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FLUCTUATIONS AND DISSIPATION IN A THERMALLY CONDUCTING, VISCOUS, HYDRODYNAMIC MEDIUM

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BY

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TABLE OF CONTENTS

LIST OF DIAGRAMS

NOMENCLATURE

variables

ε	energy density
h	enthalpy density
p	pressure
q	heat energy density
ρ	mass density
S	entropy
T	temperature
u	internal energy density
$\vec{V}$	velocity vector
x_i	general thermodynamic fluctuation
x	response fluctuation

parameters

α_T	coefficient of thermal expansion
c_1	isentropic speed of sound
c_p	constant pressure specific heat
c_v	constant volume specific heat
C^l_{mn}	general thermodynamic deviative
D_1	thermal-viscous damping coefficient
D_2	thermal diffusion coefficient (thermometric conductivity)
D_3	thermal diffusion coefficient
D_L	longitudinal viscous damping coefficient
D_T	transverse viscous damping coefficient
J^{ij}_{mn}	general thermodynamic jacobian
κ	thermal conductivity

η	first coefficient of viscosity
ζ	second coefficient of viscosity
r_c	characteristic radius of thermal-viscous diffusion and wave interaction
r_T	characteristic radius of transverse viscous diffusion

functions

$*$	spatial and temporal convolution
$*_r$	spatial convolution
δ_{ij}	Kronecker delta
$\delta(\ )$	delta function
$\delta(\vec{r})=\delta(r_x)\delta(r_y)\delta(r_z)$	three dimensional delta function
$d^3\vec{r} = dr_x dr_y dr_z$	three dimensional differential
DTV	transformed thermal-viscous Green's function denominator
DM	transformed thermal-viscous model Green's function denominator
f	forcing function
$\vec{f}$	spatial forcing function
$\bar{\bar{f}}$	temporal forcing function
G	Green's function
I	integral
$\vec{k}$	wavenumber vector
$k_{1,2,3,}$	wavenumber pole
L( )	operator of field equation
L{ }	Laplace transform
L_c\{ \}	complex Laplace transform
N	transformed thermal-viscous Green's function numerator
$P^L_{jm}(\vec{k})$	longitudinal polarization tensor
$P^T_{jm}(\vec{k})$	transverse polarization tensor

$\vec{r}$	spatial vector
$R_{()()}$	correlation function
$S_{()()}$	power spectrum
t	time
U	unit step function
ω	frequency
$z = \omega + ie$	complex Laplace frequency
$\omega_{1,2,3,}$	frequency pole

ABSTRACT

FLUCTUATIONS AND DISSIPATION IN A THERMALLY CONDUCTING, VISCOUS, HYDRODYNAMIC MEDIUM

by

KARL SUNDKVIST

University of New Hampshire, May 1979

The behavior of small amplitude fluctuations from equilibrium in a compressible fluid medium with small viscosity and thermal conductivity is characterised by the fluctuations' Green's functions.

The hydrodynamic model is comprised of the mass, momentum and energy conservation equations and the thermodynamic coupling of low frequency, small wavenumber fluctuations. From the hydrodynamic model, field equations governing the fluctuations of the thermodynamic properties and of the velocity are derived. The transverse component of velocity obeys a transverse viscous diffusion equation whereas the longitudinal velocity and thermodynamic property fluctuations all satisfy the same scalar thermal-viscous field equation. A vectorial thermal-viscous field equation for the total velocity vector fluctuation simultaneously includes the transverse viscous diffusion and longitudinal thermal-viscous effects.

The impulse Green's functions in transformed domain are derived from the spatially and temporally Fourier transformed field equations. A scalar diffusive type impulse Green's function is obtained for the transverse velocity fluctuation and a scalar thermal-viscous type impulse Green's function is obtained for the fluctuations of longitudinal velocity and thermodynamic pro-

perties. A tensorial thermal-viscous velocity Green's function is derived and separated into transverse and longitudinal components. The longitudinal component consists of the scalar thermal-viscous impulse Green's function and a longitudinal polarization tensor whereas the transverse component consists of both transverse viscous and thermal diffusive Green's functions and a transverse polarization tensor.

Initial condition Green's functions are derived from the Hilbert transformed scalar field equations via Kubo's formula. A scalar diffusion type initial condition Green's function is obtained for the response of transverse velocity fluctuation to its initial condition. Numerous scalar thermal-viscous type initial condition Green's functions are derived which relate the responses of the thermodynamic properties or the longitudinal velocity to various initial condition terms.

The diffusion and thermal-viscous type initial condition Green's functions exhibit the same denominators as the respective type impulse Green's functions. These denominators determine the Green's functions wavenumber and frequency poles which characterize the type of fluctuation response. The diffusion type of Green's functions possess two wavenumber poles and one frequency pole which are of diffusive form. The thermal-viscous type Green's functions possess four wavenumber and three frequency poles. These poles are modeled, for small viscous and thermal effects, as two wavenumber and one frequency diffusion type poles plus two wavenumber and two frequency damped acoustic wave type poles. From these approximate poles, a thermal-viscous model Green's functions are constructed which display the coupled effects of thermal diffusion and a thermally and viscously damped acoustic wave. This thermal-viscous model Green's function is applied to the longitudinal component of the tensorial thermal-viscous velocity of Green's function. Both transverse and longitudinal components display thermal diffusion however in the transverse component it is coupled with a transverse viscous diffusion whereas in the longitudinal component it is coupled with a thermally and viscously damped acoustic wave.

The information contained by the wavenumber and frequency poles of the Green's functions in transformed domain is more apparent in the Green's functions representations in alternate domains. These are derived by performing an inverse spatial transform or both on the Green's functions in wavenumber and frequency domain. The Green's function in each domain is related to the fluctuation response in space and time domain due to a particular forcing or initial condition. Some of these responses are compared to previous plane wave solutions for a viscous, thermally conducting medium. The Green's function approach is more general and consistant and yields solutions which exhibit thermal diffusion as well as the thermal-viscous effect on attenuation and propagation velocity of acoustic waves.

I INTRODUCTION

The purpose of this project is to characterize the behavior of small amplitude fluctuations from equilibrium in a compressible fluid medium with small but significant viscosity and thermal conductivity. This is accomplished by developing the impulse and initial Green's functions for the fluctuations.

Historically, interest in acoustics has led to investigation of the effects of viscosity and thermal conductivity. From conservation of mass and momentum and assuming isentropic sound speed Stokes(1) determined that the first order effect of viscosity was a viscous attenuation of acoustic waves. Rayleigh(2) allowed for modification, by thermal conduction, of the thermal fluctuations due to an acoustic wave and coupled them with the pressure and density fluctuations. Neglecting viscosity, the first order effect of thermal conductivity alone was a thermal attenuation. Noting that the viscosity and thermal conductivity are often of the same order of importance, Kirchhoff(3) simultaneously included viscosity as considered by Stokes and thermal conductivity as considered by Rayleigh. The resulting solution displayed, to first order, an acoustic wave term which was attenuated by both viscosity and thermal conductivity and a thermal diffusion term. Landau and Lifshitz(4) also determined the attenuation of an acoustic wave by examining the dissipation of energy due to viscosity and thermal conductivity. In all of these investigations, exponential solutions of given frequency as wavenumber were assumed and thus the results were limited.

In a much more fundamental investigation, Kadanoff & Martin(5) considered a hydrodynamic model which included the effects of viscosity and thermal conductivity. Their interest involved correlation functions of statistical physics which were to be compatible in the relatively low frequency and small wavenumber limit with the theory of fluid mechanics. However their correlation function approach to their hydrodynamic model suggested this Green's function investigation of a viscous thermally conducting compressible fluid medium.

The hydrodynamic model consists of the conservation equations for mass, vectorial momentum and energy which are coupled through the thermodynamic interrelationship between properties resulting from assumption of local thermodynamic equilibrium. Homogeneous field for the fluctuations of variables are derived and transformed by Fourier or Hilbert transformations yielding impulse Green's functions or initial condition Green's functions respectively. Two coupled field equations for two thermodynamic property fluctuations, as utilized by Kadanoff and Martin, and a field equation for a single property fluctuation are Hilbert transformed and initial condition Green's functions relating various property responses to various initial condition terms are derived. The single thermodynamic flucturation field equation and a tensorial field equation for the velocity vector are Fourier transformed. From these the impulse Green's function for the property fluctuations and the tensorial impulse Green's function for the velocity fluctuation, which simultaneously includes the longitudinal and transverse velocity components, are derived. The impulse Green's functions relate the fluctuation response to the forcing.

The Green's functions in transformed domain characterize the medium via the wavenumber and frequency poles. The information contained is often more visible or applicable when the Green's functions is in an alternate spatial or temporal domain, therefore the Green's functions are expressed in all four domains. In order to perform the inverse transformations consistantly the poles of the Green's function are approximated yielding a model Green's function which exhibits damped wave behavior combined with diffusive behavior. For the sake of the model, the overdamped and critically damped regimes are investigated as well as the underdamped regimes which is applicable to hydrodynamics.

The information contained within the model Green's functions is utilized to determine the fluctuation response to various forcings or initial conditions. This displays the use for and the physical meaning of the Green's functions in a thermal-viscous medium. The responses, as derived from the model Green's function, to some particular forcings correspond to the previously derived solutions. In addition, the Green's functions may be used to derive responses to statistical forcing situations and therefore represent a much more general characterization of the viscous thermally conducting hydrodynamic medium.

II Governing Equations

Conservation Equations

Fluctuations of the thermodynamic properties of a compressible, viscous, thermally conducting, single component fluid will be investigated through the use of Green's function techniques. A linearized set of field equations, required for such an analysis, may be derived from the conservation equations for mass density (ρ), vectorial momentum density ($\rho\vec{v}$), and energy density ($\varepsilon = u + \frac{1}{2}\rho v^2$) which exhibit the effects of compressibility, viscosity and thermal conduction. Inherent in these conservation equations are the fluid properties; mass density (ρ), temperature (T), pressure (p), internal energy density (u) and vectorial velocity ($\vec{v}$) for which fluctuations will be sought.

Although the fluid properties appear too numerous to be soluble by the conservation equations alone, the properties are not independent. The equilibrium state of the fluid may be specified by any two of the properties plus the vectorial velocity, which would be uniform and constant. In this case the fluid properties would be identical to the thermodynamic properties. However, dynamic (rather than static) solutions are desired, which implies that the fluid is not in thermodynamic equilibrium. This difficulty is surmounted by considering the fluid to be comprised of small regions, each of which are in quasi-static equilibrium. This local equilibrium exists when the fluctuations in the properties have time and length scales which are long compared to the relaxation time and characteristic length of the local region. Hence the fluid properties, which are essentially the thermodynamic properties, are interrelated locally through the standard thermodynamic relations and the conservation equations are sufficient for finding solutions to the fluctuations in these properties.

The conservation laws for mass density, vectorial momentum density ($\rho\vec{v}$) and energy density ($\varepsilon = u + \frac{1}{2}\rho v^2$) relate the time changes of each property to their respective flux vectors. In differential form these conservation equations are

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{F}^{\rho} = 0 \quad , \quad \frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\vec{F}^{(\rho\vec{v})} = 0 \quad , \text{ and } \quad \frac{\partial\varepsilon}{\partial t} + \nabla\cdot\vec{F}^{\varepsilon} = 0 \; . \qquad (2.1a,b,c)$$

The physical meaning of the conservation laws is more apparent in the control volume form. This may be obtained by integrating the differential form over a control volume and applying Green's formula to the divergence of the flux vector, resulting in the form $\int_{C.V.} dV \, \frac{\partial(\;)}{\partial t} + \int_{C.S.} d\vec{A}\cdot\vec{F}^{(\;)} = 0$. (2.2)

If the amount of the property is to be conserved then the time rate of increase within the control volume, represented by the first integral, plus the time rate of flow out of the control volume, represented by the second integral, must equal zero. From the physical meaning of the second integral it is apparent that the flux vector is the amount of property passing through a unit area perpendicular to its direction in a unit time.

The mass, momentum and energy flux vectors may be expressed in terms of the fluid properties. Since mass only moves by particle motion, the mass flux vector is simply $\vec{F}^{(\rho)} = \rho\vec{v}$, the momentum density. Momentum may be advected, as mass is, and it may be transfered through the action of a force. Thus the vectorial momentum flux vector, actually a second order tensor, may be expressed as an advection tensor minus the stress tensor,

$$F_j^{(\rho vi)} = \Pi_{ij} = \rho v_i v_j - \sigma_{ij} \; . \qquad (2.3a)$$

For a viscous fluid the stress tensor is composed of pressure, shear viscosity, and dillitational viscosity effects, as shown by the respective terms on the left side of the equation

$$\sigma_{ij} = -\delta_{ij}p + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij}\left(\zeta + \frac{2}{3}\eta\right)\frac{\partial V_n}{\partial x_n} \qquad (2.3b)$$

where η is the first coefficient of viscosity or the dynamic viscosity and ζ is the second coefficient of viscosity or bulk viscosity.

Energy may be advected, transfered by stress or transfered by thermal conduction. With these effects considered the energy flux vector is

$$\vec{F}^{\varepsilon} = \vec{V}\varepsilon + \vec{V}\cdot\vec{\vec{\sigma}} - \kappa\nabla T \tag{2.4}$$

where κ is the coefficient of thermal conductivity.

Utilizing these expressions for the flux vectors the differental forms of the conservation equations become in index notation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho V_j) = 0 \tag{2.5a}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}[\rho v_i v_j + \delta_{ij}p - \eta(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) + \delta_{ij}(\zeta + \frac{2}{3}\eta)\frac{\partial V_n}{\partial x_n}] = 0 \tag{2.5b}$$

$$\frac{\partial\varepsilon}{\partial t} + \frac{\partial}{\partial x_j}[v_j\varepsilon + V_j p - v_i\eta(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - v_j(\zeta + \frac{2}{3}\eta)\frac{\partial V_n}{\partial x_n} - \kappa\frac{\partial T}{\partial x_j}] = 0 \tag{2.5c}$$

Linearization

The conservation equations are not soluble by analytical methods primarily because of terms which are nonlinear in the thermodynamic properties and coefficients. If the fluctuations in the thermodynamics properties are sufficiently small in magnitude then terms which are of second or higher order in the fluctuations or their derivatives may be neglected. Furthermore, the coefficients of viscosity and thermal conductivity, which are functions of the thermodynamic properties, also will exhibit small fluctuations. Thus any term which is of second or higher order in the property fluctuations or the coefficient fluctuations may be neglected leaving the conservation equations in linearized form.

This linearization may be expressed more explicitly by expanding each property and coefficient as the sum of its equilibrium value, denoted by the subscript zero, and its fluctuation, denoted as a function of space and time:

$$\rho = \rho_0+\rho(\vec{r},t)\ ,\ T = T_0 + T(\vec{r},t)\ ,\ p = p_0 + p(\vec{r},t)\ ,\ u = u_0 + u(\vec{r},t),\ \vec{v} = \vec{v}_0 + \vec{v}(\vec{r},t),$$

$$\eta = \eta_0 + \eta(\vec{r},t)\ ,\ \zeta = \zeta_0 + \zeta(\vec{r},t)\ ,\ \kappa = \kappa_0 + \kappa(\vec{r},t)\ . \tag{2.6}$$

An additional simplification may be obtained by utilizing a reference frame in which the equilibrium velocity, $\vec{v}_0$, is zero. Thus there is no equilibrium advection of any of the properties, to first order. In terms of equilibrum values and to first order in fluctuations the linearized conservations equations simplify to

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \rho_0\nabla\cdot\vec{v}(\vec{r},t) = 0 \tag{2.7a}$$

$$\rho_0 \frac{\partial}{\partial t} \vec{v}(\vec{r},t) + \nabla p(\vec{r},t) - \eta_0\nabla^2\vec{v}(\vec{r},t) - (\zeta_0 + \frac{1}{3}\eta_0)\nabla\ \nabla\cdot\vec{v}(\vec{r},t) = 0 \tag{2.7b}$$

$$\frac{\partial}{\partial t} u(\vec{r},t) + (u_0+p_0)\nabla\cdot\vec{v}(\vec{r},t) - \kappa_0\nabla^2\ T(\vec{r},t) = 0 \tag{2.7c}$$

Some physical aspects of the linearization will be noted. In particular the fluctuations of the coefficients of viscosity and thermal conductivity are neglected, as they do not appear in the linearized equations. The momentum density is significant only as the equilibrium density times the velocity fluctuation and the energy density is essentially the internal energy since the kinetic energy is negligible when there is no equilibrium velocity. Similarly momentum is not advected but is only transported by the fluctuations in pressure and viscous stress. Energy is advected in the form of equilibrium enthalpy ($h_o = u_o + p_o$) by the fluctuation of velocity; the pressure term is all that remains of the stress tensor in the linearized energy equation. Thus viscous effects are relegated to the momentum equation by linearization. However, all the intended effects (compressibility, viscosity and thermal conductivity) have survived the process of linearizing the conservation equations to an analytically soluble form.

Field Equations

The linearized conservation equations are coupled through the thermodynamically related fluctuations in the fluid properties and the fluctuations in velocity. Specification of the thermodynamic process of fluctuation allows derivation of the field equations for all fluctuations. Two fluctuation processes will be considered, the isentropic process approximation of classical acoustics and the nonisentropic local thermodynamic equilibrium process appropriate for a compressible, thermal-viscous fluid. The field equations for each fluctuation will be developed for both processes and a pair of coupled field equations for two general thermodynamic properties will be derived for the thermal-viscous process.

The common linearized hydrodynamic equations for mass density and vectorial momentum density, equations 2.7a and 2.7b, and the linearized energy density equation, equation 2.7c, are all coupled by the divergence of the velocity regardless of the fluctuation process. Thus it is the velocity divergence that is interrelated with the thermodynamic fluctuations. Therefore, the vectorial velocity is conventionally separated into longitudinal and transverse components denoted by the respective superscripts L and T ,

$$\vec{v}(\vec{r},t) = \vec{v}^{L}(\vec{r},t) + \vec{v}^{T}(\vec{r},t) \ . \tag{2.8}$$

Since the divergence of the transverse velocity and the curl of the longitudinal velocity are both equal to zero, the longitudinal velocity fluctuation is related to the velocity divergence and thus to the thermodynamic fluctuations and the transverse velocity fluctuation is related to the curl of the velocity or the vorticity.

The transverse velocity only occurs in the momentum density equation, equation 2.7b. Since the curl of a gradient equals zero the curl of the vectorial momentum equation yields the transverse momentum equation, a field equation for vorticity fluctuation,

$$[\frac{\partial}{\partial t} - \frac{\eta}{\rho} \nabla^2]\nabla \times \vec{v}^T(\vec{r},t) = 0 \quad . \tag{2.9a}$$

Since the transverse velocity fluctuation contributes entirely to the vorticity fluctuation it also satisfies the same diffusion field equation,

$$[\frac{\partial}{\partial t} - \frac{\eta}{\rho} \nabla^2]\vec{v}^T(\vec{r},t) = 0 \quad , \tag{2.9b}$$

where the transverse viscous diffusion coefficient is defined as

$$D_T = \frac{\eta}{\rho} \quad . \tag{2.10}$$

The transverse velocity and the vorticity are independent of the other fluid properties and thus are independent of the fluctuation process.

The divergence of the vectorial momentum density equation, equation 7b, yields the longitudinal momentum equation,

$$[\frac{\partial}{\partial t} - (\frac{\zeta + \frac{4}{3}\eta}{\rho})\nabla^2]\nabla \cdot \vec{v}^L(\vec{r},t) + \frac{1}{\rho}\nabla^2 p(\vec{r},t) = 0 \quad . \tag{2.11}$$

The longitudinal velocity is coupled to the pressure by longitudinal momentum; to the mass density by the mass density equation, 2.7a, and to the internal energy and temperature by the energy 2.7c regardless of fluctuation process. Since the mass density and longitudinal momentum equations are utilized for either process considered, a useful form of the longitudinal momentum equation results from eliminating the velocity divergence between these two equations, 2.7a and 2.11. The resulting form of the longitudinal momentum equation relates the mechanical fluctuations, the mass density and pressure fluctuations, regardless of the fluctuation process, by the equation,

$$[- \frac{\partial^2}{\partial t^2} + (\frac{\zeta + \frac{4}{3}\eta}{\rho}) \frac{\partial}{\partial t} \nabla^2]\rho(\vec{r},t) + \nabla^2 p(\vec{r},t) = 0 \quad . \tag{2.12}$$

The additional information required to develop the field equations involves the specification of the fluctuation process.

Acoustic field equations

For conventional acoustic analysis the mass density and pressure are assumed to fluctuate so rapidly that no heat transfer occurs and the viscous dissipation is assumed to be thermodynamically unimportant. The fluctuation process is then approximately adiabatic and reversible, and thus isentropic. The thermodynamics of the process results in the isentropic speed of sound relation between the mass density and pressure fluctuations,

$$p(\vec{r},t) = \left(\frac{\partial p}{\partial \rho}\right)_s \rho(\vec{r},t) = c_l^2 \rho(\vec{r},t) \quad . \tag{2.13}$$

This isentropic fluctuation process equation provides the information necessary to suppliment the longitudinal momentum equation, equation 12, in order to derive the field equations governing mass density and pressure fluctuations. Both fluctuations obey the same field equation,

$$\left[- \frac{\partial^2}{\partial t^2} + \left(\frac{\zeta + \frac{4}{3}\eta}{\rho}\right) \frac{\partial}{\partial t} \nabla^2 + c_l^2 \nabla^2\right] \begin{Bmatrix} \rho(\vec{r},t) \\ p(\vec{r},t) \end{Bmatrix} = 0 \quad , \tag{2.14}$$

which represents viscous damped propagation at a speed c_l with a longitudinal viscous diffusion coefficient defined as,

$$D_L = \frac{\zeta + \frac{4}{3}\eta}{\rho} \quad . \tag{2.15}$$

The time derivative of the field equation for mass density, equation 2.14, allows the time derivative of the mass density to be replaced by the divergence of the velocity via the mass density conservation equation, equation 2.7a, and results in the field equation for the velocity divergence,

$$\left[- \frac{\partial^2}{\partial t^2} + D_L \frac{\partial}{\partial t} \nabla^2 + c_l^2 \nabla^2\right] \nabla \cdot \vec{v}^L(\vec{r},t) = 0 \quad . \tag{2.16a}$$

Since the longitudinal velocity fluctuation contributes entirely to the velocity divergence fluctuation they both obey the same field equation. This field equation is identical to that which governs the mass density fluctuation and in this case also governs the pressure fluctuation,

$$[-\frac{\partial^2}{\partial t^2} + D_L \frac{\partial}{\partial t} \nabla^2 + c_1^2 \nabla^2]\vec{v}^L(\vec{r},t) = 0 \quad . \tag{2.16b}$$

Since the field equations for the transverse and longitudinal velocity fluctuations have been derived separately, the information as to how the two components relate and form the total velocity fluctuation has been lost. For this information it is necessary to derive the field equation for the total velocity fluctuation. The pressure and mass density fluctuations are related by the isentropic speed of sound in equation 2.11, thus the pressure gradient term in the vectorial momentum density equation, equation 2.7b, may be replaced by $c_1^2 \nabla \rho(\vec{r},t)$. The time derivative of the momentum equation allows this mass density term to be replaced by a velocity divergence through the mass density equation, equation 2.7a, and results in the field equation for the vectorial velocity fluctuation,

$$[-\frac{\partial^2}{\partial t^2} + D_T \frac{\partial}{\partial t} \nabla^2 + (D_L - D_T) \frac{\partial}{\partial t} \nabla\nabla \cdot + c_1^2 \nabla\nabla\cdot]\vec{v}(\vec{r},t) = 0 \quad . \tag{2.17}$$

This field equation contains all the information previously derived for the velocity since taking it's curl yields the field equations for the vorticity and the transverse velocity, equations 2.9a and 2.9b, and taking its divergence yields the field equations for the velocity divergence and the longitudinal velocity, equations 2.16a and 2.16b. In addition, this vectorial velocity fluctuation field equation will presently be shown to relate the transverse and longitudinal components of velocity.

Thermal-viscous field equations

The fluctuation process may not be considered isentropic when significant energy transfer occurs due to thermal conduction, or when viscous dissipation is not negligible. Following the analysis of Kadanoff and Martin[6], the thermal conduction of energy may be predicted by the linearized energy density conservation equation which introduces additional property fluctuations. If the fluctuations may be considered to be in local thermodynamic equilibrium, then the interrelationships between the thermodynamic properties may be represented by linearized McLauren expansions of dependent properties in terms of independent properties. The energy balance and thermodynamic interrelationships of the fluctuations complete the description of the fluctuation process and allow derivation of the field equations for a thermo-viscous fluid.

The linearized energy density equation, 2.7c, contains an energy advection term which depends on the divergence of the velocity. The velocity divergence may be replaced with a mass density fluctuation term via the mass density equation, equation 2.7a, resulting in the form of the energy equation

$$\frac{\partial}{\partial t}\left[u(\vec{r},t) - \frac{u+p}{\rho}\,\rho(\vec{r},t)\right] - \kappa\,\nabla^2\,T(\vec{r},t) = 0 \quad . \tag{2.18}$$

Since the term involving the laplacian of the temperature represents a heat flux, the bracketed term should represent a heat energy density and for convenience will be defined as

$$q(\vec{r},t) \equiv u(\vec{r},t) - \frac{u+p}{\rho}\,\rho(\vec{r},t) \quad . \tag{2.19}$$

The heat energy density fluctuation may be related to the entropy fluctuation by the first law of thermodynamics. Since to first order in fluctuations the energy density conservation, equation 2.7c, is independent of both kinetic energy and viscous dissipation, to the same order of approximation these effects may be neglected in the first law of thermodynamics,

$$TdS = d(uV) + pdV \quad . \tag{2.20a}$$

Also for a constant mass the volume and density are related by

$$\frac{dV}{V} = -\frac{d\rho}{\rho} \tag{2.21}$$

which allows the first law to be expressed as

$$\frac{T}{V}\, dS = du - \frac{u+p}{\rho}\, d\rho \quad . \tag{2.20b}$$

The first law may be linearized, as the conservation equations were, by expressing the properties in terms of equilibrium values and fluctuations and then neglecting terms of second and higher order of the fluctuations. Thus the first order approximation of the first law of thermodynamics is

$$\frac{T_o}{V_o}\, dS(\vec{r},t) = du(\vec{r},t) - \frac{u_o+p_o}{\rho_o}\, d\rho(\vec{r},t) \quad . \tag{2.22}$$

A comparison between this linearized form of the first law and the definition of the heat energy density fluctuation, equation 2.15, allows changes in the heat energy density fluctuation to be related to changes in the entropy fluctuation as

$$dq(\vec{r},t) = \frac{T_o}{V_o}\, dS(\vec{r},t) \quad . \tag{2.23}$$

In terms of the heat energy density and temperature fluctuations, the linearized energy density conservation equation is

$$\frac{\partial}{\partial t}\, q(\vec{r},t) - \kappa\, \nabla^2\, T(\vec{r},t) = 0 \quad . \tag{2.24}$$

This energy equation, which represents the effects of thermal conduction, and the longitudinal momentum equation, equation 2.12 which includes viscous effects, comprise two compressible conservation equations in terms of four thermodynamic property fluctuations. To complete the analysis, the interrelations between the thermodynamic properties must be included.

When in local equilibrium, the fluid may be considered to be a simple compressible substance with the thermodynamic state specified by two independent properties. Thus any of the four relevent thermodynamic properties (mass density, temperature, pressure and heat energy density) is a function of any of the remaining three properties. For convenience of notation the thermodynamic properties will be denoted in general by x_ℓ. The subscript may take the values 1,2,3 or 4 to denote the particular property according to the definitions:

$$x_1 \equiv \rho \quad , \quad x_2 \equiv T \quad , \quad x_3 \equiv p \quad , \quad x_4 \equiv q \quad . \tag{2.25}$$

In this notation any fluctuation, $x_\ell(\vec{r},t)$, may be expanded in a McLauren series about equilibrium, in terms of two other fluctuations, $x_m(\vec{r},t)$ and $x_n(\vec{r},t)$. Retaining only the terms which are of the first order in the fluctuations, the linearized expansion becomes

$$x_\ell(\vec{r},t) = \left(\frac{\partial x_\ell}{\partial x_m}\right)_{x_n} x_m(\vec{r},t) + \left(\frac{\partial x_\ell}{\partial x_n}\right)_{x_m} x_n(\vec{r},t) \quad , \quad m \neq \ell \neq n \quad . \tag{2.26}$$

Since the fluctuations were expanded about equilibrium the thermodynamic derivatives involve only the equilibrium values of the properties and thus are not functions of space and time. Also since the differential of heat energy density is related to the entropy for a constant mass, equation 2.23, all thermodynamic derivatives will be taken at constant mole number. For ease of notation the thermodynamic derivatives will be denoted in general by

$$C^\ell_{mn} = \left(\frac{\partial x_\ell}{\partial x_m}\right)_{x_n} \quad n \neq m \quad . \tag{2.27}$$

The thermodynamic derivatives are not independent but are related by the calculus of two variables differentiation rules,

$$C^{\ell}_{mn} = \frac{1}{C^{m}_{\ell n}} \quad , \quad C^{\ell}_{mn}\, C^{n}_{\ell m}\, C^{m}_{n\ell} = -1 \tag{2.28a,b}$$

and by Maxwells relations. In fact, only three of the non-trivial derivatives are independent so any may be expressed in terms of these three. The three that will be chosen as independent are the isentropic speed of sound, the constant pressure specific heat and the constant volume specific heat:

$$C_1 = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_q} \tag{2.29a}$$

$$C_p = \frac{T}{v}\left(\frac{\partial S}{\partial T}\right)_p = \left(\frac{\partial q}{\partial T}\right)_p \tag{2.29b}$$

$$C_v = \frac{T}{v}\left(\frac{\partial S}{\partial T}\right)_v = \left(\frac{\partial q}{\partial T}\right)_v \quad . \tag{2.29c}$$

The coefficient of thermal expansion,

$$\alpha_T = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = \sqrt{\frac{C_p(C_p-C_v)}{\rho T C_v C_1^2}} \quad , \tag{2.29d}$$

will also be used to express the non-trivial thermodynamic derivatives:

$$
\begin{aligned}
C^1_{23} &= \left(\frac{\partial \rho}{\partial T}\right)_p = -\rho\alpha_T & \qquad C^2_{31} &= \left(\frac{\partial T}{\partial p}\right)_\rho = \frac{C_\rho}{\rho C_v C_1^2 \alpha_T} \\
C^1_{24} &= \left(\frac{\partial \rho}{\partial T}\right)_s = \frac{C_p}{T C_1^2 \alpha_T} & C^2_{34} &= \left(\frac{\partial T}{\partial p}\right)_s = \frac{T\alpha_T}{C_p} \\
C^1_{32} &= \left(\frac{\partial \rho}{\partial p}\right)_T = \frac{C_p}{C_v C_1^2} & C^2_{41} &= \frac{v}{T}\left(\frac{\partial T}{\partial s}\right)_\rho = \frac{1}{C_v} \\
C^1_{34} &= \left(\frac{\partial \rho}{\partial p}\right)_s = \frac{1}{C_1^2} & C^2_{43} &= \frac{v}{T}\left(\frac{\partial T}{\partial s}\right)_p = \frac{1}{C_p} \\
C^1_{42} &= \frac{v}{T}\left(\frac{\partial \rho}{\partial s}\right)_T = -\frac{C_p}{T C_v C_1^2 \alpha_T} & C^3_{41} &= \frac{v}{T}\left(\frac{\partial p}{\partial s}\right)_p = \frac{\rho C_1^2 \alpha_T}{C_p} \\
C^1_{43} &= \frac{v}{T}\left(\frac{\partial \rho}{\partial s}\right)_p = \frac{\rho C_p}{C_v C_1^2 \alpha_T} & C^3_{42} &= \frac{v}{T}\left(\frac{\partial p}{\partial s}\right)_T = \frac{-1}{T\alpha_T}
\end{aligned}
\tag{2.30}
$$

The inverse of the above thermodynamic derivatives are defined by the inversion rule, equation 2.28a, and the remaining trivial derivatives that may be encountered are of the forms:

$$C^{\ell}_{n\ell} = 0 \quad , \quad C^{\ell}_{\ell n} = 1 \quad , \quad C^{\ell}_{nn} \equiv 0 \ . \qquad (2.31a,b,c)$$

The last two forms are defined so that the expansion of a fluctuation into itself and any other fluctuation yields itself. An the notation defined above the linearized expansion of any fluctuation into two others is expressed as

$$x_{\ell}(\vec{r},t) = C^{\ell}_{mn}\, x_m(\vec{r},t) + C^{\ell}_{nm}\, x_n(\vec{r},t) \qquad (2.32)$$

The interrelationship between thermodynamic fluctuations, as represented by this linearized expansion, completes the information provided by the longitudinal momentum and energy equations and allows the field equations to be derived for a compressible, thermal-viscous hydrodynamic medium.

Two coupled field equations for any two independent thermodynamic fluctuations may be derived by expanding all fluctuations in the longitudinal momentum equation, equation 2.12, and the energy equation, equation 2.24, in terms of the same two independent fluctuations, $x_m(\vec{r},t)$ and $x_n(\vec{r},t)$. Recalling that the general thermodynamic derivatives in the linearized expansion are independent of space and time, and thus not operated on by spacial and temporal derivatives these coupled field equations may be expressed in general as,

$$[C^1_{mn}(-\frac{\partial^2}{\partial_t^2} + D_L\nabla^2 \frac{\partial}{\partial t}) + C^3_{mn}\nabla^2]x_m(\vec{r},t) + [C^1_{nm}(-\frac{\partial^2}{\partial t^2} + D_L\nabla^2 \frac{\partial}{\partial t}) + C^3_{nm}\nabla^2]x_n(\vec{r},t)=0 \ , \qquad (2.33a)$$

$$[-C^2_{mn}\kappa\nabla^2 + C^4_{mn}\frac{\partial}{\partial t}]x_m(\vec{r},t) + [-C^2_{nm}\kappa\nabla^2 + C^4_{nm}\frac{\partial}{\partial t}]x_n(\vec{r},t) = 0 \ . \qquad (2.33b)$$

The field equations for any simple thermodynamic fluctuation may be derived from the coupled field equations, equations 2.33a and 2.33b, by algebraicly eliminating one of the fluctuations, $x_n(\vec{r},t)$. When this is accomplished the resulting thermodynamic derivative terms occur as Jacobians of various thermodynamic properties which are defined as,

$$J^{ij}_{mn} = C^i_{mn}C^j_{nm} - C^i_{nm}C^j_{mn} = \begin{vmatrix} \left(\frac{\partial x_i}{\partial x_m}\right)_{x_n} & \left(\frac{\partial x_i}{\partial x_n}\right)_{x_m} \\ \left(\frac{\partial x_j}{\partial x_m}\right)_{x_n} & \left(\frac{\partial x_j}{\partial x_n}\right)_{x_m} \end{vmatrix} , \quad (2.34)$$

which allows the field equation for a general fluctuation to be expressed as

$$[-J^{14}_{mn}\frac{\partial^3}{\partial t^3} + (D_L J^{14}_{mn} + \kappa J^{12}_{mn})\nabla^2 \frac{\partial^2}{\partial t^2} + J^{34}_{mn}\nabla^2 \frac{\partial}{\partial t} - D_L \kappa J^{12}_{mn}\nabla^4 \frac{\partial}{\partial t} - \kappa J^{32}_{mn}\nabla^4]x_m(\vec{r},t) = 0 . \quad (2.35)$$

Since $x_n(\vec{r},t)$ has been eliminated the field equation should be independent of n. This may be shown to be so by multiplying the field equation by J^{mn}_{14} and manipulation of the Jacobian relations

$$J^{ij}_{mj} = C^i_{mj} , \quad (2.36a)$$

$$J^{ij}_{mn} = J^{ji}_{nm} = -J^{ij}_{nm} = -J^{ji}_{mn} \quad (2.36b)$$

$$J^{ij}_{mn}J^{kl}_{ij} = J^{kl}_{mn} . \quad (2.36c)$$

The resulting field equation,

$$[-\frac{\partial^3}{\partial t^3} + (D_L + \kappa C^2_{41})\nabla^2 \frac{\partial^2}{\partial t^2} + C^3_{14}\nabla^2 \frac{\partial}{\partial t} - D_L \kappa C^2_{41}\nabla^4 \frac{\partial}{\partial t} - \kappa J^{32}_{14}\nabla^4]x_m(\vec{r},t) = 0 , \quad (2.37)$$

is not only independent of n but is also independent of m. That means that the fluctuations of all the thermodynamic properties (mass density, temperature, pressure and heat energy density) obey the same field equation. As for the acoustic analysis, the velocity divergence and the mass density are related by the mass density equation, equation 2.7a, and therefore the velocity divergence and longitudinal velocity fluctuations also obey the same field equation, equation 2.37. Since there are no free scripts, n or m, in the operator of the field equation the thermodynamic derivatives may be expressed in terms of the adiabatic speed of sound and the constant pressure or volume specific heats. In these terms the field equation governing the fluctuations of the general thermodynamic properties, the velocity divergence and the longitudinal velocity vector may be expressed as

$$\left[-\frac{\partial^3}{\partial t^3} + \left(D_L + \frac{\kappa}{C_V}\right)\frac{\partial^2}{\partial t^2}\nabla^2 + C_1^2\frac{\partial}{\partial t}\nabla^2 - D_L\frac{\kappa}{C_V}\frac{\partial}{\partial t}\nabla^4 - C_1^2\frac{\kappa}{C_p}\nabla^4\right]\begin{Bmatrix} x_m(\vec{r},t) \\ \nabla\cdot\vec{v}^L(\vec{r},t) \\ \vec{v}^L(\vec{r},t)\end{Bmatrix} = 0 \,. \qquad (2.38)$$

The transverse velocity fluctuation obeys the same field equation as for the acoustic case, equation 2.9b, because it is independent of the thermodynamic variables. However, in order to relate the transverse and longitudinal velocity components the field equation governing the total velocity vector fluctuation must be derived. This may be accomplished by eliminating the pressure fluctuation from the momentum density equation, equation 2.7b. This requires a more mathematically involved procedure for this nonisentropic process than was required for the acoustic case because thermodynamic process involves the energy density equation and the thermodynamic property expansion rather than the simpler isentropic pressure and mass density relationship, equation 2.13.

The energy density equation, equation 2.24, may be expressed in terms of the mass density and pressure fluctuations by expanding the temperature and heat energy density fluctuations with the linearized thermodynamic expansion, equation 2.32. Then the time derivative of the energy equation allows its mass density term to be replaced by a velocity term through the mass density equation, equation 2.7a, and results in the pressure and velocity relationship,

$$(C_{31}^4 \frac{\partial}{\partial t} - C_{31}^2 \kappa \nabla^2) \frac{\partial}{\partial t} p(\vec{r},t) = (C_{13}^4 \frac{\partial}{\partial t} - C_{13}^2 \kappa \nabla^2) \rho \nabla \cdot \vec{v}(\vec{r},t) \quad . \tag{2.39}$$

The pressure fluctuation in the momentum density equation, equation 2.7b, may be eliminated by applying the term that operates on the pressure in equation 2.39 to the entire momentum density equation, taking the gradient of equation 2.39 and then elmininating the then identical pressure terms of the two resulting equations. This yields the velocity fluctuation field equation.

$$\left\{ \begin{array}{l} -C_{31}^4 \rho \frac{\partial^3}{\partial t^3} + (C_{31}^4 \eta + C_{31}^2 \rho\kappa) \frac{\partial^2}{\partial t^2} \nabla^2 + C_{31}^4 (\zeta + \frac{1}{3}\eta) \frac{\partial^2}{\partial t^2} \nabla\nabla \cdot - C_{31}^2 \eta\kappa \frac{\partial}{\partial t} \nabla^4 - \\ -C_{31}^2 (\zeta + \frac{1}{3}\eta)\kappa \frac{\partial}{\partial t} \nabla^2 \nabla\nabla \cdot - C_{13}^4 \rho \frac{\partial}{\partial t} \nabla\nabla \cdot + C_{13}^2 \rho\kappa \nabla^2 \nabla\nabla \ . \end{array} \right\} \vec{v}(\vec{r},t) = 0 \tag{2.40}$$

Expressing the thermodynamic derivatives in terms of specific heats and the isentropic speed of sound, and the viscosities in terms of the viscous diffusion coefficients, the field equation for the vectorial velocity fluctuation is

$$\{ -\frac{\partial^3}{\partial t^3} + (D_T + \frac{\kappa}{C_v}) \frac{\partial^2}{\partial t^2} \nabla^2 + (D_L - D_T) \frac{\partial^2}{\partial t^2} \nabla\nabla \cdot - D_T \frac{\kappa}{C_v} \frac{\partial}{\partial t} \nabla^4 - (D_L - D_T) \frac{\kappa}{C_v} \frac{\partial}{\partial t} \nabla^2 \nabla\nabla \cdot +$$

$$C_1^2 \frac{\partial}{\partial t} \nabla\nabla \cdot - C_1^2 \frac{\kappa}{C_p} \nabla^2 \nabla\nabla \cdot \} \vec{v}(\vec{r},t) = 0 \quad . \tag{2.41}$$

As expected, the divergence of this equation results in the previously derived field equations for velocity divergence and longitudinal velocity fluctuations, equation 2.38. However the curl of the velocity field equation,

$$\{ -\frac{\partial^3}{\partial t^3} + (D_T + \frac{\kappa}{C_v}) \frac{\partial^2}{\partial t^2} \nabla^2 - D_T \frac{\kappa}{C_v} \frac{\partial}{\partial t} \nabla^4 \} \nabla \times \vec{v}^T(\vec{r},t) = 0 \quad , \tag{2.42a}$$

does not appear to be equivalent to the field equation which governs the vorticity and transverse velocity fluctuations,

$$\left[\frac{\partial}{\partial t} - D_T\nabla^2\right] \begin{Bmatrix} \nabla \times \vec{v}^T(\vec{r},t) \\ \vec{v}^T(\vec{r},t) \end{Bmatrix} = 0 \quad . \tag{2.43}$$

which was previously derived, equation 2.9.

This discrepancy may be resolved by factoring the operator of equation 2.42 as

$$-\left[\frac{\partial}{\partial t} - D_T\nabla^2\right] \left(\frac{\partial}{\partial t} - \frac{\kappa}{C_v} \nabla^2\right) \frac{\partial}{\partial t} \nabla \times \vec{v}^T(\vec{r},t) = 0 \quad . \tag{2.44}$$

Thus it is apparent that a solution of equation 2.43 is also a solution of equations 2.42 and 2.44 and is the true vorticity fluctuation or transverse velocity fluctuation. The other possible solution of equations 42 and 44, that which satisfies the thermal diffusion operator in parantheses, merely indicates the complexity of the relationship between the transverse and longitudinal velocity components. As for the acoustic case, this relationship will be shown in a following chapter.

Low Temperature Field Equations

The linearized conservation equations for longitudinal momentum density and energy density, equations 2.10 and 2.20, are coupled through the thermodynamic derivaties which linearly relate the thermodynamic properties. At low temperatures some of the thermodynamic derivatives are negligibly small, for this first order in fluctuation analysis, and simplified field equations may be derived.

The Nernst postulate, entropy is equal to zero at zero temperature, implies that at low temperatures the following thermodynamic derivatives are essentially equal to zero[7]

$$\left(\frac{\partial s}{\partial \rho}\right)_T = \frac{-VC_v C_1^2 \alpha_T}{C_p} = 0 \tag{2.45a}$$

$$\left(\frac{\partial s}{\partial p}\right)_T = -V\alpha_T = 0 \tag{2.45b}$$

$$\left(\frac{\partial s}{\partial T}\right)_\rho = \frac{VC_v}{T} = 0 \tag{2.45c}$$

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{VC_p}{T} = 0 \tag{2.45d}$$

and from Maxwell's relations the derivatives;

$$\left(\frac{\partial \rho}{\partial T}\right)_p = -\rho\alpha_T = 0 \tag{2.45e}$$

$$\left(\frac{\partial p}{\partial T}\right)_\rho = \frac{\rho C_v C_1^2 \alpha_T}{C_p} = 0\ ; \tag{2.456f}$$

are also negligible. Only two thermodynamic properties are independent to first order approximation in the fluctuations. The first two vanishing derivatives, equations 2.45a and 2.45b, imply that entropy is a function of temperature. The last two vanishing derivatives imply that mass density and pressure are interrelated and independent of temperature.

At low temperature, the pressure is a function only of mass density thus the fluctuations in pressure and mass density may be related by the low temperature speed of sound, C_o, as

$$p(\vec{r},t) = \frac{dp}{d\rho}\,\rho(\vec{r},t) = C_o^2\rho(\vec{r},t) \quad . \tag{2.46}$$

This pressure-mass density relation is analagous to the isentropic speed of sound relation, equation 2.11, utilized in conventional acoustic analysis and allows an analogous solution of the longitudinal momentum equation, equation 2.10. Thus for low temperatures, as for an isentropic process, the fluctuations of mass density, and therefor those of the longitudinal velocity vector, and the pressure all satisfy a viscously damped sound field equation.

$$[-\frac{\partial^2}{\partial t^2} + D_o\,\nabla^2\,\frac{\partial}{\partial t} + C_o^2\nabla^2]\begin{Bmatrix}\rho(\vec{r},t)\\ p(\vec{r},t)\\ \vec{v}_\ell(\vec{r},t)\end{Bmatrix} \quad , \tag{2.47}$$

where D_o is the sound damping coefficient defined by equation 2.13.

Since the heat energy density is related to the entropy by equation 2.23, the low temperature relation between heat energy density and temperature becomes

$$q(\vec{r},t) = \frac{dq}{dT}\,T(\vec{r},t) = \frac{T_o}{V_o}\frac{dS}{dt}\,T(\vec{r},t). \tag{2.48}$$

It follows that the energy equation, 2.24, reduces to an equation of thermal diffusion

$$[\frac{\partial}{\partial t} - D_{to}\nabla^2]\;T(\vec{r},t) = 0 \tag{2.49}$$

where the low temperature thermal diffusion coefficient is

$$D_{To} = \kappa\,\frac{V_o}{T_o}\,\frac{dT}{dS}\;. \tag{2.50}$$

III Transformed Solutions

Field equations have been developed for the fluctuations in the general thermodynamic properties, x_m, and in the transverse, longitudinal and total components of the vectorial velocity; $\vec{v}^T$, $\vec{v}^L$ and $\vec{v}$. Although these field equations characterize the dynamical behavior of fluctuations in the field medium, specific solutions for the fluctuations require additional information such as the forcing of or the initial conditions of the fluctuations. If the forcing is known the forced response may be evaluated by a convolution of the forcing and the impulse response or impulse Green's function. If the initial conditions are known the free response or relaxation may be evaluated by a convolution of the initial conditions and the initial condition Greens function. Thus the impulse and initial condition Green's functions both characterize the dynamical behavior of the fluctuations and may be considered to be general solutions of the problem.

Derivation of the Green's functions involves transformation of the field equations from the space and time domain to the wavenumber and frequency domain. The impulse Green's function may be derived in wavenumber and frequency domain by applying Fourier transforms in space and time to the field equations which each involve one fluctuation only. These field equations are homogeneous and thus display no forcing term so a forced solution cannot be evaluated. However, an impulsive forcing will be assumed allowing an artificial impulse Green's function, which nevertheless is characteristic of the medium, to be derived.

The initial condition Green's function may be derived in wavenumber and frequency domain by applying a spacial Fourier transform and a complex temporal Laplace transform. The Laplace transform of the time derivatives in the field equations results in the initial conditions which need be specified in order to obtain a specific solution. Thus the homogeneous field equations for one fluctuation and the two coupled field equations, equations 2.33a and 2.33b, result in different specifications of the initial state. The initial condition Green's functions are more physically meaningful than the impulse Green's functions because their excitations are more realistic.

Impulse Green's Functions

Field equations which consist of differential operators in space and time operating on one fluctuating property may be used to derive the impulse Greens function for that fluctuation. For both the acoustic and thermal-viscous approximations such equations have been for the fluctuations of thermodynamic properties, vorticity, velocity divergence and of the transverse, longitudinal and total component of vectorial velocity. Since these equations are homogeneous, artificial forcing terms will be assumed in order to excite the system.

The impulse Greens function for a specific fluctuation will be shown to be related to the operator of the governing field equation. The operators in the field equations derived are either scalars or second order tensors which require the impulse Green's functions to be either scalars or second order tensors respectively. The scalar operators and scalar impulse Green's functions apply for all scalar fluctuations and to the vectorial fluctuations which do not change in direction as the system relaxes. The scalar fluctuations, x, should be forced by scalar forcing terms and thus the artificially forced scalar field equation in space and time domain is of the form,

$$L(\vec{r},t)\ x(\vec{r},t) = f(\vec{r},t) \quad . \tag{3.1a}$$

The vectorial fluctuations whose directions are not effected by the medium will be in the direction that they are forced and therefore their forcing should be vectorial and their vectorially forced field equations should be of the form

$$L(\vec{r},t)\ \vec{v}(\vec{v},t) = \vec{f}(\vec{r},t) \quad . \tag{3.1b}$$

The scalar operator effects only the magnitude of the fluctuation and not the direction. Therefore, the magnitude obeys the scalar equation form, equation 3.1a, and the fluctuation is the same as that of the forcing. The direction of only the total velocity fluctuation is effected by the medium since it is the only vectorial fluctuation with a tensorial operator in its field equation.

This is because the transverse velocity fluctuation and the longitudinal velocity fluctuation, which are in directions that are perpendicular and parallel respectively to the direction of propagation, decay at different rates.

In order that the differential operator may interact with the direction as well as the magnitude of the velocity fluctuation it should be a second order tensor, $L_{ij}(\vec{r},t)$, where the subscript indices refer to the spatial directions and may take on the value 1,2 or 3. Thus the vectorially forced vectorial field equation in space and time domain using index notation with summation convention is of the form,

$$L_{ij}(\vec{r},t)\ v_j(\vec{r},t) = f_i(\vec{r},t) \quad . \tag{3.1c}$$

The total velocity fluctuation will generally be in a changing direction different from the direction that it was forced in.

Scalar impulse Green's function theory

The impulse Green's function in space and time domain, $G(\vec{r},t)$, for the scalar fluctuation, $X(\vec{r},t)$, or the constant direction vectorial fluctuation, $\vec{v}(\vec{r},t)$, is defined as the response to a forcing that is impulsive in space and time. Thus the equation governing the scalar impulse Green's function for the forced fluctuations governed by the equation forms, 1a and 1b, is of the form,

$$L(\vec{r},t)\ G(\vec{r}-\vec{r}',t-t') = \delta(\vec{r}-\vec{r}')\ \delta(t-t') \quad . \tag{3.2}$$

where the scalar operator, $L(\vec{r},t)$, is the same as in equations 1a or 1b. For all field points in space and time, $\vec{r}$ and t, except the excitation point, $\vec{r}'$ and t', the scalar impulse Green's function, $G(\vec{r}-\vec{r}',t-t')$, is a solution to the homogeneous or unforced form of the field equation, equation 3.1a or 3.1b, which represents the free relaxation of the scalar fluctuation, x, or of the magnitude of the constant direction fluctuation, $\vec{v}$.

A transform technique allows the differential equation is space and time domain, equation 3.2, to be transformed into an algebraic equation in wavenumber and frequency domain which may be easily solved for the impulse Green's function in that domain, $G(\vec{\kappa},\omega)$. The impulse Green's function and the delta functions in space and time domain may be replaced with their spatial and temporal Fourier transform representations,

$$G(\vec{r}-\vec{r}',t-t') = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t-t')}\, G(\vec{k},\omega) \ , \tag{3.3a}$$

$$\delta(\vec{r}-\vec{r}') = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')} \underset{(1)}{} \tag{3.3b}$$

$$\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \underset{(1)}{} \quad . \tag{3.3c}$$

The corresponding inverse spatial and temporal Fourier transformation of the impulse Green's function is

$$G(\vec{k},\omega) = \int_{-\infty}^{\infty} d^3\vec{r}\; e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \int_{-\infty}^{\infty} dt\; e^{-i\omega(t-t')}\, G(\vec{r}-\vec{r}',t-t') \quad . \tag{3.4}$$

Then the differential operator, $L(\vec{r},t)$, operates only on the transform kernal, since all other terms are independent of space and time, producing the algebraic equation in wavenumber vector, $\vec{k}$, and frequency, ω, domain;

$$L(\vec{k},\omega)\, G(\vec{k},\omega) = 1 \; . \tag{3.5}$$

Thus the transform technique has effectively transformed the space and time differential operator, $L(\vec{r},t)$, to an algebraic operator which may be defined as

$$L(\vec{k},\omega) = \frac{L(\vec{r},t)\,[e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\; e^{i\omega(t-t')}]}{[e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\; e^{i\omega(t-t')}]} \tag{3.6}$$

Since the transforms of the delta functions are unity, which implies that all wavenumber components and frequencies are uniformly excited, the transformed impulse Green's function is simply the inverse of the operator in the transformed domain,

$$G(\vec{k},\omega) = [L(\vec{k},\omega)]^{-1} \quad . \tag{3.7}$$

Thus the impulse Green's function characterizes the fluctuations behavior in the particular medium as does the operator of the homogeneous field equation.

When specific wavenumbers and/or frequencies are excited by a forcing function, $f(\vec{r},t)$, its transform, $f(\vec{k},\omega)$, will be a function of wavenumber and frequency other than unity. In general the transforms of the field equations with scalar operators, equations 3.1a and 3.1b, will be of the forms

$$L(\vec{k},\omega)\; x(\vec{k},\omega) = f(\vec{k},\omega) \tag{3.8a}$$

or

$$L(\vec{k},\omega)\; \vec{v}(\vec{k},\omega) = \vec{f}(\vec{k},\omega) \quad . \tag{3.8b}$$

An inverse transformation results in the forced fluctuation in space and time domain being equal to the spatial and temporal convolution, defined by an asterisk, of the impulse Green's function and the forcing function,

$$x(\vec{r},t) = G(\vec{r},t)*f(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}',t-t')f(\vec{r}',t') \tag{3.9a}$$

or

$$\vec{v}(\vec{r},t) = G(\vec{r},t)*\vec{f}(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}',t-t')\vec{f}(\vec{r}',t') \tag{3.9b}$$

The effect of the convolution is to multiply on impulse response eminating from some point in space and time by the forcing at that point and then sum, vectorially if necessary, all such responses which eminate from all forced points in space and time.

Tensorial impulse Green's function thoery

In order that the Green's function may interact with the direction as well as the magnitude of the forcing the impulse Green's function must be a second order tensor. Using index notation and the summation convention the tensorial impulse Green's function in space and time domain, $G_{jm}(\vec{r},t)$, for the vectorial fluctuation, $v_j(\vec{r},t)$, is related to the tensorial impulse forcing by

$$L_{ij}(\vec{r},t)\; G_{jm}(\vec{r},t) = \delta_{im}\; \delta(\vec{r}-\vec{r}\,')\; \delta(t-t') \tag{3.10}$$

and thus may be considered as the free response to a tensorial impulsive forcing.

The tensorial equation may be transformed to wavenumber and frequency domain, as was the scalar equation 3.2, by a special and temporal Fourier transform yielding

$$L_{ij}(\vec{k},\omega)\; G_{jm}(\vec{k},\omega) = \delta_{im} \quad . \tag{3.11}$$

The transformed differential operator is an algebraic function of wavenumber components and frequency defined by

$$L_{ij}(\vec{k},\omega) = \frac{L_{ij}(\vec{r},t)\; [e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\; e^{i\omega(t-t')}]}{[e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\; e^{i\omega(t-t')}]} \tag{3.12}$$

Since the transformed operator is algebraic its inverse, $[L_{ni}(\vec{k},\omega)]^{-1}$, satisfies the relation

$$[L_{ni}(\vec{k},\omega)]^{-1}\; L_{ij}(\vec{k},\omega) = \delta_{nj} \quad . \tag{3.13}$$

Thus a multiplication of equation 3.11 by the inverse operator considering the relation of equation 3.13 yields

$$\delta_{nj}\; G_{jm}(\vec{k},\omega) = [L_{ni}(\vec{k},\omega)]^{-1}\; \delta_{im} \quad . \tag{3.14a}$$

Contraction of the indices by the Kronecker deltas results in the transformed tensorial impulse Green's function equal to the inverse of the transformed operator,

$$G_{nm}(\vec{k},\omega) = [L_{nm}(\vec{k},\omega)]^{-1} \quad , \tag{3.14b}$$

as was true of the scalar impulse Green's function. Thus the tensorial impulse Green's function may be considered as a solution to the homogeneous field equation which characterizes the fluctuations when all wavenumbers and frequencies have been uniformly excited.

In general the transform of the vectorial forcing will be a function wavenumber and frequency since specific wavenumbers and/or frequencies will have been excited more than others. Then the transform of the forced field equation, equation 3.16, will be

$$L_{ij}(\vec{k},\omega)\; v_j(\vec{k},\omega) \;= f_i(\vec{k},\omega) \; . \tag{3.15}$$

This equation may be multiplied by $G_{mi}(\vec{k},\omega)$ which, by equation 3.14b, is equal to $[L_{mi}(\vec{k},\omega)]^{-1}$. Using the inverse operator form on the left side of the equation and the impulse Green's function on the right side the transformed forced field equation is

$$[L_{mi}(\vec{k},\omega)]^{-1}\; L_{ij}(\vec{k},\omega)\; v_j(\vec{k},\omega) = G_{mi}(\vec{k},\omega)\; f_i(\vec{k},\omega) \quad . \tag{3.16}$$

The inverse operator, operator contraction results in a Kronecker delta, as in equation 3.13, which when contracted with the velocity results in expression for vectorial responses in wavenumber and frequency domain

$$v_m(\vec{k},\omega) = G_{mi}(\vec{k},\omega)\; f_i(\vec{k},\omega) \quad . \tag{3.17}$$

An inverse transformation to space and time domain results in the convolution formula a tensorial impulse Green's function,

$$v_m(\vec{r},t) = G_{mi}(\vec{r},t) * f_i(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}\,' \int_{-\infty}^{\infty} dt'\; G_{mi}(\vec{r}-\vec{r}\,',t-t')\; f_i(\vec{r}\,',t') \; . \tag{3.18}$$

This convolution vectorially sums the forced responses eminating from each forced point in space and time while properly accounting for the changing directions of the forced responses.

Acoustic impulse Green's functions

The theory necessary for the evaluation of the scalar and tensorial impulse Green's functions in wavenumber and frequency domain from the field equations in space and time domain has been presented and shows that a simple procedure may be followed to evaluate the impulse Green's function for each of the fluctuations. The procedure for evaluating the impulse Green's function for a specific fluctuation is to first identify the differential operator in space and time domain from the appropriate field equation which was developed in a prevous chapter. A scalar operator implies that the impulse Green's function will be a scalar and a tensorial operator implies that a tensorial impulse Green's function will be appropriate. The scalar or tensorial operator may be effectively transformed to wavenumber and frequency domain by performing the operation indicated by equation 3.6 or 3.12 respectively. The impulse Green's function in wavenumber and frequency domain for the specific fluctuation is then evaluated as the inverse of the operator, as indicated by equation 3.7 for a scalar impulsive Green's function or by equation 3.14b for a tensorial impulsive Green's function.

For the acoustic, or isentropic, fluctuation approximation the field equations have been developed for scalar fluctuations of mass density, pressure and velocity divergence and for vertical fluctuations of vorticity and the transverse, longitudinal and total components of velocity. The operators in all field equations are scalars except for the second order tensor operator in the field equation governing the vectorial velocity fluctuation. As a result the impulse Green's functions for all fluctuations are scalars except the impulse Green's function for the vectorial velocity fluctuation which is a second order tensor.

The vectorial fluctuations of vorticity and transverse velocity obey the same field equation, equations 2.9a and 2.9b, which represents a transverse momentum diffusion independent of the fluctuation process. These field equations, identify the scalar operator in space and time domain,

$$L^{\vec{v}T}(\vec{r},t) = [\frac{\partial}{\partial t} - D_T \nabla^2] \quad , \tag{3.19a}$$

which applies to the vorticity and transverse velocity fluctuations. This differential operator may be effectively transformed, as shown by equation 3.6, to the algebraic operator in wavenumber and frequency domain,

$$L^{\vec{v}T}(\vec{k},\omega) = [i\omega + D_T k^2] \quad . \tag{3.19b}$$

The scalar impulse Green's function in wavenumber and frequency domain is, by equation 3.7, equal to the inverse of the operator in the same domain. Thus the scalar impulse Green's function for the vectorial fluctuations of vorticity and transverse velocity is

$$G^{\nabla x\vec{v}}(\vec{k},\omega) = G^{\vec{v}T}(\vec{k},\omega) = \frac{1}{i\omega + D_T k^2} \quad , \tag{3.20}$$

which represents the diffusive relaxation of transverse momentum. Since the impulse Green's function is a scalar, as is the operator, the direction of the fluctuations will be the same as the direction of their forcing, from equation 3.8b, and will not change as the fluctuations relax. The transverse velocity fluctuation is always perpendicular to the direction of propagation, the propagation direction is given by the direction of the wavenumber vector $\vec{k}$. The impulse Green's function, which results from forcing at a point in space, is spherically symmetric since it is a function of wavenumber squared. Thus any variation in the transverse velocity occurs in direction of propagation and the vorticity fluctuation vector perpendicular to both the transverse velocity fluctuation vector and the propagation direction.

The scalar fluctuations of mass density, pressure and velocity divergence and the vectorial fluctuation of longitudinal velocity all obey the same field equation, equations 2.14, 2.16a and 2.16b, which represents a damped wave prop gation. The scalar differential operator appropriate for these fluctuations,

$$L^{\rho}(\vec{r},t) = [-\frac{\partial^2}{\partial t^2} + D_L \frac{\partial}{\partial t} \nabla^2 + C_1^2 \nabla^2] \quad , \tag{3.21a}$$

may be transformed to the algebraic operator in wavenumber and frequency domain,

$$L^{\rho}(\vec{k},\omega) = [\omega^2 - i\omega D_L k^2 - C_1^2 k^2] \quad . \tag{3.21b}$$

The inverse of this operator equals the scalar impulse Green's function for the fluctuations of mass density, pressure, velocity divergence and longitudinal velocity vector in wavenumber and frequency domain,

$$G^{\rho}(\vec{k},\omega) = G^{P}(\vec{k},\omega) = G^{\nabla\cdot\vec{v}}(\vec{k},\omega) = G^{\vec{v}L}(\vec{k},\omega) = \frac{1}{\omega^2 - i\omega D_L k^2 - C_1^2 k^2} \quad . \tag{3.22}$$

This impulse Green's function represents a spherically symmetric relaxation of the fluctuations in the form of a damped sound wave propagating radially outward from the spacial point of excitation. The longitudinal velocity vector fluctuation is in the direction of propagation.

The vectorial velocity fluctuation obeys the field equation, equation 2.17, which simultaneously relates the transverse and longitudinal velocity components. The differential operator of this field equation has terms which include the gradient of the divergence, and are naturally second order tensors, as well as scalar terms. To be consistant the scalar terms should be multiplied by a Kronecker delta of the proper indices resulting in the tensorial differential operator in space and time domain,

$$L_{jm}^{\vec{v}}(\vec{r},t) = [-\frac{\partial^2}{\partial t^2} + D_T \frac{\partial}{\partial t} \nabla^2]\delta_{jm} + [(D_L - D_T) \frac{\partial}{\partial t} + C_1^2] \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} \quad . \tag{3.23a}$$

This tensorial operator may be transformed, as shown by equation 3.12, to the algebraic operator in wavenumber and frequency domain,

$$L^{\vec{v}}_{jm}(\vec{k},\omega) = [\omega^2 - i\omega D_T k^2]\delta_{jm} - [i\omega(D_L - D_T) + C_1^2]k_j k_m \quad , \tag{3.23b}$$

which is a second order tensor of the form

$$A\delta_{jm} - Bk_j k_m \quad . \tag{3.24a}$$

It will be convenient to express the inverse of this tensor form, equation 3.24a, in the form

$$\frac{1}{A}\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) + \frac{1}{A-Bk^2}\,\frac{k_j k_m}{k^2} \, . \tag{3.24b}$$

Applying this inversion to the operator, equation 3.23b, results in the tensorial impulse Green's function for the vectorial velocity in the form

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = \frac{1}{-i\omega[i\omega + D_T k^2]}\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) + \frac{1}{[\omega^2 - i\omega D_L k^2 - C_1^2 k^2]}\left(\frac{k_j k_m}{k^2}\right) \quad , \tag{3.25}$$

which is separated into transverse and longitudinal parts. This form of tensorial Green's function has been presented by A. Yildiz[8] for an elastic medium and by M. Yildiz[9] for a thermoviscoelastic medium.

The tensorial terms,

$$P^T_{jm}(\vec{k}) = \left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) \tag{3.26a}$$

and

$$P^L_{jm}(\vec{k}) = \left(\frac{k_j k_m}{k^2}\right) \quad , \tag{3.26b}$$

are defined respectively as the longitudinal and transverse polarization tensors in wavenumber domain. The divergence in wavenumber domain of the transverse polarization tensor equals zero,

$$k_j\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) = 0 \quad , \tag{3.28a}$$

as does the curl of the longitudinal polarization tensor,

$$\epsilon_{inj}\, k_n \left(\frac{k_j k_m}{k^2}\right) = 0 \; , \tag{3.27b}$$

where ϵ_{inj} is the permutation tensor. The scalar terms multiplying the transverse and longitudinal polarization tensors are related to the scalar impulse Green's functions for transverse and longitudinal velocity fluctuations as shown in equations 3.20 and 3.22. Thus the tensorial impulse Green's function for velocity may be expressed as

$$G_{jm}^{\vec{v}T}(k,\omega) = \frac{-1}{i\omega}\, G^{\vec{v}T}(k,\omega)\; P_{jm}^{T}(\vec{k}) - G^{\vec{v}L}(k,\omega)\; P_{jm}^{L}(\vec{k}), \tag{3.28}$$

which obviously relates the transverse and longitudinal velocity fluctuations in magnitude, by teh scalar Green's functions, and direction, by the polarization tensors.

Thermal-viscous impulse Green's functions

For the thermal-viscous fluctuation approximation the field equations have been developed for scalar fluctuations of the general thermodynamic properties and the velocity divergence for vectorial fluctuations of the vorticity and the transverse, longitudinal and total components of the velocity. As for the acoustic approximation the field equations for all fluctuations exhibit scalar operators except the equation governing the vectorial velocity fluctuation which displays a second order tensor operator. The procedure that will be used to evaluate the impulse Green's functions for the fluctuations from their field equations is the same procedure as outlined and used in the previous section for the acoustic fluctuations.

The vectorial fluctuations of vorticity and transverse velocity are the results of transverse momentum diffusion which is independent of the fluctuation process. Therefore the scalar impulse Green's function for the fluctuations of vorticity and transverse velocity,

$$G^{\nabla x\vec{v}}(\vec{k},\omega) = G^{\vec{v}T}(\vec{k},\omega) = \frac{1}{i\omega+D_T k^2}\ , \qquad (3.29)$$

is exactly the same as derived in the acoustic section. The magnitude of the fluctuations diffuses outward in a spherically symmetric pattern about the point of excitation. The vorticity and transverse velocity fluctuation vectors are in mutually perpendicular directions which are also perpendicular to the direction of diffusion, $\vec{k}$.

The scalar fluctuations of thermodynamic properties and velocity divergence and the vectorial fluctuation of longitudinal velocity all satisfy the same field equation, equation 2.38, which represents a more complicated process than damped wave propagation or thermal diffusion alone. The scalar differential operator in space and time domain displayed by the field equation is

$$L^{x_m}(\vec{r},t) = [-\frac{\partial^3}{\partial t^3} + (D_L + \frac{\kappa}{C_v})\frac{\partial^2}{\partial t^2}\nabla^2 + C_1^2\frac{\partial}{\partial t}\nabla^2 - D_L\frac{\kappa}{C_v}\frac{\partial}{\partial t}\nabla^4 - C_1^2\frac{\kappa}{C_p}\nabla^4], \qquad (3.30a)$$

and its transformation in wavenumber and frequency domain is

$$L^{x_m}(\vec{k},\omega) = [i\omega^3+\omega^2(D_L + \frac{\kappa}{C_v})k^2 - i\omega C_1^2 k^2 - i\omega D_L \frac{\kappa}{C_v} k^4 - C_1^2 \frac{\kappa}{C_p} k^2] . \quad (3.30b)$$

The inverse of the transformed operator equals the scalar impulse Green's function in wavenumber and frequency domain for the fluctuations of thermodynamic properties, velocity divergence and longitudinal velocity vector,

$$G^{x_m}(\vec{k},\omega) = G^{\nabla\cdot\vec{v}}(\vec{k},\omega) = G^{\vec{v}L}(\vec{k},\omega) = \frac{1}{i\omega^3+\omega^2(D_L + \frac{\kappa}{C_v})k^2 - i\omega C_1^2 k^2 - i\ D_L \frac{\kappa}{C_v} k^4 - C_1^2 \frac{\kappa}{C_p} k^4} . \quad (3.31)$$

The impulse Green's function also represents a spherically symmetric relaxation of the fluctuations which propagates radially outward, which is also the direction ofthe longitudinal velocity vector, from the spatial point of excitation. The form of this relaxation will be further investigated in a later chapter.

The vectorial velocity fluctuation obeys the field equation, equation 2.41, which relates the transverse and longitudinal velocity components through the tensorial differential operator in space and time domain,

$$L^{\vec{v}}_{jm}(\vec{r},t) = [- \frac{\partial^3}{\partial t^3} + (D_T + \frac{\kappa}{C_v}) \frac{\partial^2}{\partial t^2} \nabla^2 - D_T \frac{\kappa}{C_v} \frac{\partial}{\partial t} \nabla^4]\delta_{jm} + [(D_L-D_T) \frac{\partial^2}{\partial t^2} - (D_L-D_T) \frac{\kappa}{C_v} \frac{\partial}{\partial t} \nabla^2 + C_1^2 \frac{\partial}{\partial t} - C_1^2 \frac{\kappa}{C_p} \nabla^2] \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} . \quad (3.32a)$$

The transform in wavenumber and frequency domain of this operator,

$$L^{\vec{v}}_{jm}(\vec{k},\omega) = [i\omega^3+\omega^2(D_T + \frac{\kappa}{C_v})k^2 - i\omega D_T \frac{\kappa}{C_v} k^4]\delta_{jm} - [-\omega^2(D_L-D_T)+i\omega(D_L-D_T)k^2+i\omega(D_L-D_T)k^2+ +i\omega C_1^2 + C_1^2 \frac{\kappa}{C_p} k^2]k_j k_m , \quad (3.32b)$$

may be inverted in the form shown by equations 3.24 resulting in the form of the tensorial impulse Green's function for the vectorial velocity fluctuation,

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = \frac{1}{+i\omega(i\omega + \frac{\kappa}{C_v} k^2)[i\omega + D_T k^2]} (\delta_{jm} - \frac{k_j k_m}{k^2}) + + \frac{-1}{[i\omega^3+\omega^3(D_L + \frac{\kappa}{C_v})k^2 - i\omega D_L \frac{\kappa}{C_v} k^4 - i\omega C_1^2 k^2 + C_1^2 \frac{\kappa}{C_p} k^4]} (\frac{k_j k_m}{k^2}) . \quad (3.33)$$

In terms of the polarization tnesors, equations 3.26, and the scalar impulse Green's functions for the transverse and longitudinal velocity fluctuations, equations 3.20 and 3.31, the tensorial impulse Green's function may be expressed as

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = \frac{-1}{i\omega(i\omega + \frac{\kappa}{C_v} k^2)} G^{\vec{v}T}(\vec{k},\omega)\ P^T_{jm}(\vec{k}) - G^{\vec{v}L}(k,\omega)\ P^L_{jm}(\vec{k}) \ . \quad (3.34)$$

This tensorial impulse Green's function for the thermal-viscous velocity fluctuation is similar to that derived for the acoustic velocity fluctuation, equations 3.25 and 3.28. The noteable difference is that thermal conductivity has complicated the scalar impulse Green's function for the longitudinal velocity fluctuation, equation 3.32, and has complicated the transverse velocity component of the tensorial Green's function by the additional thermal diffusion pole, which is in parenthesis in equations 3.33 and 3.34.

Although the transverse velocity fluctuation once excited, diffuses viscously as a transverse momentum relaxation independent of fluctuation process, the thermal diffusion which occurs during thermal-viscous fluctuations also excites the transverse velocity fluctuation. This explains the discrepancy which was discovered in the derivation of the field equations for the transverse velocity fluctuations, equations 2.42, 2.43 and 2.44.

Initial Condition Green's Functions

Homogeneous differential field equations that govern the property fluctuations were derived in the first chapter. In the previous section artificial forcing terms were assumed in order to mathematically excite the system and permit derivation of the impulse Green's functions. The impulse Green's function is the relaxation of a fluctuation from the specific excited state resulting from a forcing that is impulsive in space and time, equation 3.2, and correlates the response of the fluctuation with the arbitrary forcing, equations 3.8 and 3.9. In this section the excited state of the system will be specified by a set of initial condition terms, each of which may be considered as a forcing. The fluctuations relaxation, $x(\vec{k},\omega)$, from an initial state of one non-zero initial condition term $Y(\vec{k},t{=}0)$ is correlated to that initial condition term by the corresponding initial condition Green's function,

$$xy(\vec{k},\omega) = G_{xy}(\vec{k},\omega)\ Y(\vec{k},t{=}0) \quad . \tag{3.35}$$

Since this analysis is linear the fluctuations response to an arbitrary initial state is simply the sum of the responses due to each initial condition term. The initial condition terms which are appropriate for each field equation are displayed when a complex Laplace transformation of time is applied to them rather than a temporal Fourier transformation. Applying a spatial Fourier transformation completes the Hilbert transform and yields field equations supplying the information necessary for the derivation of the initial condition Greens functions by Kubo's formula.

Kubo's formula[10,11] was derived on the basis of instantaneously exciting a system from rest to an initial condition and relates the ratio of response over initial condition, which may be determined from the Hilbert transformed field equations, to the initial condition Green's function,

$$\lim_{z\to\omega} \frac{x(\vec{k},z)}{Y(\vec{k},t=0)} = \frac{1}{iz}\,[G_{xy}(\vec{k},z) - G_{xy}(\vec{k},z=0)] \quad . \tag{3.36}$$

The complex frequency,

$$z = \omega - i\varepsilon \quad , \tag{3.37}$$

is the complex Laplace transform variable and for causal processes ω is the temporal Fourier transform variable. Consider the common Laplace transform,

$$L\{f(t)\} = F(s) = \int_0^\infty dt\; e^{-st}\, f(t) \quad . \tag{3.38a}$$

The inverse transform may be represented by the Mellin inversion integral,

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds\; e^{st}\, F(s) \quad , \tag{3.38b}$$

where γ must be chosen to be larger than the real parts of the s poles of F(s). The complex Laplace transform and inversion may be obtained by substituting iz for s with ε assuring integral convergence,

$$L_c\{f(t)\} = F(z) = \int_0^\infty dt\; e^{-izt}\, f(t) \quad , \tag{3.39a}$$

and

$$L_c^{-1}\{f(z)\} = f(t) = \frac{1}{2\pi i} \int_{-i\gamma-\infty}^{-i\gamma+\infty} idz\; e^{izt}\, F(z) \quad , \tag{3.39b}$$

where the imaginary parts of the complex frequency, z, poles of F(z) must be larger than $-\gamma$.

The complex Laplace transforms of the time derivatives supply the initial condition terms through the relations,

$$L_c\{\frac{\partial}{\partial t} f(t)\} = iz\, f(z) - f(t=0) \quad , \tag{3.40a}$$

$$L_c\{\frac{\partial^2}{\partial t^2} f(t)\} = -z^2 f(z) - iz\, f(t=0) - \frac{\partial}{\partial t} f(t=0) \quad , \tag{3.40b}$$

$$L_c\{\frac{\partial^3}{\partial t^3} f(t)\} = -iz^3 f(z) + z^2 f(t=0) - iz\frac{\partial}{\partial t} f(t=0) - \frac{\partial^2}{\partial t^2} f(t=0) \quad . \tag{3.40c}$$

The Hilbert transform of the field equations also involves a Fourier spatial transform,

$$f(\vec{k}) = \int_{-\infty}^{\infty} dr\ e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\ f(\vec{r}-\vec{r}') \quad , \tag{3.41a}$$

and inverse transform,

$$f(\vec{r}-\vec{r}') = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\ e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\ f(\vec{k}) \quad . \tag{3.41.b}$$

The ratio of response over initial condition in Hilbert space may be determined from the Hilbert transformed field equations and is required to derive the initial condition Green's function by Kubo's formula. However, Kubo's formula applies in the limit as z approaches ω, or as ε approaches zero. In this limit the complex Laplace transform and inversion become

$$\lim_{z\to\omega} \quad L_c\{f(t)\} = f(\omega) = \int_0^{\infty} dt\ e^{-i\omega t}\ f(t) \quad , \tag{3.42a}$$

and

$$\lim_{z\to\omega} \quad L_c^{-1}\{f(z)\} = L_c^{-1}\{f(\omega)\} = f(t) = \int_{-i\gamma-\infty}^{-i\gamma+\infty} \frac{d\omega}{2\pi}\ e^{i\omega t}\ f(\omega) \quad , \tag{3.42b}$$

which are similar to the temporal Fourier transform and inversion. In fact if f(t) is a causal function of time, it only exists for positive time and may be considered to be equal to zero for negative time which allows the transform, equation 3.42a, to be expressed as

$$f(\omega) = \int_{-\infty}^{\infty} dt\ e^{-i\omega t}\ f(t) \quad . \tag{3.43a}$$

This is identically the temporal Fourier transform. The frequency poles of a temporal Fourier transform, $f(\omega)$, of a causal function of time have positive imaginary parts, therefore it is sufficient to choose γ equal to zero in the inverse transform, equation 3.42b. Thus the inversion becomes exactly the inverse temporal Fourier transform,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \; e^{i\omega t} \, f(\omega) \quad . \tag{3.43b}$$

Since the response to an initial condition is causal, the initial condition Green's function derived by Kubo's formula, equation 3.36, and the fluctuation response to an initial condition, equation 3.35, are effectively in Fourier wavenumber and frequency, $\vec{k}$ and ω, domain.

An inverse spatial and temporal Fourier transform of the fluctuation response, equation 3.35, results in a space and time relation of the response equal to a spatial convolution, defined by *r, of the initial condition Green's function and the forcing initial condition,

$$x(\vec{r},t) = G_{xy}(\vec{r},t) *r \; Y(\vec{r},t{=}0) = \int_{-\infty}^{\infty} d^3\vec{r}' \; G_{xy}(\vec{r}-\vec{r}',t) \; Y(\vec{r}',t{=}0) \quad . \tag{3.44}$$

Since the transformed initial condition forcing term, equation 3.35, is independent of frequency, excites all frequencies equally, it may be considered to be composed of a delta function in time domain. Thus the temporal convolution is unnecessary as excitation occurs at only one point in time. The spatial convolution sums the responses from each spatial point which was initially excited by the initial condition term under consideration. If there is more than one initial condition term, the response to the initial state is the sum of the responses from the initial condition terms.

Homogeneous equations in one fluctuation

Homogeneous differential field equations which govern a single fluctuation each may be used to develop the initial value Green's functions when the initial state is defined by the initial values of the fluctuation and its time derivatives.

The field equations for a single general fluctuation, x, in space and time domain are of the homogeneous form,

$$L(\vec{r},t)\ x(\vec{r},t) = 0 \quad . \tag{3.45}$$

In order to obtain an algebraic equation with initial condition forcing terms the field equation may be Hilbert transformed to wavenumber and complex frequency domain resulting in

$$L(\vec{k},z)\ x(\vec{k},z) = \overline{L}(\vec{k},z,t{=}0)\ x(\vec{k},t{=}0) \quad . \tag{3.46a}$$

The initial condition forcing is represented by an operator, which is algebraic in wavenumber and complex frequency but also includes time derivatives evaluated at the initial time, operating on the initial wavenumber distribution of the fluctuation. In general the initial condition forcing may be expanded in terms of initial time derivatives of the fluctuation resulting in the forced field equation form,

$$L(\vec{k},z)\ x(\vec{k},z) = \overline{L}_0(\vec{k},z)\ x(\vec{k},t{=}0) + \overline{L}_1(\vec{k},z)\ \frac{\partial}{\partial t}\ x(\vec{k},t{=}0) + \overline{L}_2(\vec{k},z)\ \frac{\partial^2}{\partial t^2}\ x(\vec{k},t{=}0) + \ldots \ , \tag{3.46b}$$

where the highest order of time derivative in the initial condition is one less than the highest order of time derivative in the differential operator of the homogeneous equation, equation 3.45.

Since all initial time derivatives of the fluctuation that are exhibited in equation 3.46b are required to specify the initial state of the system, each may be considered as an independent forcing. The response to a particular initial forcing term with the initial state specified in this manner is the initial condition Green's function which may be evaluated by applying Kubo's formula as

$$G(\vec{k},\omega) = G(\vec{k},\omega{=}0) - \lim_{z\to\omega} iz \frac{\text{Response } (\vec{k},z)}{\text{Forcing } (\vec{k},t{=}0)} \quad . \tag{3.47}$$

Thus the initial condition Greens function depends on the specific response, initial forcing term, and the manner that the initial state was specified. Specific initial condition Green's functions will be denoted by subscripts as

$$G_{\text{response, forcing, other initial conditions}}(\vec{k},\omega) \ . \tag{3.48}$$

For example the response of x to an initial condition of x when all initial time derivatives are zero is related to the initial condition Green's function,

$$G_{xx\dot{x}\ddot{x}\ldots}(\vec{k},\omega) = G_{xx\dot{x}\ddot{x}\ldots}(\vec{k},\omega{=}0) - i\omega \frac{x(\vec{k},\omega)}{x(\vec{k},t{=}0)} = 1 - i\omega \frac{\overline{L}_0(\vec{k},\omega)}{L(\vec{k},\omega)} \quad , \tag{3.49}$$

where the Green's function at zero frequency is simply the ratio of the static response to the initial forcing condition and the ratio of response to initial forcing is evaluated from equation 3.46b when all other initial forcing terms are zero. Similarly the response of x to a first time derivative initial forcing term, $\dot{x}$, is related to the initial condition Green's function,

$$G_{x\dot{x}x\ddot{x}\ldots}(\vec{k},\omega) = G_{x\dot{x}x\ddot{x}\ldots}(\vec{k},\omega{=}0) - i\omega \frac{x(\vec{k},\omega)}{\frac{\partial}{\partial t} x(\vec{k},t{=}0)} = 0 - i\omega \frac{\overline{L}_1(\vec{k},\omega)}{L(\vec{k},\omega)} \quad . \tag{3.50}$$

The static, zero frequency, Greens function for the response of a fluctuation to any time derivative of that fluctuation must be equal to zero since consideration of one time derivative forcing term requires that all other initial conditions, including the fluctuation itself, be equal to zero.

The total free response of the fluctuation, x, is the sum of the responses from each initial condition term in equation 3.46b which would be in wavenumber and frequency domain,

$$x(\vec{k},\omega) = G_{xx\dot{x}\ddot{x}\ldots}(\vec{k},\omega)\; x(\vec{k},t=0) + G_{\dot{x}xx\ddot{x}\ldots}(\vec{k},\omega)\; \frac{\partial}{\partial t}\, x(\vec{k},t=0)\; +$$

$$+\; G_{\ddot{x}xx\dot{x}\ldots}(\vec{k},\omega)\; \frac{\partial^2}{\partial t^2}\, x(\vec{k},t=0) + \ldots \quad , \tag{3.51a}$$

and would be, by the spatial convolution of equation 3.44, in space and time domain

$$x(\vec{k},\omega) = G_{xx\dot{x}\ddot{x}\ldots}(\vec{r},t) *r\; x(\vec{r},t=0) + G_{\dot{x}xx\ddot{x}\ldots}(\vec{r},t) *r\; \frac{\partial}{\partial t}\, x(\vec{r},t=0)\; +$$

$$+\; G_{\ddot{x}xx\dot{x}\ldots}(\vec{r},t) *r\; \frac{\partial^2}{\partial t^2}\, x(\vec{r},t=0) + \ldots \quad . \tag{3.51b}$$

Specific initial condition Green's functions may be evaluated by algebraic manipulation of the operators defined by the Hilbert transforms of the particular field equations developed in chapter 1.

The field equation for the transverse velocity and vorticity fluctuations, equation 2.9, is a diffusion equation which is independent of the fluctuation process. The Hilbert transform of this field equation, specifically for the magnitude of the transverse velocity fluctuation

$$[iz + D_T k^2] v^T(\vec{k},z) = v^T(\vec{k},t=0) \quad , \tag{3.52}$$

is the form of equation 3.46b, displaying operators and one initial condition term. The one initial condition Green's function for both the vorticity and the transverse velocity may be evaluated, via equation 3.49, as

$$G_{v^T v^T}(\vec{k},\omega) = 1 - i\omega\, \frac{(1)}{i\omega + D_T k^2} = \frac{D_T k^2}{i\omega + D_T k^2} \quad . \tag{3.53}$$

The initial condition Green's function for these fluctuations differs only by the numerator from the corresponding impulse Green's function, equation 3.20, which reflects the fact that the forcing term is the fluctuation's initial condition rather than an artifically applied impulse in space and time. The initial condition Green's function also represents a spherically symmetric relaxation mode of the fluctuations. Since the initial state is specified by one initial condition term the total response in wavenumber and frequency domain of the transverse velocity is simply

$$v^T(\vec{k},\omega) = G_{v^T v^T}(\vec{k},\omega)\; v^T(\vec{k},t=0) \tag{3.54}$$

which is analogous to the total vorticity response relation.

In the acoustic approximation the fluctuations of mass density, pressure, velocity divergence and longitudinal velocity obey the same damped wave field equation, equations 3.14 and 3.16. Since the operator includes second order time derivatives the Hilbert transform of the field equation,

$$[z^2 - izD_L k^2 - C_T^2 k^2]\rho(\vec{k},z) = [(-iz - D_L k^2) - \tfrac{\partial}{\partial t}]\rho(\vec{k},t=0) \quad , \tag{3.55}$$

displays initial condition terms of zeroth and first time derivatives of the fluctuation which necessitates two initial condition Green's functions. The Green's function which correlates the fluctuation to its initial wavenumber distribution is of the form shown by equation 3.49,

$$G_{\rho\dot{\rho}}(\vec{k},\omega) = 1 - i\omega \frac{(-i\omega - D_L k^2)}{\omega^2 - i\omega D_L k^2 - C_T^2 k^2} = \frac{-C_T^2 k^2}{\omega^2 - i\omega D_L k^2 - C_T^2 k^2} \quad , \tag{3.56}$$

and the Green's function which correlates the fluctuation with its initial time derivative, as related by equation 3.50, is

$$G_{\dot{\rho}\rho}(\vec{k},\omega) = 0 - i\omega \frac{(-1)}{\omega^2 - i\omega D_L k^2 - C_T^2 k^2} = \frac{i\omega}{\omega^2 - i\omega D_L k^2 - C_T^2 k^2} \quad . \tag{3.57}$$

The denominators of both initial condition Green's functions are identical to that of the corresponding impulse Green's function, equation 3.22, although the numerators differ.

The total response of the fluctuation in wavenumber and frequency domain is the sum of the responses to each initial condition, equation 3.51a; specifically the total mass density fluctuation is

$$\rho(\vec{k},\omega) = G_{\rho\rho\dot{\rho}}(\vec{k},\omega)\ \rho(\vec{k},t=0) + G_{\rho\dot{\rho}\rho}(\vec{k},\omega)\ \frac{\partial}{\partial t}\rho(\vec{k},t=0) \quad . \tag{3.58}$$

Analogous relations are valid for the acoustic fluctuations of pressure, velocity divergence and longitudinal velocity which all are correlated to their initial conditions by identical corresponding initial condition Green's functions.

In the thermal-viscous fluctuation approximation the fluctuations of the general thermodynamic properties, the velocity divergence and the longitudinal velocity satisfy the same field equation, equation 3.38. The Hilbert transform of this field equation for any of the above fluctuations, x,

$$[iz^3+z^2(D_L+\frac{\kappa}{C_v})k^2-izC_1^2k^2-izD_Lk^4-C_1^2\frac{\kappa}{C_p}k^4]x(\vec{k},z) =$$

$$= [-\frac{\partial^2}{\partial t^2}+\{-iz-(D_L+\frac{\kappa}{C_v})k^2\}\frac{\partial}{\partial t}+\{z^2-iz(D_L+\frac{\kappa}{C_v})k^2-C_1^2k^2-D_Lk^4\}]x(\vec{k},t=0) \tag{3.59}$$

exhibits zeroth, first and second order initial time derivatives of the fluctuation as initial conditions and therefore requires three initial condition Greens functions. The correlation between the fluctuation and its initial condition terms is related to the Hilbert transformed field equations operators as shown by equation 3.49 and may be evaluated specifically as,

$$G_{xx\dot{x}\ddot{x}}(\vec{k},\omega) = 1 - i\omega\frac{\{\omega^2-i\omega(D_L+\frac{\kappa}{C_v})k^2-C_1^2k^2-D_Lk^4\}}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_Lk^4-C_1^2\frac{\kappa}{C_p}k^4} =$$

$$= \frac{-C_1^2\frac{\kappa}{C_p}k^4}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_Lk^4-C_1^2\frac{\kappa}{C_p}k^4} \tag{3.60}$$

The Green's functions relating the fluctuation to its first and second initial time derivatives are, in terms of the appropriate operators, of the form of equation 3.50 and may be evaluated respectively as

$$G_{x\dot{x}x\ddot{x}}(\vec{k},\omega) = 0 - i\omega \frac{\{-i\omega-(D_L+\frac{\kappa}{C_v})k^2\}}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L k^4-C_1^2\frac{\kappa}{C_p}k^4} =$$

$$= \frac{-\omega^2+i\omega(D_L+\frac{\kappa}{C_v})k^2}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L k^4-C_1^2\frac{\kappa}{C_p}k^4} \tag{3.61}$$

and

$$G_{x\ddot{x}x\dot{x}}(\vec{k},\omega) = 0 - i\omega \frac{\{-1\}}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L k^4-C_1^2\frac{\kappa}{C_p}k^4} =$$

$$= \frac{i\omega}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L k^4-C_1^2\frac{\kappa}{C_p}k^4} \quad . \tag{3.62}$$

Again the denominators of all initial condition Green's functions and of the corresponding impulse Green's function, equation 3.31, are identical and the numerators differ because the numerators of the Green's functions merely reflect the excitation of the system whereas the denominator characterizes the system.

The total response in wavenumber and frequency domain is correlated to the initial state as by equation 3.51a,

$$x(\vec{k},\omega) = G_{xx\dot{x}\ddot{x}}(\vec{k},\omega)\; x(\vec{k},t=0)+G_{x\dot{x}x\ddot{x}}(\vec{k},\omega)\frac{\partial}{\partial t}\, x(\vec{k},t=0)+G_{x\ddot{x}x\dot{x}}(\vec{k},\omega)\frac{\partial^2}{\partial t^2}\, x(\vec{k},t=0) \quad . \tag{3.63}$$

Coupled thermal-viscous field equations

The longitudinal momentum density and the energy density equations are coupled through thermodynamic relationships in the thermal-viscous fluctuation approximation, as shown by equations 2.33a and 2.33b. Such equations provide an alternate specification of the initial state of the system and thus allow derivation of additional initial condition Green's functions.

The longitudinal momentum equation in terms of mass density and pressure fluctuations, equation 2.12, and the energy equation in terms of heat energy density and temperature fluctuations, equation 2.24, Hilbert transform as

$$[z^2-izD_L k^2]\rho(\vec{k},z)-k^2p(\vec{k},z) = [-iz-D_L k^2]\rho(\vec{k},t=0) - \frac{\partial}{\partial t}\rho(\vec{k},t=0) \tag{3.64a}$$

and

$$izq(\vec{k},z) + \kappa k^2 \, T(\vec{k},z) = q(\vec{k},t=0) \quad . \tag{3.64b}$$

These coupled equations display the initial state in terms of the initial mass density and heat energy density and the initial time derivative of the mass density. Although three initial condition terms are required to specify the initial state, as for the thermal-viscous field equation in terms of one fluctuation, no initial second time derivatives are required. Furthermore the initial time derivative of the mass density may be related to the initial divergence of the longitudinal velocity by the mass density equation, equation 2.7a. The Hilbert transform of the time derivative of the mass density equation may be subtracted from the product of iz and the Hilbert transform of the mass density equation yielding the relation

$$\frac{\partial}{\partial t}\rho(\vec{k},t=0) = i\rho\vec{k}\cdot\vec{v}^L(\vec{k},t=0) \quad . \tag{3.65}$$

Since the longitudinal velocity is in the direction of the wavenumber vector a zero initial time derivative of mass density corresponds to a zero initial longitudinal velocity. Also a Fourier spatial transform of the expansion of one thermodynamic fluctuation in terms of two others, equation 2.32, evaluated at the initial time allows the initial time of the mass density to be related to the initial time derivatives of any two independent thermodynamic fluctuations as

$$\frac{\partial}{\partial t}\rho(\vec{k},t{=}0) = C^1_{ij}\frac{\partial}{\partial t}x_i(\vec{k},t{=}0) + C^1_{ji}\frac{\partial}{\partial t}x_j(\vec{k},t{=}0) \quad . \tag{3.66}$$

Thus a zero initial time derivative of mass density fluctuation corresponds to zero initial time derivatives of two other independent fluctuations or zero initial longitudinal velocity. The fluctuations and other initial condition terms in the thermodynamically coupled field equations, equations 3.64a and 3.64b, may be expanded similarly. It is useful to expand the fluctuations and the zero order time derivative initial conditions in separate pairs of independent properties and for simplicity not expand the initial time derivative of mass density. The resulting coupled equations, which correspond to the Hilbert transform of the general coupled equations 2.33a and 2.33b, are

$$L^1_{mn}(\vec{k},z)x_m(\vec{k},z)+L^1_{nm}(\vec{k},z)x_n(\vec{k},z) = (-iz-D_Lk^2)[C^1_{ij}x_i(\vec{k},t{=}0)+C^1_{ji}x_j(\vec{k},t{=}0)]-\frac{\partial}{\partial t}x_1(\vec{k},t{=}0) \tag{3.67a}$$

$$L^2_{mn}(\vec{k},z)x_m(\vec{k},z)+L^2_{nm}(\vec{k},z)x_n(\vec{k},z) = C^4_{ij}\,x_i(\vec{k},t{=}0)+C^4_{ji}x_j(\vec{k},t{=}0) \tag{3.67b}$$

where the operators are defined by

$$L^1_{mn}(\vec{k},z) = C^1_{mn}(z^2-izD_Lk^2) - C^3_{mn}k^2 \tag{3.68a}$$

$$L^2_{mn}(\vec{k},z) = C^2_{mn}\,\kappa k^2 + C^4_{mn}\,iz \tag{3.68b}$$

and for proper expansions $m \neq n$ and $i \neq j$.

The fluctuations, x_n, may be eliminated as the coupled equations are solved for the general fluctuation, x_m, in terms of the initial conditions,

$$x_m(\vec{k},z) = \{[L^2_{nm}(-iz-D_L k^2)C^1_{ij}-L^1_{nm}C^4_{ij}]x_i(\vec{k},t=0)+$$

$$[L^2_{nm}(-iz-D_L k^2)C^1_{ji}-L^1_{nm}C^4_{ji}]x_j(\vec{k},t=0)-L^2_{nm}\frac{\partial}{\partial t}x_1(\vec{k},t=0)\} \tag{3.69}$$

$$\{J^{14}_{mn}[is^3+z^2(D_L+\frac{\kappa}{C_v})k^2-izC_1^2k^2-izD_L\frac{\kappa}{C_v}k^4 - C_1^2\frac{\kappa}{C_p}k^4]\}^{-1}$$

This relationship may be used to evaluate the ratio of response, $x_m(\vec{k},z)$, to the forcing initial condition term which is required by Kubo's formula, equation 3.47, to derive the corresponding initial condition Green's function. Note that the functional part of the denominator, in square brackets, is identically the denominator of all previously derived thermal-viscous Green's functions as is expected since only the numerator of the Green's functions is effected by the initial conditions. Two separate types of initial condition terms appear in equation 3.69, an initial fluctuation, x_i, and the initial time derivative of mass density.

The general response fluctuation, $x_m(\vec{k},z)$, may be expressed as a function of the general forcing initial fluctuation, $x_i(\vec{k},t=0)$, when the initial state specifications is completed by $x_j(\vec{k},t=0)$ and $\frac{\partial}{\partial t}x_1(\vec{k},t=0)$ equal to zero,

$$x_m(\vec{k},z) = \frac{J^{mn}_{14}[L^2_{nm}(-iz-D_L k^2)C^1_{ij}-L^1_{nm}C^4_{ij}]x_i(\vec{k},t=0)}{iz^3+z^2(D_L+\frac{\kappa}{C_v})k^2+izC_1^2k^2-izD_L\frac{\kappa}{C_v}k^4-C_1^2\frac{\kappa}{C_p}k^4} \quad . \tag{3.70}$$

From Kubo's formula, equation 3.47, the initial condition Green's function which correlates the response fluctuation with the forcing initial fluctuation is given by

$$G_{mijv^L}(\vec{k},\omega) = G_{mijv^L}(\vec{k},\omega=0) - \lim_{z\to\omega}\frac{x_m(\vec{k},z)}{x_i(\vec{k},t=0)} \tag{3.71}$$

where the Green's function's subscripts denote the index of the response, the index of the forcing, the index of the zero initial fluctuations respectively. For simplicity the fact that the initial time derivative of mass density is zero is denoted by the final subscript which denotes equivalently, equation 3.65, that the initial longitudinal velocity is zero. The zero frequency Green's function equals the ratio of static response to initial condition forcing which are related thermodynamically as

$$G_{mijv^L}(\vec{k},\omega=0) = \left(\frac{\partial x_m}{\partial x_i}\right)_{xj} = C^m_{ij} = C^m_{ij}\, C'_{ij} \tag{3.72}$$

Thus this type of initial condition Green's function may be expressed in general as

$$G_{mijv^L}(\vec{k},\omega) = \left\{C^m_{ij} - \frac{i\omega\, J^{mn}_{14}\,[L^2_{nm}(-i\omega-D_L k^2)-L'_{nm}C^4_{ij}]}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L\,\frac{\kappa}{C_v}\,k^4-C_1^2\,\frac{\kappa}{C_p}\,k^4}\right\} C'_{ij} \tag{3.73}$$

which may be shown to be independent of the index n, representing the eliminated fluctuation. Since the dependence on the forcing initial condition, represented by the index i, is limited to the multiplying term C'_{ij} the initial condition Green's functions for different forcing initial fluctuations are related as

$$G_{mijv^L}(\vec{k},\omega) = C^i_{1j}\; G_{m1jv^L}(\vec{k},\omega) \quad . \tag{3.74}$$

Therefore of the fourty-eight initial condition Green's functions derivable from equation 3.73, all combinations of m, i and j where $i \neq j$, sixteen need be evaluated and the remaining follow simply from equation 3.74. For ease of notation the numerators of the Green's functions will be defined by

$$G_{mijv^L}(\vec{k},\omega) = \frac{N_{mijv^L}}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L\,\frac{\kappa}{C_v}\,k^4-C_1^2\,\frac{\kappa}{C_p}\,k^4} \tag{3.75}$$

and thus the sixteen of this type of initial condition Green's functions, all combinations of m and j, may be represented by the numerators,

$$N_{112v^L} = \frac{-C_v C_1^2}{C_p} k^2 (i\omega + \frac{\kappa}{C_v} k^2)$$

$$N_{113v^L} = -C_1^2 \frac{\kappa}{C_p} k^4$$

$$N_{114v^L} = -C_1^2 k^2 (i\omega + \frac{\kappa}{C_p} k^2)$$

$$N_{212v^L} = \frac{-TC_v C_1^4 \alpha_T}{C_p{}^2} k^2 (i\omega)$$

$$N_{213v^L} = -\frac{\kappa}{\rho C_v \alpha_T} k^2 (\omega^2 - i\omega D_L k^2 - C_1^2 \frac{C_v}{C_p} k^2)$$

$$N_{214v^L} = \frac{TC_1^2 \kappa \alpha_T}{C_p C_v} k^2 [\omega^2 - i\omega (C_1^2 \frac{C_v}{\kappa} + D_L k^2) - C_1^2 \frac{C_v}{C_p} k^2]$$

$$N_{312v^L} = \frac{-C_v C_1^2}{C_p} k^2 (i\omega + \frac{\kappa}{C_p} k^2)$$

$$N_{313v^L} = -C_1^2 \frac{\kappa}{C_p} k^2 (\omega^2 - i\omega k^2 D_o)$$

$$N_{314v^L} = C_1^2 k^2 \{\omega^2 (\frac{\kappa}{C_v} - \frac{\kappa}{C_p}) - i\omega [C_1^2 + D_o (\frac{\kappa}{C_v} - \frac{\kappa}{C_p}) k^2] - C_1^2 \frac{\kappa}{C_p} k^2\}$$

$$N_{412v^L} = \frac{TC_v C_1^4 \kappa \alpha_T}{C_p{}^2} k^4$$

$$N_{413v^L} = -\frac{\kappa}{\rho \alpha_T} k^2 (\omega^2 - i\omega D_L k^2 - C_1^2 k^2)$$

$$N_{414v^L} = \frac{TC_1^2 \kappa \alpha_T}{C_p} k^2 (\omega^2 - i\omega D_L k^2)$$

$$N_{121vL} = \frac{-\rho C_v C_1^2 \alpha_T}{C_p} k^2(i\omega)$$

$$N_{221vL} = \frac{\kappa}{C_v} k^2 \{\omega^2 - i\omega[C_1^2 \frac{C_v}{\kappa} (1 - \frac{C_v}{C_p}) + D_L k^2] - C_1^2 \frac{C_v}{C_p} k^2 \}$$

$$N_{321vL} = \frac{\rho C_1^2 \kappa \alpha_T}{C_p} k^2[\omega^2 - i\omega(C_1^2 \frac{C_v}{\kappa} + D_L k^2) - C_1^2 \frac{C_v}{C_p} k^2]$$

$$N_{421vL} = \kappa k^2(\omega^2 - i\omega D_L k^2 - C_1^2 \frac{C_v}{C_p} k^2) \quad . \tag{3.76}$$

These numerators and the relation of equation 3.74 complete the set of initial condition Green's functions that may be derived from equation 3.70. However, for that manner of initial state specification, forcing by the initial time derivative of mass density has yet to be considered.

The general response fluctuation, $x_m(\vec{k},z)$ in equation 3.69, is a function of the forcing initial time derivative of mass density when the initial fluctuations, $xi(\vec{k},t=0)$ and $xj(\vec{k},t=0)$ are equal to zero,

$$x_m(\vec{k},z) = \frac{-J_{14}^{mn} L_{nm}^2 \frac{\partial}{\partial t} x_1(\vec{k},t=0)}{iz^3+z^2(D_L+\frac{\kappa}{C_v})k^2-izC_1^2k^2-izD_L \frac{\kappa}{C_v} k^4-C_1^2 \frac{\kappa}{C_p} k^4} \quad . \tag{3.77}$$

Since two independent thermodynamic properties specify the thermodynamic state and x_i and x_j are independent, all initial fluctuations must equal zero. Therefore the zero frequency Green's function, which correlates the forcing initial condition to the static response, must be equal to zero and the general form of the initial condition Green's function is

$$G_{m\dot{\rho}ij}(\vec{k},\omega) = i\omega \frac{J_{14}^{mn} L_{nm}^2}{i\omega^3+\omega^2(D_L+\frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L \frac{\kappa}{C_v} - C_1^2 \frac{\kappa}{C_p} k^4} \quad . \tag{3.78}$$

The four initial condition Green's functions which correlate each of the thermodynamic fluctuations to the forcing initial time derivative of mass density may be represented, through equation 3.75, by the numerators

$$N_{1\dot{\rho}ij}(\vec{k},\omega) = i\omega(i\omega + \frac{\kappa}{C_v} k^2)$$

$$N_{2\dot{\rho}ij}(\vec{k},\omega) = - \frac{TC_1^2\alpha_T}{C_p} (\omega^2)$$

$$N_{3\dot{\rho}ij}(\vec{k},\omega) = C_1^2 \; i\omega(i\omega + \frac{\kappa}{C_p} k^2)$$

$$N_{4\dot{\rho}ij}(\vec{k},\omega) = - \frac{C_p\kappa}{TC_1^2\alpha_T} k^2(i\omega) \quad . \tag{3.79}$$

These initial condition Green's functions along with those defined by equations 3.74 and 3.76 for a complete set for correlating the response of a thermodynamic fluctuation to the initial state specified by the initial time derivative of mass density and two independent initial fluctuations. The total response is simply the sum of the responses from each initial condition term of equation 3.69, which in wavenumber and frequency domain is

$$x_m(\vec{k},\omega) = G_{mijv^L}(\vec{k},\omega)\; x_i(\vec{k},t=0)+G_{mjiv^L}(\vec{k},\omega)\; x_j(\vec{k},t=0)+G_{m\dot{\rho}ij}(\vec{k},\omega)\; \frac{\partial}{\partial t}\, x_1(\vec{k},t=0) \quad . \tag{3.80}$$

Since the longitudinal velocity is related to the thermodynamic properties, specifically to the mass density by equation 2.7a, initial condition Green's functions for initial forcing by and response of longitudinal velocity fluctuations may be related to previously derived Green's functions. The spatial Fourier transform of the mass density equation evaluated at time equal to zero yields equation 3.65 which relates the initial time derivative of mass density to the initial longitudinal velocity. Since the longitudinal velocity is in the direction of propigation, the direction of the wavenumber vector, equation 3.65 becomes

$$\frac{\partial}{\partial t}\rho(\vec{k},t=0) = i\rho k v^L(\vec{k},t=0) \quad . \tag{3.81}$$

Thus forcing initial condition terms of the initial time derivative of mass density may be replaced by initial longitudinal velocity terms which allow the respective initial condition Green's functions, which correlate the response with the initial condition forcing, to be related. The response, x_m, to initial time derivative of mass density forcing with a zero static initial state, the lost term on the right of equation 3.80, may be expressed in the equivalent forms

$$G_{m\dot{\rho}ij}(\vec{k},\omega)\frac{\partial}{\partial t}x_1(\vec{k},t=0) = G_{m\dot{\rho}ij}(\vec{k},\omega)[i\rho k v^L(\vec{k},t=0)] = G_{mv^L ij}(\vec{k},\omega)\ v^L(\vec{k},t=0) \quad . \tag{3.82}$$

The second form, a direct result of equation 3.81, and the third form, a definition of the initial condition Green's function, yield the relationship between the corresponding initial time derivative of mass density forced and the initial longitudinal velocity forced Green's functions,

$$i\rho k\ G_{m\dot{\rho}ij}(\vec{k},\omega) \ = \ G_{mv^L ij}(\vec{k},\omega) \quad . \tag{3.83}$$

Similarly the initial condition Green's function correlating mass density response to initial first time derivative of mass density initial condition with zero zeroth and second initial time derivatives, in the second term on the right side of equation 3.63 when $x = \rho$, relates to a corresponding initial longitudinal velocity forced Green's function as,

$$i\rho k\ G_{\rho\dot{\rho}\rho\ddot{\rho}}(\vec{k},\omega) = G_{\rho v^L \rho\ddot{\rho}}(\vec{k},\omega) \quad . \tag{3.84}$$

A relationship between the mass density and longitudinal velocity response fluctuations in wavenumber and frequency domain may be obtained by a spatial and temporal Fourier transform of the mass density equation, equation 2.7a,

$$i\omega\, \rho(\vec{k},\omega) = i\rho k\, v^L(\vec{k},\omega) \;. \tag{3.85}$$

Consider a mass density response and its equivalent longitudinal velocity response to any initial condition forcing, $Y(\vec{k},t=0)$, and any zero initial conditions,

$$\rho(\vec{k},\omega) = G_{\rho Y}(\vec{k},\omega)\; Y(\vec{k},t=0) = \frac{\rho k}{\omega}\, v^L(\vec{k},\omega) \;. \tag{3.86a}$$

By definition the longitudinal velocity response is correlated to the same forcing initial state as

$$v^L(\vec{k},\omega) = G_{v^L Y}(\vec{k},\omega)\; Y(\vec{k},t=0) \;. \tag{3.86b}$$

Therefore the initial condition Green's functions for mass density response and the Green's functions for longitudinal velocity with the same initial state are related as

$$\frac{\omega}{\rho k}\, G_{\rho Y}(\vec{k},\omega) = G_{v^L Y}(\vec{k},\omega) \;. \tag{3.87}$$

IV Green's Functions in Alternate Domains

Impulse and initial condition Green's functions, which correlate the response fluctuations to the forcing and initial conditions respectively, have been derived in wavenumber and frequency domain for both acoustic and thermal-viscous fluctuation processes. These Green's functions represent the relaxation of the fluctuations from specific modes of excitation and thus characterize the behavior of the medium. The numerator of a particular Green's function reflects the specific mode of excitation by biasing the response whereas the denominator, which represents the transformed differential operator, determines the type of fluctuation behavior.

Three basic types of Green's functions, corresponding to these distinct denominators in wavenumber and frequency domain, have been derived and are the diffusion type, the damped wave type, and the thermal-viscous type. The tensorial Green's functions for the total velocity of both acoustic and thermal-viscous fluctuations include transverse and longitudinal parts which represent different behaviors. The transverse parts are primarily diffusive and the longitudinal parts are primarily damped wavelike or thermal-viscous, depending on the fluctuation process, although both parts are complicated by additional zero wavenumber or zero frequency poles. As a result the tensorial total velocity Green's functions for the acoustic and thermal-viscous fluctuations will each be treated separately from the three basic Green's function types.

Wavenumber and frequency domain

The denominator of a Green's function in wavenumber and frequency domain is the dominant factor in characterizing the mediums response to some form of excitation because the Green's function posesses singularities corresponding to the wavenumber and frequency roots of the denominator. The number, complex form and arrangement of these roots determines the type of Green's function and the basic form of the response. Specifically, the complex wavenumber and frequency poles of the Green's function are both functions of the systems parameters and respectively the frequency and wavenumber. Thus for an arbitrary fluctuation frequency the wavenumber poles are the related fluctuation wavenumbers and vice versa. In this manner the wavenumber or the frequency poles of the Green's function determine the specific wavenumber-frequency relation of the fluctuation and contain the majority of the information pertaining to the relaxation of an excited medium.

Although a Green's function in wavenumber and frequency domain contains all the information regarding the relaxation of the corresponding fluctuation from a specific mode of excitation it may not be in the most visable or useful form. More information may be extracted from and in some instances more useful forms may be obtained of the Green's function when it is transformed to alternate domains. Since wavenumber may be transformed to space by an inverse spatial Fourier transformation and frequency may be transformed to time by an inverse temporal Fourier transformation, the Green's function may be obtained in the alternate space and frequency, wavenumber and time, and space and time domains by the appropriate transformations.

Space and frequency domain

The Green's functions in space and frequency domain may be derived by performing inverse spatial Fourier transformations of the wavenumber and frequency domain representations,

$$G(\vec{r},\omega) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} \; e^{-i\vec{k}\cdot\vec{r}} \, G(\vec{k},\omega) \quad . \tag{4.1}$$

Since this involves integration over all real wavenumber space, the wavenumber poles of the Green's functions will be of primary importance in the evaluation.

The only dependence of any of the Green's functions on the direction of the wavenumber vector is limited to the permutation tensors, equations 3.26, of the tensorial velocity Green's functions. All other Green's functions represent spherically symmetric relaxations which are even functions of the wavenumber magnitude, k. The triple integration over wavenumber space of functions of this form has been simplified to a single integration, equation A-1.9, by utilizing spherical coordinates which allow two integrations to be performed, and symmetry. Thus these nondirectional Green's functions in space and frequency domain may be expressed more simply as

$$G(\vec{r},w) = \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \, G(\vec{k},w) \quad , \tag{4.2}$$

which may be evaluated in particular by complex contour integration. The tensorial velocity Green's functions in wavenumber and frequency domain are of the form

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = P^{T}_{jm}(\vec{k}) \; G^{T}(\vec{k},\omega) + P^{L}_{jm}(\vec{k}) \; G^{L}(\vec{k},\omega) \quad , \tag{4.3}$$

where the longitudinal and transverse are, as all other Green's functions, even functions of the wavenumber magnitude and independent of the wavenumber direction.

With the permutation tensors expressed explicitly as defined by equations 3.26 the tensorial velocity Green's functions in space and frequency domain become

$$\vec{G}^{v}_{jm}(\vec{r},\omega) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} \; e^{-i\vec{k}\cdot\vec{r}} \, [(\delta_{jm} - \frac{k_j k_m}{k^2}) G^T(\vec{k},\omega) + (\frac{k_j k_m}{k^2}) G^L(\vec{k},\omega)] \; . \tag{4.4}$$

Since a spatial derivative of the integral representation interacts only with the transform kernal as

$$\frac{\partial}{\partial r_j} \; e^{-i\vec{k}\cdot\vec{r}} = -ik_j \tag{4.5}$$

the wavenumber component terms, the directional terms, may be replaced by spatial derivatives permitting the Green's function to be expressed as

$$\vec{G}^{v}_{jm}(\vec{r},\omega) = \delta_{jm} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} \; e^{-i\vec{k}\cdot\vec{r}} \, G^T(\vec{k},\omega) + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} \; e^{-i\vec{k}\cdot\vec{r}} \; \frac{G^T(\vec{k},\omega)}{k^2} \; -$$

$$- \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} \; e^{-i\vec{k}\cdot\vec{r}} \; \frac{G^L(\vec{k},\omega)}{k^2} \; . \tag{4.6}$$

Thus the wavenumber vector directionality of the permutation tensors has been replaced by directional spatial derivatives leaving only even functions of wavenumber to be integrated over wavenumber space. These integrations may be simplified, as were the nondirectional Green's functions, resulting in the space and frequency domain representations of the tensorial velocity Green's functions,

$$\vec{G}^{v}_{jm}(\vec{r},\omega) = \delta_{jm} \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \, G^T(\vec{k},\omega) + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \; \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \, \frac{G^T(\vec{k},\omega)}{k^2} \; -$$

$$- \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \; \frac{G^L(\vec{k},\omega)}{k^2} \; . \tag{4.7}$$

Wavenumber and time domain

The wavenumber and time domain Green's functions may be derived by performing an inverse temporal Fourier transformation on the Greens functions in wavenumber and time domain,

$$G(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{i\omega t}\, G(\vec{k},\omega) \quad . \tag{4.8}$$

The required integration to be performed over all real frequencies will display the primary importance of the complex frequency poles of the Green's function in wavenumber and frequency domain. Recall that causality implies that the imaginary parts of the Green's functions frequency poles must be positive. This is true of all Green's functions except for the transverse parts of the tensorial velocity Green's functions in wavenumber and frequency domain, which posess zero frequency poles. Such entirely real frequency poles are in conflict with physical condition that the system must be causal. The conflict may be resolved by assuming that the zero frequency poles actually posess a positive imaginary part which was too small to be evaluated in this approximation and thus only has the effect of satisfying the causality condition. Otherwise, the tensorial velocity Green's function transforms as all other Green's functions since the polarization tensors are functions of wavenumber vector only,

$$G^{\vec{v}}_{jm}(\vec{k},t) = P^{T}(\vec{k}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G^{T}(\vec{k},\omega) + P^{L}(\vec{k}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G^{L}(\vec{k},\omega) = \tag{4.9a}$$

$$= P^{T}(\vec{k})\, G^{T}(\vec{k},t) + P^{L}(\vec{k}) \cdot G^{L}(\vec{k},t) \quad . \tag{4.9b}$$

Space and time domain

The Green's functions in space and time domain may be derived by performing either inverse temporal transformations of the space and frequency domain representations or inverse spatial transformations of the wavenumber and time domain Green's functions,

$$G(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{i\omega t}\ G(\vec{r},\omega) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}}\ G(\vec{k},t) \quad . \qquad (4.10a,b)$$

For the derived Green's functions the space and frequency domain representation are more difficult to integrate over real frequencies than the wavenumber and time Green's functions are to integrate over real wavenumber space so the latter method will be utilized. Thus derivation of the Green's functions in space and time domain is analogous to the derivation of the Green's functions in space and frequency domain. Only the permutation tensors will be directional functions of the wavenumber vector and their directionality may be replaced by directional spatial derivatives, equation 4.5. All remaining wavenumber space integrands will be functions of the wavenumber magnitude, k, only and the required triple integrations may be simplified to a single integral each. By analogy with equations 4.2 and 4.7, the nondirectional Green's function in space and time is most simply,

$$G(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\,k e^{ikr}\ G(\vec{k},t) \quad , \qquad (4.11)$$

and the tensorial velocity Green's function in the same domain is

$$G^{\vec{v}}_{jm}(\vec{r},t) = \delta_{jm}\frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\,k e^{ikr}\ G^T(\vec{k},t) + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\ \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\,k e^{ikr}\ \frac{G^T(\vec{k},t)}{k^2} -$$

$$- \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\ \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\,k e^{ikr}\ \frac{G^L(\vec{k},t)}{k^2} \quad . \qquad (4.12)$$

Diffusion Type Green's Functions

Wavenumber and frequency domain

Diffusion type field equations have been derived which govern the fluctuations of transverse velocity and vorticity, equations 2.9, and the low temperature thermal-viscous temperature fluctuation, equation 2.48. From these field equations both impulse and initial condition Green's functions in wavenumber and frequency domain have been derived; for the transverse velocity they are shown as equations 3.20 and 3.53.

$$G^{\vec{v}^T}_{jm}(\vec{k},\omega) = \frac{1}{i\omega+D_T k^2}\,\delta_{jm} \tag{4.13}$$

$$G_{v^T v^T}(\vec{k},\omega) = \frac{D_T k^2}{i\omega+D_T k^2} \quad . \tag{4.14}$$

The tensorial part of the impulse Green's function indicates that the response was in the direction that it was forced as is implied by the scalar initial condition Green's function. Furthermore the Kronecker delta is independent of domain and required only for convolution, equation 3.18, so that a scalar impulse Green's function is sufficient for representing the fluctuation behavior in alternate domains,

$$G^{v^T}(\vec{k},\omega) = \frac{1}{i\omega+D_T k^2} \quad . \tag{4.15}$$

Only the transverse velocity Green's functions need be evaluated in alternate domains since the velocity Green's functions are identical and the low temperature thermal-viscous temperature Green's functions differ only by the diffusion coefficient. The scalar impulse and initial condition Green's functions differ only by the numerators, which reflect the mode of excitation. The denominators derive from the field equation operators and thus display the diffusive form by the two opposite complex wavenumber roots,

$$k_1 = (1-i)\sqrt{\frac{\omega}{2D_T}} \tag{4.16a}$$

$$k_2 = (-1+i)\sqrt{\frac{\omega}{2D_T}} \tag{4.16b}$$

and the single imaginary frequency root,

$$\omega_1 = iD_T k^2 \tag{4.16c}$$

Space and frequency domain

The diffusive Green's functions may be derived in space and frequency domain as the simplified inverse spatial Fourier transformations of the non-directional Green's functions in wavenumber and frequency domain, equation 4.2. The simplified integral forms of the impulse and initial condition Green's functions are

$$G^{v^T}(\vec{r},\omega) = \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dk k e^{ikr} \frac{1}{D_T(k-k_1)(k-k_2)} = \frac{1}{i4\pi^2 r D_T} \; {}_dI_1^0 \qquad (4.17a)$$

$$G_{v^T v^T}(\vec{r},\omega) = \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dk k e^{ikr} \frac{D_T k^2}{D_T(k-k_1)(k-k_2)} = \frac{1}{i4\pi^2 r} \; {}_dI_1^1 \quad . \qquad (4.17b)$$

The required integration form ${}_dI_1^n$ is evaluated in Appendix A-2, equations A-2.8a, for the complex wavenumber pole forms of equation 4.16 by applying the Cauchey residue theorem to a complex contour integration as

$${}_dI_1^n = \int_{-\infty}^{\infty} dk \frac{k^{2n+1} e^{ikr}}{(k-k_1)(k+k_1)} = \frac{r}{|r|} \pi i (k_1)^{2n} e^{-ik_1|r|} \quad , \quad n \geq 0 \quad , \qquad (4.18)$$

which displays the importance of the wavenumber poles and yields the Green's functions in space and frequency domain,

$$G^{v^T}(\vec{r},\omega) = \frac{1}{4\pi|r|} \frac{1}{D_T} e^{-(1+i)\sqrt{\frac{\omega}{2D_T}}\,|r|} \qquad (4.19a)$$

$$G_{v^T v^T}(\vec{r},\omega) = -i\omega \, G^{v^T}(\vec{r},\omega) \quad . \qquad (4.19b)$$

The term, $\frac{1}{4\pi|r|}$, represents a decrease in the magnitude of the fluctuation with radial distance due to spherical spreading.

The exponential term represents the radial distance and frequency dependence of the diffusive fluctuation decay and oscillation as determined by the wavenumber poles. An examination of the integration form, equation 4.18, and the wavenumber pole, equation 4.16a, indicates that the complex diffusive wavenumber should be defined as the product of i and the first wavenumber pole,

$$k_d = ik_1 = k_d' + ik_d'' \quad , \tag{4.20a}$$

where the real and imaginary parts,

$$k_d' = \sqrt{\frac{\omega}{2D_T}} \quad \text{and} \quad k_d'' = \sqrt{\frac{\omega}{2D_T}} \quad , \tag{4.20b,c}$$

are the dissapition and oscillation wavenumbers respectively. Since they are equal the diffusive fluctuation relaxes with an exponential radial decay exhibiting little radial oscillation. The relaxation form remains unchanged but penetrates farther in the radial direction when either the frequency is decreased or the diffusion coefficient is increased. Zero frequency yields no exponential decay or oscilation, only spherical spreading. In terms of the diffusion wavenumber components, the diffusive Green's functions are,

$$G^{V^T}(\vec{r},\omega) = \frac{1}{4\pi|r|D_T} e^{-k_d'|r|} e^{-ik_d''|r|} \tag{4.21a}$$

$$G_{V^T V^T}(\vec{r},\omega) = -i\omega\, G^{V^T}(\vec{r},\omega) \tag{4.21b}$$

where the difference is due to the different excitation modes.

Wavenumber and time domain

The diffusive Green's functions in wavenumber and time domain are equal to inverse temporal Fourier transformations of the wavenumber and frequency domain representations, as shown by equation 4.8, resulting in the frequency integrations,

$$G_{v^T}(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{i\omega t}\ \frac{-i}{(\omega-\omega_1)} = -i\ {}_dI_2^0 \tag{4.22a}$$

$$G_{v^Tv^T}(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{i\omega t}\ \frac{-iD_Tk^2}{(\omega-\omega_1)} = -iD_Tk^2\ {}_dI_2^0 \quad . \tag{4.22b}$$

The required frequency integration may be evaluated by a complex contour integration, equation A-3.7a, for which the positive imaginary frequency pole results in the unit step in time, U(t), and the exponential term,

$${}_dI_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} = U(t)i\ (\omega_1)^n\ e^{i\omega_1 t} \quad . \tag{4.23}$$

The resulting causal Green's functions in wavenumber and time domain,

$$G^{v^T}(\vec{k},t) = U(t)\ e^{-D_Tk^2t} \tag{4.24a}$$

$$G_{v^Tv^T}(\vec{k},t) = D_Tk^2\ G^{v^T}(\vec{k},t) \quad , \tag{4.24b}$$

display the wavenumber and time fluctuation relaxation characteristics resulting from the frequency pole. The exponential term of the integration form, equation 4.21, and the frequency root, equation 4.16c, suggest that the diffusive frequency be defined as the opposite of the product of i and the diffusive frequency root,

$$\omega_d = -i\omega_1 = \omega_d' + i\omega_d'' \quad , \tag{4.25a}$$

with the real and imaginary parts,

$$\omega_d' = D_Tk^2 \qquad \text{and} \qquad \omega_d'' = 0 \quad . \tag{4.25b,c}$$

The complex diffusive frequency indicates that diffusive fluctuations relax exponentially in time with no temporal oscillation and decay more rapidly as the wavenumber or diffusion coefficient is increased.

The diffusive Green's functions may be expressed in terms of the diffusive frequency as,

$$G^{v^T}(\vec{k},t) = U(t)\, e^{-\omega_d' t} \qquad (4.26a)$$

$$G_{v^T v^T}(\vec{k},t) = D_T k^2\, G^{v^T}(\vec{k},t) \quad , \qquad (4.26b)$$

which reflect the different modes of excitation by their differences.

Space and time domain

The diffusive Green's functions in space and time domain may be most easily evaluated as the inverse spatial Fourier transforms of the wavenumber and time domain representations, equations 4.26 which are non-directional, as shown in simplified form by equation 4.11. An alternate method of derivation would involve the inverse temporal transforms, as shown by equation 4.10a, of the space and frequency Green's functions, equations 4.19, which are complicated by the appearance of terms involving the square root of the frequency that would require branch cuts. The simplified inverse spatial Fourier transform method produces the Green's functions in the wavenumber integration forms

$$G^{v^T}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkk e^{ikr}\, U(t) e^{-D_T tk^2} = \frac{U(t)}{i4\pi^2 r}\,[d I_3^0]\Big|^{\delta=D_T t} \tag{4.27a}$$

$$G_{v^T v^T}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkk e^{ikr}\, U(t) k^2 e^{-D_T tk^2} = \frac{U(t)}{i4\pi^2 r}\,[d I_3^1]\Big|^{\delta=D_T t} \quad . \tag{4.27b}$$

The required integration forms are related by equation A-4.5 to the results of known integrations as

$$d I_3^n = \int_{-\infty}^{\infty} dk e^{ikr}\, k^{2n+1}\, e^{-\delta k^2} = \left(\frac{1}{\delta}\right)^n \frac{\partial^n}{d\delta^n}\left[\frac{-r\sqrt{\pi}}{2}\,\delta^{-3/2}\, e^{-r^2/4\delta}\right] \quad . \tag{4.28}$$

The diffusive Green's functions in space and time domain may be expressed, after some mathematical manipulations, as

$$G^{v^T}(\vec{r},t) = \frac{i\, U(t)}{\pi^{3/2}(4D_T t)^{3/2}}\; e^{-\frac{r^2}{4D_T t}} \tag{4.29a}$$

$$G_{v^T v^T}(\vec{r},t) = \frac{4}{(4D_T t)}\left[\frac{3}{2} - \frac{r^2}{(4D_T t)}\right] G^{v^T}(\vec{r},t) \quad . \tag{4.29b}$$

The space and time dependance of the exponential term is a result of the exponential term is a result of the exponential term of the wavenumber and time domain representation which was in turn a result of the frequency pole of the wavenumber and frequency domain Green's function. The time dependence term,

$$r_T = \sqrt{4D_T t} \quad , \tag{4.30}$$

represents a radius characteristic of the transverse viscous diffusion. As this radius increases the exponential radial decay rate decreases and the multiplying term decreases. The Green's function displays a maximum at zero radius which decreases as time increases and the radial form spreads.

Damped Wave Type Green's Functions

Wavenumber and frequency domain

Damped wave type field equations have been derived which govern the acoustic fluctuations of mass density, pressure, longitudinal velocity and velocity divergence, equations 2.14, 2.16, and 2.45. Although the acoustic and the low temperature thermal-viscous field equations exhibit different damping coefficients and sound speeds, their forms are identical. Therefore the set of impulse and initial condition Green's functions for the acoustic mass density, equations 3.22, 3.56 and 3.57,

$$G^{\rho}(\vec{k},\omega) = \frac{1}{\omega^2 - i\omega D_L k^2 - C_1^2 k^2} \tag{4.31a}$$

$$G_{\rho\dot{\rho}}(\vec{k},\omega) = \frac{-C_1^2 k^2}{\omega^2 - i\omega D_L k^2 - C_1^2 k^2} \tag{4.31b}$$

$$G_{\dot{\rho}\rho}(\vec{k},\omega) = \frac{i\omega}{\omega^2 - i\omega D_L k^2 - C_1^2 k^2} \tag{4.31c}$$

represent all of the damped wave type field equation forms. Two initial condition Green's functions, equations 4.31b and 4.31c, are required because two initial condition modes of excitation result from a field equation operator which posesses second order time derivatives. The different numerators for each of the three Green's functions in wavenumber and frequency domain reflect the different excitation modes and the identical denominators for all result from the common damped wave type field equation operator. The damped wave form is displayed by the denominators two opposite complex wavenumber roots,

$$k_1 = -k_2 = \frac{|\omega|}{\sqrt{2}\,\sqrt{C_1^4+D_L^2\omega^2}} \left(\sqrt{C_1^2 + \sqrt{C_1^4+D_L^2\omega^2}} - i\sqrt{-C_1^2 + \sqrt{C_1^4+D_L^2\omega^2}}\right) , \quad (4.32a,b)$$

and thus frequency roots,

$$\omega_1 = \sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4} + i \frac{D_L}{2} k^2 \quad (4.32c)$$

$$\omega_2 = -\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4} + i \frac{D_L}{2} k^2 , \quad (4.32d)$$

which are the poles of the wavenumber and frequency domain Green's functions.

Space and frequency domain

The damped wave Green's functions in space and frequency domain are, by equation 4.2, equal to the simplified inverse spatial Fourier transformations of the wavenumber and frequency domain representations,

$$G^{\rho}(\vec{r},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\,\frac{-1}{(C_1^2+i\omega D_L)(k-k_1)(k-k_2)} = \frac{-1}{i4\pi^2 r(C_1^2+i\omega D_L)}\; d^{I_1^0} \tag{4.33a}$$

$$G_{\rho\rho\dot{\rho}}(\vec{r},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\,\frac{C_1^2 k^2}{(C_1^2+i\omega D_L)(k-k_1)(k-k_2)} = \frac{C_1^2}{i4\pi^2 r(C_1^2+i\omega D_L)}\; d^{I_1^1} \tag{4.33b}$$

$$G_{\rho\dot{\rho}\rho}(\vec{r},\omega) = i\omega\; G^{\rho}(\vec{k},\omega) \tag{4.33c}$$

The wavenumber integration form and the complex wavenumber pole forms are the same as was required for the analogous diffusive Green's functions transformations so the same integration evaluation, equation 4.18, is valid. This suggests that an analogous complex damped wave wavenumber be defined for the specific damped wave type wavenumber roots of equation 4.32,

$$k_W = ik_1 = k_W' + ik_W'' \quad , \tag{4.34a}$$

with specific real and imaginary parts

$$k_W' = \frac{|\omega|}{\sqrt{2}\,\sqrt{C_1^4+D_L^2\omega^2}}\sqrt{-C_1^2+\sqrt{C_1^4+D_L^2\omega^2}} \qquad k_W'' = \frac{|\omega|}{\sqrt{2}\,\sqrt{C_1^4+D_L^2\omega^2}}\sqrt{C_1^2+\sqrt{C_1^4+D_L^2\omega^2}} \;. \tag{4.34b,c}$$

In these terms the space and frequency domain Green's functions may be evaluated as

$$G^{\rho}(\vec{r},\omega) = \frac{-1}{4\pi|r|(C_1^2+iD_L\omega)}\; e^{-k_W'|r|}\; e^{-ik_W''|r|} \tag{4.35a}$$

$$G_{\rho\rho\dot{\rho}}(\vec{r},\omega) = \frac{-C_1^2\omega^2}{(C_1^2+iD_L\omega)} \quad G^{\rho}(\vec{r},\omega) \tag{4.35b}$$

$$G_{\rho\dot{\rho}\rho}(\vec{r},\omega) = i\omega \; G^{\rho}(\vec{r},\omega) \quad . \tag{4.35c}$$

The damped wave Green's functions exhibit spherical spreading and exponential decay and oscillation in the radial direction as do the diffusive Green's functions and in addition possess the frequency pole,

$$\omega = i \frac{C_1^2}{D_L} \quad . \tag{4.36}$$

This frequency pole along with the different functional frequency dependence of the damped wave wavenumber suggest that the damped wave Green's functions in space and time domain would be of a form quite different from that of the diffusive Green's functions, equations 4.29.

The real part of the complex wavenumber is always less than the imaginary part, equations 4.34b and c, which yields damped wave fluctuation relaxations which are more oscilatory as they decay in the radial direction than are the diffusive fluctuations. The oscilatory nature is more pronounced as the frequency or damping coefficient decreases since the relative difference in the real and imaginary parts of the complex wavenumber will increase. However the real part will decrease, and therefore the fluctuation relaxation will penetrate farther when the frequency is decreased or the damping coefficient is increased.

Wavenumber and time domain

The damped wave Green's functions in wavenumber and time domain may be represented in integral form as inverse temporal Fourier transformations of the wavenumber and frequency domain Green's functions,

$$G^{\rho}(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{1}{(\omega-\omega_1)(\omega-\omega_2)} = {}_wI_2^0 \qquad (4.37a)$$

$$G_{\rho\rho\dot{\rho}}(\vec{k},t) = -C_1^2 k^2 \, G^{\rho}(\vec{k},t) \qquad (4.37b)$$

$$G_{\rho\dot{\rho}\rho}(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{i\omega}{(\omega-\omega_1)(\omega-\omega_2)} = i\,{}_wI_2^1 \quad . \qquad (4.37c)$$

The frequency integration form, equation A-3.7c, exhibits causality as a result of positive imaginary parts of both frequency poles,

$${}_wI_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = \frac{i\,U(t)}{(\omega_1-\omega_2)} [(\omega_1)^n e^{i\omega_1 t} - (\omega_2)^n e^{i\omega_2 t}] \quad . \qquad (4.38)$$

(36)

Three distinct forms of the frequency poles exist corresponding to a positive, zero or negative term under the radical in equations 4.32c and d which relate to temporally underdamped, critically damped, or overdamped Green's functions. For each condition the complex frequency components may be defined and utilized to express the frequency poles.

Temporally underdamped Green's functions occur when the damping coefficient or wavenumber are small as shown by the inequality,

$$C_1^2 > \frac{D_L^2}{4} k^2 \quad . \qquad (37)$$

The components of the complex frequency,

$$\omega_W' = \frac{D_L}{2} k^2 \qquad\qquad \omega_W'' = \sqrt{\left|C_1^2 k^2 - \frac{D_L^2}{4} k^4\right|} \quad , \qquad (4.40a,b)$$

allow the underdamped frequency poles to be expressed as

$$\omega_1 = \omega_W'' + i\omega_W' \qquad \omega_2 = -\omega_W'' + i\omega_W' \quad , \tag{4.41a,b}$$

which exhibit both real and imaginary parts corresponding to temporal decay and oscillation. The resulting temporally underdamped Green's functions are

$$G^{\rho}(\vec{k},t) = -U(t)\, e^{-\omega_W' t} \frac{\sin(\omega_W'' t)}{\omega_W''} \tag{4.42a}$$

$$G_{\rho\dot{\rho}}(\vec{k},t) = -C_1^2 k^2\, G^{\rho}(\vec{k},t) \tag{4.42b}$$

$$G_{\dot{\rho}\rho}(\vec{k},t) = -U(t)\, e^{-\omega_W' t} \left[\cos(\omega_W'' t) - \frac{\omega_W'}{\omega_W''}\sin(\omega_W'' t)\right] \quad . \tag{4.42c}$$

The real part of one complex frequency, equation 38a, relates to an exponential decay in time which occurs more rapidly as the damping coefficient or wavenumber is increased. Meanwhile the imaginary part of the complex frequency, equation 4.40 which corresponds to the frequency of sinusoidal oscilation, decreases, resulting in a longer time period of oscilations which are decaying more rapidly in time. This less oscilatory, more rapid decay in time trend continues as the damping coefficient or wavenumber are increased until the critical damping condition is met.

The critical temporal damping condition,

$$C_1^2 = \frac{D_L^2}{4} k^2 \quad , \tag{4.43}$$

corresponds to zero imaginary component of the complex frequency and the resulting coincident, critically damped frequency poles

$$\omega_1 = \omega_2 = i\omega_W' \quad . \tag{4.44}$$

The critically damped wave Green's functions,

$$G^{\rho}(\vec{k},t) = -U(t)\ t\ e^{-\omega_W' t} \tag{4.45a}$$

$$G_{\rho\rho\dot{\rho}}(\vec{k},t) = -C_1^2 k^2\ G^{\rho}(\vec{k},t) \tag{4.45b}$$

$$G_{\rho\rho\dot{\rho}}(\vec{k},t) = U(t)\ [1 - \frac{D_L}{2} k^2 t]e^{-\omega_W' t} \quad , \tag{4.45c}$$

display no oscilation, only the exponential temporal decay due to the real part of the complex frequency, equation 4.40a. For the critical condition, equation 4.43, the real part of the complex frequency is also equal to the product of the adiabatic speed of sound and the wavenumber,

$$\omega_W' = \frac{D_L}{2} k^2 = C_1^2\ k \quad . \tag{4.46}$$

Temporally overdamped Green's functions occur for larger damping coefficient or wavenumber according to the inequality,

$$C_1^2 < \frac{D_L^2}{4}\ k^2 \quad . \tag{4.47}$$

For this condition, the term under the radial in the frequency poles, equations 4.32c and d, is negative and thus the radical term is imaginary. Consequently the complex frequency consists of two real parts,

$$\omega_W' = \frac{D_L}{2} k^2 \qquad\qquad \omega_{W_0}' = \sqrt{\left|\frac{D_L^2}{4} k^4 - C_1^2 k^2\right|} \quad , \tag{4.48a,b}$$

the first of which is identically the real part of the underdamped complex frequency, equation 4.40a, and the second of which corresponds to the imaginary part of the underdamped complex frequency, equation 4.40b. The resulting over-damped wave Green's functions,

$$G^{\rho}(\vec{k},t) = \frac{-U(t)}{2\omega'_{w_o}} [e^{-(\omega'_w-\omega'_{w_o})t} - e^{-(\omega'_w+\omega'_{w_o})t}] \tag{4.49a}$$

$$G_{\rho\rho\dot{\rho}}(\vec{k},t) = -C_1^2 k^2 \, G^{\rho}(\vec{k},t) \tag{4.49b}$$

$$G_{\rho\dot{\rho}\rho}(\vec{k},t) = \frac{-U(t)}{2} [(1 + \frac{\omega'_w}{\omega'_{w_o}})e^{-(\omega'_w-\omega'_{w_o})t} + (1 - \frac{\omega'_w}{\omega'_{w_o}})e^{-(\omega'_w+\omega'_{w_o})t}] \ , \tag{4.49c}$$

exhibit two exponentially decaying functions of time. The first exponential term is dependent on the difference of the two real parts of the complex frequency and therefore decays more slowly in time than the second exponential which is dependent on the sum of the real parts. Furthermore as the damping coefficient or wavenumber increase the first exponential decays even more slowly in time since

$$\frac{\partial}{\partial D_L} (\omega'_w - \omega'_{w_o}) < 0 \qquad , \qquad \frac{\partial}{\partial k} (\omega'_w - \omega'_{w_o}) < 0 \tag{4.50a,b}$$

whereas the second exponential decays more rapidly.

The temporally underdamped, critically damped and overdamped Green's functions in wavenumber and time domain have been expressed explicitly in order to display the corresponding different forms. However, transformation of these Green's functions to space and time domain requires integration over all wavenumber space which would be facilitated by defining single functional forms that are valid for all wavenumbers. Since the frequency poles, equations 4.32c and d, are functionally valid for all wavenumber and are of the underdamped form, Green's functions derived directly from them are

$$G^{\rho}(\vec{k},t) = -U(t) \, e^{-\frac{D_L}{2} k^2 t} \, \frac{\sin(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4} \, t)}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}} \tag{4.51a}$$

$$G_{\rho\dot{\rho}\rho}(\vec{k},t) = -C_1^2 k^2 \, G^{\rho}(\vec{k},t) \tag{4.51b}$$

$$G_{\rho\dot{\rho}\rho}(\vec{k},t) = -U(t)e^{-\frac{D_L}{2}k^2 t}\left[\cos\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}\; t\right) - \frac{\frac{D_L}{2}k^2}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}} \sin\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}\; t\right)\right], \tag{4.51c}$$

which are functionally valid for all wavenumber. When the radical term equals zero these Green's functions reduce to their critically damped form since

$$\lim_{x\to 0} \frac{\sin(xt)}{x} = t \tag{4.52}$$

and when the radical term is imaginary these Green's functions take on their overdamped form since

$$\cos(ixt) = \cosh(xt) \quad \text{and} \quad \frac{\sin(ixt)}{ix} = \frac{\sinh(xt)}{x} \; . \tag{4.53a,b}$$

Space and time domain

The damped wave Green's functions in space and time domain may be expressed in integral form as the simplified inverse spatial Fourier transformations

$$G^{\rho}(\vec{r},t) = \frac{-1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k e^{ikr}\, U(t) e^{-\frac{D_L}{2} t k^2} \frac{\sin\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}\; t\right)}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}} = \frac{-U(t)t}{i4\pi^2 r}\left[{}_W I_5^0\right]\Bigg|_{\rho_3 = \frac{2C_1}{D_L}}^{\delta = \frac{D_L}{2} t}$$

(4.54a)

$$G_{\rho\rho\dot{\rho}}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k e^{ikr}\, U(t) C_1^2 k^2 e^{-\frac{D_L}{2} t k^2} \frac{\sin\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}\; t\right)}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}} = \frac{C_1^2 U(t)t}{i4\pi^2 r}\left[{}_W I_5^1\right]\Bigg|_{\rho_3 = \frac{2C_1}{D_L}}^{\delta = \frac{D_L}{2} t}$$

(4.54b)

$$G_{\rho\dot{\rho}\rho}(\vec{r},t) = \frac{-1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k e^{ikr}\, U(t) e^{-\frac{D_L}{2} t k^2} \left[\cos\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}\; t\right) - \right.$$

$$\left. - \frac{\frac{D_L}{2} k^2}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}} \sin\left(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4} k^4}\; t\right)\right] = \frac{-U(t)}{i4\pi^2 r}\left[{}_W I_4^0\right]^{\delta = \frac{D_L}{2} t} + \frac{\frac{D_L}{2} U(t) t}{i4\pi^2 r}\left[{}_W I_5^1\right]^{\delta = \frac{D_L}{2} t}$$

(4.54c)

The required integrals have been evaluated in series form and may be evaluated numerically. However, such forms are not easily compared with the closed form diffusive or thermal-viscous Green's functions in space and lime domain and therefore will not be included.

Thermal-Viscous Type Green's Functions

Wave and frequency domain

Thermal-viscous type field equations have been derived for the fluctuations of mass density, temperature, pressure, heat energy density, longitudinal velocity and velocity divergence. Each of these fluctuations satisfy the same field equation separately, equation 2.38, and any two of the thermodynamic fluctuations satisfy the coupled field equations, equation 2.33, simultaneously. Furthermore the longitudinal velocity and velocity divergence are related by the mass density equation, equation 2.7a, to the mass density which is in turn thermodynamically coupled to the other thermodynamic fluctuations by the expansion equation 2.32. This results in a common impulse Green's function, equation 3.31, for all fluctuations,

$$G^{Xm}(\vec{k},w) = \frac{1}{i\omega^3+\omega^2(D_L + \frac{\kappa}{C_v})k^2-i\omega C_1^2k^2-i\omega D_L \frac{\kappa}{C_v} k^4-C_1^2 \frac{\kappa}{C_p} k^4} , \qquad (4.55)$$

and a multitude of initial condition Green's functions. As for the diffusive and damped wave type fluctuations, all thermal-viscous Green's functions display the same denominator which the numerators differ as a result of the mode of excitation. The field equation for a single fluctuation results in possible initial conditions of the zeroth, first and second initial time derivatives and the numerator forms of equations 3.60, 3.61, 3.62. The additional numerator forms of equations 3.76 and 3.79, result from the initial conditions of the coupled field equations. Of the many distinct numerators that have been derived, only five functional forms of the wavenumber and frequency exist,

$$N_0 = 1 \qquad N_1 = i\omega k^2 \qquad N_2 = k^4 \qquad N_3 = \omega^2k^2 \qquad N_4 = i\omega k^4 \qquad (4.56\ a,b,c,d,e)$$

Thus any thermal-viscous Green's function may be constructed from the system parameters, the functional numerator forms and the thermal-viscous denominator defined as,

$$DTV = i\omega^3 + \omega^2(D_L + \frac{\kappa}{C_v})k^2 - i\omega C_1^2 k^2 - i\omega D_L \frac{\kappa}{C_v} k^4 - C_1^2 \frac{\kappa}{C_p} k^2 \quad , \tag{4.57}$$

The thermal-viscous Green's functions may be represented concisely in wavenumber and frequency domain by the five basic functional forms,

$$G_o(\vec{k},\omega) = \frac{N_o}{DTV} \qquad G_1(\vec{k},\omega) = \frac{N_1}{DTV} \qquad G_2(\vec{k},\omega) = \frac{N_2}{DTV} \qquad G_3(\vec{k},\omega) = \frac{N_3}{DTV}$$

$$G_4(k,\omega) = \frac{N_4}{DTV} \quad . \tag{4.58a,b,c,d,e}$$

of which the first, $G_o(\vec{k},\omega)$, is identically the impulse Green's function and the remaining may be used to construct any of the initial condition Green's functions.

The denominator, which displays the thermal-viscous form, is a quartic functions of wavenumber and a cubic function of frequency and therefore possesses form wavenumber roots and three frequency roots. The wavenumber roots may be derived as the solutions to the equation

$$A_1 k^4 + A_2 k^2 + A_3 = 0 \tag{4.59a}$$

where $A_1 = i\omega D_L \frac{\kappa}{C_v} - C_1^2 \frac{\kappa}{C_p}$, $A_2 = \omega^2(D_L + \frac{\kappa}{C_v}) - i\omega C_1^2$, $A_3 = i\omega_3$. (4.59b,c,d)

The general solutions yield two pairs of opposite complex wavenumber roots

$$k_1 = -k_2 = -\left\{\frac{1}{2A_1}\left[-A_2-(A_2^2-4A_1A_3)^{1/2}\right]\right\}^{1/2} \tag{4.60c,d}$$

$$k_3 = -k_4 = -\left\{\frac{1}{2A_1}\left[-A_2+(A_2^2-4A_1A_3)^{1/2}\right]\right\}^{1/2} \quad . \tag{4.60c,d}$$

The frequency roots are the solutions to the cubic equation,

$$\omega^3 + B_1\omega^2 + B_2\omega + B_3 = 0 \tag{4.61a}$$

where

$$B_1 = -i(D_L + \frac{\kappa}{C_v})k^2 \quad , \quad B_2 = -C_1^2 k^2 - D_L \frac{\kappa}{C_v} k^4 \quad B_3 = iC_1^2 \frac{\kappa}{C_p} k^4 \quad . \qquad (4.61b,c,d)$$

The general solutions are known to be

$$\omega_1 = \frac{B_1}{3} - \frac{(B_6+B_7)}{2} + i\,\frac{(B_6-B_7)}{2}\sqrt{3} \qquad \omega_2 = \frac{B_1}{3} - \frac{(B_6+B_7)}{2} - i\,\frac{(B_6-B_7)}{2}\sqrt{3}$$

$$\omega_3 = \frac{B_1}{2} + B_6 + B_7 \qquad (4.63a,b,c)$$

here

$$B_4 = B_2 - \frac{B_1^2}{3} \qquad B_5 = \frac{2B_1^3}{27} - \frac{B_1 B_2}{3} + B_3$$

$$B_6 = [\frac{-B_5}{2} + (\frac{B_5^2}{4} + \frac{B_4^3}{27})^{1/2}]^{1/3} \qquad B_7 = [\frac{-A_5}{2} - (\frac{B_5^2}{4} + \frac{B_4^3}{27})^{1/2}]^{1/3} \quad . \qquad (63d,e,f,g)$$

The exact wavenumber and frequency roots of the denominator are such complicated complex functions of the system parameters and respectively the frequency and wavenumber that it is not practical to use them as the poles of the Green's functions for transformation purposes.

In order to analytically perform the integrations required to transform the Green's functions to space and frequency domain, equation 4.2, and wavenumber and time domain, equation 4.8, it is necessary to approximate the wavenumber and frequency poles. The thermal-viscous Green's functions have been formulated from the assumption of local thermodynamic equilibrium which suggests a small frequency and wavenumber approximation for the wavenumber and frequency roots. Utilizing binomial expansions and the respective smallness criteria,

$$k^2 << \frac{C_1^2}{D^2} \qquad \text{and} \qquad \omega << \frac{C_1^2}{D} \quad , \qquad (4.63, 4.64)$$

the wavenumber and frequency roots may be approximated as

$$k_1 = -k_2 \approx \frac{\omega}{C_1} - i\,\frac{\omega^2}{2C_1^3}\left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right) \qquad k_3 = -k_4 \approx (1-i)\sqrt{\frac{\omega}{2\,\frac{\kappa}{C_p}}} \tag{4.65a,b}$$

$$\omega_1 \approx C_1 k + \frac{i}{2}\left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right) k^2 \qquad \omega_2 \approx -C_1 k + \frac{i}{2}\left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right) k^2 \qquad \omega_3 \approx i\,\frac{\kappa}{C_p}\,k^2$$

(4.65c,d,e)

The first pair of wavenumber roots, k_1 and k_2, and the first two frequency roots, w_1 and w_2, are similar to the damped wave type roots, equations 4.32. Indeed, the small frequency and wavenumber approximation of the damped wave roots results in the thermal viscous roots with the thermal diffusion coefficients, $\frac{\kappa}{C_p}$ and $\frac{\kappa}{C_v}$, absent. This indicates that the first pair of thermal-viscous wavenumber roots and the first two thermal-viscous frequency roots are related to a damped wave with a thermally modified longitudinal viscous damping coefficient, or thermal-viscous damping coefficient, defined as

$$D_1 = \left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right) . \tag{4.66}$$

The remaining pair of wavenumber roots, k_3 and k_4, and the remaining frequency root, w_3, are of the same form as the diffusion type roots, equations 4.16. The difference is that these thermal-viscous roots display a thermal diffusion coefficient, or thermometric conductivity, defined as

$$D_2 = \frac{\kappa}{C_p} \tag{4.67}$$

rather than the transverse viscous diffusion coefficient. Thus the approximate roots indicate that the thermal viscous denominator is approximately the product of a damped wave type denominator and a diffusion type denominator. However, the approximate wavenumber poles are inconsistant with the frequency poles because a denominator fabricated from the approximate wavenumber roots, $(k-k_1)(k-k_2)(k-k_3)(k-k_4)$, is not proportional to one fabricated from the approximate frequency roots. Furthermore, the inconsistancy is significant because the wavenumber fabricated denominator possesses five frequency roots and the frequency fabricated denominator possesses six wavenumber roots. An inconsistancy in the number of roots is unsatisfactory because the transformations of the Green's functions to alternate domains are critically dependent on them.

Rather than use the approximate roots, equation 4.65, an attempt to derive alternate, consistant, approximate wavenumber and frequency roots from the quastic and cubic solution forms, equations 4.60 and 4.63, it is more fruitful to extrapolate the information provided by the approximate roots. That is, assume that the thermal-viscous denominator may be more accurately approximated as the product of a damped wave type denominator, with thermal-viscous damping coefficient, and a diffusion type denominator, with a thermal diffusion coefficient. This thermal-viscous model denominator,

$$DM = [\omega^2 - i\omega D_1 k^2 - C_1^2 k^2](i\omega + D_2 k^2) \tag{4.68a}$$

$$= i\omega^3 + \omega^2\left(D_L + \frac{\kappa}{C_v}\right)k^2 - i\omega C_1^2 k^2 - i\omega\left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right)\frac{k}{C_p}\, k^4 - C_1^2 \frac{\kappa}{C_p}\, k^4 \quad , \tag{4.68b}$$

only differs from the true thermal viscous denominator, DTV, equation , by one term. This term for the model denominator is

$$-i\omega\left(D_L - \frac{\kappa}{C_p} + \frac{\kappa}{C_v}\right)\frac{\kappa}{C_p}\, k^4 \tag{4.69a}$$

and for the true denominator is

$$-i\omega D_L \frac{\kappa}{C_v}\, k^4 \quad . \tag{4.69b}$$

These two terms are equal when the specific heats are equal

$$C_p = C_v \tag{4.70a}$$

or when the longitudinal viscous damping coefficient is equal to the thermal diffusion coefficient

$$D_L = \frac{\kappa}{C_p} \quad . \tag{4.70b}$$

For either of the above cases the model denominator is equal to the true denominator. Otherwise, the model denominator differs by one term only which displays the correct functional form of wavenumber and frequency with the correct complex form of coefficients. Thus the model denominator is superior to the approximate wavenumber or frequency fabricated denominators and possesses manageable wavenumber and frequency roots.

Since the model denominator is the product of a damped wave type denominator and a diffusion type denominator, it exhibits both damped wave type roots and diffusion type roots. Replacing the longitudinal viscous damping coefficient in the damped wave type roots, equation 4.32, with the thermal-viscous damping coefficient and replacing the transverse viscous diffusion coefficient in the diffusion type roots, equations 4.16, with the thermal diffusion coefficient yields the thermal-viscous model denominator roots,

$$k_{1_m} = -k_{2_m} = \frac{|\omega|}{\sqrt{2}\sqrt{C_1^4+\omega^2 D_1^2}}\left(\sqrt{C_1^2 + \sqrt{C_1^4+\omega^2 D_1^2}} - i\sqrt{-C_1^2 + \sqrt{C_1^4+\omega^2 D_1^2}}\;\right) \tag{4.71a,b}$$

$$k_{3_m} = -k_{4_m} = (1-i)\sqrt{\frac{\omega}{2D_2}} \tag{4.71c,d}$$

$$\omega_{1_m} = \sqrt{C_1^2k^2 - \frac{D_1^2}{4}k^4} + i\,\frac{D_1}{2}k^2 \qquad \omega_{2_m} = -\sqrt{C_1^2k^2 - \frac{D_1^2}{4}k^4} + i\,\frac{D_1}{2}k^2 \qquad \omega_{3_m} = iD_2k^2 \quad .$$

(4.71e,f,g)

The thermal-viscous model denominator is useful since it provides a means of approximating the Green's function in wavenumber and frequency domain which due to the relatively simple forms of the model wavenumber and frequency poles, may be readily transformed analytically to alternate domains. The true thermal-viscous Green's functions may be transformed to either space and frequency domain by a wavenumber integration or to wavenumber and time domain by a frequency integration. However the exact wavenumber and frequency poles, which are major factors determining the results of the respective integrations, are of such complicated forms that they render a further frequency or wavenumber transformation of the Green's functions to space and time domain analytically impossible. Furthermore the complication is unmerited because the entire thermal-viscous analysis is based on linearly approximated equations. The model Green's functions in wavenumber and frequency corresponding to the five basic thermal-viscous Green's function forms result when the model denominator is substituted for the true denominator,

$$G_{0_m}(\vec{k},\omega) = \frac{N_0}{DM} \quad G_{1_m}(\vec{k},\omega) = \frac{N_1}{DM} \quad G_{2_m}(\vec{k},\kappa) = \frac{N_2}{DM} \quad G_{3_m}(\vec{k},\omega) = \frac{N_3}{DM} \quad G_{4_m}(\vec{k},\omega) = \frac{N_4}{DM} \; . \tag{4.72a,b,c,d,e}$$

Space and frequency domain

The thermal-viscous Green's functions in space and frequency domain may be expressed as the simplified inverse spatial Fourier transformations of the wavenumber and frequency domain representations, as shown by equation 4.2. The resulting wavenumber integral forms of the true Green's functions are

$$G_0(\vec{r},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\frac{1}{-D_2(C_1^2+i\omega D_1)(k^2-k_1^2)(k^2-k_3)^2} = \frac{-1}{i4\pi^2 rD_2(C_1^2+i\omega D_1)}\ I_1^0 \quad (4.73a)$$

$$G_1(\vec{r},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\frac{i\omega k^2}{-D_2(C_1^2+i\omega D_1)(k^2-k_1^2)(k^2-k_3^2)} = \frac{-i\omega}{i4\pi^2 rD_2(C_1^2+i\omega D_1)}\ I_1^1 \quad (4.73b)$$

$$G_2(\vec{r},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\frac{k^4}{-D_2(C_1^2+i\omega D_1)(k^2-k_1^2)(k^2-k_3^2)} = \frac{-1}{4\pi^2 rD_2(C_1^2+i\omega D_1)}\ I_1^2 \quad (4.73c)$$

$$G_3(\vec{r},\omega) = -i\omega G_1(\vec{r},\omega) \quad (4.73d)$$

$$G_4(\vec{r},\omega) = i\omega G_2(\vec{r},\omega) \quad . \quad (4.73e)$$

The required integration form is evaluated by complex contour integration, equation A-2.8b, assuming that the wavenumber poles are located in the complex wavenumber plane as the model wavenumber poles are,

$$I_1^n = \int_{-\infty}^{\infty} dk\,\frac{k^{2n+1}e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \frac{r}{|r|}\,\frac{i\pi}{(k_1^2-k_3^2)}\,[k_1^{2n}\,e^{-ik_1|r|} - k_3^{2n}\,e^{-ik_3|r|}\,] \quad . \quad (4.74)$$

By substituting the model wavenumber roots for the actual wavenumber roots in the integration form the thermal-viscous Green's functions may be approximated in space and frequency domain by the thermal-viscous model Green's functions.

Since the integration shows exponential dependence on the wavenumber roots and the thermal-viscous model wavenumber roots, equations 4.71a,b,c,d, are analogous to the damped wave wavenumber roots, equations 4.32a,b, and the diffusive wavenumber roots, 4.16a,b; analogous complex model wavenumbers may be defined. Replacing the longitudinal viscous damping coefficient in the complex damped wave wavenumber, equation 4.32, with the thermal viscous damping coefficient yields the complex model wave wavenumber,

$$k_{w_m} = ik_{1_m} = k'_{w_m} + ik''_{w_m} \tag{4.75a}$$

with the real and imaginary parts

$$k'_{w_m} = \frac{|\omega|}{\sqrt{2}\,\sqrt{C_1^4+D_1^2\omega^2}}\sqrt{-C_1^2+\sqrt{C_1^4+D_1^2\omega^2}} \qquad k''_{w_m} = \frac{|\omega|}{\sqrt{2}\,\sqrt{C_1^4+D_1^2\omega^2}}\sqrt{C_1^2+\sqrt{C_1^4+D_1^2\omega^2}} \quad . \tag{4.75b,c}$$

Similarly the complex model diffusion wavenumber is equal to the complex diffusion wavenumber, equation 4.20, with the thermal, rather than the transverse viscous, diffusion coefficient,

$$k_{d_m} = ik_{3_m} = k'_{d_m} + ik''_{d_m} \quad , \quad k'_{d_m} = \sqrt{\frac{\omega}{2D_2}} \quad , \quad k''_{d_m} = \sqrt{\frac{\omega}{2D_2}} \quad . \tag{4.76a,b,c}$$

The damped wave and diffusion aspects of the thermal-viscous model Green's functions in space and time domain may be displayed by expressing the Green's functions in terms of components of the complex model wave and diffusion wavenumbers,

$$G_{o_m}(\vec{r},\omega) = \frac{-i}{4\pi|r|\omega[C_1^2-i(D_1-D_2)\omega]}\left[e^{-k'_{w_m}|r|}e^{-ik''_{w_m}|r|} - e^{-k'_{d_m}|r|}e^{-ik''_{d_m}|r|}\right] \tag{4.77a}$$

$$G_{1_m}(\vec{r},\omega) = \frac{1}{4\pi|r|[C_1^2-i(D_1-D_2)\omega]}\left|\frac{-C_1^4\omega^2+iC_1^2(2D_1-D_2)\omega^3+D_1(D_1-D_2)\omega^4}{(C_1^4+D_1^2\omega^2)}\, e^{-k'_{w_m}|r|}\, e^{-ik''_{w_m}|r|} - \frac{C_1^2\omega^2+i(D_1-D_2)\omega^3}{D_2}\, e^{-k'_{d_m}|r|}\, e^{-ik''_{d_m}|r|}\right|$$

(4.77b)

$$G_{2_m}(\vec{r},\omega) = \frac{1}{4\pi|r|[C_1^2-i(D_1-D_2)\omega]}\left|\frac{iC_1^6\omega^3+C_1^4(3D_1-D_2)\omega^4-iC_1^2(3D_1-2D_2)\omega^5-D_1^2(D_1-D_2)\omega^6}{(C_1^4+D_1^2\omega^2)^2}\, e^{-k'_{w_m}|r|}\, e^{-ik''_{w_m}|r|} - \frac{C_1^2\omega^2+i(D_1-D_2)\omega^3}{D_2}\, e^{-k'_{d_m}|r|}\, e^{-ik''_{d_m}|r|}\right|$$

(4.77c)

$$G_{3_m}(\vec{r},\omega) = -i\omega\, G_{1_m}(\vec{r},\omega) \qquad (4.77d)$$

$$G_{4_m}(\vec{r},\omega) = i\omega\, G_{2_m}(\vec{r},\omega) \quad . \qquad (4.77e)$$

Each Green's function possesses a separate exponential dependences on the model wave wavenumber and the model diffusion wavenumber which are analogous to the exponential terms of the damped wave Green's functions, equations 4.35, and the diffusive Green's functions, equations 4.21, respectively. The wave type exponentials are more oscilatory as they decay in the radial direction than the diffusion type exponentials. A decrease in the frequency or the thermal-viscous damping coefficient tends to increase the oscillation in the radial direction and decrease the radial decay of the wave type exponentials. The diffusion type exponentials also penetrates farther in the radial direction as the frequency is decreased however they do so without changing oscilatory form and the same tendency is produced by an increase in the thermal diffusion coefficient.

The interaction of the wave type and diffusion type behaviors of the model Green's functions is displayed by their associated coefficients, D_1 and D_2, respectively. Coupling of the two effects is shown by the appearance of both diffusion coefficients in the coefficients of each exponential term and in the frequency pole,

$$\omega = \frac{C_1^2}{i(D_1-D_2)} \quad , \tag{4.78}$$

which is common to all Green's functions. This frequency pole in particular indicates that the relative magnitudes of the thermal-viscous damping and the thermal diffusion coefficients are important since the sign of the frequency pole depends on which coefficient is larger. The Green's functions in space and time domain are equal to integrations over all frequencies and therefore may be expected to reflect which diffusion coefficient is larger.

The model Green's functions in space and frequency also shown spherical spreading and the effect of the excitation modes as do the damped wave type and diffusive type Green's functions do.

Wavenumber and time domain

The thermal-viscous Green's functions in wavenumber and time domain are equal to the inverse temporal Fourier transformations of the Green's functions in wavenumber and time domain, equation 4.8, resulting in the integral forms

$$G_0(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{1}{i[(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)]} = \frac{1}{i2\pi} I_2^0 \tag{4.79a}$$

$$G_1(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{i\omega k^2}{i[(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)]} = \frac{k^2}{2\pi} I_2^1 \tag{4.79b}$$

$$G_2(\vec{k},t) = k^4\, G_0(\vec{k},t) \tag{4.79c}$$

$$G_3(\vec{k},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{\omega^2 k^2}{i[(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)]} = \frac{k^2}{i2\pi} I_2^2 \tag{4.79d}$$

$$G_4(\vec{k},t) = k^2\, G_1(\vec{k},t) \tag{4.79c}$$

Since all of the frequency poles possess positive imaginary parts the integration form, equation A-3.78, is a causal function of time,

$$I_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)} = \frac{iU(t)}{(\omega_1-\omega_3)(\omega_2-\omega_3)} \left\{ \frac{1}{(\omega_1-\omega_2)} [(\omega_1^n)(\omega_2-\omega_3)e^{i\omega_1 t} - -(\omega_2)^n(\omega_1-\omega_3)e^{i\omega_2 t}] + (\omega_3)^n e^{i\omega_3 t} \right\}. \tag{4.80}$$

Rather than to evaluate the thermal viscous Green's functions by substituting the complicated true frequency roots, equations 4.62, in the above integration form it is more useful to use the model frequency roots, equations 4.71e,f and g, and evaluate the thermal-viscous model Green's functions, which are simpler approximations of the true Green's functions.

The thermal-viscous model frequency roots, ω_{1_m} and ω_{2_m} of equations 4.71e and f, are analogous to the damped wave type frequency roots, equations 4.32c and d, and the model frequency root, ω_{3_m} of equation 4.71g, is analogous to the diffusion type frequency root, equation 4.16c. Therefore, real and imaginary parts of the complex frequency may be defined for the model frequency roots, as was done for the wave type and diffusion type frequency roots, which will display the exponential temporal decay and oscilation characteristics of the model Green's functions corresponding to a combination of wave and diffusion behavior. The first and second frequency roots are similar to the damped wave frequency roots and therefore display underdamped, critically damped and over-damped forms which should be examined separately. The third frequency root corresponds to the diffusion frequency root and may be defined in terms of the complex model diffusion frequency as was done for the diffusion frequency in equation 4.25a,

$$\omega_{d_m} = -i\omega_{3_m} = \omega'_{d_m} + i\omega''_{d_m} \quad . \tag{4.81a}$$

The real and imaginary parts of the complex model frequency are defined by replacing the transverse viscous diffusion coefficient of equations 4.25b and c, the components of the complex diffusion frequency, with the thermal diffusion coefficient, D_2, yielding

$$\omega'_{d_m} = D_2 k^2 \qquad \text{and} \qquad \omega''_{d_m} = 0 \quad . \tag{4.81b,c}$$

The complex model wave frequency components may be defined for each damping form by replacing the longitudinal viscous damping coefficient, D_L, of the corresponding complex damped worse components with the thermal-viscous damping ccefficient, D_1.

The temporally underdamped model Green's function forms occur when the thermal-viscous damping coefficient or wavenumber are small as shown by the inequality,

$$C_1^2 > \frac{D_1^2}{4} k^2 \quad . \tag{4.82}$$

The resulting form of the complex model wave frequencys real and imaginary parts are,

$$\omega'_{w_m} = \frac{D_1}{2} k^2 \qquad \omega''_{w_m} = \sqrt{\left|C_1^2 k^2 - \frac{D_1^2}{4} k^4\right|} \quad , \tag{4.84a,b}$$

which relate to the underdamped model wave frequency poles as

$$\omega_{1_m} = \omega''_{w_m} + i\omega'_{w_m} \qquad \omega_{2_m} = -\omega''_{w_m} + i\omega'_{w_m} \; , \tag{4.85a,b}$$

by analogy with the underdamped wave type equations 4.40 and 4.41. The resulting temporally underdamped model Green's functions display both underdamped wave and diffusion characteristics,

$$G_{0_m}(\vec{k},t) = \frac{U(t)}{[C_1^2 - D_2(D_1-D_2)k^2]} \left\{ e^{-\omega'_{w_m} t} \left[\frac{1}{k^2} \cos(\omega''_{w_m} t) + \left(\frac{D_1}{2} - D_2\right) \frac{\sin(\omega''_{w_m} t)}{\omega''_{w_m}}\right] - \frac{1}{k^2} e^{-\omega'_{d_m} t} \right\} \tag{4.86a}$$

$$G_{1_m}(\vec{k},t) = \frac{-U(t)}{[C_1^2 - D_2(D_1-D_2)k^2]} \left\{ e^{-\omega'_{w_m} t} \left[D_2 k^2 \cos(\omega''_{w_m} t) + \left(C_1^2 k^2 - \frac{D_1 D_2}{2} k^4\right) \frac{\sin(\omega''_{w_m} t)}{\omega''_{w_m}}\right] - D_2 k^2 e^{-\omega'_{d_m} t} \right\} \tag{4.86b}$$

$$G_{2_m}(\vec{k},t) = k^4 \, G_{0_m}(\vec{k},t) \tag{4.86c}$$

$$G_{3_m}(\vec{k},t) = \frac{U(t)}{[C_1^2 - D_2(D_1-D_2)k^2]} \left\{ e^{-\omega'_{w_m} t} \left[(C_1^2 k^2 - D_1 D_2 k^4) \cos(\omega''_{w_m} t) + \left(-C_1^2\left(\frac{D_1}{2} + D_2\right) k^4 + \frac{D_1 D_2}{2} k^6\right) \frac{\sin(\omega''_{w_m} t)}{\omega''_{w_m}}\right] + k^4 e^{-\omega'_{d_m} t} \right\} \tag{4.86d}$$

$$G_{4_m}(\vec{k},t) = k^2\, G_{1_m}(k,t) \quad . \tag{4.86e}$$

The complex model diffusion frequency possess only a real part, ω'_{d_m}, which is conspicuous in the Green's functions as governing the temporal decay of the diffusion type exponentials. The complex diffusion frequency, equation 4.81, shows that these terms decay, without temporal oscilations, more rapidly in time as the thermal diffusion coefficient or the wavenumber is increased. The complex model wave frequency, equations 4.84a and b, appropriate for underdamped Green's functions possesses both real and imaginary parts which govern the temporal decay and oscilations respectively of the wave type exponentials. As the thermal-viscous damping coefficient or the wavenumber are increased and the critical damping condition is approached the real part of the complex model wave frequency increases and the imaginary part decreases towards a value of zero. Correspondingly the wave type terms decay more rapidly and become less oscilatory in time.

The critically damped model Green's function forms are appropriate when the models critical damping condition,

$$C_1^2 = \frac{D_1^2}{4}\, k^2 \quad , \tag{4.87}$$

is satisfied. For this condition the imaginary part of the complex model wave frequency is equal to zero, although the real part is still represented by equation 4.84a, and the resulting coincident critically damped model wave frequency poles are

$$\omega_1 = \omega_2 = i\omega'_{w_m} \quad . \tag{4.88}$$

The complex model diffusion frequency is uneffected by the temporal damping condition of the wave frequency roots and therefore remains as defined in equation 4.81. As a result the critically damped model Green's functions in wavenumber and time domain display a critically damped wave form and an unaltered diffusion form,

$$G_{o_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{[\frac{1}{k^2}+(\frac{D_1}{2}-D_2)t]e^{-\omega'_{w_m}t} - \frac{1}{k^2}e^{-\omega'_{d_m}t}\right\} \quad (4.89a)$$

$$G_{1_m}(\vec{k},t) = \frac{-U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{[D_2k^2+(C_1^2k^2-\frac{D_1D_2}{2}k^4)t]\,e^{-\omega'_{w_m}t} - D_2k^2\,e^{-\omega'_{d_m}t}\right\} \quad (4.89b)$$

$$G_{2_m}(\vec{k},t) = k^4\,G_{o_m}(\vec{k},t) \quad (4.89c)$$

$$G_{3_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{[(C_1^2k^2-D_1D_2k^4)+(-C_1^2(\frac{D_1}{2}+D_2)k^4+\frac{D_1^2D_2}{2}k^6)t]e^{-\omega'_{w_m}t}+k^4e^{-\omega'_{d_m}t}\right\} \quad (4.89d)$$

$$G_{4_m}(\vec{k},t) = k^2\,G_{1_m}(\vec{k},t) \quad . \quad (4.89e)$$

Exponential temporal decay without oscilation is characteristic of the critically damped wave as well as the diffusion behavior.

The temporally overdamped model Green's function forms occur for larger thermal-viscous damping coefficient or wavenumber which satisfy the inequality,

$$C_1^2 < \frac{D_1^2}{4}k^2 \quad . \quad (4.90)$$

In this case the radical term of the model wave frequency roots, equations 4.71e and f, becomes imaginary resulting in another real part of the complex model wave frequency,

$$\omega w'_{w_{m_o}} = \sqrt{\left|\frac{D_1^2}{4}k^4 - C_1^2k^2\right|} \quad , \quad (4.91)$$

in addition to the one defined by equation 4.84a and the overdamped model wave frequency roots may be expressed as,

$$\omega_1 = i(\omega'_{w_m} + \omega'_{w_{m_o}}) \qquad \omega_2 = i(\omega'_{w_m} - \omega'_{w_{m_o}}) \quad . \tag{4.92a,b}$$

Two separate overdamped wave type decays terms result in addition to the diffusion type decay term in each of the temporally overdamped model Green's functions,

$$G_{o_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{\frac{1}{2}\left[\frac{1}{k^2}+\frac{(\frac{D_1}{2}-D_2)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}-\omega'_{w_{m_o}})t}+\frac{1}{2}\left[\frac{1}{k^2}-\frac{(\frac{D_1}{2}-D_2)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}+\omega'_{w_{m_o}})t} - \frac{1}{k^2}e^{-\omega_{d_m}t}\right\} \tag{4.93a}$$

$$G_{1_m}(\vec{k},t) = \frac{-U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{\frac{1}{2}\left[D_2k^2+\frac{(C_1^2k^2-\frac{D_1D_2}{2}k^4)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}-\omega'_{w_{m_o}})t} + \frac{1}{2}\left[D_2k^2-\frac{(C_1^2k^2-\frac{D_1D_2}{2}k^4)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}+\omega'_{w_{m_o}})t} - D_2k^2e^{-\omega'_{d_m}t}\right\} \tag{4.93b}$$

$$G_{2_m}(\vec{k},t) = k^4\, G_{o_m}(\vec{k},t) \tag{4.93c}$$

$$G_{3_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{\frac{1}{2}\left[(C_1^2k^2-D_1D_2k^4)+\frac{(-C_1^2(\frac{D_1}{2}+D_2)k^4+\frac{D_1^2D_2}{2}k^6)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}-\omega'_{w_{m_o}})t} + \frac{1}{2}\left[(C_1^2k^2-D_1D_2k^4)+\frac{(-C_1^2(\frac{D_1}{2}+D_2)k^4+\frac{D_1^2D_2}{2}k^6)}{\omega'_{w_{m_o}}}\right]e^{-(\omega'_{w_m}+\omega'_{w_{m_o}})t} + k^4\,e^{-\omega'_{d_m}t}\right\} \tag{4.93d}$$

$$G_{4_m}(\vec{k},t) = k^2 G_{1_m}(\vec{k},t) \quad . \tag{4.93e}$$

The first exponential wave term decays more slowly in time than the second because it depends on the difference of the real parts of the complex model wave frequency rather than the sum. As the thermal-viscous damping coefficient or wavenumber increase the first wave type exponential decays more slowly in time, by analogy with the inequalities of equation 4.50, and the second wave type exponential decays more rapidly.

In order to display the underdamped, critically damped and overdamped forms of the wave type terms the thermal-viscous model Green's functions in wavenumber and time domain have been derived for each of the temporal damping conditions. Since the damping condition changes with changing wavenumber and transformation of the model Green's functions to space and time requires integration over all real wavenumbers it would be advantageous to express each model Green's function in a functional form which is valid for all wavenumber. The model frequency roots, equations 4.71e, f, and g, are functionally valid for all wavenumbers and are shown in the underdamped form. The thermal-viscous model Green's functions in wavenumber and time domain in terms of the functional forms of the frequency roots are functionally valid for all wavenumbers,

$$G_{0_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{e^{-\frac{D_1}{2}k^2t}\left[\frac{1}{k^2}\cos\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right)+\left(\frac{D_1}{2}-D_2\right)\frac{\sin\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right)}{\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}}\right]-\frac{1}{k^2}e^{-D_2k^2t}\right\} \tag{4.94a}$$

$$G_{1_m}(\vec{k},t) = \frac{-U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{e^{-\frac{D_1}{2}k^2t}\left[D_2k^2\cos\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right) + \right.\right.$$

$$\left.\left. + \left(C_1^2k^2-\frac{D_1D_2}{2}k^4\right)\frac{\sin\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right)}{\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}}\right]-D_2k^2e^{-D_2k^2t}\right\}$$

(4.94b)

$$G_{2_m}(\vec{k},t) = k^4\, G_{o_m}(\vec{k},t) \qquad (4.94c)$$

$$G_{3_m}(\vec{k},t) = \frac{U(t)}{[C_1^2-D_2(D_1-D_2)k^2]}\left\{e^{-\frac{D_1}{2}k^2t}\left[(C_1^2k^2-D_1D_2k^4)\cos\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right) + \right.\right.$$

$$\left. + \left(-C_1^2\left(\frac{D_1}{2}+D_2\right)k^4+\frac{D_1^2D_2}{2}k^6\right)\frac{\sin\left(\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}\;t\right)}{\sqrt{C_1^2k^2-\frac{D_1^2}{4}k^4}}\right] +$$

$$\left. + k^4\, e^{-D_2k^2t}\right\} \qquad (4.94d)$$

$$G_{4_m}(\vec{k},t) = k^2\, G_{1_m}(\vec{k},t) \quad . \qquad (4.94e)$$

The wave related terms of these model Green's functions automatically take on the critically damped form when the radical term is equal to zero, by equation 4.52, and take on the overdamped form when the radical term becomes imaginary, by equations 4.53. As has been discussed, the magnitude of the thermal-viscous damping coefficient effects the damping form and temporal exponential decay of the wave type terms of the model Green's functions and the magnitude of the thermal diffusion coefficient effects the exponential decay of the diffusion type terms. Therefore changing the relative magnitudes of the coefficients changes the relative importance of the related effect. The temporal exponential decay occurs more slowly for the underdamped wave terms if the thermal-viscous damping coefficient is decreased and occurs most slowly for an overdamped wave term when the same diffusion coefficient is increased. Decreasing the thermal diffusion coefficient always causes the temporal exponential decay of the diffusion terms to occur more slowly.

The wave type and diffusion type effects are coupled by the occurence of both of the related coefficients in the coefficients of the exponential wave and diffusion terms and in the wavenumber poles of the model Green's functions given by,

$$k^2 = \frac{C_1^2}{D_2(D_1-D_2)} \,. \tag{4.95}$$

Since these wavenumber poles will be imaginary if the thermal diffusion coefficient is larger than the thermal-viscous damping coefficient and will be real otherwise the relative magnitudes of the diffusion coefficients will have a basic effect on the model Green's functions in space and time domain which are inverse spatial Fourier transformations of the wavenumber and time domain representations. A similar effect was suggested by the frequency pole, equation 4.78, of the space and frequency domain model Green's functions.

Space and time domain

The thermal-viscous Green's functions in space and time domain are equal, by equation 4.10, to either the inverse temporal Fourier transformation of the Green's functions in space and frequency domain or the inverse spatial Fourier transformation of the Green's functions in wavenumber and time domain. These Green's functions to be transformed are expressed in integral form by equations 4.73 and 4.79. The integral forms may be evaluated, as was done for the thermal-viscous model Green's functions, by using the true wavenumber or frequency roots, equations 4.60 or 4.62. Because of the complexity of these roots the resulting thermal-viscous Green's functions are impossible to analytically integrate to space and time domain. These thermal-viscous Green's functions are unnecessarily complicated, considering that the entire analysis is approximate, and therefore, the thermal-viscous model Green's functions will be substituted. The inverse temporal transformations of the model Green's functions in space and frequency domain, equations 4.77 are also difficult to perform because the form of the complex model wave and model diffusion wavenumbers, equations 4.75 and 4.76, are complicated and introduce branch points. However, the inverse spatial Fourier transformations of the thermal-viscous model Green's functions in wavenumber and time domain may be evaluated analytically.

The thermal-viscous model Green's functions in space and time domain are euqal to the inverse spatial Fourier transformations of the wavenumber and time domain representations, equations 4.94. Since these Green's functions are even functions of the wavenumber magnitude the transformations may be simplified to the form of equation 4.11 and the transformed Green's functions may be expressed in the integral forms,

$$G_{o_m}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\,\frac{-U(t)}{D_2(D_1-D_2)(k^2-k_5^2)}\left\{e^{-\delta_1 k^2}\left[\frac{1}{k^2}\cos(\delta_1 k\sqrt{\rho_3^2-k^2})+\right.\right.$$

$$\left.\left.+\left(\frac{D_1}{2}-D_2\right)t\,\frac{\sin(\delta_1 k\sqrt{\rho_3^2-k^2})}{\delta_1 k\sqrt{\rho_3^2-k^2}}\right]-\frac{1}{k^2}e^{-\delta_2 k^2}\right\} =$$

$$= \frac{-U(t)}{i4\pi^2 rD_2(D_1-D_2)}\left\{[I_4^{(-1)}]^{\delta=\delta_1}+\left(\frac{D_1}{2}-D_2\right)t\,[I_5^0]^{\delta=\delta_1}-[I_3^{(-1)}]^{\delta=\delta_2}\right\} \qquad (4.96a)$$

$$G_{1_m}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\,\frac{U(t)}{D_2(D_1-D_2)(k^2-k_5^2)}\left\{e^{-\delta_1 k^2}\left[D_2k^2\cos(\delta_1 k\sqrt{\rho_3^2-k^2})+\right.\right.$$

$$\left.\left.+\left(C_1^2k^2-\frac{D_1D_2}{2}k^4\right)t\,\frac{\sin(\delta_1 k\sqrt{\rho_3^2-k^2})}{\delta_1 k\sqrt{\rho_3^2-k^2}}\right]-D_2k^2\,e^{-\delta_2 k^2}\right\} =$$

$$= \frac{U(t)}{i4\pi^2 rD_2(D_1-D_2)}\left\{D_2[I_4^{(1)}]^{\delta=\delta_1}+C_1^2t[I_5^{(1)}]^{\delta=\delta_1}-\frac{D_1D_2}{2}t[I_5^{(2)}]^{\delta=\delta_1}-D_2[I_3^{(1)}]^{\delta=\delta_1}\right\}$$

(4.96b)

$$G_{2_m}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}[k^4G_{o_m}(\vec{k},t)] = \frac{-U(t)}{ir\pi^2 rD_2(D_1-D_2)}\left\{[I_4^{(1)}]^{\delta=\delta_1}+\left(\frac{D_1}{2}-D_2\right)t[I_5^{(2)}]^{\delta=\delta_1}-\right.$$

$$\left.-[I_3^{(0)}]^{\delta=\delta_2}\right\} \qquad (4.96c)$$

$$G_{3_m}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\,\frac{-U(t)}{D_2(D_1-D_2)(k^2-k_5^2)}\left\{e^{-\delta_1 k^2}[(C_1^2 k^2 - D_1 D_2 k^4)\cos(\delta_1 k\sqrt{\rho_3^2-k^2}) + \right.$$

$$\left.(-C_1^2(\frac{D_1}{2}+D_2)k^4 + \frac{D_1^2 D_2}{2}k^6)t\,\frac{\sin(\delta_1 k\sqrt{\rho_3^2-k^2})}{\delta_1 k\sqrt{\rho_3^2-k^2}}\,] + k^4\, e^{-\delta_2 k^2}\right\} =$$

$$= \frac{-U(t)}{i4\pi^2 rD_2(D_1-D_2)}\left\{C_1^2[I_4^{(1)}]^{\delta=\delta_1} - D_1 D_2[I_4^{(2)}]^{\delta=\delta_1} - C_1^2(\frac{D_1}{2}+D_2)[I_5^{(2)}]^{\delta=\delta_1} + \frac{D_1^2 D_2}{2}[I_5^{(3)}]^{\delta=\delta_1}\right.$$

$$\left. + [I_3^{(2)}]^{\delta=\delta_2}\right\} \quad (4.96d)$$

$$G_{4_m}(\vec{r},t) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}[k^2 G_{1_m}(\vec{k},t)] =$$

$$= \frac{U(t)}{i4\pi^2 rD_2(D_1-D_2)}\left\{D_2[I_4^{(2)}]^{\delta=\delta_1} + C_1^2 t[I_5^{(2)}]^{\delta=\delta_1} - \frac{D_1 D_2}{2}t[I_5^{(3)}]^{\delta=\delta_1} - D_2[I_3^{(2)}]^{\delta=\delta_2}\right\}$$

(4.96e)

where $\delta_1 = \frac{D_1}{2}t$, $\delta_2 = D_2 t$ and $\rho_3 = \frac{2C_1}{D_1}$. (4.97a, b,c)

In order to perform the required integrations the wavenumber poles of the Green's functions in wavenumber and time domain, which are solutions to equation 4.95

$$k_5 = -k_6 = \frac{C_1}{\sqrt{D_2(D_1-D_2)}} , \quad (4.98)$$

must be examined. The poles may be purely real or purely imaginary depending upon the relative magnitudes of the thermal or thermal-viscous coefficients. For a larger thermal diffusion coefficient, the thermal damping condition is

$$D_2 > D_1 \quad , \tag{4.99a}$$

and the wavenumber poles are imaginary,

$$k_5 = -k_6 = i\,\rho_2 \quad , \tag{4.99b}$$

where
$$\rho_2 = \frac{C_1}{\sqrt{D_2(D_2-D_1)}} \quad . \tag{4.99c}$$

The other condition of predominant thermal-viscous damping,

$$D_1 > D_2 \quad , \tag{4.100a}$$

results in the real wavenumber poles

$$k_5 = -k_6 = \rho_1 \quad , \tag{4.100b}$$

where
$$\rho_1 = \frac{C_1}{\sqrt{D_2(D_1-D_2)}} \ . \tag{4.100c}$$

Not only are the complex forms of the wavenumber poles for each type of damping dominance different, but the transition from one form to the other occurs discontinuously. As the thermal diffusion coefficient approaches the thermal-viscous coefficient from a larger value, the thermal damping wavenumber poles, equations 4.99, move along the imaginary wavenumber axis towards plus and minus infinity. In contrast, the thermal-viscous damping wavenumber poles, equations 4.100, move along the real wavenumber axis towards plus and minus infinity when the thermal approaches the thermal-viscous damping coefficient from a smaller value. These discontinuously changing wavenumber poles necessitate separate evolution of the required integrals for each type of damping dominance.

For the condition of larger thermal than thermal-viscous coefficient the integration forms have been evaluated by a transform technique and complex contour integrations in Appendix A-5. The evaluations for non-negative values of n, no zero wavenumber poles, are given by equation A-5.66, 74b and 77b.

$$ {}_2I_3^{(n)} = \int_{-\infty}^{\infty} dk\, e^{ikr} \frac{k^{(2n+1)} e^{-\delta k^2}}{(k-i\rho_2)(k+i\rho_2)} = \frac{i\pi}{2} (i\rho_2)^{2n} e^{\delta\rho_2^2} \Big[e^{-\rho_2 r} \,\mathrm{erfc}\Big(\frac{r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big) - e^{\rho_2 r} \,\mathrm{erfc}\Big(\frac{-r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big)\Big] \qquad (4.101a) $$

$$ {}_2I_4^{(n)} = \int_{-\infty}^{\infty} dk\, e^{ikr} \frac{k^{(2n+1)} e^{-\delta k^2} \cos(\delta k\sqrt{\rho_3^2-k^2})}{(k-i\rho_2)(k+i\rho_2)} = \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\, {}_2I_3^{(n)} \qquad (4.101b) $$

$$ {}_2I_5^{(n)} = \int_{-\infty}^{\infty} dk\, e^{ikr} \frac{k^{(2n+1)} e^{-\delta k^2} \sin(\delta k\sqrt{\rho_3^2-k^2})}{(k-i\rho_2)(k+i\rho_2)\delta k\sqrt{\rho_3^2-k^2}} = \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}}\, {}_2I_3^{(n)} \qquad (4.101c) $$

and the evaluations of the integrals with simple wavenumber poles at zero, n = -1, are given by equations A-5.70b and 75b

$$ {}_2I_3^{(-1)} = \int_{-\infty}^{\infty} dk\, e^{ikr} \frac{e^{-\delta k^2}}{k(k-i\rho_2)(k+i\rho_2)} = \frac{i\pi}{2} \frac{1}{(i\rho_2)^2} \Big\{ e^{\delta\rho_2^2} \Big[e^{-\rho_2 r} \,\mathrm{erfc}\Big(\frac{r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big) - e^{\rho_2 r} \,\mathrm{erfc}\Big(\frac{-r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big)\Big] - 2\,\mathrm{erfc}\Big(\frac{r}{2\sqrt{\delta}}\Big) \Big\} \qquad (4.101d) $$

$$ {}_2I_4^{(-1)} = \int_{-\infty}^{\infty} dk\, e^{ikr} \frac{e^{-\delta k^2} \cos(\delta k\sqrt{\rho_3^2-k^2})}{k(k-i\rho_2)(k+i\rho_2)} = \frac{i\pi}{2} \frac{1}{(i\rho_2)^2} \Big\{ \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\, e^{\delta\rho_2^2} \Big[e^{-\rho_2 r} \,\mathrm{erfc}\Big(\frac{r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big) - e^{\rho_2 r} \,\mathrm{erfc}\Big(\frac{-r}{2\sqrt{\delta}} + \sqrt{\delta}\,\rho_2\Big)\Big] - 2\,\mathrm{erf}\Big(\frac{r}{2\sqrt{\delta}}\Big) \Big\} \qquad (4.101e) $$

where the Gaussian error function and complimentary error function are defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z dx e^{-x^2} \qquad \text{and} \qquad \mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty dx e^{-x^2} \quad . \tag{4.102a,b}$$

When the thermal-viscous damping coefficient is larger than thermal diffusion coefficient the wavenumber poles, k_5 and k_6, are real and the integration forms, as evaluated in Appendix A-5.86, 91b and 94b, for non-negative n, are

$$ {}_1I_3^{(n)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{k^{(2n+1)}e^{-\delta k^2}}{(k-\rho_1)(k+\rho_1)} = \frac{i\pi}{2}\rho_1^{2n} e^{-\delta\rho_1^2}[e^{i\rho_1 r}\mathrm{erf}(\frac{r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1) - $$

$$ - e^{-i\rho_1 r}\mathrm{erf}(\frac{-r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1)] \tag{4.103a}$$

$$ {}_1I_4^{(n)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{k^{(2n+1)}e^{-\delta k^2}\cos(\delta k\sqrt{\rho_3^2-k^2})}{(k-\rho_1)(k+\rho_1)} = \cosh(\delta_1\rho_1\sqrt{\rho_1^2-\rho_3^2})\ {}_1I_3^{(n)} \tag{4.103b}$$

$$ {}_1I_5^{(n)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{k^{(2n+1)}e^{-\delta k^2}\sin(\delta k\sqrt{\rho_3^2-k^2})}{(k-\rho_1)(k+\rho_1)\delta k\sqrt{\rho_3^2-k^2}} = \frac{\sinh(\delta_1\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta_1\rho_1\sqrt{\rho_1^2-\rho_3^2}}\ {}_1I_3^{(n)} \tag{4.103c}$$

and for equations A-5.89a and 92b for simple wavenumber poles at zero, n = -1, are

$$ {}_1I_3^{(-1)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}}{k(k-\rho_1)(k+\rho_1)} = \frac{i\pi}{2}\frac{1}{\rho_1^2}\Big\{e^{-\delta\rho_1^2}[e^{i\rho_1 r}\mathrm{erf}(\frac{r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1) - $$

$$ -e^{-i\rho_1 r}\mathrm{erf}(\frac{-r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1)] - 2\,\mathrm{erf}(\frac{r}{2\sqrt{\delta}})\Big\} \tag{4.103d}$$

$$ {}_1I_4^{(-1)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}\cos(\delta k\sqrt{\rho_3^2-k^2})}{k(k-\rho_1)(k+\rho_1)} = \frac{i\pi}{2}\frac{1}{\rho_1^2}\Big\{\cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})e^{-\delta\rho_1^2}[e^{i\rho_1 r}\mathrm{erf}(\frac{r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1) $$

$$ -e^{-i\rho_1 r}\mathrm{erf}(\frac{-r}{2\sqrt{\delta}} + i\sqrt{\delta}\,\rho_1)] - 2\,\mathrm{erf}(\frac{r}{2\sqrt{\delta}})\Big\} \quad . \tag{4.103e}$$

The forms of the thermal-viscous model Green's functions in space and time domain depend on the relative magnitudes of the thermal and the thermal-viscous coefficients as do the forms of the transformation integrals.

For the larger thermal than thermal-viscous coefficient the model Green's functions, equations 4.96, may be evaluated by utilizing the proper integral evaluations, equations 4.101, as

$$G_{o_m}(\vec{r},t) = \frac{-U(t)}{8\pi r C_1^2} \left\{ \begin{array}{l} e^{\frac{\pm(2D_1t-4D_2t)+2D_1t}{r_C^2}} \left[e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{-r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C}\right) - e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C}\right)\right] - \\ \quad - e^{\frac{4D_2t}{r_C^2}} \left[e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{-r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C}\right) - e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C}\right)\right] - \\ \quad -2 \left[\operatorname{erf}\left(\frac{r}{\sqrt{2D_1t}}\right) - \operatorname{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)\right] \end{array} \right\} \tag{4.104a}$$

$$G_{1_m}(\vec{r},t) = \frac{C_1^2 U(t)}{8\pi r D_2^2 (D_1-D_2)^2} \left\{ \begin{array}{l} e^{\frac{\pm(2D_1t-4D_2t)+2D_1t}{r_C^2}} \left[e^{\frac{2r}{r_C}} \operatorname{efc}\left(\frac{-r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C}\right) - e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C}\right)\right] - \\ \quad - e^{\frac{4D_2t}{r_C^2}} \left[e^{\frac{2r}{r_C}} \operatorname{efc}\left(\frac{-r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C}\right) - e^{\frac{2r}{r_C}} \operatorname{erfc}\left(\frac{r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C}\right)\right] \end{array} \right\} \tag{4.104b}$$

$$G_{2_m}(\vec{r},t) = \frac{1}{D_2} G_{1_m}(\vec{r},t) \tag{4.104c}$$

$$G_{3_m}(\vec{r},t) = \frac{-C_1^2}{(D_1-D_2)} G_{1_m}(\vec{r},t) \tag{4.104d}$$

$$G_{4_m}(\vec{r},t) = \frac{-C_1^2}{D_2(D_1-D_2)} \; G_{1_m}(\vec{r},t) \quad . \tag{4.104e}$$

The characteristic radius of the diffusion and wave interaction, defined as

$$r_c = \frac{\sqrt{4D_2|D_1-D_2|}}{C_1} \quad , \tag{4.105}$$

is a result of the wavenumber poles, equation 4.98, of the wavenumber and time domain Green's functions. Due to an indeterminate sign of the argument of the hyperbolic functions in the integral evaluations, equations 4.101b and 4.101c, the sign of the exponential of the Green's function term,

$$e^{\pm\frac{(2D_1t-4D_2t)}{r_c^2}} \quad , \tag{4.106}$$

is undefined. However, the Green's functions represent relaxations of fluctuations of which all wavenumbers decay in time, as shown by the wavenumber and time domain Green's function equations 4.86, 4.89 and 4.93, and of which all frequencies decay in space, as shown by the Green's functions in space and frequency domain in equations 4.77. Therefore in order to be physically realistic, the model Green's functions in space and time domain should decay to zero as time or radial distance becomes infinite. The temporal infinite boundary condition, the Green's function must approach zero in the limit as time approaches infinity, may be applied to determine the sign of the exponential term of equation 4.106. The terms of the Green's functions which include the exponential of undefined sign are of the temporal form,

$$e^{At} \, \mathrm{erfc}\left(\frac{B}{\sqrt{t}} + \sqrt{Ct}\right) \quad , \tag{4.107a}$$

which satisfies the temporal infinite boundary condition if the constants satisfy the inequality,

$$(A-C) \leq 0 \quad . \tag{4.107b}$$

With the constants appropriate for the undetermined Green's function term the inequality reduces to

$$\pm (D_1 - 2D_2) \leq 0 \tag{4.107c}$$

which is satisfied by the negative sign only since the Green's functions in question are appropriate larger thermal diffusion coefficient than thermal-viscous damping coefficient. Thus by physical reasoning the sign of the exponential term, equation 4.106, of the model Green's functions is determined to be negative. The other terms of the model Green's functions which are of the form of equation 4.106 satisfy the temporal boundary condition inequality, equation 4.107b, as $0 \leq 0$. The remaining time function terms are of the form,

$$\operatorname{erf}\left(\frac{B}{\sqrt{t}}\right) \quad , \tag{4.108}$$

which equals zero in the limit as time approaches infinity since the error function of zero arguement is equal to zero. The spatial infinite boundary condition, that the Green's functions must be equal to zero in the limit as the radial distance approaches infinity, should also be satisfied. The model Green's function terms of the spatial form,

$$\frac{1}{r} e^{Ar} \operatorname{erfc}(Br + C) \quad , \tag{4.109a}$$

satisfy the spatial infinite boundary condition unless

$$A > 0 \qquad \text{and} \qquad B > 0 \tag{4.109b,c}$$

which is not the case. The remaining spatial terms are simultaneously, of the form,

$$\frac{1}{r}\,\mathrm{erf}(Br) \quad , \tag{4.110}$$

which also satisfies the spatial infinite boundary condition since the error function of a real argument is bounded by plus or minus unity. Thus these thermal-viscous model Green's functions in space and time domain for larger thermal than thermal-viscous coefficient satisfy the temporal, with the proper sign, and spatial infinite boundary conditions.

For the condition of a thermal-viscous damping coefficient which is larger than the thermal diffusion coefficient, the integral evaluations shown by equations 4.103 may be used to evaluate the model Green's functions

$$G_{o_m}(\vec{r},t)=\frac{U(t)}{8\pi r C_1^2}\left\{\begin{aligned} & e^{\frac{\pm(2D_1t-4D_2t)-2D_1t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ & -\,e^{\frac{-4D_2t}{r_C^2}}\left[e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right]- \\ & -2\left[\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}\right)-\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)\right] \end{aligned}\right\} \tag{4.111a}$$

$$G_{1_m}(\vec{r},t)=\frac{C_1^2U(t)}{8\pi r D_2^2(D_1-D_2)^2}\left\{\begin{aligned} & e^{\frac{\pm(2D_1t-4D_2t)-2D_1t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ & -\,e^{\frac{-4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right] \end{aligned}\right\} \tag{4.111b}$$

$$G_{2_m}(\vec{r},t) = \frac{-1}{D_2} G_{1_m}(\vec{r},t) \qquad (4.111c)$$

$$G_{3_m}(\vec{r},t) = \frac{C_1^2}{(D_1-D_2)} G_{1_m}(\vec{r},t) \qquad (4.111d)$$

$$G_{4_m}(\vec{r},t) = \frac{C_1^2}{D_2(D_1-D_2)} G_{1_m}(\vec{r},t) \qquad (4.111e)$$

The model Green's functions for the condition of larger thermal-viscous damping coefficient also contain terms of undetermined exponent sign, equation 4.106, which were exhibited by the Green's functions for the condition of larger thermal diffusion coefficients. Again, the sign may be determined by the temporal infinite boundary condition. The terms which contain the exponent of undefined sign are of the temporal form,

$$e^{At} \operatorname{erf}\left(\frac{B}{\sqrt{t}} + i\sqrt{Ct}\right) . \qquad (4.112a)$$

This temporal form satisfies the temporal infinite boundary condition if the constants satisfy the inequality,

$$(A + C) \leq 0 \quad , \qquad (4.112b)$$

which for the Green's function term in question is

$$\pm (D_1 - 2D_2) \leq 0 \quad . \qquad (4.112c)$$

This inequality is satisfied by the positive sign when the thermal-viscous diffusion coefficient is less than or equal to twice the thermal diffusion coefficient and is satisfied by the negative sign when the thermal-viscous damping coefficient is greater than or equal to twice the thermal diffusion coefficient. Thus the sign of the exponential is determined to be different depending on the relative magnitudes of the thermal-viscous damping coefficient and twice the thermal diffusion coefficient and therefore the model Green's function are required to be expressed separately for each of these conditions.

The other Green's function terms which are of the form of equation 4.112a satisfy the inequality equation 4.112b since $0 \leq 0$ and the remaining Green's function terms are of the form of equation 4.108 which also is equal to zero for infinite time.

The model Green's functions should also satisfy the spatial infinite boundary condition that they be equal to zero at infinite radial distance. This condition is satisfied for the terms of the form

$$\frac{1}{r} e^{iAr} \operatorname{erf}(Br + iC) \tag{4.113}$$

since both the exponential of imaginary exponent and the error function of complex argument with larger real than imaginary parts are bounded. The remaining terms are of the form of equation 4.110 which also satisfies the spatial infinite boundary condition. Thus these model Green's functions for larger thermal-viscous than thermal coefficient satisfy the spatial infinite boundary condition and the temporal infinite boundary condition with the appropriate sign depending on the relative magnitudes of the thermal-viscous damping coefficient and twice the thermal diffusion coefficient.

Due to the inverse transformation forms and the temporal infinite boundary conditions, the forms of the thermal-viscous model Green's functions in space and time domain are different for each of these parameter regions depending on the relative magnitudes of the thermal, D_2, and thermal-viscous, D_1, coefficients. The parameter regions and corresponding Green's function forms are:

$D_2 > D_1$ (4.114a)

$$G_{o_m}(\vec{r},t) = \frac{-U(t)}{8\pi r C_1^2}\left\{ \begin{array}{l} e^{\frac{4D_2 t}{r_C^2}}\left[e^{\frac{2r}{r_C}}\,\mathrm{erfc}\left(\frac{-r}{\sqrt{2D_1 t}} + \frac{\sqrt{2D_1 t}}{r_C}\right) - e^{\frac{2r}{r_C}}\,\mathrm{erfc}\left(\frac{r}{\sqrt{2D_1 t}} + \frac{\sqrt{2D_1 t}}{r_C}\right)\right] - \\ -e^{\frac{4D_2 t}{r_C^2}}\left[e^{\frac{2r}{r_C}}\,\mathrm{erfc}\left(\frac{-r}{\sqrt{4D_2 t}} + \frac{\sqrt{4D_2 t}}{r_C}\right) - e^{\frac{2r}{r_C}}\,\mathrm{erfc}\left(\frac{r}{\sqrt{4D_2 t}} + \frac{\sqrt{4D_2 t}}{r_C}\right)\right] - \\ -2\left[\mathrm{erf}\left(\frac{r}{\sqrt{2D_1 t}}\right) - \mathrm{erf}\left(\frac{r}{\sqrt{4D_2 t}}\right)\right] \end{array} \right\} \quad (4.114b)$$

$$G_{1_m}(\vec{r},t) = \frac{C_1^2 U(t)}{8\pi r D_2^2 (D_1 - D_2)^2}\left\{ \begin{array}{l} e^{\frac{4D_2 t}{r_C^2}}\left[e^{\frac{2r}{r_C}}\mathrm{erfc}\left(\frac{-r}{\sqrt{2D_1 t}} + \frac{\sqrt{2D_1 t}}{r_C}\right) - e^{\frac{2r}{r_C}}\mathrm{erfc}\left(\frac{r}{\sqrt{2D_1 t}} + \frac{\sqrt{2D_1 t}}{r_C}\right)\right] - \\ -e^{\frac{4D_2 t}{r_C^2}}\left[e^{\frac{2r}{r_C}}\mathrm{erfc}\left(\frac{-r}{\sqrt{4D_2 t}} + \frac{\sqrt{4D_2 t}}{r_C}\right) - e^{\frac{2r}{r_C}}\mathrm{erfc}\left(\frac{r}{\sqrt{4D_2 t}} + \frac{\sqrt{4D_2 t}}{r_C}\right)\right] \end{array} \right\} \quad (4.114c)$$

$$G_{2_m}(\vec{r},t) = \frac{1}{D_2} G_{1_m}(\vec{r},t) \;,\quad G_{3_m}(\vec{r},t) = \frac{-C_1^2}{(D_1 - D_2)} G_{1_m}(\vec{r},t) \;,\quad G_{4_m}(\vec{r},t) = \frac{-C_1^2}{D_2(D_1 - D_2)} G_{1_m}(\vec{r},t) \quad (4.114d,e,f)$$

$2D_2 \geq D_1 > D_2$ (4.115a)

$$G_{o_m}(\vec{r},t)=\frac{U(t)}{8\pi r C_1^2}\left\{\begin{array}{l} e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ -e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right]- \\ \quad -2\left[\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}\right)-\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)\right] \end{array}\right\} \quad (4.115b)$$

$$G_{1_m}(\vec{r},t)=\frac{C_1^2U(t)}{8\pi rD_2^2(D_1-D_2)^2}\left\{\begin{array}{l} e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ -e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right] \end{array}\right\} \quad (4.115c)$$

$$G_{2_m}(\vec{r},t)=\frac{-1}{D_1}\,G_{1_m}(\vec{r},t)\quad,\quad G_{3_m}(\vec{r},t)=\frac{C_1^2}{(D_1-D_2)}\,G_{1_m}(\vec{r},t)\quad,\quad G_{4_m}(\vec{r},t)=\frac{C_1^2}{D_2(D_1-D_2)}\,G_{1_m}(\vec{r},t) \quad (4.115d,e,f)$$

$$D_1 > 2D_2 \quad (4.116a)$$

$$G_{o_m}(\vec{r},t)=\frac{U(t)}{8\pi r C_1^2}\left\{\begin{array}{l} e^{\frac{C_1^2 t}{D_2}}\left[e^{-i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ -e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right]- \\ -2\left[\operatorname{erf}\left(\frac{r}{\sqrt{2D_1t}}\right)-\operatorname{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)\right] \end{array}\right\} \quad (4.116b)$$

$$G_{1_m}(\vec{r},t)=\frac{C_1^2U(t)}{8\pi r D_2^2(D_1-D_2)^2}\left\{\begin{array}{l} e^{\frac{C_1^2 t}{D_2}}\left[e^{-i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\,\frac{\sqrt{2D_1t}}{r_C}\right)\right]- \\ -e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\operatorname{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\,\frac{\sqrt{4D_2t}}{r_C}\right)\right] \end{array}\right\} \quad (4.116c)$$

$$G_{2_m}(\vec{r},t)=\frac{-1}{D_2}\,G_{1_m}(\vec{r},t) \quad , \quad G_{3_m}(\vec{r},t)=\frac{C_1^2}{(D_1-D_2)}\,G_{1_m}(\vec{r},t) \quad , \quad G_{4_m}(\vec{r},t)=\frac{C_1^2}{D_2(D_1-D_2)}\,G_{1_m}(\vec{r},t) \quad (4.116d,e,f)$$

The five basic forms of the thermal-viscous model Green's functions in wavenumber and frequency domain result in only two space and time functional forms, one corresponding to the impulse Green's function and one representing all of the initial condition Green's functions. The impulse Green's function, equations 4.114b, 4.115b, and 4.116b, contains the initial condition Green's function forms, shown by equations

4.114c, 4.115c and 4.116c, terms which result from the zero wavenumber poles of the wavenumber and time domain impulse Green's functions. These additional terms represent the fluctuations relaxation from the zero wavenumber, infinite wavelength component of the impulsive excitation. The modes of excitation resulting in the initial condition Green's functions excite all wavenumbers and frequencies differently, and do not excite the zero wavenumber, as is evident from equations 4.56, 4.57, and 4.58.

The different Green's functions are of different wavenumber and frequency distribution. The Green's functions in space and frequency domain show that spatial decay and oscilation behavior is a function of frequency and the wavenumber and time domain representations show that the temporal behavior is different for different wavenumbers. However all Green's functions in space and time domain are of the same form, with the exception of the terms in the impulse Green's functions which result from zero wavenumber.

All of the model Green's functions are real even functions of the radial distance and are nonzero for positive time only. The inverse radial distance dependence common to all terms is a result of spherical spreading. Exponential time and radial distance functions multiply Gaussian error functions with arguements that are mixed functions of time and distance producing terms representing complicated spatial and temporal behavior. The reoccuring time dependencies,

$$\sqrt{2D_1 t} \quad \text{and} \quad \sqrt{4D_2 t} \quad , \tag{4.117a,b}$$

represent the radii characteristic of the thermal-viscous and the thermal effects respectively and are of the form of the diffusion time dependence, equation 4.29. Error function terms which are functions of the thermal-viscous characteristic radius derived from the wave related exponential terms in wavenumber and time domain. Similar error function terms which depend on the thermal characteristic radius result from the inverse transformation of the diffusion type exponentials.

Acoustic Tensorial Type Green's Functions

Wavenumber and frequency domain

The field equation which governs the acoustic fluctuations of vectorial velocity, equation 2.17, includes the diffusion type effect of the transverse velocity and the damped wave type effect of the longitudinal velocity. These effects as well as the interaction between the transverse and longitudinal velocity are displayed by the tensorial impulse Green's function, equation 3.25, in wavenumber and frequency domain,

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = \frac{-1}{i\omega[i\omega+D_T k^2]}\left(\delta_{jm} - \frac{k_j k_m}{k^2}\right) + \frac{1}{[\omega^2 - i\omega D_L k^2 - C_1^2 k^2]}\left(\frac{k_j k_m}{k^2}\right) \quad . \tag{4.118}$$

The scalar transverse Green's function corresponds to a viscous diffusion type impulse Green's function, equation 4.15, complicated by an additional zero frequency pole,

$$G^T(\vec{k},\omega) = \frac{-1}{iw}\, G^{vT}(\vec{k},\omega) \quad , \tag{4.119a}$$

and possesses the non zero wavenumber and frequency poles of equations 4.16. The scalar longitudinal Green's function is exactly the damped wave type impulse Greens function of equation 4.31a,

$$G^L(\vec{k},\omega) = G^{\rho}(\vec{k},\omega) \quad , \tag{4.119b}$$

and therefore possesses the wavenumber and frequency poles of equations 4.32. Due to the transverse and longitudinal polarization tensors, equations 4.27, additional zero wavenumber poles are introduced to both the transverse and longitudinal scalar Green's functions.

Space and frequency domain

The tensorial acoustic velocity Green's function in space and frequency domain is equal to the inverse spatial Fourier transformation of the wavenumber and frequency domain representation. Expressing the directional wavenumbers in terms of directional spatial derivatives results in spherically symmetric integrands which allows the inverse transformation to be simplified to the form of equation 4.7. In terms of the diffusion and damped wave type Green's function and the resulting integral forms, the tensorial Green's function becomes

$$G^{\vec{v}}_{jm}(\vec{r},\omega) = \delta_{jm}\frac{1}{i4\pi^2 r}\left(\frac{-1}{i\omega}\right)\int_{-\infty}^{\infty} dk k e^{ikr}\, G^{\vec{v}T}(\vec{k},\omega) + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{1}{i4\pi^2 r}\left(\frac{-1}{i\omega}\right)\int_{-\infty}^{\infty} dk k e^{ikr}\,\frac{G^{vT}(\vec{k},\omega)}{k^2} -$$

$$-\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk k e^{ikr}\,\frac{G^{p}(\vec{k},\omega)}{k^2}$$

$$= \delta_{jm}\frac{1}{4\pi^2 r\omega}\, {}_dI_1^{(0)} + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{1}{4\pi^2 r\omega}\, {}_dI_1^{(-1)} + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{1}{4\pi^2 r(C_1^2+i\omega D_L)}\, {}_dI_1^{(-1)} \,, \tag{4.120}$$

which involves the integral form evaluation shown by equation 4.18 and the integral which includes zero wavenumber poles,

$${}_dI_1^{(-1)} = \int_{-\infty}^{\infty} dk\,\frac{e^{ikr}}{k(k-k_1)(k+k_1)} = \frac{r}{|r|}\frac{\pi i}{(k_1)^2}\left[e^{-ik_1|r|} - 1\right] \,, \tag{4.121}$$

as evaluated in Appendix A-2, equation A-2.14a.

Utilizing the complex diffusion and damped wave wavenumbers defined by equations 4.20 and 4.32, the tensorial acoustic velocity Green's functions may be evaluated in space and frequency domain as,

$$G_{jm}^{\vec{v}}(\vec{r},\omega) = \delta_{jm} \frac{i}{4\pi|r|\omega D_T} e^{-k_d'|r|} e^{-ik_d''|r|} + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{-1}{4\pi|r|\omega^2}[e^{-k_d'|r|} e^{-ik_d''|r|} -1] +$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{-1}{4\pi|r|\omega^2(C_l^2+i\omega D_L)}[e^{-k_w'|r|} e^{-ik_w''|r|} - 1] \tag{4.122}$$

Wavenumber and time domain

The tensorial acoustic velocity Green's functions in wavenumber and time domain may be represented in integral form as the inverse temporal Fourier transformation of the wavenumber and frequency domain representations. As shown by equation 4.9, the polarization tensors are functions of wavenumber only and are independent of the temporal integrations resulting in the integral representation of the tensorial Green's function,

$$G_{jm}^{\vec{v}}(\vec{k},t) = P^T(\vec{k}) \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} e^{i\omega t} \frac{G^{vT}(\vec{k},\omega)}{-iw} + P^L(\vec{k}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G^{\rho}(\vec{k},\omega)$$

$$= P^T(\vec{k}) \;\; {}_dI_2^{(-1)} + P^L(\vec{k})\, G^{\rho}(\vec{k},t) \quad . \qquad (4.123)$$

The integral form required for the inverse transformation of the transverse Green's functions exhibits a zero frequency pole which mathematically results in a non causal Green's function which is unsatisfactory. In order that the Green's functions be physically reasonable the zero frequency poles were displaced to the positive imaginary side of the origin resulting in the integral evaluation of equation A-3.7b,

$${}_dI_2^{(-1)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \; e^{i\omega t} \frac{1}{\omega(\omega-\omega_1)} \; = \; \frac{i\; U(t)}{\omega_1} [e^{i\omega_1 t} - 1] \quad . \qquad (4.124)$$

The transverse Green's function may be evaluated by applying this integration evaluation and complex diffusive frequency defined by equations 4.25. The longitudinal Green's function transformation is exactly as has been evaluated for the damped wave type impulse Green's functions which exhibits temporally underdamped, critically damped and overdamped forms shown by equations 4.42a, 4.45a and 4.49a respectively. Expressing the longitudinal Green's function in the functional form, equation 4.51a, which is valid for all temporal damping conditions, the tensorial acoustic velocity Green's function in wavenumber and time domain is

$$G^{\vec{v}}_{jm}(\vec{k},t) = (\delta_{jm} - \frac{k_j k_m}{k^2})\frac{U(t)}{D_T k^2}[e^{-D_T k^2 t} - 1] - (\frac{k_j k_m}{k^2})U(t)\, e^{-\frac{D_L}{2}k^2 t}\, \frac{\sin(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}\, t)}{\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}} \tag{4.125}$$

Space and time domain

The tensorial acoustic velocity Green's function in space and time domain may be expressed as either the inverse temporal or inverse spatial Fourier transformation of the space and frequency domain on wavenumber and time domain representations respectively. Since the inverse temporal transformations are more difficult for both the transverse and longitudinal Green's functions the inverse spatial transformation will be utilized. The simplification of this transformation is analagous to that performed for the evaluation of the tensorial Green's function in space and frequency domain and results in the simplified form shown by equation 4.12. The integral form of the tensorial Green's function in space and time domain is

$$G^{\vec{v}}_{jm}(\vec{r},t) = \delta_{jm} \frac{U(t)}{i4\pi^2 rD_T} \int_{-\infty}^{\infty} dkke^{ikr} \frac{(e^{-D_T k^2 t}-1)}{k^2} + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{i4\pi^2 rD_T}\int_{-\infty}^{\infty} dkke^{ikr}\frac{(e^{-D_T k^2 t}-1)}{k^4} +$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{i4\pi^2 r}\int_{-\infty}^{\infty} dkke^{ikr}\, e^{-\frac{D_L}{2}k^2 t}\, \frac{\sin(\sqrt{C_1^2 k^2 - \frac{D_L^2}{4}k^4}\; t)}{k^2 \sqrt{C_1^2 k^2 - \frac{D_1^2}{4} k^4}}$$

$$= \delta_{jm}\frac{U(t)}{i4\pi rD_T}([_d I_3^{(-1)}]^{\delta=D_T t} - 2\pi_t I_3^{(-1)}) + \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{i4\pi rD_T}([_d I_3^{(-2)}]^{\delta=D_T t} - 2\pi_t I_3^{(-2)}) +$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{i4\pi^2 r}[_w I_5^{(-1)}]^{\delta=\frac{D_L}{2}t} . \qquad (4.126)$$

The integrals required for the transverse part have been evaluated in Appendix A-4, euations A-4.13a, 13b, 28a and 28b. The integral required for the longitudinal part, which is similar to the integrals required to evaluate the acoustic Green's function in space and time domain, has been evaluated in series form and may be evaluated numerically. However these forms are not useful for comparison with the closed form results obtained for the thermal-viscous tensorial Green's function in space and time domain and therefore will not be included.

Thermal-Viscous Tensorial Type Green's Functions

Wavenumber and frequency domain

The thermal-viscous fluctuations of vectorial velocity obey the field equation, equation 2.41, which couples a diffusion type transverse effect with a thermal-viscous type longitudinal effect. The resulting tensorial thermal-viscous impulse Green's function in wavenumber and frequency domain, equation 3.33

$$G^{\vec{v}}_{jm}(\vec{k},\omega) = \frac{-1}{i\omega(i\omega + \frac{\kappa}{C_v} k^2)[i\omega + D_T k^2]} (\delta_{jm} - \frac{k_j k_m}{k^2}) +$$

$$+ \frac{-1}{[i\omega^3 + \omega^2(D_L + \frac{\kappa}{C_v})k^2 - i\omega D_L \frac{\kappa}{C_v} (k^4 - i\omega C_1^2 \frac{\kappa}{C_p} k^4]} (\frac{k_j k_m}{k^2}) \quad , \qquad (4.127)$$

displays the transverse and longitudinal effects separately as the products of scalar Green's functions and the polarization tensors. The scalar transverse Green's function displays a thermal diffusion term, in parenthesis, in addition to the viscous diffusion term and zero frequency pole exhibited by the acoustic fluctuation counterpart,

$$G^T(\vec{k},\omega) = \frac{-1}{i\omega(i\omega + D_3 k^2)} \, G^{vT}(\vec{k},\omega) \qquad (4.128)$$

where $\qquad D_3 = \frac{\kappa}{C_v} \qquad (4.129)$

is the constant volume thermal diffusion coefficient. The poles of the transverse Green's functions include the zero frequency pole, the viscous diffusion wavenumber and frequency poles of equations 4.16 and the analagous thermal diffusion wavenumber and frequency poles

$$k_3 = (1-i)\sqrt{\frac{\omega}{2D_3}} \qquad k_4 = (-1+i)\sqrt{\frac{\omega}{2D_3}} \qquad \omega_2 = i\,D_3\,k^2 \ . \qquad (4.130a,b,c)$$

The scalar longitudinal Green's function is equal to the opposite of the thermal-viscous type impulse Green's function defined by equation 4.58a,

$$G^L(\vec{k},\omega) = -G_o(\vec{k},\omega) \ . \qquad (4.131a)$$

The corresponding wavenumber and frequency poles, equations 4.60 and 4.63, are such complicated functions that it is more useful to approximate them by the thermal-viscous model poles, equations 4.71, and approximate the longitudinal Green's function by the thermal-viscous model Green's function,

$$G^{Lm}(\vec{k},\omega) = -G_{o_m}(\vec{k},\omega) = \frac{1}{[\omega^2 - i\omega D_1 k^2 - C_1^2 k^2](i\omega + D_2 k^2)} \ . \qquad (4.131b)$$

Space and frequency domain

The tensorial thermal-viscous velocity Green's function in space and frequency domain may be expressed as a simplified inverse spatial Fourier transformation, equation 4.7, by replacing the directional wavenumbers with directional spatial derivatives. The resulting integral representation of the tensorial Green's function, using the approximate longitudinal model Green's function, is

$$G_{jm}^{\vec{v}m}(\vec{r},\omega) = \delta_{jm} \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \frac{-1}{i\omega D_3 D_T (k^2 + \frac{i\omega}{D_3})(k^2 + \frac{i\omega}{D_T})} +$$

$$+ \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \frac{-1}{i\omega D_3 D_T k^2 (k^2 + \frac{i\omega}{D_3})(k^2 + \frac{i\omega}{D_T})} -$$

$$- \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} \frac{1}{i4\pi^2 r} \int_{-\infty}^{\infty} dkke^{ikr} \frac{-G_{o_m}(\vec{k},\omega)}{k^2}$$

$$= \delta_{jm} \frac{1}{4\pi^2 r\omega D_3 D_T} [I_1^{(0)}]_T + \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} \frac{1}{4\pi^2 r\omega D_3 D_T} [I_1^{(-1)}]_T +$$

$$+ \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_m} \frac{1}{4\pi^2 r\omega D_2 (C_1^2 + i\omega D_1)} [I_1^{(-1)}]_L \qquad (4.132)$$

where the subscripts T or L on the bracketed integral forms denote affiliation with transverse or longitudinal Green's functions. One integral form evaluation is shown by equation 4.74 and, as also evaluated in Appendix A-2, equation A-2.14b the other integral form which includes zero wavenumber poles is evaluated as

$$I_1^{(-1)} = \int_{-\infty}^{\infty} dk \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = \frac{r}{|r|} \frac{i\pi}{(k_1^2-k_3^2)} \left\{ \frac{1}{k_1^2} [e^{-ik_1|r|}-1] - \frac{1}{k_3^2} [e^{-ik_3|r|}-1] \right\} . \tag{4.133}$$

This integral form is appropriate for both transverse and longitudinal Green's function evaluations since the corresponding wavenumber poles appropriate for each lie in the same quadrant of the complex wavenumber plane. However, evaluation of the transverse or longitudinal Green's function terms require that the respective pole definitions, as described above, be used. The tensorial thermal-viscous velocity Green's function may be expressed as

$$G_{jm}^{\vec{v}m}(\vec{r},\omega) = \delta_{jm} \frac{1}{4\pi|r|(D_T-D_3)\omega^2} [e^{-(1+i)\sqrt{\frac{\omega}{2D_T}}|r|} - e^{-(1+i)\sqrt{\frac{\omega}{2D_3}}|r|}] +$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \frac{i}{4\pi|r|\omega^3(D_T-D_3)} \left\{ D_T[e^{-(1+i)\sqrt{\frac{\omega}{2D_T}}|r|}-1] - D_3[e^{-(1+i)\sqrt{\frac{\omega}{2D_3}}|r|}-1] \right\} +$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \frac{-1}{4\pi|r|\omega^2[C_1^2+i\omega(D_1-D_2)]} \left\{ \frac{(C_1^2+i\omega D_1)}{\omega} [e^{-k'_{w_m}|r|-ik''_{w_m}|r|}-1] - \right.$$

$$\left. - iD_2[e^{-(1+i)}-1] \right\} \tag{4.134}$$

where the real and imaginary parts of the complex model wave wavenumber are defined by equations 4.75b and c.

Wavenumber and time domain

The tensorial thermal-viscous velocity Green's function in wavenumber and time domain equals the inverse temporal Fourier transformation of the Green's function in wavenumber and frequency domain, equation 4.9, which using the approximate model longitudinal Green's function, is

$$G_{jm}^{\vec{v}m}(\vec{k},t) = P^T(\vec{k}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \frac{-i}{\omega(\omega - iD_3k^2)[\omega - iD_Tk^2]} - P^L(k) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G_{o_m}(\vec{k},\omega)$$

$$= P^T(\vec{k})\ (-i)\ {}_{\omega}I_2^{(-1)} - P^L(\vec{k})\ G_{o_m}(\vec{k},t) \qquad (4.135)$$

The inverse transformation of the transverse Green's function is mathematically non causal because of the zero wavenumber pole. However the zero frequency pole will be assumed to be displaced to the positive imaginary side of the origin so that the integration will be physically reasonable,

$${}_{\omega}I_2^{(-1)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega(\omega-\omega_1)(\omega-\omega_2)} = \frac{U(t)}{(\omega_1-\omega_2)} \left\{ \frac{1}{\omega_1} [e^{i\omega_1 t} - 1] - \frac{1}{\omega_2} [e^{i\omega_2 t} - 1] \right\} . \qquad (4.136)$$

The inverse transformation of the longitudinal Green's function, the thermal-viscous impulse Green's function, has been evaluated and exhibits temporally underdamped, critically damped and overdamped forms, equations 4.86a, 4.89a, and 4.93a. The tensorial thermal-viscous Green's functions in wavenumber and time domain, with the longitudinal Green's function expressed in the functional form which is valid for all cases of temporal damping, is

$$G_{jm}^{\vec{v}m}(\vec{k},t) = (\delta_{jm} - \frac{k_j k_m}{k^2}) \frac{i\, U(t)}{(D_T - D_3)k^4} \left\{ \frac{1}{D_T} [e^{-D_T k^2 t} - 1] - \frac{1}{D_3} [e^{-D_3 k^2 t} - 1] \right\} +$$

$$+ (\frac{k_j k_m}{k^2}) \frac{-U(t)}{[C_1^2 - D_2(D_1 - D_2)k^2]} \left\{ e^{-\frac{D_1}{2} k^2 t} [\frac{1}{k^2} \cos(\sqrt{C_1^2 k^2 - \frac{D_1^2}{4} k^4}\, t) + \right.$$

$$\left. + (\frac{D_1}{2} + D_2) \frac{\sin(\sqrt{C_1^2 k^2 - \frac{D_1^2}{4} k^4}\, t)}{\sqrt{C_1^2 k^2 - \frac{D_1^2}{4} k^4}}] - D_2 k^2 e^{-D_2 k^2 t} \right\} \tag{4.137}$$

Space and time domain

The tensorial thermal-viscous velocity Green's function in space and time domain may be derived as either the inverse temporal Fourier transformation of the space and frequency domain representation or the inverse spatial transformation of the Green's function in wavenumber and time domain. Both the transverse and longitudinal Green's functions are most easily attained via the inverse spatial Fourier transformation which may be simplified, as for the space and frequency domain tensorial Green's function evaluation to the form of equation 4.12. The resulting integral form of the tensorial Green's function in space and time domain is

$$G^{\vec{v}m}_{jm}(\vec{r},t) = \delta_{jm}\frac{U(t)}{4\pi^2 r(D_T-D_3)}\int_{-\infty}^{\infty} dk\,k e^{ikr}\frac{\left[\frac{1}{D_T}\left(e^{-D_T k^2 t}-1\right)-\frac{1}{D_3}\left(e^{-D_3 k^2 t}-1\right)\right]}{k^4} +$$

$$+\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{4\pi^2 r(D_T-D_3)}\int_{-\infty}^{\infty} dk\,k e^{ikr}\frac{\left[\frac{1}{D_T}\left(e^{-D_T k^2 t}-1\right)-\frac{1}{D_3}\left(e^{-D_3 k^2 t}-1\right)\right]}{k^6} +$$

$$-\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\,k e^{ikr}\frac{-G_{o_m}(\vec{k},t)}{k^2}$$

$$= \delta_{jm}\frac{U(t)}{4\pi^2 r(D_T-D_3)}\left\{\frac{1}{D_T}\left([{}_dI_3^{(-2)}]^{\delta=D_T t}-2\pi{}_tI_3^{(-2)}\right)-\frac{1}{D_3}\left([{}_dI_3^{(-2)}]^{\delta=D_3 t}-2\pi{}_tI_3^{(-2)}\right)\right\} +$$

$$+\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{4\pi^2 r(D_T-D_3)}\left\{\frac{1}{D_T}\left([{}_dI_3^{(-3)}]^{\delta=D_T t}-2\pi{}_tI_3^{(-3)}\right)-\frac{1}{D_3}\left([{}_dI_3^{(-3)}]^{\delta=D_3 t}-2\pi{}_tI_3^{(-3)}\right)\right\}+$$

$$-\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{U(t)}{i4\pi^2 r D_2(D_1-D_2)}\left\{[I_4^{(-2)}]^{\delta=\delta_1}+\left(\frac{D_1}{2}-D_2\right)t[I_5^{(-1)}]^{\delta=\delta_1}-[I_3^{(-2)}]^{\delta=\delta_2}\right\}.$$

(4.138)

The integral evaluations required for the evaluation of the transverse Green's function, derived in Appendix A-4, are those required for the transverse tensorial acoustic Green's function, shown by equations A-4.13b and A-4.28b, and the evaluations for higher order zero wavenumber poles, equations A-4.13c and A-4.28c

$$ {}_tI_3^{(-3)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, k\, e^{ikr} \frac{1}{k^6} = \frac{r}{|r|}\frac{ir^4}{48} \tag{4.139a}$$

$$ {}_dI_3^{(-3)} = \int_{-\infty}^{\infty} dk k e^{ikr} \frac{e^{-\delta k^2}}{k^6} = \frac{i\pi}{2}\left(\delta^2+\delta r^2+\frac{r^4}{12}\right)\mathrm{erf}\left(\frac{r}{2\sqrt{\delta}}\right)+\frac{i\sqrt{\pi\delta}}{12}\, r(10\delta+r^2)e^{-r^2/4\delta} \quad . \tag{4.139b}$$

The longitudinal Green's function requires integral evaluations that are similar to those corresponding to the thermal-viscous impulse Green's functions but possess zero wavenumber poles of two higher orders of magnitude. Similarly the nonzero wavenumber poles, equation 4.98, may be either imaginary or real depending on which diffusion coefficient, thermal or thermal-viscous, is larger and the integrations must be evaluated for both cases. For the condition of larger thermal than thermal-viscous diffusion coefficients the integral forms have been evaluated in Appendix A-5, equations A-5.73b, 76b and 78b

$$ {}_2I_3^{(-2)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}}{k^3(k-i\rho_2)(k+i\rho_2)} = \frac{i\pi}{2\rho_2^4}\Big\{ e^{\delta\rho_2^2}\Big[e^{-\rho_2 r}\mathrm{erfc}\left(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}\right)-e^{\rho_2 r}\mathrm{erfc}\left(\frac{r+2\delta\rho_2}{2\sqrt{\delta}}\right)\Big]-$$

$$ -\left[2+\rho_2^2\left(\delta+\frac{r^2}{2}\right)\right]\mathrm{erf}\left(\frac{r}{2\sqrt{\delta}}\right)-4\sqrt{\frac{\delta}{\pi}}\;\rho_2^2\; e^{-r^2/4\delta}\Big\} \tag{4.140a}$$

$$ {}_2I_4^{(-2)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}\cos(\delta k\sqrt{\rho_3^2-k^2})}{k^3(k-i\rho_2)(k+i\rho_2)} = \frac{i\pi}{2\rho_2^4}\left\{\cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})e^{\delta\rho_2^2}[e^{-\rho_2 r}\mathrm{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - \right.$$

$$\left. - e^{\rho_2 r}\mathrm{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] - (1+\frac{1}{2}\delta^2\rho_2^2\rho_3^2+\rho_2^2\delta+\frac{1}{2}\rho_2^2 r^2)\mathrm{erf}(\frac{r}{2\sqrt{\delta}}) - 4\sqrt{\frac{\delta}{\pi}}\rho_2^2 r e^{-r^2/4\delta}\right\}$$

(4.140b)

$$ {}_2I_5^{(-1)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}\sin(\delta k\sqrt{\rho_3^2-k^2})}{k(k-i\rho_2)(k+i\rho_2)\delta k\sqrt{\rho_3^2-k^2}} = \frac{-i\pi}{2\rho_2^2}\left\{\frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} e^{\delta\rho_2^2}[e^{-\rho_2 r}\mathrm{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) \right.$$

$$\left. - e^{\rho_2 r}\mathrm{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] - 2\,\mathrm{erf}\,(\frac{r}{2\sqrt{\delta}})\right\} \quad (4.140c)$$

where ρ_2 and ρ_3 are defined by equations 4.99c and 4.97c. For the condition of larger thermal-viscous diffusion coefficient than thermal diffusion coefficient the integral forms have also been evaluated in Appendix A-5, equations A-5.90b, 93b and 95b

$$ {}_1I_3^{(-2)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}}{k^3(k-\rho_1)(k+\rho_1)} = \frac{-i\pi}{2\rho_1^4}\left\{e^{-\delta\rho_1^2}[e^{-i\rho_1 r}\mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{i\rho_1 r}\mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})] + \right.$$

$$\left. + [2-\rho_1^2(\delta+\frac{r^2}{2})]\mathrm{erf}(\frac{r}{2\sqrt{\delta}}) - 4\sqrt{\frac{\delta}{\pi}}\rho_1^2 r e^{-r^2/r\delta}\right\} \quad (4.141a)$$

$$ {}_1I_4^{(-2)} = \int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}\cos(\delta k\sqrt{\rho_3^2-k^2})}{k^3(k-\rho_1)(k+\rho_1)} = \frac{-i\pi}{2\rho_1^4}\left\{\cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})e^{-\delta\rho_1^2}[e^{-i\rho_1 r}\mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}}) - \right.$$

$$\left. - e^{i\rho_1 r}\mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})] + [2-\rho_1^2(\delta^2\rho_3^2+\delta+\frac{r^2}{2})]\mathrm{erf}(\frac{r}{2\sqrt{\delta}})\ 4\sqrt{\frac{\delta}{\pi}}\rho_1^2 r e^{-r^2/4\delta}\right\} \quad (4.141b)$$

$${}_1I_5^{(-1)}=\int_{-\infty}^{\infty} dk e^{ikr} \frac{e^{-\delta k^2}\sin(\delta k\sqrt{\rho_3^2-k^2})}{k^3(k-\rho_1)(k+\rho_1)\delta k\sqrt{\rho_3^2-k^2}} = \frac{-i\pi}{2\rho_1^2}\left\{\frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}} e^{-\delta\rho_1^2}[e^{-i\rho_1^2}\operatorname{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})-\right.$$

$$\left.-e^{i\rho_1 r}\operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})]+2\operatorname{erf}(\frac{r}{2\sqrt{\delta}})\right\} \qquad (4.141c)$$

where ρ_1 is defined by equation 4.100c.

The longitudinal Green's function evaluations for each of the diffusion coefficient conditions possess exponents of mathematically undetermined sign as did the thermal viscous impulse Green's functions. Analogously, the physical infinite temporal and spatial boundary conditions may be applied to determine the proper exponent sign for each of three diffusion parameter regions defined by the relative magnitudes of the thermal and the thermal-viscous diffusion coefficients.

The tensorial thermal-viscous velocity Green's function in space and time domain may be evaluated, utilizing the transverse and longitudinal integral evaluations, as

$$G_{jm}^{\vec{v}m}(\vec{r},t)=\delta_{jm}\frac{i\,U(t)}{32\pi r(D_T-D_3)}\left\{\frac{1}{D_T}\left\{[(4D_Tt)+2r^2]\operatorname{erf}(\frac{r}{\sqrt{4D_Tt}})+\sqrt{\frac{4D_Tt}{\pi}}\frac{r}{2}e^{-\frac{r^2}{4D_Tt}}-\frac{r}{|r|}4r^2\right\}-\right.$$

$$\left.-\frac{1}{D_3}\left\{[(4D_3t)+2r^2]\operatorname{erf}(\frac{r}{\sqrt{4D_3t}})+\sqrt{\frac{4D_3t}{\pi}}\frac{r}{2}e^{-\frac{r^2}{4D_3t}}-\frac{r}{|r|}4r^2\right\}\right\}+$$

$$+\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m}\frac{-i\,U(t)}{32\pi r(D_T-D_3)}\left\{\frac{1}{D_T}\left\{[\frac{(4D_Tt)^2}{4}+(4D_Tt)r+\frac{r^4}{3}]\operatorname{erf}(\frac{r}{\sqrt{4D_Tt}})+\sqrt{\frac{4D_Tt}{\pi}}\frac{r}{12}[10(4D_Tt)+\right.\right.$$

$$+4r^2]e^{-\frac{r^2}{4D_Tt}}-\frac{r}{|r|}\frac{2r^4}{3}\right\}-$$

$$- \frac{1}{D_3} \left\{ [\frac{(4D_3t)^2}{4} + (4D_3t)r + \frac{r^4}{3}]\mathrm{erf}(\frac{r}{\sqrt{4D_3t}}) + \sqrt{\frac{4D_3t}{\pi}} \frac{r}{12}[5(4D_3t)+4r^2]e^{-\frac{r^2}{4D_3t}} - \frac{r}{|r|}\frac{2r4}{3} \right\} \Bigg\}$$

$$+ \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_m} \overline{G}_{o_m}(\vec{r},t) \tag{4.142}$$

where the three diffusion parameter regions and the corresponding longitudinal Green's functions are

$$D_2 > D_1 \tag{4.143a}$$

$$\overline{G}_{o_m}(\vec{r},t) = \frac{-U(t)D_2(D_2-D_1)}{8\pi r C_1^4} \Bigg\{ e^{\frac{4D_2t}{r_C^2}} [e^{\frac{2r}{r_C}}\mathrm{erfc}(\frac{-r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C}) - e^{\frac{2r}{r_C}}\mathrm{erfc}(\frac{r}{\sqrt{2D_1t}} + \frac{\sqrt{2D_1t}}{r_C})] -$$

$$-e^{\frac{4D_2t}{r_C^2}} [e^{\frac{2r}{r_C}} \mathrm{erfc}(\frac{-r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C}) - e^{\frac{2r}{r_C}} \mathrm{erfc}(\frac{r}{\sqrt{4D_2t}} + \frac{\sqrt{4D_2t}}{r_C})] +$$

$$-\left\{1 + \frac{1}{r_C^2}[2r^2+2C_1^2t^2-(2D_1t)+2(4D_2t)]\right\}\mathrm{erf}(\frac{r}{\sqrt{2D_1t}}) + \left\{2 + \frac{1}{r_C^2}[2r^2+(4D_2t)]\right\}\mathrm{erf}(\frac{r}{\sqrt{4D_2t}}) -$$

$$- \frac{8}{\sqrt{\pi}} \frac{r}{r_C^2} [\sqrt{2D_1t}\; e^{-\frac{r^2}{2D_1t}} - \sqrt{4D_2t}\; e^{-\frac{r^2}{4D_2t}}] \Bigg\} \tag{4.143b}$$

$$2D_2 \geq D_1 > D_2 \tag{4.144a}$$

$$\bar{G}_{o_m}(\vec{r},t)=\frac{-U(t)D_2(D_1-D_2)}{8\pi r C_1^4}\left\{ e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\frac{\sqrt{2D_1t}}{r_C}\right)\right]-\right.$$

$$-e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\frac{\sqrt{4D_2t}}{r_C}\right)\right]+$$

$$+\left\{2-\frac{1}{r_C^2}[4r^2+4C_1^2t-(2D_1t)+2(4D_2t)]\right\}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}\right)-\left\{2-\frac{1}{r_C^2}[2r^2+(4D_2t]\right\}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)-$$

$$\left.-\frac{8}{\sqrt{\pi}}\frac{r}{r_C^2}\left[\sqrt{2D_1t}\,e^{-\frac{r^2}{2D_1t}}-\sqrt{4D_2t}\,e^{-\frac{r^2}{4D_2t}}\right]\right\} \tag{4.144b}$$

$$D_1 \geq 2D_2 \tag{4.145a}$$

$$\bar{G}_{o_m}(\vec{r},t)=\frac{-U(t)D_2(D_1-D_2)}{8\pi r C_1^4}\left\{ e^{-\frac{C_1^2t}{D_2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{2D_1t}}+i\frac{\sqrt{2D_1t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}+i\frac{\sqrt{2D_1t}}{r_C}\right)\right]-\right.$$

$$-e^{\frac{4D_2t}{r_C^2}}\left[e^{-i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{-r}{\sqrt{4D_2t}}+i\frac{\sqrt{4D_2t}}{r_C}\right)-e^{i\frac{2r}{r_C}}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}+i\frac{\sqrt{4D_2t}}{r_C}\right)\right]+$$

$$+\left\{2-\frac{1}{r_C^2}[4r^2+4C_1^2t-(2D_1t)+2(4D_2t)]\right\}\mathrm{erf}\left(\frac{r}{\sqrt{2D_1t}}\right)-\left\{2-\frac{1}{r_C^2}[2r^2+(4D_2t)]\right\}\mathrm{erf}\left(\frac{r}{\sqrt{4D_2t}}\right)-$$

$$\left.-\frac{8}{\sqrt{\pi}}\frac{r}{r_C^2}\left[\sqrt{2D_1t}\,e^{-\frac{r^2}{2D_1t}}-\sqrt{4D_2t}\,e^{-\frac{r^2}{4D_2t}}\right]\right\} \tag{4.145b}$$

V. Application of Green's Functions

The Green's functions contain the information about the medium which is necessary to determine the response to a given forcing situation or initial condition. Although the Green's functions are spherically symmetric, the spatial forcing distribution may excite responses of different geometries. The forcing may also be either deterministic or statistical yielding either deterministic or statistical responses respectively.

Several specific deterministic forcing situations will be considered in order to illustrate the physical meaning of the Green's functions in each of the possible domains. In addition to the spherically symmetric responses which are natural to the Green's functions, plane responses will be derived. Some of these plane responses may be directly compared with the previously derived plane wave responses.

The statistical formulation for responses to nonstationary random temporal forcing and to random spatial forcing is outlined but no specific results are presented.

Deterministic Responses

The response of a fluctuation in space and time domain may be determined from its forcing and the impulse Green's function of the medium by two methods. The response is equal to the spatial and temporal convalution of the impulse Green's function and the forcing as given by equation 3.9a,

$$x(\vec{r},t) = G(\vec{r},t) * f(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}',t-t') \; f(\vec{r}',t'). \tag{5.1}$$

From equations 3.7 and 3.8a it is obvious that the response in wavenumber and frequency domain is equal to the product of the impulse Green's function and the forcing in those domains,

$$x(\vec{k},\omega) = G(\vec{k},\omega) \; f(\vec{k},\omega) \; , \tag{5.2}$$

and therefore the response in space and time domain may also be expressed as the inverse spatial and temporal Fourier transformation of this product,

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \, e^{-i\vec{k}\cdot r} \int_{-\infty}^{\infty} \frac{d}{2\pi} \, e^{i\omega t} \; G(\vec{k},\omega) \; f(\vec{k},\omega) \; . \tag{5.3}$$

The relaxation of a fluctuation from a specific initial condition is equal to the relaxations due to each of the initial condition terms. The relaxation due to each initial condition term may be determined from that initial condition term and the initial condition Green's function by two methods. The relaxation is equal to the spatial convaluation of the initial condition Green's function and the non-zero initial condition as given by equation 3.44,

$$x(\vec{r},t) = G_{xy}(\vec{r},t) *_r Y(\vec{r},t{=}0) = \int_{-\infty}^{\infty} d^3\vec{r}' \; G_{xy}(\vec{r}-\vec{r}',t) \; Y(\vec{r},t{=}0) \; . \tag{5.4}$$

Since the relaxation in wavenumber and frequency domain is equal to the product of the initial condition Green's function and the initial condition term, equation 3.35

$$x(\vec{k},\omega) = G_{xy}(\vec{k},\omega) \; Y(\vec{k},t{=}0) \; , \tag{5.5}$$

the relaxation in space and time domain may also be expressed as the inverse spatial and temporal Fourier transform of this product,

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G_{xy}(\vec{k},\omega)\; Y(\vec{k},t=0)\ . \tag{5.6}$$

The initial condition term represents a spatial distribution at time t=0 and therefore is not a function of time or frequency. However, an analogy between the relaxation from an initial condition, equation 5.5, and the response to a forcing, equation 5.2, shows that the initial condition and impulse Green's functions are analogous and that the initial condition term is analogous to a forcing with no frequency dependence. If there is no coupling between the space forcing and the time forcing then the forcing may be expressed in wavenumber and frequency domain as the product of the wavenumber forcing and the frequency forcing,

$$f(\vec{k},\omega) = \bar{f}(\vec{k})\ \bar{\bar{f}}(\omega)\ . \tag{5.7}$$

If the frequency forcing is unity, all frequencies are excited equally and the wavenumber forcing is analogous to the initial condition term in wavenumber domain,

$$f(\vec{k},\omega) = \bar{f}(\vec{k})(1) \ \leftrightarrow\ Y(\vec{k},t=0)\ . \tag{5.8}$$

Performing an inverse spatial and temporal Fourier transform yields the forcing in space and time domain,

$$f(\vec{r},t) = \bar{f}(\vec{r})\bar{\bar{f}}(t) = \bar{f}(\vec{r})\ \delta(t)\ , \tag{5.9}$$

where the uniform frequency forcing corresponds to impulsive temporal forcing. If the inverse spatial Fourier transformation of the initial conditon term is to remain analogous to spatial forcing it must be multiplied by a temporal impulse to remain analogous to the total forcing,

$$f(\vec{r},t) = \bar{f}(\vec{r})\ \delta(t) \ \leftrightarrow\ Y(\vec{r},t=0)\ \delta(t)\ . \tag{5.10}$$

This analogy is substantiated by expressing the response as a spatial and temporal convolution of the impulse Green's function and the forcing, equation 5.1. When the temporal forcing is impulsive, the temporal convolution may be evaluated as

$$x(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}',t-t') \; \tilde{f}(\vec{r}') \; \delta(t') = \int_{-\infty}^{\infty} d^3\vec{r}' \; G(\vec{r}-\vec{r}',t) \; \tilde{f}(\vec{r}') \tag{5.11}$$

which is analogous to the relaxation as a spatial convolution, equation 5.4. Thus when the forcing is impulsive in time, the response from a spatial forcing is analogous to the relaxation from an initial spatial distribution.

In order to illustrate the physical meaning and application of the Green's functions in the Fourier domains, the response in space and time domain will be obtained for various specific forcing distributions. When the temporal forcing distribution is impulsive the response corresponds to a relaxation and the initial condition Green's functions are illustrated.

The forcings that will be considered will be spatially and temporally uncoupled. Two types of temporal forcing, impulsive in time and single frequency sinusoidal in time. The temporally impulsive forcing and its temporal Fourier transform

$$\bar{\tilde{f}}(t) = \delta(t) \qquad , \quad \bar{\tilde{f}}(\omega) = 1 \tag{5.12a,b}$$

represent uniform forcing of all frequencies at time t=0. Sinusoidal temporal forcing at a single frequency, ω_0, is represented by the temporal forcing in time domain and in frequency domain as

$$\bar{\tilde{f}}(t) = e^{i\omega_0 t} \qquad , \quad \bar{\tilde{f}}(\omega) = 2\pi\delta(\omega-\omega_0) \; . \tag{5.13a,b}$$

The spatial forcing determines the mode of propagation. Because the Green's functions are spherically symetric, spherical propagation is natural to consider. Another common mode which will also be considered is plane propagation. For each mode, both impulsive in space and single wavenumber sinusoidal in space forcings will be considered. Spherical propagation may be excited by forcing at a point in space or by forcing the wavenumber vector in the radial direction. Impulsive forcing at $\vec{r}$=0 and its spatial Fourier transformation,

$$\vec{f}(\vec{r}) = \delta(\vec{r}) \quad , \quad \vec{f}(\vec{k}) = 1 \tag{5.14a,b}$$

forces all wavenumbers equally. Sinusoidal radial forcing at a single wavenumber in the radial direction, k_{ro}, in space domain and in wavenumber domain is,

$$\vec{f}(\vec{r}) = e^{-i\vec{k}_{ro}\cdot\vec{r}} = e^{-ik_{ro}r}, \ \vec{f}(\vec{k}) = (2\pi)^3\delta(\vec{k}-\vec{k}_{ro}) \ . \tag{5.15a,b}$$

Plane propagation may be excited by forcing uniformly over a plane or by forcing the wavenumber vector uniformly perpendicular to that plane. Impulsive forcing at $r_x=0$, over the yz plane, and it's spatial Fourier transformation

$$\vec{f}(\vec{r}) = \delta(r_x) \quad , \quad \vec{f}(\vec{k}) = 2\pi\delta(k_y)\ 2\pi\delta(k_z) \tag{5.16a,b}$$

force all wavelengths uniformly in the x direction but do not allow the wavenumber vector to have any component in the y or z directions. The result is plane propagation in the x direction as when only one wavenumber, k_{xo}, is sinusoidally forced in the x direction by the spatial forcing in space domain and in wave number domain,

$$\vec{f}(\vec{r}) = e^{-i\vec{k}_{xo}\cdot\vec{r}} = e^{-k_{xo}r_x}, \ \vec{f}(\vec{k}) = (2\pi)^3\ \delta(k_x-k_{xo})\delta(k_y)\delta(k_z) \ . \tag{5.17a,b}$$

i Spherical sinusoid spatial and sinusoid temporal forcing

The product of sinusoidal temporal forcing, equation 5.13, and sinusoidal radial forcing, equation 5.15, results in a forcing of one frequency, ω_o, and one wavenumber in the radial direction $\vec{k}_{ro}$. In space and time domain and wavenumber and frequency domain this forcing is

$$f(\vec{r},t) = e^{-k_{ro}r}\, e^{i\omega_o t}, \qquad f(\vec{k},\omega) = (2\pi)^3\delta(\vec{k}-\vec{k}_{ro})\; 2\pi\delta(\omega-\omega_o). \tag{5.18a,b}$$

The spherical response to this forcing may be expressed in terms of the impulse Green's function by evaluation of the inverse transformation of equation 5.3 after substituting the forcing representation of equation 5.18b

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G(\vec{k},\omega)\; (2\pi)^3\delta(\vec{k}-\vec{k}_{ro})\; 2\pi\delta(\omega-\omega_o) =$$

$$= e^{-ik_{ro}r}\, e^{i\omega_o t}\, G(\vec{k}_{ro},\omega_o) \;. \tag{5.19}$$

Thus the response to sinusoidal radial and sinusoidal temporal forcing is the product of the forcing and the wavenumber and frequency domain Green's function evaluated at the forced wavenumber and the forced frequency. Since the Green's functions in wavenumber and frequency domain are complex algebraic functions of the wavenumber and frequency, the response is an unattenuated spherical wave which propagates radially with a phase velocity ω_o/k_{ro}. The impulse Green's function in wavenumber and frequdncy domain modifies the amplitude and phase of the response and shows the tendency for particular wavenumbers and frequencies to be forced.

ii Plane sinusoid spatial and sinusoid temporal forcing

Forcing of one frequency, ω_0, and one wavenumber in the x direction, k_{xo}, may be obtained from the product of sinusoidal temporal forcing, equation 5.13, and plane sinusoidal spatial forcing, equation 5.17, as

$$f(\vec{r},t)\ e^{-k_{xo}r_x}\ e^{i\omega_o t} \quad , \quad f(\vec{k},\omega) = (2\pi)^3\delta(k_x-k_{xo})\delta(k_y)\delta(k_z)\ 2\pi\delta(\omega-\omega_o) \tag{5.20a,b}$$

The plane response to this forcing may be expressed in a simplified form of equation 5.3 by evaluating the inverse transformation upon substitution of the forcing form of equation 5.20b,

$$x(\vec{r}_x,t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\ G(k,\omega)(2\pi)^3\delta(k_x-k_{xo})\delta(k_y)\delta(k_z)\ 2\pi\delta(\omega-\omega_o) =$$

$$= e^{-ik_{xo}r_x}\ e^{i\omega_o t}\ G(\vec{k}_{xo},\omega_o). \tag{5.21}$$

As for the spherical forcing, the response to plane sinusoidal forcing of wavenumber and sinusoidal forcing of frequency is the product of the forcing and the impulse Green's function in wavenumber and space domain evaluated at the forced wavenumber and frequency. The response is an unattenuated plane wave traveling in the x direction with a group velocity of ω_o/k_{xo}. As for the corresponding spherical forcing the Green's function in wavenumber and frequency domain shows the tendency for particular wavenumbers and frequencies to be propagated.

iii· Spherical impulse spatial and sinusoid temporal forcing

The product of sinusoidal temporal forcing, equation 5.13, and impulsive forcing at a point in space, equation 5.14, yields the radial forcing at frequency ω_0,

$$f(\vec{r},t) = \delta(\vec{r})\, e^{i\omega_0 t} \quad , \quad f(\vec{k},\omega) = 2\pi\delta(\omega-\omega_0) \; . \tag{5.22a,b}$$

This forcing represents a common method of exciting spherical propagation. By substituting the forcing expression of equation 5.22b in the inverse transformation of equation 5.3 and performing the transformation integrations, the response in space and time domain becomes

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G(\vec{k},\omega)\; 2\pi\delta(\omega-\omega_0) =$$

$$= e^{i\omega_0 t}\, G(\vec{r},\omega_0) \tag{5.23}$$

The response in space and time domain is equal to the product of a sinusoidal temporal oscilation at the forced frequency and the impulse Green's function in space and frequency domain evaluated at the forced frequency. The Green's function determines the spatial characteristics of the response. The diffusion type, equation 4.21a, and the wave type, equation 4.35a, each contain exponential terms with arguments which are complex functions of the magnitude of the radius vector. They yield spatially attenuated spherical wave responses. The thermal viscous type model Green's function, equation 4.77a, exhibits both diffusion type and wave type terms and yields the response,

$$x(\vec{r},t) = \frac{-i\, e^{i\omega_0 t}}{4\pi|r|\omega_0[C_1^2-i(D_1-D_2)\omega_0]} \left[e^{-(k'_{wm}+ik''_{wm})|r|} - e^{-(k'_{dm}+ik''_{dm})|r|} \right]\Bigg|^{\omega=\omega_0} . \tag{5.24}$$

The real and imaginary parts of the complex model wave wavenumber, k_{wm}, and the complex model diffusion wavenumber as defined by equations 4.75 and 4.76 respectively are the product of i and the wavenumber poles of the Green's function in wavenumber and frequency domain. The response exhibits spherical spreading of both the wave related term and the diffusion related term. The wave related response is attenuated exponentially by k'_{wm} as it propagates radially at the phase velocity ω_o/k''_{wm} and the diffusion related response is attenuated exponentially by k'_{dm} as it propagates radially at the phase velocity ω_o/k''_{dm}.

iv· Plane impulse spatial and sinusoid temporal forcing

Plane forcing at a single frequency, ω_0, may be obtained from the product of sinusoidal temporal forcing, equation 5.13, and impulsive forcing at a plane, equation 5.16, as

$$f(\vec{r},t) = \delta(r_x)\, e^{i\omega_0 t} \quad , \quad f(\vec{k},\omega) = 2\pi\delta(k_y)\, 2\pi\delta(k_z)\, 2\pi\delta(\omega-\omega_0). \qquad (5.25a,b)$$

The response due to the forcing of equation 5.25, may be expressed as an inverse spatial and temporal Fourier transformation of the product of the impulse Green's function and the forcing as shown by equation 5.3. Since all but one of the transformation integrations may be readily performed, the response may be simplified as

$$x(\vec{r},t) = \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}e^{-ik_xr_x}\int_{-\infty}^{\infty}\frac{dk_y}{2\pi}e^{-ik_yr_y}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}e^{-ik_zr_z}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}G(\vec{k},\omega)2\pi\delta(k_y)2\pi\delta(k_z)2\pi\delta(\omega-\omega_0) =$$

$$= \int_{-\infty}^{\infty}\frac{dk_x}{2\pi}\, e^{-ik_xr_x}\, e^{i\omega_0 t}\, G(\vec{k}_x,\omega_0) \qquad (5.26)$$

Since the Green's function is a three dimensional function in wavenumber domain, the one dimensional inverse Fourier transformation integral of equation 5.26 does not transform the Green's function to space domain. The three dimensional inverse spatial transformation has been simplified to a single integral in Appendix A-1, equation A-1.9, for Green's function that are even functions of the wavenumber magnitude, k, as

$$G(\vec{r},\omega) = \int_{-\infty}^{\infty}\frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}}\, G(\vec{k},\omega) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk k e^{ikr}\, G(\vec{k},\omega). \qquad (5.27a)$$

Because the Green's function is symmetric the simplified inverse transformation may also be expressed as

$$G(\vec{r},\omega) = \frac{-1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk k e^{-ikr}\, G(\vec{k},\omega) \qquad (5.27b)$$

which posesses an integrand that differ from the integrand of equation 5.26 by the term k. This term may be eliminated by reexpressing the integral as shown

$$G(\vec{r},\omega) = \frac{-1}{i4\pi^2 r} \left(\frac{1}{-i}\right) \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk\ e^{-ikr}\ G(\vec{k},\omega)\ . \tag{5.27c}$$

Therefore the desired integration form of the Green's function in wavenumber domain may be expressed as

$$\int_{-\infty}^{\infty} dk\ e^{-ikr}\ G(\vec{k},\omega) = -\int dr\ 4\pi^2 r G(\vec{r},\omega) \tag{5.28}$$

which is related to, but not equal to, the Green's function in space domain. Thus with a wavenumber vector which is entirely in the x direction, $\vec{k}_{xo}$, the response may be expressed as

$$x(\vec{r}_x,t) = -2\pi\ e^{i\omega_o t} \int dr_x\ r_x\ G(\vec{r}_x,\omega_o)\ . \tag{5.29}$$

As occurs with radial impulse forcing, the plane response in space and time domain is equal to the product of a sinusoidal temporal oscilation and an indefinite integral over the x direction of space of the Green's function in space and frequency domain. Since the spatial dependence of these Green's functions is limited to an inverse of distance spherical spreading and complex exponential functions of distance, the indefinite spatial integral is easily evaluated and has the effect of eliminating the spherical spreading inherent in the Green's functions. For a thermal-viscous medium the thermal viscous model impulse Green's function in space and frequency domain, equation 4.77a, may be substituted in equation 5.29 to determine the response

$$x(\vec{r}_x,t) = -2\pi e^{i\omega_o t} \int dr_x r_x \frac{-i}{4\pi|r_x|\omega_o[C_1^2-i(D_1-D_2)\omega_o]} \left[e^{-(k'_{wm}+ik''_{wm})|r_x|} - e^{-(k'_{wm}+ik''_{wm})|r_x|} \right]_{\omega=\omega_o} \tag{5.30a}$$

The indefinite integral is readily evaluated and yields a spatially attenuated plane wave response

$$x(\vec{r}_x,t) = \frac{e^{i\omega_o t}}{2\omega_o[C_1^2-i(D_1-D_2)\omega_o]} \left[\frac{e^{-(k'_{wm}+ik''_{wm})|r_x|}}{(k'_{wm}+ik''_{wm})} - \frac{e^{-(k'_{dm}+ik''_{dm})|r_x|}}{(k'_{dm}+ik''_{dm})} \right]^{\omega=\omega_o} . \tag{5.30b}$$

The plane wave response is similar to the spherical wave response, equation 5.24, without spherical spreading. The acoustic wave related response is attenuated exponentially in space by the term

$$e^{-k'_{wm}|r_x|} = e^{-\frac{|\omega_o||r_x|}{\sqrt{2}\sqrt{C_1^4+D_1^2\omega_o^2}}\sqrt{-C_1^2+\sqrt{C_1^4+D_1^2\omega_o^2}}} \tag{5.31a}$$

as it propagates in the x direction at the phase velocity

$$\frac{\omega_o}{k''_{wm}} = \frac{\sqrt{2}\sqrt{C_1^4+D_1^2\omega_o^2}}{\sqrt{C_1^2+\sqrt{C_1^4+D_1^2\omega_o^2}}} \ . \tag{5.31b}$$

The diffusion related response is attenuated exponentially in space by the term

$$e^{-k'_{dm}|r_x|} = e^{-\sqrt{\frac{\omega_o}{2D_2}}\,|r_x|} \tag{5.32a}$$

as it propagates in the x direction at the phase velocity

$$\frac{\omega o}{k''_{dm}} = \sqrt{2D_2\omega_o} \ . \tag{5.32b}$$

v Comparison with previous temporally sinusoidal plane wave solutions

The effects of viscosity and thermal conductivity on the propigation of plane acoustic waves have been investigated by Stokes, Rayleigh, Kirchhoff, and Landau and Lifshitz. Their solutions may be compared with the plane response derived from the impulse Green's functions and the forcing which is a plane impulse in space and sinusoidal in time, equation 5.30.

Stokes(12) investigated the effect of viscosity alone on plane acoustic waves by utilizing the same basic equations as were used to derive the acoustic wave type Green's function. By combining the conservation of mass equation, the conservation of momentum equation with first viscosity only, and the adiabatic speed of sound relation between pressure and density and by assuming the sinusoid temporally forced plane wave solution form,

$$e^{i\omega_o t} e^{ik_x r_x}, \tag{5.33}$$

the solution form for small viscosity was found to be approximately

$$e^{i\omega_o t} e^{-i\frac{\omega_o}{C_1} r_x} e^{\frac{\omega_o^2}{2C_1^3} \frac{4\eta}{3\rho} r_x}. \tag{5.34}$$

Rayleigh(13) also investigated the effect of thermal conductivity alone on plane acoustic waves. The conservation of mass equation and the conservation of momentum equation in the absence of viscosity were suplemented by a pressure and density relation which allows the speed of sound to be effected by changes of density and temperature and by an equation which relates the time change of temperature to the time change density and the thermal conduction of energy. The assumption of the frequency forced plane wave solution form given by equation 5.33 yielded the small conductivity approximate solution.

$$e^{i\omega_o t} e^{-i\frac{\omega_o}{C_1} r_x} e^{-\frac{\omega_o^2}{2C_1^3} (\frac{\kappa}{C_p} - \frac{\kappa}{C_v}) r_x}. \tag{5.35}$$

As for viscosity, the first order effect of thermal conductivity was a spatial attenuation and a decrease of the propagation velocity was of second order.

Kirchhoff [14, 15] investigated the combined effects of viscosity and thermal conductivity by utilizing the same equations as Rayleigh used in his thermal conductivity investigation and by including the viscous terms of the momentum conservation equation. By assuming that all variables had sinusoid temporally forced plane wave solution form, equation 4.33, a characteristic equation for the wavenumber was derived,

$$i\omega_o^3 + \left\{-\omega_o^2\left(\frac{4\eta}{3\rho} + \frac{\zeta}{\rho} + \frac{\kappa}{C_v}\right) - i\omega_o C_1^2\right\} k_x^2 + \left\{i\omega_o\left(\frac{4\eta}{3\rho} + \frac{\zeta}{\rho}\right)\frac{\kappa}{C_v} - C_1^2\frac{\kappa}{C_p}\right\} k_x^4 = 0. \tag{5.36}$$

This characteristic equation is exactly the denominator of the true thermal viscous Green's functions in wavenumber and frequency domain set equal to zero; the equation for determining the wavenumber poles, equation 4.60.

For small viscosity and thermal conductivity the solution form was found to be approximately

$$Ae^{i\omega_o t}\, e^{\mp i\frac{\omega_o}{C_1} r_x}\, e^{\mp\frac{\omega_o^2}{2C_1^3}\left[\left(\frac{4\eta}{3\rho} + \frac{\zeta}{\rho}\right) + \left(\frac{\kappa}{C_v} - \frac{\kappa}{C_p}\right)\right] r_x} +$$

$$+ Be^{i\omega_o t}\, e^{\mp i\sqrt{\frac{\omega_o C_p}{2\kappa}}\, r_x}\, e^{\mp\sqrt{\frac{\omega_o C_p}{2\kappa}}\, r_x} . \tag{5.37}$$

The first term represents an acoustic wave which is spatially attenuated by the product of the exponential viscous attenuation found by Stokes, equation 5.34 (the second coefficient of viscosity has been included also) and the exponential thermal conduction attenuation derived by Rayleigh, equation 5.35. Similarly in this approximation the propagation velocity is uneffected. The second term represents thermal diffusion.

Landau and Lifshitz [16] investigated the attenuation of acoustic waves due to viscosity and thermal conductivity by examining the wave energy. The time rate of dissapation of the mechnaical energy, $\dot{E}_{mech}$, of the acoustic wave was derived by considering the rate of entropy change due to small viscosity and thermal conductivity. This was divided by twice the total energy of an undamped wave to obtain the coefficient for exponential temporal decay of the fluctuation for small dissapation,

$$\frac{\dot{E}_{mech}}{2\overline{E}} = \frac{-k^2}{2} \left[\left(\frac{4}{3} \frac{\eta}{\rho} + \frac{\zeta}{\rho} \right) + \left(\frac{\kappa}{C_v} - \frac{\kappa}{C_p} \right) \right] . \tag{5.38}$$

This coefficient is applicable for the time decay of a given wavenumber fluctuation and will be discussed in later section. It was assumed that the propagation velocity was not effected by the dissipation therefore the temporal coefficient of decay was divided by the propigation velocity yielding the exponential spatial attenuation

$$e^{\frac{\dot{E}_{mech}}{2\overline{E}\, C_1} r_x} = e^{\frac{k^2}{2C_1} \left[\left(\frac{4}{3} \frac{\eta}{\rho} + \frac{\zeta}{\rho} \right) + \left(\frac{\kappa}{C_v} - \frac{\kappa}{C_p} \right) \right] r_x} \tag{5.39}$$

for small viscosity and thermal conductivity. This spatial attenuation is exactly the attenuation of the acoustic wave term derived by Kirchhoff, equation 5.37.

The plane wave solution form for exponential temporal forcing at frequency ω_0 derived by Kirchhoff is comparable to the plane response derived from the thermal viscous model Green's function, equation 5.30. Both display spatially attenuating and propagating acoustic wave and thermal diffusion terms, the spatial attenuation and propagation characteristics of the thermal diffusion term, equation 5.32, are identical. However, there is a descrepancy in the spatial characteristics of the acoustic wave terms. Although Kirchhoff's characteristic equation for wavenumber is identical to the equation used to determine the true wavenumber poles of the Green's functions in wavenumber and time domain, the approximation of the wavenumbers by Kirchhoff and for the model Green's function are different. The model Green's function approximation includes the effect of viscosity and thermal conductivity on the propagation velocity, equation 5.31, which was necessary to obtain compatible wavenumber and frequency poles. Either by further approximating the model Green's function or by using the approximate wavenumber poles, equation 4.65, the spatial attenuation becomes

$$e^{-k_w' |r_x|} \approx e^{-\frac{\omega_o^2}{2C_1^3} D_1 |r_x|} = e^{-\frac{\omega_o^2}{2C_1^3} [(\frac{4}{3} \frac{\eta}{\rho} + \frac{\zeta}{\rho}) + (\frac{\kappa}{C_v} - \frac{\kappa}{C_p})] |r_x|} \qquad (5.40a)$$

and the phase velocity becomes

$$\frac{\omega_o}{k_w''} \approx C_1 \qquad (5.40b)$$

which are exactly those of Kirchhoff. Thus the model Green's function yields results which are consistant with those previously derived for small viscosity and thermal conductivity. In addition the thermal viscous model Green's function yields the effect on the acoustic propigation velocity which is necessary for the Green's functions in alternate domains to be compatible. As will be seen, this results with a compatible response to forcing at a given wavenumber.

Spherical sinusoid spatial and impulse temporal forcing

The product of impulse temporal forcing, equation 5.12, and radial sinusoidal forcing, equation 5.15, yields equal forcing of all frequencies and forcing of one wavenumber, $\vec{k}_{ro}$, in the radial direction,

$$f(\vec{r},t) = e^{-ik_{ro}r}\delta(t) \quad , \quad f(\vec{k},\omega) = (2\pi)^3\delta(\vec{k}-\vec{k}_{ro}). \tag{5.41a,b}$$

The spherical response in terms of the impusle Green's function results from evaluation of the inverse Fourier transformation of equation 3 for this forcing,

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, G(\vec{k},\omega)\; (2\pi)^3\delta(\vec{k}-\vec{k}_{ro}) =$$

$$= e^{-ik_{ro}r}\, G(\vec{k}_{ro},t) \tag{5.42}$$

Due to the impulse Green's function, initial condition Green's function analogy for impulsive temporal forcing, equation 5.10, if the initial condition term is spherically sinusoidal,

$$Y(\vec{r},t=0) = e^{-ik_{ro}r} \quad , \quad Y(\vec{k},t=0) = (2\pi)^3\delta(\vec{k}-\vec{k}_{ro}) \tag{5.43a,b}$$

then the relaxation may be expressed as

$$x(\vec{r},t) = e^{-ik_{ro}r}\, G_{xy}(\vec{k}_{ro},t) \quad . \tag{5.44}$$

Thus the response to or relaxation from sinusoidal radial forcing is the product of a sinusoidal radial oscilation and the Green's function in wavenumber and time domain evaluated at the forced wavenumber. The Green's function determines the temporal characteristics of the response or relaxation. The diffusion type Green's functions, equations 4.24, exhibit exponential temporal decay only. The wave type Green's functions may exhibit exponential temporal decay and oscilation, equations 4.42, or a single exponential decay, equations 4.45, or two separate exponential temporal decays, equation 4.49, depending on the respective temporal damping conditions, underdamped, critically damped, or overdamped. The thermal-viscous type Green's functions, equations 4.94, simultaneously exhibit both the diffusion type and wave type terms. Using the thermal viscous model impulse Green's function with temporally underdamped wave type behavior, equation 4.86a, the response becomes

$$x(\vec{r},t) = \frac{U(t)e^{-ik_{ro}r}}{[C_1^2-D_2(D_1-D_2)k_{ro}^2]}\left\{e^{-\omega'_{wm}t}\left[\frac{1}{k^2}\cos(\omega''_{wm}t)+\left(\frac{D_1}{2}-D_2\right)\frac{\sin(\omega''_{wm}t)}{\omega''_{wm}}\right]-\frac{1}{k^2}e^{-\omega'_{dm}t}\right\}_{k=k_{ro}} \tag{5.45}$$

The real and imaginary ports of the underdamped complex model wave frequency, ω_{wm}, and the complex model diffusion frequency ω_{dm}, as defined by equations 4.84 and 4.81 are the product of i and the frequency poles of the Green's function in wavenumber and frequency domain. They are functions of the parameters of the medium and the forcing wavenumber which also determine the temporal damping condition. The wave related response is attenuated temporally with exponential dependence on ω'_{wm} as it propagates radially with the phase velocity ω''_{wm}/k_{ro}. The diffusion related response decays temporally with exponential dependence on ω'_{dm} without propagation.

Plane sinusoid spatial and impulse temporal forcing

Equal forcing of all frequencies and forcing of one wavenumber, $\vec{k}_{xo}$, in the x direction may be obtained from the product of impulse temporal forcing, equation 5.12, and plane sinusoid forcing, equation 5.17, as

$$f(\vec{r},t) = e^{-ik_{xo}r_x}\,\delta(t) \quad , \quad f(\vec{k},\omega) = (2\pi)^3\delta(k_x-k_{xo})\delta(k_y)\delta(k_z) \qquad (5.46a,b)$$

The plane response may be expressed in terms of the impulse Green's function in wavenumber and time domain by evaluation of the inverse transformation of equation 5.3 upon substitution of the forcing,

$$x(\vec{r}_x,t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} G(\vec{k},\omega)(2\pi)^3\delta(k_x-k_{xo})\delta(k_y)\delta(k_z) =$$

$$= e^{ik_{xo}r_x}\, G(\vec{k}_{xo},t) \qquad (5.47)$$

By the impulse Green's function, initial condition Green's function analogy, the initial condition term

$$Y(\vec{r},t{=}0) = e^{-ik_{xo}r_x} \quad , \quad Y(\vec{k},t{=}0) = (2\pi)^3\delta(k_x-k_{xo})\delta(k_y)\delta(k_z) \qquad (5.48a,b)$$

yields the plane relaxation

$$x(\vec{r}_x,t) = e^{-ik_{xo}r_x}\, G_{xy}(\vec{k}_{xo},t). \qquad (5.49)$$

The response or relaxation in space and time domain from plane sinusoidal oscilation in space is equal to the product of the spatial oscilation and the Green's function in wavenumber and time domain. As occurs with radial sinusoidal forcing and impulsive temporal forcing, the Green's function determines the temporal characteristics of the response or relaxation via the wavenumber poles of the Green's function in wavenumber and frequency domain.

Both the temporally underdamped wave type behavior and the diffusive type behavior are displayed when the temporally underdamped thermal viscous model impulse Green's function is used to determine the response

$$x(\vec{r}_x,t)=\frac{U(t)e^{-ik_{xo}r_x}}{[C_1^2-D_2(D_1-D_2)k_{xo}^2]}\left\{e^{\frac{D_1}{2}k_{xo}^2 t}\left[\frac{1}{k_{xo}^2}\cos\left(\sqrt{C_1^2k_{xo}^2-\frac{D_1^2}{4}k_{xo}^4}\,t\right)+\left(\frac{D_1}{2}-D_2\right)\frac{\sin\left(\sqrt{C_1^2k_{xo}^2-\frac{D_1^2}{4}k_{xo}^4}\,t\right)}{\sqrt{C_1^2k_{xo}^2-\frac{D_1^2}{4}k_{xo}^4}}\right]\frac{1}{k_{xo}^2}e^{-D_2k^2t}\right\} . \tag{5.50}$$

The plane wave response is similar to the spherical wave response given by equation 5.44. The underdamped wave type response decays exponentially in time by the term

$$e^{-\omega'_{wm}t} = e^{\frac{D_1}{2}k_{xo}^2 t} = e^{\frac{k_{xo}^2}{2}[\frac{4}{3}\frac{\eta}{\rho}+\frac{\zeta}{\rho})+(\frac{\kappa}{C_v}-\frac{\kappa}{C_p})]t} \tag{5.51a}$$

as it propigates in the x direction at the phase velocity

$$\frac{\omega''_{wm}}{k_{xo}} = \sqrt{C_1^2-\frac{D_1^2}{4}k_{xo}^2} \quad . \tag{5.51b}$$

The diffusion type response decays exponentially in time by the term

$$e^{-\omega'_{dm}t} = e^{-D_2k_{xo}^2 t} \tag{5.52}$$

without propagation.

The temporal attenuation of the wave response to wavenumber forcing, equation 5.51a, is exactly the temporal attenuation derived by Landau and Lifshitz and previously discussed, equation 5.38. Their approximation of uneffected propagation velocity which they used to derive the spatial attenuation of a response to frequency forcing, equation 5.39, is an approximation of the propagation velocity, equation 5.51b. The model Green's function yields compatible spatial characteristics of the frequency forced acoustic response, equation 5.31, and temporal characteristics of the wavenumber forced acoustic response, equation 5.51, when the propagation velocity is effected by small viscosity and thermal conductivity.

Spherical impulse spatial and impulse temporal forcing

Forcing which is the product of impulsive temporal forcing, equation 5.12, and impulsive forcing at a point in space, equation 5.14,

$$f(\vec{r},t) = \delta(\vec{r})\delta(t) \quad , \quad f(\vec{k},\omega) = 1 \tag{5.53a,b}$$

forces all frequencies and radial wavenumbers equally. The response to this forcing, as may be evaluated by the inverse transformation of equation 5.3

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G(\vec{k},\omega) = \tag{5.54}$$

$$= G(\vec{r},t)$$

is simply the Green's function in space and time domain.

From the impulse Green's function, initial condition Green's function analogy for impulsive temporal forcing, equation 10, if the initial condition term is impulsive at a point in space

$$Y(\vec{r},t = 0) = \delta(\vec{r}) \quad , \quad Y(\vec{k},t = 0) = 1 \tag{5.55a,b}$$

then the relaxation is equal to the impulse Green's function as may also be evaluated from the spatial convolution of equation 5.4,

$$x(\vec{r},t) = \int_{-\infty}^{\infty} d^3\vec{r}\,' \, G_{xy}(\vec{r}-\vec{r}\,',t)\delta(\vec{r}\,') = G_{xy}(\vec{r},t) \tag{5.56}$$

Plane impulse spatial and impulse temporal forcing

The product of impulsive temporal forcing, equation 5.12, and impulsive forcing at a plane results in the forcing

$$f(\vec{r},t) = \delta(r_x)\delta(t) \quad , \quad f(\vec{k},\omega) = 2\pi\delta(k_y)2\pi\delta(k_z) \tag{5.57a,b}$$

forces all frequencies and all wavenumbers in the x direction equally but does not allow the wavenumber vector to have any components in the y or z directions. The plane response to this forcing, as expressed by equation 5.3, may be simplified to the form of a single integral,

$$x(\vec{r},t) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} G(\vec{k},\omega)\ 2\pi\delta(k_y)2\pi\delta(k_z) =$$

$$= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-ik_x r_x} G(\vec{k}_x,t) \quad . \tag{5.58}$$

This one dimensional inverse spatial Fourier transformation of a three dimensional Green's function which is an even function of the magnitude of the wavenumber has been related to an indefinite integral of the Green's function in space domain by equation 5.29, thus the plane response in space and time domain may be expressed as an indefinite integral over the x direction of space of the impulse Green's function in space and time domain,

$$x(\vec{r}_x,t) = -2\pi \int dr_x\, r_x\, G(\vec{r}_x,t) \tag{5.59}$$

As a result of the impulse Green's function, initial condition Green's function analogy for impulsive temporal forcing, equation 5.10, an initial condition term which is impulsive at a plane,

$$Y(\vec{r},t=0) = \delta(r_x) \quad , \quad Y(\vec{k},t=0) = 2\pi\delta(k_y)2\pi\delta(k_z) \tag{5.60a,b}$$

results in a relaxation which may be expressed as

$$x(\vec{r}_x,t) = -2\pi \int dr_x\, r_x\, G_{xy}(\vec{r}_x,t) \tag{5.61}$$

Statistical Responses

When the fluctuation response is random, due to random forcing, the response autocorrelation is related by the Green's functions to the forcing correlation. Representing a statistical average by the brackets, < > , the response autocorrelation is defined as

$$R_{xx}(\vec{r},\vec{r}';t,t') = \langle x(\vec{r},t)\, x^*(\vec{r}',t')\rangle \tag{5.62}$$

The response representations of equations 5.1 or 5.3 allow the response autocorrelation to be expressed as

$$R_{xx}(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty} d^3\vec{r}''\int_{-\infty}^{\infty} d^3\vec{r}'''\int_{-\infty}^{\infty} dt''\int_{-\infty}^{\infty} dt''' G(\vec{r}-\vec{r}'',t-t'')G^*(\vec{r}-\vec{r}''',t'-t''')\cdot \cdot\langle f(\vec{r}'',t'')f(\vec{r}''',t''')\rangle \tag{5.63}$$

or as

$$R_{xx}(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty}\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}\int_{-\infty}^{\infty}\frac{d^3\vec{k}'}{(2\pi)^3}e^{i\vec{k}'\cdot\vec{r}'}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t'}G(\vec{k},\omega)G^*(\vec{k}',\omega')\cdot \cdot\langle f(\vec{k},\omega)f^*(\vec{k}',\omega')\rangle \tag{5.64}$$

where the forcing corrélation is

$$R_{ff}(\vec{r}'',\vec{r}''';t'',t''') = \langle f(\vec{r}'',t'')f^*(\vec{r}''',t'')\rangle \tag{5.65}$$

and the forcing power spectrum is

$$S_{ff}(\vec{k},\vec{k}';\omega,\omega') = \langle f(\vec{k},\omega)f^*(\vec{k}',\omega')\rangle \quad . \tag{5.66}$$

The transformation between correlation and power spectrum is given by the pair

$$R(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty}\frac{d^3\vec{k}}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{r}}\int_{-\infty}^{\infty}\frac{d^3\vec{k}'}{(2\pi)^3}e^{i\vec{k}'\cdot\vec{r}'}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{=i\omega' t'}S(\vec{k},\vec{k}';\omega,\omega') \tag{5.67a}$$

$$S(\vec{k},\vec{k}';\omega,\omega')=\int_{-\infty}^{\infty} d^3\vec{r}\; e^{i\vec{k}\cdot\vec{r}}\int_{-\infty}^{\infty} d^3\vec{r}'\; e^{-i\vec{k}'\cdot\vec{r}'}\int_{-\infty}^{\infty} dt\; e^{-i\omega t}\int_{-\infty}^{\infty} dt' e^{i\omega' t'}R(\vec{r},\vec{r}';t,t'). \tag{5.67b}$$

Temporally nonstationary and random forcing

In order to consider the characteristics of fluctuations of a given wavelength to temporally nonstationary and random forcing the forcing will be separated into the product of spatial forcing and temporal forcing. The spatial forcing will be the sinusoidal radial forcing of equation 5.15. The nonstationary random temporal forcing considered will be the product of a nonstationary envelope, $f_E(t)$, and random noise, $f_N(t)$, as shown

$$\tilde{f}(t) = f_E(t)f_N(t) \quad , \quad \tilde{f}(\omega) = f_E(\omega)f_n(\omega) \tag{5.68a,b}$$

Substituting the forcing in the response autocorrelation form of equation 5.64 and performing the spatial integrations yields

$$R_{xx}(\vec{r},\vec{r}';t,t')=e^{i\vec{k}_{ro}\cdot(\vec{r}'-\vec{r})}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega't'}G(\vec{k}_{ro},\omega)G^*(\vec{k}_{ro},\omega')<\tilde{f}(\omega)\tilde{f}^*(\omega')> \tag{5.69}$$

where the spatial correlation is trivial. Considering one point in space the correlation in time may be expressed as

$$R(t,t')=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega t}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega't'}G(\vec{k}_{ro},\omega)G^*(\vec{k}_{ro},\omega')S_{\tilde{f}\tilde{f}}(\omega,\omega') \ . \tag{5.70}$$

Thus the response autocorrelation in time is related to the impulse Green's function in wavenumber and frequency domain and the power spectrum of the temporal forcing. Caughey [17], and Barnoski and Maurer [18] have considered this type of temporal analysis with nonstationary random excitation and showed that the mean square response could be expressed as

$$R_{xx}(\vec{r},\vec{r};t,t) = <x^2(\vec{r},t)> = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_{f_Nf_N}(\omega)\left|\Lambda(t,\omega)\right|^2 \tag{5.71}$$

where

$$\Lambda(t,\omega) = \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}e^{i\omega't}G(\vec{k}_{ro},\omega')F_E(\omega+\omega') \ . \tag{5.72}$$

They considered both unit step and rectangular step envelopes,

$$F_E(t) = U(t)$$
$$F_E(t) = U(t) - U(t-t_o) \ , \qquad (5.73a,b)$$

with both white and correlated noises,

$$R_{f_N f_N}(t,t') = R_o \delta(t-t')$$
$$R_{f_N f_N}(t,t') = R_o \ e^{-\alpha|t-t'|} \cos(\rho(t-t')) \qquad (5.74a,b)$$

for an acoustic wave type Green's function. The mean square responses to these nonstationary random temporal forcings have been derived but will not be included here.

Random spatial forcing or initial condition

In order to consider the relaxation characteristics of a fluctuation due to spatially random forcing of all frequencies at an instant in time or due to a spatially random initial condition the forcing will be separated into the product of a random spatial forcing and the impulsive temporal forcing of equation 5.12. Substituting this forcing into the response autocorrelation form of equation 5.63 and evaluating the temporal integrals yields

$$R_{xx}(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty} d^3\vec{r}''\int_{-\infty}^{\infty} d^3\vec{r}'''G(\vec{r}-\vec{r}'',t)G(\vec{r}'-\vec{r}''',t')\langle\bar{f}(\vec{r}'')\bar{f}(\vec{r}''')\rangle. \tag{5.75}$$

Due to the impulse Green's function, initial value Green's function analogy for impulsive temporal forcing the correlation of the relaxations due to single initial condition terms may be expressed as

$$R_{x_1,x_2}(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty} d^3\vec{r}''\int_{-\infty}^{\infty} d^3\vec{r}'''G_{x_1y_1}(\vec{r}-\vec{r}'',t)G_{x_2y_2}(\vec{r}'-\vec{r}''',t')\ Y_1(\vec{r}'',t-0)Y_2(\vec{r}''',t=0) \tag{5.76}$$

Also, the tensorial correlation may be expressed as

$$R_{v_iv_j}(\vec{r},\vec{r}';t,t')=\int_{-\infty}^{\infty} d^3\vec{r}''\int_{-\infty}^{\infty} d^3\vec{r}'''G_{im}(\vec{r}-\vec{r}'',t)G_{jn}(\vec{r}'-\vec{r}''',t')\langle\bar{f}_m(\vec{r}'')\bar{f}_n(\vec{r}''')\rangle\ . \tag{5.76}$$

If the spatial forcing of equation 5.75 is completely random, the forcing correlation is

$$R_{\bar{f}\bar{f}}(\vec{r}'',\vec{r}''') = \langle\bar{f}(\vec{r}'')\bar{f}(\vec{r}''')\rangle = F_0\delta(\vec{r}''-\vec{r}''') \tag{5.77}$$

and all wavenumbers are excited. One spatial integral may be evaluated and the other may be reexpressed by integration variable substitution yielding

$$R_{xx}(\vec{r},\vec{r}';t,t') = \int_{-\infty}^{\infty} d^3\vec{r}''\ G(\vec{r}'',t)G(\vec{r}''+\vec{r}-\vec{r}',t')\quad . \tag{5.78}$$

The spatial response autocorrelation at one time becomes

$$R_{xx}(\vec{r},\vec{r}') = \int_{-\infty}^{\infty} d^3\vec{r}''\ G(\vec{r}'',t)G(\vec{r}''+\vec{r}-\vec{r}',t)\quad . \tag{5.79}$$

This spatial autocorrelation is easily evaluated for a diffusive Green's function since the diffusive Green's function in space and time domain is a Gaussian function in space.

Although the closed form of the thermal viscous Green's function in space and time domain is awkward, the tripple spatial integration of the autocorrelation has been partially performed. By expressing the error functions of the Green's function in integral form and changing integration order only single integration forms remain which may perhaps be evaluated numerically. Hopefully evaluation could yield viscous and thermal conduction effects for compressible turbulence.

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APPENDIX A-1

A three dimensional inverse Fourier spatial transformation requires the evaluation of a triple integral. Due to the inherent spherical symmetry of the functions to be transformed the triple integration may be simplified to a single integration.

Consider the inverse transformation integral expressed in terms of a cartesian coordinate system,

$$F(\vec{r}) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} e^{-ik_x r_x} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} e^{-ik_y r_y} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{-ik_z r_z} F(\vec{k}) =$$

$$= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{-i\vec{k}\cdot\vec{r}} F(\vec{k}) \quad . \qquad \text{(A-1.1)}$$

Since the integrations are to be performed over the entire wavenumber space, the coordinate directions are arbitrary. It is convenient to align positive z axis along the distance vector, $\vec{r}$, from the point of excitation to the receiver as shown in figure A-1.1. The orientation of the coordinate axis allows the integral to be simplified when expressed in terms of a spherical coordinate system. When the spherical coordinate directions k, θ, ϕ are defined as shown in figure A-1.1, the inverse Fourier transformation may be expressed as

$$F(\vec{r}) = \int_0^{\infty} \frac{dk}{2\pi} \int_0^{\pi} \frac{kd\theta}{2\pi} \int_0^{2\pi} \frac{k\sin\theta d\phi}{2\pi} e^{-ikr\cos\theta} F(\vec{k}) \quad . \qquad \text{(A-1.2)}$$

Since the functions to be transformed are spherically symmetric they must be independent of the non-radial directions, θ and ϕ, and two of the three required integrations may be performed. The integration with respect to ϕ is simply

$$\int_0^{2\pi} d\phi = \phi \Big|_{\phi=0}^{\phi=2\pi} = 2\pi \qquad \text{(A-1.3)}$$

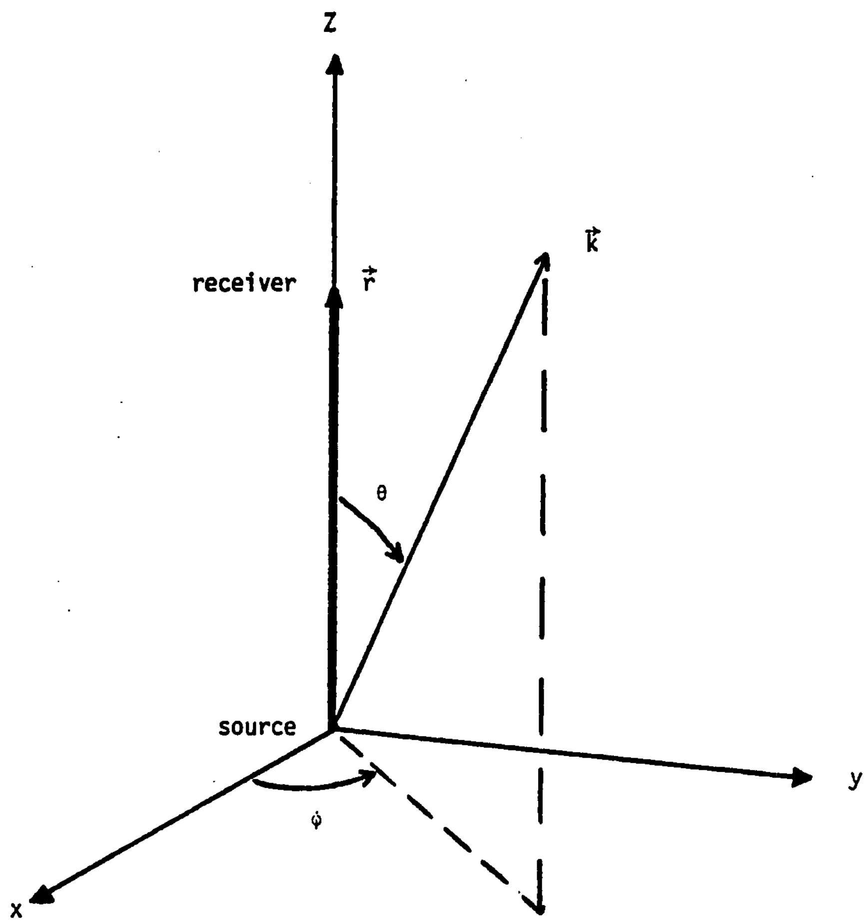

Figure A-1.1

Spherical geometry of wavenumber vector

and the integration with respect to θ may be evaluated as

$$\int_0^\pi d\theta \sin\theta \, e^{-ikr\cos\theta} = \frac{1}{ikr} \, e^{-ikr\cos\theta} \Big|_{\theta=0}^{\theta=\pi} =$$

$$= \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right) = \frac{2}{kr}\sin(kr) \quad . \qquad \text{(A-1.4)}$$

Thus the inverse transformation requires only one integration to be performed and simplifies to

$$F(\vec{r}) = \frac{1}{2\pi^2 r} \int_0^\infty dk \, k \sin(kr) \; F(\vec{k}) \quad . \qquad \text{(A-1.5)}$$

As a result of spherical symmetry, $F(\vec{k})$ must be an even function of k because the sign of k, which changes the direction of $\vec{k}$ by 180°, must not have any effect. Since the term k sin (kr) is an even function of k, the entire integrand of equation A-1.5 is an even function of k. This allows the integration limits to be changed so that the integration is to be performed along the entire real k axis, rather than just the positive real k axis, and the transformation becomes

$$F(\vec{r}) = \frac{1}{4\pi^2 r} \int_{-\infty}^\infty dk \, k \sin(kr) \; F(\vec{k}) \quad . \qquad \text{(A-1.6)}$$

These limits facilitate the use of the Cauchey Residue theorem for the evaluation of the integral since useful complex contours include the entire real axis.

In many instances the integration along part of the complex contour is equal to zero due to an exponential term of the integrand. Therefore it is convenient to expand the sinusoidal term in an exponential form. Since $F(\vec{k})$ is an even function of k, it is most convenient to replace the sinusoidal term with the identity

$$\sin(kr) = \frac{1}{i} e^{ikr} - \frac{1}{i}\cos(kr) \qquad \text{(A-1.7)}$$

and express the inverse transformation in terms of two integrals as

$$F(\vec{r}) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k\, e^{ikr} F(\vec{k}) - \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k \cos(kr)\, F(\vec{k}) \quad . \qquad \text{(A-1.8)}$$

The second integration of equation 8 is equal to zero because its integrand is an odd function of k. Thus the spherical symmetry of the transformed function, $F(\vec{k})$ only an even function of k, allows the inverse spatial Fourier transformation to be expressed in the simplified form,

$$F(\vec{r}) = \frac{1}{i4\pi^2 r}\int_{-\infty}^{\infty} dk\, k\, e^{ikr}\, F(\vec{k}) \quad . \qquad \text{(A-1.9)}$$

APPENDIX A-2

The inverse spatial Fourier transformations from wavenumber and frequency domain to space and frequency domain of all of the Green's functions involve wavenumber integrations which may be evaluated by contour integrations on the complex wavenumber plane since the integrals possess wavenumber poles. The two poles, k_1 and k_2, of either the diffusive or damped wave type Green's functions are opposite complex wavenumbers so the required integrations are of the form

$$_dI_1^n = \int_{+\infty}^{\infty} dk \, \frac{k^{2n+1} \, e^{ikr}}{(k-k_1)(k+k_1)} \quad , \qquad \text{(A-2.1)}$$

where n is an integer which is greater than or equal to zero. The acoustic tensorial type Green's function possesses a zero wavenumber pole in addition to the complex pair of poles and requires evaluation of the same form with n equal to minus one. The thermal-viscous type Green's functions possesses two pairs of opposite complex wavenumber poles; k_1, k_2 and k_3, k_4; and requires evaluation of the integral form

$$I_1^n = \int_{-\infty}^{\infty} dk \, \frac{k^{2n+1} \, e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} \qquad \text{(A-2.2)}$$

for transformation to space and frequency domain. This form with n equal to minus one, which possesses a zero wavenumber pole, is appropriate for the transformation of the thermal-viscous tensorial type Green's function.

The wavenumber poles of the diffusive type, equations 4.16, the damped wave type, equations 4.32, and the model thermal-viscous type, equations 4.71, Green's functions as well as their common integration contours, C_1 and C_2, are shown in figures A-2.1, A-2.2 and A-2.3 respectively. The contours consist of a real segment, along which the integrations of interest are to be evaluated, and a closing semicircular segment. For each of the Green's functions, the odd numbered wavenumber poles possess negative imaginary parts and their opposites, the even numbered poles, possess positive imaginary parts. Thus the odd numbered poles are enclosed by the complex contour C_1 in the negative imaginary wavenumber half plane and the even numbered poles are enclosed by the complex contour C_2 in the positive imaginary half plane. According to the Cauchey residue theorem an integration counterclockwise around a closed contour is equal to the product of $2\pi i$ and the sume of the residues of the enclosed poles. Thus the integration around the contour C_1 may be expressed for the diffusive and damped wave type Green's functions as

$$\int_{C_1} dk\, \frac{k^{2n+1} e^{ikr}}{(k-k_1)(k+k_1)} = \int_A^{-A} dk\, \frac{k^{2n+1} e^{ikr}}{(k-k_1)(k+k_1)} + \int_\pi^{2\pi} iAe^{i\alpha} d\alpha\, \frac{(Ae^{i\alpha})^{2n+1} e^{irA\cos\alpha - rA\sin\alpha}}{(Ae^{i\alpha}-k_1)(Ae^{i\alpha}+k_1)} =$$

$$= 2\pi i \text{ Residue } (k=k_1) \qquad \text{(A-2.3a)}$$

and may be expressed for the thermal-viscous type Green's functions as

$$\int_{C_1} dk\, \frac{k^{2n+1} e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \int_A^{-A} dk\, \frac{k^{2n+1} e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} + \int_\pi^{2\pi} iAe^{i\alpha} d\alpha\, \frac{(Ae^{i\alpha})^{2n+1} e^{irA\cos\alpha - rA\sin\alpha}}{(A^2e^{i2\alpha}-k_1^2)(A^2e^{i2\alpha}-k_3^2)} =$$

$$= 2\pi i \,[\text{Residue } (k=k_1) + \text{Residue } (k=k_3)] \quad . \qquad \text{(A-2.3b)}$$

In the limit as the circular contour radius, A, becomes infinite the integration along the real wavenumber axis is identically the required integration and the integration along the closing half circle equals zero if

$$r \sin \alpha > 0 \qquad \text{(A-2.4)}$$

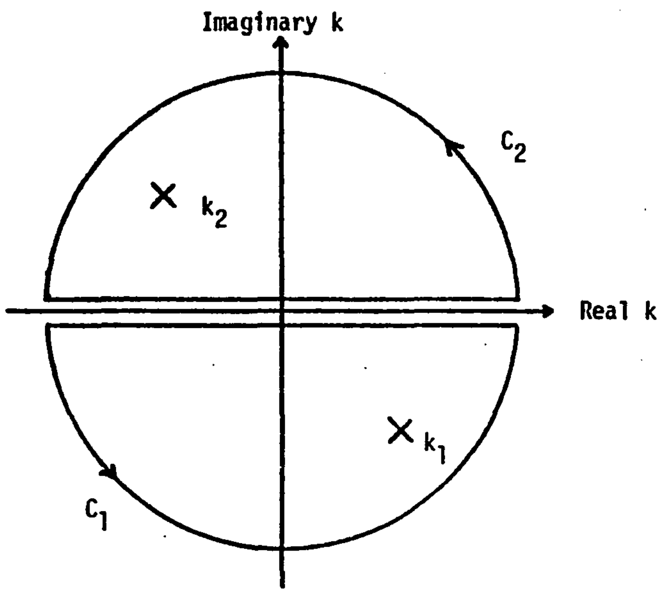

Figure A-2.1

Complex integration contours with diffusion wavenumber poles

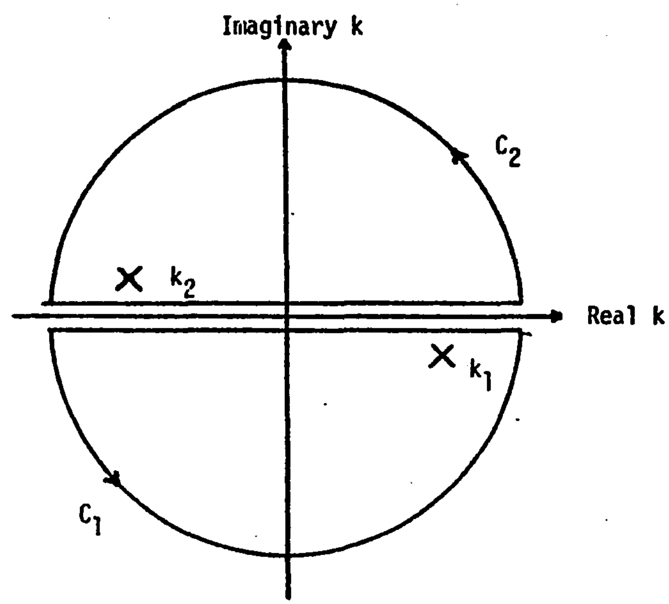

Figure A-2.2

Complex integration contours with acoustic wavenumber poles

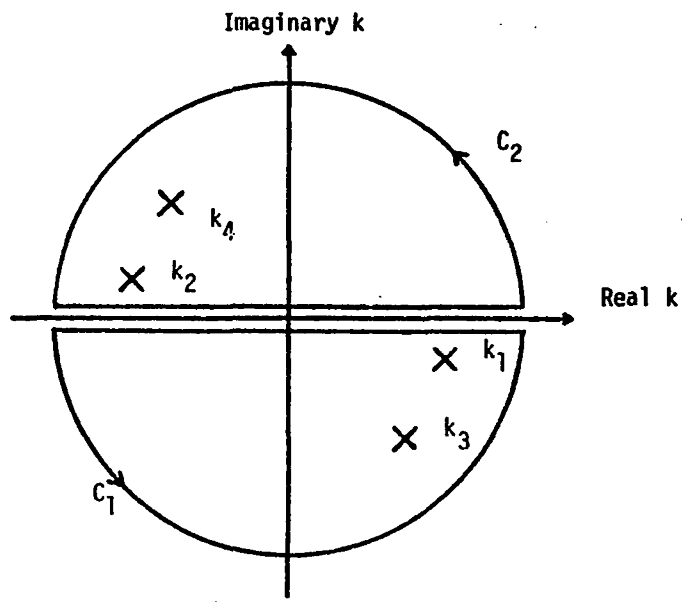

Figure A-2.3

Complex integration contours with

thermal-viscous wavenumber poles

which is true for negative values of r since sin α is negative for any point in that half plane.

For negative values of r the required integrations are equal to the product of $-2\pi i$ and the sum of the residues of the odd numbered poles. The diffusive and damped wave type integrations may be evaluated as

$$\int_{-\infty}^{\infty} dk \frac{k^{2n+1}e^{ikr}}{(k-k_1)(k+k_1)} = -i\pi\, k_1^{2n}\, e^{ik_1r} \quad , \quad r < 0 \tag{A-2.5a}$$

and the thermal-viscous type integration becomes

$$\int_{-\infty}^{\infty} dk \frac{k^{2n+1}e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \frac{-i\pi}{(k_1^2-k_3^2)}[k_1^{2n}\, e^{ik_1r} - k_3^{2n}\, e^{ik_3r}] \quad , \quad r < 0 \quad . \tag{A-2.5b}$$

Integrations along the contour C_2 yield the integral evaluations for positive values of r. Since the enclosed, even numbered, wavenumber poles are opposites of the odd numbered poles the diffusive and damped wave type integration may be expressed as

$$\int_{C_2} dk \frac{k^{2n+1}e^{ikr}}{(k-k_1)(k+k_1)} = \int_{-A}^{A} dk \frac{k^{2n+1}e^{ikr}}{(k-k_1)(k+k_1)} + \int_0^{\pi} iAe^{i\alpha}d\alpha \frac{(Ae^{i\alpha})^{2n+1}e^{irA\cos\alpha - rA\sin\alpha}}{(k-k_1)(k+k_1)} =$$

$$= 2\pi i \text{ Residue } (k = k_2 = -k_1) \tag{A-2.6a}$$

and the thermal-viscous type integration becomes

$$\int_{C_2} dk \frac{k^{2n+1}e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \int_{-A}^{A} dk \frac{k^{2n+1}e^{ikr}}{(k^2-k_1^2)(k^2k_3^2)} + \int_0^{\pi} iAe^{i\alpha}d\alpha \frac{(Ae^{i\alpha})^{2n+1}e^{irA\cos\alpha - rA\sin\alpha}}{(k^2-k_1^2)(k^2-k_3^2)} =$$

$$= 2\pi i \,[\text{Residue } (k = k_2 = -k_1) + \text{Residue } (k = k_4 = -k_3)] \; . \tag{A-2.6b}$$

In the limit as the semicircular closing contour radius becomes infinite the condition of equation A-2.4 is satisfied and the integration along the closing contour equals zero for positive values of r. The required integrations are equal to the product of $2\pi i$ and the sum of the residues of the poles enclosed. In terms of the odd numbered wavenumber poles the diffusive and damped wave type integrations may be evaluated as

$$\int_{-\infty}^{\infty} dk \frac{k^{2n+1} e^{ikr}}{(k-k_1)(k+k_1)} = i\pi k_1^{2n} \; e^{-ik_1 r} \quad , \qquad r > 0 \tag{A-2.7a}$$

and the thermal-viscous type integration may be evaluated as

$$\int_{-\infty}^{\infty} dk \frac{k^{2n+1} e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \frac{i\pi}{(k_1^2-k_3^2)} [k_1^{2n} \; e^{-ik_1 r} - k_3^{2n} \; e^{-ik_3 r}] \quad , \quad r > 0 \; . \tag{A-2.7b}$$

The required integrations have been evaluated separately for positive and negative values of r and may be expressed functionally for all values of r as

$$_dI_1^n = \int_{-\infty}^{\infty} dk \frac{k^{2n+1} e^{ikr}}{(k-k_1)(k+k_1)} = \frac{r}{|r|} \, i\pi \; k_1^{2n} \; e^{-ik_1|r|} \quad , \quad n \geq 0 \tag{A-2.8a}$$

$$I_1^n = \int_{-\infty}^{\infty} dk \frac{k^{2n+1} e^{ikr}}{(k^2-k_1^2)(k^2-k_3^2)} = \frac{r}{|r|} \frac{i\pi}{(k_1^2-k_3^2)} [k_1^{2n} \; e^{-ik_1|r|} - k_3^{2n} \; e^{-ik_3|r|}] \; , \; n \geq 0 \; . \tag{A-2.8b}$$

The acoustic tensorial and thermal-viscous tensorial type Green's functions also possess zero wavenumber poles which must be avoided by the contour. These poles may be either included or excluded from the interior of the closed contour by small semicircular contours without effecting the result of the required integration, the principal part of the integration along the real wavenumber axis. The integration contours, C_3 and C_4, and the appropriate wavenumber poles are shown by figures A-2.4, A-2.5 and A-2.6 where the nonzero wavenumber poles

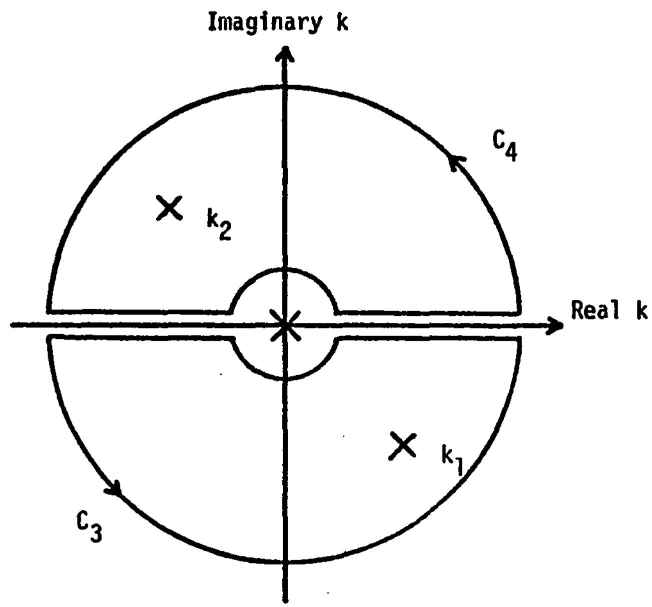

Figure A-2.4

Complex integration contours with zero and diffusion wavenumber poles

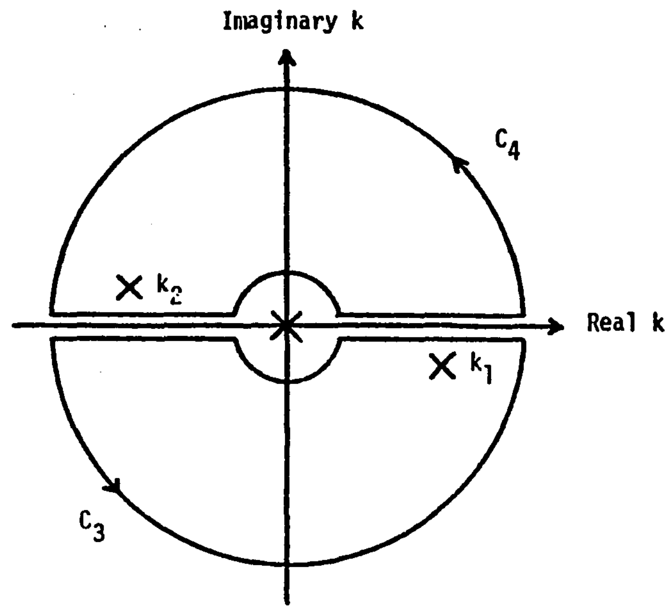

Figure A-2.5

Complex integration contours with zero and acoustic wavenumber poles

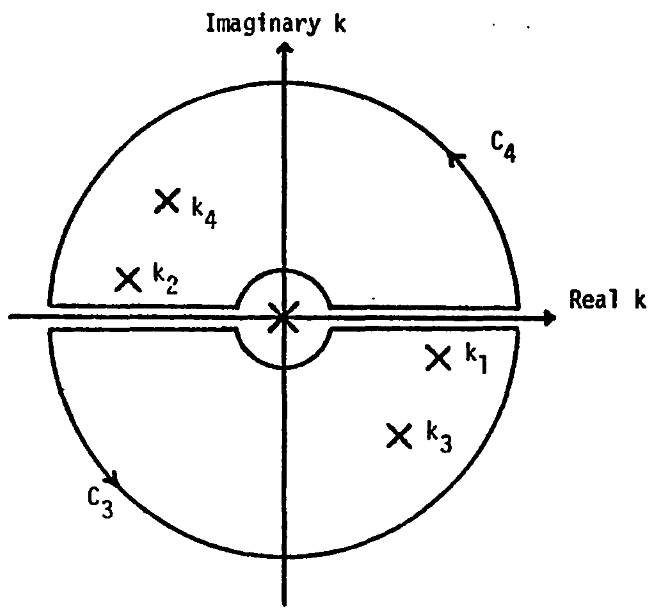

Figure A-2.6

Complex integration contours with zero and thermal-viscous wavenumber poles appropriate for the transverse acoustic tensorial, longitudinal acoustic tensorial and longitudinal thermal-viscous tensorial type Green's functions are exactly those of the diffusive, damped wave and thermal-viscous types.

In the limit as A becomes infinite the integration along the closing contour equals zero for negative values of r, according to equation A-2.4, and in the limit as ε approaches zero the real wavenumber integration equals its principal value resulting in the integration around the contour C_3 for the transverse and longitudinal acoustic tensorial Green's functions,

$$\int_{C_3} dk \frac{e^{ikr}}{k(k-k_1)(k+k_1)} = P\int_{\infty}^{-\infty} dk \frac{e^{ikr}}{k(k-k_1)(k+k_1)} + 0 + \operatorname*{limit}_{\varepsilon\to 0} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha} d\alpha \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon e^{i\alpha}-k_1)(\varepsilon e^{i\alpha}+k_1)} =$$

$$= 2\pi i \text{ Residue } (k = k_1) \quad , \qquad \text{(A-2.9a)}$$

and for the longitudinal thermal-viscous tensorial Greens function,

$$\int_{C_3} dk \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = P\int_{\infty}^{-\infty} dk \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} + 0 + \operatorname*{limit}_{\varepsilon\to 0} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha} d\alpha \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon^2 e^{i2\alpha}-k_1^2)(\varepsilon^2 e^{i2\alpha}-k_3^2)} =$$

$$= 2\pi i \left[\text{Residue } (k = k_1) + \text{Residue } (k = k_3)\right] \quad . \qquad \text{(A-2.9b)}$$

The integrations around the zero wavenumber poles are equal to the product of $-\pi i$ and the residue of the pole in the limit as ε approaches zero or may be integrated directly as

$$\operatorname*{limit}_{\varepsilon\to 0} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha} d\alpha \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon e^{i\alpha}-k_1)(\varepsilon e^{i\alpha}+k_1)} = \int_{2\pi}^{\pi} i d\alpha \frac{1}{(-k_1)(k_1)} = \frac{i\pi}{k_1^2} \qquad \text{(A-2.10a)}$$

$$\lim_{\varepsilon\to 0} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha} d\alpha \, \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon^2 e^{i2\alpha}-k_1^2)(\varepsilon^2 e^{i2\alpha}-k_3^2)} = \int_{2\pi}^{\pi} i d\alpha \, \frac{1}{(-k_1^2)(-k_3^2)} = \frac{-i\pi}{k_1^2 k_3^2} \quad . \qquad \text{(A-2.10b)}$$

Thus the required integrations for the acoustic and thermal-viscous type tensorial Green's functions may be evaluated for negative values of r to be

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k-k)(k+k_1)} = \frac{-i\pi}{k_1^2} [e^{ik_1 r}-1] \quad , \quad r > 0 \qquad \text{(A-2.11a)}$$

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = \frac{-i\pi}{(k_1^2-k_3^2)} \left\{ \frac{1}{k_1^2} [e^{ik_1 r}-1] - \frac{1}{k_3^2} [e^{ik_3 r}-1] \right\} , \quad r > 0 \; . \qquad \text{(A-2.11b)}$$

In the limit as A becomes infinite and ε approaches zero, integration along the closing half circle of contour C_4 is equal to zero for positive values of r and the integrations for the acoustic tensorial and longitudinal thermal-viscous tensorial type Green's functions become

$$\int_{C_4} dk \, \frac{e^{ikr}}{k(k-k_1)(k+k_1)} = P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k-k_1)(k+k_1)} + \lim_{\varepsilon\to 0} \int_{\pi}^{0} i\varepsilon e^{i\alpha} d\alpha \, \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon e^{i\alpha}-k_1)(\varepsilon e^{i\alpha}+k_1)} =$$

$$= 2\pi i \;\; \text{Residue } (k = k_2 = -k_1) \qquad \text{(A-2.12a)}$$

$$\int_{C_4} dk \, \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} + \lim_{\varepsilon\to 0} \int_{\pi}^{0} i\varepsilon e^{i\alpha} d\alpha \, \frac{e^{ir\varepsilon e^{i\alpha}}}{\varepsilon e^{i\alpha}(\varepsilon^2 e^{i2\alpha}-k_1^2)(\varepsilon^2 e^{i2\alpha}-k_3^2)} =$$

$$= 2\pi i \, [\text{Residue } (k = k_2 = -k_1) + \text{Residue } (k = k_4 = -k_3)] \; . \qquad \text{(A-2.12b)}$$

The integrations around the zero wavenumber poles may be integrated directly, as shown by equations A-2.10, and are equal to the product of $-\pi i$ and residue of the poles resulting in the required integral evaluations for positive values of r,

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k-k_1)(k+k_1)} = \frac{i\pi}{k_1^2} [e^{-ik_1 r} - 1] \quad , \quad r > 0 \tag{A-2.13a}$$

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = \frac{i\pi}{(k_1^2-k_3^2)} \left\{ \frac{1}{k_1^2} [e^{-ik_1 r} - 1] - \frac{1}{k_3^2} [e^{-ik_3 r} - 1] \right\} , \; r > 0 \, . \tag{A-2.13b}$$

From equations A-2.11 and A-2.13 the integrations with simple zero wavenumber poles required for evaluation of the acoustic tensorial and the longitudinal thermal-viscous tensorial Green's functions may be expressed functionally for all values of r as

$${}_dI_1^{(-1)} = \int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k-k_1)(k+k_1)} = \frac{r}{|r|} \frac{i\pi}{k_1^2} [e^{-ik_1|r|} - 1] \tag{A-2.14a}$$

$$I_1^{(-1)} = \int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k(k^2-k_1^2)(k^2-k_3^2)} = \frac{r}{|r|} \frac{i\pi}{(k_1^2-k_3^2)} \left\{ \frac{1}{k_1^2} [e^{-ik_1|r|} - 1] - \frac{1}{k_3^2} [e^{-ik_3|r|} - 1] \right\} . \tag{A-2.14b}$$

Appendix A-3

The inverse temporal transformation from wavenumber and frequency domain to wavenumber and time domain of all of the Green's functions involve frequency integrations which may be evaluated similarly. The integrands possess complex frequency poles which allow the required integrations to be evaluated by the Cauchey residue theorem as a segment of the integration around a contour in the complex frequency plane. The integration form required for evaluation of the diffusive type Green's function possesses one frequency pole,

$$_dI_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} , \tag{A-3.1}$$

where n is an integer which is greater than or equal to zero. The same integration with n equal to minus one yielding an additional, zero frequency, pole is required for evaluation of the transverse acoustic tersorial type Green's function. The damped wave type Green's function in wavenumber and frequency domain presses two non-zero frequency poles and a corresponding transformation in integration form

$$_\omega I_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} \tag{A-3.2}$$

with n greater than or equal to zero. The transverse thermal-viscous tensorial type Green's function possesses two diffusive type frequency poles and a zero frequency pole resulting in the same integration form to be evaluated for n equal to minus one. The integration form required for the thermal-viscous type Green's functions,

$$I_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)} \tag{A-3.3}$$

displays two integrand poles, in square brackets, that are similar to those of the damped wave type and one which is similar to the diffusion type pole.

All non-zero frequency poles possess positive imaginary parts which are required to yield causal Green's functions in wavenumber and time domain. The zero frequency poles mathematically produce non causal Green's functions. In order that the Green's functions satisfy the physical condition of causality the zero frequency poles will be displaced to the positive imaginary side of the origin. The frequency poles of the diffusive type, equation 4.16c, the transverse acoustic tensorial type, equation 4.16c and zero frequency, the damped wave type, equations 4.30, the transverse thermal-viscous tensorial type, equation 4.16c, 4.130c and zero frequency, and the model thermal-viscous type, equations 4.71, Green's functions as well as their common integration respectively. Each contour consists of a segment along the real frequency axis, along which the required integrations are to be evaluated, and a closing semicircular segment. Since all of the frequency poles lie in the positive imaginary half plane, including the displaced zero frequency poles, the contour C_1, which closes in the negative imaginary half plane, encloses no poles and the contour C_2, which closes in the positive imaginary half plane encloses all of the frequency poles of each of the Green's functions.

Integrations counterclockwise around contour C_1 are according to the Cauchey residue theorem, equal to the product of $2\pi i$ and the sum of the resi- of the enclosed poles which is equal to zero since no poles are enclosed. The integration around the contour for the diffusive type and, since the zero frequency pole is displaced, the transverse acoustic tensorial type Green's functions, n equal to minus one, may be expressed as

$$\int_{C_1} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} = \int_{A}^{-A} \frac{d}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} + \int_{\pi}^{2\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{(Ae^{i\alpha}-\omega_1)} \tag{A-3.4a}$$

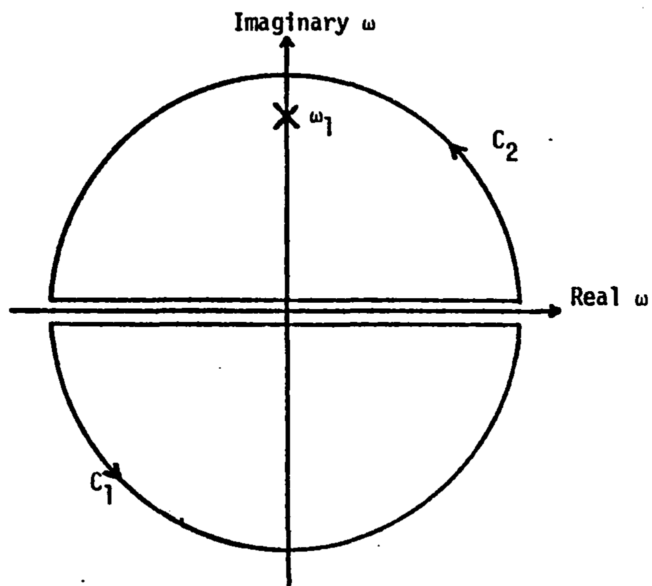

Figure A-3.1

Complex integration contours with diffusion frequency pole

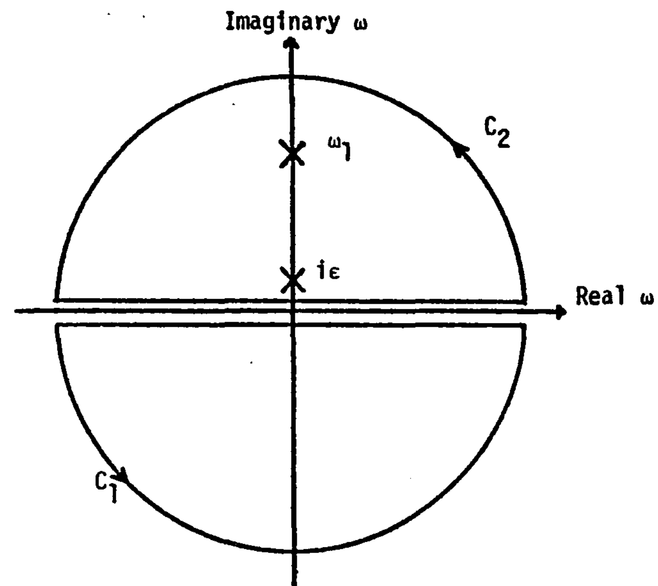

Figure A-3.2

Complex integration contours with transverse acoustic tensorial frequency poles

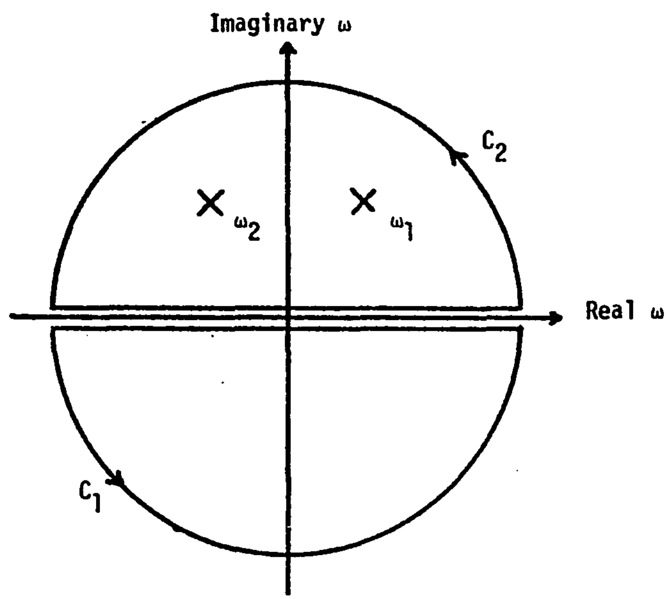

Figure A-3.3

Complex integration contours with acoustic frequency poles

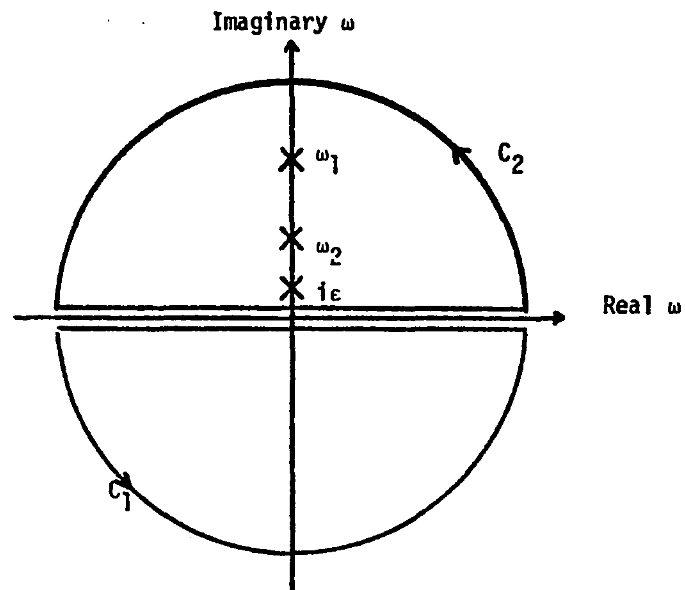

Figure A-3.4

Complex integration contours with transverse thermal-viscous tensorial frequency poles

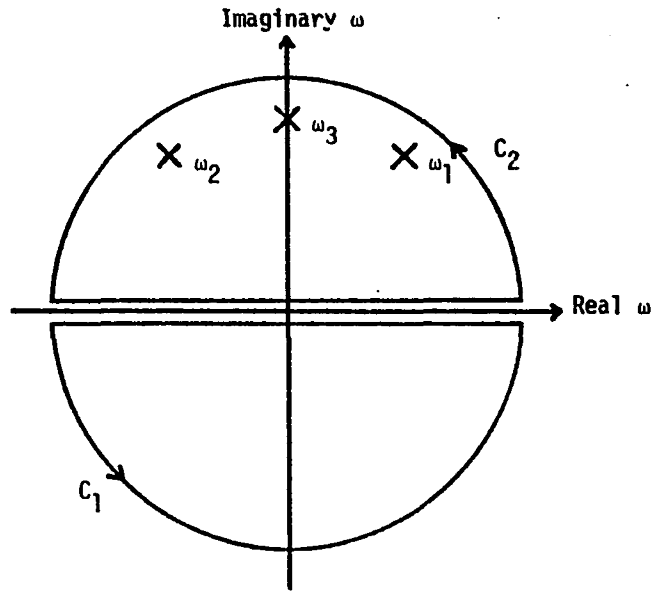

Figure A-3.5

Complex integration contours with

thermal-viscous frequency poles

The damped wave type contour integration and, due to a displaced zero frequency pole, the transverse thermal-viscous tensorial countour integration, n equal to minus one, become

$$\int_{C_1} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = \int_{A}^{-A} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} + \int_{\pi}^{2\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{(Ae^{i}-\omega_1)(Ae^{i}-\omega_2)} \qquad \text{(A-3.4b)}$$

The contour integration corresponding to the thermal-viscous type Green's function is

$$\int_{C_1} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)} = \int_{A}^{-A} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)}$$

$$+ \int_{\pi}^{2\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{[(Ae^{i\alpha}-\omega_1)(Ae^{i\alpha}-\omega_2)](Ae^{i\alpha}-\omega_3)} \qquad \text{(A-3.4c)}$$

In the limit as the clsoing contour radius, A, becomes infinite the integration along the real frequency axis becomes the required integration and the integration along the closing half circule equals zero if

$$t \sin\alpha > 0. \qquad \text{(A-3.5)}$$

This condition holds over the closing contour in the negative imaginary half plane, where $\sin\alpha$ is negative, for negative values of t. Therefore for negative time all required integrations and thus all Green's functions in wavenumber and time domain are equal to zero. If the zero frequency poles had not been displaced the integrations required for the transverse acoustic and thermal-viscous tensorial Green's functions would have been non-zero for negative time and the Green's functions would have been non causal.

Integrations counterclockwise around the contour C_2 are equal to the product of $2\pi i$ and the sum of the residues of the enclosed poles and yield the required integral evolutions for positive values of t. All of the frequency poles of each of the Green's functions in wavenumber and frequency domain lie in or have been displaced to the positive imaginary half plane and are enclosed by the contour C_2. The contour integration for the diffusive type Green's function may be expressed as

$$\int_{C_2} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} = \int_{-A}^{A} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} + \int_0^{\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{(Ae^{i\alpha}-\omega_1)} =$$

$$= 2\pi i \text{ Residue } (\omega=\omega_1), \ n \geq 0, \qquad \text{(A-3.6a)}$$

and similarly the integration for the transverse acoustic tensorial type Green's function, with a displaced zero frequency poles, becomes

$$\int_{C_2} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega(\omega-\omega_1)} = \int_{-A}^{A} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega(\omega-\omega_1)} + \int_0^{\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{e^{itA\cos\alpha - tA\sin\alpha}}{Ae^{i\alpha}(Ae^{i\alpha}-\omega_1)} =$$

$$= 2\pi i \text{ [Residue } (\omega=\omega_1) + \text{Residue } (\omega=0), \ n \geq 0, \qquad \text{(A-3.6b)}$$

The damped wave type integration around contour C_2 is

$$\int_{C_2} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = \int_{-A}^{A} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} + \int_0^{\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{(Ae^{i\alpha}-\omega_1)(Ae^{i\alpha}-\omega_2)} =$$

$$= 2\pi i \text{ [Residue } (\omega=\omega_1) + \text{Residue } (\omega=\omega_2)], \ n \geq 0, \qquad \text{(A-3.6c)}$$

and the contour integration for the transverse thermal-viscous tensorial type Green's function with a displaced zero frequency pole is

$$\int_{C_2} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = \int_{-A}^{A} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega(\omega-\omega_1)(\omega-\omega_2)} + \int_0^{\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{e^{itA\cos\alpha - tA\sin\alpha}}{Ae^{i\alpha}(Ae^{i\alpha}-\omega_1)(Ae^{i\alpha}-\omega_2)} =$$

$$= 2\pi i \;[\text{Residue } (\omega=\omega_1) + \text{Residue } (\omega=\omega_2) + \text{Residue } (\omega=0)]. \qquad \text{(A-3.6d)}$$

The thermal-viscous type contour integration becomes

$$\int_{C_2} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)} = \int_{-A}^{A} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_3)}$$

$$+ \int \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^n e^{itA\cos\alpha - tA\sin\alpha}}{[(Ae^{i\alpha}-\omega_1)(Ae^{i\alpha}-\omega_2)](Ae^{i\alpha}-\omega_3)} =$$

$$= 2\pi i \;[\text{Residue } (\omega=\omega_1) + \text{Residue } (\omega=\omega_2) + \text{Residue } (\omega=\omega_3)]. \qquad \text{(A-3.6e)}$$

In the limit as the closing contour radius, A, becomes infinite the integration alone the closing half circle in the positive imaginary frequency half plane is equal to zero when the condition of equation A-3.5 is satisfied, for positive values of t. In the same limit the integration along the real frequency axis becomes the required integration which, for positive time, is equal to the product of $2\pi i$ and the sum of the residues. Since the required integrations are equal to zero for negative values of t, each of the integration evaluations may be expressed functionally for all time as the product of the unit step function of time, U(t), and th corresponding positive time evaluation. Thus the integration required for the diffusive type Green's function may be evaluated in terms of the frequency poles as

$$_dI_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)} = U(t)i(1_1)^n e^{i\omega_1 t}, \; n \geq 0, \qquad \text{(A-3.7a)}$$

and evaluation of the transverse acoustic tensorial type integration, n equals minus one, also includes the zero frequency residue yielding

$$ {}_{d}I_{2}^{(-1)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega(\omega-\omega_1)} = U(t)\, \frac{i}{\omega_1} [e^{i\omega_1 t} - 1] \quad . \tag{A-3.7b} $$

The damped wave type integration becomes

$$ {}_{w}I_{2}^{n} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)} = U(t)\, \frac{i}{(\omega_1-\omega_2)} [(\omega_1)^n e^{i\omega_1 t} - (\omega_2)^n e^{i\omega_2 t}],\ n \geq 0, \tag{A-3.7c} $$

and the transverse thermal-viscous tensorial type integration, with a zero frequency residue also, is

$$ {}_{w}I_{2}^{(-1)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi\omega(\omega-\omega_1)(\omega-\omega_2)} e^{i\omega t} = U(t)\, \frac{i}{(\omega_1-\omega_2)} \left\{ \frac{1}{\omega_1} [e^{i\omega_1 t}-1] - \frac{1}{\omega_2} [e^{i\omega_2 t}-1] \right\} \tag{A-3.7d} $$

The thermal-viscous type Green's function integration may be evaluated, in terms of the frequency poles, as

$$ I_2^n = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^n e^{i\omega t}}{[(\omega-\omega_1)(\omega-\omega_2)](\omega-\omega_2)} = $$

$$ = U(t)\, \frac{i}{(\omega_1-\omega_3)(\omega_2-\omega_3)} \left\{ \begin{array}{l} \frac{1}{(\omega_1-\omega_2)} [(\omega_1)^n(\omega_2-\omega_3)e^{i\omega_1 t} - (\omega_2)^n(\omega_1-\omega_3)e^{i\omega_2 t}] + \\ \quad + (\omega_3)^n e^{i\omega_3 t} \\ \qquad\qquad\qquad n \geq 0. \end{array} \right\}, \tag{A-3.7e} $$

APPENDIX A-4

The inverse spatial Fourier transformations from wavenumber and time domain to space and time domain of the diffusive, transverse acoustic tensorial and transverse thermal-viscous tensorial type Green's functions all involve wavenumber integrations of the form

$$ {}_dI_3^n = \int_{-\infty}^{\infty} dk\ e^{ikr}\ k^{2n+1}\ e^{-\delta k^2} \tag{A-4.1} $$

and the transverse tensorial type Green's functions also involve integrations of the form

$$ {}_tI_3^n = \int_{-\infty}^{\infty} dk\ e^{ikr}\ k^{2n+1} \tag{A-4.2} $$

The required diffusive type integrations correspond to nonnegative values of the integer n and may be evaluated by relating them to a well known integration form. All of the transverse tensorial type integrations correspond to values of n equal to minus one, minus two or minus three and thus the integrands possess zero wavenumber poles of order one, three or five respectively. As a result, the integration forms ${}_tI_3^n$ may be evaluated by contour integrations in the complex wavenumber plane. The exponential dependence on wavenumber squared in the transverse tensorial integration forms ${}_dI_3^n$ precludes evaluation by similar contour integrations however these integrations may be evaluated by a convolution technique.

The diffusive type integrations, equation A-4.1 with non negative n, may be related to a known integration and thus evaluated. Since the δ and r dependence of the integration occurs only in exponential terms of the integrand which also involve k^2 and k respectively the term k^{2n+1} of the integrand may be conveniently expressed in terms of partial derivatives with respect to δ and r yielding the equivalent integration form

$$ {}_dI_3^n = (-1)^n \frac{\partial^n}{\partial\delta^n} \left[\frac{1}{i}\frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk\, e^{ikr}\, e^{-\delta k^2}\right], \; n \geq 0 \quad . \qquad \text{(A-4.3)} $$

The integration to be performed,

$$ I_o = \int_{-\infty}^{\infty} dk\, e^{ikr}\, e^{-\delta k^2} \quad , \qquad \text{(A-4.4a)} $$

may, by symmetry arguements, be expressed in terms of a known integration form. For the wavenumber integration limits from negative to positive infinity, only the even part of the integrand will contribute to the integration which will be equal to twice the integration from zero to infinity of the even part of the integrand. Thus the integration, I_0, is equal to twice a known integration, tabulated in Gradshteyn and Ryzhik ( ) and is evaluated as

$$ I_0 = 2 \int_0^{\infty} dk \cos(rk)\, e^{-\delta k^2} = 2\left(\sqrt{\frac{\pi}{4\delta}}\; e^{-\frac{r^2}{4\delta}}\right) \quad . \qquad \text{(A-4.4b)} $$

Utilizing this evaluation in equation A-4.3 and performing the partial differentiation with respect to r, the diffusive type integration may be expressed as

$$ {}_dI_3^n = (i)^{2n+1} \frac{\partial^n}{\partial\delta^n} \left[\frac{r\sqrt{\pi}}{2}\; \delta^{-\frac{3}{2}}\; e^{-\frac{r^2}{4\delta}}\right], \quad n \geq 0 \quad , \qquad \text{(A-4.5)} $$

which is easily evaluated for a given integer n.

The transverse tensorial type integrations, equation A-4.2 with n equal to minus one, minus two or minus three, may be evaluated by contour integrations. The integrands possess zero wavenumber poles which will be avoided and excluded from the interior of the closed contour in the complex wavenumber plane by small semicircular paths of radius ϵ as shown by figure A-4.1. The remainder

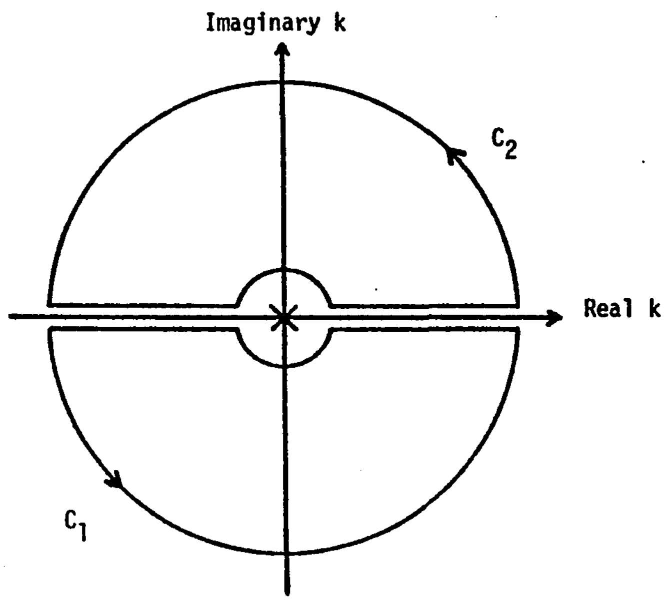

Figure A-4.1

Complex integration contours with

zero wavenumber pole

of the integration contours consist of the principal part of the integrations along the real axis (in the limit as ε approaches zero) and the semicircular closing contours of radius A in the negative or positive imaginary half planes. Since no wavenumber poles are included within the contour, the integrations around the contour are equal to zero and may be expressed as

$$\int_{C_1} dk \frac{e^{ikr}}{k^{(-2n-1)}} = P\int_{A}^{-A} dk \frac{e^{ikr}}{k^{(-2n-1)}} + \int_{\pi}^{2\pi} iAe^{i\alpha}d\alpha \frac{e^{irA\cos\alpha - rA\sin\alpha}}{(Ae^{i\alpha})^{(-2n-1}} +$$

$$+ \underset{\varepsilon\to 0}{\text{limit}} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha}d\alpha \frac{e^{ir\varepsilon e^{i\alpha}}}{(Ae^{i\alpha})^{(-2n-1)}} = 0 \quad , n = -1,-2,-3 \quad . \qquad \text{(A-4.6)}$$

In the limit as the closing contour radius, A, becomes infinite the integrations along the real wavenumber axis are the required integrations and integrations along the closing half circle equals zero if

$$r \sin\alpha > 0 \quad . \qquad \text{(A-4.7)}$$

This condition exists for negative values of r since $\sin\alpha$ is negative in the negative imaginary half plane. In the limit as ε approaches zero the integrations clockwise half way around the zero wavenumber poles are equal to minus one half the zero wavenumber pole contributions, the product of $-\pi i$ and the Residue. Thus for negative values of r the required integrations maybe expressed as

$$P\int_{-\infty}^{\infty} dk \frac{e^{ikr}}{k^{(-2n-1)}} = \underset{\varepsilon\to 0}{\text{limit}} \int_{2\pi}^{\pi} i\varepsilon e^{i\alpha}d\alpha \frac{e^{ir\varepsilon e^{i\alpha}}}{(\varepsilon e^{i\alpha})^{(-2n-1)}} = -\pi i \text{ Residue } (k = 0) \quad . \qquad \text{(A-4.8)}$$

Since the residue of a pole of order m at wavenumber equal to zero given by

$$\text{Residue } (k = 0) = \frac{1}{(m-1)!} \lim_{k\to 0} \frac{\partial^{(m-1)}}{\partial k^{(m-1)}} [(k^m)(\text{integrand})] \, , \tag{A-4.9}$$

the required integrations for negative values of r may be evaluated as

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k} = -\pi i \lim_{k\to 0} e^{ikr} = -i\pi \, , \quad r < 0 \tag{A-4.10a}$$

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k^3} = -\pi i \, \frac{1}{2!} \lim_{k\to 0} \frac{\partial^2}{\partial k^2} \, e^{ikr} = \frac{i\pi r^2}{2} \, , \quad r < 0 \tag{A-4.10b}$$

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k^5} = -\pi i \, \frac{1}{4!} \lim_{k\to 0} \frac{\partial^4}{\partial k^4} \, e^{ikr} = \frac{-i\pi r^4}{24} \, , \quad r < 0 \, . \tag{A-4.10c}$$

The integrations counterclockwise around the contour C_2 also exlcude the zero wavenumber poles and may be expressed as

$$\int_{C_2} dk \, \frac{e^{ikr}}{k^{(-2n-1)}} = P\int_{-A}^{A} dk \, \frac{e^{ikr}}{k^{(-2n-1)}} + \int_0^{\pi} iAe^{i\alpha} d\alpha \, \frac{e^{irA\cos\alpha - rA\sin\alpha}}{(Ae^{i\alpha})^{(-2n-1)}} +$$

$$+ \lim_{\varepsilon\to 0} \int_{\pi}^{0} i\varepsilon e^{i\alpha} d\alpha \, \frac{e^{ir\varepsilon e^{i\alpha}}}{(\varepsilon e^{i\alpha})^{(-2n-1)}} = 0 \, , \; n = -1,-2,-3 \, . \tag{A-4.11}$$

The integrations around the closing semicircular path are equal to zero for positive values of r, by equation A-4.7, in the limit as its radius, A, becomes infinite and in the limit as ε approaches zero the integrations clockwise half way around the zero wavenumber pole is equal to minus one half the pole contribution. Thus the required integration becomes

$$P\int_{-\infty}^{\infty} dk \, \frac{e^{ikr}}{k^{(-2n-1)}} = -\lim_{\varepsilon\to 0} \int_{\pi}^{0} i\varepsilon e^{i\alpha} d\alpha \, \frac{e^{ir\varepsilon e^{i\alpha}}}{(\varepsilon e^{i\alpha})^{(-2n-1)}} = \pi i \text{ Residue } (k = 0) \tag{A-4.12}$$

for positive values of r which is just the opposite as for negative values of r, equation A-4.8. There the required integrations may be expressed for all values of r as

$$ {}_tI_3^{(-1)} = P\int_{-\infty}^{\infty} dk\, \frac{e^{ikr}}{k} = \frac{r}{|r|}\, i\pi \qquad \text{(A-4.13a)} $$

$$ {}_tI_3^{(-2)} = P\int_{-\infty}^{\infty} dk\, \frac{e^{ikr}}{k^3} = \frac{-r}{|r|}\, \frac{i\pi r^2}{2} \qquad \text{(A-4.13b)} $$

$$ {}_tI_3^{(-3)} = P\int_{-\infty}^{\infty} dk\, \frac{e^{ikr}}{k^5} = \frac{r}{|r|}\, \frac{i\pi r^4}{24} \quad . \qquad \text{(A-4.13c)} $$

The transverse tensorial type integrations, equation A-4.1 with n equal to minus one, minus two or minus three, may be evaluated by a convolution technique. The integral form is similar to that of equation A-4.2 but includes an additional exponential term, $e^{-\delta k^2}$. The integrations may not be evaluated by integrations along the contours of figures A-4.1 and A-4.2 because this additional exponential term causes the integral evaluations along the semicircular closing contours to be non zero in the limit as its radius becomes infinite. However the complicating exponential term may be handled by an alternate expression of the integration form. The integration, which results from a three dimensional inverse Fourier transformation between $\vec{k}$ and $\vec{r}$ domains with a kernal $e^{-i\vec{k}\cdot\vec{r}}$, is of the form of a one dimensional inverse Fourier transformation between k and r domains with a kernal e^{ikr}. Such an inverse transformation relates the integration in r domain, that integration to be evaluated, to its k domain representation as

$$ {}_dI_3^n(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\; e^{ikr}\;\; {}_dI_3^n\,(k) \qquad \text{(A-4.14a)} $$

where

where

$$ {}_dI_3^n\,(k) = 2\pi k^{2n+1}\, e^{-\delta k^2} = \int_{-\infty}^{\infty} dr\, e^{-ikr}\; {}_dI_3^n\,(r) \quad . \qquad \text{(A-4.14b)} $$

The k domain integral representation may be expressed in terms of the k domain representations, similarly derived from equation A-4.4.a and equation A-4.2, of two evaluated integrations as

$$ {}_dI_3^n\,(\vec{k}) = \frac{1}{2\pi}\, I_0(k)\; {}_tI_3^n\,(k) \qquad \text{(A-4.15a)} $$

where

$$ I_0(k) = 2\pi e^{-\delta k^2} \qquad \text{and} \qquad {}_tI_3^n(k) = 2\pi k^{2n+1} \quad . \qquad \text{(A-4.15b,c)} $$

Therefore the desired integration may be reexpressed in terms of a convolution of two evaluated integrations as

$$ {}_dI_3^n(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikr}\, \frac{I_0(k)\; {}_t\bar{I}_3^n(k)}{2\pi} = \frac{1}{2\pi}\, I_0(r) * {}_tI_3^n(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta\; I_0(\zeta)\; {}_tI_3^n(r-\zeta) \ . $$

(A-4.16)

The evaluation of the integral $I_0(r)$ is given by equation A-4.4a and the evaluations of the integrals ${}_tI_3^n(r)$ for values of n equal to minus one, minus two and minus three are given by equations A-4.13 resulting in the respective convolution forms

$$ {}_dI_3^{(-1)}(r) = \frac{i}{2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\; e^{-\frac{\zeta^2}{4\delta}}\; \frac{(r-\zeta)}{|r-\zeta|} \qquad \text{(A-4.17a)} $$

$$ {}_dI_3^{(-2)}(r) = \frac{-i}{4} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\; e^{-\frac{\zeta^2}{4\delta}}\; \frac{(r-\zeta)}{|r-\zeta|}\, (r-\zeta)^2 \qquad \text{(A-4.17b)} $$

$$ {}_dI_3^{(-3)}(r) = \frac{i}{48} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\; e^{-\frac{\zeta^2}{4\delta}}\; \frac{(r-\zeta)}{|r-\zeta|}\, (r-\zeta)^4 \quad . \qquad \text{(A-4.17c)} $$

These convolution integrations may be expressed in terms of the general integration form

$$I^m = \int_{-\infty}^{\infty} d\zeta \; e^{-\frac{\zeta^2}{4\delta}} \; \frac{(r-\zeta)}{|r-\zeta|} \; \zeta^m \tag{A-4.18}$$

as

$$_dI_3^{(-1)}(r) = \frac{i}{2}\sqrt{\frac{\pi}{\delta}} \; I^0 \tag{A-4.19a}$$

$$_dI_3^{(-2)}(r) = \frac{-i}{4}\sqrt{\frac{\pi}{\delta}} \; [I^2 - 2r\,I^1 + r^2\,I^0] \tag{A-4.19b}$$

$$_dI_3^{(-3)}(r) = \frac{i}{48}\sqrt{\frac{\pi}{\delta}} \,[I^4 - 4r\,I^3 + 6r^2\,I^2 - 4r^3\,I^1 + r^4\,I^0] \quad . \tag{A-4.19c}$$

The integration form of equation A-4.18 need be evaluated for integer values of m equal to zero through four inclusive.

The integration I^0 may be expressed in terms of the Gaussian error function. It is convenient to account for the sign change of the integrand at ζ equal to r by separating the integration into two integrals as

$$I^0 = \int_r^{\infty} d\zeta \; e^{-\frac{\zeta^2}{4\delta}} \,(-1) + \int_{-\infty}^{r} d\zeta \; e^{-\frac{\zeta^2}{4\delta}} \,(+1) \quad . \tag{A-4.20}$$

By performing the integration variable substitution,

$$\eta = \frac{\zeta}{2\sqrt{\delta}} \quad , \tag{A-4.21}$$

the integrals may be expressed in terms of the Gaussian error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x d\eta \; e^{-\eta^2} = -\mathrm{erf}(-x) \quad , \tag{A-4.22}$$

as

$$I^0 = -2\sqrt{\delta}\int_{\frac{r}{2\sqrt{\delta}}}^{\infty} d\eta\, e^{-\eta^2} + 2\sqrt{\delta}\int_{-\infty}^{\frac{r}{2\sqrt{\delta}}} d\eta\, e^{-\eta^2} =$$

$$= 2\sqrt{\delta}\ \frac{\sqrt{\pi}}{2}\left\{-[\mathrm{erf}(\infty) - \mathrm{erf}(\frac{r}{2\sqrt{\delta}})]+[\mathrm{erf}(\frac{r}{2\sqrt{\delta}}) - \mathrm{erf}(-\infty)]\right\} = 2\sqrt{\pi\delta}\ \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) \ . \tag{A-4.23}$$

The integration I^1 may be evaluated by similarly accounting for the sign change of the integrand and then integrating directly as

$$I^1 = \int_r^{\infty} d\zeta\, e^{-\frac{\zeta^2}{4\delta}}(-\zeta) + \int_{-\infty}^{r} d\zeta\, e^{-\frac{\zeta^2}{4\delta}}\ (+\zeta)\ =$$

$$= 2\delta\, e^{-\frac{\zeta^2}{4\delta}}\Big|_{\zeta=r}^{\zeta=\infty} - 2\delta\, e^{-\frac{\zeta^2}{4\delta}}\Big|_{\zeta=-\infty}^{\zeta=r} = -4\delta\, e^{-\frac{r^2}{4\delta}} \ . \tag{A-4.24}$$

The remaining integrals, equation A-4.18 with m equal to two, three and four, may be related to the integrals I^0 or I^1 by the differential recursion formula,

$$I^{m+2} = 4\delta^2 \frac{\partial}{\partial\delta} I^m \ . \tag{A-4.25}$$

Since the differential of the error function may be derived by Liebnitz rule as

$$d[\mathrm{erf}(x)] = d[\frac{2}{\sqrt{\pi}}\int_0^x d\eta\, e^{-\eta^2}] = \frac{2}{\sqrt{\pi}}\ e^{-x^2}\,dx \ , \tag{A-4.26}$$

the integrals may be easily evaluated as

$$I^2 = 4\delta^2 \frac{\partial}{\partial\delta} I^0 = 4\sqrt{\pi}\ \delta^{\frac{3}{2}}\ \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) - 4\delta\ r\ e^{-\frac{r^2}{4\delta}} \qquad \text{(A-4.27a)}$$

$$I^3 = 4\delta^2 \frac{\partial}{\partial\delta}\ I^1 = -4\delta\ (4\delta + r^2)\ e^{-\frac{r^2}{4\delta}} \qquad \text{(A-4.27b)}$$

$$I^4 = 4\delta \frac{\partial}{\partial\delta} I^2 = 24\sqrt{\pi}\ \ \delta^{\frac{5}{2}}\ \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) - 4\delta r\ (6\delta + r^2)\ e^{-\frac{r^2}{4\delta}}\ . \qquad \text{(A-4.27c)}$$

Therefore the transverse tensorial type integrations may be evaluated, by means of algebraic manipulation of the integrations of equations A-4.23, A-4.24 and A-4.27 as illustrated by equations A-4.19, to be

$${}_dI_3^{(-1)} = i\pi\ \mathrm{erf}\ (\frac{r}{2\sqrt{\delta}}) \qquad \text{(A-4.28a)}$$

$${}_dI_3^{(-2)} = -i\pi\ (\delta + \frac{r^2}{2})\ \mathrm{erf}\ (\frac{r}{2\sqrt{\delta}}) - i\ \sqrt{\pi\delta}\ \ r\ e^{-\frac{r^2}{4\delta}} \qquad \text{(A-4.28b)}$$

$${}_dI_3^{(-3)} = \frac{i\pi}{2}\ (\delta^2 + \delta r^2 + \frac{r^4}{12})\ \mathrm{erf}\ (\frac{r}{2\sqrt{\delta}}) + \frac{i\sqrt{\pi\delta}}{12}\ (10\delta r + r^3)\ e^{-\frac{r^2}{4\delta}}\ . \qquad \text{(A-4.28c)}$$

Since the integral form in equation A-4.1 reduces to the integral form in equation A-4.2 in the limit as δ approaches zero, in the same limit the evaluations of equations A-4.28 reduce to the evaluations of equations A-4.13.

APPENDIX A-5

The inverse spatial Fourier transformations from wavenumber and time domain to space and time domain of the thermal-viscous and the longitudinal thermal-viscous type Green's functions involve integrations of the forms

$$I_3^n = \int_{-\infty}^{\infty} dke^{ikr} \frac{k^{2n+1} e^{-\delta k^2}}{(k^2-k_5^2)} \tag{A-5.1a}$$

$$I_4^n = \int_{-\infty}^{\infty} dke^{ikr} \frac{k^{2n+1} e^{-\delta k^2} \cos(\delta k \sqrt{\rho_3^2-k^2})}{(k^2-k_5^2)} \tag{A-5.1b}$$

$$I_5^n = \int_{-\infty}^{\infty} dke^{ikr} \frac{k^{2n+1} e^{-\delta k^2} \sin(\delta k \sqrt{\rho_3^2-k^2})}{(k^2-k_5^2) \quad \delta k \sqrt{\rho_3^2-k^2}} \tag{A-5.1c}$$

where the integer n may be non negative, equal to minus one or, in equations 1a and 1b, equal to minus two. The integrands possess wavenumber poles at plus and minus k_5 as defined by equation 4.98 and zero wavenumber poles when n is negative. Integrand poles suggest that the integrations may be evaluated by the Cauchey residue theorem applied to closed contoursim the complex wavenumber plane. By this theorem the integration along the real wavenumber axis, which is to be evaluated, plus the integration along a closing contour is related to the sum of the residues of the enclosed poles. However, the exponential term $e^{-\delta k^2}$, which is common to all integrands, inhibits evaluation of the integration along a closing contour because of its wavenumber squared dependence and thus prevents evaluation of the required integrals by this method.

Since all integrands possess the exponential term e^{ikr}, the unmanageable combination of the wavenumber squared dependent exponential term and the nonzero wavenumber poles may be effectively separated by a convolution technique. The integrals, which result from three dimensional inverse Fourier transformations between $\vec{k}$ and $\vec{r}$ domains with kernals $e^{-i\vec{k}\cdot\vec{r}}$, are of the forms of one dimensional inverse Fourier transformations between k and r domains with kernals e^{ikr}. Such inverse transformations relate the integrals in r domain, those integrals to be evaluated, to their k domain representations, each of which is a part of the required integrand of one of equations A-5.1, as

$$I^{n}_{(\ )}(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikr} \quad I^{n}_{(\ )}(k) \tag{A-5.2a}$$

$$I^{n}_{(\ )}(k) = \int_{-\infty}^{\infty} dr\; e^{-ikr} \quad I^{n}_{(\ )}(r) \quad . \tag{A-5.2b}$$

The wavenumber squared dependent exponential may be separated by expressing each of the k domain integral representations as the product of two k domain representations,

$$I^{n}_{(\ )}(k) = I_{0}(k) \quad I^{n}_{1(\ )}(k) \tag{A-5.3}$$

where

$$I_{0}(k) = 2\pi\; e^{-\delta k^{2}} \tag{A-5.4}$$

and the specific separated k domain representations corresponding to each of equations A-5.1 are

$$I_{13}^{n}(k) = \frac{k^{2n+1}}{(k^2-k_5^2)} \tag{A-5.5a}$$

$$I_{14}^{n}(k) = \frac{k^{2n+1}\cos\left(\delta k\sqrt{\rho_3^2-k^2}\right)}{(k^2-k_5^2)} \tag{A-5.5b}$$

$$I_{15}^{n}(k) = \frac{k^{2n+1}\sin\left(\delta k\sqrt{\rho_3^2-k^2}\right)}{(k^2-k_5^2)\,\delta k\sqrt{\rho_3^2-k^2}} \tag{A-5.5c}$$

The required integrals, r domain representations, may be recovered from the separated r domain representations by the convolutions,

$$I^{n}{}_{(\ )}(r) = I_o(r) * I_{1(\ )}^{n}(r) = \int_{-\infty}^{\infty} d\zeta\ I_o(\zeta)\ I_{1(\ )}^{n}(r-\zeta), \tag{A-5.6}$$

which are alternate expressions of equations A-5.1. Evaluation of the required integrals by the convolution technique requires that the separated representations be evaluated, by inverse transformation, in r domain. The wavenumber squared dependent exponential inverse transformation has been evaluated in Appendix A-4, equation A-4.4 as

$$I_o(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikr}\, 2\pi\, e^{-\delta k^2} = \sqrt{\frac{\pi}{\delta}}\quad e^{-\frac{r^2}{4\delta}} \tag{A-5.7}$$

The remaining separated r domain integrals,

$$I_{13}^{n}(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikr}\ \frac{k^{2n+1}}{(k^2-k_5^2)} \tag{A-5.8a}$$

$$I_{14}^{n}(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikr}\ \frac{k^{2n+1}\cos\left(\delta k\sqrt{\rho_3^2-k^2}\right)}{(k^2-k_5^2)} \tag{A-5.8b}$$

$$I_{15}^{n}(r) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikr} \frac{k^{2n+1} \sin(\delta k\sqrt{\rho_3^2-k^2})}{(k^2-k_5^2)\ \delta k\sqrt{\rho_3^2-k^2}} , \qquad \text{(A-5.8c)}$$

may be evaluated by contour integration in the complex wavenumber plane. It is convenient to express the sinusoidal terms of the integrands in series form,

$$\cos(\delta k\sqrt{\rho_3^2-k^2}) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}\ k^{2j}\ (-\rho_3^2+k^2)^j \qquad \text{(A-5.9a)}$$

$$\frac{\sin(\delta k\sqrt{\rho_3^2-k^2})}{\delta k\sqrt{\rho_3^2-k^2}} = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j+1)!} k^{2j}(-\rho_3^2+k^2)^j , \qquad \text{(A-5.9b)}$$

so that the separated r domain integrals, equations A-5.8, may be expressed in terms of one integral form,

$$I^{(n)}(j) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k^2-k_5^2)} , \qquad \text{(A-5.10)}$$

as

$$I_{13}^{n}(r) = I^{(n)}(0) \qquad \text{(A-5.11a)}$$

$$I_{14}^{n}(r) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}\ I^{(n)}(j) \qquad \text{(A-5.11b)}$$

$$I_{15}^{n}(r) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j+1)!}\ I^{(n)}(j) . \qquad \text{(A-5.11c)}$$

The integrand of interest, shown by equation A-5.10, possesses wavenumber poles at plus and minus k_5 and may possess zero wavenumber poles for certain values of the integers n and j. Since n is greater than or equal to negative two and j is non negative for the integration of interest, the integrand possesses a zero wavenumber pole of order one when n equals negative one and j equals zero and when n equals negative two and j equals one and possesses a zero wavenumber pole of order three when n equals negative two and j equals zero. The nonzero wavenumber poles, $\pm k_5$, may be either imaginary, equations 4.99, or real, equations 4.100, depending on the relative magnitudes of the thermal and thermal-viscous diffusion coefficients. Each possible case of imaginary or real nonzero wavenumber poles and zero, first or third order zero wavenumber poles requires separate evaluation.

For the case of imaginary nonzero wavenumber poles in the absence of zero wavenumber poles contours C_1 and C_2, shown by figure A-5.1, which each consist of a segment along the real wavenumber axis, along which the integral of equation A-5.10 is to be evaluated, and a closing half circle will be utilized. The existence of zero wavenumber poles, which must be avoided by the contours and will be excluded from the contours interior by small semicircular segments, requires tht contours C_3 and C_4, shown by figure A-5.2, be utilized for the integral evoluation. If the nonzero wavenumber poles are real, they must be avoided by the integration contour as were the zero wavenumber poles. The results of the evaluation are unchanged by including or excluding the real wavenumber poles from the interior of the contour. Since excluding the real poles simplifies the evaluation, they will be excluded by small semicircular contour segments. The contours in the absence of zero wavenumber poles, C_5 and C_6, are shown by figure A-5.3 and the contours with zero wavenumber poles, C_7 and C_8, are shown by figure A-5.4.

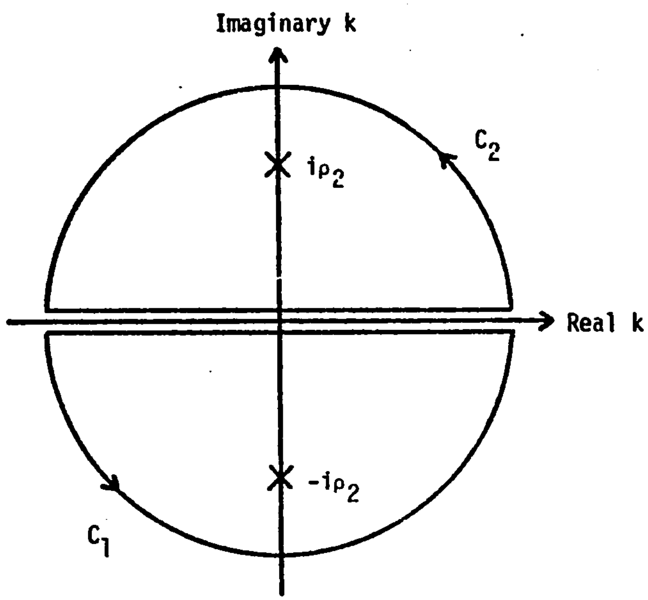

Figure A-5.1

Complex integration contours with thermal-viscous wavenumber poles when $D_2 > D_1$

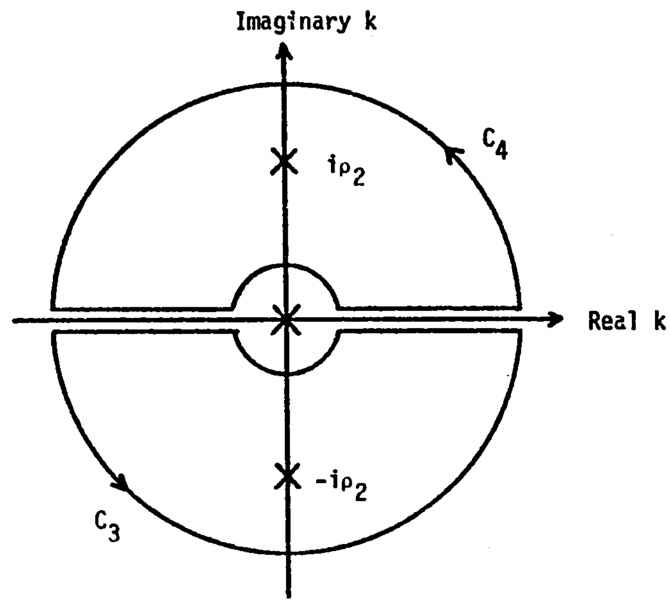

Figure A-5.2

Complex integration contours with zero and thermal-viscous wavenumber poles when $D_2 > D_1$

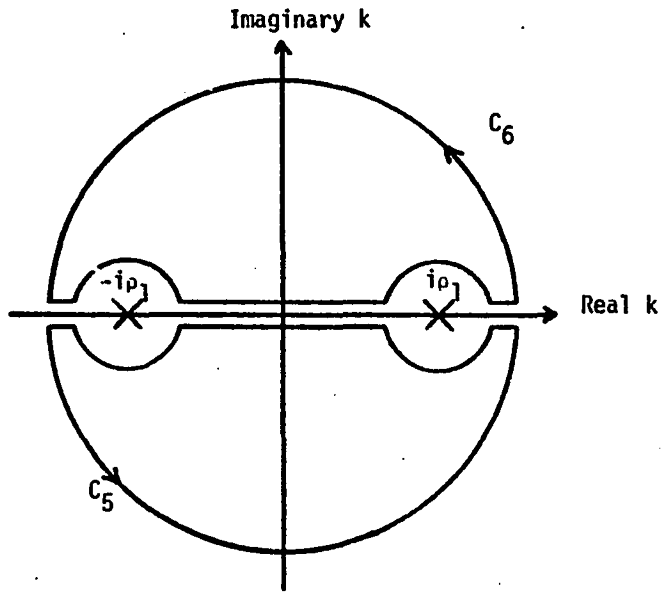

Figure A-5.3

Complex integration contours with thermal-viscous wavenumber poles when $D_1 > D_2$

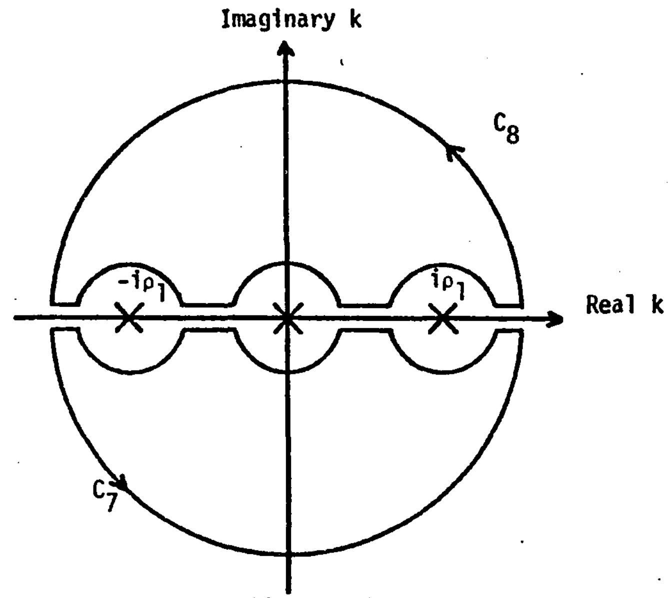

Figure A-5.4

Complex integration contours with zero and thermal-viscous number poles when $D_1 > D_2$

For the condition of larger thermal than thermal-viscous diffusion coefficient,

$$D_2 > D_1 \quad , \tag{A-5.12a}$$

the nonzero wavenumber poles are imaginary,

$$k_5 = -k_6 = i\rho_2 \quad , \tag{A-5.12b}$$

and the integrations will be identified by a presubscript 2,

$${}_2I^{(n)(j)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} \quad . \tag{A-5.13}$$

In the absence of zero wavenumber poles, which corresponds to the sum of n and j being non negative, integration counterclockwise around the contours C_1 and C_2 is appropriate and equal to the product of $2\pi i$ and the residue of the enclosed pole. Thus the integration around the contour C_1 may be expressed as

$$\int_{C_1} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} = \int_{A}^{-A} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} +$$

$$+ \int_{\pi}^{2\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^{2n+2j+1}\ (-\rho_3^2+A^2e^{i2\alpha})^j e^{irA\cos\alpha - rA\sin\alpha}}{(Ae^{i\alpha}-i\rho_2)(Ae^{i\alpha}+i\rho_2)} =$$

$$= 2\pi i \text{ Residue } (k = -i\rho_2) \tag{A-5.14}$$

In the limit as the semicircular closing contour radius, A, becomes infinite the integration along the real wavenumber axis is identically the required integration and the integration along the closing half circle equals zero if

$$r \sin \alpha > 0 \quad . \tag{A-5.15}$$

This condition is satisfied along the closing contour of C_1, where $\sin \alpha$ is negative, for negative values of r. Thus the required integration may be evaluated for negative values of r as

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} = \frac{-i}{2}\ (-i\rho_2)^{2n+2j}\ (-\rho_3^2-\rho_2^2)^j\ e^{\rho_2 r} \quad ,$$

$$r < 0 \quad . \tag{A-5.16}$$

Integration along the contour C_2 yields the integral evaluation for positive values of r and may be expressed as,

$$\int_{C_2} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} = \int_{-A}^{A} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} +$$

$$+ \int_0^{\pi} \frac{iAe^{i\alpha}d\alpha}{2\pi} \frac{(Ae^{i\alpha})^{2n+2j+1}\ (-\rho_3^2+A^2e^{i2\alpha})^j\ e^{irA\cos\alpha - rA\sin\alpha}}{(Ae^{i\alpha}-i\rho_2)(Ae^{i\alpha}+i\rho_2)} =$$

$$= 2\pi i \text{ Residue } (k = i\rho_2) \tag{A-5.17}$$

In the limit as the semicircular closing contour radius becomes infinite the integration along it becomes equal to zero for positive values of r, since the condition of equation A-5.15 is satisfied, and the required integration may be evaluated as

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} = \frac{i}{2}\ (i\rho_2)^{2n+2j}\ (-\rho_3^2-\rho_2^2)^j\ e^{-\rho_2 r} \quad , \quad r > 0. \tag{A-5.18}$$

The integration, in the absence of wavenumber poles, has been evaluated separately for negative and positive values of r and may be expressed functionally for all values of r as

$$_2I^{(n)(j)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}\ (-\rho_3^2+k^2)^j\ e^{ikr}}{(k-i\rho_2)(k+i\rho_2)} = \frac{r}{|r|}\frac{i}{2}(i\rho_2)^{2n+2j}(-\rho_2^2-\rho_3^2)^j e^{-\rho_2|r|} ,$$

$$n + j \geq 0 \quad . \qquad \text{(A-5.19)}$$

When the integrand of equation A-5.13 possesses a pole at zero wavenumber in addition to the imaginary wavenumber poles the contours C_3 and C_4 are appropriate for evaluation of the integral. Each contour encloses an imaginary wavenumber pole, as in the absence of a zero wavenumber pole, and excludes the zero wavenumber pole by a small semicircular contour segment of radius ε. In the limit as A becomes infinite the integration along the closing contour is equal to zero if the condition of equation A-5.15 is satisfied and in the limit as ε approaches zero the real wavenumber integration equals its principal value. In these limits the integration counter clockwise around C_3 for negative values of r may be expressed as

$$\int_{C_3} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j\ e^{ikr}}{k^{-2n-2j-1}\ (k-i\rho_2)(k+i\rho_2)} = P\int_{\infty}^{-\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j\ e^{ikr}}{k^{-2n-2j-1}\ (k-i\rho_2)(k+i\rho_2)} + 0 +$$

$$+ \lim_{\varepsilon\to 0} \int_{2\pi}^{\pi} \frac{i\varepsilon e^{i\alpha}d\alpha}{2\pi} \frac{(-\rho_3^2+\varepsilon^2 e^{i2\alpha})^j\ e^{ir\varepsilon e^{i\alpha}}}{(\varepsilon e^{i\alpha})^{-2n-2j-1}\ (\varepsilon e^{i\alpha}-i\rho_2)(\varepsilon e^{i\alpha}+i\rho_2)} =$$

$$= 2\pi i\ \text{Residue}\ (k = -i\rho_2) \quad , \quad n + j < 0 \quad , \quad r < 0 \quad . \qquad \text{(A-5.20)}$$

In the limit as ε approaches zero the integration clockwise along the semicircular contour avoiding the zero wavenumber pole is equal to the product of

$-\pi i$ and the residue of the zero wavenumber pole. Thus the principal value of the integral with zero wavenumber poles for negative values of r may be expressed in terms of the residues as

$$P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1} \; (k-i\rho_2)(k+i\rho_2)} = -2\pi i \text{ Residue } (k = -i\rho_2) - \pi i \text{ Residue } (k=0),$$

$$n + j < 0 \quad , \quad r < 0 \quad . \tag{A-5.21}$$

In the limit as A approaches infinity the condition of equation A-5.15 is satisfied by positive values of r for integration along the closing contour of C_4 and therefore the integration along that segment is equal to zero. The integration clockwise around the contour C_4 for positive values of r then becomes

$$\int_{C_4} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1} \; (k-i\rho_2)(k+i\rho_2)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1} \; (k-i\rho_2)(k+i\rho_2)} + 0 +$$

$$+ \lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{i\varepsilon e^{i\alpha} d\alpha}{2\pi} \frac{(-\rho_3^2+\varepsilon^2 e^{i2\alpha})^j \; e^{ir\varepsilon e^{i\alpha}}}{(\varepsilon e^{i\alpha})^{-2n-2j-1} \; (\varepsilon e^{i\alpha}-i\rho_2)(\varepsilon e^{i\alpha}+i\rho_2)} =$$

$$= 2\pi i \text{ Residue } (k = i\rho_2) \quad , \quad n + j < 0 \quad , \quad r > 0 \quad . \tag{A-5.22}$$

The integration along the semicircular contour avoiding the zero wavenumber pole is, as in equation A-5.20, in the clockwise direction and therefore equal to the product of $-\pi i$ and the zero wavenumber pole residue allowing the principal value of the integration for positive values of r to be expressed as

$$P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1} \; (k-i\rho_2)(k+i\rho_2)} = 2\pi i \text{ Residue } (k = i\rho_2) + \pi i \text{ Residue } (k = 0),$$

$$n + j < 0 \quad , \quad r > 0 \quad . \tag{A-5.23}$$

Simple poles are encountered at zero wavenumber when the sum of n and j equals minus one. For the integrations of interest this occurs for n equal to minus one and j equal to zero and for n equal to minus two and j equal to one. The zero wavenumber pole residue is then simply

$$\text{Residue } (k = 0) = \frac{(-\rho_3^2)^j}{2\pi\ \rho_2^2} \quad , \quad n + j = -1 \quad . \qquad \text{(A-5.24)}$$

Also, a zero wavenumber pole of order three is encountered when n equals minus two and j equals zero. Its residue may be evaluated as

$$\text{Residue } (k=0) = \lim_{k \to 0} \frac{1}{2!} \frac{\partial^2}{\partial k^2} \frac{e^{ikr}}{2\pi(k-i\rho_2)(k+i\rho_2)} = \frac{1}{2\pi\ \rho_2^4} \left(1 + \frac{\rho_2^2 r^2}{2}\right) ,$$

$$n = -2 \quad , \quad j = 0 \quad . \qquad (25)$$

Thus the principal value integral for negative values of r, equation A-5.21, may be evaluated when the integrand possesses a simple zero wavenumber pole as

$$P \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j\ e^{ikr}}{k(k-i\rho_2)(k+i\rho_2)} = \frac{i}{2\rho_2^2} (-\rho_3^2-\rho_2^2)^j\ e^{\rho_2 r} - \frac{i}{2\rho_2^2} (-\rho_3^2)^j \ , \ n + j = 1 \ ,$$

$$r < 0 \quad . \qquad \text{(A-5.26)}$$

For a zero wavenumber pole of order three the principal value integral encountered becomes

$$P \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikr}}{k^3(k-i\rho_2)(k+i\rho_2)} = \frac{-i}{2\rho_2^4}\ e^{\rho_2 r} + \frac{i}{2\rho_2^4} \left(1 + \frac{\rho_2^2 r^2}{2}\right) \ , \ n = -2 \ , \ j = 0 \ ,$$

$$r < 0 \ . \qquad \text{(A-5.27)}$$

For positive values of r the principal value integral, equation A-5.23, may be evaluated when the integrand possesses a simple zero wavenumber pole as

$$P\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{(-\rho_3^2+k^2)^j\,e^{ikr}}{k(k-i\rho_2)(k+i\rho_2)}=\frac{-i}{2\rho_2^2}\,(-\rho_3^2-\rho_2^2)^j\,e^{-\rho_2 r}+\frac{i}{2\rho_2^2}\,(-\rho_3^2)^j\ ,\quad n+j=-1\ ,$$

$$r>0\ , \qquad \text{(A-5.28)}$$

and when the integrand possesses a zero wavenumber pole of order three as

$$P\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ikr}}{k^3(k-i\rho_2)(k+i\rho_2)}=\frac{i}{2\rho_2^4}\,e^{-\rho_2 r}-\frac{i}{2\rho_2^4}\,(1+\frac{\rho_2^2r^2}{2})\ ,\quad n=-2\ ,\ j=0\ ,$$

$$r>0\ . \qquad \text{(A-5.29)}$$

Although the principal value integrals have been evaluated separately for negative and positive values of r they may be expressed functionally for all r. For a simple zero wavenumber pole in the integrand, refering to equations A-5.26 and A-5.28, the integral may be expressed when n equals minus one and j equals zero as

$${}_2I^{(-1)(0)}=P\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ikr}}{k(k-i\rho_2)(k+i\rho_2)}=\frac{r}{|r|}\,\frac{-i}{2\rho_2^2}\,[e^{-\rho_2|r|}-1]\ , \qquad \text{(A-5.30)}$$

and may be expressed when n equals minus two and j equals one as

$${}_2I^{(-2)(1)}=P\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{(-\rho_3^2+k^2)\,e^{ikr}}{k(k-i\rho_2)(k+i\rho_2)}=\frac{r}{|r|}\,\frac{i}{2\rho_2^2}\,[(\rho_2^2+\rho_3^2)\,e^{-\rho_2|r|}-\rho_3^2]\ . \qquad \text{(A-5.31)}$$

When n equals minus two and j equals zero the integrand possesses a third order zero wavenumber pole and the integral evolution, refering to equations A-5.27 and A-5.29, becomes

$${}_2I^{(-2)(0)}=P\int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ikr}}{k^3(k-i\rho_2)(k+i\rho_2)}=\frac{r}{|r|}\,\frac{i}{2\rho_2^4}\,[e^{-\rho_2|r|}-(1+\frac{\rho_2^2r^2}{2})]\ . \qquad \text{(A-5.32)}$$

Thus the integral form shown by equation A-5.13, which applies to the condition of larger thermal than thermal-viscous diffusion coefficient, has been evaluated, as given by equations A-5.19, A-5.30, A-5.31 and A-5.32, for all required values of the integers n and j.

The integral form shown by equation A-5.10 need also be evaluated for the condition of larger thermal-viscous then thermal diffusion coefficient,

$$D_1 > D_2 \quad , \tag{A-5.33a}$$

which yields real nonzero wavenumber poles,

$$k_5 = -k_6 = \rho_1 \quad , \tag{A-5.33b}$$

and the corresponding integration form identified by the presubscript 1,

$$_1I^{(n)(j)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \; \frac{k^{2n+2j+1} \; (-\rho_3^2+k^2)^j \; e^{ikr}}{(k-\rho_1)(k+\rho_1)} \quad . \tag{A-5.34}$$

In the absence of zero wavenumber poles, the sum of n and j non negative, integration around the contours C_5 and C_6 is appropriate and equal to zero since the wavenumber poles are excluded from the contour interiors by semi-circular contour segments of radius ε. In the limit as ε approaches zero the integration along the real wavenumber axis is equal to its principal value and in the limit as the radius A approaches infinity the integration along the closing contours of C_5 and C_6 are equal to zero if the condition of equation A-5.15 is satisfied, as was shown for contours C_1 and C_2. In these limits the integration along C_5 for negative values of r may be expressed as

$$\int_{C_5} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j e^{ikr}}{(k-\rho_1)(k+\rho_1)} = P \int_{\infty}^{-\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j\ e^{ikr}}{(k-\rho_1)(k+\rho_1)} + 0 +$$

$$+ \lim_{\varepsilon\to 0} \int_{2\pi}^{\pi} \frac{i\varepsilon e^{i\alpha}d\alpha}{2\pi} \frac{(-\rho_1+\varepsilon e^{i\alpha})^{2n+2j+1}(-\rho_3^2+(-\rho_1+\varepsilon e^{i\alpha})^2)^j\ e^{ir(-\rho_1+\varepsilon e^{i\alpha})}}{((-\rho_1+\varepsilon e^{i\alpha})-\rho_1)((-\rho_1+\varepsilon e^{i\alpha})+\rho_1)} +$$

$$+ \lim_{\varepsilon\to 0} \int_{2\pi}^{\infty} \frac{i\varepsilon e^{i\alpha}d\alpha}{2\pi} \frac{(\rho_1+\varepsilon e^{i\alpha})^{2n+2j+1}(-\rho_3^2+(\rho_1+\varepsilon e^{i\alpha})^2)^j\ e^{ir(\rho_1+\varepsilon e^{i\alpha})}}{((\rho_1+\varepsilon e^{i\alpha})-\rho_1)((\rho_1+\varepsilon e^{i\alpha})+\rho_1)} = 0 ,$$

$$r < 0 \quad . \tag{A-5.35}$$

as ε approaches zero each integration clockwise along the semicircle avoiding the real wavenumber pole is equal to the product of $-\pi i$ and the residue of that pole. Thus the principal value integration for negative values of r may be expressed in terms of the residues as

$$P \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j\ e^{ikr}}{(k-\rho_1)(k+\rho_1)} = -\pi i \text{ Residue } (k = -\rho_1) - \pi i \text{ Residue } (k = \rho_1) ,$$

$$r < 0 \quad . \tag{A-5.36}$$

As its radius approaches infinity the integration along the closing contour of C_6 equals zero for positive values of r allowing the contour integration to be expressed as

$$\int_{C_6} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j\ e^{ikr}}{(k-\rho_1)(k+\rho_1)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j\ e^{ikr}}{(k-\rho_1)(k+\rho_1)} + 0 +$$

$$+ \lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{i\varepsilon e^{i\alpha}d\alpha}{2\pi} \frac{(-\rho_1+\varepsilon e^{i\alpha})^{2n+2j+1}(-\rho_3^2+(-\rho_1+\varepsilon e^{i\alpha})^2)^j\ e^{ir(-\rho_1+\varepsilon e^{i\alpha})}}{((-\rho_1+\varepsilon e^{i\alpha})-\rho_1)((-\rho_1+\varepsilon e^{i\alpha})+\rho_1)} +$$

$$+ \lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{i\varepsilon e^{i\alpha}d\alpha}{2\pi} \frac{(\rho_1+\varepsilon e^{i\alpha})^{2n+2j+1}(-\rho_3^2+(\rho_1+\varepsilon e^{i\alpha})^2)^j\ e^{ir(\rho_1+\varepsilon e^{i\alpha})}}{((\rho_1+\varepsilon e^{i\alpha})-\rho_1)((\rho_1+\varepsilon e^{i\alpha})+\rho_1)} = 0 \quad ,$$

$$r > 0 \quad . \qquad \text{(A-5.37)}$$

Again each pole is avoided by a semicircular contour segment clockwise along which the integration is equal to minus one half the contribution of the pole had it been included in the contour.

In terms of the residues of the real wavenumber poles the principal value integration for positive values of r is

$$P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j\ e^{ikr}}{(k-\rho_1)(k+\rho_1)} = \pi i\ \text{Residue}\ (k = -\rho_1) + \pi i\ \text{Residue}\ (k = \rho_1) \quad ,$$

$$r > 0 \quad , \qquad \text{(A-5.38)}$$

which is the negative of the evaluation for negative values of r, equation A-5.36. Therefore the principal value integration, in the absence of zero wavenumber poles, may be expressed for all values of r as

$$_1I^{(n)(j)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^{2n+2j+1}(-\rho_3^2+k^2)^j e^{ikr}}{(k-\rho_1)(k+\rho_1)} = \frac{r}{|r|} \frac{i}{4} (\rho_1)^{2n+2j}(-\rho_3^2+\rho_1^2)^j(e^{-i\rho_1 r}+e^{i\rho_1 r}) =$$

$$= \frac{r}{|r|} \frac{i}{2} (\rho_1)^{2n+2j} (-\rho_3^2+\rho_1^2)^j \cos(\rho_1 r) \quad , \quad n + j \geq 0 \quad . \qquad \text{(A-5.39)}$$

When the integrand of equation A-5.34 possesses a pole at wavenumber equal to zero in addition to the nonzero real wavenumber poles the contours C_7 and C_8, which avoid the zero wavenumber pole as well as the nonzero poles by semicircular contour segments, are appropriate. Integration along the closing contours of C_7 and C_8 is equal to zero for negative and positive values of r respectively in the limit as A approaches infinity, as for the contours C_5 and C_6. In the limit as ε approaches zero the integration along the real wavenumber axis is the principal value integral and the clockwise integration around each pole is equal to the product of $-\pi i$ and the residue of the wavenumber pole. Thus the integration counterclockwise around C_7 for negative values of r becomes

$$\int_{C_7} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1}(k-\rho_1)(k+\rho_1)} = P\int_{\infty}^{-\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1}(k-\rho_1)(k+\rho_1)} + 0 -$$

$$-\pi i \text{ Residue } (k = -\rho_1) - \pi i \text{ Residue } (k = \rho_1) - \pi i \text{ Residue } (k = 0) = 0 \; ,$$

$$n + j < 0 \quad , \quad r < 0 \quad . \qquad \text{(A-5.40)}$$

and the integration counterclockwise around C_8 is, for positive values of r,

$$\int_{C_8} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1}(k-\rho_1)(k+\rho_1)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1}\;(k-\rho_1)(k+\rho_1)} + 0 -$$

$$- \pi i \text{ Residue } (k = -\rho_1) - \pi i \text{ Residue } (k = \rho_1) - \pi i \text{ Residue } (k = 0) = 0 \; ,$$

$$n + j < 0 \quad , \quad r > 0 \quad . \qquad \text{(A-5.41)}$$

Thus the principal value integral may be expressed for all values of r in terms of the residues as

$$P\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(-\rho_3^2+k^2)^j \; e^{ikr}}{k^{-2n-2j-1} \; (k-\rho_1)(k+\rho_1)} =$$

$$= \frac{r}{|r|} \pi i[\text{Residue } (k = -\rho_1) + \text{Residue } (k = \rho_1) + \text{Residue } (k = 0)] \; ,$$

$$n + j < 0 \; . \tag{A-5.42}$$

The sum of the residues of the simple, nonzero wavenumber poles may be evaluated for all values of the integers n and j to be

$$\text{Residue } (k = -\rho_1) + \text{Residue } (k = \rho_1) = \frac{(-\rho_3^2+\rho_1^2)^j}{4\pi(\rho_1)^{-2n-2j}} (e^{-i\rho_1 r} + e^{i\rho_1 r}) \tag{A-5.43}$$

as was utilized in the integral evaluation in the absence of wavenumber poles, equation A-5.39.

The effect of the additional, zero wavenumber pole on the integral evaluation is displayed by its residue. For the integrals of interest the wavenumber poles may be of order one or of order three depending on the values of n and j. Simple zero wavenumber poles are encountered when the sum of n and j is equal to minus one resulting in the residue evaluation

$$\text{Residue } (k = 0) = \frac{-(-\rho_3^2)^j}{2\pi \; \rho_1^2} \quad , \quad n + j = -1 \quad . \tag{A-5.44}$$

A zero wavenumber pole of order three occurs for the integrations of interest only when n equals minus two and j equals zero which yields the residue evaluation

$$\text{Residue } (k = 0) = \lim_{k \to 0} \frac{1}{2!} \frac{\partial^2}{\partial k^2} \frac{e^{ikr}}{2\pi(k-\rho_1)(k+\rho_1)} = \frac{-1}{2\pi \; \rho_1^4} (1 - \frac{\rho_1^2 r^2}{2}) \; , \; n = -2 \; , \; j = 0 \; . \tag{A-5.45}$$

Thus when the integrands possess a zero wavenumber pole in addition to the real nonzero poles the integrals of interest may be evaluated by substituting the proper residue evaluation in equation A-5.42. For the integer values of n equal to minus one and j equal to zero the integral evaluates as

$$ {}_1I^{(-1)(0)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{e^{ikr}}{k(k-\rho_1)(k+\rho_1)} = \frac{r}{|r|}\,\frac{i}{2}\,\frac{1}{\rho_1^2}\,[\cos(\rho_1 r) - 1]\ , \qquad \text{(A-5.46)} $$

and also for a simple zero wavenumber pole with n equal to minus two and j equals to n the integral evaluation is

$$ {}_1I^{(-2)(1)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{(-\rho_3^2+k^2)\ e^{ikr}}{k(k-\rho_1)(k+\rho_1)} = \frac{r}{|r|}\,\frac{i}{2}\,\frac{1}{\rho_1^2}\,[(\rho_1^2-\rho_3^2)\ \cos(\rho_1 r) + \rho_3^2]\ . \qquad \text{(A-5.47)} $$

when the integrand possesses a zero wavenumber pole of third order, n equal to minus two and j equal to zero, the integral evaluates as

$$ {}_1I^{(-2)(0)} = P\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{e^{ikr}}{k^3(k-\rho_1)(k+\rho_1)} = \frac{r}{|r|}\,\frac{i}{2}\,\frac{1}{\rho_1^4}\,[\cos(\rho_1 r) - 1 + \frac{\rho_1^2 r^2}{2}]\ . \qquad \text{(A-5.48)} $$

This completes the evaluution of the integral form shown by equation A-5.34 for all required values of the integers n and j, equations A-5.39, A-5.46, A-5.47 and A-5.48, which correspond to the condition of larger thermal-viscous than thermal diffusion coefficient.

Thus the general integral form shown by equation A-5.10 has been evaluated for all of the required values of the integers n and j for each condition of imaginary, equation A-5.13, or real, equation A-5.34, non zero wavenumber poles. The r domain integrals of equation A-5.8, which are required for the convolution evaluation, are related to the general integral form by equations A-5.11. Since distinct evaluations of the general integral were derived for imaginary and real poles of the integrand, the r domain integrals must also be derived separately.

The nonzero wavenumber poles of the integrands are imaginary for the condition of larger thermal diffusion coefficient than thermal-viscous damping coefficient and the integrals will be identified by presubscript 2. For this condition the r domain integral which is defined by equation A-5.8a is related to the general integration form by equation A-5.11a. In the absence of zero wavenumber poles this integral may be evaluated, utilizing equation A-5.19, as

$$ {}_2I^{n}_{13}(r) = {}_2I^{(n)(j=0)} = \frac{r}{|r|}\frac{i}{2}(i\rho_2)^{2n} e^{-\rho_2|r|} \quad , \quad n \geq 0 \quad . \tag{A-5.49a} $$

When the integrand possesses a simple zero wavenumber pole the general integral evaluation of equation A-5.30 applies and yields

$$ {}_2I^{(-1)}_{13}(r) = {}_2I^{(-1)(0)} = \frac{r}{|r|}\frac{i}{2}\frac{-1}{\rho_2^2}(e^{-\rho_2|r|} - 1) \tag{A-5.49b} $$

and for a third order pole at zero wavenumber equation A-5.32 may be used for the evaluation

$$ {}_2I^{(-2)}_{13}(r) = {}_2I^{(-2)(0)} = \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_2^4}(e^{-\rho_2|r|} - 1 - \frac{\rho_2^2 r^2}{2}) \quad . \tag{A-5.49c} $$

The r domain integral defined by equation A-5.8b is expressed in equation A-5.11b as a series of the general integrals. When the integrand does not possess a pole at wavenumber equal to zero and the integral may be evaluated in series form as

$$ {}_2I^{(n)}_{14}(r) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}\, {}_2I^{(n)(j)} = \frac{r}{|r|}\frac{i}{2}(i\rho_2)^{2n} e^{-\rho_2|r|} \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}(\pm\rho_2\sqrt{\rho_2^2+\rho_3^2})^{2j} \, , $$

$$ n \geq 0 \quad . \tag{A-5.50} $$

Noting that the series is a hyperbolic cosine function and that the preceding group of terms is identically the integral evolution of equation A-5.49a, the integration may be expressed in closed form as

$$ {}_2I_{14}^{(n)}(r) = {}_2I_{13}^{(n)}(r) \quad \cosh\left(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}\right) \quad , \quad n \geq 0 \quad . \qquad \text{(A-5.50b)} $$

If the integrand posesses a simple zero wavenumber pole its effect will be exhibited by an additional term of the first integral in the series

$$ {}_2I_{14}^{(-1)}(r) = {}_2I^{(-1)(0)} + \sum_{j=1}^{\infty} \frac{\delta^{2j}}{(2j)!} \; {}_2I^{(n=-1)(j)} = $$

$$ = \frac{r}{|r|} \frac{i}{2} \frac{-1}{\rho_2^2} \left[-1 + e^{-\rho_2|r|} \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!} \left(\pm \rho_2\sqrt{\rho_2^2+\rho_3^2}\right)^{2j}\right] \quad . \qquad \text{(A-5.51a)} $$

Again the series represents a hyperbolic cosine function and the r domain integral evaluation becomes

$$ {}_2I_{14}^{(-1)}(r) = \frac{-1}{\rho_2^2} \; {}_2I_{13}^{(n=0)} \cosh\left(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}\right) + \frac{r}{|r|} \frac{i}{2} \frac{1}{\rho_2^2} \qquad \text{(A-5.51b)} $$

A third order zero wavenumber pole of the integrand results in the series form evaluation,

$$ {}_2I_{14}^{(-2)}(r) = {}_2I^{(-2)(0)} + \frac{\delta^2}{2!} \, {}_2I^{(-2)(1)} + \sum_{j=2}^{\infty} \frac{\delta^{2j}}{(2j)!} \, {}_2I^{(n=-2)(j)} = $$

$$ = \frac{r}{|r|} \frac{i}{2} \frac{1}{\rho_2^4} \left[-1 - \frac{\rho_2^2 r^2}{2} - \frac{\delta^2\rho_2^2\rho_3^2}{2} + e^{-\rho_2|r|} \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!} \left(\pm\rho_2\sqrt{\rho_2^2+\rho_3^2}\right)^{2j}\right] , \qquad \text{(A-5.52a)} $$

which may be reexpressed as

$$ {}_2I_{14}^{(-2)}(r) = \frac{1}{\rho_2^4}\ {}_2I_{13}^{(n=0)}(r)\cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_2^2}) - \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_2^4}\left(1+\frac{\delta^2\rho_2^2\rho_3^3}{2}+\frac{\rho_2^2r^2}{2}\right). \tag{A-5.52b} $$

The r domian integral defined by equation A-5.8c and expressed in series form by equation A-5.11c becomes, in the absence of zero wave number poles,

$$ {}_2I_{15}^{(n)}(r) = \sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\,{}_2I^{(n)}(j) = \frac{r}{|r|}\frac{i}{2}(i\rho_2)^{2n}e^{-\rho_2|r|}\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\left(\pm\rho_2\sqrt{\rho_2^2+\rho_3^2}\right)^{2j}, $$

$$ n \geq 0 \tag{A-5.53a} $$

The series form is that of a hyperbolic sine function divided by its argument which allows the integral to be expressed as

$$ {}_2I_{15}^{(n)}(r) = {}_2I_{13}^{(n)}(r)\,\frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}}\ ,\ n \geq 0 \tag{A-5.53b} $$

If the integrand posesses a simple zero wavenumber pole the series form evaluation of the integral is

$$ {}_2I_{15}^{(-1)}(r) = {}_2I^{(-1)(0)} + \sum_{j=1}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\,{}_2I^{(n=-1)(j)} = $$

$$ = \frac{r}{|r|}\frac{i}{2}\frac{-1}{\rho_2^2}\left[-1+e^{-\rho_2|r|}\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\left(\pm\rho_2\sqrt{\rho_2^2+\rho_3^2}\right)^{2j}\right] \tag{A-5.54a} $$

and may be expressed in closed form as

$$ {}_2I_{15}^{(-1)}(r) = \frac{-1}{\rho_2^2}\,{}_2I_{13}^{(n=0)}(r)\,\frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} + \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_2^2} \tag{A-5.54b} $$

This completes the evaluation of the r domain integrals, with imaginary non-zero wavenumber poles of the integrand, that are required for convolution.

The nonzero wavenumber poles of the integrands are real for the condition of larger thermal-viscous damping coefficient than thermal diffusion coefficient and the integrals will be identified by the presubscript 1. For this condition the r domain integrals of equations A-5.8 are related by equations A-5.11 to the general integral form of equation A-5.34. The r domain integral which is defined by equation A-5.8a may be evaluated, in the absence of zero wavenumber poles, by its relation to the general integral evaluation shown by equation A-5.39

$$ {}_1I_{13}^{n}(r) = {}_1I^{(n)(j=0)} = \frac{r}{|r|}\,\frac{i}{2}\,\rho_1^{2n}\cos(\rho_1 r) \quad , \quad n \geq 0 \quad . \tag{A-5.55a} $$

when the integrand possesses a simple pole of zero wavenumber the general integral evaluation given by equation A-5.46 may be used to evaluate the integral as

$$ {}_1I_{13}^{(-1)}(r) = {}_1I^{(-1)(0)} = \frac{r}{|r|}\,\frac{i}{2}\,\frac{1}{\rho_1^2}\left[\cos(\rho_1 r) - 1\right] \tag{A-5.55b} $$

and, for a third order zero wavenumber pole of the integrand, the general integral evaluation shown by equation A-5.48 applies and yields

$$ {}_1I_{13}^{(-2)}(r) = {}_1I^{(-2)(0)} = \frac{r}{|r|}\,\frac{i}{2}\,\frac{1}{\rho_1^4}\left[\cos(\rho_1 r) - 1 + \frac{\rho_1^2 r^2}{2}\right] \quad . \tag{A-5.55c} $$

The r domain integral defined by equation A-5.8b is related to a series of general integrals by equation A-5.11b. In the absence of zero wavenumber poles of the integrand, the series form evaluation of the integral is

$$ {}_1I_{14}^{(n)}(r) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}\,{}_1I^{(n)(j)} = \frac{r}{|r|}\,\frac{i}{2}\,\rho_1^{2n}\cos(\rho_1 r) \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!}\left(\pm\rho_1\sqrt{\rho_1^2-\rho_3^2}\right)^{2j} \; , $$

$$ n \geq 0 \quad . \tag{A-5.56a} $$

As for the condition of imaginary nonzero wavenumber poles of the integrand, equation A-5.50b, the series represents a hyperbolic cosine function and the closed form evaluation of the integral is

$$ {}_1I_{14}^{(n)}(r) = {}_1I_{13}^{(n)}(r) \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) \quad , \quad n \geq 0 \quad . \tag{A-5.56b} $$

If the integrand possesses a simple zero wavenumber pole the appropriate series form evaluation is

$$ {}_1I_{14}^{(-1)}(r) = {}_1I^{(-1)(0)} + \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!} \; {}_1I^{(n=-1)(j)} = $$

$$ = \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^2}[-1 + \cos(\rho_1 r) \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j)!} (\pm\rho_1\sqrt{\rho_1^2-\rho_3^2})^{2j}] \tag{A-5.57a} $$

which may also be expressed as

$$ {}_1I_{14}^{(-1)}(r) = \frac{1}{\rho_1^2} \; {}_1I_{13}^{(n=0)} \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) - \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^2} \quad . \tag{A-5.57b} $$

A third order wavenumber pole of the integrand requires that the series form evaluation of the integral be

$$ {}_1I_{14}^{(-2)}(r) = {}_1I^{(-2)(0)} + \frac{\delta}{2!} \, {}_1I^{(-2)(1)} + \sum_{j=2}^{\infty} \frac{\delta^{2j}}{(2j)!} \; {}_1I^{(n=-2)(j)} = $$

$$ = \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^4}[-1 + \frac{\rho_1^2 r^2}{2} + \frac{\delta^2\rho_1^2\rho_3^2}{2} + \cos(\rho_1 r) \sum_{j=0}^{\infty} (\pm\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})^{2j}] \tag{A-5.58a} $$

and results in the closed form evaluation

$$ {}_1I_{14}^{(-2)}(r) = \frac{1}{\rho_1^4} \, {}_1I_{13}^{(n=0)}(r) \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) - \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^4}(1 - \frac{\delta^2\rho_1^2\rho_3^2}{2} - \frac{\rho_1^2 r^2}{2}) \quad . \tag{A-5.58b} $$

The r domain integral defined by equation A-5.8c is related by equation A-5.11c to a series of general integrals. If the integrand does not possess poles at wavenumber equal to zero the integral may be evaluated in series form as

$$ {}_1I_{15}^{(n)}(r) = \sum_{j=0}^{\infty} \frac{\delta^{2j}}{(2j+1)!}\ {}_1I^{(n)(j)} = \frac{r}{|r|}\frac{i}{2}\rho_1^{2n}\cos(\rho_1 r)\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\left(\pm\rho_1\sqrt{\rho_1^2-\rho_3^2}\right)^{2j} , $$

$$ n \geq 0 . \qquad \text{(A-5.59a)} $$

Since the resulting series form is that of a hyperbolic sine function divided by its argument, the integral evaluation may be expressed as

$$ {}_1I_{15}^{(n)}(r) = {}_1I_{13}^{(n)}(r)\ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}} , \quad n \geq 0 . \qquad \text{(A-5.59b)} $$

If the integrand possess a simple zero wavenumber pole the integral may be evaluated in series form as

$$ {}_1I_{15}^{(-1)}(r) = {}_1I^{(-1)(0)} + \sum_{j=1}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\ {}_1I^{(n=-1)(j)} = $$

$$ = \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^2}\left[-1 + \cos(\rho_1 r)\sum_{j=0}^{\infty}\frac{\delta^{2j}}{(2j+1)!}\left(\pm\rho_1\sqrt{\rho_1^2-\rho_3^2}\right)^{2j}\right] \qquad \text{(A-5.60a)} $$

which may also be expressed as

$$ {}_1I_{15}^{(-1)}(r) = \frac{1}{\rho_1^2}\ {}_1I_{13}^{(n=0)}(r)\ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}} - \frac{r}{|r|}\frac{i}{2}\frac{1}{\rho_1^2} . \qquad \text{(A-5.60b)} $$

These integral evaluations, for real wavenumber poles of the integrand, complete the evaluation of the separated r domain integrals, defined by equations A-5.8, and allow the required convolutions to be performed.

Each of the three integral types to be evaluated, equations A-5.1 where the integer n determines the existance of an order of the integrand's zero wavenumber poles, may be expressed by application of equation A-5.6 as a convolution of two functions of r. These functions are $I_0(r)$, as defined and evaluated in equation A-5.7 and the form of the three integrals of equation A-5.8, $I^n_{1(\ )}(r)$, which corresponds to the particular one of the three integral forms to be evaluated. The evaluations of the integral forms of equation A-5.8 and therefore the convolutions forms of the desired integrals are critically dependent on the wavenumber poles of the integrands. The existance of an order of the zero wavenumber poles is determined by the integer n and the condition of imaginary or real nonzero wavenumber poles depends on the relative magnitudes of the thermal diffusion coefficient and thermal-viscous damping coefficient..

For the condition of larger thermal than thermal-viscous coefficients the nonzero wavenumber poles are imaginary. The integrals whose integrands possess these wavenumber poles will be identified by a presubscript 2, thus the convolution form of the desired integrals becomes

$$ {}_2I^n_{(\ )} = I_0(r) * {}_2I^n_{1(\ )}(r) = \int_{-\infty}^{\infty} d\zeta \; I_0(r-\zeta) \;\; {}_2I^n_{1(\ )}(\zeta) \quad . \tag{A-5.61} $$

In the absence of nonzero wavenumber poles of the integrands, n nonnegative, the first type of integral, equation A-5.1a, may be expressed as a convolution of the integral evaluations of equations A-5.7 and A-5.49a,

$$ {}_2I^n_3 = I_0(r) * {}_2I^n_{13}(r) = (i\rho_2)^{2n} \frac{i}{2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \; e^{-\frac{(r-\zeta)^2}{4\delta}} \; \frac{\zeta}{|\zeta|} \, e^{-\rho_2|\zeta|} , \quad n \geq 0 \; . \tag{A-5.62a} $$

Since the integrand of the convolution integral is discontinuous at ζ equal to zero, the convolution separates into a sum of two integrals as

$$ {}_2I_3^n = (i\rho_2)^{2n} \frac{i}{2} \sqrt{\frac{\pi}{\delta}} \left[\int_0^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{-\rho_2 \zeta} - \int_{-\infty}^{0} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{\rho_2 \zeta} \right] . \qquad \text{(A-5.62b)} $$

By completing the squares of the ζ terms in the arguements of the integrand exponentials and by performing the integration variable substitutions

$$ \eta_1 = \frac{(\zeta - r + 2\delta\rho_2)}{2\sqrt{\delta}} \qquad \text{and} \qquad \eta_2 = \frac{(-\zeta + r + 2\delta\rho_2)}{2\sqrt{\delta}} \qquad \text{(A-5.63a,b)} $$

in the first and second integrals respectively, the convolution may be expressed as

$$ {}_2I_3^n = (i\rho_2)^{2n} \, i\sqrt{\pi} \left[e^{(\delta\rho_2^2 - \rho_2 r)} \int_{\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}}^{\infty} d\eta_1 e^{-(\eta_1)^2} + e^{(\delta\rho_2^2 + \rho_2 r)} \int_{\infty}^{\frac{r+2\delta\rho_2}{2\sqrt{\delta}}} d\eta_2 \, e^{-(\eta_2)^2} \right] . \qquad \text{(A-5.64)} $$

Since the integrals have been expressed in the form of the complementary Gaussian error function,

$$ \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} d\eta \, e^{-\eta^2} , \qquad \text{(A-5.65)} $$

the first type integral with imaginary integrand wavenumber poles only may be expressed in terms of known functions as

$$ {}_2I_3^n = (i\rho_2)^{2n} \frac{i\pi}{2} e^{\delta\rho_2^2} \left[e^{-\rho_2 r} \operatorname{erfc}\left(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}\right) - e^{\rho_2 r} \operatorname{erfc}\left(\frac{r+2\delta\rho_2}{2\sqrt{\delta}}\right) \right] , \; n \geq 0 . \qquad \text{(A-5.66)} $$

When the integrand possesses a simple pole at wavenumber equal to zero the integral evaluation of equation A-5.49b is the appropriate second function of the convolution and yields

$$ {}_2I_3^{(-1)} = I_0(r) * {}_2I_{13}^{(-1)}(r) = \frac{-i}{2\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} (e^{-\rho_2|\zeta|} - 1) = $$

$$ = \frac{-i}{2\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} e^{-\rho_2|\zeta|} + \frac{i}{2\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \; . \tag{A-5.67} $$

The first term contains the same convolution integral form as in the absence of zero wavenumber poles and therefore this term may be related to that integral. The integral of the second term may also be related to a previously evaluated integral by performing the variable substitution,

$$ \zeta' = r - \zeta \; , \tag{A-5.68} $$

which yields the second term expression,

$$ \frac{i}{2\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta' \; e^{-\frac{(\zeta')^2}{4\delta}} \; \frac{r-\zeta'}{|r-\zeta'|} \; . \tag{A-5.69} $$

Thus the first term may be related to the integral form of equation A-5.62a and the second term may be related to the integral of equation A-4.17a,

$$ {}_2I_3^{(-1)} = \frac{-1}{\rho_2^2} \; {}_2I_3^{n=0} + \frac{1}{\rho_2^2} \; {}_dI_3^{(-1)}(r) \; . \tag{A-5.70a} $$

Utilizing the first term integral evaluation, equation A-5.66, and the second term integral evaluation, equation A-4.28a, the first type integral with a simple zero wavenumber pole of the integrand in addition to imaginary wavenumber poles may be expressed as

$$ {}_2I_3^{(-1)} = \frac{-1}{\rho_2^2} \{ {}_2I_3^{n=0} - i\pi \, \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) \} = $$

$$ = \frac{-i\pi}{2\rho_2^2} \{ e^{\delta\rho_2^2} [e^{-\rho_2 r} \, \mathrm{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - e^{\rho_2 r} \, \mathrm{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] - 2 \, \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) \} . \quad \text{(A-5.70b)} $$

When the integrand of the first type integral, equation A-5.1a, possesses a third order pole at zero wavenumber, n equal to minus two, the integral evaluation of equation A-5.49c must be used as the second convolved function and results in the convolution expression

$$ {}_2I_3^{(-2)} = I_0(r) * {}_2I_{13}^{(-2)} = \frac{i}{2\rho_2^4} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} [(e^{-\rho_2|\zeta|} - 1) - \frac{\rho_2^2\zeta^2}{2}] = $$

$$ = \frac{i}{2\rho_2^4} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} (e^{-\rho_2|\zeta|} - 1) - \frac{i}{4\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \zeta^2 $$

(A-5.71)

The first term contains the same convolution integral form as for a simple wavenumber pole, equation A-5.67, and by the integration variable substitution of equation A-5.68 the second term becomes

$$ \frac{-i}{4\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta' \, e^{-\frac{(\zeta')^2}{4\delta}} \frac{(r-\zeta')^3}{|r-\zeta'|} \quad \text{(A-5.72)} $$

which contains the integral form of equation A-4.17b; thus the convolution may be related to previously evaluated integration forms as

$$ {}_2I_3^{(-2)} = \frac{-1}{\rho_2^2} \, {}_2I_3^{(-1)} + \frac{1}{\rho_2^2} \, {}_dI_3^{(-2)}(r) \quad . \quad \text{(A-5.73a)} $$

By substituting the integral evaluations of equations A-5.70b and A-4.28b the first type integral with third order zero wavenumber and imaginary wavenumber poles of the integrand may be evaluated as

$$ {}_2I_3^{(-2)} = \frac{1}{\rho_2^4} \cdot \left\{ {}_2I_3^{n=0} - i\pi(1+\rho_2^2\delta + \frac{\rho_2^2 r^2}{2})\operatorname{erf}(\frac{r}{2\sqrt{\delta}}) - i\rho_2^2\sqrt{\pi\delta}\, r\, e^{-\frac{r^2}{4\delta}} \right\} = $$

$$ = \frac{i\pi}{2\rho_2^4} \left\{ \begin{array}{l} e^{\delta\rho_2^2}[e^{-\rho_2 r} \operatorname{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - e^{\rho_2 r} \operatorname{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] - \\ -(2 + 2\delta\rho_2^2 + \rho_2^2 r^2) \operatorname{erf}(\frac{r}{2\sqrt{\delta}}) - 2\sqrt{\frac{\delta}{\pi}}\, \rho_2^2 r\, e^{-\frac{r^2}{4\delta}} \end{array} \right\} . \qquad \text{(A-5.73b)} $$

The second type of integral to be evaluated, equation A-5.1b, may be expressed when there are no zero wavenumber poles of the integrand, as a convolution of the integral evaluations of equations A-5.7 and A-5.50,

$$ {}_2I_4^n = I_0(r) * {}_2I_{14}^n(r) = I_0(r) * {}_2I_{13}^n(r) \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}) \ , \ n \geq 0 \ . \qquad \text{(A-5.74a)} $$

Since the arguement of the hyperbolic cosine function is independent of r the function is independent of the convolution integration and the second type integral is simply related to the first type integral evaluation of equation A-5.62a as,

$$ {}_2I_4^n = \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\ {}_2I_3^n = $$

$$ = (i\rho_2)^{2n}\ \frac{i\pi}{2} \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\ e^{\delta\rho_2^2}\ [e^{-\rho_2 r}\operatorname{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - e^{\rho_2 r} \operatorname{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] \ , $$

$$ n \geq 0 \ . \qquad \text{(A-5.74b)} $$

A simple zero wavenumber pole of the integrand, n equal to minus one, requires that the second convolved integral be that of equation A-5.51 which also differs from the corresponding first type integral convolution by the same hyperbolic cosine function as shown,

$$ {}_2I_4^{(-1)} = I_0(r) * {}_2I_{14}^{(-1)}(r) = $$

$$ = I_0(r) * \left[\frac{-1}{\rho_2^2}\ {}_2I_{13}^{n=0}(r) \cosh\left(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}\right) + \frac{r}{|r|}\frac{i}{2\rho_2^2}\right] = $$

$$ = \frac{-1}{\rho_2^2} \cosh\left(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}\right)\ {}_2I_3^{n=0} + \frac{i}{2\rho_2^2}\sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\ e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \ . \qquad \text{(A-5.75a)} $$

The first term of the convolution is similar to the convolution in the absence of zero wavenumber poles and the second term is identical to the second term of the first type integral with a simple zero wavenumber pole, equation A-5.67. As a result, the first term evaluation is similar to that of equation A-5.74 and the second term evaluation is the second term of equation A-5.70 yielding the evaluation of the second type integral with a zero wavenumber pole of the integrand,

$$ {}_2I_4^{(-1)} = \frac{-1}{\rho_2^2}\ {}_2I_4^{n=0} + \frac{1}{\rho_2^2}\ {}_dI_3^{(-1)}(r) = $$

$$ = \frac{-i\pi}{2\rho_2^2}\left\{\cosh\left(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}\right)e^{\delta\rho_2^2}\left[e^{-\rho_2 r}\operatorname{erfc}\left(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}\right)-e^{\rho_2 r}\operatorname{erfc}\left(\frac{r+2\delta\rho_2}{2\sqrt{\delta}}\right)\right]-2\operatorname{erf}\left(\frac{r}{2\sqrt{\delta}}\right)\right\} \ . $$

(A-5.75b)

When the integrand of the second type integral possesses a third order pole at zero wavenumber, n equal to minus two, the integral evaluation of equation A-5.52 is appropriate as the second convolved function and results in the convolution form,

$$ {}_2I_4^{(-2)} = I_0(r) * {}_2I_{14}^{(-2)}(r) = $$

$$ = I_0(r) * [\frac{1}{\rho_2^4}\, {}_2I_{13}^{n=0}(r) \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}) - \frac{r}{|r|}\frac{i}{2\rho_2^4}(1 + \frac{\delta^2\rho_2^2\rho_3^2}{2} + \frac{\rho_2^2 r^2}{2})] = $$

$$ = \frac{1}{\rho_2^4}\cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\ {}_2I_3^{n=0} - \frac{i}{2\rho_2^4}(1 + \frac{\delta^2\rho_2^2\rho_3^2}{2})\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\ \frac{\zeta}{|\zeta|} - $$

$$ - \frac{i}{4\rho_2^2}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\ \frac{\zeta}{|\zeta|}\ \zeta^2 \ . \qquad \text{(A-5.76a)} $$

In the last convolution representation above, the first term is similar to the convolution in the absence of zero wavenumber poles, the second term contains the same integral as in the last term of equation A-5.67 and the third term contains the same integral as the last term of equation A-5.71. Therefore the first term is similar to the evaluation of equation A-5.74 and the second and third term integrals may be evaluated as the last terms of equations A-5.70 and A-5.73 respectively. Thus the second type integral with a third order zero wavenumber pole may be evaluated as

$$ {}_2I_4^{(-2)} = \frac{1}{\rho_2^4}\ {}_2I_4^{n=0} - \frac{1}{\rho_2^4}(1 + \frac{\delta^2\rho_2^2\rho_3^2}{2})\ {}_dI_3^{(-1)}(r) + \frac{1}{\rho_2^2}\ {}_dI_3^{(-2)}(r) = $$

$$ = \frac{i\pi}{2\rho_2^4}\left\{\begin{array}{l} \cosh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})\ e^{\delta\rho_2^2}[e^{-\rho_2 r}\ \mathrm{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - e^{\rho_2 r}\ \mathrm{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] - \\ -(2 + 2\delta\rho_2^2 + \delta^2\rho_2^2\rho_3^2 + \rho_2^2 r^2)\ \mathrm{erf}\,(\frac{r}{2\sqrt{\delta}}) - 2\sqrt{\frac{\delta}{\pi}}\ \rho_2^2 r\ e^{-\frac{r^2}{4\delta}} \end{array}\right\} . \qquad \text{(A-5.76b)} $$

The third type of integral to be evaluated, equation A-5.1c, is equal to a convolution of the integral evaluations of equations A-5.7 and A-5.53 in the absence of zero wavenumber poles of the integrand,

$$ {}_2I_5^n = I_0(r) * {}_2I_{15}^n(r) = I_0(r) * {}_2I_{13}^n(r) \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} , \quad n \geq 0 \ . \tag{A-5.77a} $$

Since the arguement of the hyperbolic sine function is independent of the convolution integration, in the absence of zero wavenumber poles the third type integral evaluation is simply related to that of the first type integral as

$$ {}_2I_5^n = \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} \ {}_2I_3^n = $$

$$ = (i\rho_2)^{2n} \frac{i\pi}{2} \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} \ e^{\delta\rho_2^2} [e^{-\rho_2 r} \operatorname{erfc}(\frac{-r+2\delta\rho_2}{2\sqrt{\delta}}) - e^{\rho_2 r} \operatorname{erfc}(\frac{r+2\delta\rho_2}{2\sqrt{\delta}})] , $$

$$ n \geq 0 \ . \tag{A-5.77b} $$

When the integrand of the integral possesses a simple pole at wavenumber equal to zero, n equal to minus one, the appropriate second convolved function is the integral evaluation of equation A-5.54 which yields the convolution form,

$$ {}_2I_5^{(-1)} = I_0(r) * {}_2I_{15}^{(-1)}(r) = $$

$$ = I_0(r) * [\frac{-1}{\rho_2^2} \ {}_2I_{13}^{n=0}(r) \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} + \frac{r}{|r|} \frac{i}{2\rho_2^2}] = $$

$$ = \frac{-1}{\rho_2^2} \frac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}} \ {}_2I_3^{n=0} + \frac{i}{2\rho_2^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \ e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \ . \tag{A-5.78a} $$

The first term of the convolution is similar to the convolution in the absence of zero wavenumber poles and the second term is identically the second term of equation A-5.67. Therefore the third type integral with a simple zero wavenumber pole may be evaluated by evaluating the two terms of the convolution form, equation A-5.78a, as in equations A-5.77 and A-5.70 respectively,

$$
{}_2I_5^{(-1)} = \frac{-1}{\rho_2^2}\ {}_2I_5^{n=0} + \frac{1}{\rho_2^2}\ {}_dI_3^{(-1)}(r)
$$

$$
= \frac{-i\pi}{2\rho_2^2}\left\{\begin{array}{l} \dfrac{\sinh(\delta\rho_2\sqrt{\rho_2^2+\rho_3^2})}{\delta\rho_2\sqrt{\rho_2^2+\rho_3^2}}\, e^{\delta\rho_2^2}[e^{-\rho_2 r}\,\mathrm{erfc}(\dfrac{-r+2\delta\rho_2}{2\sqrt{\delta}})-e^{\rho_2 r}\,\mathrm{erfc}(\dfrac{r+2\delta\rho_2}{2\sqrt{\delta}})] - \\ \qquad - 2\,\mathrm{erf}\,(\dfrac{r}{2\sqrt{\delta}}) \end{array}\right\} . \qquad \text{(A-5.78b)}
$$

This completes the evaluation of the desired integrals, equations A-5.1, for all necessary zero wavenumber poles of the integrand when the nonzero wavenumber poles are imaginary, which corresponds to the condition of larger thermal diffusion coefficient than thermal-viscous damping coefficient.

The integrals of equation A-5.1 need also be evaluated for the condition of larger thermal-viscous then thermal diffusion coefficient which results in real nonzero wavenumber poles of the integrand. These integrals will be identified by a presubscript 1 and may be evaluated by the convolution form

$$
{}_1I^n_{(\ )} = I_0(r) * {}_1I^n_{1(\ )}(r) = \int_{-\infty}^{\infty} d\zeta\ I_0(r-\zeta)\ \ {}_1I^n_{1(\ )}(\zeta) \quad . \qquad \text{(A-5.79)}
$$

The first type of integral, equation A-5.1a, may be expressed in the absence of zero wavenumber poles of the integrand as a convolution of the integral evaluations of equations A-5.7 and A-5.55a,

$$ {}_1I_3^n = I_0(r) * {}_1I_{13}^n(r) = \rho_1^{2n} \, \frac{i}{2}\sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \cos(\rho_1\zeta) \ , \ n \geq 0 \ . \quad \text{(A-5.80a)} $$

The discontinuity in the integrand of the convolution may be eliminated by expressing the integral as the sum of two continuous integrals. Furthermore expanding the cosine function in exponential form, the convolution may be expressed as the sum of four integrals,

$$ {}_1I_3^n = \rho_1^{2n} \frac{i}{4}\sqrt{\frac{\pi}{\delta}} \left\{ \begin{array}{l} \int_0^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{i\rho_1\zeta} + \int_0^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{-i\rho_1\zeta} - \\ - \int_{-\infty}^{0} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{i\rho_1\zeta} - \int_{-\infty}^{0} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} e^{-i\rho_1\zeta} \end{array} \right\} . \quad \text{(A-5.80b)} $$

By completing the squares of the ζ terms in the arguments of the integrand exponentials and by performing the integration variable substitutions,

$$ \eta_1 = \frac{\zeta-r-i2\delta\rho_1}{2\sqrt{\delta}} \ , \quad \eta_2 = \frac{\zeta-r+i2\delta\rho_1}{2\sqrt{\delta}} \ , \quad \eta_3 = \frac{-\zeta+r+i2\delta\rho_1}{2\sqrt{\delta}} \ \text{and} \ \eta_4 = \frac{-\zeta+r-i2\delta\rho_1}{2\sqrt{\delta}} \quad \text{(A-5.81a, b,c,d)} $$

in the first, second, third and fourth integrals respectively, the convolution becomes

$$ {}_1I_3^n = \rho_1^{2n} \frac{i\sqrt{\pi}}{2} e^{-\delta\rho_1^2} \left\{ \begin{array}{l} e^{i\rho_1 r} \int_{\frac{-r-i2\delta\rho_1}{2\sqrt{\delta}}}^{\frac{\infty-i2\delta\rho_1}{2\sqrt{\delta}}} d\eta_1 \, e^{-(\eta_1)^2} + e^{-i\rho_1 r} \int_{\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}}}^{\frac{\infty+i2\delta\rho_1}{2\sqrt{\delta}}} d\eta_2 \, e^{-(\eta_2)^2} + \\ + e^{i\rho_1 r} \int_{\frac{\infty+i2\delta\rho_1}{2\sqrt{\delta}}}^{\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}} d\eta_3 \, e^{-(\eta_3)^2} + e^{-i\rho_1 r} \int_{\frac{\infty-i2\delta\rho_1}{2\sqrt{\pi}}}^{\frac{r-i2\delta\rho_1}{2\sqrt{\delta}}} d\eta_4 \, e^{-(\eta_4)^2} \end{array} \right\} . \quad \text{(A-5.82)} $$

Each of these integrals may be expressed as the difference of two Gaussian error functions of complex arguement,

$$\mathrm{erf}(x+iy) = \frac{2}{\sqrt{\pi}} \int_0^{x+iy} d\eta \, e^{-\eta^2} \quad , \tag{A-5.83}$$

therefore the convolution becomes,

$$ {}_1I_3^n = \rho_1^{2n} \frac{i\pi}{4} e^{-\delta\rho_1^2} \left\{ \begin{array}{l} e^{i\rho_1 r}[\mathrm{erf}(\frac{\infty-i2\delta\rho_1}{2\sqrt{\delta}})-\mathrm{erf}(\frac{-r-i2\delta\rho_1}{2\sqrt{\delta}})]+e^{-i\rho_1 r}[\mathrm{erf}(\frac{\infty+i2\delta\rho_1}{2\sqrt{\delta}})- \\ \mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})]+e^{i\rho_1 r}[\mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})-\mathrm{erf}(\frac{\infty+i2\delta\rho_1}{2\sqrt{\delta}})] + \\ +e^{-i\rho_1 r}[\mathrm{erf}(\frac{r-i2\delta\rho_1}{2\sqrt{\delta}})-\mathrm{erf}(\frac{\infty-i2\delta\rho_1}{2\sqrt{\delta}})] \end{array} \right\} . \tag{A-5.84}$$

Simplification is possible because the Gaussian error function is an odd function of its arguement and is equal to unity for an infinite real part of its arguement larger than the imaginary part of its arguement,

$$\mathrm{erf}(-x-iy) = -\mathrm{erf}(x+iy) \quad , \tag{94a}$$

$$\lim_{x\to\infty} \mathrm{erf}(x+iy) = 1 \quad \text{if} \quad |x| > |y| \quad \text{for real } x,y \ . \tag{A-5.85b}$$

All terms which are specifically infinite in equation A-5.84 result from the convolution being performed over an infinite medium and must be larger than any other terms which may become infinite but must be within the medium. As a result all error functions with specifically infinite real parts of the arguement are equal to unity and utilizing the odd property of the error function, equation A-5.85a, the first type integral with only nonzero, real wavenumber poles of its integrand maybe evaluated as

$$ {}_1I_3^n = \rho_1^{2n} \frac{i\pi}{2} e^{-\delta\rho_1^2} [e^{i\rho_1 r} \operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{-i\rho_1 r} \operatorname{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] \quad . \tag{A-5.86} $$

An alternate form may be derived through use of the error function property that the error function is equal to the conjugate of the error function of conjugate arguement,

$$ \begin{aligned} \operatorname{erf}(x+iy) &= \text{Real } [\operatorname{erf}(x+iy)] + i \text{ Imaginary } [\operatorname{erf}(x+iy)] \\ &= \text{Real } [\operatorname{erf}(x-iy)] - i \text{ Imaginary } [\operatorname{erf}(x-iy)] \end{aligned} \tag{A-5.87} $$

By expressing the exponential of imaginary arguement and the error function of complex arguement in terms of real and imaginary parts and by using the conjugate property of the error function the first type integral may be expressed as

$$ {}_1I_3^n = \rho_1^{2n}\, i\pi\, e^{-\delta\rho_1^2} \left\{\cos(\rho_1 r) \text{ Real}[\operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})] - \sin(\rho_1 r) \text{ Imaginary}[\operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})]\right\} \tag{A-5.88} $$

which is purely imaginary.

A simple zero wavenumber pole of the first type integral's integrand requires that the integral evaluation A-5.55b be utilized as the second convolution function in equation A-5.88. This yields the convolution form of the desired integral,

$$ {}_1I_3^{(-1)} = I_0(r) * {}_1I_{13}^{(-1)}(r) = \frac{i}{2\rho_1^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} (\cos(\rho_1\zeta)-1) = $$

$$ = \frac{i}{2\rho_1^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \cos(\rho_1\zeta) - \frac{i}{2\rho_1^2} \sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \tag{A-5.89a} $$

The integral in the first term is identically the integral in the absence of zero wavenumber poles and the second term is identically the second term of equation A-5.67 which is typical in the presence of a zero wavenumber pole. Therefore the evaluation of the first term is similar to the evaluation given by equation A-5.86 and the second term evaluation is identically the second term of equation A-5.70 and the first type integral with a simple zero wavenumber pole of the integrand may be evaluated as

$$ {}_1I_3^{(-1)} = \frac{1}{\rho_1^2}\, {}_1I_3^{n=0} - \frac{1}{\rho_1^2}\, {}_dI_3^{(-1)}(r) = $$

$$ = \frac{i\pi}{2\rho_1^2}\left\{e^{-\delta\rho_1^2}\left[e^{i\rho_1 r}\,\mathrm{erf}\left(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}\right) - e^{-i\rho_1 r}\,\mathrm{erf}\left(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}}\right)\right] - 2\,\mathrm{erf}\left(\frac{r}{2\sqrt{\delta}}\right)\right\} . \qquad \text{(A-5.89b)} $$

When the integrand of the first type integral possesses a third order pole of the integrand, n equal to minus two, the integral evaluation of equation A-5.55c is appropriate as the second of the convolved functions and yields the convolution expression

$$ {}_1I_3^{(-2)} = I_0(r) * {}_1I_{13}^{(-2)}(r) = \frac{i}{2\rho_1^4}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\,\frac{\zeta}{|\zeta|}\left[(\cos(\rho_1\zeta)-1)+\frac{\rho_1^2\zeta^2}{2}\right] = $$

$$ = \frac{i}{2\rho_1^4}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\,\frac{\zeta}{|\zeta|}(\cos(\rho_1\zeta)-1) + \frac{i}{4\rho_1^2}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\,\frac{\zeta}{|\zeta|}\zeta^2 . \qquad \text{(A-5.90a)} $$

The first term contains the same convolution integral form as for a simple zero wavenumber pole, equation A-5.89, and the second term is identically the second term of equation A-5.71 which is typical in the presence of a third order zero wavenumber pole. By evaluating each term as done previously the first type integral with third order zero wavenumber poles of the integrand may be expressed as

$$ {}_1I_3^{(-2)} = \frac{1}{\rho_1^2}\ {}_1I_3^{(-1)} - \frac{1}{\rho_1^2}\ {}_dI_3^{(-2)}(r) = $$

$$ = \frac{i\pi}{2\rho_1^4} \cdot \left\{ \begin{array}{l} e^{-\delta\rho_1^2}[e^{i\rho_1 r}\ \mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{-i\rho_1 r}\ \mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] + \\ \\ +(-2+2\delta\rho_1^2+\rho_1^2 r^2)\ \mathrm{erf}(\frac{r}{2\sqrt{\delta}}) + 2\sqrt{\frac{\pi}{\delta}}\ \rho_1^2 r^2\ e^{-\frac{r^2}{4\delta}} \end{array} \right\} \qquad \text{(A-5.90b)} $$

The second type of integral to be evaluated, equation A-5.1b, may be expressed, in the absence of zero wavenumber poles, as a convolution of the integral evaluations of equations A-5.7 and A-5.56,

$$ {}_1I_4^n = I_0(r) * {}_1I_{14}^n(r) = I_0(r) * {}_1I_{13}^n(r)\ \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) \quad , \quad n \geq 0 \quad . \qquad \text{(A-5.91a)} $$

The hyperbolic cosine function is independent of the convolution integration so the second type integral is simply related to the first type integral as

$$ {}_1I_4^n = \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})\ {}_1I_3^n = $$

$$ = \rho_1^{2n}\ \frac{i\pi}{2}\ \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) \cdot e^{-\delta\rho_1^2}[e^{i\rho_1 r}\ \mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{-i\rho_1 r}\ \mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] \ , $$

$$ n \geq 0 \quad . \qquad \text{(A-5.91b)} $$

When the integrand possesses a simple zero wavenumber pole, n equal to minus one, the second convolved function should be the integral in equation A-5.57 which yields the convolution form,

$$ {}_1I_4^{(-1)} = I_0(r) * {}_1I_{14}^{(-1)}(r) = $$

$$ = I_0(r) * [\frac{1}{\rho_1^2} \; {}_1I_{13}^{n=0} \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) - \frac{r}{|r|}\frac{i}{2\rho_1^2}] = $$

$$ = \frac{1}{\rho_1^2}\cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) \; {}_1I_3^{n=0} - \frac{i}{2\rho_1^2}\sqrt{\frac{\pi}{\delta}} \int_{-\infty}^{\infty} d\zeta \, e^{-\frac{(r-\zeta)^2}{4\delta}} \frac{\zeta}{|\zeta|} \; . \qquad \text{(A-5.92a)} $$

The first term of the convolution is similar to the first term in the absence of zero wavenumber poles as evaluated in equation A-5.91. The second term is typical of integrals with simple zero wavenumber poles as in equation A-5.67 and evaluated in equation A-5.70. As a result, the first type integral with a simple zero wavenumber pole of the integrand may be evaluated as

$$ {}_1I_4^{(-1)} = \frac{1}{\rho_1^2} \; {}_1I_4^{n=0} - \frac{1}{\rho_1^2} \; {}_dI_3^{(-1)}(r) = $$

$$ = \frac{i\pi}{2\rho_1^2}\left\{ \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) e^{-\delta\rho_1^2} [e^{i\rho_1 r} \operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{-i\rho_1 r} \operatorname{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] - 2\operatorname{erf}(\frac{r}{2\sqrt{\delta}}) \right\} . $$

(A-5.92b)

A third order pole at zero wavenumber of the integrand, n equal to minus two, requires that the second convolved function be that of equation A-5.58 and results with the convolution form,

$$ {}_1I_4^{(-2)} = I_0(r) * {}_1I_{14}^{(-2)}(r) = $$

$$ = I_0(r) * [\frac{1}{\rho_1^4}\, {}_1I_{13}^{n=0}(r) \cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}) - \frac{r}{|r|}\frac{i}{2\rho_1^4}(1-\frac{\delta^2\rho_1^2\rho_3^2}{2} - \frac{\rho_1^2 r^2}{2})] = $$

$$ = \frac{1}{\rho_1^4}\cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})\ {}_1I_3^{n=0} - \frac{i}{2\rho_1^4}(1-\frac{\delta^2\rho_1^2\rho_3^2}{2})\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\frac{\zeta}{|\zeta|} + $$

$$ + \frac{i}{4\rho_1^2}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\, e^{-\frac{(r-\zeta)^2}{4\delta}}\frac{\zeta}{|\zeta|}\zeta^2 \ . \qquad \text{(A-5.93a)} $$

In the last convolution representation above, the first term is similar to the convolution in the absence of zero wavenumber poles and the integrals in the second and third terms are the same as contained in equations A-5.67 and A-5.71 respectively. Evaluating the first term as in equation A-5.91 and evaluating the integrals contained in the second and third terms as in equations A-5.70 and A-5.73 respectively, the second type integral with a third order zero wavenumber pole of the integrand may be expressed as

$$ {}_1I_4^{(-2)} = \frac{1}{\rho_1^4}\ {}_1I_4^{n=0} - \frac{1}{\rho_1^4}(1-\frac{\delta^2\rho_1^2\rho_3^2}{2})\ {}_dI_3^{(-1)}(r) - \frac{1}{\rho_1^2}\ {}_dI_3^{(-2)}(r) = $$

$$ = \frac{i\pi}{2\rho_1^4}\left\{\begin{aligned} &\cosh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})e^{-\delta\rho_1^2}[e^{i\rho_1 r}\operatorname{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}) - e^{-i\rho_1 r}\operatorname{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] + \\ &+ (-2 + 2\delta\rho_1^2 + \delta^2\rho_1^2\rho_3^2 + \rho_1^2 r^2)\operatorname{erf}(\frac{r}{2\sqrt{\delta}}) + 2\sqrt{\frac{\delta}{\pi}}\ \rho_1^2 r\, e^{-\frac{r^2}{4\delta}} \end{aligned}\right\}. \qquad \text{(A-5.93b)} $$

The third type of integral, equation A-5.1c, in the absence of zero wavenumber poles of the integrand may be expressed as a convolution of the integral evaluations of equations A-5.7 and A-5.59.

$$ {}_1I_5^n = I_0(r) * {}_1I_{15}^n(r) = I_0(r) * {}_1I_{13}^n(r)\ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}} \ , \quad n \geq 0 \ . \qquad \text{(A-5.94a)} $$

Since the arguement of the hyperbolic sine function is independent of the convolution integration, the third type integral evaluation in the absence of zero wavenumber poles is

$$ {}_1I_5^n = \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}}\ {}_1I_3^n = $$

$$ = \rho_1^n\,\frac{i\pi}{2}\ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}}\ e^{-\delta\rho_1^2}[e^{i\rho_1 r}\,\mathrm{erf}(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}})-e^{-i\rho_1 r}\,\mathrm{erf}(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}})] \ , $$

$$ n \geq 0 \ . \qquad \text{(A-5.94b)} $$

When the integrand of the third type integral possesses a simple pole at zero wavenumber, n equal to minus one, the integral evaluation of equation A-5.60 is the appropriate second function of the convolution representation,

$$ {}_1I_5^{(-1)} = I_0(r) * {}_1I_{15}^{(-1)}(r) = $$

$$ = I_0(r) * [\frac{1}{\rho_1^2}\ {}_1I_{13}^{n=0}(r)\ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}} - \frac{r}{|r|}\,\frac{i}{2\rho_1^2}] = $$

$$ = \frac{1}{\rho_1^2}\,\frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}}\ {}_1I_3^{n=0} - \frac{i}{2\rho_1}\sqrt{\frac{\pi}{\delta}}\int_{-\infty}^{\infty} d\zeta\ e^{-\frac{(r-\zeta)^2}{4\delta}}\ \frac{\zeta}{|\zeta|} \ . \qquad \text{(A-5.95a)} $$

The first term of the convolution is similar to the convolution in the absence of zero wavenumber poles, evaluated in equation A-5.94, and the integral contained in the second term is typical in the presence of simple zero wavenumber poles as in equation A-5.67 and evaluated in equation A-5.70. Thus the third type integral with a simple zero wavenumber pole of the integrand may be evaluated as

$$ {}_1I_5^{(-1)} = \frac{1}{\rho_1^2}\ {}_1I_5^{n=0} - \frac{1}{{}^2_1}\ {}_dI_3^{(-1)}(r) = $$

$$ = \frac{i\pi}{2\rho_1^2} \left\{ \frac{\sinh(\delta\rho_1\sqrt{\rho_1^2-\rho_3^2})}{\delta\rho_1\sqrt{\rho_1^2-\rho_3^2}}\, e^{-\delta\rho_1^2}\left[e^{i\rho_1 r}\,\mathrm{erf}\left(\frac{r+i2\delta\rho_1}{2\sqrt{\delta}}\right) - e^{-i\rho_1 r}\,\mathrm{erf}\left(\frac{-r+i2\delta\rho_1}{2\sqrt{\delta}}\right)\right] - 2\,\mathrm{erf}\left(\frac{r}{2\sqrt{\delta}}\right) \right\} . \qquad \text{(A-5.95b)} $$

This completes the evaluation of the desired integrals, equations A-5.1, for all necessary zero wavenumber poles of the integrand when the nonzero wavenumber poles are real, which corresponds to the condition of larger thermal-viscous damping coefficient than thermal diffusion coefficient, and along with the previous evaluations for imaginary nonzero wavenumber poles completes all required evaluations of these integrals.