume that the curved member is excited harmonically with a natural frequency Ω and let,

$$\left.\begin{aligned} w(\theta,t) &= W(\theta)\ e^{i\Omega t} \\ \psi(\theta,t) &= \Psi(\theta)\ e^{i\Omega t} \\ u(\theta,t) &= U(\theta)\ e^{i\Omega t} \end{aligned}\right\} \qquad (151)$$

Using the same procedure as shown in Chapter II, we have

$$W^{VI} + k_{11}\,W^{IV} + k_{22}\,W'' + k_{33}\,W = 0 \qquad (152)$$

where

$$\left.\begin{aligned} k_{11} &= 2 + b^2r^2 + b^2s^2 \\ k_{22} &= 1 + 2\,b^2r^2 - b^2s^2 - b^2 + b^4r^2s^2 \\ k_{33} &= b^2 + b^2r^2 - b^4r^2s^2 \end{aligned}\right\} \qquad (153)$$

The solution of equation (152) may be expressed by equation (41) as

$$W(\theta) = \sum_{n=1}^{6} a_n e^{\xi_n \theta} \tag{154}$$

and the solution of $\Psi(\theta)$ can also be expressed by equation (47) as

$$R\Psi(\theta) = \sum_{n=1}^{6} q_n a_n e^{\xi_n \theta} \tag{155}$$

From equations (146) and (154) one has

$$U(\theta) = W'(\theta) = \sum_{n=1}^{6} \xi_n a_n e^{\xi_n \theta} \tag{156}$$

where

$$q_n = \frac{s^2 \xi_n^4 + \xi_n^2 (1 + 2s^2 + b^2 s^4) + (1 + s^2 - b^2 s^4)}{(1 + s^2 - b^2 r^2 s^2)} \tag{157}$$

and a_n, ξ_n, b, r and s are defined already.

3. Derivation of Dynamic Stiffness Matrix

Referring to Fig. 3 and following the same procedure as shown in Chapter II, we obtain

$$M(\theta) = -\frac{EI}{R^2} \sum_{n=1}^{6} q_n a_n \xi_n e^{\xi_n \theta} \tag{158}$$

$$F(\theta) = \frac{EI}{R^3} \sum_{n=1}^{6} Z_n a_n e^{\xi_n \theta} \tag{159}$$

$$T(\theta) = -\frac{EI}{R^3} \sum_{n=1}^{6} a_n (Z_n + b^2) \xi_n e^{\xi_n \theta} \tag{160}$$

where

$$Z_n = \frac{b^2 (s^2 - r^2) - \xi_n^2 (1 + b^2(r^2 + s^2)) - \xi_n^4}{(1 + s^2 - b^2 r^2 s^2)} \qquad (161)$$

Using the boundary conditions given by equations (60) and (61) one has

$$[D] = [E] [X] \qquad (162)$$

$$[F] = \frac{EI}{R^3} [C] [X] \qquad (163)$$

where [D] and [X] are defined by equation (64), and matrices [E] and [C] are given in Appendix G.

Premultiplying equation (162) by $[E]^{-1}$ and substituting into equation (163) yields

$$[F] = \frac{EI}{R^3} [C] [E]^{-1} [D] = [ss] [D] \qquad (164)$$

where [ss] is the dynamic stiffness matrix of inextensional curved member and is given by

$$[ss] = \frac{EI}{R^3} [C] [E]^{-1} \qquad (165)$$

4. <u>Fixed-end Moments and Thrusts</u>

Referring again to Fig. 4 and following the same approach as given in Chapter IV, one obtains

$$[D_1] = [E_1] [X_{AC}] \qquad (166)$$

$$[F_1] = \frac{EI}{R^3} [C_1] [X_{AC}] \qquad (167)$$

$$[D_1] = [E_2] [X_{BC}] \qquad (168)$$

$$[F_2] = \frac{EI}{R^3} [C_2] [X_{BC}] \qquad (169)$$

where $[D_1]$, $[F_1]$, $[F_2]$, $[X_{AC}]$ and $[X_{BC}]$ are defined by equations (73), (75), (79), (73) and (77), respectively.

Matrices $[E_1]$, $[C_1]$, $[E_2]$ and $[C_2]$ are given in Appendix H.

Using the dynamic equilibrium and the continuity at point c, finally the fixed-end moment may be expressed in the same form as equation (108) and (109).

CHAPTER VI

NUMERICAL EXAMPLES

Example 1: A three-span symmetrical circular curved beam of constant cross section undergoing in-plane vibration as shown in Fig. 9 is analyzed for natural frequencies.

The boundary conditions are

$$v_A = v_B = v_C = v_D = 0 \quad ; \quad h_A = h_B = h_C = h_D = 0 \qquad (170)$$

and the conditions of dynamic equilibrium at A, B, C and D give

$$\left.\begin{aligned} M_{AB} &= 0 \\ M_{BA} + M_{BC} &= 0 \\ M_{CB} + M_{CD} &= 0 \\ M_{DC} &= 0 \end{aligned}\right\} \qquad (171)$$

due to symmetry, Figures 4 and 9 give

$$\rho = \eta \qquad (172)$$

Since the beam has three identical spans, we have

$$[S]_a = [S]_b = [S]_c = [S] \; ; \; [A]_a = [A]_b = [A]_c = [A] \; ;$$
$$[B]_a = [B]_b = [B]_c = [B] \qquad (173)$$

From equations (170) and (171) one has

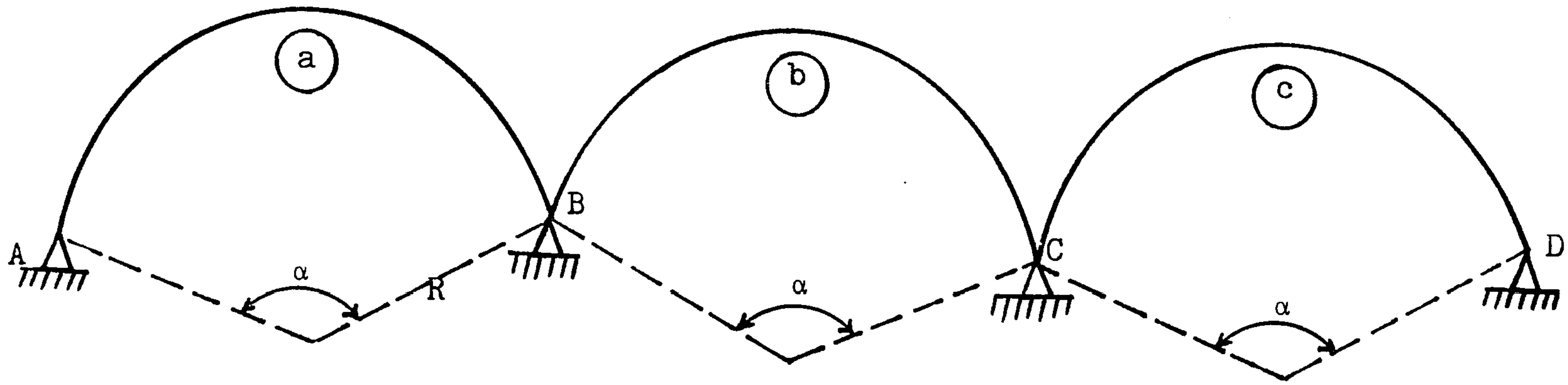

Figure 9. A three-span circular curved beam

$$[D]_a = \begin{bmatrix} \theta_A R \\ \theta_B R \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D]_b = \begin{bmatrix} \theta_B R \\ \theta_C R \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D]_c = \begin{bmatrix} \theta_C R \\ \theta_D R \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (174)$$

$$[F]_a = \begin{bmatrix} M_{AB/R} \\ M_{BA/R} \\ V_{AB} \\ V_{BA} \\ H_{AB} \\ H_{BA} \end{bmatrix}, \quad [F]_b = \begin{bmatrix} M_{BC/R} \\ M_{CB/R} \\ V_{BC} \\ V_{CB} \\ H_{BC} \\ H_{CB} \end{bmatrix}, \quad [F]_c = \begin{bmatrix} M_{CD/R} \\ M_{DC/R} \\ V_{CD} \\ V_{DC} \\ H_{CD} \\ H_{DC} \end{bmatrix} \quad (175)$$

Using equation (70) we can write

$$[F]_a = [S]\,[D]_a\,,\; [F]_b = [S]\,[D]_b\,,\; [F]_c = [S]\,[D]_c \quad (176)$$

Introducing equations (174) and (175) into equations (176) gives

$$\left.\begin{aligned} &\frac{M_{AB}}{R} = S_{11}\,\theta_A R + S_{12}\,\theta_B R = 0 \\ &\frac{M_{BA}}{R} + \frac{M_{BC}}{R} = S_{21}\,\theta_A R + (S_{11} + S_{22})\,\theta_B R + S_{12}\,\theta_C R = 0 \end{aligned}\right\} \quad (177)$$

$$\frac{M_{CB}}{R} + \frac{M_{CD}}{R} = S_{21}\ \theta_B R + (S_{11} + S_{22})\ \theta_C R + S_{12}\ \theta_D R = 0$$

$$\frac{M_{DC}}{R} = S_{21}\ \theta_C R + S_{22}\ \theta_D R = 0$$

Equations (177) may be rearranged in the following matrix form:

$$[F_{11}] = [S_1]\ [D_{11}] = [0] \qquad (178)$$

where

$$[F_{11}] = \begin{bmatrix} M_{AB}/R \\ (M_{BA}/R)+(M_{BC}/R) \\ (M_{CB}/R)+(M_{CD}/R) \\ M_{DC}/R \end{bmatrix}$$

$$[S_1] = \begin{bmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{21} & (S_{11}+S_{22}) & S_{12} & 0 \\ 0 & S_{21} & (S_{11}+S_{22}) & S_{12} \\ 0 & 0 & S_{21} & S_{22} \end{bmatrix} \qquad (179)$$

$$[D_{11}] = \begin{bmatrix} \theta_A R \\ \theta_B R \\ \theta_C R \\ \theta_D R \end{bmatrix}$$

Equating the determinate of the stiffness matrix $[S_1]$ in equation (178) to zero yields the following frequency equation:

$$\begin{Vmatrix} S_{11} & S_{12} & 0 & \\ S_{21} & (S_{11}+S_{22}) & S_{12} & \\ 0 & S_{21} & (S_{11}+S_{22}) & S_{12} \\ 0 & 0 & S_{21} & S_{22} \end{Vmatrix} = 0 \qquad (180)$$

Equation (180) is the frequency equation and will be used to find the natural frequencies of the curved beam.

For a given beam, the values of r, s and α are known and the frequencies can be determined from equation (180). In order to show the effects of rotary inertia and shear deformation on the natural frequencies of the beam, the cross section of the beam is assumed to be a rectangle. The shear coefficient k for a rectangular section can be computed from the following expression given by Cowper[20]

$$k = \frac{10\ (1 + \mu)}{12 + 11\mu} \qquad (181)$$

where μ, the Poisson's ratio, equals to 0.3 for steel.

From equations (30) and (31) one has

$$\frac{s^2}{r^2} = \frac{E}{kG} \tag{182}$$

The relation between modulus of elasticity and modulus of rigidity is given by [27]

$$G = \frac{E}{2\,(1+\mu)} \tag{183}$$

Substituting equations (181) and (183) into equation (182) and setting $\mu = 0.3$, we obtain

$$s = 1.75\ r \tag{184}$$

A computer algorithm has been written to find the elements of equation (180), and then to evaluate the determinant, based upon input values for the frequency parameter b, the rotary inertia parameter r and the central angle α.

Using this algorithm, the values of b were obtained for $\alpha = 60^{\circ}$, 120° and 180°. The first four modes of vibration, with r varying from 0 to 0.10, are shown in Fig. 10. Figures 11 and 12 and in Tables 1, 2 and 3 show the effect of extensional deformation on the natural frequencies of the beam for $\alpha = 60^{\circ}$ and 180°, respectively. A comparison of natural frequencies for the beam having a rectangular section and a 24 W= 110 section can be found in Fig. 13 and in Table 4.

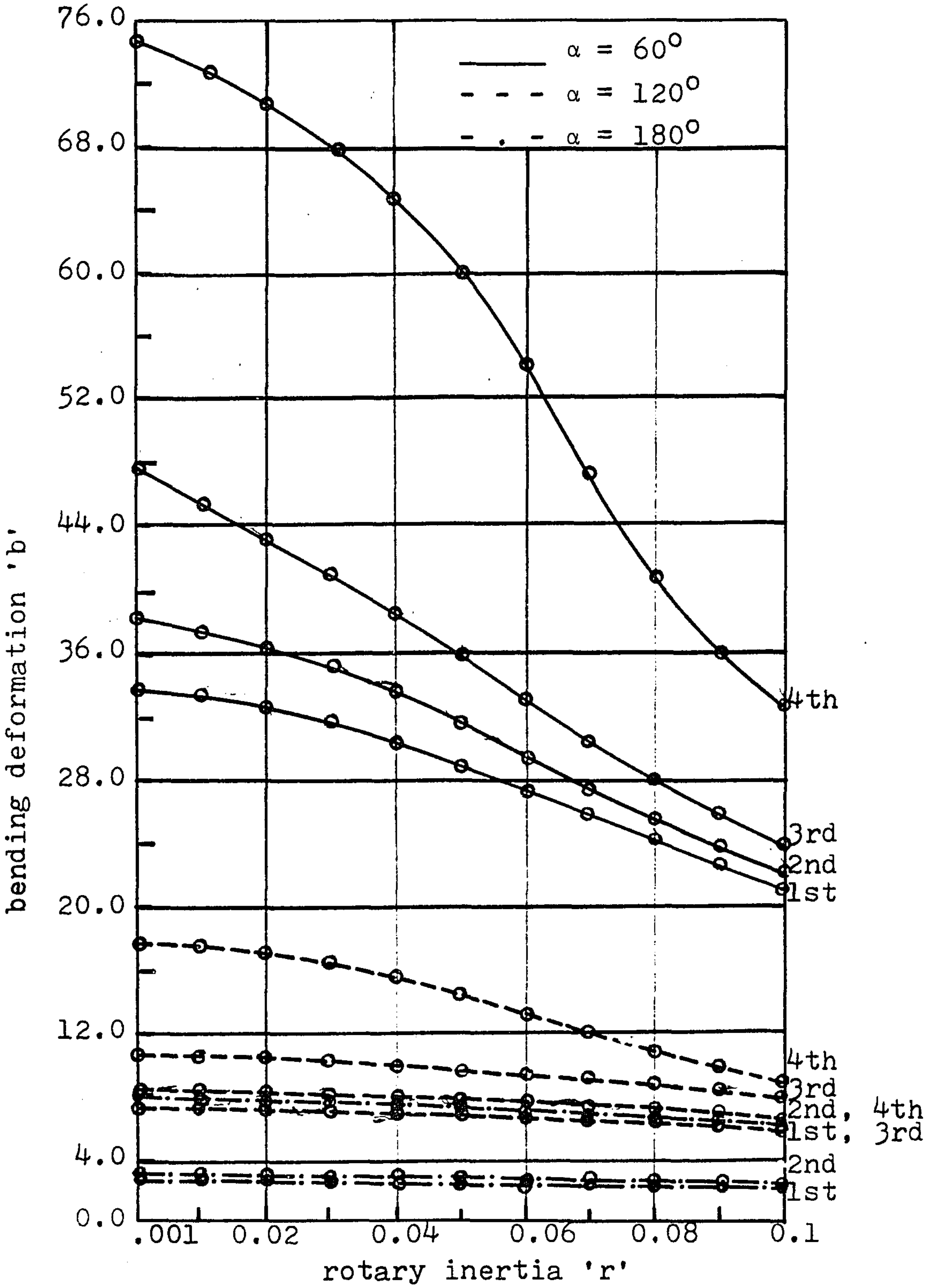

Figure 10. Natural frequencies of a three-span curved beam owing to rotary inertia and shear deformation. α = central angle.

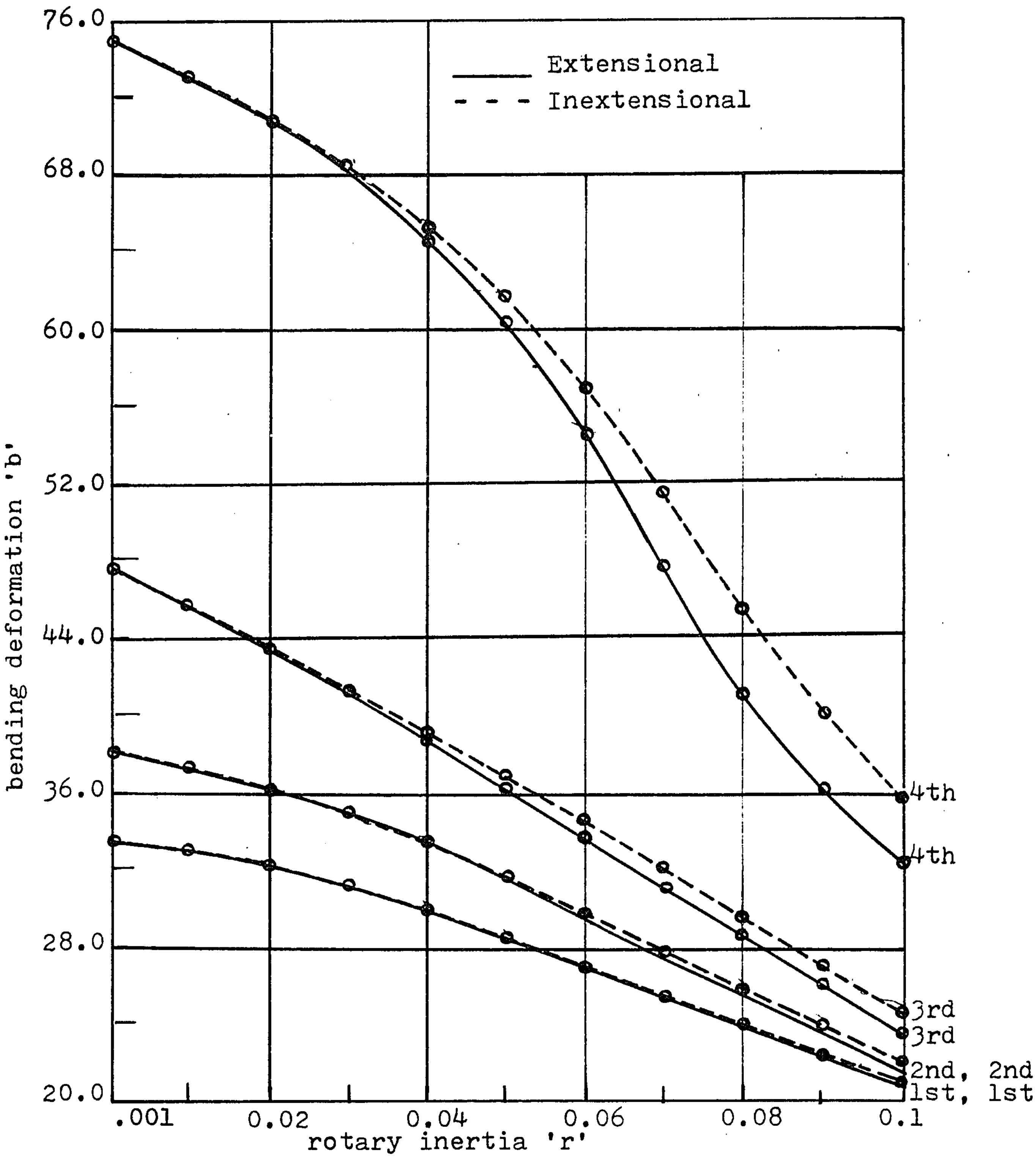

Figure 11. Effect of extensional deformation on natural frequencies of a three-span curved beam for $\alpha = 60^{\circ}$

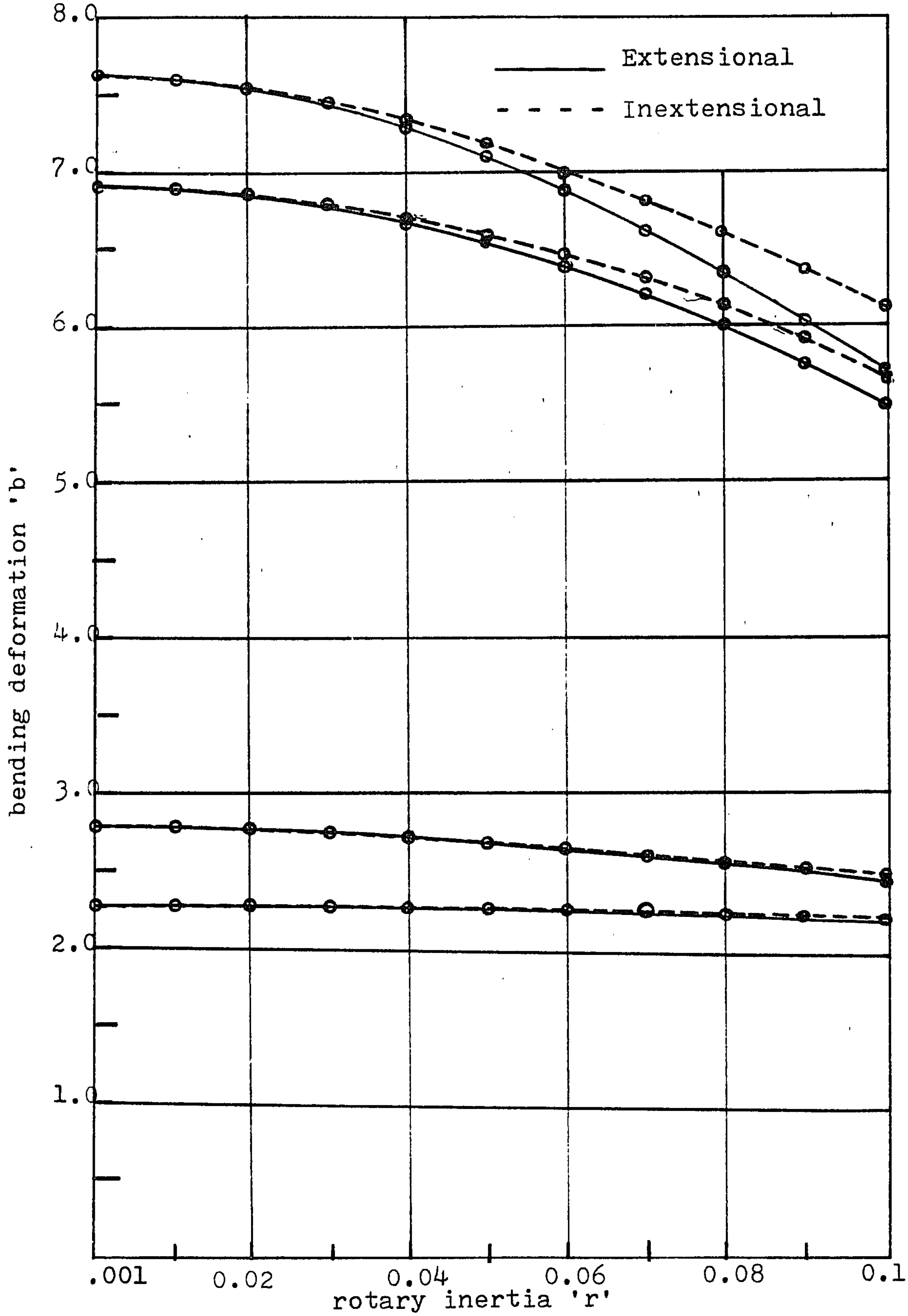

Figure 12. Effect of extensional deformation on natural frequencies of a three-span curved beam for $\alpha = 180^{\circ}$

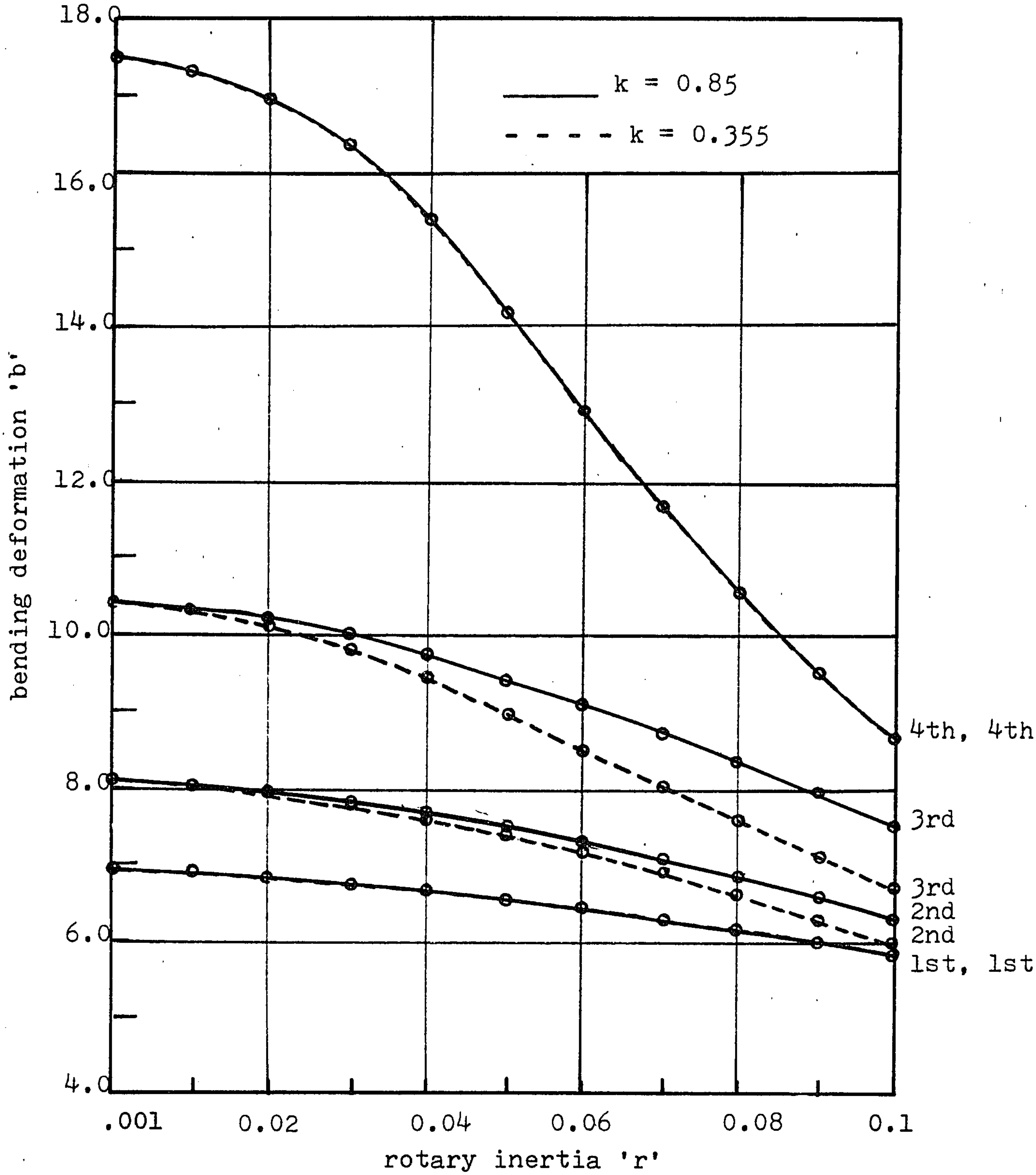

Figure 13. Effect of shape factors on natural frequencies of a three-span curved beam. $\alpha = 120^{o}$

Table 1. Effect of extensional deformation on natural frequencies for $\alpha = 60^{0}$

	Extensional				Inextensional			
r	1	2	3	4	1	2	3	4
.001	33.62	38.31	47.83	75.04	33.62	38.31	47.83	75.04
0.01	33.37	36.74	45.70	72.62	33.38	36.85	45.74	73.00
0.02	32.62	36.24	43.19	70.80	32.63	36.35	43.55	71.25
0.03	31.49	34.93	40.95	68.00	31.50	35.08	41.81	68.54
0.04	30.11	33.78	38.53	64.82	30.12	34.00	39.53	65.50
0.05	28.58	31.55	35.95	60.10	28.59	31.82	36.90	61.90
0.06	26.99	29.40	33.16	54.60	27.00	29.80	34.51	57.10
0.07	25.42	27.36	30.35	48.00	25.54	27.85	32.01	51.28
0.08	23.90	25.45	27.93	41.83	24.01	26.01	29.52	45.01
0.09	22.47	23.72	25.52	35.99	22.66	23.80	27.00	40.00
0.1	21.12	22.13	23.63	32.28	21.31	22.63	25.02	35.90

Table 2. Effect of extensional deformation on natural frequencies for $\alpha = 180^{\circ}$

	Extensional				Inextensional			
r	1	2	3	4	1	2	3	4
.001	2.27	2.77	6.92	7.63	2.27	2.77	6.92	7.63
0.01	2.26	2.76	6.91	7.60	2.26	2.76	6.91	7.61
0.02	2.26	2.75	6.86	7.53	2.26	2.75	6.87	7.55
0.03	2.25	2.74	6.78	7.42	2.25	2.74	6.80	7.45
0.04	2.23	2.71	6.67	7.27	2.24	2.71	6.71	7.32
0.05	2.21	2.68	6.53	7.08	2.22	2.68	6.59	7.16
0.06	2.19	2.64	6.38	6.87	2.20	2.65	6.46	6.99
0.07	2.17	2.60	6.20	6.63	2.18	2.61	6.32	6.80
0.08	2.14	2.55	6.01	6.38	2.15	2.56	6.17	6.60
0.09	2.11	2.51	5.75	6.12	2.12	2.52	5.91	6.40
0.1	2.07	2.45	5.50	5.75	2.09	2.47	5.68	6.05

Table 3. % Effect of axial deformation on the natural frequencies

	α = 60°		α = 180°	
	r	mode	r	mode
1% or less	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	1	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	1
	.001, .01, .02 .03, .04, .05	2	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	2
	.001, .01, .02	3	.001, .01, .02, .03, .04, .05, .06	3
	.001, .01, .02	4	.001, .01, .02, .03, .04	4
10% or less	.001, .01, .02 .03, .04, .05, .06, .07, .08, .09, .1	1	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	1
	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	2	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	2
	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	3	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	3
	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	4	.001, .01, .02, .03, .04, .05, .06, .07, .08, .09, .1	4

Table 4. Effect of shape factors on natural frequencies for $\alpha = 120^{o}$

	Rectangular Section ($k = 0.85$)				24 W= 110 ($k = 0.355$)			
r	1	2	3	4	1	2	3	4
0.001	6.93	8.08	10.41	17.50	6.93	8.08	10.41	17.50
0.01	6.91	8.05	10.37	17.38	6.91	8.04	10.34	17.37
0.02	6.87	7.98	10.24	17.03	6.86	7.94	10.14	17.00
0.03	6.80	7.87	10.04	16.44	6.78	7.78	9.82	16.35
0.04	6.70	7.71	9.77	15.56	6.66	7.58	9.42	15.43
0.05	6.58	7.52	9.45	14.38	6.53	7.34	8.96	14.23
0.06	6.44	7.31	9.10	13.01	6.37	7.08	8.48	12.90
0.07	6.30	7.08	8.72	11.66	6.20	6.08	8.00	11.62
0.08	6.13	6.84	8.33	10.47	6.01	6.52	7.53	10.50
0.09	5.96	6.59	7.93	9.46	5.82	5.82	7.09	9.58
0.1	5.79	6.34	7.53	8.61	5.62	5.97	6.68	8.78

<u>Example 2</u>: This example is intended to illustrate how the moments at the interior supports of a continuous circular curved beam are affected by rotary inertia, shear deformation, central angle, and axial deformation when subjected to a concentrated dynamic load. The load is applied at the center of an exterior span CD of the same three-span beam considered in the previous example (see Fig. 14).

The boundary conditions are the same as those given in equation (170).

From Fig. 14, the method of superposition gives

$$M_{CD} = \bar{M}_{CD} + M^{F}{}_{CD} \tag{185}$$

$$M_{DC} = \bar{M}_{DC} - M^{F}{}_{DC} \tag{186}$$

where $M^{F}{}_{CD}$ is the fixed-end moments for a concentrated load acting on span C (Fig. 14), and $\bar{M}_{CD}$ is the interior moment from the condition given in Fig. 15b.

The conditions of dynamic equilibrium for this case are

$$\left.\begin{aligned} M_{AB} &= 0 \\ M_{BA} + M_{BC} &= 0 \\ M_{CB} + M_{CD} &= 0 \\ M_{DC} &= 0 \end{aligned}\right\} \tag{187}$$

Introducing equations (185) and (186) into equations (187) yields

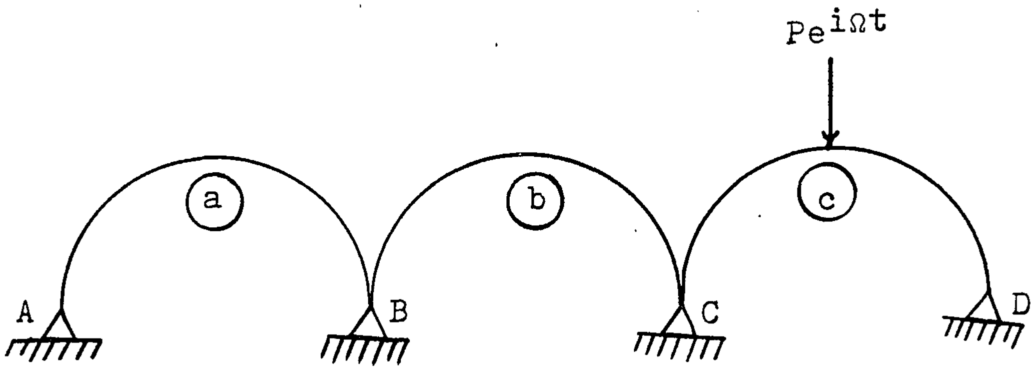

Figure 14. A three-span curved beam subjected to a dynamic concentrated load

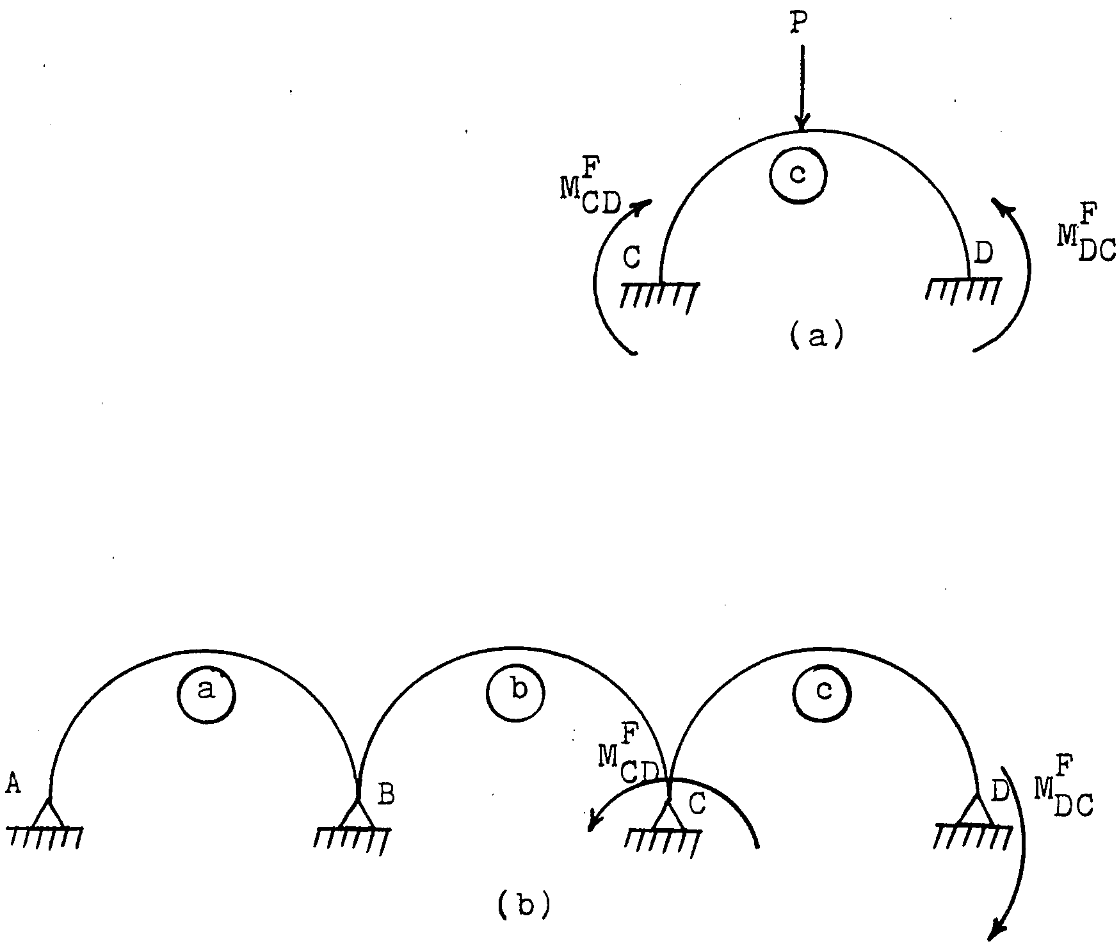

Figure 15. Combined moments and forces of a curved beam system with common factor $e^{i\Omega t}$ omitted

$$\left.\begin{aligned} &M_{AB} = 0 \\ &M_{BA} + M_{BC} = 0 \\ &M_{CB} + \bar{M}_{CD} + M^F_{CD} = 0 \\ &\bar{M}_{DC} - M^F_{DC} = 0 \end{aligned}\right\} \quad (188)$$

Referring to equations (177), equations (188) may be written as

$$\left.\begin{aligned} &\frac{M_{AB}}{R} = S_{11}\ \theta_A R + S_{12}\ \theta_B R = 0 \\ &\frac{M_{BA}}{R} + \frac{M_{BC}}{R} = S_{21}\ \theta_A R + (S_{11} + S_{22})\theta_B R + S_{12}\ \theta_C R = 0 \\ &\frac{M_{CB}}{R} + \frac{M_{CD}}{R} = S_{21}\ \theta_B R + (S_{11} + S_{22})\theta_C R + S_{12}\ \theta_D R = -\frac{M^F_{CD}}{R} \\ &\frac{M_{DC}}{R} = S_{21}\ \theta_C R + S_{22}\ \theta_D R = \frac{M^F_{DC}}{R} \end{aligned}\right\} \quad (189)$$

Equations (189) may be rearranged in the following matrix form:

$$[D_{11}] = [S_1]^{-1}\ [F_{22}] \qquad (190)$$

where

$$[F_{22}] = \begin{bmatrix} 0 \\ 0 \\ -M^F_{CD}/R \\ M^F_{DC}/R \end{bmatrix} \qquad (191)$$

and $[D_{11}]$ and $[S_1]$ are defined already.

A computer algorithm has been written for finding the unknown displacements θ_A, θ_B, θ_C and θ_D from equation (190) then the moment M_{CD} can be evaluated from equations (176).

The computer algorithm uses the following subroutines which are obtained from the IMSL library as

ZPOLR - This subroutine finds the roots of the characteristic equation.

LEQ2C - This subroutine computes the inverse of a complex matrix.

Using this algorithm the values of f_{CD}, the moment coefficient of M_{CD}, can be obtained for $\alpha = 60^\circ$, 120° and 180° with b varying from 0 to 100. The results are shown in Figures 16, 17 and 18.

Figure 19 shows a comparison of joint moment for the beam having different sections.

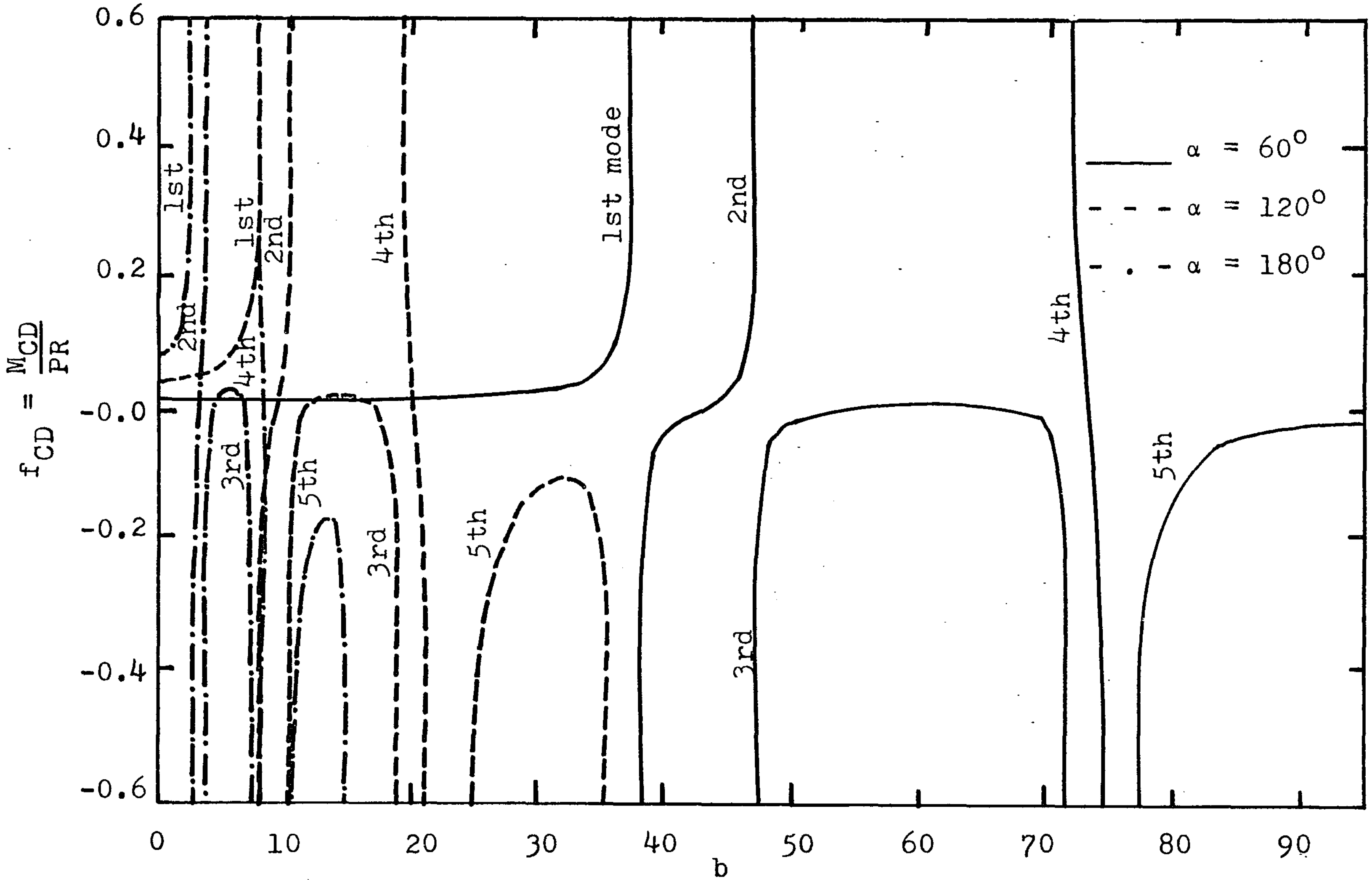

Figure 16. Variation of f_{CD} with b for r = 0.01
(b = bending deformation, r = rotary inertia,
α = central angle)

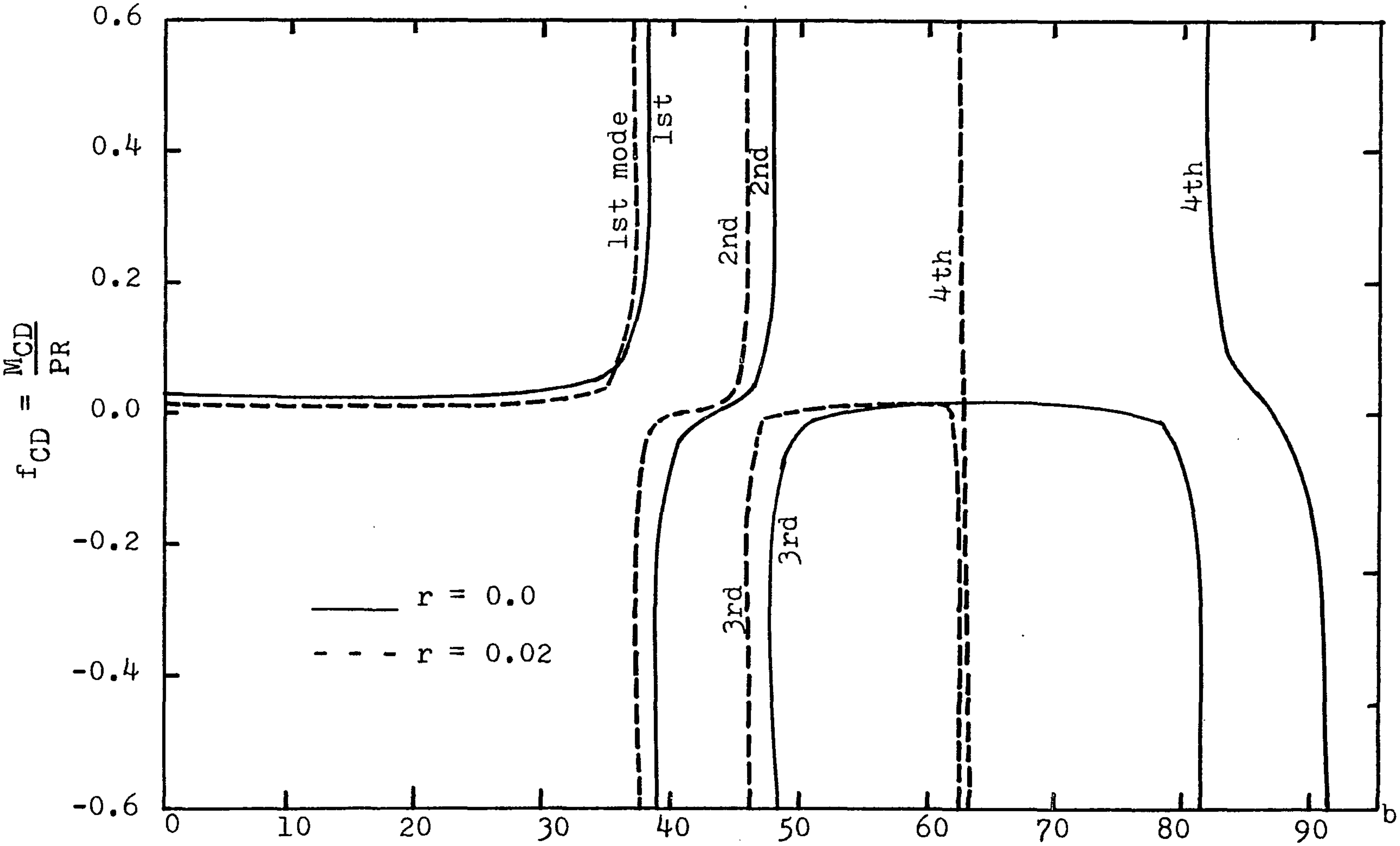

Figure 17. Variation of f_{CD} with b for $\alpha = 60^o$

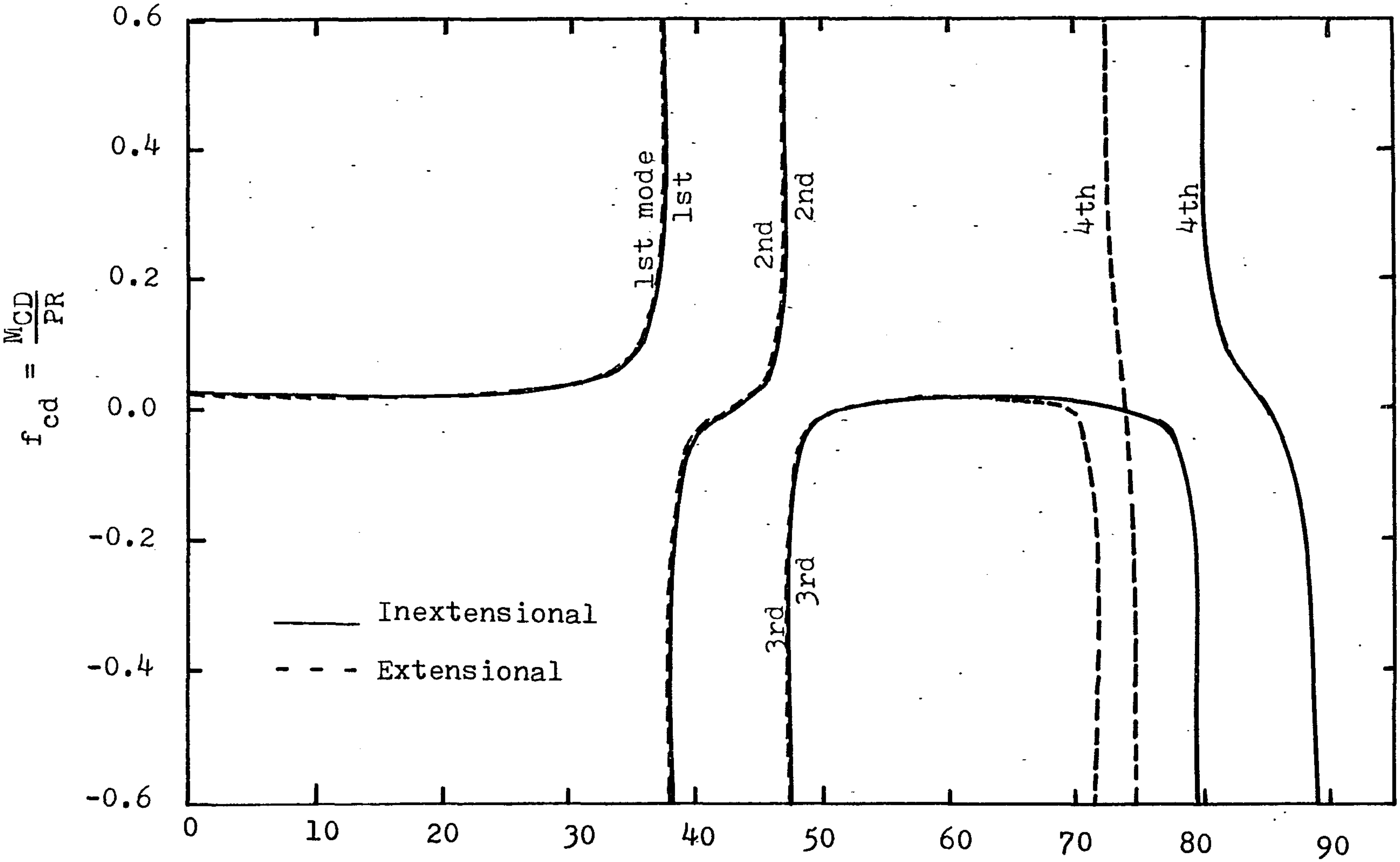

Figure 18. Effect of extensional deformation for f_{CD} vs b with $\alpha = 60^{o}$ and r = 0.01

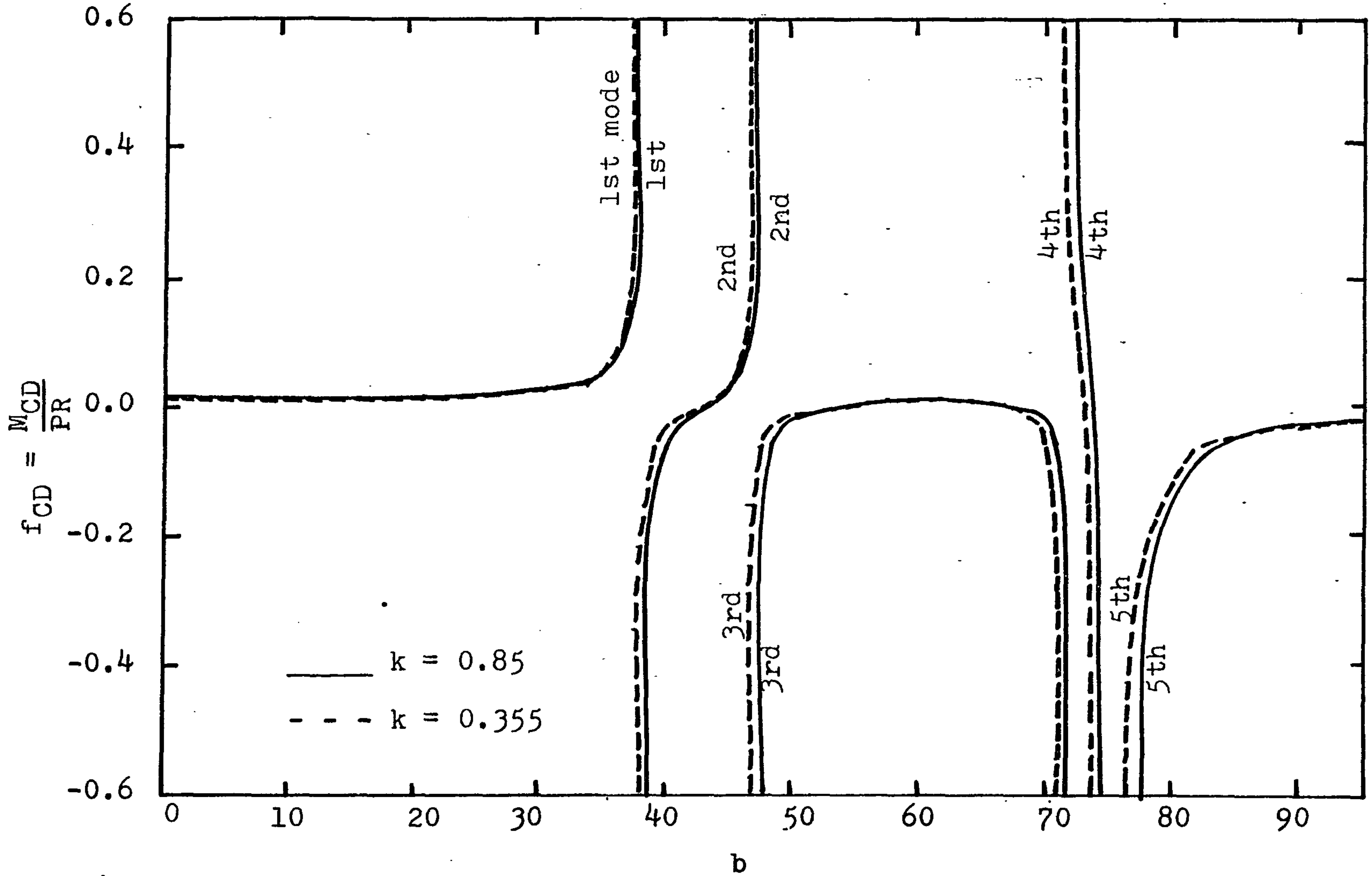

Figure 19. Effect of shape factor on joint moment of a three-span curved beam. $\alpha = 60^{o}$ and $r = 0.01$.

CHAPTER VII

DISCUSSIONS AND CONCLUSIONS

The dynamic stiffness matrix formulation for circular curved members of constant cross section, including the effects of rotary inertia, transverse shear deformation and axial deformation, has been presented in this dissertation for the dynamic analysis of continuous circular curved beams. Two examples of the three-span circular curved beam have been given to illustrate the application of the proposed method. In the first example, the beam is undergoing free vibrations. It can be seen in Fig. 10 that the natural frequency decreases as the value of the rotary inertia parameter r and the shear deformation parameter s increase and the effect is greater at smaller central angle.

The effect of axial deformation on the natural frequencies can be seen in Figures 11 and 12 and in Tables 1, 2 and 3 for two different values of central angle ($\alpha = 60^{\circ}$ and $\alpha = 180^{\circ}$). It is observed that the effect of axial deformation tends to decrease the natural frequencies of the beam and becomes significant with increasing r and s. This effect is more pronounced for higher modes and for smaller angles. For example, in Fig. 11, when r = 0.04

the natural frequency decreases by 0.033% and 1.03% for first and fourth modes, respectively, and when r = 0.10 the natural frequency decreases by 0.89% and 10% for first and fourth modes, respectively. Similarly, in Fig. 12, when r = 0.04 the natural frequency decreases by 0.44% and 0.683% for first and fourth modes, respectively, and when r = 0.10 the natural frequency decreases by 0.95% and 4.96% for first and fourth modes, respectively.

The second example illustrates the same beam subjected to forced vibrations. Since the joint moments are very sensitive to the load frequency, thus the moment coefficient f is calculated with the bending deformation b being taken for every increment of 0.01. The numerical results given in Figures 16, 17 and 18 show the effect of rotary inertia, shear and axial deformation on the joint moments of the beam. From the curves shown in Fig. 16, it is seen that the modes shift from the right to the left as the central angle becomes larger, i.e., the member becomes longer. This means as the central angle α increases, the same joint moment can be obtained at lower load frequency. This phenomenon indicates that the dynamic stiffness of the member decreases as the central angle increases. It is also noticed, from Fig. 17, that the natural frequencies decrease as r and s increase. This means as the rotary inertia r increases, the same joint moment can be obtained at lower load frequency. Thus for higher values of r and s, resonance will occur at lower frequencies. Figure 18 reveals that the effect of axial deformation is to give the joint moment at the lower frequency and this effect becomes pronounced for higher modes.

As a result of the present study, the following major conclusions can be drawn:

1 - The natural frequency decreases for increases of rotary inertia and transverse shear deformation.

2 - The frequencies decrease when axial deformation is taken into consideration. This effect becomes significant with increasing rotary inertia and transverse shear deformation. It is more pronounced for higher modes and for smaller angles.

3 - The effect of different shape factors on the natural frequencies and moment coefficients is insignificant and can be neglected (Figures 13 and 19, Table 4).

Although both free and forced vibrations have been considered in the present study, future investigations should involve other forced vibration such as non-harmonic forced vibrations and continuous curved beams with variable sections. The proposed method could be extended to the analysis of non-circular curved beams.

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APPENDIX A.

DERIVATION OF THE EQUATIONS OF MOTION

Taking the differential element shown in Fig. 1 as a free body, we have

(I) Equilibrium of forces in the radial direction

$$-\bar{F} \cos \frac{d\theta}{2} + \bar{T} \sin \frac{d\theta}{2} - f_{I1}\, ds + \left(\bar{F} + \frac{\partial \bar{F}}{\partial s} ds\right) \cos \frac{d\theta}{2}$$

$$+ \left(\bar{T} + \frac{\partial \bar{T}}{\partial s} ds\right) \sin \frac{d\theta}{2} = 0$$

where ds is the arc length of the differential element. For small $d\theta$, $\sin \frac{d\theta}{2} \simeq \frac{d\theta}{2}$ and $\cos \frac{d\theta}{2} \simeq 1$.

Neglecting higher order terms and dividing through by $ds = Rd\theta$, one obtains

$$\frac{\partial \bar{F}}{\partial \theta} + \bar{T} = f_{I1} R \qquad \text{(a)}$$

(II) Equilibrium of forces in the tangential direction

$$-\bar{F} \sin \frac{d\theta}{2} - \bar{T} \cos \frac{d\theta}{2} - f_{I2}\, ds - \left(\bar{T} + \frac{\partial \bar{T}}{\partial s} ds\right) \sin \frac{d\theta}{2}$$

$$+ \left(\bar{T} + \frac{\partial \bar{T}}{\partial s} ds\right) \cos \frac{d\theta}{2} = 0$$

which yields

$$\frac{\partial \bar{T}}{\partial \theta} - \bar{F} = f_{I2} R \qquad \text{(b)}$$

(III) Equilibrium of moments about C

$$\bar{M} - (\bar{M} + \frac{\partial \bar{M}}{\partial s} ds) + \bar{F}R \sin \frac{d\theta}{2} + (\bar{F} + \frac{\partial \bar{F}}{\partial s} ds) R \sin \frac{d\theta}{2}$$

$$- \bar{T} \sin \frac{d\theta}{2} (R \sin \frac{d\theta}{2}) + \bar{T} \cos \frac{d\theta}{2} (R - R \cos \frac{d\theta}{2})$$

$$- (\bar{T} + \frac{\partial \bar{T}}{\partial s} ds) \sin \frac{d\theta}{2} (R \sin \frac{d\theta}{2})$$

$$- (\bar{T} + \frac{\partial \bar{T}}{\partial s} ds) \cos \frac{d\theta}{2} (R - R \cos \frac{d\theta}{2}) - m_I ds = 0$$

which leads to

$$- \frac{\partial \bar{M}}{\partial \theta} + \bar{F}R = m_I R \qquad \text{(c)}$$

APPENDIX B

GENERAL SOLUTION OF THE EQUATIONS OF MOTION

Equations (28) and (29) may be rewritten, respectively, as:

$$\left.\begin{aligned} L_1U - L_2W &= 0 \\ L_3U - L_4W &= 0 \end{aligned}\right\} \qquad \text{(d)}$$

where L_1, L_2, L_3 and L_4 are differential operators with the following constant coefficients:

$$\left.\begin{aligned} L_1 &= \frac{s^2}{r^2} D^2 + \frac{s^2}{r^2} - b^2 \cdot s^2 \\ L_2 &= \frac{s^2}{r^2} D + \frac{s^2}{r^2} D^3 + b^2 \cdot s^2 D \\ L_3 &= s^2 D^3 + \frac{s^4}{r^2} D^3 + r^2 \cdot s^2 \cdot b^2 D - \frac{s^2}{r^2} D + s^4 \cdot b^2 D \\ L_4 &= \frac{s^4}{r^2} D^4 - \frac{s^2}{r^2} D^2 + s^4 \cdot b^2 D^2 + s^4 \cdot b^2 D^2 - b^2 \cdot s^2 \\ &\quad + r^2 \cdot s^4 \cdot b^4 - s^2 D^2 - r^2 \cdot s^2 \cdot b^2 \end{aligned}\right\} \text{(e)}$$

Apply the operator L_3 to the first equation of (d) and the operator L_1 to the second equation of (d), one has

$$(L_1L_4 - L_3L_2)W = 0 \qquad \text{(f)}$$

Also, apply the operators L_4 and L_2 to the first and second equations of (d) gives:

$$(L_1L_4 - L_2L_3)U = 0 \qquad (g)$$

Substituting equations (e) into equation (f) yields:

$$(L_1L_4 - L_3L_2)W = (\frac{s^2}{r^2} D^2 + \frac{s^2}{r^2} - b^2 \cdot s^2)\ (\frac{s^2}{r^2} D^4$$

$$- \frac{s^2}{r^2} D^2 + s^4 \cdot b^2 D^2 + s^4 \cdot b^2 D^2 - b^2 \cdot s^2 + r^2 \cdot s^4 \cdot b^4$$

$$- s^2D^2 - r^2 \cdot s^2 \cdot b^2)W - (s^2D^3 + \frac{s^4}{r^2} D^3 + r^2 \cdot s^2 \cdot b^2 D$$

$$- \frac{s^2}{r^2} D + s^4 \cdot b^2 D)\ (\frac{s^2}{r^2} D + \frac{s^2}{r^2} D^3 + b^2 \cdot s^2 D)W \qquad (h)$$

Equation (h) may be rewritten as:

$$(- \frac{s^4}{r^2})\ D^6W + (-2 \frac{s^4}{r^2} - 2 s^4 \cdot b^2 - b^2 \cdot \frac{s^6}{r^2})\ D^4W$$

$$+ (\frac{s^4}{r^2} \cdot b^2 - 2 s^6 \cdot b^4 - s^4 \cdot b^2 + \frac{s^6}{r^2} \cdot b^2 - \frac{s^4}{r^2} - b^4 \cdot s^4 \cdot r^2)\ D^2W$$

$$+ (- \frac{s^4}{r^2} \cdot b^2 + s^6 \cdot b^4 - s^4 \cdot b^2 + b^4 \cdot s^4 - b^6 \cdot s^6 \cdot r^2$$

$$+ b^4 \cdot s^4 \cdot r^2)W = 0 \qquad (i)$$

Dividing equation (i) by $- \frac{s^4}{r^2}$ gives:

$$\frac{d^6W}{d\theta^6} + (2 + 2b^2 \cdot r^2 + b^2 \cdot s^2)\ \frac{d^4W}{d\theta^4} + (1 - b^2 +$$

$$2\, s^2 \cdot r^2 \cdot b^4 + b^2 \cdot r^2 - s^2 \cdot b^2 + b^4 \cdot r^4)\, \frac{d^2 W}{d\theta^2} + (b^2 -$$

$$s^2 \cdot r^2 \cdot b^4 + r^2 \cdot b^2 - b^4 \cdot r^2 + b^6 \cdot r^4 \cdot s^2 - b^4 \cdot r^4) W = 0 \qquad (j)$$

Equation (j) may be rewritten as

$$W^{VI} + k_1\, W^{IV} + k_2\, W'' + k_3\, W = 0 \qquad (k)$$

where k_1, k_2 and k_3 are defined in equations (33).

The solution of equation (k) may be expressed as:

$$W(\theta) = \sum_{n=1}^{6} a_n\, e^{\xi_n \theta} \qquad (l)$$

Also, substituting equation (e) into equation (g) yields:

$$U^{VI} + k_1\, U^{IV} + k_2\, U'' + k_3\, U = 0 \qquad (m)$$

The solution of the equation (m) takes the form of:

$$U(\theta) = \sum_{n=1}^{6} z_n\, e^{\xi_n \theta} \qquad (n)$$

Substituting equation (l) and (n) into equation (29) yields the relation between the constants a's and z's as follows:

$$\frac{s^2}{r^2} \Sigma z_n \xi_n^2 e^{\xi_n \theta} + \frac{s^2}{r^2} \Sigma z_n e^{\xi_n \theta} - b^2 \cdot s^2 \Sigma z_n e^{\xi_n \theta} -$$

$$\frac{s^2}{r^2} \Sigma a_n \xi_n e^{\xi_n \theta} - \frac{s^2}{r^2} \Sigma a_n \xi_n^3 e^{\xi_n \theta} - b^2 \cdot s^2 \Sigma a_n \xi_n e^{\xi_n \theta} = 0$$

$$(o)$$

or

$$z_n = \lambda_n a_n \qquad (p)$$

where

$$\lambda_n = \frac{\xi_n + \xi_n^3 + b^2 r^2 \xi_n}{\xi_n^2 - b^2 r^2 + 1} \qquad (q)$$

Substituting equation (p) into equation (n) yields

$$U(\theta) = \sum_{n=1}^{6} \lambda_n a_n e^{\xi_n \theta} \qquad (r)$$

Introducing equations (l) and (r) into equations (35) gives

$$R\Psi(\theta) = \sum_{n=1}^{6} a_n c_n e^{\xi_n \theta} \qquad (s)$$

where

$$c_n = \lambda_n \xi_n + \frac{s^2}{r^2} \lambda_n \xi_n - \frac{s^2}{r^2} \xi_n^2 - b^2 \cdot s^2 + 1 \qquad (t)$$

APPENDIX C

$$
[A] = \begin{bmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
c_1 e^{\xi_1 \alpha} & c_2 e^{\xi_2 \alpha} & c_3 e^{\xi_3 \alpha} & c_4 e^{\xi_4 \alpha} & c_5 e^{\xi_5 \alpha} & c_6 e^{\xi_6 \alpha} \\
(\lambda_1 \sin\rho - \cos\rho) & (\lambda_2 \sin\rho - \cos\rho) & (\lambda_3 \sin\rho - \cos\rho) & (\lambda_4 \sin\rho - \cos\rho) & (\lambda_5 \sin\rho - \cos\rho) & (\lambda_6 \sin\rho - \cos\rho) \\
(\lambda_1 \sin\eta + \cos\eta) e^{\xi_1 \alpha} & (\lambda_2 \sin\eta + \cos\eta) e^{\xi_2 \alpha} & (\lambda_3 \sin\eta + \cos\eta) e^{\xi_3 \alpha} & (\lambda_4 \sin\eta + \cos\eta) e^{\xi_4 \alpha} & (\lambda_5 \sin\eta + \cos\eta) e^{\xi_5 \alpha} & (\lambda_6 \sin\eta + \cos\eta) e^{\xi_6 \alpha} \\
(\lambda_1 \cos\rho + \sin\rho) & (\lambda_2 \cos\rho + \sin\rho) & (\lambda_3 \cos\rho + \sin\rho) & (\lambda_4 \cos\rho + \sin\rho) & (\lambda_5 \cos\rho + \sin\rho) & (\lambda_6 \cos\rho + \sin\rho) \\
(\sin\eta - \lambda_1 \cos\eta) e^{\xi_1 \alpha} & (\sin\eta - \lambda_2 \cos\eta) e^{\xi_2 \alpha} & (\sin\eta - \lambda_3 \cos\eta) e^{\xi_3 \alpha} & (\sin\eta - \lambda_4 \cos\eta) e^{\xi_4 \alpha} & (\sin\eta - \lambda_5 \cos\eta) e^{\xi_5 \alpha} & (\sin\eta - \lambda_6 \cos\eta) e^{\xi_6 \alpha}
\end{bmatrix}
$$

$$
[B] = \begin{bmatrix}
-c_1 \xi_1 & -c_2 \xi_2 & -c_3 \xi_3 & -c_4 \xi_4 & -c_5 \xi_5 & -c_6 \xi_6 \\
c_1 \xi_1 e^{\xi_1 \alpha} & c_2 \xi_2 e^{\xi_2 \alpha} & c_3 \xi_3 e^{\xi_3 \alpha} & c_4 \xi_4 e^{\xi_4 \alpha} & c_5 \xi_5 e^{\xi_5 \alpha} & c_6 \xi_6 e^{\xi_6 \alpha} \\
(m_1 \sin\rho + d_1 \cos\rho) & (m_2 \sin\rho + d_2 \cos\rho) & (m_3 \sin\rho + d_3 \cos\rho) & (m_4 \sin\rho + d_4 \cos\rho) & (m_5 \sin\rho + d_5 \cos\rho) & (m_6 \sin\rho + d_6 \cos\rho) \\
(m_1 \sin\eta - d_1 \cos\eta) e^{\xi_1 \alpha} & (m_2 \sin\eta - d_2 \cos\eta) e^{\xi_2 \alpha} & (m_3 \sin\eta - d_3 \cos\eta) e^{\xi_3 \alpha} & (m_4 \sin\eta - d_4 \cos\eta) e^{\xi_4 \alpha} & (m_5 \sin\eta - d_5 \cos\eta) e^{\xi_5 \alpha} & (m_6 \sin\eta - d_6 \cos\eta) e^{\xi_6 \alpha} \\
(-m_1 \cos\rho + d_1 \sin\rho) & (-m_2 \cos\rho + d_2 \sin\rho) & (-m_3 \cos\rho + d_3 \sin\rho) & (-m_4 \cos\rho + d_4 \sin\rho) & (-m_5 \cos\rho + d_5 \sin\rho) & (-m_6 \cos\rho + d_6 \sin\rho) \\
(m_1 \cos\eta + d_1 \sin\eta) e^{\xi_1 \alpha} & (m_2 \cos\eta + d_2 \sin\eta) e^{\xi_2 \alpha} & (m_3 \cos\eta + d_3 \sin\eta) e^{\xi_3 \alpha} & (m_4 \cos\eta + d_4 \sin\eta) e^{\xi_4 \alpha} & (m_5 \cos\eta + d_5 \sin\eta) e^{\xi_5 \alpha} & (m_6 \cos\eta + d_6 \sin\eta) e^{\xi_6 \alpha}
\end{bmatrix}
$$

APPENDIX D

$$[A_1] = \begin{bmatrix} c_1\, e^{\xi_1\theta} & c_2\, e^{\xi_2\theta} & c_3\, e^{\xi_3\theta} & c_4\, e^{\xi_4\theta} & c_5\, e^{\xi_5\theta} & c_6\, e^{\xi_6\theta} \\ \lambda_1\, e^{\xi_1\theta} & \lambda_2\, e^{\xi_2\theta} & \lambda_3\, e^{\xi_3\theta} & \lambda_4\, e^{\xi_4\theta} & \lambda_5\, e^{\xi_5\theta} & \lambda_6\, e^{\xi_6\theta} \\ e^{\xi_1\theta} & e^{\xi_2\theta} & e^{\xi_3\theta} & e^{\xi_4\theta} & e^{\xi_5\theta} & e^{\xi_6\theta} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[B_1] = \begin{bmatrix} c_1\xi_1 e^{\xi_1\theta} & c_2\xi_2 e^{\xi_2\theta} & c_3\xi_3 e^{\xi_3\theta} & c_4\xi_4 e^{\xi_4\theta} & c_5\xi_5 e^{\xi_5\theta} & c_6\xi_6 e^{\xi_6\theta} \\ d_1\, e^{\xi_1\theta} & d_2\, e^{\xi_2\theta} & d_3\, e^{\xi_3\theta} & d_4\, e^{\xi_4\theta} & d_5\, e^{\xi_5\theta} & d_6\, e^{\xi_6\theta} \\ m_1\, e^{\xi_1\theta} & m_2\, e^{\xi_2\theta} & m_3\, e^{\xi_3\theta} & m_4\, e^{\xi_4\theta} & m_5\, e^{\xi_5\theta} & m_6\, e^{\xi_6\theta} \\ -c_1\,\xi_1 & -c_2\,\xi_2 & -c_3\,\xi_3 & -c_4\,\xi_4 & -c_5\,\xi_5 & -c_6\,\xi_6 \\ -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & -d_6 \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \end{bmatrix}$$

$$[A_2] = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ c_1\, e^{\xi_1\phi} & c_2\, e^{\xi_2\phi} & c_3\, e^{\xi_3\phi} & c_4\, e^{\xi_4\phi} & c_5\, e^{\xi_5\phi} & c_6\, e^{\xi_6\phi} \\ \lambda_1\, e^{\xi_1\phi} & \lambda_2\, e^{\xi_2\phi} & \lambda_3\, e^{\xi_3\phi} & \lambda_4\, e^{\xi_4\phi} & \lambda_5\, e^{\xi_5\phi} & \lambda_6\, e^{\xi_6\phi} \\ e^{\xi_1\phi} & e^{\xi_2\phi} & e^{\xi_3\phi} & e^{\xi_4\phi} & e^{\xi_5\phi} & e^{\xi_6\phi} \end{bmatrix}$$

$$[B_2] = \begin{bmatrix} -c_1\, \xi_1 & -c_2\, \xi_2 & -c_3\, \xi_3 & -c_4\, \xi_4 & -c_5\, \xi_5 & -c_6\, \xi_6 \\ -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & -d_6 \\ -m_1 & -m_2 & -m_3 & -m_4 & -m_5 & -m_6 \\ c_1\xi_1 e^{\xi_1\phi} & c_2\xi_2 e^{\xi_2\phi} & c_3\xi_3 e^{\xi_3\phi} & c_4\xi_4 e^{\xi_4\phi} & c_5\xi_5 e^{\xi_5\phi} & c_6\xi_6 e^{\xi_6\phi} \\ d_1\, e^{\xi_1\phi} & d_2\, e^{\xi_2\phi} & d_3\, e^{\xi_3\phi} & d_4\, e^{\xi_4\phi} & d_5\, e^{\xi_5\phi} & d_6\, e^{\xi_6\phi} \\ m_1\, e^{\xi_1\phi} & m_2\, e^{\xi_2\phi} & m_3\, e^{\xi_3\phi} & m_4\, e^{\xi_4\phi} & m_5\, e^{\xi_5\phi} & m_6\, e^{\xi_6\phi} \end{bmatrix}$$

APPENDIX E

$$[\bar{B}_1] = \frac{EI}{R^3} \begin{bmatrix} c_1\xi_1 e^{\xi_1\theta} & c_2\xi_2 e^{\xi_2\theta} & c_3\xi_3 e^{\xi_3\theta} & c_4\xi_4 e^{\xi_4\theta} & c_5\xi_5 e^{\xi_5\theta} & c_6\xi_6 e^{\xi_6\theta} \\ d_1\ e^{\xi_1\theta} & d_2\ e^{\xi_2\theta} & d_3\ e^{\xi_3\theta} & d_4\ e^{\xi_4\theta} & d_5\ e^{\xi_5\theta} & d_6\ e^{\xi_6\theta} \\ m_1\ e^{\xi_1\theta} & m_2\ e^{\xi_2\theta} & m_3\ e^{\xi_3\theta} & m_4\ e^{\xi_4\theta} & m_5\ e^{\xi_5\theta} & m_6\ e^{\xi_6\theta} \end{bmatrix}$$

$$[\bar{B}_2] = \frac{EI}{R^3} \begin{bmatrix} -c_1\ \xi_1 & -c_2\ \xi_2 & -c_3\ \xi_3 & -c_4\ \xi_4 & -c_5\ \xi_5 & -c_6\ \xi_6 \\ -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & -d_6 \\ -m_1 & -m_2 & -m_3 & -m_4 & -m_5 & -m_6 \end{bmatrix}$$

APPENDIX F

$$[\bar{T}\bar{T}] = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ \lambda_1 e^{\xi_1\theta} & \lambda_2 e^{\xi_2\theta} & \lambda_3 e^{\xi_3\theta} & \lambda_4 e^{\xi_4\theta} & \lambda_5 e^{\xi_5\theta} & \lambda_6 e^{\xi_6\theta} \\ e^{\xi_1\theta} & e^{\xi_2\theta} & e^{\xi_3\theta} & e^{\xi_4\theta} & e^{\xi_5\theta} & e^{\xi_6\theta} \\ c_1 e^{\xi_1\theta} & c_2 e^{\xi_2\theta} & c_3 e^{\xi_3\theta} & c_4 e^{\xi_4\theta} & c_5 e^{\xi_5\theta} & c_6 e^{\xi_6\theta} \end{bmatrix}$$

$$[c\,\xi] = [\,c_1\xi_1 \quad c_2\xi_2 \quad c_3\xi_3 \quad c_4\xi_4 \quad c_5\xi_5 \quad c_6\xi_6\,]$$

$$[c\,\xi\, e^{\xi\theta}] = [c_1\xi_1 e^{\xi_1\theta} \quad c_2\xi_2 e^{\xi_2\theta} \quad c_3\xi_3 e^{\xi_3\theta} \quad c_4\xi_4 e^{\xi_4\theta} \quad c_5\xi_5 e^{\xi_5\theta} \quad c_6\xi_6 e^{\xi_6\theta}]$$

$$[m] = [\,m_1 \quad m_2 \quad m_3 \quad m_4 \quad m_5 \quad m_6\,]$$

$$[d] = [\,d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6\,]$$

$$[m\, e^{\xi\theta}] = [\,m_1 e^{\xi_1\theta} \quad m_2 e^{\xi_2\theta} \quad m_3 e^{\xi_3\theta} \quad m_4 e^{\xi_4\theta} \quad m_5 e^{\xi_5\theta} \quad m_6 e^{\xi_6\theta}\,]$$

$$[d\, e^{\xi\theta}] = [\,d_1 e^{\xi_1\theta} \quad d_2 e^{\xi_2\theta} \quad d_3 e^{\xi_3\theta} \quad d_4 e^{\xi_4\theta} \quad d_5 e^{\xi_5\theta} \quad d_6 e^{\xi_6\theta}\,]$$

$$[E] = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ q_1 e^{\xi_1 \alpha} & q_2 e^{\xi_2 \alpha} & q_3 e^{\xi_3 \alpha} & q_4 e^{\xi_4 \alpha} & q_5 e^{\xi_5 \alpha} & q_6 e^{\xi_6 \alpha} \\ (\xi_1 \sin\rho - \cos\rho) & (\xi_2 \sin\rho - \cos\rho) & (\xi_3 \sin\rho - \cos\rho) & (\xi_4 \sin\rho - \cos\rho) & (\xi_5 \sin\rho - \cos\rho) & (\xi_6 \sin\rho - \cos\rho) \\ e^{\xi_1 \alpha}(\xi_1 \sin\eta + \cos\eta) & e^{\xi_2 \alpha}(\xi_2 \sin\eta + \cos\eta) & e^{\xi_3 \alpha}(\xi_3 \sin\eta + \cos\eta) & e^{\xi_4 \alpha}(\xi_4 \sin\eta + \cos\eta) & e^{\xi_5 \alpha}(\xi_5 \sin\eta + \cos\eta) & e^{\xi_6 \alpha}(\xi_6 \sin\eta + \cos\eta) \\ (\xi_1 \cos\rho + \sin\rho) & (\xi_2 \cos\rho + \sin\rho) & (\xi_3 \cos\rho + \sin\rho) & (\xi_4 \cos\rho + \sin\rho) & (\xi_5 \cos\rho + \sin\rho) & (\xi_6 \cos\rho + \sin\rho) \\ e^{\xi_1 \alpha}(\sin\eta - \xi_1 \cos\eta) & e^{\xi_2 \alpha}(\sin\eta - \xi_2 \cos\eta) & e^{\xi_3 \alpha}(\sin\eta - \xi_3 \cos\eta) & e^{\xi_4 \alpha}(\sin\eta - \xi_4 \cos\eta) & e^{\xi_5 \alpha}(\sin\eta - \xi_5 \cos\eta) & e^{\xi_6 \alpha}(\sin\eta - \xi_6 \cos\eta) \end{bmatrix}$$

$$
[C] = \begin{bmatrix}
-q_1\xi_1 & -q_2\xi_2 & -q_3\xi_3 & -q_4\xi_4 & -q_5\xi_5 & -q_6\xi_6 \\
q_1\xi_1 e^{\xi_1\alpha} & q_2\xi_2 e^{\xi_2\alpha} & q_3\xi_3 e^{\xi_3\alpha} & q_4\xi_4 e^{\xi_4\alpha} & q_5\xi_5 e^{\xi_5\alpha} & q_6\xi_6 e^{\xi_6\alpha} \\
z_1(\sin\rho+\xi_1\cos\rho) +b^2\xi_1\cos\rho & z_2(\sin\rho+\xi_2\cos\rho) +b^2\xi_2\cos\rho & z_3(\sin\rho+\xi_3\cos\rho) +b^2\xi_3\cos\rho & z_4(\sin\rho+\xi_4\cos\rho) +b^2\xi_4\cos\rho & z_5(\sin\rho+\xi_5\cos\rho) +b^2\xi_5\cos\rho & z_6(\sin\rho+\xi_6\cos\rho) +b^2\xi_6\cos\rho \\
[z_1(\sin\eta-\xi_1\cos\eta) -b^2\xi_1\cos\eta]e^{\xi_1\alpha} & [z_2(\sin\eta-\xi_2\cos\eta) -b^2\xi_2\cos\eta]e^{\xi_2\alpha} & [z_3(\sin\eta-\xi_3\cos\eta) -b^2\xi_3\cos\eta]e^{\xi_3\alpha} & [z_4(\sin\eta-\xi_4\cos\eta) -b^2\xi_4\cos\eta]e^{\xi_4\alpha} & [z_5(\sin\eta-\xi_5\cos\eta) -b^2\xi_5\cos\eta]e^{\xi_5\alpha} & [z_6(\sin\eta-\xi_6\cos\eta) -b^2\xi_6\cos\eta]e^{\xi_6\alpha} \\
z_1(\xi_1\sin\rho-\cos\rho) +b^2\xi_1\sin\rho & z_2(\xi_2\sin\rho-\cos\rho) +b^2\xi_2\sin\rho & z_3(\xi_3\sin\rho-\cos\rho) +b^2\xi_3\sin\rho & z_4(\xi_4\sin\rho-\cos\rho) +b^2\xi_4\sin\rho & z_5(\xi_5\sin\rho-\cos\rho) +b^2\xi_5\sin\rho & z_6(\xi_6\sin\rho-\cos\rho) +b^2\xi_6\sin\rho \\
[z_1(\xi_1\sin\eta+\cos\eta) +b^2\xi_1\sin\eta]e^{\xi_1\alpha} & [z_2(\xi_2\sin\eta+\cos\eta) +b^2\xi_2\sin\eta]e^{\xi_2\alpha} & [z_3(\xi_3\sin\eta+\cos\eta) +b^2\xi_3\sin\eta]e^{\xi_3\alpha} & [z_4(\xi_4\sin\eta+\cos\eta) +b^2\xi_4\sin\eta]e^{\xi_4\alpha} & [z_5(\xi_5\sin\eta+\cos\eta) +b^2\xi_5\sin\eta]e^{\xi_5\alpha} & [z_6(\xi_6\sin\eta+\cos\eta) +b^2\xi_6\sin\eta]e^{\xi_6\alpha}
\end{bmatrix}
$$

APPENDIX H

$$[E_1] = \begin{bmatrix} q_1 e^{\xi_1 \theta} & q_2 e^{\xi_2 \theta} & q_3 e^{\xi_3 \theta} & q_4 e^{\xi_4 \theta} & q_5 e^{\xi_5 \theta} & q_6 e^{\xi_6 \theta} \\ \xi_1 e^{\xi_1 \theta} & \xi_2 e^{\xi_2 \theta} & \xi_3 e^{\xi_3 \theta} & \xi_4 e^{\xi_4 \theta} & \xi_5 e^{\xi_5 \theta} & \xi_6 e^{\xi_6 \theta} \\ e^{\xi_1 \theta} & e^{\xi_2 \theta} & e^{\xi_3 \theta} & e^{\xi_4 \theta} & e^{\xi_5 \theta} & e^{\xi_6 \theta} \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[C_1] = \begin{bmatrix} q_1\xi_1 e^{\xi_1\theta} & q_2\xi_2 e^{\xi_2\theta} & q_3\xi_3 e^{\xi_3\theta} & q_4\xi_4 e^{\xi_4\theta} & q_5\xi_5 e^{\xi_5\theta} & q_6\xi_6 e^{\xi_6\theta} \\ \xi_1 e^{\xi_1\theta}(z_1+b^2) & \xi_2 e^{\xi_2\theta}(z_2+b^2) & \xi_3 e^{\xi_3\theta}(z_3+b^2) & \xi_4 e^{\xi_4\theta}(z_4+b^2) & \xi_5 e^{\xi_5\theta}(z_5+b^2) & \xi_6 e^{\xi_6\theta}(z_6+b^2) \\ z_1 e^{\xi_1\theta} & z_2 e^{\xi_2\theta} & z_3 e^{\xi_3\theta} & z_4 e^{\xi_4\theta} & z_5 e^{\xi_5\theta} & z_6 e^{\xi_6\theta} \\ -q_1\xi_1 & -q_2\xi_2 & -q_3\xi_3 & -q_4\xi_4 & -q_5\xi_5 & -q_6\xi_6 \\ -\xi_1(z_1+b^2) & -\xi_2(z_2+b^2) & -\xi_3(z_3+b^2) & -\xi_4(z_4+b^2) & -\xi_5(z_5+b^2) & -\xi_6(z_6+b^2) \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix}$$

$$[E_2] = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ q_1\, e^{\xi_1 \phi} & q_2\, e^{\xi_2 \phi} & q_3\, e^{\xi_3 \phi} & q_4\, e^{\xi_4 \phi} & q_5\, e^{\xi_5 \phi} & q_6\, e^{\xi_6 \phi} \\ \xi_1\, e^{\xi_1 \phi} & \xi_2\, e^{\xi_2 \phi} & \xi_3\, e^{\xi_3 \phi} & \xi_4\, e^{\xi_4 \phi} & \xi_5\, e^{\xi_5 \phi} & \xi_6\, e^{\xi_6 \phi} \\ e^{\xi_1 \phi} & e^{\xi_2 \phi} & e^{\xi_3 \phi} & e^{\xi_4 \phi} & e^{\xi_5 \phi} & e^{\xi_6 \phi} \end{bmatrix}$$

$$[C_2] = \begin{bmatrix} -q_1\ \xi_1 & -q_2\ \xi_2 & -q_3\ \xi_3 & -q_4\ \xi_4 & -q_5\ \xi_5 & -q_6\ \xi_6 \\ -\xi_1(z_1+b^2) & -\xi_2(z_2+b^2) & -\xi_3(z_3+b^2) & -\xi_4(z_4+b^2) & -\xi_5(z_5+b^2) & -\xi_6(z_6+b^2) \\ -z_1 & -z_2 & -z_3 & -z_4 & -z_5 & -z_6 \\ q_1\ \xi_1\ e^{\xi_1\phi} & q_2\ \xi_2\ e^{\xi_2\phi} & q_3\ \xi_3\ e^{\xi_3\phi} & q_4\ \xi_4\ e^{\xi_4\phi} & q_5\ \xi_5\ e^{\xi_5\phi} & q_6\ \xi_6\ e^{\xi_6\phi} \\ \xi_1\ e^{\xi_1\phi}(z_1+b^2) & \xi_2\ e^{\xi_2\phi}(z_2+b^2) & \xi_3\ e^{\xi_3\phi}(z_3+b^2) & \xi_4\ e^{\xi_4\phi}(z_4+b^2) & \xi_5\ e^{\xi_5\phi}(z_5+b^2) & \xi_6\ e^{\xi_6\phi}(z_6+b^2) \\ z_1\ e^{\xi_1\phi} & z_2\ e^{\xi_2\phi} & z_3\ e^{\xi_3\phi} & z_4\ e^{\xi_4\phi} & z_5\ e^{\xi_5\phi} & z_6\ e^{\xi_6\phi} \end{bmatrix}$$