Spring 1975

NORMED NEAR ALGEBRAS AND FINITE DIMENSIONAL NEAR ALGEBRAS OF CONTINUOUS FUNCTIONS

JOEL W. IRISH

NORMED NEAR ALGEBRAS AND FINITE DIMENSIONAL NEAR ALGEBRAS OF CONTINUOUS FUNCTIONS

by

JOEL W. IRISH

M. S. University of New Hampshire, 1969

A THESIS

Submitted to the University of New Hampshire

In Partial Fulfillment of

The Requirements for the Degree of

Doctor of Philosophy

Graduate School

Department of Mathematics

May, 1975

This thesis has been examined and approved.

Robert J. Silverman

Thesis director, Robert J. Silverman, Prof. of Mathematics

M. Evans Munroe

M. Evans Munroe, Prof. of Mathematics

Shepley L. Ross

Shepley L. Ross, Prof. of Mathematics

Homer F. Bechtell

Homer F. Bechtell, Asso. Prof. of Mathematics

Eric A. Nordgren

Eric A. Nordgren, Asso. Prof. of Mathematics

4/9/75

Date

ACKNOWLEDGEMENTS

The author wishes to thank his advisor, Dr. Robert Silverman, for his constant guidance and encouragement. Also, a special thanks to his wife, Janet, whose understanding and long hours of typing made possible the completion of this paper.

TABLE OF CONTENTS

ABSTRACT

NORMED NEAR ALGEBRAS AND FINITE DIMENSIONAL
NEAR ALGEBRAS OF CONTINUOUS FUNCTIONS

by

JOEL W. IRISH

A real near algebra system is a real algebra which lacks the left distributive property and the scalar peoperty $x(ty) = t(xy)$. The emphasis of this paper is in the area of analysis and function near algebras rather that the study of algebraic properties.

A real normed near algebra is a near algebra over the field $\mathbb{R}$ such that the linear space is a normed space and $\|xy\| \leq \|x\| \|y\|$. This condition is not strong enought to force continuity of multiplication in both variables; but it does result in the following.

The right multiplication operator is a continuous function. Every normed near algebra without identity can be isomorphically isometrically embedded in a normed near algebra with identity provided $x(ty) = t(xy)$ for all $t \geq 0$.

A D-normed near algebra is a normed near algebra such that $\|bx - by\| \leq K_b \|x - y\|$ for all x, y and b and for some $K_b \geq 0$. A strongly D-normed near algebra occurs whenever

$K_b = \|b\|$ for all b. These conditions are a weak distributive property of the norm. Several results follow directly from the D-norm condition.

Multiplication is continuous in both variables separately and simultaneously in a D-normed near algebra.

Every strongly D-normed near algebra is a near algebra with continuous inverse.

Every strongly D-normed near algebra can be embedded in a complete strongly D-normed near algebra.

The left regular representation of a normed algebra is generalized for normed near algebras with the bounded left multiplication operators (not necessarily linear) as the representation space. For D-normed near algebras, the representation is into the space of Lipschitz left multiplication operators.

The existence of left continuity of multiplication and left modules result in several important theorems. For a near algebra N and left module A, let $[n,a,b] = n(a+b) - na - nb$ denote the distributor of a, b $\in$ A with respect to n $\in$ N. For nonempty sets S_1, S_2, $S_3 \subseteq N$, let $[S_1,S_2,S_3]$ be the subspace of N^+ generated by the distributor elements $[s_1,s_2,s_3]$. Define $D^o(A) = A$ and $D^k(A) = [N,D^{k-1}(A),N]$; then N is D-weakly distributive (D-w-d) of length $k > 0$ if $D^k(N) = 0$ and $D^{k-1}(N) \neq 0$.

Every left continuous semisimple D-w-d near algebra is a semisimple algebra.

If N is a normed near algebra and there exists a non-zero

left module M which is left distributive with respect to N, then M is a sub algebra and there is a representation of N into the space of bounded linear operators on M.
If N is a D-w-d normed near algebra of length $k > 0$, then there exists a representation of N as a space of bounded linear operators.
Every normed semisimple near algebra which contains a non-zero left distributive left module is a semisimple algebra.

Every finite dimensional positive homogeneous normed near algebra with orthogonal idempotent basis is a D-normed near algebra and multiplication is continuous.

The major result of this paper deals with the finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$, the continuous functions on $\mathbb{R}^n$, which contain $\mathcal{B}(\mathbb{R}^n)$. In addition, the finite dimensional sub near algebras of $T_C(\mathbb{R})$ are completely determined.

Every one-dimensional sub near algebra of $T_C(\mathbb{R})$ has the form $N_{(a,b)} = \langle aJ + bK \rangle$, $a^2 + b^2 \neq 0$, where $J(x) = (x + |x|)/2$ and $K(x) = (x - |x|)/2$. $N_{(a,b)} \cong N_{(c,d)}$ iff $ad = bc$ or $ac = bd$.
The near algebra of positive homogeneous Lipschitz functions is the only two-dimensional sub near algebra of $T_C(\mathbb{R})$.
There are no k-dimensional sub near algebras of $T_C(\mathbb{R})$ for $k \geq 3$.
For $n \geq 2$, there are no finite dimensonal sub near algebras of $T_C(\mathbb{R}^n)$ which properly contain $\mathcal{B}(\mathbb{R}^n)$, the linear operators on $\mathbb{R}^n$.

INTRODUCTION

The study of near algebras and normed near algebras is motivated by the desire to generalize the results of normed algebras and the possible applications to non-linear operators. For example, quantum mechanical models have been considered in which the operators form only a near algebra. Although the discussion of stronger multiplicative conditions on the norm to insure continuity of multiplication is central to this paper, the principle result is Theorem 4.7 which is found in Chapter IV and does not require a norm. Theorem 4.7 resulted from an attempt to construct near algebras of continuous functions on $\mathbb{R}$. From these considerations it was possible to completely determine all the finite dimensional near algebras of continuous functions on $\mathbb{R}$ and, in Theorem 4.7, to prove that there is no finite dimensional sub near algebra to $T_C(\mathbb{R}^n)$ which properly contains the algebra of bounded linear operators on $\mathbb{R}^n$ for $n \geq 2$.

A near ring is an algebraic system with two binary operations satisfying all of the axioms for a ring, except possibly one distributive law; and a near field is a near ring in which every nonzero element has a multiplicative inverse. Near fields are useful in the study of certain non Desarguesian planes [18, 25] while near rings appear to have

Desarguesian planes [18, 25] while near rings appear to have

application to the study of nonlinear operators and in characterizing endomorphisms of a group. Finite near fields were first considered by Dickson [15]. Zassenhaus [33] determined all finite near fields. It was Kalscheuer [19] who proved that the twisted quaternions are the only finite dimensional real near fields (Fastkörper) with continuous multiplication. More recently Blackett [9] considered a class of simple and semisimple near rings and Deskins [14] studied near ring radicals. In [2, 3, 4] Beidleman studied near ring modules and the ideal structure of near rings and in [2] he organized many of the results on near rings.

A near algebra is a near ring which admits a field as a right operator domain. Brown [10] studied the structure of certain classes of near algebras. He investigated the multiplicative semigroup structure of a near algebra and some necessary and sufficient conditions for the admissibility of division in a near algebra. He discussed the concepts of a distributor $[a,b,c] = a(b + c) - ab - ac$ and of a distributor chain. Semisimple near algebras are also introduced and a topological near algebra is defined as a near algebra whose linear space structure is a Banach space. The principle result arising from the topological property is that any semisimple topological near algebra such that right multiplication is differentiable at the origin is a semisimple algebra.

Yamamuro [29, 30, 31, 31] has written several papers on near algebras of bounded functions (f maps bounded sets to bounded sets) on a Banach space. The near algebra has a metric induced by the seminorms of the bounded functions. His results are, for the most part, generalizations of the results on two sided ideals studied by Calkin [12]. For example, he proved in [32] that the closed ideal C is minimal amongest all closed ideals which contain all linear mappings of finite rank. C is the set of all compact and continuous functions on the Banach space.

Neither Brown nor Yamamuro define a topological near algebra in the usual sense. Although Brown used the left or right continuity of multiplication, he did not incorporate the continuity of multiplication in his definition of a topological near algebra. Yamamuro made no explicit reference to the continuity of the operations at all. As mentioned earlier, however, Kalscheuer [19] did consider topological near algebras. If the scalar field allowed a linear topology for the linear space structure, then the division near algebra was called continuous (stelig Fastkörper) whenever multiplication satisfied the following limit property, provided the limit existed, $\lim a_i b_i = \lim a_i \lim b_i$. The previously mentioned example of "twisted quaternions" consists of the vector space of quaternions over $\mathbb{R}$ with the usual norm topology with a product defined by the formula $a * b = b * a = 0$ for $a = 0$ and $a * b = a \cdot \delta_y(a) \cdot b \cdot (\delta_y(a))^{-1}$ for $a \neq 0$ where $\delta_y(a) = \cos(\frac{1}{y}\log Na)) + i \sin(\frac{1}{y}(\log Na))$ for $0 \neq y \in \mathbb{R}$. Here $\cdot$ denotes the usual quaternion multiplication, and Na denotes the norm of a.

More recently Beidleman and Cox [5] defined a topological near ring as a near ring with Hausdorff topology such that addition and multiplication are coordinate-wise continuous. The authors noted that this definition is weaker than that of Kaplansky [20] who insisted that addition and multiplication be continuous on the product space. The results of Beidleman and Cox are concerned with the closure of ideals and radicals and tend to generalize the results of Kaplansky.

This brings us to the present paper. We wish to develop the theory of a normed near algebra. Moving in the direction of Beidleman and Cox, we search for conditions on the norm which will insure at least a topological near algebra in the norm topology. Although the usual multiplicative property for a normed algebra, $\|xy\| \leq \|x\| \, \|y\|$, is not sufficient to insure the continuity of left multiplication, we define such a near algebra to be a <u>normed near algebra</u> and prove right multiplication is continuous as is scalar multiplication and addition. If left multiplication is continuous, then multiplication is continuous on the product space.

To insure continuity in both variables we define a <u>D-norm</u>. A norm has the D-norm condition if and only if $\|xy\| \leq \|x\| \, \|y\|$ and, for all b, x, and y, there exists $K_b \geq 0$ such that $\|bx - by\| \leq K_b \|x - y\|$. The D-norm condition is a left distributive property of the norm and is a special case of the normed near algebra condition.

Chapter I presents the basic definitions and properties of a near algebra. We also investigate several difficulties that the missing distributive property creates for near algebra arithmetic.

In Chapter II we begin the study of normed near algebras. In section one we study the norm and D-norm conditions and various continuity properties of multiplication that result. We also show that the adjoining of an identity is not a straightforward generalization of the normed algebra case. The completion of a normed near algebra requires special conditions on the norm.

Section two is devoted to representation theory. The left regular representation is generalized and the near algebra of Lipschitz functions generalizes the space of bounded linear operators. The distributive conditions of Brown [11] are used to strengthen the representations. We prove an important theorem similar to that of Brown [10] stated earlier. Every normed semisimple near algebra which contains a nonzero left distributive left module is a semisimple algebra.

In section three we investigate finite dimensional normed near algebras. Positive homogeneity and an orthogonal idempotent basis insure continuity of multiplication.

Chapter III is devoted to a study of the near algebras of Lipschitz and locally Lipschitz functions and is primarily concerned with the development of these special examples. The locally Lipschitz functions are a special case of the bounded functions studied by Yamamuro [32]. The principle result of this chapter is that all strongly D-normed near algebras are near algebras with continuous inverse.

Chapter IV, one of the most important chapters, does not rely heavily on the norm properties. We determine all finite dimensional near algebras of continuous functions on $\mathbb{R}$ and via representation theory all one dimensional near algebras with continuous multiplication. Extending these ideas to $\mathbb{R}^n$ we prove the main result of this paper: There are no finite dimensional near algebras of continuous functions on $\mathbb{R}^n$ which properly contain the bounded linear operators.

Finally we represent every finite dimensional near algebra with a special annihilator condition on the basis set as a function near algebra in $T(\mathbb{R}^n)$. This generalizes the matrix representation found in [1].

CHAPTER I

BASIC CONCEPTS

This chapter presents the basic definitions and establishes some notation and a few introductory results. Not all of the results are new but they are included for completeness and to demonstrate the oddities of near algebra arithmetic. Standard terms in algebra and ring theory, which are not defined here, may be found in MacLane Birkhoff [24], terms in functional analysis can be found in Wilansky [28] and Naimark [26] and topological terms can be found in Kelley [21]. The definitions and notations are chosen to be generalizations of ring theory terms.

1.1 Definition. A (right) near algebra over a field F is a linear space N over F on which a multiplication is defined such that

i) N forms a semigroup under multiplication,

ii) Multiplication is right distributive over addition: For all a, b, and c belonging to N, $(a + b)c = ac + bc$

iii) $(ta)b = t(ab)$ for all $a, b \in N$ and $t \in F$.

This is the definition presented by Brown [10] except that we have assumed right distributivity instead of left distributivity. The linear space structure of N will be denoted by N^+ and the additive identity of N^+ by 0. If N has a multiplicative identity it will be denoted by e. A near algebra satisfies all the algebra axioms with the possbile exception of the left distributive law and the scalar property $a(tb) = t(ab)$.

An immediate example of a near algebra which is not an algebra is the set of all operators on a linear space V into itself over the field F. For future reference, let $T(V) = \{ f \mid f: V \rightarrow V, f \text{ is a function from } V \text{ into } V \}$ denote this near algebra with the pointwise operations of sum and scalar multiplication, $(f + g)(x) = f(x) + g(x)$ and $(tf)(x) = t(f(x))$, and function composition as the multiplication, $(f \circ g)(x) = f(g(x))$. Henceforth, when it is convenient and no confusion arises, juxtaposition will be used for function composition.

<u>1.2 Definition.</u> A subset $N_1 \subseteq N$ is called a <u>sub near algebra</u> of the near algebra N if and only if N_1 is a near algebra when the operations of N are restricted to N_1.

As in the case of an algebra, we have the following lemma.

<u>1.1 Lemma.</u> $N_1 \subseteq N$ is a sub near algebra of N if $sa + tb \in N_1$ and $ab \in N_1$ whenever $a, b \in N_1$ and $s, t \in F$.

For example, $T_0(V) = \{ f \mid f \in T(V) \text{ and } f(0) = 0 \}$ is a sub near algebra of T(V). If V is a linear topological space, then $T_C(V) = \{ f \mid f \in T_0(V) \text{ and } f \text{ is continuous on } V \}$ is an important sub near algebra of T(V).

Some special sub near algebras of $T(\mathbb{R})$ will be considered to demonstrate the oddities of near algebra arithmetic. These oddities often cause a problem in generalizing some of the algebra properties. First consider the additive identity, which in an algebra has the property that $0a = a0 = 0$. All near algebras have the property that $0a = 0$; however, if we choose the constant function $f(x) = c \neq 0$ as an element in $T(\mathbb{R})$, then $f \cdot 0 = f \neq 0$. We make the following definition which was presented for near rings by Berman and Silverman [6].

<u>1.3 Definition.</u> A near algebra N is a <u>near-c-algebra</u> if N has the property that $a0 = 0$ for every $a \in N$. $T_o(\mathbb{R})$ is an example of a near-c-algebra.

Although the multiplicative identity behaves properly, the additive inverse of the identity does not always behave as expected. Consider $T(\mathbb{R})$ as a near algebra with multiplicative identity I; then, $f(-I) \neq (-I)f$ for all $f \in T(\mathbb{R})$. For example, let $f(x) = x^2$, then $f(-I) = f \neq -f$. This is in keeping with the fact that, in general, $f(tg) \neq t(fg)$ for $t \in \mathbb{R}$. The set $O(\mathbb{R}) = \{ f \mid f \in T(\mathbb{R}) \text{ and } f(-x) = -f(x)\}$ of odd functions on $\mathbb{R}$ forms a sub near algebra of $T(\mathbb{R})$ such that $f(-I) = (-I)f$ and yet $f(tg) \neq t(fg)$ for all $t \in \mathbb{R}$.

As a further consideration, let $\mathbb{R}^2$ be considered as a real linear space and let $H(\mathbb{R}^2) = \{ f \mid f \in T(\mathbb{R}^2) \text{ and } f(tx) = tf(x) \text{ for all } t \in \mathbb{R}\}$ be the set of homogeneous functions on $\mathbb{R}^2$. $H(\mathbb{R}^2)$ is a near algebra which is not an algebra such that $f(tg) = t(fg)$ for $t \in \mathbb{R}$. Clearly $f(-I) = (-I)f$. This suggests other examples which satisfy certain weaker scalar multiplication properties. For instance, $H_p(\mathbb{R}^2) = \{ f \mid f \in T(\mathbb{R}^2) \text{ and } f(tx) = t(f(x)) \text{ for all } t \in \mathbb{R},\ t \geq 0\}$ is the near algebra of positive homogeneous functions on $\mathbb{R}^2$; hence, $f(tg) = t(fg)$ whenever $t \geq 0$. Finally, for the near algebra $H_a(\mathbb{R}^2) = \{ f \mid f \in T(\mathbb{R}^2)$ and $f(tx) = |t| f(x)$ for all $t \in \mathbb{R} \}$, the property $f(tg) = |t| (fg)$ holds for all $t \in \mathbb{R}$.

Commutative near algebras are obviously algebras. In general, we define the concept of a center and generalize the examples of Berman and Silverman [6].

<u>1.4 Definition.</u> Let N be a near algebra. The <u>center</u> of N is the set $N_c = \{ a \mid a \in N$ and $ax = xa$ for all $x \in A\}$. For an algebra the center is a nonempty commutative subalgebra; however, this is not always the case for a near algebra. The sub near algebra $K(\mathbb{R})$ of constant functions in $T(\mathbb{R})$ has no center. The center of a near-c-algebra is nonempty; however, it is not necessarily a sub near algebra. For example, let $N = \{ f \mid f \in T_0(\mathbb{R}^2)\ f : \mathbb{R}^2 \to \mathbb{R} \times \{0\}$ and $f(B) = (0,0)\}$ where $B = \{ (x,y) \mid \sqrt{x^2 + y^2} \leq 1 \}$. N is easily shown to be a sub near-c-algebra of $T_0(\mathbb{R}^2)$. Define $f \in T_0(\mathbb{R}^2)$ by the formula $f(x,y) = \begin{cases} (0,0), & \text{if } (x,y) \in B \cup \{ (x,0) \mid x \in \mathbb{R}\} \\ (\frac{1}{x},0), & \text{if } (x,y) \notin B \cup \{ (x,0) \mid x \in \mathbb{R}\} \end{cases}$. Then, $f \in N$ and $fg = gf = 0$ for all $g \in N$. However, for $g \in N$ defined by the formula $g(x,y) = \begin{cases} (0,0), & \text{if } (x,y) \in B \\ (x,0), & \text{if } (x,y) \notin B \end{cases}$, $(2f)g = 2(fg) = 0$ while $g(2f)(1,0) = (2,0)$. Thus, $f \in N_c$ but $2f \notin N_c$.

Although the center, N_c, of a near-c-algebra N may not be a sub near algebra, the set of elements $N_{N_c} = \{ x \mid x \in N$ and $xa = ax$ for all $a \in N_c\}$, which commute with the center, does form a sub near algebra. Also, if N_c is a sub near algebra, it must be a commutative sub algebra.

Let us consider, more generally, the existence of commutative sub near algebras and the generation of sub near algebras. It is well known that every element x of an algebra A generates a commutative subalgebra A(x) which contains x. In fact, every subalgebra and hence every element is contained in a maximal commutative subalgebra. This is not the case for a near algebra.

Let E be a normed linear space and let Lip(E) $= \{ f \mid f \in T_0(E) \text{ and } \|f(x) - f(y)\| \leq K\|x - y\| \text{ for all } x, y \in E \text{ and for some } K \geq 0 \}$ be the space of Lipschitz functions on E. This space is a near algebra and will be developed more fully in a later chapter. Lip(E) contains the non-commutative algebra of bounded linear operators, $\mathcal{B}(E)$, as a subalgebra; hence, the center of $\mathcal{B}(E)$ or any algebra generated by a single element of $\mathcal{B}(E)$ is a commutative sub near algebra of Lip(E). On the other hand, let $E = \mathbb{R}$ and let

$$f(x) = \begin{cases} x^2, & \text{for } 0 \leq x \leq 1 \\ 1, & \text{for } x > 1 \\ 0, & \text{for } x < 0 \end{cases}$$

. By checking the various cases, it can be shown that f belongs to Lip($\mathbb{R}$). For example, if $0 \leq x, y \leq 1$, then $|f(x) - f(y)| = |x^2 - y^2|$ $= |x + y| \, |x - y| \leq 2\,|x - y|$. The function f does not belong to a commutative sub near algebra or even a sub algebra of Lip($\mathbb{R}$), since

$$f(2f)(x) = \begin{cases} 4x^4, & \text{for } 0 \leq x \leq \sqrt{2}/2 \\ 1, & \text{for } x > \sqrt{2}/2 \\ 0, & \text{for } x < 0 \end{cases} \neq$$

$$2ff(x) = \begin{cases} 2x^4, & \text{for } 0 \leq x \leq 1 \\ 2, & \text{for } x > 1 \\ 0, & \text{for } x < 0 \end{cases}$$. Therefore, some elements of

Lip($\mathbb{R}$) generate commutative sub near algebras while others do not. If a commutative sub near algebra exists, then the maximality condition generalizes in the following theorem. The proof is analogous to that of an algebra and can be found in [26, p. 155].

1.1 Theorem. Let N be a near algebra. If M is a commutative sub near algebra, then M is contained in a maximal commutative sub near algebra.

Also, if commutativity is not a concern, the following statement which is valid for an algebra is valid for a near algebra and the proof is similar.

1.1 Proposition. The nonvoid intersection of any collection of sub near algebras of a near algebra N is again a sub near algebra of N. In particular, for $S \subseteq N$, $S \neq \emptyset$, $(S) = \bigcap \{M \mid S \subseteq M$ and M is a sub near algebra of $N\}$ is the minimal sub near algebra containing S.

If A is an algebra and S is a nonempty subset of A, then the minimal sub algebra of A containing S is characterized as the set of all finite sums of the form $\Sigma t_k a_k$ where a_k is the product of a finite number of elements in S. Let us denote this characterization by F(S). Although it is easily shown that $F(S) \subseteq (S)$ for a near algebra, the following example shows that the above characterization, F(S) = (S), is not valid in general for a near algebra.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then f belongs to $T(\mathbb{R})$, the near algebra of all functions on the space of real numbers where composition of functions is the multiplication operation. It can be easily shown by induction that $f^n(x) = x^{2^n}$, $n = 1,2,\cdots$. Thus, $F(f) = \left\{g \middle| g : \mathbb{R} \rightarrow \mathbb{R} \text{ and } g(x) = \sum_{k=1}^{n} a_k x^{2^k} \text{ for all } n = 1,2,\cdots\right\}$. However, using $f^n(x) = x^{2^n}$, induction, and appropriate sums and products of f, it can be shown that any sub near algebra, M, of $T(\mathbb{R})$ which contains f must also contain $g_n(x) = x^{2n}$ for all $n = 1,2,\cdots$. The set $H = \left\{h \middle| h : \mathbb{R} \rightarrow \mathbb{R},\ h(x) = \sum_{k=1}^{n} a_k x^{2k} \text{ for } n = 1,2,\cdots, a_k \in \mathbb{R}\right\}$ which also contains f is a near algebra itself; thus, $H = (f)$. Therefore, as asserted, $F(f) \neq (f)$ since $g(x) = x^6 \in (f)$ while $g \notin F(f)$.

In some special cases, as the next example illustrates, $(S) = F(S)$ and, yet, (S) is still not an algebra.

Let $N = T(V)$, V a real linear space, and let c be a fixed element of V. Define $f : V \rightarrow V$ by $f(x) = c$ for all $x \in V$. Let $S = \{f\}$; then, as stated earlier $F(S) \subseteq (S)$. Conversely, for any natural number k, $f^k = f$; thus, $\sum t_k f^k = tf$. Therefore, $F(S) = \{tf \mid t \in \mathbb{R}\}$. Clearly $F(S)$ is a linear subspace of N and $(tf)(sf) = tf \in F(S)$. Therefore, by Lemma 1.1, $F(S)$ is a sub near algebra of $T(V)$ which contains f; thus, $(S) \subseteq F(S)$.

The above discussion serves to demonstrate the importance of both distributive properties. The remainder of this chapter is devoted to some basic definitions and results and cites references where further studies have been made.

The concept of a multiplicative inverse and its uniqueness agree with that of an algebra. However, the terminology associated with near ring ideals is not as completely standardized. For example, the definitions and terms of Berman and Silverman [6] differ slightly from that of Brown [10] or Beidleman [2]. If only near-c-rings are allowed, the definitions are equivalent. We state the following definitions from [6] in the context of a near algebra for reference purposes.

1.5 Definition. A set L (R) of elements of the near algebra N is called a left (right) ideal or left (right) module of N if

(1) L (R) is a linear subspace of the linear space N^+ and

(2) $x \in L$ $(x \in R)$ and $a \in N$ implies $ax \in L$ $(xa \in R)$.

1.6 Definition. A set I of elements of the near algebra N is called an ideal of N if

(1) I is a linear subspace of the linear space N^+ and

(2) $(a + x)(b + y) - ab \in I$ for all $x, y \in I$ and $a, b \in N$.

Given the above definition of an ideal we can characterize the ideals as kernels of near algebra homomorphisms in a manner completely analogous to that of ring theory. If f is a mapping from the near algebra N to the near algebra M over the field F, then the usual definitions of homomorphism and isomorphism apply for the near algebra systems. Similarly, if I is an ideal in the near algebra N, then N/I, the set of additive cosets a + I, determines a near algebra with respect to the usual induced operations of addition, scalar multiplication, and product of cosets.

1.2 Theorem. Under a near algebra homomorphism of the near algebra N into the near algebra M, the complete inverse image K of the zero, 0, in M is an ideal in the near algebra N. i.e. the kernel of the near algebra homomorphism is an ideal in N.

1.3 Theorem. Every ideal I of the near algebra N induces a homomorphism of N onto the quotient space N/I of additive cosets and I is the kernel of this homomorphism. Hence, every ideal is the kernel of a homomorphism and conversely.

We close this chapter with a few results from near ring theory which extend naturally to near algebras. The theory of simple and semisimple near rings has been explored by Blackett [9] and Beidleman [2]. A sampling of the results is listed below.

1.7 Definition. A near algebra N is said to be semisimple if N satisfies the descending chain condition on left modules and has no nonzero nilpotent left modules. N is said to be simple if N is a semisimple, nonzero near algebra with no proper ideals.

1.8 Definition. The J-radical of a near algebra N, denoted by

J(N), is defined to be the intersection of all left annihilator ideals $L(R) = \{x \mid x \in N, xR = 0\}$ where R ranges over the minimal left modules of N.

Betsch [8] has shown the following which carries over to near algebras.

1.4 Theorem. If N is a near algebra satisfying the descending chain condition on left modules, N is a semisimple near algebra if and only if $J(N) = 0$ and that $N/J(N)$ is semisimple.

Blackett [9] has proved and the proof carries over directly to near algebras that:

1.5 Theorem. A semisimple near algebra can be decomposed into a ring theoretic sum of simple near algebras.

Much of the emphasis in current literature on near ring systems deals with the ideal structure and characterization of a radical. With the notable exceptions of Kalscheuer [19] and more recently Brown [10] and Beidleman and Cox [5] there seems to be little study of near algebras or the topological properties of near algebras. We now address ourselves to this problem in the study of normed near algebras.

CHAPTER II

NORMED NEAR ALGEBRAS

The Norm and Related Properties

In this section we define and discuss some properties of a normed near algebra. Near algebras with a topology determined by a norm appear in the literature, but nowhere have we found the concept of a normed near algebra discussed in detail. For example, Yamamuro [32] considers a near algebra of bounded functions defined on a Banach space which has a uniform topology determined by seminorms. However, Yamamuro is more interested in the ideal structure of these special function near algebras than the properties of the seminorms. Brown [10] also considers a near algebra such that the underlying linear space is a Banach space; he, however, does not explore in any detail the abstract properties of a normed near algebra. Finally, Beidleman and Cox [5] have considered topological near rings.

We investigate, in the first section, various multiplicative conditions for the norm along with their effect on the continuity of multiplication. The first condition imposed is the usual multiplicative property for a normed algebra. A near algebra with norm such that $\|xy\| \leq \|x\| \|y\|$ is called a normed near algebra and many examples of such a near algebra exist. However, although every normed near algebra is a near-c-algebra and right multiplication is continuous, left multiplication is not continuous, a property that normed algebras enjoy. Therefore, we strengthen the usual norm property.

We define a D-normed algebra N as a normed near algebra such that for every $b \in N$ there exists $K_b \geq 0$ and $\|bx - by\| \leq K_b \|x - y\|$ for all $x, y \in N$. The D-norm condition is a left distributive property for the norm. The D-norm condition is sufficient to insure continuity of left multiplication. If $K_b = \|b\|$ for all $b \in N$, then the D-norm is called a strong D-norm. A very important example of a strong D-normed near algebra is the set of positive homogeneous Lipschitz functions on $\mathbb{R}$.

We define a Banach near algebra and a D-Banach near algebra and show that the quotient near algebra of additive cosets modulo a closed ideal is a Banach (D-Banach) near algebra. We also show that the kernel of a continuous homomorphism is a closed ideal and every closed ideal induces a continuous homomorphism.

Finally, although the usual argument for adjcining an identity does not generalize, we prove that every normed near algebra can be embedded in a near algebra with identity. The completion of a normed near algebra is also a standard result for normed algebras. We have been able to prove that a strong D-normed near algebra can be completed; the general normed case remains an open question.

We begin by considering the definition of a normed near algebra. One of the defining properties of a normed algebra is the multiplicative property of the norm, $\| xy \| \leq \| x \| \| y \|$. It is readily shown that from this property multiplication is a continuous function in both variables simultaneously. This condition does not force such a result in the case of a near algebra as will be demonstrated presently. We will begin with this condition, however, and then strengthen it to get the desired continuity property.

2.1 Definition. A nonempty set N is called a (right) normed near algebra if

(1) N is a (right) near algebra over the field $\mathbb{R}$ or $\mathbb{C}$

(2) There exists a norm on N, denoted by $\| \cdot \|$, such that $(N^+, \| \cdot \|)$ is a normed linear space, and

(3) $\| x \cdot y \| \leq \| x \| \| y \|$ for each $x, y \in N$.

Unless otherwise specified we will be considering real normed near algebras. For example, every normed algebra is a normed near algebra.

The following are some examples of normed near algebras which are not algebras.

(1) Let E be a normed linear space and let $T_B(E) = \{ f \mid f \in T(E)$ and $\| f(x) \| \leq M \| x \|$ for all $x \in E$ and for some $M \geq 0 \}$. $T_B(E)$ is easily shown to be a sub near algebra of T(E) and we can define a norm on $T_B(E)$ by $\| f \| = \sup \left\{ \frac{\| f(x) \|}{\| x \|} \,\middle|\, x \in E,\ x \neq 0 \right\}$. This is the definition of the usual sup norm defined on the bounded linear operators extended to all bounded operators on E. Consequently, this

defines a norm with the property that $\|fg\| \leq \|f\|\,\|g\|$.

(2) Let $\mathcal{D} = \{f \mid f \in T_0(\mathbb{R}),\ f'(x)$ exists and $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, $M \geq 0\}$. $\mathcal{D}$ is a sub near algebra of $T(\mathbb{R})$ and with the norm defined by the following: $\|f\| = \sup\{|f'(x)| \mid x \in \mathbb{R}\}$, $\mathcal{D}$ becomes a normed near algebra with $\|fg\| \leq \|f\|\,\|g\|$.

(3) Let E be a real normed linear space and let $\mathrm{Lip}(E) = \{f \mid f \in T_0(E)$ and $\|f(x) - f(y)\| \leq K_f \|x - y\|$ for all $x, y \in E$ and for some $K_f \in \mathbb{R}$, $K_f \geq 0\}$. Define $\|f\| = \inf\{K_f \mid \|f(x) - f(y)\| \leq K_f \|x - y\|$ for all $x, y \in E\}$. We will call this the <u>induced Lipschitz norm.</u> This example is central to later discussions; hence, we will develop it in more detail than the other examples. Let t and s be arbitrary elements of $\mathbb{R}$ and let $f, g \in \mathrm{Lip}(E)$. $\|(tf + sg)(x) - (tf + sg)(y)\| = \|t(f(x) - f(y)) + s(g(x) - g(y))\| \leq |t|\ K_f \|x - y\| + |s|\ K_g \|x - y\| = (|t|\ K_f + |s| K_g) \|x - y\|$ for all $x, y \in E$. Therefore, $tf + sg \in \mathrm{Lip}(E)$. Also $\|(fg)(x) - (fg)(y)\| \leq K_f \|g(x) - g(y)\| \leq K_f K_g \|x - y\|$ for all $x, y \in E$; hence, $fg \in \mathrm{Lip}(E)$. Thus, by Lemma 1.1, Lip(E) is a near algebra.

We now show that $\|\cdot\|$ is a norm and $\|fg\| \leq \|f\|\,\|g\|$. Since all $K_f \geq 0$ and there exists at least one K_f for each $f \in \mathrm{Lip}(E)$, $\|f\| \geq 0$ and $\|f\|$ is well defined. If $\|f(x) - f(y)\| > \|f\|\,\|x - y\|$ for some $x \neq y$, then $\|f\|\,\|x - y\| < K_f \|x - y\|$ for all K_f. Thus, $\|f\| < K_f$ and hence, for some $\varepsilon > 0$, $\|f\| + \varepsilon < K_f$ for all K_f satisfying the required Lipschitz condition. This contradicts the infinum property of $\|f\|$. Therefore $\|f(x) - f(y)\| \leq \|f\| \|x - y\|$ for all x, y $\in$ E.

If $\|f\| = 0$, then $\|f(x) - f(0)\| = \|f(x)\| \leq \|f\| \|x\| = 0$ for all x $\in$ E. Therefore, f = 0. Conversly, if f = 0, then choose 0 as one of the K_f's and hence $\|f\| = 0$. Therefore, $\|f\| = 0$ if and only if f = 0.

For $t \in \mathbb{R}$, $t \neq 0$, $\|(tf)(x) - (tf)(y)\| \leq \|tf\| \|x - y\|$ for all x, y $\in$ E; thus, $\|f(x) - f(y)\| \leq \frac{\|tf\|}{|t|} \|x - y\|$ and $\|f\| \leq \frac{\|tf\|}{|t|}$, $t \neq 0$. Also, for all x, y $\in$ E, $\|f(x) - f(y)\| \leq \|f\| \|x - y\|$; hence, $\|(tf)(x) - (tf)(y)\| = |t| \|f(x) - f(y)\| \leq |t| \|f\| \|x - y\|$ for all x, y $\in$ E. Thus, $\|tf\| \leq |t| \|f\|$, $t \neq 0$. Combining these two results we have $\|tf\| = |t| \|f\|$, $t \neq 0$. For $t = 0$, $\|0f\| = 0 \|f\|$; therefore, $\|tf\| = |t| \|f\|$ for all $t \in \mathbb{R}$.

Finally, for f, g $\in$ Lip(E), $\|(f + g)(x) - (f + g)(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| \leq \|f\| \|x - y\| + \|g\| \|x - y\| = (\|f\| + \|g\|) \|x - y\|$. Thus, $\|f + g\| \leq \|f\| + \|g\|$. Therefore, $\|f\|$ is a norm for the linear space Lip $(E)^+$.

To show that the multiplicative property holds, let f, g $\in$ Lip(E). $\|(fg)(x) - (fg)(y)\| = \|f(g(x)) - f(g(y))\| \leq \|f\| \|g(x) - g(y)\| \leq \|f\| \|g\| \|x - y\|$; hence $\|fg\| \leq \|f\| \|g\|$. Therefore (Lip(E), $\|\cdot\|$) is a normed near algebra.

The following proposition shows that we can assume the additive identity behaves properly with respect to multiplication in the discussion of normed near algebras.

<u>2.1 Proposition.</u> A normed near algebra is a near-c-algebra.

Proof: $0 \leq \|a0\| \leq \|a\|\,\|0\| = 0$; thus, $\|a0\| = 0$ and $a \cdot 0 = 0$. ▯

We may also assume that the norm of the identity, if it exists, is one.

<u>2.1 Theorem.</u> If N is a normed near algebra and e is the multiplicative identity, then there is an equivalent norm on N such that the norm of e is one.

Proof: For any norm on N, $\|e\| = \|ee\| \leq \|e\|\,\|e\|$; thus $1 \leq \|e\|$.

Let $b \in N$ be arbitrary but fixed and define $R_b: N \to N$ by $R_b(x) = xb$. That is, R_b is the right multiplication operator. Although not all the algebra axioms hold, we can still show that R_b is a bounded linear operator on N^+. $R_b(x + y) = (x + y)b = xb + yb = R_b(x) + R_b(y)$ and $R_b(tx) = (tx)b = t(xb) = t(R_b(x))$. $\|R_b(x)\| = \|xb\| \leq \|b\|\,\|x\|$. Since $R_b \in \mathcal{B}(N^+)$ for all $b \in N$, let $|||b||| = \|R_b\|$. The remainder of the proof is analogous to the normed algebra proof. See [22]. ▯

For convenience in terminology we make the following definition.

<u>2.2 Definition.</u> A near algebra N with a linear Hausdorff topology is defined to be:

(1) A <u>right continuous near algebra</u> if the multiplication is continuous with respect to the first factor. That is, all right multiplication operators are continuous functions.

(2) A <u>left continuous near algebra</u> if the multiplication is continuous with respect to the second factor for an arbitrary but fixed first factor. That is, all left multiplication operators are continuous functions.

(3) A <u>continuous near algebra</u> if the multiplication is continuous with respect to both variables simultaneously. That is, multiplication is continuous as a function on the product space.

As an example of a left and a right continuous near algebra which is not necessarily a normed near algebra, we consider the example given in [32] by Yamamuro. Let B(E) $= \{f \mid f \in T_C(E)$ and f maps bounded sets to bounded sets$\}$ where E is a Banach space. Yamamuro defines a collection of seminorms on B(E) by $\|f\|_n = \sup \{ \|f(x)\| \mid \|x\| \leq n \}$ and constructs a metric $d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g\|_k}{1 + \|f - g\|_k}$. One can show that multiplication is continuous in both variables separately for $B(\mathbb{R})$.

The collection of seminorms is also a sufficient collection which determines a locally convex linear topology. B(E) is right continuous and $B(\mathbb{R})$ is both left and right continuous.

Finally, let N be any locally convex linear topological space and define a product on N by the formula xy = yx = 0 if x = 0 and xy = x if x $\neq$ 0. N becomes a near algebra and the right multiplication operator is the identity; hence, it is continuous.

2.2 Proposition. Every normed near algebra is a right continuous near algebra.

Proof: The proof of Theorem 2.1 shows that an arbitrary right multiplication operator is a bounded linear operator on any normed near algebra. In Naimark [26 P. 74] it is shown that every bounded linear operator is continuous. ▯

2.3 Proposition. Every normed near algebra N is a continuous near algebra if and only if N is left continuous.

Proof: Assume multiplication is left continuous and let x, $y \in N$. If $\{x_n\}$ and $\{y_n\}$ are sequences in N such that $x_n \to x$ and $y_n \to y$, then $\| x_n y_n - xy \| = \| x_n y_n - xy_n + xy_n - xy \| \leq \| x_n y_n - xy_n \| + \| xy_n - xy \|$. Although the left distributive property does not hold, $\| xy_n - xy \|$ can be made arbitrarily small for sufficiently large n by the continuity of left multiplication. Thus, for $\varepsilon > 0$, $\| x_n y_n - xy \| \leq \| x_n - x \| \| y_n \| + \| xy_n - xy \| < \varepsilon$ for sufficently large n and N is a continuous near algebra.

Suppose N is a continuous near algebra and let b be a fixed element of N. For arbitrary $x \in N$ let $\{x_n\}$ be a sequence which converges to x. Let $b_n = b$ be the constant b sequence. Since N is a continuous near algebra, $b_n x_n = bx_n \to bx$ and multiplication is left continuous. Therefore, N is a left continuous near algebra. ▯

The following examples show that right continuity alone is not a sufficient condition for a normed near algebra to be a continuous near algebra, while the previous proposition shows that left continuity is sufficient.

In example (1) above, we showed that $T_B(E)$ was a near algebra with a norm such that $\|fg\| \leq \|f\|\,\|g\|$. Let $E = \mathbb{R}$ and let $f(x) = \begin{cases} 1, & |x| \geq 1 \\ 0, & |x| < 1 \end{cases}$ and let $g(x) = \begin{cases} 1, & |x| > 1 \\ 0, & |x| \leq 1 \end{cases}$. A quick calculation shows that $|f(x)| \leq |x|$ and $|g(x)| \leq |x|$ for all $x \in \mathbb{R}$; hence, $f, g \in T_B(\mathbb{R})$. We now define $g_n = \frac{(n-1)}{n} g$ for all $n \in \mathbb{N}$. Clearly, $g_n \in T_B(\mathbb{R})$ for all n and g_n converges to g in the norm. Now, $(fg_n)(x) = f\left(\left(\frac{n-1}{n}\right)g(x)\right)$ $= \begin{cases} f\left(\frac{n-1}{n}\right), & |x| > 1 \\ f(0), & |x| \leq 1 \end{cases} = 0$. Therefore, $fg_n = 0$ for all n; hence, $fg_n \rightarrow 0$. However, $(fg)(x) = \begin{cases} f(1), & |x| > 1 \\ f(0), & |x| \leq 1 \end{cases} = \begin{cases} 1 & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$ $= g(x)$; hence, $fg = g \neq 0$. Thus, fg_n does not converge to fg and left multiplication is not continuous despite the fact that $\|fg\| \leq \|f\|\,\|g\|$.

As another example, consider the space of Lipschitz functions on a normed linear space E. In example (3) above, we proved that Lip(E) is a normed near algebra; we now wish to prove that Lip(E) is not, in general, a left continuous near algebra. The first proof of this fact was to construct a rather complicated counter-example in the space Lip($\mathbb{R}$) consisting of the function

$g(x) = \begin{cases} -x-\pi, & x \leq -\pi \\ \sin x, & -\pi \leq x \leq 2\pi \\ x-2\pi, & 2\pi \leq x \end{cases}$ and the sequence of functions

$$g_n(x) = \begin{cases} -x-\left(\frac{n}{n-1}\right)\pi, & x \leq \left(\frac{-n}{n-1}\right)\pi \\ \frac{n}{n-1}\sin\frac{n-1}{n}x, & \left(\frac{-n}{n-1}\right)\pi \leq x \leq \left(\frac{n}{n-1}\right)2\pi \\ x-\left(\frac{n}{n-1}\right)2\pi, & \left(\frac{n}{n-1}\right)2\pi \leq x \end{cases}$$

such that g_n converges to g in the norm topology of Lip($\mathbb{R}$). For the fixed function $f(x) = \begin{cases} ax, & x \geq 0 \\ bx, & x < 0 \end{cases}$, $a \neq b$, which belongs to Lip($\mathbb{R}$), we were able to show fg_n does not converge to fg and hence, Lip($\mathbb{R}$) is not left continuous. Later a much less complicated example was discovered.

Let $\mathbb{R}$ be the space of real numbers and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x \text{ for } x \leq 1 \\ 1 \text{ for } x > 1 \end{cases}$. A quick check of several cases shows that $f \in \text{Lip}(\mathbb{R})$. Let $g_n = \left(\frac{n-1}{n}\right)I$ where I is the identity function, then $g_n \in \text{Lip}(\mathbb{R})$ for all $n \in \mathbb{N}$ and g_n converges to I in the Lipschitz norm.

If we assume multiplication is left continuous, then fg_n must converge to $fI = f$. Thus, if $\varepsilon = 1/4$, there exists $n_o \in \mathbb{N}$ such that $|fg_n - f| < 1/4$ whenever $n > n_o$. For all $x, y \in \mathbb{R}$, $|(fg_n - f)(x) - (fg_n - f)(y)| =$

$\left|f\left(\left(\frac{n-1}{n}\right)x\right) - f(x) - f\left(\left(\frac{n-1}{n}\right)y\right) + f(y)\right| \leq \frac{1}{4}|x - y|$. Choose $m > \max(n_o, 4)$ and let $u = \frac{m-1}{m} < 1$ and $v = \frac{m+1}{m} > 1$. For this pair of elements $u, v \in \mathbb{R}$ and $m > \max(n_o, 4)$,

$|(fg_m - f)(u) - (fg_m - f)(v)| =$

$$\left|\frac{m^2 - 2m + 1}{m^2} - \frac{(m-1)}{m} - \frac{m^2 - 1}{m^2} + 1\right| = \left|\frac{2}{m^2} - \frac{1}{m}\right| < \frac{1}{4}|u - v| = \frac{1}{2m}.$$

Therefore, dividing by m, it follows that $\left|\frac{2}{m} - 1\right| < \frac{1}{2}$ which contradicts the choice of $m > 4$. Thus, fg_n does not converge to f and Lip($\mathbb{R}$) is not left continuous.

Although a normed near algebra has been shown, in general, not to be left continuous, it does contain a left continuous near algebra.

2.4 Proposition. Let N be a normed near algebra and for each $b \in N$ let L_b be the left multiplication operator on N with respect to b. Let $N_{LC} = \{ b \mid b \in N \text{ and } L_b \text{ is continuous on } N\}$, then N_{LC} is a closed continuous sub near algebra of N.

Proof: $N_{LC} \neq \emptyset$, since $0 \in N_{LC}$. If $a, b \in N_{LC}$ and t is a scalar, then $a + b$, ta, and $ab \in N_{LC}$ since $L_{a+b} = L_a + L_b$, $L_{ta} = tL_a$, and $L_{ab} = L_aL_b$ are continuous whenever L_a and L_b are continuous. Thus, N_{LC} is a continuous sub near algebra of N.

We now show $N_{LC} = \overline{N}_{LC}$, the closure of N_{LC}. Let $a \in \overline{N}_{LC}$ and let $x \in N$, $x \neq 0$. Choose $\varepsilon > 0$. There exists a sequence $\{a_n\} \subseteq N_{LC}$ such that $a_n \to a$ and, for each n, L_{a_n} is continuous on N. For each $n \in \mathbb{N}$, choose $\delta_n \leq \delta_1$ such that $\| a_n u - a_n x \| < \varepsilon/2$ whenever $\| u - x \| < \delta_n$. Also, choose $m \in \mathbb{N}$ such that $\| a_m - a \| < \varepsilon/2((2\|x\|) + \delta_1)$. Thus,
$\| L_a u - L_a x \| = \| au - a_m u + a_m u - a_m x + a_m x - ax \|$
$\leq \| a_m - a \| \, \| u \| + \| a_m u - a_m x \| + \| a_m - a \| \, \| x \| =$
$(\| a_m - a \|)(\| x \| + \| u \|) + \| a_m u - a_m x \| < \varepsilon$ whenever $\| u - x \| < \delta_m$. Therefore, L_a is continuous and $a \in N_{LC}$. $\square$

As an example, $Lip_{LC}(E)$ is a closed continuous sub near algebra of Lip(E) which contains the bounded linear operators on the normed linear space E.

We now present a sufficient condition for continuous left multiplication.

2.3 Definition. A normed near algebra N is a $\dot{D}$-normed near algebra if and only if for each $b \in N$ there exists $K_b \in \mathbb{R}$,

$K_b \geq 0$, such that for all $x, y \in N$, $\|bx - by\| \leq K_b \|x - y\|$. Clearly, if N is a normed algebra, then N is a D-normed near algebra since $\|bx - by\| = \|b(x - y)\| \leq \|b\| \|x - y\|$.

The following are examples of D-normed near algebras which are not algebras.

(1) Let E be a normed linear space over $\mathbb{R}$ and let f be a Lipschitz functional such that $f(tx) = tf(x)$ for all $t \in \text{Range}(f)$ and $0 \in \text{Range}(f)$. Define a multiplication on E by $xy = f(y)x$ for all $x, y \in E$. Since E is a nomred linear space, we need only to check the multiplicative properties of the norm and semigroup structure. Clearly, multiplication is closed and for $x, y, z \in E$, $(xy)z = f(z)(xy) = f(z)f(y)x$ $=f(f(z)y)x = f(yz)x = x(yz)$. Also, $(x + y)z = xz + yz$, but, in general, $x(y + z) \neq xy + xz$. Thus, E becomes a near algebra. Finally, $\|xy - xz\| = |f(y) - f(z)| \|x\|$ $\leq \|x\| K_f \|y - z\|$ and, since $f(0) = 0$, $\|xy\| \leq K_f \|x\| \|y\|$. Thus, there is an equivalent norm on E such that E is a D-normed near algebra. In particular, let $f(x) = \|x\|$.

(2) Let $N = T_B(\mathbb{R}) \cap \mathfrak{D}(\mathbb{R})$ where $T_B(\mathbb{R})$ and $\mathfrak{D}(\mathbb{R})$ are given as examples after Definition 2.1. Let $\|f\|$ $= \sup \left\{ \frac{|f(x)|}{|x|} \;\middle|\; x \neq 0 \right\}$ be the norm on N and let $\|f\|_\infty$ $= \sup \left\{ |f'(x)| \;\middle|\; x \in \mathbb{R} \right\}$. N is a normed near algebra. Also, by the mean value theorem, $\frac{|fg(x) - fh(x)|}{|x|} = |f'(c)| \frac{|g(x) - h(x)|}{|x|}$ $\leq \|f\|_\infty \frac{|g(x) - h(x)|}{|x|} \leq \|f\|_\infty \|g - h\|$; thus $\|fg - fh\| \leq$ $\|f\|_\infty \|g - h\|$.

(3) The space of positive homogeneous functions on

on the real line, $H_p(\mathbb{R})$, with the norm of $Lip(\mathbb{R})$ is a D-normed near algebra. Since the example is of importance later in the theory, we will develop some of its properties. Recall that $H_p(E)$ is a near algebra for an arbitrary normed linear space E; hence, $H_p(\mathbb{R})$ is a near algebra. We will show $H_p(\mathbb{R}) = Lip_p(\mathbb{R})$, the positive homogeneous Lipschitz functions on $\mathbb{R}$.

We first show that $H_p(\mathbb{R})$ is a finite dimensional near algebra. Let $J(x) = \begin{cases} x, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$ and let $K(x) = \begin{cases} 0, & \text{for } x > 0 \\ x, & \text{for } x \leq 0 \end{cases}$. Clearly J and K are independent functions on $\mathbb{R}$ and, if $f \in H_p(\mathbb{R})$, then $f(x) = \begin{cases} f(x\cdot 1) & , \text{for } x \geq 0 \\ f((-x)(-1)), & \text{for } x < 0 \end{cases}$ $= \begin{cases} xf(1) & , \text{for } x \geq 0 \\ -xf(-1), & \text{for } x < 0 \end{cases} = f(1)J(x) + (-f(-1))K(x)$. Thus, $\{J,K\}$ forms a basis for $H_p(\mathbb{R})$. It is easy to show that $J, K \in Lip(\mathbb{R})$; hence, any linear combination of J and K belongs to $Lip(\mathbb{R})$. Therefore, $H_p(\mathbb{R}) = Lip_p(\mathbb{R})$ and inherits the norm of $Lip(\mathbb{R})$. For $f \in H_p(\mathbb{R})$, the norm of f can be calculated in terms of the coefficients in its basis expansion.

Let $f = aJ + bK$ and let $M = \max\{|a|,|b|\}$. Choose $x, y \in \mathbb{R}$. If $x, y \geq 0$ or $x, y \leq 0$, then $|f(x) - f(y)| = |a||x - y|$ or $|f(x) - f(y)| = |b||x - y|$, respectively. If $x \geq 0$, $y \leq 0$, then $|f(x) - f(y)| = |ax - by| \leq |a||x| + |b||y| \leq M(|x| + |y|) \leq M|x - y|$ since x and y are of opposite signs. Similarly for $x \leq 0$, $y \geq 0$. Therefore, in all cases, $|f(x) - f(y)| \leq M|x - y|$; hence, $\|f\| \leq M$.

Conversely, by the property of the norm of f, $|f(x) - f(y)| \leq \|f\| \, |x - y|$ for all x, $y \in \mathbb{R}$. Then, $|f(1) - f(0)| = |a| \leq \|f\|$ and $|f(-1) - f(0)| = |b| \leq \|f\|$. Therefore, both $|a|$ and $|b|$ are less than $\|f\|$ and $\|f\| = \max\{|a|, |b|\}$ where f = aJ + bK. We immediately get that $\|J\| = 1$ and $\|K\| = 1$.

The following algebraic properties can be demonstrated by the tedious checking of cases.

(1) For $a, b \geq 0$ $\quad J(aJ + bK) = aJ$

$K(aJ + bK) = bK$

(2) For $a \geq 0$, $b \leq 0$ $\quad J(aJ + bK) = aJ + bK$

$K(aJ + bK) = 0$

(3) For $a \leq 0$, $b \geq 0$ $\quad J(aJ + bK) = 0$

$K(aJ + bK) = aJ + bK$

(4) For $a \leq 0$, $b \leq 0$ $\quad J(aJ + bK) = bK$

$K(aJ + bK) = aJ$

Some immediate consequences of these facts are the properties:

(1) J and K form and orthogonal idempotent basis:
$JJ = J$, $JK = 0$, $KJ = 0$, $KK = K$

(2) $J(-K) = -K$, $J(-J) = 0$, $K(-J) = -J$, $K(-K) = 0$

(3) Although not left distributive for all $f \in H_p(\mathbb{R})$, J and K have the distributive properties:
$J(aJ + bK) = J(aJ) + J(bK)$ and $K(aJ + bK) = K(aJ) + K(bK)$.

Using the property of the norm derived above and the various properties listed here, one can again check the numerous cases involved and show that $\|fg - fh\| \leq \|f\| \, \|g - h\|$.

Therefore, $H_p(\mathbb{R})$ is a D-normed near algebra with several "nice" algebraic properties. In addition to these algebraic properties, one can show, either directly by a somewhat lengthy argument or by the following proposition, that left multiplication is continuous in $H_p(\mathbb{R})$ when considered as the normed near algebra $Lip_p(\mathbb{R})$. It is interesting to note, that, although $Lip(\mathbb{R})$ is not a continuous near algebra in the Lipschitz norm, it does contain a continuous sub near algebra, $Lip_p(\mathbb{R})$. This raises an interesting but still unanswered question as to the characterization of $Lip_{LC}(\mathbb{R})$, the largest continuous sub near algebra of Lip $(\mathbb{R})$. In addition to $Lip_p(\mathbb{R})$, $Lip(\mathbb{R}) \cap \mathcal{D}(\mathbb{R})$ with the Lipschitz norm is another continuous sub near algebra of $Lip(\mathbb{R})$.

<u>2.5 Proposition.</u> If N is a D-normed near algebra, then N is a left continuous near algebra.

Proof: Suppose b is an arbitrary but fixed element of N, $K_b \neq 0$, and let $x \in N$. Let $\{x_n\}$ be a sequence in N such that x_n converges to x. For $\varepsilon > 0$, choose $n_o \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon / K_b$ whenever $n > n_o$, then $\|bx_n - bx\| \leq K_b \|x_n - x\| \leq K_b(\varepsilon / K_b) = \varepsilon$ whenever $n > n_o$. Therefore, bx_n converges to bx for $K_b \neq 0$. If $K_b = 0$, then $\|bx_n - bx\| = 0 < \varepsilon$. Therefore, bx_n converges to bx for all b and for all x which implies multiplication is left continuous. $\square$

Examples (1) and (3) above indicate that a stronger multiplicative condition is possible in some cases.

<u>2.4 Definition.</u> A normed near algebra N is a <u>strongly D-normed near algebra</u> if and only if N is a D-normed near algebra and, for each $b \in N$, $K_b = \|b\|$.

<u>2.5 Definition.</u> A normed (D-normed) right near algebra N is a <u>complete normed (D-normed) right near algebra</u> if the normed linear space $(N^+, \| \ \|)$ is a Banach space. We will call a complete normed (D-normed) near algebra a <u>Banach (D-Banach) near algebra.</u>

Some examples of Banach and D-Banach near algebras are given below.

(1) Lip(E), the near algebra of Lipschitz functions on a normed space E, forms a Banach near algebra in the Lipschitz norm provided E is a Banach space.

Assume $\{f_n\}$ is a Cauchy sequence in the norm of Lip(E). For $\varepsilon > 0$ and $x \in E$, $x \neq 0$, choose $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\| < \frac{\varepsilon}{\|x\|}$ for all $n, m > n_0$. By the definition of the norm of Lip(E), $\|(f_n - f_m)(u) - (f_n - f_m)(v)\|$ $\leq \|f_n - f_m\| \, \|u - v\| < \frac{\varepsilon}{\|x\|} \|u - v\|$ for all $u, v \in E$ and for all $n, m > n_o$. Choose $u = x$, $v = 0$, then $\|f_n(x) - f_m(x)\| \leq \varepsilon$ for $n, m > n_o$. Therefore, $\{f_n(x)\}$ is a Cauchy sequence in E for each arbitrary but fixed $x \in E$, $x \neq 0$. Since E is a Banach space, $f_n(x)$ converges for each $x \in E$; thus, let $f(x) = \lim_{n \to \infty} f_n(x)$. Clearly, $f: E \to E$ since $\lim_{n \to \infty} f(x)$ is uniquely determined in E for arbitrary x. We need to show $f \in$ Lip(E) and f_n converges to f in the Lipschitz norm.

The proof of this result is similar to that of showing the bounded linear operators form a Banach space [26, p.76]. Assume $\varepsilon > 0$ and show that there exists an m_o which is independent of x and y such that $\|(f - f_m)(x) - (f - f_m)(y)\| < \varepsilon \|x - y\|$ for all x, $y \in E$ whenever $m > m_o$. This implies that $f - f_{m_o} \in Lip(E)$; hence, $f = (f - f_{m_o}) + f_{m_o} \in Lip(E)$. Also, $\|f - f_m\| < \varepsilon$ whenever $m > m_o$; thus, $f_n \rightarrow f$.

For $\varepsilon > 0$, choose m_o such that $\|f_n - f_m\| < \varepsilon$ whenever m, $n > m_o$. Then, for arbitrary but fixed x, $y \in E$ and for an arbitrary $\delta > 0$, choose n_1 (δ, x, y) such that $n > n_1$ implies $\|f(x) - f_n(x)\| < \delta$ and $\|f(y) - f_n(y)\| < \delta$. For $n > \max(m_o, n_1)$ and $m > m_o$, $\|f(x) - f_m(x) - (f(y) - f_m(y))\|$ $= \|f(x) - f_n(x) + f_n(x) - f_m(x) - f(y) + f_n(y) - f_n(y) + f_m(y)\|$ $\leq \delta + \|f_n - f_m\| \|x - y\| + \delta \leq 2\delta + \varepsilon \|x - y\|$. Since x, y were arbitrary and δ was arbitrary, $\|f(x) - f_m(x) - (f(y) - f_m(y))\| \leq \varepsilon \|x - y\|$ for all x, $y \in E$ whenever $m > m_o$. This completes the demonstration that Lip(E) is a Banach near algebra.

(2) $T_B(E)$ is a Banach near algebra in the sup norm, provided E is a Banach space. The completeness of this norm also follows in a manner analogous to that for the space of bounded linear operators on E. See [26, p. 76].

(3) Every Banach algebra is a D-Banach near algebra.

(4) In the above example of a D-normed near algebra where multiplication was defined by $ab = \|b\| a$, if E is a Banach space, then the resulting near algebra is a D-Banach near algebra.

As in the case for normed algebras certain properties hold for normed near algebras because of their linear space structure. The following are two such properties. The proofs can be found in [26].

2.6 Proposition. Every continuous homomorphism of the normed near algebra N into the normed near algebra N' is a bounded linear operator.

2.7 Proposition. Every continuous isomorphism of a Banach near algebra N onto a Banach near algebra N' is a topological isomorphism.

We conclude this section with several important generalizations from normed algebras.

2.2 Theorem. In a Banach (D-Banach) near algebra N, the quotient N/I of additive cosets modulo a closed ideal I is a Banach (D-Banach) near algebra.

Proof: Except for the definition of an ideal the proof is similar to that of a normed algebra. See [26]. Since I is closed, N/I is a complete normed linear space and we have indicated in Chapter I that N/I is a near algebra. It remains to show that the multiplicative properties of the induced norm continue to hold where $\| a + I \| = \inf \{ \|a + x\| \mid x \in I \}$.

Assume N is a Banach near algebra and let a + I, b + I $\in$ N/I. For $\varepsilon > 0$, choose u, v $\in$ I such that $\|a + u\| \leq \|a + I\| + \varepsilon$ and $\|b + v\| \leq \|b + I\| + \varepsilon$. Let $J = \{ z \mid z \in N \text{ and } z = (a + x)(b + y) - ab \text{ for } x, y \in I \}$ and let $z_0 = (a + u)(b + v) - ab \in J \subseteq I$. Thus, $\|(a + I)(b + I)\| = \|ab + I\| \leq \inf \{ \|ab + x\| \mid x \in J \} \leq \|ab + z_0\| = \|(a + u)(b + v)\| \leq (\|a + I\| + \varepsilon)(\|b + I\| + \varepsilon)$ for all $\varepsilon > 0$. Therefore, $\|(a + I)(b + I)\| \leq \|a + I\| \, \|b + I\|$.

Similarly assume N is a D-Banach near algebra and let a + I, b + I, and c + I $\in$ N/I. For $\varepsilon > 0$, choose v $\in$ I such that $\|(b - c) + v\| \leq \|(b - c) + I\| + \varepsilon$. Let $J = \{z \mid z \in N$ and $z = a(b + x) - a(c + y) - (ab - ac)$ for $x, y \in I\}$; then, $J \subseteq I$ and $z_o = a(b + 2v) - a(c + v) - (ab - ac) \in J$. Thus, $\|(a + I)(b + I) - (a + I)(c + I)\| = \inf\{\|(ab - ac) + x\| \mid x \in I\} \leq \inf\{\|(ab - ac) + x\| \mid x \in J\} \leq \|(ab - ac) + z_o\| = \|a(b + 2v) - a(c + v)\| \leq K_a\|(b - c) + v\| \leq K_a(\|(b - c) + I\| + \varepsilon)$. Therefore, $\|(a + I)(b + I) - (a + I)(c + I)\| \leq K_a\|(b + I) - (c + I)\|$. $\square$

Using the above results and Theorems 1.2 and 1.3 of Chapter I, we have the following statement analogous to that for an algebra.

2.3 Theorem. Under a continuous homomorphism of the Banach (D-Banach) near algebra N onto the Banach (D-Banach) near algebra N', the kernel I of the homomorphism is a closed ideal in N and the near algebra N' is topologically isomorphic to N/I. Conversely, every closed ideal I of the near algebra N induces a continuous homomorphism of the near algebra N onto N/I.

The following theorem is a straightforward generalization of Berman and Silverman [7].

2.4 Theorem. Every near algebra without identity can be embedded isomorphically into a near algebra with identity.

The corresponding result for a normed near algebra or Banach near algebra is presented with the additional hypothesis of positive homogeneity. However, many near algebras with identities do not satisfy this property;

hence, it is not a necessary condition.

<u>2.5 Theorem.</u> Every normed (Banach) near algebra, N, without identity can be isomorphically embedded in a normed (Banach) near algebra with identity provided $x(ty) = t(xy)$ for all $x, y \in N$ and $t \geq 0$.

Proof: Let N be a normed near algebra such that $x(ty) = t(xy)$ for all $x, y \in N$ and $t \geq 0$. Let $B(N) = \{f \mid f \in T(N)$ and $\|f(x)\| \leq M$ for all $x \in E$, for some $M \geq 0\}$. B(N) becomes a near algebra under the pointwise operations of sum, product, and scalar multiplication. Note, the product is pointwise and not composition of functions.

Let $\|f\| = \sup\{\|f(x)\| \mid x \in N\}$. It is easy to show that $\| \; \|$ is a norm. Also $\|f \cdot g\| = \sup\{\|f(x) \cdot g(x)\| \mid x \in N\}$ $\leq \sup\{\|f(x)\| \, \|g(x)\| \mid x \in N\} \leq \|f\| \, \|g\|$. Therefore, $(B(N), \| \; \|)$ is a normed near algebra.

If we assume N is a Banach near algebra, then B(N) is also a Banach near algebra. Assume $\{f_n\}$ is a Cauchy sequence in B(N) and let $\varepsilon > 0$ be arbitrary. For sufficiently large n_o, $\|f_n(x) - f_m(x)\| < \varepsilon$ whenever $n, m > n_o$ for all $x \in N$; hence, $\{f_n(x)\}$ is a Cauchy sequence in the complete space N for all $x \in N$. Let $f(x) = \lim_{n \to \infty} f(x)$ for each $x \in N$. Thus, for an arbitrary $x \in N$ and $\varepsilon > 0$, there exists $n(x, \varepsilon)$ such that $\|f(x) - f_n(x)\| < \varepsilon$ whenever $n > n(x, \varepsilon)$. Choosing $m > n_o$ and $n > \max(n_o, n(x, \varepsilon))$, we have $\|f(x) - f_m(x)\|$ $= \|f(x) - f_n(x) + f_n(x) - f_m(x)\| \leq \|f(x) - f_n(x)\|$ $+ \|f_n(x) - t_m(x)\| < 2\varepsilon$. Therefore, $\|f - f_m\| < 2\varepsilon$ whenever $m > n_o$ and $f = f_m + (f - f_m) \in B(N)$. Thus, B(N) is a Banach near algebra.

For each $a \in N$ define $\mu(a)$ to be the constant 'a' function from N to N, $\mu(a)x = a$. Consequently, $\|\mu(a)x\| = \|a\|$ for all $x \in N$ and $\mu(a) \in B(N)$ for each $a \in N$. Therefore, $\mu: N \to B(N)$. By direct calculation, one can show μ is a near algebra homomorphism which is also one-to-one. Furthermore, $\|\mu(a)\| = \sup\{\|\mu(a)x\| \mid x \in N\} = \sup\{\|a\| \mid x \in N\} = \|a\|$. Therefore, μ is an isometric isomorphism of N into B(N).

Let $N^2 = T_B(B(N)) = \{F \mid F \in T(B(N))$ and $\|F(f)\| \leq M\|f\|$ for all $f \in B(N)$ and some $M \geq 0\}$. N^2 has been shown to be a normed near algebra with sup norm, pointwise sums and scalar products, and composition as the multiplication. It is also a Banach near algebra if B(N) is a Banach near algebra. N^2 contains the bounded linear operators on B(N) and hence N^2 contains the identity operator I.

Let $\mu(N) = S \subseteq B(N)$ and define a function φ as follows: For each $f \in S$, $\varphi(f)g = \begin{cases} f \cdot g, & \text{for } g \in S \\ \|g\| f, & \text{for } g \in B(N) - S \end{cases}$ where $f \cdot g$ is the pointwise product of B(N). Since $S \subseteq B(N)$, it is clear that $\varphi(f): B(N) \to B(N)$ for each $f \in S$. Suppose $g \in B(N)$; if $g \in S$ then $\|\varphi(f)g\| = \|f \cdot g\| \leq \|f\| \|g\|$ and if $g \in B(N) - S$, then $\|\varphi(f)g\| = \|\|g\| f\| = \|f\| \|g\|$. Therefore, $\|\varphi(f)g\| \leq M\|g\|$ where $M = \|f\|$ and $\varphi: S \to N^2$.

A direct calculation shows that φ is linear. However, one must show that B(N) satisfies the positive homogeneous condition in order to show that φ is multiplicative. Since

N has the positive homogeneous condition, $f\cdot(tg)(x) = f(x)\cdot(tg(x)) = t(f\cdot g)(x)$ for $t \geq 0$ and $x \in N$. Thus $f\cdot(tg) = t(f\cdot g)$ for $f, g \in B(N)$ and $t \geq 0$.

For $f, h \in S$, $\varphi(f\cdot h)g = \begin{cases}(f\cdot h)g \\ \|g\|(f\cdot h)\end{cases} =$

$= \begin{cases} f\cdot(h\cdot g), & \text{for } g \in S \\ f\cdot(\|g\|h), & \text{for } g \in B(N) - S\end{cases}$ since B(N) satisfies the positive homogeneous condition. For any $g \in B(N)$, $\varphi(h)g \in S$; thus, $[\varphi(f)\circ\varphi(h)](g) = \varphi(f)(\varphi(h)g) = f\cdot(\varphi(h)g) =$

$= \begin{cases} f\cdot(h\cdot g), & \text{for } g \in S \\ f\cdot(\|g\|h), & \text{for } g \in B(N) - S\end{cases}$. Therefore $\varphi(f\cdot h) = \varphi(f)\circ\varphi(h)$ and φ is a near algebra homomorphism.

Suppose $\varphi(f) = \varphi(h)$ and choose $g \in B(N) - S$, $g \neq 0$. $\varphi(f)g = \|g\|f = \varphi(h)g = \|g\|h$; thus, $\|g\|f = \|g\|h$ which implies $f = h$ and φ is one-to-one.

For $f \in S$ and arbitrary $g \in B(N)$, $g \neq 0$, $\dfrac{\|\varphi(f)g\|}{\|g\|} =$

$= \begin{cases} \dfrac{\|f\cdot g\|}{\|g\|}, & \text{for } g \in S \\ \dfrac{\|\,\|g\|f\,\|}{\|g\|}, & \text{for } g \in B(N) - S\end{cases} \leq \|f\|$ for all $g \in B(N)$.

Therefore, by the definition of the sup norm $\|\varphi(f)\| \leq \|f\|$. However, for $g \in B(N) - S$, $g \neq 0$, $\dfrac{\|\varphi(f)g\|}{\|g\|} = \dfrac{\|\,\|g\|f\,\|}{\|g\|} =$

$= \dfrac{\|g\|\,\|f\|}{\|g\|} = \|f\| \leq \|\varphi(f)\|$. Therefore, $\|\varphi(f)\| = \|f\|$. This completes the demonstration that φ is an isometric isomorphism of S into N^2. The composition of φ and μ is an isometric isomorphism of N into N^2 and N^2 has an identity. $\square$

It is important to point out that the standard procedure for adjoining the identity does not work because the formally constructed near algebra multiplication is not associative. This is a result of the missing left distributive property. Let N be a near algebra without identity and formally construct $N' = \left\{ te + x \middle| x \in N \text{ and } t \in F \right\}$ with the usual operations. If we consider elements $e + x_1$, $e + x_2$, and $e + x_3$, then

$[(e + x_1)(e + x_2)](e + x_3) =$

$e + (x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3)$ while

$(e + x_1)[(e + x_2)(e + x_3)] =$

$e + (x_1 + x_2 + x_3 + x_2x_3 + x_1(x_2 + x_3 + x_2x_3))$. Since, in general, $x_1(x_2 + x_3 + x_2x_3) \neq x_1x_2 + x_1x_3 + x_1x_2x_3$, the multiplication is not associative.

We now turn our attention to the completion of a normed near algebra.

2.6 Theorem. Every non-complete strongly D-normed near algebra can be embedded in a complete strongly D-normed near algebra.

Proof: The proof is similar to that of a normed algebra. See [26, p. 176]. Let N be a non-complete strongly D-normed near algebra and let M be the standard completion of the normed space N^+ in terms of equivalence classes of Cauchy sequences. The strongly D-normed condition forces the product of two Cauchy sequences to be a Cauchy sequence and can be used to show that the product does not depend on the chosen representatives. Passing to the limit in the operations on N, one can show that M is a near algebra which satisfies the strong D-norm property. $\square$

Representation Theory

The representation of algebras and normed algebras by a system of linear transformations on a linear vector space has proved useful in the discussion of algebras. Since linear transformations are too restrictive for a near algebra representation, we extend the concept of a representation space to allow for a larger class of transformations. Berman and Silverman [7] have considered some very general representations and embedding theorems. We have already demonstrated in this paper that a normed near algebra without identity can be embedded isometrically and isomorphically into a function space with identity.

Given a normed algebra A, one of the most natural representations is the left regular representation of A by the space of bounded linear left multiplication operators on A. We generalize this representation for a normed near algebra and define the <u>left regular representation</u> of a normed near algebra N into the space of bounded left multiplication operators on N contained in $T_B(N)$. These left multiplication operators are not linear, in general. Using the multiplicative identity conditions given in [27], we show that the left regular representation can be an isometric isomorphism.

If N is a D-normed near algebra, then there is a representation of N into the space of Lipschitz functions on N.

This important representation is an isomorphism whenever the multiplicative identity conditions are present. The strong D-normed condition results in an isometry.

In [10, 11] Brown defines a distributor element and a sequence of distributive left modules. For a near algebra N, $D^0(N) = N$ and $D^k(N) = [N, D^{k-1}(N), N]$, the subspace of N^+ generated by certain distributor elements. A near algebra N is D-weakly distributive (D-w-d) of length k, if $D^k(N) = 0$ and $D^{k-1}(N) \neq 0$ for $k > 0$. We show that every left continuous semisimple D-w-d near algebra is a semisimple algebra. For a D-w-d normed near algebra N there exists a representation of N as a space of bounded linear operators which leads to the principle result of this section. We prove that every normed semisimple near algebra which contains a nonzero left distributive left module is a semisimple algebra.

The following theorem generalizes the representation of a normed algebra into the space of bounded linear operators.

<u>2.7 Theorem.</u> Every normed near algebra N has a representation in the space $T_B(N)$, the bounded operators on N.

Proof: For each $b \in N$, define $L_b(x) = bx$ for all $x \in N$. L_b is the left multiplication operator. As in the proof for an algebra, L_b is a function from N to N but L_b is not linear since left distributivity does not hold. However, $\| L_b(x) \| = \| bx \| \leq \| b \| \| x \|$ for all $x \in N$; hence, $L_b \in T_B(N)$. We now define the representation $\Theta : N \rightarrow T_B(N)$ to be $\Theta(b) = L_b$. Clearly Θ is a near algebra homomorphism since right distributivity does hold. This homomorphism will be an isomorphism into $T_B(N)$ if and only if zero is the left annihilator of N. Such a representation is called a <u>faithful representation</u> by Rickart [27]. $\square$

<u>2.6 Definition.</u> Let N be a normed near algebra and let BL(N) be the sub near algebra of bounded left multiplication operators on N contained in $T_B(N)$. The representation $\Theta: N \longrightarrow BL(N)$ defined by $\Theta(b) = L_b$ is called the <u>left regular representation of N</u>.

Using the ideas of Rickart [27] we can make the representation norm preserving whenever certain multiplicative identity conditions exists.

<u>2.7 Corollary.</u> Let N be a normed near algebra satisfying at least one of the properties:

i) There exists a right identity e_r, such that $\|e_r\| \leq 1$,

ii) For each $b \in N$ there exists e_b such that $\|e_b\| \leq 1$ and $be_b = b$, or

iii) There exists a net $\{e_\alpha : \alpha \in \Lambda\}$ such that $\|e_\alpha\| \leq 1$ for all α and for each $b \in N$ $\lim_\alpha be_\alpha = b$.

Then the left regular representation is an isometric isomorphism.

Proof: We have shown that the left regular representaion $\Theta(b) = L_b$ is a near algebra homomorphism on N. For $x \in N$, $\|\Theta(b)x\| = \|L_b(x)\| = \|bx\| \leq \|b\|\,\|x\|$; hence, $\|\Theta(b)\| = \sup\left\{ \frac{\|\Theta(b)x\|}{\|x\|} \,\middle|\, x \neq 0\right\} \leq \|b\|$. If conditions i) or ii) hold, then choose $u = e_b$ or e_r. Thus, $\|b\| \leq \frac{\|bu\|}{\|u\|} = \frac{\|\Theta(b)u\|}{\|u\|} \leq \sup\left\{ \frac{\|\Theta(b)x\|}{\|x\|} \,\middle|\, x \neq 0\right\} = \|\Theta(b)\|$. Therefore, $\|\Theta(b)\| = \|b\|$. Suppose condition iii) holds and let $\varepsilon > 0$

be arbitrary. There exists α such that $\|b - be_\alpha\| < \varepsilon$; consequently, $\|\Theta(b)e_\alpha\| = \|be_\alpha\| \geq \|b\| - \varepsilon$. Therefore, for all $\varepsilon > 0$, $\|b\| - \varepsilon \leq \frac{\|be_\alpha\|}{\|e_\alpha\|} \leq \|\Theta(b)\|$, which implies $\|b\| \leq \|\Theta(b)\|$. Therefore, in each case, Θ is an isometry. Being a linear isometry Θ is one-to-one. □

Two important observations are in order at this point. First, we observe that the following statement found in [27, p. 4] remains valid for near algebras.

<u>2.8 Theorem.</u> If N is a normed near algebra such that $\|b\| = \sup\left\{\frac{\|bx\|}{\|x\|} \,\middle|\, x \neq 0\right\}$, then the left regular representation is an isometric isomorphism.

This allows a more "natural" adjunction of an identity to N without the positive homogeneous condition. Identify N with its image in $T_B(N)$ and let N_1 be the sub near algebra generated by N and the identity of $T_B(N)$. Consequently,

<u>2.9 Theorem.</u> Every normed near algebra such that $\|b\| = \sup\left\{\frac{\|bx\|}{\|x\|} \,\middle|\, x \neq 0\right\}$ can be embedded in a normed near algebra with identity.

Second, the right regular representation can not be generalized for near algebras. For, if R_b is the right multiplication operator, then the representation $\varphi(b) = R_b$, is not linear.

We now consider some representation theorems for D-normed near algebras and, using the results of Brown [10, 11], some representation theorems concerning D-w-d normed near algebras.

<u>2.10 Theorem.</u> If N is a D-normed near algebra, then there

exists a representation of N into Lip(N), the space of Lipschitz operators on N.

Proof: If L_b is defined to be the left multiplication operator, then by the D-normed condition $\| L_b(x) - L_b(y) \| = \| bx - by \| \leq K_b \| x - y \|$ for all x, y $\in$ N. Therefore, $L_b \in$ Lip(N) for all b $\in$ N. We can define $\varphi : N \to \mathrm{Lip}(N)$ by $\varphi(b) = L_b$ and, as in Theorem 2.7, φ is a near algebra homomorphism. This is a faithful representation if and only if zero is the left annihilator of N. $\square$

We can now prove a corollary analogous to Corollary 2.7 in which Lip(N) has the Lipschitz norm.

<u>2.10 Corollary.</u> Let N be a D-normed near algebra satisfying at least one of the following properties:

i) There exists a right identity e_r, such that $\| e_r \| \leq 1$,

ii) For each b $\in$ N there exists e_b such that $\| e_b \| \leq 1$ and $be_b = b$, or

iii) There exists a net $\{ e_\alpha : \alpha \in \Lambda \}$ such that $\| e_\alpha \| \leq 1$ for all α and for each b $\in$ N $\lim_\alpha be_\alpha = b$.

Then the representation is an isomorphism into Lip(N) with Lipschitz norm. If N is strongly D-normed, then the representation is also an isometry.

Proof: Let N be such a normed near algebra and let b $\in$ N. With Lipschitz norm, $\| \varphi(b)(x) - \varphi(b)(y) \| \leq \| \varphi(b) \| \| x - y \|$ for all x, y $\in$ N. If condition i) or ii) is satisfied, then let x = e_r or e_b, respectively. Thus, $\| \varphi(b)x - \varphi(b)(0) \| = \| bx \| = \| b \| \leq \| \varphi(b) \| \| x \| \leq \| \varphi(b) \|$. If condition iii) holds, then let $\varepsilon > 0$ be arbitrary. Choose $\alpha \in \Lambda$ such that $\|b - be_\alpha\| < \varepsilon$, then $\|b\| - \varepsilon < \|be_\alpha\|$. For $x = e_\alpha$, $\|b\| - \varepsilon < \|be_\alpha\| = \|\varphi(b)e_\alpha - \varphi(b)(0)\| \leq \|\varphi(b)\| \|e_\alpha\|$ $\leq \|\varphi(b)\|$. Since this holds for arbitrary $\varepsilon > 0$, $\|b\|$ $\leq \|\varphi(b)\|$. Thus, in each case, $\|b - a\| \leq \|\varphi(b - a)\|$ $= \|\varphi(b) - \varphi(a)\|$ and $b = a$ whenever $\varphi(b) = \varphi(a)$.

If N is strongly D-normed, then, for $b \in N$, $\|\varphi(b)(x) - \varphi(b)(y)\| = \|bx - by\| \leq \|b\| \|x - y\|$ for all $x, y \in N$. Thus, $\|\varphi(b)\| \leq \|b\|$ and φ is an isometry. ∎

As an example, let E be a Banach space and define multiplication on E by the formula $xy = \|y\|x$ for all $x, y \in E$. We have previously shown $(E, \| \|)$ to be a strongly D-normed near algebra. There is no multiplicative identity, but condition ii) of Corollary 2.10 is satisfied. That is, for each $b \in E$, $b \neq 0$, let $e_b = b/\|b\|$. Therefore, $\varphi: E \rightarrow \text{Lip}(E)$ is an isometric isomorphism into Lip(E).

Let N be a near algebra and V be a left module of N (V is a subspace of N^+ and $NV \subseteq V$). In [10] Brown called the element $n(a + b) - na - nb$ of V the distributor of $a, b \in V$ with respect to $n \in N$ and denoted it by $[n, a, b]$. For $A \subseteq V$, $A \neq \emptyset$, and for B, a sub near algebra of N, he denoted by $D_B(A)$ the subspace of V generated by $\{[n, a, b] \mid a, b \in A, n \in B\}$. One can show that $D_B(A)$ $= \{\sum [n_i, a_i, b_i] \mid a_i, b_i \in A, n_i \in B\}$. A is said to be left distributive if $D_N(A) = 0$; that is, $n(a + b) - na - nb$ $= 0$ for all $a, b \in A$ and $n \in N$. For a sub near algebra A, let $D_A(A) = D(A)$. For nonempty sets $S_1, S_2, S_3 \subseteq N$, let $[S_1, S_2, S_3]$ be the subspace of N^+ generated by

$\{[s_1, s_2, s_3] \mid s_i \in S_i\}$; therefore, for $A \subseteq N^+$, $A \neq \emptyset$, and a sub near algebra B, $D_B(A) = [B, A, A]$. Finally, let A be a subspace of N^+ and define $D^0(A) = A$ and $D^k(A) = [N, D^{k-1}(A), N]$.

In [11] it is shown that $D^k(N)$ is an ideal and a left module in N for $k \geq 0$. N is defined to be D-weakly distributive (D-w-d) of length k if $D^k(N) = 0$ and $D^{k-1}(N) \neq 0$ for $k > 0$. If N is D-w-d then N is a near-c-algebra ($N0 = 0$).

Using the above properties and definitions we prove the following results for left continuous near algebras and normed near algebras.

2.1 Lemma. Every D-w-d near algebra of length $k > 0$ with no proper ideals is left distributive.

Proof: Let N be a D-w-d near algebra of length $k > 0$. Then, $D^k(N) = 0$ and $D^{k-1}(N)$ is an ideal. Therefore, $D^{k-1}(N) = N$ and the D-w-d chain reduces to $D^0(N) = N$ and $D^1(N) = 0$. This states that N is left distributive. $\square$

2.8 Proposition. Let N be a near algebra with a linear topology and let M be a nonzero left module which is left distributive with respect to N. If the multiplication on M is left continuous, then M is a sub algebra and the left multiplication operator, L_b, is linear on M for each $b \in N$.

Proof: M is a sub near algebra of N since M is a linear subspace of N^+ and $MM \subseteq M$. Being left distributive with respect to N, M would be an algebra except for the scalar property $x(ty) = t(xy)$. However, since M is left continuous, the left multiplication operator, L_b, is a continuous additive function from the linear topological space M^+ to M^+ for each $b \in N$. Therefore, L_b is real homogeneous [13, p. 12] and, hence, linear on M for each $b \in N$. This also implies M is a sub algebra since $b(tx) = L_b(tx) = tL_b(x) = t(bx)$ for all $b, x \in M$ and $t \in \mathbb{R}$. $\square$

<u>2.9 Proposition.</u> Every left continuous D-w-d near algebra with no proper ideals is an algebra.

Proof: Let N be such a near algebra. By Lemma 2.1, N is left distributive and, by Proposition 2.8, N is an algebra. $\square$

<u>2.9 Corollary.</u> Every left continuous D-w-d simple near algebra is a simple algebra.

It is clear from Proposition 2.8 that every left continuous left distributive near algebra is an algebra. In [11, p. 540] it is shown that a semisimple near algebra is D-w-d if and only if it is left distributive. Combining this result with Proposition 2.8, we get the following important theorem.

<u>2.11 Theorem.</u> Every left continuous semisimple D-w-d near algebra is a semisimple algebra.

If the linear topology on the near algebra N is determined by a norm, then we have similar results without assuming left continuity of the multiplication on N.

<u>2.12 Theorem.</u> If N is a normed near algebra and there exists a nonzero left module M which is left distributive with respect to N, then M is a sub algebra and there is a representation of N into the space of bounded linear functions on M.

Proof: Since M is left distributive, we have, for $b \in N$ and $x \in M$, $0 = b0 = b(x+(-x)) = bx + b(-x)$. Therefore, $b(-x) = -(bx)$. Let $b \in N$ and let L_b denote the left multiplication operator on N with respect to b. For an arbitrary $x_0 \in M$,

$\| L_b(x) - L_b(x_o) \| = \| bx - bx_o \| = \| bx + b(-x_o) \|$ $= \| b(x - x_o) \| \leq \| b \| \| x - x_o \|$. This inequality implies that L_b is continuous on M for all $b \in N$ and, by Proposition 2.8, M is an algebra and L_b is linear on M. Also $\| L_b(x) \|$ $= \| bx \| \leq \| b \| \| x \|$; hence, L_b is a bounded linear function on M. The desired representation $\phi : N \to \mathcal{B}(M)$ is defined by $\phi(b) = L_b$ for each $b \in N$. $\square$

2.12.1 Corollary. Every left distributive normed near algebra is an algebra.

2.12.2 Corollary. Every normed near algebra which contains a left distributive left module whose left annihilator is zero is an algebra.

Proof: Let N be such a near algebra with module M. By the theorem, $\phi : N \to \mathcal{B}(M)$ is a homomorphism. The kernel of ϕ is the left annihilator of M; thus, ϕ is one-to-one and N is an algebra. $\square$

2.13 Theorem. If N is a D-w-d normed near algebra of length $k > 0$, then there exists a representation of N as a space of bounded linear functions.

Proof: Let N be such a D-w-d normed near algebra. Then, $D^k(N) = 0$ and $D^{k-1}(N) \neq 0$ and, by the properties stated above, $D^{k-1}(N)$ is a nonzero left module such that $D_N(D^{k-1}(N))$ $= [N, D^{k-1}(N), D^{k-1}(N)] \subseteq [N, D^{k-1}(N), N] = D^k(N) = 0$. Therefore, $D_N(D^{k-1}(N)) = 0$ and, consequently, $D^{k-1}(N)$ is a nonzero left distributive left module. The result follows from Theorem 2.12. $\square$

2.13 Corollary. If N is a D-w-d normed near algebra of length

$k > 0$ such that the left annihilator of $D^{k-1}(N)$ is zero, then N is a normed algebra.

Proof: The result follows from the theorem and Corollary 2.12.2. $\square$

2.14 Theorem. If N is a D-w-d normed near algebra of length $k > 0$ with no proper ideals, then N is a normed algebra.

Proof: By Lemma 2.1, N is left distributive and, by Corollary 2.12.1, N is an algebra. $\square$

2.14 Corollary. Every D-w-d simple normed near algebra is a simple algebra.

Brown showed in [10] that every semisimple near algebra with identity whose linear space is a Banach space such that the left multiplication operator L_b is differentiable on N at 0 for each $b \in N$ is a semisimple algebra. We have shown, in Theorem 2.11, that every left continuous semisimple D-w-d near algebra is a semisimple algebra. For a normed near algebra we have the following important result.

2.15 Theorem. Every normed semisimple near algebra which contains a nonzero left distributive left module is a semisimple algebra.

Proof: Let N be such a near algebra and let M be the left module which is left distributive with respect to N. By Theorem 1.5, $N = \bigoplus_{i=1}^{k} N_i$ where each N_i is a simple near algebra. Let $M_i = N_i \cap M$. Each M_i is a left module of N_i and is left distributive with respect to N_i. By Theorem 2.12, there exists a near algebra homomorphism $\varphi_i: N_i \to \mathcal{B}(M_i)$ and, since N_i is simple, $\ker \varphi_i = 0$ for each i. Therefore, φ_i is an isomorphism into $\mathcal{B}(M_i)$ and N_i is an algebra. Thus,

$N = \bigoplus_{i=1}^{k} N_i$ is a semisimple algebra. $\square$

The following example shows that the existence of a left distributive left module is not sufficient to have an algebra.

Let $V = \mathbb{R}\times\{0\}$ and let $N = \{f \mid f \in T_B(\mathbb{R}^2),\ f|_V : V \to V$ and $f|_V$ is linear$\}$. By direct calculation it is easy to show that N is a sub near algebra of $T_B(\mathbb{R}^2)$ and with the sup norm becomes a normed near algebra. Define f by the formula

$$f(x,y) = \begin{cases} (x,y), & \text{for } y = 0 \\ (x,y), & \text{for } y \neq 0 \text{ and } \|(x,y)\| \leq 1 \\ (1,1), & \text{for } y \neq 0 \text{ and } \|(x,y)\| > 1 \end{cases}.$$

Then, $f|_V$ is linear and maps V into V. Also, $\|f(x,y)\| \leq \sqrt{2}\,\|(x,y)\|$; thus, $f \in N$. Now $f(2f)(1,1) = (1,1)$, while $2(ff(1,1)) = (2,2)$. Therefore, $f(2f) \neq 2ff$ and N is not an algebra.

Let $M = \{f \mid f \in N \text{ and } f(\mathbb{R}^2 - V) = 0\}$. It is easy to show that M is a linear subspace of N^+. Suppose $f \in N$ and $g \in M$ then $fg(\mathbb{R}^2 - V) = f(0) = 0$; hence, $fg \in M$ and M is a left module. Suppose $f \in N$ and $g, h \in M$. For $(x,y) \in \mathbb{R}^2$, $f(g + h)(x,y) = f(g(x,y)) + f(h(x,y)) = (fg)(x,y) + (fh)(x,y)$ since $g(x,y)$ and $h(x,y)$ belong to V and f is linear on V. Therefore, $f(g + h) = fg + fh$ and M is a left module which is left distributive with respect to N, but N is not an algebra.

Finite Dimensional Near Algebras

This section presents some properties of finite dimensional near algebras. We abstract the properties of $\mathrm{Lip}_p(\mathbb{R})$ and show that a positive homogeneous finite dimensional normed near algebra with orthogonal idempotent basis is a left continuous near algebra. In particular, we define a D-distributive property and show that positive homogeneity and an orthogonal idempotent basis insure left continuity of multiplication. We then prove the stronger result that a positive homogeneous normed near algebra with orthogonal idempotent basis such that $x_i(-x_j) = -K_{ij}x_j$, $K_{ij} \geq 0$, becomes a strongly D-normed near algebra with an equivalent norm. Eventually, we remove the D-distributive condition and the condition $x_i(-x_j) = -K_{ij}x_j$ and prove that all positive homogeneous normed near algebras with an orthogonal idempotent basis can be given an equivalent strong D-norm. The strong D-norm insures continuous left multiplication. Finally, we prove that every finite dimensional homogeneous near algebra with orthogonal idempotent basis is an algebra.

Unless otherwise stated, we will assume all near algebras are near-c-algebras. In particular, for function algebras we will assume $f(0) = 0$.

Although $\mathrm{Lip}(\mathbb{R})$ is not a left continuous near algebra, we did show that $\mathrm{Lip}_p(\mathbb{R})$ is left continuous, in fact, strongly

D-normed. $Lip_p(\mathbb{R})$ is a finite dimensional near algebra with basis elements J and K which are orthogonal idempotents. Also, J and K are positive homogeneous and have the property that $J(sJ + tK) = J(sJ) + J(tK)$ and $K(sJ + tK) = K(sJ) + K(tK)$ for all s, $t \in \mathbb{R}$. We generalize these concepts to an arbitrary finite dimensional near algebra.

2.7 Definition. A near algebra N is said to be finite dimensional ifand only if N^+ is a finite dimensional linear space. N has an orthogonal idempotent basis $\{x_1, x_2, \cdots, x_n\}$ if and only if $x_i x_j = \delta_{ij} x_j$ where δ_{ij} is the Kronecker delta.

2.8 Definition. A real near algebra N is said to be positive homogeneous if and only if $x(ty) = t(xy)$ for all x, $y \in N$ and $t \in \mathbb{R}$, $t \geq 0$. N is said to be homogeneous if and only if $x(ty) = t(xy)$ for all x, $y \in N$ and $t \in \mathbb{R}$.

In [16, 17] Frohlich defined a distributively generated near ring R as one in which the additive group of R has a set U of generators such that $u(x + y) = ux + uy$ for all $u \in U$ and all x, $y \in R$. We make the following generalization to near algebras which is a slightly weaker property.

2.9 Definition. A nonempty subset U of a near algebra N is said to have the D-distributive property if $u(\sum t_i u_i) = \sum u(t_i u_i)$ for each $u \in U$ and every finite linear combination of elements in U. A near algebra N is said to be D-distributively generated (D-d-g) if the linear space N^+ is generated by a D-distributive set.

$Lip_p(\mathbb{R})$ is an example of a D-distributively generated near algebra since we have shown the basis elements J and K have the properties $J(tJ + sK) = J(tJ) + J(sK)$ and

$K(tJ + sK) = K(tJ) + K(sK)$. However, if we assume $f = aJ + bK \in \mathrm{Lip}_p(\mathbb{R})$ such that $f(g + h) = fg + fh$ for all $g, h \in \mathrm{Lip}_p(\mathbb{R})$, then $f(2K + (-3K)) = f(2K) + f(-3K)$ implies $a = b$. Therefore, $f = aI$ and $\mathrm{Lip}_p(\mathbb{R})$ is not distributively generated.

2.10 Proposition. Let N be a finite dimensional normed near algebra. If N has an orthogonal idempotent basis, then the max norm on N defined by $|||x||| = \max\{|t_i| \mid x = \sum t_i x_i\}$ is equivalent to the given norm and satisfies the multiplicative property $|||xy||| \leq k^2 |||x|||\ |||y|||$ where k is the dimension of N.

Proof: Let N be a k-dimensional normed near algebra with basis $\{x_i \mid i = 1,2,\cdots,k\}$. We may assume $\|x_i\| = 1$. Let $|||x||| = \max\{|t_i| \mid i = 1,2,\cdots,k\}$ for each $x = \sum t_i x_i \in N$.

For arbitrary $x = \sum t_i x_i$, $\|x\| = \|\sum t_i x_i\|$ $< \sum \|t_i x_i\| \leq \sum |t_i| \leq k \max\{|t_i| \mid i = 1,2,\cdots,k\}$ $= k\,|||x|||$. Also, since $\|xx_j\| \leq \|x\|\,\|x_j\| = \|x\|$, $\|xx_j\| = \|t_j x_j\| = |t_j| \leq \|x\|$ for each $j = 1,2,\cdots,k$. Therefore, $|||x||| = \max\{|t_i| \mid i = 1,2,\cdots,k\} \leq \|x\|$ $\leq k\,|||x|||$ and $|||xy||| \leq \|xy\| \leq \|x\|\,\|y\|$ $\leq k^2\,|||x|||\ |||y|||$. ▯

2.16 Theorem. Let N be a positive homogeneous finite dimensional normed near algebra. If N has an orthogonal idempotent basis which has the D-distributive property, then N is left continuous.

Proof: Let N be such a k-dimensional near algebra with basis $\{x_i \mid i = 1,2,\cdots,k\}$. By Proposition 2.10, choose the equivalent max norm on N where $\|x\| = \max\{|t_i| \mid i = 1,2,\cdots,k\}$, for $x = \sum t_i x_i$, and $\|xy\| \leq k^2 \|x\| \|y\|$.

Let $\{y_n\}$ be a sequence in N with $y_n = \sum t_{ni}x_i$ and let $y \in N$ with $y = \sum t_i x_i$. Assume $t_{ni} \xrightarrow[n]{} t_i$ for each $i = 1,2,\cdots,k$. Then, $\|y_n - y\| = \|\sum(t_{ni} - t_i)x_i\| \leq \sum|t_{ni} - t_i|$. Since $t_{ni} \xrightarrow[n]{} t_i$ for each i and there are only a finite number of i's, we choose n_o sufficiently large so that, for $\varepsilon > 0$, $\sum|t_{ni} - t_i| < \varepsilon$ whenever $n > n_o$. Therefore, $y_n \to y$.

Conversely, assume $y_n \to y$ and let $\varepsilon > 0$, then there exists n_o such that $\|y_n - y\| = \|\sum(t_{ni} - t_i)x_i\| = \max\{|t_{ni} - t_i| \mid i = 1,2,\cdots,k\} < \varepsilon$ whenever $n > n_o$. Thus, for arbitrary $i = 1,2,\cdots,k$, $|t_{ni} - t_i| < \varepsilon$ whenever $n > n_o$ and $t_{ni} \xrightarrow[n]{} t_i$. Therefore, $t_{ni} \xrightarrow[n]{} t_i$, for $i = 1,2,\cdots,k$, if and only if $y_n \to y$.

Let $x \in N$ and let x_i be an arbitrary basis element. Assume $t_n x_i \xrightarrow[n]{} tx_i$. If $x = 0$, then $0\,(t_n x_i) = 0 \to 0 = 0(tx_i)$. If $x \neq 0$, then, by the above, with $y_n = t_n x_i$ and $y = tx_i$, we have that $t_n \to t$. For $\varepsilon > 0$, choose n_o such that $t_n - t < \varepsilon/(k^2\|x\|)$ whenever $n > n_o$. Let $\theta(a)$ be the algebraic sign of $a \in \mathbb{R}$. If $t \neq 0$, we can choose n_o sufficiently large so that $\theta(t) = \theta(t_n)$ for $n \geq n_o$. Thus, $\|x(t_n x_i) - x(tx_i)\| = \||t_n|(x(\theta(t)x_i)) - |t|(x(\theta(t)x_i))\| \leq |t_n - t|\,\|x(\theta(t)x_i)\| \leq k^2|t_n - t|\,\|x\| < \varepsilon$, for $n > n_o$. If $t = 0$, then $\|x(t_n x_i) - x(0)\| = \|x(t_n x_i)\| \leq k^2|t_n|\,\|x\| < \varepsilon$, for $n > n_o$. Therefore, $x(t_n x_i) \xrightarrow[n]{} x(tx_i)$ for arbitrary $i = 1,2,\cdots,k$.

Let $x \in N$ and let $\{y_n\}$ be a sequence in N such that

$y_n \to y$. We wish to prove that multiplication is left continuous in the max norm by showing $xy_n \to xy$. Assume $x = \sum t_i x_i$, $y_n = \sum s_{ni} x_i$ and $y = \sum s_i x_i$ are the basis expansions where $y_n \to y$. By the first property above, $s_{ni} \underset{n}{\longrightarrow} s_i$ for all $i = 1,2,\cdots,k$. Thus, $\| s_{ni}x_m - s_i x_m \| = | s_{ni} - s_i |$ which implies $s_{ni}x_m \underset{n}{\longrightarrow} s_i x_m$ for all i, $m = 1,2,\cdots,k$. In particular, $s_{ni}x_i \underset{n}{\longrightarrow} s_i x_i$ and, by the second property above, for i, $j = 1,2,\cdots,k$, $x_j(s_{ni}x_i) \underset{n}{\longrightarrow} x_j(s_i x_i)$. Since the indices i and j can assume only a finite number of integer values, we can choose, for $\varepsilon > 0$, an n_o such that $\| x_j(s_{ni}x_i) - x_j(s_i x_i) \| < \varepsilon$ for all i, $j = 1,2,\cdots,k$ whenever $n > n_o$.

Using the D-distributive property, $\| xy_n - xy \| = \| \sum_j t_j \left(\sum_i (x_j(s_{ni}x_i) - x_j(s_i x_i)) \right) \|$. Applying the triangle inequality, $\| xy_n - xy \| \leq \sum_j |t_j| \sum_i \| x_j(s_{ni}x_i) - x_j(s_i x_i) \| \leq k^2 \|f\| \varepsilon$. Therefore, for sufficiently large n, $\| xy_n - xy \| < \varepsilon$ and $xy_n \to xy$ in the max norm. Since the given norm is equivalent, $xy_n \to xy$ in the given norm. $\square$

As in the case of $\mathrm{Lip}_p(\mathbb{R})$ we can show that a strong D-norm condition holds in certain cases. A partial result is given below. A stronger statement is proved later.

<u>2.11 Proposition.</u> Let N be a positive homogeneous finite dimensional normed near algebra. If N has basis $\{x_i\}$ such that $x_i x_j = \delta_{ij} x_j$ and $x_i(-x_j) = -K_{ij} x_j$ where δ_{ij} is the Kronecker delta and $K_{ij} \geq 0$, then there is an equivalent norm on N such that N is a strongly D-normed near algebra.

Proof: Let N be a k-dimensional near algebra with basis

$\{x_i\}$ satisfying the given properties. By Proposition 2.10 we may choose the max norm, $\|x\|$, which satisfies $\|xy\| \leq k^2\|x\|\,\|y\|$ and is equivalent to the given norm.

Let x_n be an arbitrary basis element, $1 \leq n \leq k$. $x_n(\sum t_i x_i) = \sum s_{ni} x_i$ and, multiplying both sides on the left by x_j, we have $x_n(t_j x_j) = s_{nj} x_j$ for $j = 1,2,\cdots,k$. For $t_j \geq 0$, $x_n(t_j x_j) = \delta_{nj} t_j x_j$ and, for $t_j < 0$, $x_n(t_j x_j) = -t_j x_n(-x_j)$ $= K_{nj} t_j x_j$. Therefore, $s_{nj} = \begin{cases} \delta_{jn} t_j, & \text{for } t_j \geq 0 \\ K_{nj} t_j, & \text{for } t_j < 0 \end{cases}$. Let us define a new function which will allow us to express s_{nj} in a more convenient form. Let $\operatorname{sgn}(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ -1, & \text{for } t < 0 \end{cases}$ and define $\theta(i, t_j) = \left[\frac{1 + \operatorname{sgn}(t_j)}{2}\right]\delta_{ij} + \left[\frac{1 - \operatorname{sgn}(t_j)}{2}\right]K_{ij}$.

A quick check shows $s_{nj} = \theta(n, t_j)t_j$ and, thus, $x_n(\sum t_j x_j)$ $= \sum \theta(n, t_j) t_j x_j$ for $n = 1,2,\cdots,k$.

For $x, y, z \in N$ with basis representation $x = \sum r_i x_i$, $y = \sum s_i x_i$, and $z = \sum t_i x_i$, we have $xy - xz = \left(\sum_i r_i x_i\right)\left(\sum_j s_j x_j\right)$ $- \left(\sum_i r_i x_i\right)\left(\sum_j t_j x_j\right) = \sum_i r_i\left(x_i\left(\sum_j s_j x_j\right)\right) - \sum_i r_i x_i \sum_j t_j x_j$

$$= \sum_i r_i \left(\sum_j [\theta(i,s_j)s_j - \theta(i,t_j)t_j] x_j\right)$$

$$= \sum_j \left[\sum_i (r_i(\theta(i,s_j)s_j - \theta(i,t_j)t_j))\right] x_j.$$

Thus, $\|xy - xz\|$

$$= \max\left\{ \left| \sum_i r_i(\theta(i,s_j)s_j - \theta(i,t_j)t_j)\right| \;\middle|\; j = 1,2,\cdots,k\right\}.$$

For fixed j, $\left|\sum_i r_i(\theta(i,s_j)s_j - \theta(i,t_j)t_j\right|$

$\leq \sum_i |r_i| \, |\theta(i,s_j)s_j - \theta(i,t_j)t_j|$

$\leq \|x\| \sum_i |\theta(i,s_j)s_j - \theta(i,t_j)t_j|$ and a check of the various cases for each $i = 1,2,\cdots,k$ shows that $|\theta(i,s_j)s_j - \theta(i,t_j)t_j|$ $\leq (1 + K_{ij})|s_j - t_j|$. For example, for fixed j, if s_j and $t_j \geq 0$, then $\theta(i,s_j) = \delta_{ij}$ and $\theta(i,t_j) = \delta_{ij}$. Thus, $|\theta(i,s_j)s_j - \theta(i,t_j)t_j| = |\delta_{ij}s_j - \delta_{ij}t_j| \leq |s_j - t_j|$ $\leq (1 + K_{ij})|s_j - t_j|$ for all $i = 1,2,\cdots,k$. Therefore, $\sum_i |\theta(i,s_j)s_j - \theta(i,t_j)t_j| \leq kK|s_j - t_j| \leq kK\|y - z\|$ where $K = \max\left\{1 + K_{ij} \middle| i, j = 1,2,\cdots,k\right\}$. Therefore, $\|xy - xz\|$ $\leq kK\|x\| \, \|y - z\|$. The desired equivalent norm is $\|x\|'$ $= kK\|x\|$. $\square$

<u>2.11.1 Corollary.</u> Every positive homogeneous finite dimensional normed near algebra with basis elements satisfying the conditions of the theorem is a left continuous near algebra.

Proof: By the proposition there is an equivalent strong D-norm and the existance of such a norm implies left continuity. $\square$

<u>2.11.2 Corollary.</u> Every positive homogeneous finite dimensional near algebra with basis elements satisfying the conditions of the theorem is a D-distributively generated near algebra.

Proof: In the theorem we showed, independent of the norm, that, if $x_n(\sum t_i x_i) = \sum s_{ni} x_i$, then multiplying both sides on the left by x_j, we have $x_n(t_j x_j) = s_{nj} x_j$, for $j = 1, 2, \cdots, k$. Thus, $x_n(\sum t_i x_i) = \sum x_n(t_i x_i)$. Since N^+ is spanned by the basis, N is D-distributively generated. $\square$

2.12 Proposition. Let N be a positive homogeneous near algebra and let V be a nonempty D-distributive subset of N. If $M = V \cup (-V)$ forms a multiplicative semi-group in N, then the linear subspace generated by V forms a D-distributively generated sub near algebra of N.

Proof: Let $x = \sum_{i=1}^{n} t_i u_i$ and $y = \sum_{j=1}^{m} s_j v_j$ be elements of the linear space, $\langle V \rangle$, generated by V where u_i, $v_j \in V$. Then,

$xy = (\sum t_i u_i)(\sum s_j v_j) = \sum t_i(\sum u_i(s_j v_j)) = \sum t_i(\sum |s_j| u_i(\pm v_j))$

$= \sum(\sum t_i |s_j| w_{ij})$ where $w_{ij} = u_i(\pm v_j) \in M$; hence, $w_{ij} \in \langle V \rangle$. Therefore, $xy \in \langle V \rangle$ and $\langle V \rangle$ is a D-distributively generated sub near algebra by Lemma 1.1. $\square$

$Lip_p(\mathbb{R})$ is one example of the above ideas. As another example, let H be an n-dimensional real Hilbert space with basis $\{x_i\}$. For each $x = \sum t_i x_i \in H$, define

$$P_i(x) = \begin{cases} J(t_i)x_i & , \text{ for } 1 \leq i \leq n \\ K(t_{i-n})x_{i-n}, & \text{for } n + 1 \leq i \leq 2n \end{cases} \quad \text{where } t_i = (x, x_i)$$

is the inner product of x and x_i. Since J and K are positive homogeneous, it is clear that P_i is positive homogeneous. Also, for $x = \sum t_i x_i$ and $y = \sum s_i x_i$, we have $\|P_i(x) - P_i(y)\| \leq |t_i - s_i|$ since J, $K \in Lip(\mathbb{R})$. By Bessel's inequality

$|t_i - s_i|^2 \leq \sum |t_i - s_i|^2 \leq \|x - y\|^2$. Thus, $\|P_i(x) - P_i(y)\| \leq \|x - y\|$ and $P_i \in Lip_p(H)$. A straightforward but tedious check shows that $P_iP_j = \delta_{ij}P_j$ and, for $1 \leq i \leq n$, $P_i(-P_j) = \begin{cases} 0 & , \text{ if } j \neq n + i \\ -P_j, & \text{if } j = n + i \end{cases}$; and $P_i(-P_j) = \begin{cases} 0 & , \text{ if } j \neq i - n \\ -P_j, & \text{if } j = i - n \end{cases}$, for $n + 1 \leq i \leq 2n$. Finally, one can show that $P_i\left(\sum_{j=1}^{2n} t_jP_j\right) = \sum_{j=1}^{2n} P_i(t_jP_j)$. Therefore, let $V = \{P_i | i = 1,2,\cdots,2n\}$; then, by Propositions 2.11 and 2.12, $\langle V \rangle$ is a strongly D-normed near algebra with basis V and norm equivalent to the Lipschitz norm.

As a final example, let V be a finite dimensional linear space and define a multiplication on V by the formula $xy = \sum s_i|t_i|x_i$ for each $x = \sum s_ix_i$ and $y = \sum t_ix_i$ belonging to V. This multiplication is closed, associative and right distributive, but not left distributive; hence, V becomes a finite dimensional near algebra with the basis $\{x_i\}$ of V. $x_ix_j = \delta_{ij}x_j$ and $x_i(-x_j) = \delta_{ij}x_j$. Also $x_i(\sum t_ix_j) = t_ix_i = \sum x_i(t_jx_j)$ and for $t \geq 0$, $x(ty) = \sum s_i|tt_i|x_i = t\sum s_i|t_i|x_i = t(xy)$. Therefore, V is a positive homogeneous finite dimensional near algebra with orthogonal idempotent basis which has the D-distributive property. Finally, define the max norm on V, $\|x\| = \max\{|t_i| \mid x = \sum t_ix_i\}$. This is a norm and, for $x, y \in V$, $\|xy\| = \|\sum s_i|t_i|x_i\| = \max\{|s_i|t_i||\}$

$= \max\{|s_i|\,|t_i|\} \leq \|x\|\,\|y\|$. Therefore, V is a normed near algebra with this norm and, by Theorem 2.16, V is left continuous. Notice, also, that V is strongly D-normed, since $\|xy - xz\| = \|\sum r_i|s_i|x_i - \sum r_i|t_i|x_i\|$ $= \|\sum r_i(|s_i| - |t_i|)x_i\| = \max\{|r_i|\,||s_i| - |t_i||\} \leq$ $\max\{|r_i|\,|s_i - t_i|\} \leq \|x\|\,\|y - z\|$.

We will now show that positive homogeneity and an orthogonal idempotent basis are sufficient conditions for the strongly D-normed condition and left continuity of multiplication in the finite dimensional case.

<u>2.13 Proposition.</u> If N is a finite dimensional positive homogeneous near algebra with orthogonal idempotent basis $\{x_i\}$, then either $x_i(-x_j) = -K_{ij}x_j$ where $K_{ij} = 0$ or 1 for all i and j or there exists a basis $\{y_i\}$ such that $y_iy_j = \delta_{ij}M_jy_j$, $M_j > 0$ and $y_i(-y_j) = -S_{ij}y_j$ where $S_{ij} \geq 0$, for $i \neq j$, and $S_{ii} \geq 0$ or $S_{ii} = -1$.

Proof: Let N be a k-dimensional positive homogeneous near algebra with orthogonal idempotent basis $\{x_i\}$. For arbitrary but fixed i, $1 \leq i \leq k$, and for $j = 1,2,\ldots,k$, $x_i(-x_j)$ $= t_1(i,j)x_1 + \cdots + t_k(i,j)x_k$. For $n \neq j$, $x_i(-x_j)x_n = 0$ $= t_n(i,j)x_n$. Therefore, $t_n(i,j) = 0$ for $n \neq j$ and, since i was arbitrary, $x_i(-x_j) = t_j(i,j)x_j$ for $i, j = 1,2,\ldots,k$.

Again, let i be arbitrary but fixed and assume there exists $j \neq i$ such that $t_j(i,j) > 0$. Then $t_j(i,j)x_j = x_i(-x_j)$ $=(x_ix_i)(-x_j) = x_i(x_i(-x_j)) = x_i(t_j(i,j)x_j) = t_j(i,j)x_ix_j = 0$

and hence $t_j(i,j) = 0$. Therefore, for arbitrary i there exists $t_j(i,j) = -K_{ij}$ for $j = 1,2,\cdots,k$, $j \neq i$ such that $K_{ij} \geq 0$ and $x_i(-x_j) = -K_{ij}x_j$. Applying x_i to this last equality, we get $K_{ij}^2 - K_{ij} = 0$. Thus, $K_{ij} = 0$ or 1 for $i \neq j$.

Let $t_i(i,i) = K_i$ and consider the two cases: $K_i \leq 0$ for all $i = 1,2,\cdots,k$ or $K_i > 0$ for some i, $1 \leq i \leq k$.

Case 1. If $K_i \leq 0$ for all $i = 1,2,\cdots,k$, then $x_i(-x_i) = K_i x_i$ and applying x_i to both sides we get $K_i^2 + K_i = 0$. Thus, $K_i = 0$ or -1 and this case results in the orthogonal idempotent basis x_i such that $x_i(-x_j) = K_{ij}x_j$ where $K_{ij} = 0$ or 1.

Case 2. If $K_{i_o} > 0$ for some i_o, then reorder the basis elements, if necessary, so that $K_i > 0$ for $1 \leq i \leq n$ and $K_i \leq 0$ for $n < i \leq k$ where $1 < n \leq k$. Let $y_i = \frac{1}{K_i}x_i$, $1 \leq i \leq n$, and let $y_i = x_i$, $n < i \leq k$. The set $\{y_i\}$ forms a new basis such that $y_i y_j = 0$ for $i \neq j$. Also, for $1 \leq i \leq n$, $y_i y_j = \frac{1}{K_i}y_i$ and, for $n < i \leq k$, $y_i x_i = x_i = y_i$. Therefore, $y_i y_j = \delta_{ij} M_j y_j$ where $M_j > 0$.

For $1 \leq i \leq n$ and $i \neq j$, if $1 \leq j \leq n$, then $y_i(-y_j)$

$= \frac{1}{K_i K_j}(x_i(-x_j)) = \frac{-K_{ij}}{K_i} y_j$. If $n < j \leq k$, then $y_i(-y_j)$

$= \frac{-K_{ij}}{K_i} x_j = \frac{-K_{ij}}{K_i} y_j$. If we now consider i, $n < i \leq k$, and the same two cases for $j \neq i$, we find that $y_i(-y_j) = -K_{ij}y_j$

in both cases. Therefore, for $i \neq j$, there exists $S_{ij} \geq 0$ such that $y_i(-y_j) = -S_{ij}y_j$.

Finally, for $i = j$, $1 \leq i \leq n$, $y_i(-y_i) = \frac{1}{K_i^2}(x_i(-x_i))$ $= \frac{1}{K_i^2}(K_i x_i) = y_i$. If $n < i \leq k$, $y_i(-y_i) = x_i(-x_i) = K_i x_i$ $= K_i y_i$. Therefore, combining these results with the above, we have $y_i(-y_j) = S_{ij}y_j$ where $S_{ij} \geq 0$ for $i \neq j$ and $S_{ii} \geq 0$ or $S_{ii} = -1$. $\square$

<u>2.17 Theorem.</u> If N is a finite dimensional positive homogeneous normed near algebra with orthogonal idempotent basis, then N can be strongly D-normed with an equivalent norm and the basis has the D-distributive property. That is, N is a D-distributively generated left continuous near algebra.

Proof: Let $\{x_i\}$ be an orthogonal idempotent basis for the k-dimensional near algebra N. By Proposition 2.13, two cases exist. If $x_i(-x_j) = -K_{ij}x_j$ where $K_{ij} = 0$ or 1, then the results follow from Proposition 2.11 and Corollary 2.11.2. Otherwise, let $\{y_i\}$ be a new basis such that $y_i y_j = \delta_{ij} M_j y_j$ where $M_j > 0$ and $y_i(-y_j) = -S_{ij}y_j$ where the basis elements are so ordered that $S_{ij} \geq 0$, for $i \neq j$ and $S_{ii} = -1$ for $1 \leq i \leq n_o$ and $S_{ii} \geq 0$ for $n_o < i \leq k$. We now choose the equivalent max norm on N and proceed in a manner similar to that of Proposition 2.11.

For arbitrary y_n, $1 \leq n \leq k$, $y_n(\sum t_i y_i) = \sum r_{ni} y_i$. If we apply y_j, $j = 1,2,\cdots,k$, then $y_n(t_j M_j y_j) = r_{nj} M_j y_j$. Since $M_j > 0$, it factors out and $y_n(t_j y_j) = r_{nj} y_j$ for

$j = 1,2,\cdots,k$ and arbitrary n. Therefore, for arbitrary n, $\sum y_n(t_jy_j) = \sum r_{nj}y_j = y_n(\sum t_jy_j)$ and the basis set has the D-distributive property. This implies multiplication is left continuous. We now wish to show that N can be strongly D-normed.

From the above, if $t_j \geq 0$, then $y_n(t_jy_j) = \delta_{ij}M_jt_jy_j$ and, if $t_j < 0$, then $y_n(t_jy_j) = -t_jy_n(-y_j) = S_{nj}t_jy_j$. Therefore, using the properties of S_{nj}, we have the following description for r_{nj}. For $t_j > 0$, $r_{nj} = \begin{cases} 0 & \text{, for } n \neq j \\ M_jt_j, & \text{for } n = j \end{cases}$ and,

$$\text{for } t_j < 0,\ r_{nj} = \begin{cases} S_{nj}t_j, & \text{for all } n \neq j \text{ or } n_o < n \leq k,\ j = n \\ -t_j & \text{, for } 1 \leq n \leq n_o,\ j = n \end{cases}.$$

For $t_j = 0$, $r_{nj} = 0$ for all n. For $1 \leq i \leq n_o$, define $\theta(i,t_j)$

$$= \left[\frac{1+\operatorname{sgn}(t_j)}{2}\right]\delta_{ij}(M_j - S_{ij}) + \frac{\operatorname{sgn}(\delta_{ij}t_j)-1}{2}$$

$+ \left[\operatorname{sgn}(\delta_{ij}t_j) + \operatorname{sgn}((\delta_{ij} - 1)t_j)\right]S_{ij}$ and for $n_o < i \leq k$,

$$\text{define } \theta(i,t_j) = \left[\frac{1 + \operatorname{sgn}(t_j)}{2}\right]\delta_{ij}M_j + \left[\frac{1 - \operatorname{sgn}(t_j)}{2}\right]S_{ij}.$$

A check of the various cases shows that $r_{nj} = \theta(n,t_j)\, t_j$ for all t_j and for $j, n = 1,2,\cdots,k$. Therefore, $y_n \sum_j t_jy_j$

$$= \sum_j \theta(n,t_j)t_jy_j \text{ for all } n = 1,2,\cdots,k.$$

For x, y, z $\in$ N with basis representation $x = \sum r_iy_i$, $y = \sum s_iy_i$ and $z = \sum t_iy_i$ and for θ defined above, we have

$$xy - xz = \sum_j \Big[\sum_i (r_i(\theta(i,s_j)s_j) - \theta(i,t_j)t_j)) \Big] y_j.$$ Thus

$$\| xy - xz \| = \max \Big\{ \Big| \sum_i r_i(\theta(i,s_j)s_j - \theta(i,t_j) \Big| \; \Big| j = 1,2,\cdots,k \Big\}.$$

For fixed j, $\left| \sum_i r_i(\theta(i,s_j)s_j - \theta(i,t_j)t_j) \right|$

$\leq \|x\| \sum_i |\theta(i,s_j)s_j - \theta(i,t_j)t_j|$. For fixed j and this new function θ, a check of the various cases shows that $|\theta(i,s_j)s_j - \theta(i,t_j)t_j| \leq (1+M_j)(1+|S_{ij}|)|s_j - t_j|$ for each $i = 1,2,\cdots,k$. As an example, consider the case for $1 \leq i \leq n_o$ and $j = i$. $\theta(i,s_j) = \begin{cases} M_j, & \text{for } s_j \geq 0 \\ -1, & \text{for } s_j < 0 \end{cases}$, similarly for $\theta(i,t_j)$. Therefore,

$$|\theta(i,s_j)s_j - \theta(i,t_j)t_j| = \begin{cases} |M_js_j - M_jt_j| & , \text{for } s_j,t_j \geq 0 \\ |M_js_j - (-t_j)|, & \text{for } s_j \geq 0,\ t_j < 0 \\ |-s_j - M_jt_j| & , \text{for } s_j < 0,\ t_j \geq 0 \\ |-s_j - (-t_j)| & , \text{for } s_j < 0,\ t_j < 0. \end{cases}$$

If we now consider each instance separately, we find that in the first case $|M_js_j - M_jt_j| \leq M_j|s_j - t_j|$. In the second case, for $s_j \geq 0$, $t_j < 0$, we have $|M_js_j + t_j| < M_js_j - t_j$ $< (1+M_j)|s_j - t_j|$. Case three, for $s_j < 0$, $t_j \geq 0$, is similar to case two with the roles s and t interchanged. Finally, for s_j, $t_j < 0$, $|-s_j - (-t_j)| = |s_j - t_j|$. Thus, in all cases $|\theta(i,s_j)s_j - \theta(i,t_j)t_j| \leq (1+M_j)(1+|S_{jj}|)|s_j - t_j|$. The other choices for i are similarly tedious and will be omitted.

Thus, for arbitrary j, $\sum_i |\theta(i,s_j)s_j - \theta(i,t_j)t_j|$ $\leq kK|s_j - t_j| \leq kK\|y - z\|$ where $K = \max\{(1+M_j)(1+|S_{ij}|) \mid i, j = 1,2,\cdots,k\}$ and, hence, $\|xy - xz\| \leq kK\|x\|\,\|y - z\|$. The desired equivalent norm is $\|x\|' = kK\|x\|$. ▯

2.18 Theorem. Every finite dimensional homogeneous near algebra with orthogonal idempotent basis is an algebra.

Proof: If N is homogeneous, then, by Theorem 2.17, N is D-distributively generated by the basis. (This result did not depend on N being normed.) Let x, y, z $\in$ N with basis representation $x = \sum r_i x_i$, $y = \sum s_i x_i$, and $z = \sum t_i x_i$.

Then, $x(y + z) = \sum_i r_i x_i \left(\sum_j (s_j + t_j)x_j\right)$

$= \sum_i r_i \left(\sum_j x_i(s_j + t_j)x_j\right) = \sum_i r_i \left(\sum_j (s_i + t_i)x_i x_j\right)$

$= \sum_i r_i s_i x_i + \sum_i r_i t_i x_i = xy + xz.$ ▯

CHAPTER III

SPECIAL FUNCTION NEAR ALGEBRAS

Lipschitz Functions

This section is devoted to the discussion of the special function near algebra of Lipschitz functions on a normed linear space E. The Lipschitz functions are an immediate generalization of the bounded linear operators which are so important in analysis. The first few results do not depend upon the algebraic structure of Lip(E) but demonstrate that the Lipschitz functions have range properties and conditions for invertibility similar to those for bounded linear operators. We show that if $f \in$ Lip(E), then f(K) is closed for all closed sets K whenever there exists $t > 0$ such that $\|f(x) - f(y)\| \geq t\|x - y\|$. A Lipschitz function f is invertible if and only if its range is dense and there exists $t > 0$ such that $\|f(x) - f(y)\| \geq t\|x - y\|$.

Although, as a near algebra, Lip(E) is not D-normed, it is a normed near algebra which is a Banach space whenever E is a Banach space. It plays a role similar to that of the bounded linear operators in representation theorems. One such representation was the normed linear space E with multiplication defined by $x \cdot y = \|y\|x$. If S is the image of E in Lip(E), we then show that $f \in T_C(E)$ is linear whenever f is left distributive with respect to all the functions in S.

The most important algebraic property of Lip(E) is that Lip(E) is a near algebra with continuous inverse and the set of invertible elements is an open set which forms a multiplicative group. Brown [10] showed this to be valid for a finite dimensional Banach near algebra. Using Lip(E) as a representation space, we can remove the finite dimensional condition and replace it by the strongly D-normed condition. We are able to show that every strongly D-normed Banach near algebra with identity is a near algebra with continuous inverse and the set of invertible elements is an open set.

3.1 Proposition. Let E be a Banach space and let $f \in$ Lip(E). If there exists $t > 0$ such that $\|f(x) - f(y)\| \geq t\|x - y\|$ for all $x, y \in E$, then f(K) is closed for all closed sets K.

Proof: Let f be such a function in Lip(E) and let K be a closed set. Let $\{y_n\}$ be a sequence in f(K) which converges to y such that $f(x_n) = y_n$. If t is the constant associated with f, then $t\|x_n - x_m\| \leq \|f(x_n) - f(x_m)\| = \|y_n - y_m\|$ for all $n, m \in \mathbb{N}$. Since $\{y_n\}$ is a Cauchy sequence, we have that $\{x_n\}$ is a Cauchy sequence and by completeness $x_n \to x$. Since K is closed, $x \in K$ and, by the continuity of f, $f(x_n) \to f(x) = y$. Thus, $y = f(x) \in f(B)$ and f(B) is closed. $\square$

3.2 Proposition. Let E be a Banach space and let $f \in$ Lip(E). f is invertible if and only if its range is dense in E and there exists $t > 0$ such that $\|f(x) - f(y)\| \geq t\|x - y\|$ for all $x, y \in E$.

Proof: Assume f is invertiable. Then, f is one-to-one and $f(E) = E$; thus, the closure $\overline{f(E)} = E$. Also,

$\|x - y\| = \|f^{-1}(f(x)) - f^{-1}(f(y))\| \leq \|f^{-1}\| \|f(x) - f(y)\|$; thus, $\frac{1}{\|f^{-1}\|}\|x - y\| \leq \|f(x) - f(y)\|$.

Conversely, assume f(E) is dense in E and $\|f(x) - f(y)\| \geq t\|x - y\|$ for some $t > 0$ and all x, y $\in$ E. By Proposition 3.1, f(E) is closed, hence $f(E) = \overline{f(E)} = E$. If f(x) = f(y), then $\|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\| = 0$ which implies x = y. Therefore, f is one-to-one and onto and f^{-1} exists. Also $\|f^{-1}(u) - f^{-1}(v)\| = \|f^{-1}(f(x)) - f^{-1}(f(y))\|$ $= \|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\| = \frac{1}{t}\|u - v\|$. Therefore $f^{-1} \in \text{Lip}(E)$. $\square$

Let E be a normed linear space and define multiplication on E by $x \cdot y = \|y\| x$. We have shown that there exists a representation ϕ: E $\rightarrow$ Lip(E) which is an isometric isomorphism into Lip(E). We let $S = \phi(E)$ and show that linearity of a function and left distributivity are related through S.

3.3 Proposition. Let E be a real normed linear space and let $f \in T_C(E)$. Then f is linear if and only if f is left distributive with respect to all functions in S.

Proof: Clearly, if f is linear then it is left distributive with respect to all functions in $T_C(E)$.

Conversely, assume f is left distributive with respect to S. Let x, y $\in$ E and choose z $\in$ E, $\|z\| = 1$. Define $f_x, f_y : E \rightarrow E$ by $f_x(u) = \|u\| x$ and $f_y(u) = \|u\| y$ for all u $\in$ E. f_x and f_y belong to S and $f_x(z) = x$ and $f_y(z) = y$, for all x, y $\in$ E. For x, y $\in$ E, $f(x + y) = f(f_x(z) + f_y(z))$ $= (f(f_x + f_y))(z) = (ff_x + ff_y)(z) = (ff_x)(z) + (ff_y)(z) =$

= f(x) + f(y). Since f is additive and continuous then f is real homogeneous. //

The proposition states that Lip(E) is an algebra of bounded linear functions if S is left distributive with respect to Lip(E). This result is similar to those following Theorem 2.12 except that S is not a left module and Lip(E) is not left continuous.

The concept of an algebra with continuous inverse is central in the study of topological and normed algebras. For example, every topological division algebra with continuous inverse is isomorphic to the field of complex numbers [26, p. 175]. Although such strong results will not hold for true near algebras, we can investigate the concept of continuous inverse.

<u>3.1 Definition.</u> A near algebra with a Hausdorff topology and an identity e is called a near algebra with continuous inverse if there exists a neighborhood of the identity, U(e), possessing the properties:

i) Every element $x \in U(e)$ has an inverse x^{-1},

ii) x^{-1} is a continuous function of x at the point $x = e$.

In [10] Brown defines a topological near algebra as a near algebra with a Banach linear space. He then proves that a finite dimensional topological near algbra is a near algebra with continuous inverse and the map $x \rightarrow x^{-1}$ is continuous on the unit group. We will examine these concepts in the setting of normed near algebras.

First, we observe that Brown's proof does not require the completeness of the norm; hence, we get the following result.

3.1 Theorem. Let N be an n-dimensional left continuous normed near algebra with identity e. Then, N is a near algebra with continuous inverse and $x \to x^{-1}$ is a continuous map on the group of invertible elements.

We will prove that the finite dimensional condition can be removed for a strongly D-normed near algebra; but, as the next example illustrates, the strong D-norm condition and finite dimensionality can not both be removed.

Consider the Banach normed near algebra $T_B(\mathbb{R})$ with sup norm. If we assume $T_B(\mathbb{R})$ is a near algebra with continuous inverse, then there exists an $\varepsilon > 0$ such that f^{-1} exists whenever $\|f - I\| < \varepsilon$. That is, f is one-to-one whenever $\|f - I\| < \varepsilon$. Let $\alpha = \min(\varepsilon/2, 1)$ and define $f: \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} x & , \text{ for } x < 0 \\ (1+\alpha)x, & \text{ for } 0 \leq x < 1 \\ 1+\alpha & , \text{ for } 1 \leq x < 1 + \alpha \\ x & , \text{ for } 1 + \alpha \leq x \end{cases}$$

By direct calculation, $|f(x)| \leq (1+\alpha)|x|$ for all x; thus, $f \in T_B(\mathbb{R})$. Also, $|f(x) - I(x)| = |x - x| = 0$ for $x < 0$ and $|f(x) - I(x)| = |f(x) - x| \leq \alpha|x|$ for $x \geq 0$; therefore, $\|f - I\| \leq \alpha < \varepsilon$. However, f is not one-to-one since it is constant on the interval $(1, 1+\alpha)$. This shows that $T_B(\mathbb{R})$ can not be a near algebra with continuous inverse.

We will now establish that Lip(E), with Lipschitz norm, is a near algebra with continuous inverse provided E is a Banach space. We will proceed to this result through a sequence of lemmas.

3.1 Lemma. Every normed linear space E with the norm topology is connected.

The proof is straightforward and will be omitted.

3.2 Lemma. Let E be a normed linear space and let $f \in \mathrm{Lip}(E)$. If $\|f - I\| < 1$, then there exists a t, $0 < t \leq 1$, such that $\|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\| \leq (2 - t)\|x - y\|$ for all x, $y \in E$. In particular, for $y = 0$, $\|x\| \leq \frac{1}{t}\|f(x)\| \leq (2 - t)\|x\|$.

Proof: Suppose $f \in \mathrm{Lip}(E)$ and $\|f - I\| < 1$. Let $t = 1 - \|f - I\|$, then $0 < t \leq 1$. For all x, $y \in E$,

$\|(f - I)(x) - (f - I)(y)\| = \|(x - y) - (f(x) - f(y))\|$
$\leq \|f - I\| \|x - y\| = (1 - t)\|x - y\|$. Therefore,
$\|x - y\| - \|f(x) - f(y)\| \leq \|(x - y) - f(x) - f(y)\|$
$\leq (1 - t)\|x - y\|$. This implies $\|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\|$.
Similarly, $\|f(x) - f(y)\| - \|(x - y)\| \leq \|f(x) - f(y) - (x-y)\|$
$\leq (1 - t)\|x - y\|$. Thus, $\|f(x) - f(y)\| \leq (2 - t)\|x - y\|$. □

3.3 Lemma. Let E be a Banach space and let $f \in \mathrm{Lip}(E)$ such that $\|f - I\| < 1$. If K is a closed subset of E, then f(K) is closed in E.

The proof follows from Lemma 3.2 and Proposition 3.1.

3.4 Lemma. Let E be a Banach space and let $f \in \mathrm{Lip}(E)$ such that $\|f - I\| < 1$. If U is an open subset of E, then f(U) is an open subset of E.

Proof: See [23, p. 131]. Let $t = 1 - \|f - I\|$. If $t = 1$, then $f = I$ and $f(U) = U$ is open, thus, we may assume $0 < t < 1$. For arbitrary $b \in E$ and $r > 0$, we adopt the notation $B(b,r) = \{x \mid x \in E, \|x - b\| \leq r\}$ for the closed ball about b with radius r and $S(b,r) = \{x \mid x \in E, \|x - b\| < r\}$ for the open ball about b with radius r.

For arbitrary $b \in E$ and arbitrary $s > 1/t$, we wish to show that $S(f(b), r/s) \subseteq f[B(b,r)]$ for all $r > 0$. Assume not, then there exists $r > 0$ such that, for $S = S(f(b),r/s)$ and $B = B(b,r)$, $S \nsubseteq f(B)$. Choose $v \in S$ such that $v \notin f(B)$ and let $\rho = \inf\{\|v - u\| \mid u \in f(B)\}$. Let $S' = S(v,\rho)$, then $S' \cap f(B) = \emptyset$. This implies $\rho \leq r/s$; for, if $r/s < \rho$, then $\|v - f(b)\| < \rho$ and $f(b) \in S' \cap f(B)$. Also, for any $P_1 > P$ there exists $f(x) \in f(B)$ such that $\|v - f(x)\| \leq P_1$.

Since $\rho < \frac{P}{1-t}$, choose $z = f(u) \in f(B)$ such that $\|v - z\| < P/1-t$. We can now construct the following inequalities. For $z = f(u)$, $\|f(u + (v - z)) - v\|$

$= \|f(u + (v - z)) - z + z - u + u - v\|$

$= \|f(u + (v - z)) - f(u) - ((u + (v - z)) - u)\|$

$= \|(f - I)(u + (v - z)) - (f - I)(u)\|$

$\leq \|f - I \quad u + (v - z) - u\|$

$\leq (1 - t)\|v - z\| < (1 - t)(P/1-t) = P$. Therefore, $\|f(u + (v - z)) - v\| < P$ and $f(u + (v - z)) \in S'$. Also, $\|(u + (v - z)) - b\| = \|u + (v - z) - b + f(b) - f(b)\|$

$= \|(f(b) - z) - (b - u) + (v - f(b)\|$

$= \|(f(b) - f(u) - (b - u) + (v - f(b))\|$

$\leq \|f - I\| \|b - u\| + \|v - f(b)\|$

$\leq (1 - t) r + r/s$ since $u \in B$ and $v \in S$. Since $(1 - t)r + r/s = r((1 - t) + 1/s) < r$, $\|(u + (v - z) - b\| < r$ which implies $u + (v - z) \in B$. Therefore, we have the contradiction that $f(u + (v - z)) \in f(B)$ and $f(u + (v - z)) \in S'$. Thus, $S(f(b), r/s) \subseteq f[B(b,r)]$ for $b \in E$, $s > 1/t$ and

$r > 0$. In addition, for $b \in E$, $s > 1/t$, and $r > 0$, $B(b,r/2) \subseteq S(b,r)$ and, hence, $S(f(b), r/2s) \subseteq f(B(b,r/2)) \subseteq f(S(b,r))$.

We are now ready to show that $f(U)$ is open. Let U be an open subset of E and let $y = f(x) \in f(U)$. Since U is open and $x \in U$, there exists $r > 0$ such that $x \in S(x,r) \subseteq U$. Choose $s > 1/t$, then, by the above, $y \in S(f(x), r/2s) \subseteq f(S(x,r)) \subseteq f(U)$. Therefore U is open. ▯

<u>3.5 Lemma.</u> Let E be a Banach space. If $f \in \text{Lip}(E)$ such that $\|f - I\| < 1$, then f is invertible in $\text{Lip}(E)$.

Proof: Let $f \in \text{Lip}(E)$ such that $\|f - I\| < 1$ and let $t = 1 - \|f - I\|$. By Lemma 3.2, for all $x, y \in E$, $\|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\|$; thus, if $f(x) = f(y)$, then $x = y$ and f is one-to-one. Since E is both open and closed, then, by Lemmas 3.3 and 3.4, $f(E)$ is both open and closed. Therefore, $f(E) = E$ since E is connected. Thus, f is both one-to-one and onto, and f^{-1} exists as a function from E to E.

Suppose $u, v \in E$, then there exist $x, y \in E$ such that $f(x) = u$ and $f(y) = v$. Applying Lemma 3.2, we have $\|f^{-1}(u) - f^{-1}(v)\| = \|x - y\| \leq \frac{1}{t}\|f(x) - f(y)\| = \frac{1}{t}\|u - v\|$. Therefore, $f^{-1} \in \text{Lip}(E)$ and $\|f^{-1}\| \leq \frac{1}{t}$. ▯

<u>3.6 Lemma.</u> Let E be a Banach space. In $\text{Lip}(E)$, $f \to f^{-1}$ is a continuous function of f at the point $f = I$.

Proof: Suppose $\varepsilon > 0$ and choose $\delta < \min\left(\frac{\varepsilon}{1+\varepsilon}, 1\right)$. For f such that $\|f - I\| < \delta$, we know f^{-1} exists by Lemma 3.5 and, for all $x, y \in E$, $\|f(x) - f(y) - (x - y)\| = \|(f - I)(x) - (f - I)(y)\| \leq \delta\|x - y\|$. Therefore,

$\|x - y\| - \|f(x) - f(y)\| \leq \delta \|x - y\|$ and $\|x - y\| \leq \frac{1}{1-\delta}\|f(x) - f(y)\|$ for all $x, y \in E$. Suppose $u, v \in E$ such that $f(x) = u$ and $f(y) = v$, then $\|f^{-1}(u) - f^{-1}(v)\| = \|x - y\| \leq \frac{1}{1-\delta}\|f(x) - f(y)\| = \frac{1}{1-\delta}\|u - v\|$. Thus, $\|f^{-1}\| \leq \frac{1}{1-\delta}$. Therefore, $\|f^{-1} - I\| = \|If^{-1} - ff^{-1}\| = \|(I - f)f^{-1}\| \leq \frac{\delta}{1-\delta} < \varepsilon$ which implies $f \to f^{-1}$ is continuous at I. ▯

Combining Lemmas 3.5 and 3.6, we have the following theorem.

<u>3.2 Theorem.</u> For every Banach space E, Lip(E) is a near algebra with continuous inverse.

<u>3.3 Theorem.</u> If E is a Banach space then the set of invertible elements in Lip(E) is an open set and forms a group under multiplication.

Proof: Let E be a Banach space and let $G = \{f \mid f \in \text{Lip}(E) \text{ and } f \text{ is invertible}\}$. Clearly, G is a multiplicative group.

Suppose $g \in G$ with inverse g^{-1}. Let $r = 1/\|g^{-1}\|$, then $S(g,r)$ is an open neighborhood of g. If $f \in S(g,r)$, then $\|f - g\| < r$ and $\|I - fg^{-1}\| = \|gg^{-1} - fg^{-1}\| = \|(g - f)g^{-1}\| < 1$. By Theorem 3.2, $fg^{-1} = h$ is invertible; hence, $f = hg \in G$ is invertible. Therefore, for each $g \in G$, $S(g, 1/\|g^{-1}\|) \subseteq G$ and G is open. ▯

<u>3.3 Corollary.</u> The set of non-invertible elements in Lip(E) is a closed subset of Lip(E).

Although we have shown that $f \to f^{-1}$ is continuous at

I, it remains an open question as to whether $f \to f^{-1}$ is continuous on the group of all invertible elements in Lip(E). A sufficient condition is given in the following proposition. If the group of invertible elements of Lip(E) is contained in $Lip_{LC}(E)$, the sub near algebra of left continuous elements, then $f \to f^{-1}$ is continuous on all invertible elements.

3.4 Proposition. Let N be a near algebra with continuous inverse and let G be the group of invertible elements. If $G \subseteq N_{LC}$, the sub near algebra of left continuous elements, then $x \to x^{-1}$ is a continuous function on G.

Proof: The proof is similar to that given in [26, p. 171]. However, we must be careful not to use left distributivity. Let x be an arbitrary element of G and let $U(z^{-1})$ be an open neighborhood of z^{-1}. Since $G \subseteq N_{LC}$, $x \to xz^{-1}$ and $x \to z^{-1}x$ are both continuous functions for all $x \in N$. Using the left continuity at e, choose an open neighborhood U(e) such that $x \in U(e)$ implies $z^{-1}x \in U(z^{-1}e) = U(z^{-1})$. The continuity of $x \to x^{-1}$ at e implies $x^{-1} \in U(e)$ whenever $x \in V(e)$ for some open neighborhood V(e). Finally, using right continuity at z, choose an open neighborhood U(z) such that $x \in U(z)$ implies $xz^{-1} \in V(zz^{-1}) = V(e)$. Combining these results, we have that $x \in U(z)$ implies $z^{-1}(xz^{-1})^{-1} = z^{-1}(zx^{-1}) = x^{-1} \in U(z^{-1})$. Therefore $x \to x^{-1}$ is continuous on G. ▯

3.5 Proposition. Let N be a near algebra with continuous inverse. If M is a sub near algebra of N which contains the identity of N and is left continuous in the relative topology, then M is a near algebra with continuous inverse and $x \to x^{-1}$ is continuous on the group of invertible elements of M with respect to the relative topology.

Proof: Let G be the group of invertible elements in N. Since G is open in N, $G' = G \cap M$ is relatively open and there exists a neighborhood of e contained in G'. Clearly $x \to x^{-1}$ is continuous at e in the relative topology. The remainder of the proof follows from Proposition 3.4. ▯

In particular, if M is a left continuous sub near algebra of Lip(E) which contains the identity of Lip(E), then $f \to f^{-1}$ is a continuous function on the set of invertible elements in M.

For a strong D-Banach near algebra we have the following important theorem which removes the finite dimensional property in the theorem of Brown [10].

<u>3.4 Theorem.</u> Let N be a strong D-Banach near algebra with identity. Then, the group G of invertible elements is an open set and the map $x \to x^{-1}$ is a continuous function on G.

Proof: By Corollary 2.10, there exists a function $\varphi: N \to \text{Lip}(N)$ which is an isometric isomorphism into Lip(N) such that $\varphi(b) = L_b$. If e is the identity of N, then $\varphi(e) = I$. Therefore, since φ is an isometry, $\varphi(N)$ is a strongly D-normed sub near algebra of Lip(N) which contains the identity I. By Proposition 3.5, the set of invertible elements in $\varphi(N)$ is open and $f \to f^{-1}$ is a continuous function on this set. Consequently, N has these properties. ▯

Locally Lipschitz Functions

In this section we investigate near algebras of functions on a normed linear space E which are Lipschitz in nature. We construct two near algebras: the near algebra of bounded locally Lipschitz functions denoted B-Lip(E) and the near algebra of locally Lipschitz functions denoted L-Lip(E). A function f belongs to B-Lip(E) whenever f is Lipschitz on every closed bounded sphere about $0 \in E$; while, f belongs to L-Lip(E) if, for each $x \in E$, there exists a neighborhood of x on which f is Lipschitz. Both of these near algebras are a generalization of the Lipschitz functions of the previous section and both are sub near algebras of $T_C(E)$, the continuous functions on E. In general, Lip(E) is a proper sub near algebra of B-Lip(E); however, the sub near algebra of positive homogeneous functions in B-Lip(E) is equal to the positive homogeneous functions in Lip(E). B-Lip(E) is a sub near algebra of L-Lip(E) and for a finite dimensional normed linear space B-Lip(E) = L-Lip(E).

The natural Lipschitz norm of Lip(E) does not extend to B-Lip(E) or L-Lip(E). We do, however, construct a collection of seminorms on B-Lip(E) which determine a locally convex linear topology on the linear structure of B-Lip(E). With this topology B-Lip(E) becomes a right continuous near algebra. Although Lip(E) $\neq$ B-Lip(E), if the seminorm of f is bounded for each seminorm, then, the Lipschitz norm of f is equal to the least upper bound of the seminorms of f. Finally, we show that the relative topology on Lip(E) determined by the topology of B-Lip(E) is strictly weaker than the Lipschitz norm topology of Lip(E).

We begin by stating the two definitions for locally Lipschitz functions. In the discussion to follow, for a normed linear space E, let $S(b,r) = \{x \mid x \in E, \|x - b\| < r\}$ and $B(b,r) = \{x \mid x \in E, \|x - b\| \leq r\}$ be the open and closed spheres about b of radius r, respectively.

3.2 Definition. Let E be a normed linear space. A function $f \in T_0(E)$ is said to be bounded locally Lipschitz if and only if, for every $x \in E$ and every bounded sphere $B(x,r)$ about x, there exists $K_f(B(x,r)) \geq 0$ such that $\|f(u) - f(v)\| \leq K_f(B(x,r))\|u - v\|$ for all $u, v \in B(x,r)$. A function $f \in T_0(E)$ is said to be locally Lipschitz if and only if, for each $x \in E$, there exists a neighborhood $U_f(x)$ and $K(U_f(x)) \geq 0$ such that $\|f(u) - f(v)\| \leq K(U_f(x))\|u - v\|$ for all $u, v \in U_f(x)$.

3.7 Lemma. Let E be a normed linear space and let $f \in T_0(E)$. Then f is bounded locally Lipschitz if and only if f is Lipschitz on every sphere $B(0,r)$ about 0.

Proof: Let $f \in T_0(E)$. Clearly, if f is bounded locally Lipschitz, then f is Lipschitz on $B(0,r)$ for all $r > 0$.

Conversely, assume f is Lipschitz on every sphere $B(0,r)$ and let $x \in E$ and $s > 0$. Choose $r > 0$ such that $B(x,s) \subseteq B(0,r)$. Thus, for $u, v \in B(x,s)$, $\|f(u) - f(v)\| \leq K_r\|u - v\|$ where K_r is the Lipschitz constant for f on B(0,r). ☐

We now define the following subsets of functions contained in $T_0(E)$ where E is a normed linear space. Let B-Lip(E) $= \{f \mid f \in T_0(E)$ and f is bounded locally Lipschitz$\}$ and call B-Lip(E) the <u>B-locally Lipschitz space</u>. Also, let L-Lip(E) $= \{f \mid f \in T_0(E)$ and f is locally Lipschitz$\}$ and call it the <u>L-locally Lipschitz space.</u>

<u>3.6 Proposition.</u> Let E be a normed linear space. Then

i) L-Lip(E) and B-Lip(E) are both sub near algebras of $T_C(E)$, the near algebra of continuous functions on E, and

ii) As near algebras, Lip(E) $\subseteq$ B-Lip(E) $\subseteq$ L-Lip(E).

Proof: The proof of ii) is immediate once i) is established. For i), we will first show that L-Lip(E) $\subseteq T_C(E)$. Let $f \in$ L-Lip(E) and let $x \in E$. Choose $\varepsilon > 0$ and let $U_f(x)$ be a neighborhood of x such that f is Lipschitz on $S(x,r) \subseteq U_f(x)$. Choose $\delta \leq \min\{\varepsilon/(K+1), r\}$ where K is the Lipschitz constant on $U_f(x)$. Thus, if $\|u - x\| < \delta$, then $\|f(u) - f(x)\| \leq K\|u - x\| < \varepsilon$ and f is continuous at x.

Now show L-Lip(E) is a near algebra. Let f, $g \in$ L-Lip(E) and let $x \in E$. If $U_f(x)$ and $U_g(x)$ are the neighborhoods of x on which f and g are Lipschitz, respectively, then $V(x) = U_f(x) \cap U_g(x)$ is a neighborhood of x and a direct calculation shows $\|(f + g)(u) - (f + g)(v)\| \leq (K(U_f(x)) + K(U_g(x)))\|u - v\|$ for all u, $v \in V(x)$. If t is a scalar and u, $v \in U_f(x)$, then $\|(tf)(u) - (tf)(v)\| = |t|\,\|f(u) - f(v)\| \leq |t|K(U_f(x))\|u - v\|$. Therefore, L-Lip(E) is a linear space.

Let $U_f(g(x))$ be a neighborhood of g(x) on which f is Lipschitz. Since g is continuous, $W(x) = g^{-1}(U_f(g(x)) \cap U_g(x))$ is a neighborhood of x and, for u, v $\in$ W(x),
$\|(fg)(u) - (fg)(v)\| \leq K(U_f(g(x)))\|g(u) - g(v)\|$
$\leq K(U_f(g(x))K(U_g(x))\|u - v\|$. Thus, fg $\in$ L-Lip(E) and L-Lip(E) is a near algebra.

The proof that B-Lip(E) is a linear space is similar to the above and is omitted. Let f, g $\in$ B-Lip(E) and let B(x,r) be any closed sphere about x for arbitrary x $\in$ E. If $y \in g(B(x,r))$, then y = g(u) for some u $\in$ B(x,r) and
$\|y - g(x)\| = \|g(u) - g(x)\| \leq K_g(B(x,r))r$. Let δ
$= K_g(B(x,r))r$, then $g(B(x,r)) \subseteq B(g(x),\delta)$. Thus, for
u, v $\in$ B(x,r), $\|(fg)(u) - (fg)(v)\|$
$\leq K_f(B(g(x),\delta))\|g(u) - g(v)\| \leq K_f(B(g(x),r))K_g(B(x,r))\|u - v\|$. []

Although Lip(E) is contained in both B-Lip(E) and L-Lip(E), they are, in general, not equal. Let f $\in T_C(\mathbb{R})$ be defined by $f(x) = x^2$; then, f $\in$ B-Lip($\mathbb{R}$) but f $\notin$ Lip($\mathbb{R}$).

<u>3.7 Proposition.</u> Let E be a normed linear space and let $L\text{-}Lip_p(E)$ be the sub near algebra of positive homogeneous functions in L-Lip(E). Then $L\text{-}Lip_p(E) = Lip_p(E)$.

Proof: Let f $\in L\text{-}Lip_p(E)$ and let $U_f(0)$ be the neighborhood of 0 on which f is Lipschitz. Choose r > 0 such that $B(0,r) \subseteq U_f(0)$. Suppose x, y $\in$ E, then let $s = \|x\| + \|y\|$ + 1 and let $u = \frac{r}{s}x$ and $v = \frac{r}{s}y$. Thus, u, v $\in$ B(0,r) and
$\|f(x) - f(y)\| = \frac{s}{r}\|f(u) - f(v)\| \leq \frac{s}{r} K(U_f(0))\|u - v\|$
$= K(U_f(0))\|x - y\|$. Therefore, f $\in Lip_p(E)$ and $L\text{-}Lip_p(E)$

$= Lip_p(E)$. □

<u>3.8 Lemma.</u> Let E be a normed linear space and let u, v $\in$ E, u $\neq$ v. If $\lambda[0,1] = \ell$ is the line segment from u to v and $S(x,r) \cap \ell \neq \emptyset$, then there exists t_0 and t_1, $0 \leq t_0 < t_1 \leq 1$, such that $\lambda([t_0,t_1]) \subseteq \overline{S(x,r)} = B(x,r)$. If $v \notin B(x,r)$, then $\lambda(t_1) \in B(x,r) - S(x,r)$ and $\lambda(t) \notin B(x,r)$ for all t, $t_1 < t \leq 1$.

Proof: Let t_0, t_1 be the minimum and maximum, respectively, of the closed bounded set $\lambda^{-1}(B(x,r) \cap \ell)$. By the convexity of $B(x,r)$, $\lambda([t_0,t_1]) \subseteq B(x,r)$.

Assume $v \notin B(x,r)$. Then, since $\lambda(1) = v$, $t_1 < 1$. By the maximum condition of t_1, $\|\lambda(t_1) - x\| = r$ and $\lambda(t) \notin B(x,r)$ for all t, $t_1 < t \leq 1$. □

<u>3.8 Proposition.</u> Let E be a finite dimensional normed linear space. Then L-Lip(E) = B-Lip(E).

Proof: By Proposition 3.6, B-Lip(E) $\subseteq$ L-Lip(E); hence, we need only show L-Lip(E) $\subseteq$ B-Lip(E).

Let f $\in$ L-Lip(E) and let B(0,r) be any closed sphere about 0. For each $x \in B(0,r)$, there exists ε_x such that f is Lipschitz on $B(x,\varepsilon_x)$ and, since B(0,r) is compact in a finite dimensional normed space, there exists x_i, i = 1,2,$\cdots$,n, such that $B(0,r) \subseteq \bigcup_{i=1}^{n} S(x_i,\varepsilon_{x_i})$. For convenience, let $S(x_i,\varepsilon_{x_i}) = S_i$, $B(x_i,\varepsilon_{x_i}) = B_i$, and $K(B_i) = K_i$.

For any u, v $\in$ B(0,r), if u, v belong to the same set B_i, then $\|f(u) - f(v)\| \leq K_i \|u - v\| \leq K \|u - v\|$ where $K = n \max\{K_i \mid i = 1,2,\cdots,n\}$. If $u \in S_i$ and $v \notin B_i$, then assume, without loss of generality, that $i = 1$ and consider the line segment ℓ from u to v defined by $\lambda(t) = (1-t)u + tv$. By Lemma 3.8, there exists t_1, $0 < t_1 < 1$, such that $\lambda([0,t_1]) \subseteq B_1$ and $\lambda(t_1) \in B_1 - S_1$. Therefore, $\lambda(t_1)$ belongs to another S_i, say S_2. By continuity of λ, there exists $t < t_1$ such that $\lambda(t) \in S_2$ and, by the above, $\lambda(t) \in S_1$; therefore, $(S_1 \cap S_2) \cap \ell \neq \emptyset$. Let $z_1 = \lambda(t_1) \in (B_1 \cap B_2) \cap \ell$.

If $v \notin B_2$, we can then apply the above procedure to z_1 and v and get t_2, $t_1 < t_2 < 1$, such that $\lambda([t_1,t_2]) \subseteq B_2$ and $\lambda(t_2) \in B_2 - S_2$. Also, $\lambda(t_2)$ must belong to another S_i, say S_3. By the same argument as above, $(S_3 \cap S_2) \cap \ell \neq \emptyset$ and $z_2 = \lambda(t_2) \in (B_2 \cap B_3) \cap \ell$.

We may continue this process as long as $v \notin B_k$ for the k^{th} step. However, there exists only a finite number of distinct sets B_k; hence, after m applications, $m \leq n$, we must have $v \in S_m \subseteq B_m$. We have constructed the points $0 = t_o < t_1 < t_2 < \cdots < t_m = 1$, where $\lambda(t_o) = z_o = u$, $\lambda(t_1) = z_1, \cdots, \lambda(t_m) = z_m = v$ all belong to ℓ and z_{k-1}, $z_k \in B_k$ for $k = 1,2,\cdots,m$.

For any k, $k = 0, 1,\cdots,m-1$, $\|z_k - z_{k+1}\| = \|\lambda(t_k) - \lambda(t_{k+1})\| = \|(1 - t_k)u + t_k v - (1-t_{k+1})u - t_{k+1}v\| = |t_{k+1} - t_k| \, \|u - v\| \leq \|u - v\|$. Thus, $\|f(u) - f(v)\| = \|f(u) - f(z_1) + f(z_1) + \cdots - f(z_{m-1}) + f(z_{m-1}) - f(v)\|$ $\leq \|f(u) - f(z_1)\| + \|f(z_1) - f(z_2)\| + \cdots + \|f(z_{m-1}) - f(v)\|$ $\leq K_1 \|u - z_1\| + K_2 \|z_1 - z_2\| + \cdots + K_m \|z_{m-1} - v\| \leq K\|u - v\|$, where $K = n \max\{K_i \mid i = 1,2,\cdots,n\}$. Therefore, $\|f(u) - f(v)\| \leq K\|u - v\|$ for all u, $v \in B(0,r)$ and $f \in$ B-Lip(E). $\square$

The preceding discussion was not dependent on the topological structure of the spaces involved. We now construct a sufficient collection of seminorms on B-Lip(E) and prove that B-Lip(E) is a right continuous near algebra in the topology determined by the seminorms.

3.9 Proposition. Let E be a normed linear space and let $f \in$ B-Lip(E). For each $r \in \mathbb{R}^+$, let $P_r(f)$ $= \inf\{K_r \mid \|f(u) - f(v)\| \leq K_r \|u - v\| \text{ for all } u, v \in B(0,r)\}$. Then the collection $\{P_r \mid r \in \mathbb{R}^+\}$ forms a sufficient system of seminorms on B-Lip(E) and defines a locally convex linear topology on the linear space structure of B-Lip(E).

Proof: Let $r \in \mathbb{R}^+$. If $f \in$ B-Lip(E), then f is Lipschitz on $B(0,r)$ and there exists $K_r = K_f(B(0,r)) \geq 0$ such that $\|f(u) - f(v)\| \leq K_r \|u - v\|$ for all $u, v \in B(0,r)$. Thus, $P_r(f)$ is well defined and $P_r(f) \geq 0$.

If $\|f(u) - f(v)\| > P_r(f)\|u - v\|$ for some u, $v \in B(0,r)$, $u \neq v$, then $P_r(f)\|u - v\| < K_r\|u - v\|$ for all K_r and; hence, $P_r(f) < K_r$ for all K_r. Therefore, $\|f(u) - f(v)\| \leq P_r(f)\|u - v\|$ for all $u, v \in B(0,r)$.

Suppose $f, g \in$ B-Lip(E) and $u, v \in B(0,r)$. $\|(f + g)u - (f + g)v\| \leq \|f(u) - f(v)\| + \|g(u) - g(v)\|$ $\leq (P_r(f) + P_r(g))\|u - v\|$; thus, $P_r(f + g) \leq P_r(f) + P_r(g)$. If t is a scalar, then $\|(tf)(u) - (tf)(v)\| = |t|\,\|f(u) - f(v)\|$ $\leq |t|P_r(f)\|u - v\|$ for all $u, v \in B(0,r)$. Thus, $P_r(tf)$ $\leq |t|P_r(f)$. Similarly, for $t \neq 0$, $\|f(u) - f(v)\|$ $= \frac{1}{|t|}\|(tf)(u) - (tf)(v)\| \leq \frac{1}{|t|}P_r(tf)\|u - v\|$. Thus, $|t|P_r(f) \leq P_r(tf)$ for $t \neq 0$ and, clearly, $|t|P_r(f) \leq P_r(tf)$ for $t = 0$. Therefore, P_r is a seminorm for each $r \in \mathbb{R}^+$.

If $f \in$ B-Lip(E) and $f \neq 0$, then choose $x \in E$, $x \neq 0$, such that $f(x) \neq 0$ and choose $r \geq \|x\|$. Thus, $0 < \|f(x)\| = \|f(x) - f(0)\| \leq P_r(f)\|x\|$. Therefore, $P_r(f) > 0$ and the collection of seminorms is sufficient.

Naimark [26] shows that such a system defines a locally convex linear topology on the linear space structure of B-Lip(E). //

<u>3.8 Definition.</u> Let E be a normed linear space and let $f \in$ B-Lip(E). For $r \in \mathbb{R}^+$ define $P_r(f) = \inf\{K_r \mid \|f(u) - f(v)\| \leq K_r\|u - v\|$ for all $u, v \in B(0,r)$ to be <u>the Lipschitz seminormed determined by r.</u>

<u>3.9 Lemma.</u> Let E be a normed linear space and let f, $g \in$ B-Lip(E). For $r \in \mathbb{R}^+$, if $P_r(g) \neq 0$, then $P_r(fg) \leq P_{rP_r(g)}(f)P_r(g)$ and if $P_r(g) = 0$, then $P_r(fg) = 0$.

Proof: Let f, $g \in$ B-Lip(E) and let $r \in \mathbb{R}^+$. If $u \in B(0,r)$, then $\|g(u) - g(0)\| \leq P_r(g)\|u\| \leq rP_r(g)$ and $g(B(0,r)) \subseteq B(0,rP_r(g))$ whenever $P_r(g) \neq 0$. For $u, v \in B(0,r)$, $\|(fg)(u) - (fg)(v)\| \leq P_{rP_r(g)}(f)\|g(u) - g(v)\| \leq P_{rP_r(g)}(f)P_r(g)\|u - v\|$.

If $P_r(g) = 0$, then $\|g(u) - g(0)\| \leq rP_r(g) = 0$ and $g(u) = g(0)$ for all $u \in B(0,r)$. Therefore, $\|f(g(u)) - f(g(v))\| = \|f(g(0)) - f(g(0))\| = 0$ and $P_r(fg) = 0$. //

<u>3.10 Proposition.</u> Let E be a normed linear space. Then B-Lip(E) is a right continuous near algebra in the locally convex linear topology determined by the sufficient collection of seminorms $\{P_r \mid r \in \mathbb{R}^+\}$.

Proof: Let f be an arbitrary but fixed element of B-Lip(E) and let U(0) be an arbitrary neighborhood of 0 in the topology of B-Lip(E) determined by $\varepsilon > 0$ and $r_j \in \mathbb{R}^+$, $j = 1,2,\cdots,n$. $U(0) = \{h | P_{r_j}(h) < \varepsilon \ \ j = 1,2,\cdots,n\}$.

If $P_{r_j}(f) = 0$ for all j, then, by Lemma 3.9, $P_{r_j}(gf) = 0 < \varepsilon$ for all $g \in$ B-Lip(E). Thus, for any neighborhood of 0, if $g \in V$, then $gf \in U(0)$.

If $P_{r_j}(f) \neq 0$ for some j, then let $J = \{j | j \in \mathbb{N}$ and $P_{r_j}(f) \neq 0\}$ and let $M = \max\{P_{r_j}(f) | j \in J\}$. Let $\delta = \varepsilon/M$ and let $s_j = r_j P_{r_j}(f)$ for $j \in J$. Then, $V(0) = \{h / P_{s_j}(h) < \delta \ \ j \in J\}$ is a neighborhood of 0 in B-Lip(E) and, for $g \in V(0)$, $P_{r_j}(gf) \leq P_{r_j P_{r_j}(f)}(g) P_{r_j}(f) = P_{s_j}(g) P_{r_j}(f) < \delta M \leq \varepsilon$. Thus, right multiplication is continuous at 0 and this is sufficient for continuity on all of B-Lip(E) since right multiplication is a linear function. ▯

Previously we have shown that if $f \in$ B-Lip(E) and f if positive homogeneous, then $f \in$ Lip(E). We now show that if $f \in$ B-Lip(E) and $P_r(f)$ is bounded for all $r \in \mathbb{R}^+$, then $f \in$ Lip(E).

<u>3.11 Proposition.</u> Let E be a normed linear space and let $f \in$ B-Lip(E). Then $f \in$ Lip(E) and the Lipschitz norm of f equals $\sup\{P_r(f) | r \in \mathbb{R}^+\}$ if and only if $\sup\{P_r(f) | r \in \mathbb{R}^+\}$ is finite.

Proof: Let $f \in$ B-Lip(E). If $f \in$ Lip(E), then, clearly, $\sup\{P_r(f) | r \in \mathbb{R}^+\} \leq \|f\|$ is finite.

Conversely, assume $\sup\{P_r(f) \mid r \in \mathbb{R}^+\} = M$ is finite. For u, $v \in E$, there exists an $r \in \mathbb{R}^+$ such that u, $v \in B(0,r)$ and; thus, $\|f(u) - f(v)\| \leq P_r(f)\|u - v\| \leq M\|u - v\|$. Therefore, $f \in \mathrm{Lip}(E)$ and $\|f\| \leq M$. Let $r \in \mathbb{R}^+$ and u, $v \in B(0,r)$, then $\|f(u) - f(v)\| \leq \|f\|\,\|u - v\|$. Thus, $P_r(f) \leq \|f\|$ for all $r \in \mathbb{R}^+$ and $\sup\{P_r(f) \mid r \in \mathbb{R}^+\} = \|f\|$. ▯

The final result of this section compares the relative linear topology of B-Lip(E) on Lip(E) with the Lipschitz norm topology of Lip(E).

<u>3.10 Lemma.</u> Let E be a normed linear space and let U and V be subsets of E. If $f \in T_C(E)$ is Lipschitz on U and on V, then f is Lipschitz on $U \cup V$ whenever the line segment from u to v intersects $U \cap V$ for all $u \in U$ and $v \in V$, $u \neq v$.

Proof: Let K_1 and K_2 be the Lipschitz constants for f on U and V, respectively, and let $K = \max\{K_1, K_2\}$. Let x, $y \in U \cup V$. Clearly, if x, $y \in U$ or x, $y \in V$, then $\|f(x) - f(y)\| \leq K\|x - y\|$.

If $x \in U$ and $y \in V$, $x \neq y$, then there exists $z \in U \cap V \cap \ell$ where ℓ is the line segment from x to y. For some t, $0 < t < 1$, $z = (1-t)x + ty$ and $\|f(x) - f(y)\|$ $= \|f(x) - f(z) + f(z) - f(y)\| \leq K_1\|x - z\| + K_2\|z - y\|$ $= K(t\|x - y\| + (1-t)\|x - y\|) = K\|x - y\|$. Therefore, f is Lipschitz on $U \cup V$ where $\|f(x) - f(y)\| \leq K\|x - y\|$ for all x, $y \in U \cup V$. ▯

<u>3.12 Proposition.</u> Let E be a normed linear space and consider Lip(E) as a sub near algebra of B-Lip(E). The relative topology on Lip(E) determined by the collection $\{P_r \mid r \in \mathbb{R}^+\}$ of seminorms on B-Lip(E) is strictly weaker than the Lipschitz

norm topology of Lip(E).

Proof: Let τ be the norm topology and let τ' be the relative topology. Let $U(0) = \{f \mid f \in \text{Lip}(E),\ P_{r_j}(f) < \varepsilon\ ,\ j = 1,2,\cdots,n\}$ be an arbitrary relative τ'-basic open neighborhood of 0. Then $V(0) = \{f \mid f \in \text{Lip}(E)\ \|f\| < \varepsilon\}$ is a τ-basic open neighborhood of 0 and if $f \in V(0)$, then $P_{r_j}(f) \leq \|f\| < \varepsilon$ for all $j = 1,2,\cdots,n$. Thus $V(0) \subseteq U(0)$ and $U(0)$ is τ-open set $\tau' \leq \tau$.

Let $V(0) = \{f \mid f \in \text{Lip}(E)\ \|f\| < \varepsilon\}$ be an arbitrary τ-open neighborhood of 0. We will show that there is no τ'-open neighborhood contained in V(0). Let U(0) = $\{f \mid f \in \text{Lip}(E),\ P_{r_j}(f) < \delta\ \ j = 1,2,\cdots,n\}$ be an arbitrary but fixed τ'-open neighborhood of 0. Let $r = \max\{r_1, r_2, \cdots r_n, 2\}$ and choose $t > \max\left\{\frac{\varepsilon r^2}{\delta},\ r\right\}$. Define the function $f : E \to E$ be the formula

$$f(x) = \begin{cases} \frac{\delta}{r}x & ,\ \text{if } x \in B(0,r) \\ \frac{\delta}{r^2}\|x\|\, x, & \text{if } x \in B(0,t) - S(0,r) \\ \frac{\delta t}{r^2}x & ,\ \text{if } x \in E - S(o,t) \end{cases}$$

If $u, v \in B(0,r)$, then $\|f(u) - f(v)\| = \frac{\delta}{r}\|u - v\|$ and f is Lipschitz on B(0,r). Also, if $u, v \in B(0,t) - S(0,r)$, then $\|f(u) - f(v)\| = \frac{\delta}{r^2}\|\|u\|u - \|v\|v\|$

$$= \frac{\delta}{r^2}\|\|u\|u - \|v\|u + \|v\|u - \|v\|v\|$$

$$\leq \frac{\delta}{r^2}(\|u\| + \|v\|)\|u - v\| \leq \frac{2\delta t}{r^2}\|u - v\|.$$ Thus, f is

Lipschitz on $B(0,t) - S(0,r)$.

Let $C(0,r) = B(0,r) - S(0,r)$, then $B(0,r) \cap (B(0,t) - S(0,r)) = C(0,r) \neq \emptyset$ and, for $u \in B(0,r)$ and $v \in B(0,t) - S(0,t)$, the line segment ℓ from u to v must intersect $C(0,r)$. Thus, by Lemma 3.10, f is Lipschitz on $B(0,t) = B(0,r) \cup (B(0,t) - S(0,r))$.

If u, $v \in E - S(0,t)$, then $\| f(u) - f(v) \| = \frac{\delta t}{r^2} \| u - v \|$. Thus, f is Lipschitz on $E-S(0,t)$. By an argument similar to the above, f is Lipschitz on $E = (E - S(0,t)) \cup B(0,t)$.

For arbitrary j, $1 \leq j \leq n$, $B(0,r_j) \subseteq B(0,r)$; thus, $\| f(u) - f(v) \| = \frac{\delta}{r} \| u - v \|$ for all u, $v \in B(0,r_j)$. Therefore, $P_{r_j}(f) < \delta$ for all j and $f \in U(0)$.

However, if $\| u \| = t$, then $\| f(u) - f(0) \| = \frac{\delta}{r^2} t \| u \| \leq \| f \| \| u \|$. Thus, $\| f \| \geq \frac{\delta t}{r^2}$ and $f \notin V(0)$. Therefore, $U(0) \nsubseteq V(0)$ and, since U(0) was arbitrary, there is no τ'-basic open neighborhood of 0 contained in V(0). Thus, $\tau' \subsetneq \tau$. //

CHAPTER IV

FINITE DIMENSIONAL FUNCTION NEAR ALGEBRAS

The Determination of Sub Near Algebras of $T_C(\mathbb{R}^n)$

In section three of Chapter II we considered finite dimensional normed near algebras and generalized the basis properties of $Lip_p(\mathbb{R})$. We now use the properties of $Lip_p(\mathbb{R})$ and the basis elements J and K to investigate the finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$, the continuous functions on $\mathbb{R}^n$.

On $\mathbb{R}$ a complete characterization of the finite dimensional sub near algebras of $T_C(\mathbb{R})$ is possible. We first determine, up to an isomorphic equivalence, all the one-dimensional sub near algebras of $T_C(\mathbb{R})$. Every one-dimensional sub near algebra of $T_C(\mathbb{R})$ has the form $N_{(a,b)} = \langle aJ + bK \rangle$ where (a,b) is any ordered pair of real numbers such that $a \neq 0$ or $b \neq 0$. If $(a,b) = s(c,d)$, then $N_{(a,b)} = N_{(c,d)}$ and if $ad = bc$ or $ac = bd$, then $N_{(a,b)} \cong N_{(c,d)}$. Otherwise, we have non-isomorphic one-dimensional sub near algebras of $T_C(\mathbb{R})$. Thus, unlike the case for an algebra, we can show there is an uncountable number of non-isomorphic one-dimensional near algebras.

We then show that every one-dimensional near algebra is isomorphic to a sub near algebra of $T_o(\mathbb{R})$ or has trivial multiplication. In addition, if multiplication is left continuous, then every one-dimensional near algebra is isomorphic to a sub near algebra of $T_C(\mathbb{R})$. In this case, the multiplication of such a one-dimensional near algebra is completely determined. Using the positive homogeneity of J and K, we show that left continuity of multiplication and positive homogeneity are equivalent in the one-dimensional case.

Higher dimensional sub near algebras of $T_C(\mathbb{R})$ are not as "numerous" as the one-dimensional near algebras. In fact, we show that $Lip_p(\mathbb{R})$ is the unique two-dimensional sub near algebra of $T_C(\mathbb{R})$ and there are no n-dimensional sub near algebras of $T_C(\mathbb{R})$ for $n \geq 3$.

Motivated by the characterization of finite dimensional sub near algebras of $T_C(\mathbb{R})$, we investigate those of $T_C(\mathbb{R}^n)$ for $n \geq 2$. The results are inconclusive for arbitrary sub near algebras of $T_C(\mathbb{R}^n)$, since, in general, there exist finite dimensional sub near algebras of all dimensions. However, if we restrict our attention to finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ which contain $\mathcal{B}(\mathbb{R}^n)$, all the linear functions on $\mathbb{R}^n$, then we arrive at the principle result of this paper: There is no finite dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which properly contains $\mathcal{B}(\mathbb{R}^n)$ for $n \geq 2$.

We now begin the discussion of the one-dimensional near algebras. Unless otherwise stated, we will assume all near algebras are near-c-algebras. In particular, for function algebras we will assume $f(0) = 0$.

In Chapter I we showed that $T_o(V)$, the set of all functions on the linear space V into V such that $f(0) = 0$, is a near algebra with composition as the multiplication. If V is a linear topological space, let $T_C(V)$ $= \{f \mid f \in T_o(V)$ and f is continuous$\}$. $T_C(V)$ has been shown to be a sub near algebra of $T_o(V)$. We wish to discuss the finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ and begin with an important theorem which completely characterizes the one-dimensional sub near algebras of $T_C(\mathbb{R})$.

<u>4.1 Theorem.</u>

i) Every one-dimensional sub near algebra of $T_C(\mathbb{R})$ is of the form: $N_{(a,b)} = \langle aJ + bK \rangle$ for some ordered pair of real numbers (a,b) where J and K are the constructed basis elements of $Lip_p(\mathbb{R})$.

ii) $N_{(a,b)} = N_{(c,d)}$ if and only if $(a,b) = (sc, sd)$ for some $s \in \mathbb{R}$, $s \neq 0$.

iii) $N_{(a,b)}$ is a one-dimensional near algebra if and only if $a^2 + b^2 \neq 0$ and $ab \leq 0$ or $a = b$. $N_{(a,b)}$ is a one-dimensional algebra if and only if $a = b$. In particular, $N_{(a,a)} = N_{(1,1)} = \langle I \rangle$ and $N_{(1,-1)} = \langle A \rangle$ where $A(x) = |x|$.

iv) $N_{(a,b)} \cong N_{(c,d)}$ if and only if $ad = bc$ or $ac = bd$ and $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$.

Proof: i) Let N be a one-dimensional sub near algebra of $T_C(\mathbb{R})$ and, for convenience in notation, let $\mathbb{R}^+$ $= \{x \mid x \in \mathbb{R}$ and $x > 0\}$ and $\mathbb{R}^- = \{x \mid x \in \mathbb{R}$ and $x < 0\}$. Also, for any $f \in N$, let R(f) denote the range of f.

Let $f \in N$, $f \neq 0$, such that $N = \langle f \rangle$. Let g be an arbitrary element of N, $g \neq 0$. Then, $f = tg$ and $fg = sf$ for some $t, s \in \mathbb{R}$ both of which depend on g. Therefore,

$fg = sf = s(tg) = ag$ where $a = st$ depends on g. Thus, for each $g \in N$, $g \neq 0$, there exists a constant $a = a(g)$ such that $f(x) = ax$ for $x \in R(g)$.

We first consider the case in which there exists $g \in N$ such that $R(g) \cap \mathbb{R}^+ \neq \emptyset$ and $R(g) \cap \mathbb{R}^- \neq \emptyset$. Let $u \in R(g) \cap \mathbb{R}^+$ and $v \in R(g) \cap \mathbb{R}^-$ and let $t = \min\{|u|, |v|\}$. Then, since g is continuous, $[-t, t] \subseteq R(g)$ and, by the above, there exists $a = a(g) \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in [-t,t]$. For $k \in \mathbb{N}$, $kg \in N$ and there exists $a(kg) \in \mathbb{R}$ such that $f(x) = a(kg)x$ for all $x \in [-kt,kt] \subseteq R(kg)$. Since $[-t,t] \subseteq [-kt,kt]$ for all $k \in \mathbb{N}$, $f(t) = at = a(kg)t$ and, thus, $a = a(kg)$ for all $k \in \mathbb{N}$. If $x \in \mathbb{R}$, then $x \in [-kt,kt]$ for some $k \in \mathbb{N}$ and $f(x) = a(kg)x = ax$. Therefore, in this case, $f = aJ + aK$ and $N = \langle aJ + aK \rangle$.

Assume for all $g \in N$ that either $R(g) \cap \mathbb{R}^+ = \emptyset$ or $R(g) \cap \mathbb{R}^- = \emptyset$. Choose $g \in N$ such that $R(g) \cap \mathbb{R}^+ \neq \emptyset$ and $R(g) \cap \mathbb{R}^- = \emptyset$. Then, there exists $u \in \mathbb{R}$ such that $g(u) = v > 0$ and $-g(u) = -v < 0$. Since g is continous and $g(0) = 0$, $[0,v] \subseteq R(g)$ and $[-v,0] \subseteq R(-g)$. As in the first case, there exists $a = a(g)$ and $b = b(-g)$ such that $f(x) = ax$ for $x \in [0,v]$ and $f(x) = bx$ for $x \in [-v,0]$. Let k be an arbitrary natural number. There exists $a(kg)$, $b(-kg) \in \mathbb{R}$ such that $f(x) = a(kg)x$ for $x \in [0,kv]$ and $f(x) = b(-kg)x$ for $x \in [-kv,0]$. Thus, using v and $-v$, we have that $a = a(kg)$ and $b = b(-kg)$ for all $k \in \mathbb{N}$. If $x \in \mathbb{R}^+$, then $x \in [0,kv]$ for some $k \in \mathbb{N}$ and $f(x) = a(kg)x = ax$ and, if $x \in \mathbb{R}^-$, then $x \in [-nv,0]$ for some $n \in \mathbb{N}$ and $f(x) = b(-ng)x = bx$. Thus, $f = aJ + bK$ and $N = \langle aJ + bK \rangle$. This completes the proof of part i).

ii) If $N_{(a,b)} = N_{(c,d)}$ then $aJ + bK = s(cJ + dK)$ for some $s \in \mathbb{R}$, $s \neq 0$. Therefore, $a = sc$ and $b = sd$ and $(a,b) = (sc, sd)$. Conversely, if $(a,b) = (sc, sd)$, $s \neq 0$, then, for $f \in N_{(a,b)}$, $f = t(aJ + bK) = ts(cJ + dK)$ and $f \in N_{(c,d)}$. Similarly, if $g \in N_{(c,d)}$, then $g = t(cJ + dK)$ $= \frac{t}{s}(scJ + sdK) = \frac{t}{s}(aJ + bK) \in N_{(a,b)}$.

We prove iii) by considering the various choices for a and b. Either $a = b$ or $a \neq b$ and, in the latter case, we need only consider the cases $a > 0$, $b > 0$ or $a > 0$, $b < 0$ or $ab = 0$ since $N_{(a,b)} = N_{(-a,-b)}$.

If $a = b$, then $N_{(a,b)} = N_{(a,a)} = N_{(1,1)} = \langle I \rangle$ is the algebra of linear functions on $\mathbb{R}$. Also, if $N_{(a,b)}$ is not an algebra, then $a \neq b$ and $ab < 0$.

If $a \neq b$ and $a > 0$, $b > 0$, then, using the properties of J and K, we have that $(aJ + bK)(aJ + bK) = a^2J + b^2K$ $= saJ + sbK$ for some $s \in \mathbb{R}$. Thus, $a = s = b$. Therefore, if $a \neq b$ and $ab \neq 0$, then $a > 0$, $b < 0$. Again, using the properties of J and K, we get $(aJ + bK)(aJ + bK) = a^2J - abK$ and $(aJ + bK)(-(aJ + bK)) = -abJ - b^2K$. If $N_{(a,b)}$ is assumed to be an algebra, then $a^2J - abK = -(-abJ - b^2K)$ and we have that $a(a-b) = 0$ and $-b(a+b) = 0$. Since the only solution is $a = b = 0$, $N_{(a,b)}$ is not an algebra. Finally, if $ab = 0$, then either $a = 0$ or $b = 0$ and $N_{(a,0)} = \langle aJ \rangle$ or $N_{(0,b)}$ $= \langle bK \rangle$, both of which are not algebras. Thus, if N is an algebra, then $a = b$ and, if $ab \leq 0$ then $a \neq b$ and N is not an algebra.

iv) Assume $N_{(a,b)} \cong N_{(c,d)}$. If $N_{(a,b)}$ is an algebra, then $N_{(c,d)}$ is an algebra and a = b and c = d. Thus ac = bd and ad = bc. If $N_{(a,b)}$ is not an algebra, then $N_{(c,d)}$ is not an algebra and we may assume $a \geq 0$, $b \leq 0$ and $c \geq 0$, $d \leq 0$. Let φ be the isomorphism from $N_{(a,b)}$ onto $N_{(c,d)}$ and let f = aJ + bK and g = cJ + dK be the respective basis elements.

By checking the various cases and using the properties of J and K and the fact that a, c ≥ 0 and b, d ≤ 0, one can show that

$$(tf)(sf) = \begin{cases} tasf, & \text{for } s \geq 0 \\ tbsf, & \text{for } s < 0 \end{cases} \quad \text{and} \quad (tg)(sg) = \begin{cases} tcsg, & \text{for } s \geq 0 \\ tdsg, & \text{for } s < 0 \end{cases}$$

for all $t \in \mathbb{R}$. For example, if $s \geq 0$, (tf)(sf)
$= t[aJ(saJ) + aJ(sbK) + bK(saJ) + bK(sbK)]$
$= t[asaJ + a(-sb)(-K)] = tasf.$

For the isomorphism φ there exists $t \in \mathbb{R}$, $t \neq 0$, such that $\varphi(f) = tg$. Using the above formulas, if $t > 0$, then $\varphi(ff) = \varphi(af) = atg = \varphi(f)\varphi(f) = (tg)(tg) = t^2cg$. Thus, a = tc. Also $\varphi(f(-f)) = \varphi(bf) = -tbg = \varphi(f)\varphi(-f)$ $= (tg)(-tg) = -t^2dg$. Thus, b = td. Therefore, ad = tcd = bc. If $t < 0$, then a similar argument shows that ac = bd.

Conversely, assume ad = bc or ac = bd and $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$. If a = b, then c = d and, by ii) and iii), $N_{(a,b)} = N_{(1,1)} = N_{(c,d)}$. If $a \neq b = 0$, then d = 0 or c = 0. If d = 0, then $c \neq 0$ and $N_{(a,0)} = N_{\frac{c}{a}(a,0)} = N_{(c,0)}$. If c = 0, then $d \neq 0$ and $N_{(a,0)} = N_{(1,0)} = \langle J \rangle$ while $N_{(0,d)} = N_{(0,1)}$ $= \langle K \rangle$. Define $\psi: N_{(1,0)} \to N_{(0,1)}$ by $\psi(sJ) = sK$. Clearly, ψ is a linear space isomorphism and

$$\Psi((sJ)(tJ)) = \begin{cases} \Psi(stJ) \\ \Psi(0) \end{cases} = \begin{cases} stK, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases} = (sK)(tK)$$

$= \Psi(sJ)\Psi(tJ)$. Therefore, $N_{(1,0)} \cong N_{(0,1)}$. A similar argument holds when $0 = a \neq b$ and $d = 0$ or $c = 0$.

Finally, if $a \neq 0$, $b \neq 0$ and $a \neq b$, then $c \neq 0$, $d \neq 0$ and $c \neq d$ and we may assume $a > 0$, $c > 0$ and $b < 0$, $d < 0$. First, we will consider the case $ad = bc$. Let $k = a/c = b/d > 0$ and define $\varphi: N_{(a,b)} \rightarrow N_{(c,d)}$ by $\varphi(f) = kg$ where $f = aJ + bK$ and $g = cJ + dK$ are the respective basis elements. Extend φ linearly so that φ becomes a linear space isomorphism of $N_{(a,b)}$ onto $N_{(c,d)}$.

$$\varphi((tf)(sf)) = \begin{cases} ktasg, & \text{for } s \geq 0 \\ ktbsg, & \text{for } s < 0 \end{cases} \quad \text{while } \varphi(tf)\varphi(sf)$$

$$= (ktg)(ksg) = k^2(tg)(sg)$$

$$= \begin{cases} k^2tcsg \\ k^2tdsg \end{cases} = \begin{cases} (a^2/c^2)tcsg \\ (b^2/d^2)tdsg \end{cases} = \begin{cases} (a/c)tasg \\ (b/d)tbsg \end{cases} = \begin{cases} ktasg, & \text{for } s \geq 0 \\ ktbsg, & \text{for } s < 0 \end{cases}.$$

Therefore, φ is a near algebra isomorphism and $N_{(a,b)} \cong N_{(c,d)}$.

If $ac = bd$, then let $w = a/b = d/c < 0$ and let $\mu: N_{(a,b)} \rightarrow N_{(c,d)}$ be defined by $\mu(f) = wg$ where $f = aJ + bK$ and $g = cJ + dK$. The proof that μ is a near algebra isomorphism is similar to the above and the details are omitted. $\square$

Theorem 4.1 leads to several important results. It is well known that the only one-dimensional algebras over a field F are F itself and a one-dimensional linear space with trivial multiplication. The following discussion indicates the the one-dimensional near algebras are more "numerous".

Let $\mathcal{N}$ be the collection of all one-dimensional sub near algebras of $T_C(\mathbb{R})$ and define an equivalence relation on

$\mathcal{N}$ by the following: $N_{(a,b)} \sim N_{(c,d)}$ if and only if $N_{(a,b)} \cong N_{(c,d)}$. Let $\widetilde{\mathcal{N}}$ be the set of equivalence classes for $a \neq b$.

4.2 Theorem. There is a one-to-one correspondence from the closed interval $[-1,0]$ to the collection of equivalence classes $\widetilde{\mathcal{N}}$. In particular, there exists an uncountable number of non-isomorphic one-dimensional sub near algebras of $T_C(\mathbb{R})$.

Proof: For each $t \in [-1,0]$, define $\varphi(t) = \widetilde{N}_{(1,t)}$, the equivalence class determined by $N_{(1,t)}$. For $s,t \in [-1,0]$, if $\widetilde{N}_{(1,t)} = \widetilde{N}_{(1,s)}$, then $N_{(1,t)} \cong N_{(1,s)}$ and, by Theorem 4.1, $s = t$ or $1 = st$. However, $st = 1$, for $s, t \in [-1,0]$, implies $s = t = -1$. Therefore, φ is one-to-one.

Let $\widetilde{N}_{(a,b)} \in \widetilde{\mathcal{N}}$. If $a = 0$ or $b = 0$, then $N_{(a,b)} \cong N_{(1,0)}$ and $\varphi(0) = \widetilde{N}_{(1,0)} = \widetilde{N}_{(a,b)}$. If $a \neq 0$ and $b \neq 0$, then $N_{(a,b)} = N_{(1,\frac{b}{a})} \cong N_{(1,\frac{a}{b})}$. Assuming $-1 \leq \frac{b}{a} \leq 0$, then $\varphi(\frac{b}{a}) = \widetilde{N}_{(a,b)}$. Otherwise, $-1 \leq \frac{a}{b} \leq 0$ and $\varphi(\frac{a}{b}) = \widetilde{N}_{(a,b)}$. Thus, φ is one-to-one and onto. ∎

One should observe that there is only one equivalence class which is an algebra and its representative can be chosen as $N_{(1,1)}$.

4.1 Proposition. Every one-dimensional sub near algebra of $T_C(\mathbb{R})$ is a positive homogeneous near algebra.

Proof: The functions J and K are positive homogeneous; hence, any linear combination is positive homogeneous. Every one-dimensional sub near algebra of $T_C(\mathbb{R})$ has a basis in terms of J and K. ∎

<u>4.2 Proposition.</u> Every one-dimensional near algebra is isomorphic to a sub near algebra of $T_o(\mathbb{R})$ or is a one-dimensional linear space with trivial multiplication.

Proof: Let N be such a near algebra with basis element e. If $e(te) = 0$ for all $t \in \mathbb{R}$, then, for $x, y \in N$, $xy = (se)(te) = s(e(te)) = s0 = 0$ and the multiplication is trivial. Otherwise, there exists $t_o \in \mathbb{R}$, $t_o \neq 0$, such that $s(t_o e) = s_o e \neq 0$. Let $z = t_o e$, then $z^2 = s_o z$ and we choose z as a new basis element.

Define a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ by assigning to each $t \in \mathbb{R}$ the uniquely determined coefficient in the expression $z(tz) = f(t)z$. The function f is not identically zero since $f(1) = s_o \neq 0$ and $f(0) = 0$; hence, $f \in T_o(\mathbb{R})$. Let $\mathcal{C}_z = \langle f \rangle$, the linear space generated by f. We wish to show $\mathcal{C}_z$ is a sub near algebra of $T_o(\mathbb{R})$.

For each $t \in \mathbb{R}$, $(z(tz))z = (f(t)z)z = f(t)z^2 = z((tz)z) = z(tz^2)$; thus, $f(t)z^2 = z(tz^2)$ for all $t \in \mathbb{R}$. For all $s \in \mathbb{R}$, we must show $f(sf) = rf$ for some $r \in \mathbb{R}$. To this end, let $t \in \mathbb{R}$ and, since $z(sz) = rz$ for some $r \in \mathbb{R}$, we have $(z(sz))(tz^2) = z(sz(tz^2)) = z(sf(t)z^2) = f(sf(t))z^2$. Also $(z(sz))(tz^2) = rz(tz^2) = rf(t)z^2$. Since $z^2 \neq 0$, then $f(sf(t)) = rf(t)$ for all $t \in \mathbb{R}$; thus, $f(sf) = rf$ and $\mathcal{C}_z$ is a one-dimensional sub near algebra of $T_o(\mathbb{R})$. Note that $r = f(s)$; hence, $f(sf) = f(s)f$.

Define $\varphi(z) = f$ and extend φ linearly so that $\varphi(tz) = t\varphi(z) = tf$. φ is a linear isomorphism of N onto $\mathcal{C}_z$. Let $x = sz$ and $y = tz$ be elements of N. Then, $\varphi(xy) = \varphi((sz)(tz)) = \varphi(s(f(t)z)) = sf(t)f = (sf)(tf) = \varphi(x)\varphi(y)$. Therefore, is a near algebra isomorphism. □

<u>4.3 Theorem.</u> Every one-dimensional left continuous near algebra is isomorphic to a sub near algebra of $T_C(\mathbb{R})$ or has trivial multiplication.

Proof: Let N be such a near algebra. By Proposition 4.2 N has trivial multiplication or $N \cong \mathcal{C}_z$ where $\mathcal{C}_z$ is a one-dimensional sub near algebra of $T_0(\mathbb{R})$ and z is a basis element in N such that $z^2 \neq 0$. The basis element of $\mathcal{C}_z$ is a function f defined by $z(tz) = f(t)\,z$ for each $t \in \mathbb{R}$. We will show f is continuous, then $\mathcal{C}_z \subseteq T_C(\mathbb{R})$.

Since N is a finite dimensional space, its topology is given by a complete norm. Let $s \in \mathbb{R}$ and let $\varepsilon > 0$. Since multiplication is left continuous, there exists $\delta > 0$ such that $\| z(tz) - z(sz) \| < \varepsilon \| z \|$ whenever $\| tz - sz \| < \delta$. That is, $\| f(t)z - f(s)z \| = | f(t) - f(s) | \, \| z \| < \varepsilon \| z \|$ whenever $| t - s | \, \| z \| < \delta$. Thus, $| f(t) - f(s) | < \varepsilon$ whenever $| t - s | < \delta / \| z \|$ and f is continuous on $\mathbb{R}$ since s was arbitrary. □

In Chapter II we showed that positive homogeneity and an orthogonal idempotent basis lead to left continuity and a strong D-normed condition. In the one-dimensional case these are equivalent.

<u>4.4 Theorem.</u> A one-dimensional normed near algebra is positive homogeneous if and only if it is left continuous.

Proof: Let N be a one-dimensional normed near algebra. If N has trivial multiplication, then N is an algebra and multiplication is continuous. If N is positive homogeneous with nontrivial multiplication, then let e be a basis element such that $\|e\| = 1$. Let $b \in N$ and suppose $\{t_n e\}$ is a sequence in N such that $t_n e \rightarrow te$. For $\varepsilon > 0$, there exists n_o such that $\|t_n e - te\| = |t_n - t| < \varepsilon$ whenever $n > n_o$. Let $\theta(t)$ $= \operatorname{sgn}(t)$. If $t \neq 0$, then choose n_o sufficiently large so that $\theta(t) = \theta(t_n)$ and $\|b(t_n e) - b(te)\|$

$= \| |t_n| b(\theta(t)e - |t| b(\theta(t)e) \| \leq |t_n - t| \|b\| < \varepsilon \|b\|$

whenever $n > n_o$. If $t = 0$, the proof is similar except $\theta(t)$ is not needed. Therefore, in each case, $b(t_n e) \rightarrow b(te)$ and multiplication is left continuous.

Conversely, if multiplication is left continuous, then, by Theorem 4.3, N has trivial multiplication or is isomorphic to a one-dimensional sub near algebra of $T_C(\mathbb{R})$. In either case, using Proposition 4.1, N is positive homogeneous.▯

Also, as a consequence of Theorem 4.3, if N is a one-dimensional left continuous near algebra with basis $\{e\}$ such that $x = se$ and $y = te$ are arbitrary elements of N, then multiplication is characterized by one of the following:

i) $xy = 0$, ii) $xy = asy$ for some fixed $a \in \mathbb{R}$,

iii) $xy = \begin{cases} bsy, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$ for some fixed $b \in \mathbb{R}$, or

iv) $xy = \begin{cases} csy, & \text{if } t \geq 0 \\ dsy, & \text{if } t < 0 \end{cases}$ for some fixed $c, d \in \mathbb{R}$. The first two multiplications result in algebras while the last two correspond to $N_{(a,0)}$ and $N_{(a,b)}$ respectively and are near algebras.

Althought there are an uncountable number of non-isomorphic one-dimensional sub near algebras of $T_C(\mathbb{R})$, there exists a unique two-dimensional sub near algebra and no n-dimensional sub near algebras of $T_C(\mathbb{R})$ for $n \geq 3$. We proceed to this important result throught a series of lemmas.

<u>4.1 Lemma.</u> Let $n \in \mathbb{N}$ and let $f_i \in T_C(\mathbb{R})$ for $i = 1,2,\cdots,n$. If the set $\{f_i\}$ is linearly independent on $\mathbb{R}$, then there exists $a_n > 0$ such that $\{f_i\}$ is linearly independent on $[-a_n,a_n]$.

Proof: We will prove this by induction on n. For $n = 2$, assume f_1, f_2 are linearly independent on $\mathbb{R}$ and choose $x_1 \in \mathbb{R}$ such that $f_1(x_1) \neq 0$. Using the linear independence, $-f_2(x_1)f_1 + f_1(x_1)f_2 = 0$ can not hold for all $x \in \mathbb{R}$; thus, let $x_2 \in \mathbb{R}$ such that $- f_2(x_1)f_1(x_2) + f_1(x_1)f_2(x_2) \neq 0$. Choose $a \in \mathbb{R}$ such that $x_1,x_2 \in [-a,a]$ and assume $t_1f_1 + t_2f_2 = 0$ on $[-a,a]$. For x_1, $x_2 \in [-a,a]$ we have $t_1f_1(x_1) + t_2f_2(x_1) = 0$ and $t_1f_1(x_2) + t_2f_2(x_2) = 0$. The determinant of coefficients is $f_1(x_1)f_2(x_2) - f_1(x_2)f_2(x_1) \neq 0$; therefore, $t_1 = t_2 = 0$ and f_1, f_2 are linearly independent on $[-a,a]$.

Assume the statement holds for n and let $f_1,f_2\cdots,f_n,f_{n+1}$ be linearly independent on $\mathbb{R}$; but dependent on every closed bounded symmetric interval of $\mathbb{R}$. By the induction hypothesis, there exists $a_n \in \mathbb{R}$, $a_n > 0$, such that $f_1,f_2,\cdots,f_n$ are linearly independent on $[-a_n,a_n]$ and hence on $[-a,a]$ for all $a \geq a_n$. However, $f_1,f_2,\cdots,f_{n+1}$ are linearly dependent on $[-a_n,a_n]$; hence, there exist scalars $t_i \in \mathbb{R}$ such that $f_{n+1} = t_1f_1+\cdots+t_nf_n$ on $[-a_n,a_n]$. Similarly, for $a \geq a_n$, there exist scalars $t_i(a) \in \mathbb{R}$ such that $f_{n+1} = t_1(a)f_1+\cdots+t_n(a)f_n$ on $[-a,a]$. Therefore, on $[-a_n,a_n]$,

$t_1(a)f_1+\cdots+t_n(a)f_n = t_1f_1+\cdots+t_nf_n$ for all $a \geq a_n$. Since the f_i are linearly independent on $[-a_n,a_n]$, then $t_i(a) = t_i$ for all $a \geq a_n$ and $i = 1,2,,\cdots,n$.

Let $x \in \mathbb{R}$, then $x \in [-a,a]$ for some $a \geq a_n$. Therefore, $f_{n+1}(x) = t_1(a)f_1(x)+\cdots+t_n(a)f_n(x)$ $= t_1f_1(x)+\cdots+t_nf_n(x)$. This contradiction implies there exists $a_{n+1} > 0$ such that $f_1,f_2,\cdots,f_{n+1}$ are linearly independent on $[-a_{n+1},a_{n+1}]$. $\square$

<u>4.2 Lemma.</u> Let $n \in \mathbb{N}$, $n \geq 2$, and let $f_i \in T_C(\mathbb{R})$ for $i = 1,2,\cdots,n$. If the set $\{f_i\}$ is linearly independent on an interval S, then there exists a set of n elements in S, $\{x_1,x_2,\cdots,x_n\} \subseteq S$, such that $\det F_n \neq 0$ where F_n is the n square matrix whose ij^{th} entry is given by $f_j(x_i)$.

Proof: For $n = 2$, assume that $\det F_2 = 0$ for all pairs $x_1,x_2 \in S$. Let $x \in S$ be arbitrary but fixed and then, for all $y \in S$, $\det F_2 = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1(y) & f_2(y) \end{vmatrix} = f_1(x)f_2(y) - f_2(x)f_1(y) = 0.$ Thus, $f_1(x) = f_2(x) = 0$ for $x \in S$; hence, $f_1 = 0$ and $f_2 = 0$ on S. This contradiction implies there exist $x_1,x_2 \in S$ such that $\det F_2 \neq 0$.

Assume for any $n < k$ that, if $f_1,f_2,\cdots,f_n$ are linearly independent, then there exists a set of n elements $\{x_1,x_2,\cdots,x_n\} \subseteq S$ such that $\det F_n \neq 0$. For $n = k$, let $f_1,f_2,\cdots,f_k$ be linearly independent on S and suppose that, for all sets of k elements in S, $\det F_k = 0$. By the induction hypothesis, however, there exists a set of k-1 elements, $\{x_1,x_2,\cdots,x_{k-1}\}$ such that $\det F_{k-1} \neq 0$. Let x be an arbitrary element of S. Then, for the set

$\{x_1, x_2, \cdots, x_{k-1}, x\}$, det $F_k = 0$ where the k^{th} row of F_k is $(f_1(x)\ f_2(x) \ldots f_k(x))$. If we expand the k^{th} row by minors, we get $A_1 f_1(x) + A_2 f_2(x) + \cdots + A_k f_k(x) = 0$ where $A_i =$ det (minor of $f_i(x)$) and does not depend on x. Since x was arbitrary, $A_1 f_1 + A_2 f_2 + \cdots + A_k f_k = 0$ on S and since the f_i are linearly independent on S, $A_1 = A_2 = \cdots = A_k = 0$. But, $A_k = \det F_{k-1} \neq 0$ which is a contradiction. Therefore, for $n = k$, there exists a set of k elements such that det $F_k \neq 0$. ▯

In what follows, if $f \in T_C(\mathbb{R})$ let R(f) denote the range of f and let $\mathbb{R}^+ = \{x \mid x > 0\}$ and $\mathbb{R}^- = \{x \mid x < 0\}$.

<u>4.3 Lemma.</u> Let $n \in \mathbb{N}$, $n \geq 2$. If N is an n-dimensional sub near algebra of $T_C(\mathbb{R})$, then there exists a function $h \in N$ such that $R(h) \cap \mathbb{R}^+ \neq \emptyset$ and $R(h) \cap \mathbb{R}^- \neq \emptyset$.

Proof: Assume not, then for all $h \in N$, $R(h) \subseteq \mathbb{R}^+ \cup \{0\}$ or $R(h) \subseteq \mathbb{R}^- \cup \{0\}$. In particular, let f, g be basis elements such that $R(f) \subseteq \mathbb{R}^+ \cup \{0\}$ and $R(g) \subseteq \mathbb{R}^+ \cup \{0\}$ and choose $x, y \in \mathbb{R}$ such that $f(x) > 0$ and $g(y) > 0$.

If $f(y) = 0$, then let $t > g(x)/f(x)$ and let $h = tf - g$. Then, $h(x) = tf(x) - g(x) = f(x)(t - g(x)/f(x)) > 0$ and $h(y) = tf(y) - g(y) = -g(y) < 0$. Thus, $h \in N$ and $R(h) \cap \mathbb{R}^+ \neq \emptyset$ and $R(h) \cap \mathbb{R}^- \neq \emptyset$.

If $f(y) > 0$, then let $t = g(y)/f(y) > 0$ and let $h = tf$. Since $h \neq g$, there exists $z \in \mathbb{R}$ such that $h(z) \neq g(z)$. If $h(z) = 0$, then let $H = 2h - g$. $H(z) = -g(z) < 0$ and $H(y) = 2h(y) - g(y) = 2tf(y) - g(y) = g(y) > 0$. Therefore, $R(H) \cap \mathbb{R}^+ \neq \emptyset$ and $R(H) \cap \mathbb{R}^- \neq \emptyset$.

Assume $h(z) \neq 0$. Let $s = \dfrac{h(z) + g(z)}{2h(z)}$ and let

$H = sh - g$. Then, $H(y) = sh(y) - g(y) = stf(y) - g(y)$ $= (s - 1)g(y)$ and $H(z) = sh(z) - g(z) =$ $\frac{h(z) + g(z)}{2h(z)} h(z) - g(z) = \frac{h(z) - g(z)}{2}$. If $H(z) > 0$ then $h(z) > g(z) \geq 0$ and; hence, $1 > \frac{h(z) + g(z)}{2h(z)} = s$. Thus, $s - 1 < 0$ and $H(y) < 0$. If $H(z) < 0$, then $h(z) < g(z)$ and $1 < s$. Therefore, $s - 1 > 0$ and $H(y) > 0$. Finally, if $H(z) = 0$ then $h(z) = g(z)$ which is a contradiction. Therefore $R(H) \cap \mathbb{R}^+ \neq \emptyset$ and $R(H) \cap \mathbb{R}^- \neq \emptyset$.

In all cases we have constructed a function whose range intersects $\mathbb{R}^+$ and $\mathbb{R}^-$ which is a contradiction. $\square$

4.4 Lemma. Let $n \in \mathbb{N}$, $n \geq 2$. If N is an n-dimensional sub near algebra of $T_C(\mathbb{R})$ with basis $\{f_i\}$ such that $R(f_k) \cap \mathbb{R}^+ \neq \emptyset$ and $R(f_k) \cap \mathbb{R}^- \neq \emptyset$ for some k, then the identity function, I, belongs to N.

Proof: Let n be an arbitrary natural number, $n \geq 2$. By reordering the basis elements, we may assume $R(f_1) \cap \mathbb{R}^+ \neq \emptyset$ and $R(f_1) \cap \mathbb{R}^- \neq \emptyset$. By Lemma 4.1, there exists $a_n > 0$ such that $f_1, f_2, \cdots, f_n$ are linearly independent on $[-a_n, a_n]$. Since $R(f_1) \cap \mathbb{R}^+ \neq \emptyset$ and $R(f_1) \cap \mathbb{R}^- \neq \emptyset$, there exists $t > 0$ such that $[-a_n, a_n] \subseteq R(tf_1)$ and consequently $f_1, f_2, \cdots, f_n$ are linearly independent on $R(sf_1)$ for any $s \geq t$. For $s \geq t$, consider the equations $f_i(sf_1) = s_{i1}f_1 + s_{i2}f_2 + \cdots + s_{in}f_n$ for $i = 1, 2, \cdots, n$ and the linear transformation $\varphi_s(f) = f(sf_1)$ from N to N. φ_s has matrix representation $S(s) = (s_{ij})$ relative to the basis $\{f_i\}$ where the s_{ij} are the coefficients in the above equations. Suppose $\varphi_s(f) = \varphi_s(g)$, then $(f - g)(sf_1) = 0$. Thus, for $y = (sf_1)(x) \in R(sf_1)$,

$(f - g)(y) = (f - g)(sf_1)(x) = 0$. However, if $f - g = (t_1 - r_1)f_1 + (t_2 - r_2)f_2 + \cdots + (t_n - r_n)f_n = 0$ on $R(sf_1)$, then $t_i = r_i$, $i = 1,2,\cdots,n$, where $f = \sum t_i f_i$ and $g = \sum r_i f_i$. Therefore, $f = g$ and φ_s is one-to-one and $S(s)^{-1}$ exists. Thus, $\det S(s) \neq 0$, for all $s \geq t$.

Let $y = (sf_1)(x)$ be an arbitrary element of $R(sf_1)$. Then, $f_i(y) = f_i(sf_1)(x) = s_{i1}f_1(x) + \cdots + s_{in}f_n(x)$ for $i = 1,2,\cdots,n$. Solving this system of equations for $f_1(x)$ we get

$$f_1(x) = \frac{\begin{vmatrix} f_1(y) & s_{12} & \cdots & s_{1n} \\ \vdots & & & \\ f_n(y) & s_{n2} & \cdots & s_{nn} \end{vmatrix}}{\det S} = \frac{\det S_1}{\det S} f_1(y) + \cdots + \frac{\det S_n}{\det S} f_n(y)$$

where S_i is the minor of $f_i(y)$. Therefore $y = (sf_1)(x) = (\alpha_1(s)f_1 + \alpha_2(s)f_2 + \cdots + \alpha_n(s)f_n)(y)$ where $\alpha_i(s)$ is the scalar $\frac{s \det S_i}{\det S}$. Since $y \in R(sf_1)$ was arbitrary, the function $h_s = \alpha_1(s)f_1 + \cdots + \alpha_n(s)f_n$ is the identity on $R(sf_1)$.

By Lemma 4.2, there exists a set of n elements $\{y_1, y_2, \cdots, y_n\}$ contained in $R(tf_1)$ such that $\det F \neq 0$ where F is the n square matrix with ij^{th} entry given by $f_j(y_i)$. For arbitrary $s \geq t$, solving the system of equations $h_s(y_i) = y_i = \alpha_i(s)f_1(y_i) + \cdots + \alpha_n(s)f_n(y_i)$, $i = 1,2,\cdots,n$, for $\alpha_i(s)$, we get

$$\alpha_i(s) = \frac{\begin{vmatrix} f_1(y_1) & \cdots & y_1 & \cdots & f_n(y_1) \\ f_1(y_2) & \cdots & y_2 & \cdots & f_n(y_2) \\ \vdots & & & & \vdots \\ f_1(y_n) & \cdots & y_n & \cdots & f_n(y_n) \end{vmatrix}}{\det F}$$

Since the y_i depend only on t, for all $s \geq t$, $\alpha_i(s) = \alpha_i(t)$ for $i = 1, 2, \cdots, n$. Therefore, $h_t = h_s = I$ on $R(sf_1)$ for all $s \geq t$.

Suppose $x \in \mathbb{R}$, then there exists an $s \geq t$ such that $x \in R(sf_1)$ and $h_t(x) = x$. Therefore, $h_t = I$. $\square$

We will now show that there is only one two-dimensional sub near algebra of $T_C(\mathbb{R})$.

4.5 Theorem. If N is a two-dimensional sub near algebra of $T_C(\mathbb{R})$, then $N = Lip_p(\mathbb{R})$.

Proof: Let N be a two-dimensional sub near algebra of $T_C(\mathbb{R})$. By Lemma 4.3 and the replacement property of basis elements, we may choose a basis such that at least one of the functions has a range which intersects both $\mathbb{R}^+$ and $\mathbb{R}^-$. By Lemma 4.4, the identity, I, belongs to N and we may choose $\{I, f\}$ as a basis for N. Choose $u \in \mathbb{R}$, $u > 0$ and let $g = f - \frac{f(u)I}{u}$. These new functions, g and I, are linearly independent and $g(u) = f(u) - f(u) = 0$. Thus, we have constructed basis elements I and g where I is the identity and $g(u) = 0$ for some $u > 0$.

For each $t \in \mathbb{R}$, $g(tg) = sI + rg$ for some $r, s \in \mathbb{R}$. Then $g(tg)(u) = 0 = su + rg(u) = su$. Thus, $s = 0$ and $\langle g \rangle$, the linear subspace generated by g, is a one-dimensional sub near algebra of $T_C(\mathbb{R})$. By Theorem 4.1, $g = aJ + bK$ for some $a, b \in \mathbb{R}$; $g(u) = 0 = au$ implies $g = bK$. Since $I = J + K$, we may choose J, K as a basis for N and $N = \mathrm{Lip}_p(\mathbb{R})$. ∥

The following important theorem states that there are no sub near algebras of $T_C(\mathbb{R})$ with dimension greater than 2.

<u>4.6 Theorem.</u> For $n \geq 3$, there are no n-dimensional sub near algebras of $T_C(\mathbb{R})$.

Proof: For $n = 3$, assume N is a three-dimensional sub near algebra of $T_C(\mathbb{R})$. Using Lemma 4.3 and 4.4 and the replacement property for basis elements, we may choose $\{I, f_2, f_3\}$ as a basis where I is the identity function. Let $u \in \mathbb{R}$, $u > 0$, and let $h_2 = \frac{f_2(u)}{u} I - f_2$ and $h_3 = \frac{f_3(u)}{u} I - f_3$. A straightforward check shows that I, h_2, h_3 are independent; hence, they form a basis. Also, $h_2(u) = h_3(u) = 0$.

For arbitrary $s, t \in \mathbb{R}$, we have $h_2(sh_2 + th_3) = \alpha_1(s,t)I + \alpha_2(s,t)h_2 + \alpha_3(s,t)h_3$. Then, $h_2(sh_2(u) + th_3(u)) = h_2(0) = 0 = \alpha_1(s,t)u$. Therefore, for all $s,t \in \mathbb{R}$, $h_2(sh_2 + th_3) = \alpha_2(s,t)h_2 + \alpha_3(s,t)h_3$. Similarly, $h_3(sh_2 + th_3) = \beta_2(s,t)h_2 + \beta_3(s,t)h_3$ for all $s, t \in \mathbb{R}$. These results combined with right distributivity imply that $\langle h_2, h_3 \rangle$, the linear space generated by h_2 and h_3, is a two-dimensional sub near algebra of $T_C(\mathbb{R})$. By Theorem 4.5, $I \in \langle h_2, h_3 \rangle$ which contradicts the fact that I, h_2 and h_3 are independent. Therefore, there is no three-dimensional sub near algebra of $T_C(\mathbb{R})$.

Assume that there are no n-dimensional sub near algebras of $T_C(\mathbb{R})$ for all $n < k$. Suppose there is a k-dimensional sub near algebra of $T_C(\mathbb{R})$. Once again, by Lemmas 4.3 and 4.4 and the replacement property for basis elements, we may choose a basis $\{I, f_2, f_3 \cdots, f_k\}$. Choose $u \in \mathbb{R}$, $u > 0$, and let $h_i = \frac{f_i(u)}{u} I - f_i$ for $i = 2,3,\cdots,k$. As before, $h_i(u) = f_i(u) - f_i(u) = 0$ for $i = 2,3,\cdots,k$ and $\{I, h_2, h_3,\cdots,h_k\}$ form a basis since they are linearly independent. We will now show that $\langle h_2, h_3,\cdots,h_k \rangle$, the linear subspace generated by $\{h_2, h_3,\cdots,h_k\}$, forms a (k-1)-dimensional near algebra.

For an arbitrary but fixed i, $2 \leq i \leq k$, $h_i(t_2h_2+\cdots+t_kh_k) = \alpha_i I + \alpha_2h_2+\cdots+\alpha_kh_k$ where α_i depends on t_j, $j = 2,\cdots,k$. Using u, we get $h_i(t_2h_2(u)+\cdots+t_kh_k(u)) = h_i(0) = 0 = \alpha_1 u$. Therefore, for arbitrary i, $2 \leq i \leq k$, $h_i(t_2h_2+\cdots+t_kh_k) = \alpha_2h_2+\cdots+\alpha_kh_k$. Thus, $\langle h_2,\cdots,h_k \rangle$ is a (k-1)-dimensional near algebra which contradicts the induction hypothesis. Therefore, there is no k-dimensional sub near algebra of $T_C(\mathbb{R})$. $\square$

Although the above theorems have completely determined the finite dimensional sub near algebras of $T_C(\mathbb{R})$, we have not been able to completely determine the finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ for $n \geq 2$. However, we shall now prove the principle result of this section which states that there is no finite dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which properly contain $\mathcal{B}(\mathbb{R}^n)$, the bounded linear functions on $\mathbb{R}^n$, for $n \geq 2$.

Before beginning the proof of this result, we will construct an example to show that there are finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$, $n \geq 2$, of arbitrarily prescribed dimension. These, however, do not contain all the linear operators on $\mathbb{R}^n$.

Let L_1 and L_2, where $L_1(x,y) = (x,0)$ and $L_2(x,y) = (y,0)$, denote two of the standard basis elements for the space of linear functions on $\mathbb{R}^2$. For each $k \in \mathbb{N}$, define $P_k(x,y) = (xy^k,0)$; then $P_k \in T_C(\mathbb{R}^2)$. For arbitrary $n \in \mathbb{R}$, let $\langle S \rangle$ be the linear subspace of $T_C(\mathbb{R}^2)$ generated by $S = \{L_1, L_2, P_1, P_2, \cdots, P_n\}$. Since L_1 acts like the identity and L_2 acts like the zero function on first coordinates, it is an easy check to show that $L_1F = F$ and $L_2F = 0$ for arbitrary $F \in \langle S \rangle$. Similarly $P_iF = 0$ for $i = 1,2,\cdots,n$. Therefore, if $F = \sum s_iP_i + t_1L_1 + t_2L_2$ and G, H are elements of $\langle S \rangle$, then $FG = \sum s_iP_iG + t_1L_1G + t_2L_2G = t_1G$. Thus, $FG = t_1G \in \langle S \rangle$. Also, $F(G + H) = t_1(G + H) = t_1G + t_1H = FG + FH$ and $F(\alpha G) = t_1(\alpha G) = \alpha FG$. Therefore, $\langle S \rangle$ is an algebra, though not an algebra of linear operators. It is easily shown that S is a set of (n+2) linearly independent functions on $\mathbb{R}^2$. Thus, $\langle S \rangle$ is an (n+2)-dimensional sub algebra of $T_C(\mathbb{R}^2)$ for arbitrary $n \in \mathbb{N}$.

For convenience in the discussion below, we will adopt the following notation. A vector or n-tuple in $\mathbb{R}^n$ will be denoted by $\overline{x} = (x_1,x_2,x_3\cdots,x_n)$ where the x_i's are the components of the n-tuple. The standard basis elements of $\mathbb{R}^n$ will be given by $\overline{e}_i = (0,0,\cdots,1,0,\cdots,0)$ where the 1 appears in the i^{th} position. The standard basis elements for $\mathcal{B}(\mathbb{R}^n)$ will be denoted by A_{ij} where $A_{ij}(\overline{x}) = x_j\overline{e}_i$. That is, A_{ij} is the linear transformation whose matrix representation has a 1 in the ij^{th} position and 0 everywhere else. Unless otherwise specified, $\mathcal{A}_n = \{A_{ij} | i,j = 1,2,\cdots,n\}$ will denote this set of n^2 linear functions. Finally, an arbitrary function $P \in T(\mathbb{R}^n)$ will often be denoted by $P = (P_1,P_2,\cdots,P_n)$ where $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the i^{th} coordinate function of P.

4.3 Proposition. Let $n \in \mathbb{N}$, $n \geq 2$. There does not exist an (n^2+1)-dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which contains $\mathcal{B}(\mathbb{R}^n)$.

Proof: Let n be an arbitrary natural number, $n \geq 2$, and assume N is an (n^2+1)-dimensional sub near algebra of $T_C(\mathbb{R}^n)$ such that $\mathcal{B}(\mathbb{R}^n) \subseteq N$. There must exist a non-linear function $P \in N$ such that $\mathcal{A}_n \cup \{P\}$ forms a basis for N. Multiplying P on the left by $A_{ij}, i,j = 1,2,\cdots,n$, we get $A_{ij}P = B_{ij} + t_{ij}P$ where B_{ij} is a linear combination of functions from $\mathcal{A}_n$. If $t_{11} \notin$ spectrum (A_{11}), then $(A_{11} - t_{11}I)^{-1}$ exists and $P = (A_{11} - t_{11}I)^{-1}B_{11}$ which is linear. Therefore, $t_{11} \in$ spectrum (A_{11}) and $t_{11} = 0$ or 1.

If $t_{11} = 0$, then $A_{11}P = B_{11}$. Thus, for all $j = 1,2,\cdots,n$, $A_{11}(A_{1j}P) = A_{1j}P = A_{11}B_{1j} + t_{1j}A_{11}P = A_{11}B_{1j} + t_{1j}B_{11}$ which is linear. However, $A_{1j}P = (0,0,\cdots,P_j,0,\cdots,0)$ which implies P_j is linear for $j = 1,2,\cdots,n$. Therefore, P is linear.

If $t_{11} = 1$, then $A_{11}P = B_{11} + P$ and $A_{1n}(A_{11}P) = 0 = A_{1n}B_{11} + A_{1n}P$ Therefore, $A_{1n}P = -A_{1n}B_{11}$. Also, $A_{1n}(A_{n1}P) = A_{11}P = A_{1n}(B_{n1}) + t_{n1}A_{1n}P = A_{1n}B_{n1} - t_{n1}A_{1n}B_{11}$. Thus, $P = A_{11}P - B_{11} = A_{1n}B_{n1} - (t_{n1}A_{1n} + I)\ B_{11}$ is linear.

In all cases P is linear. Thus, since n was artitrary, there are no (n^2+1)-dimensional sub near algebras of $T_C(\mathbb{R}^n)$ which contain $\mathcal{B}(\mathbb{R}^n)$. $\square$

We now proceed to prove a series of lemmas and to facilitate the discussion, we will make the following conventions in notation. For $k, n \in \mathbb{N}$, $k, n \geq 2$, an (n^2+k)-dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which contains $\mathcal{B}(\mathbb{R}^n)$ will be called an (n,k)-near algebra. Since an (n,k)-near algebra contains $\mathcal{B}(\mathbb{R}^n)$, we will choose $\mathcal{A}_n$ as part of the basis. Thus, when proving properties of the basis, we will be concerned with only the non-linear portion.

4.5 Lemma. Let $k, n \in \mathbb{N}$, $k, n \geq 2$. If N is an (n,k)-near algebra, then a basis $\mathcal{A}_n \cup \{P_i \mid i = 1,2,\cdots,k\}$ may be chosen such that $P_i(\bar{e}_j) = \bar{0}$ for $i = 1,2,\cdots,k$ and $j = 1,2,\cdots,n$.

Proof: Let n and k be arbitrary natural numbers; $n, k \geq 2$. Let $\{Q_i \mid i = 1,2,\cdots,k\}$ be the non-linear portion of the basis for N where $Q_i = (Q_{i1},Q_{i2},\cdots Q_{in})$. For an arbitrary but fixed s, $1 \leq s \leq k$, we will replace Q_s by a function R_s such that $R_s(\bar{e}_i) = \bar{0}$ for all $i = 1,2,\cdots,n$. This new collection $\{Q_1,Q_2,\cdots,R_s,\cdots,Q_n\}$ will be a basis and, since the construction does not depend on the particular choice of s, we can repeat the process and replace each Q_s with the corresponding R_s. The new non-linear portion of the basis $\{R_1,R_2,\cdots,R_k\}$ will have the desired property that $R_i(\bar{e}_j) = \bar{0}$ for all $j = 1,2,\cdots,n$ and $i = 1,2,\cdots,k$.

For fixed s, $1 \leq s \leq k$, we construct a sequence of functions ${}_iP_s, i = 0,1,\cdots,n$, defined recursively, such that ${}_iP_s(\bar{e}_p) = \bar{0}$ for $p \leq i$. Let ${}_0P_s = Q_s$ and define

$$ {}_iP_s = \sum_{m=1}^{n} {}_{i-1}P_{sm}(\bar{e}_i)A_{mi} - {}_{i-1}P_s \text{ for } i = 1,2,\cdots,n \text{ where} $$

${}_{i-1}P_{sm}$ is the m^{th} coordinate function of ${}_{i-1}P_s$. By direct

calculation, ${}_iP_s(\bar{e}_i) = \sum_{m=1}^{n} {}_{i-1}P_{sm}(\bar{e}_i)A_{mi}(\bar{e}_i) - {}_{i-1}P_s(\bar{e}_i)$

$$ = \sum_{m=1}^{n} {}_{i-1}P_{sm}(\bar{e}_i)\bar{e}_m - {}_{i-1}P_s(\bar{e}_i) = {}_{i-1}P_s(\bar{e}_i) - {}_{i-1}P_s(\bar{e}_i) = \bar{0} $$

for $i = 1\ 2,\cdots,n$.

Using the recursive property and the fact that $A_{mi}(\bar{e}_p) = 0$ for $p \neq i$, we can show that ${}_iP_s(\bar{e}_p) = \bar{0}$ for $p \leq i$. For example, ${}_2P_s(\bar{e}_2) = \bar{0}$ by the above and ${}_2P_s(\bar{e}_1)$

$$ = \sum_{m=1}^{n} {}_1P_{sm}(\bar{e}_2)A_{m2}(\bar{e}_1) - {}_1P_s(\bar{e}_1) = \bar{0} - \bar{0} = \bar{0}. $$ In particular,

${}_nP_s(\bar{e}_p) = \bar{0}$ for $p \leq n$.

Let S_i $= \mathcal{A}_n \cup \{R_m \mid m = 1,2,\cdots,k,\ R_m = Q_m \text{ for } m \neq s \text{ and } R_s = {}_iP_s\}$ for $i = 0,1,2,\cdots,n$. Now, $S_0 = \mathcal{A}_n \cup \{Q_m \mid m = 1,2,\cdots,k\}$ is the original basis and S_1 is the original basis with Q_s replace by ${}_1P_s$. It is a straightforward check to show S_1 is a linearly independent set and after n replacements S_n is linearly independent. Thus, S_n is a basis and $R_s(\bar{e}_i) = \bar{0}$ for $i = 1,2,\cdots,n$. //

4.6 Lemma. Let $k, n \in \mathbb{N}$, $k, n \geq 2$. If N is a (n,k)-near algebra, then a basis $\mathcal{A}_n \cup \{P_i \mid i = 1,2,\cdots,k\}$ may be chosen such that $A_{ij}P_m$, P_mA_{ij} and P_mP_r belong to $\langle P_i | i = 1,2,\cdots,k \rangle$, the linear subspace generated by $\{P_i\}$, for all i, j = 1,2,···,n and m, r = 1,2,···,k.

Proof For arbitrary $n \geq 2$ and $k \geq 2$, we may choose, by Lemma 4.5, a basis whose non-linear portion has the property that $P_i(\bar{e}_j) = \bar{0}$ for i = 1,2,···,k and j = 1,2,···,n. Then, for arbitrary α, β and μ, $A_{\alpha\beta}P_\mu = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}A_{ij} + \sum_{i=1}^{k} t_iP_i$. Using the property of $\bar{e}_m$, m = 1,2,···,n, we have $\bar{0} = A_{\alpha\beta}(\bar{0})$ $= A_{\alpha\beta}P_\mu(\bar{e}_m) = \sum_{i=1}^{n} t_{im}\bar{e}_i + \bar{0}$. Therefore, $t_{im} = 0$ for i = 1,2,···,n and m = 1,2,···,n. Thus, $A_{\alpha\beta}P_\mu$ $= \sum_{i=1}^{k} t_iP_i \in \langle P_i \; i = 1,2,\cdots,k \rangle$. A similar direct computation holds for P_mA_{ij} and P_mP_r. $\square$

<u>4.1 Definition.</u> A function $P: \mathbb{R}^n \to \mathbb{R}^n$ which projects onto the i^{th} coordinate axis, $P = (0,0,\cdots P_i,\cdots,0)$, is called an <u>$i^{th}$ coordinate function.</u> Let S be a subset of $T(\mathbb{R}^n)$. If, for each $P \in S$, P is an i^{th} coordinate function for some i, $1 \leq i \leq n$, then S is said to be a <u>collection of coordinate functions.</u>

<u>4.7 Lemma.</u> The sum and the scalar multiple of i^{th} coordinate functions are i^{th} coordinate functions. Composition on the left by an i^{th} coordinate function with any function in $T_C(\mathbb{R}^n)$ is also an i^{th} coordinate function.

Proof: Since sums and scalar multiples are done componentwise, the first two results are immediate. If P is an i^{th} coordinate function, $P = (0,0,\cdots,P_i 0,0,\cdots,0)$ and $f \in T_C(\mathbb{R}^n)$, $f = (f_1,f_2,\cdots,f_n)$, then $Pf(\overline{x})$ $= (0,0,\ldots,P_i(f_1(\overline{x}),f_2(\overline{x}),\ldots,f_n(\overline{x})),0,\ldots,0)$ is an i^{th} coordinate function. $\square$

<u>4.8 Lemma.</u> Let $k, n \in \mathbb{N}$; $k, n \geq 2$. If N is an (n,k)-near algebra, then there exists a basis $\mathcal{A}_n \cup \{P_i \mid i = 1,2,\cdots,k\}$ such that $\{P_i \mid i = 1,2,\cdots,k\}$ is a collection of coordinate functions. Further, not all the P_i's can be m^{th} coordinate functions for any fixed m, $1 \leq m \leq n$.

Proof: For arbitrary $k, n \geq 2$, choose a basis $\mathcal{A}_n \cup \{P_i \mid i = 1,2,\cdots,k\}$ such that $P_i(\overline{e}_j) = \overline{0}$ for all $i = 1,2,\cdots,k$ and $j = 1,2,\cdots,n$. Let m be an arbitrary but fixed natural number, $1 \leq m \leq k$, and consider the sets $S_{im} = \mathcal{A}_n \cup \{P_1,P_2,\cdots,P_{m-1}, A_{ii}P_m, P_{m+1},\cdots,P_k\}$ for each $i = 1,2,\cdots,n$.

If the sets S_{im} are all linearly dependent sets of functions for $i = 1,2,\cdots,n$, then there exists scalars t_{ij} such that $A_{ii}P_m = B_i + t_{i1}P_1 + t_{i2}P_2+\cdots+t_{im-1}P_{m-1} + t_{im+1}P_{m+1}+\cdots+t_{ik}P_k$ where B_i is an appropriate combination of linear functions. Also, by Lemma 4.6, $A_{ii}P_m = s_{i1}P_1 + s_{i2}P_2+\cdots+s_{im}P_m+\cdots+s_{ik}P_k$ for $i = 1,2,\cdots,n$. For $i = 1,2,\cdots,n$, if we equate the two expressions for $A_{ii}P_m$, then we get $s_{im} = 0$ because of the linear independence of the P_i. However, if we add the latter expressions for $A_{ii}P_m$, we get $A_{11}P_m + A_{22}P_m+\cdots+A_{nn}P_m = (A_{11} + A_{22}+\cdots+A_{nn})P_m =$

$$= IP_m = P_m = \left(\sum_{i=1}^{n} s_{i1}\right) P_1 + \cdots + \left(\sum_{i=1}^{n} s_{im}\right) P_m + \cdots + \left(\sum_{i=1}^{n} s_{ik}\right) P_k .$$

Since the P_i are linearly independent, this implies $\sum_{i=1}^{n} s_{im}$ $- 1 = 0$. This contradicts the fact that $s_{im} = 0$ for $i = 1,2,\cdots,n$; thus, the assumption that the sets S_{im} are all linearly dependent sets must be false. Therefore, there exists α such that $S_{\alpha m}$ is a linearly independent set and may be chosen as a new basis.

Clearly, $A_{\alpha\alpha} P_m(\overline{e}_j) = A_{\alpha\alpha}(\overline{0}) = \overline{0}$ for $j = 1,2,\cdots,n$; hence, this new basis satisfies the conditions of our original basis. Therefore, since m was arbitrary, we can repeat this process and, for each $m = 1,2,\cdots,k$, there exists i_m such that P_m can be replaced by $R_m = A_{i_m i_m} P_m$. The resulting basis is a collection of coordinate functions since $A_{ii}P$ is an i^{th} coordinate function for all i and any P.

Finally, if there exists a j such that the R_m's are all j^{th} coordinate functions, choose $i \neq j$ and apply A_{ij} to $R_1 = (0,0,\cdots,R_{1j},0,\cdots,0)$. For $\overline{x} \in \mathbb{R}^n$, we get $A_{ij}R_1(\overline{x})$

$$= R_{1j}(\overline{x})\overline{e}_i = \sum_{m=1}^{k} t_m R_m(\overline{x}) = \sum_{m=1}^{k} t_m R_{mj}(\overline{x})\ \overline{e}_j .$$ Since $i \neq j$,

$R_{1j}(\overline{x}) = 0$ for all $x \in \mathbb{R}^n$ which implies $R_1 = 0$. This contradicts the fact that R_1 is a basis element. $\square$

<u>4.9 Lemma.</u> Let $k, n \in \mathbb{N}$, $k, n \geq 2$. If N is an (n,k)-near algebra, then $k = qn$ for some $q \geq 1$ and there exists a basis

of the form $\mathcal{A}_n \cup \{P_i | i = 1,2,\cdots,q\} \cup \{A_{21}P_i | i = 1,2,\cdots,q\}$ $\cup \cdots \cup \{A_{ni}P_i | i = 1,2,\cdots,q\}$ where P_i is a first coordinate function for $i = 1,2,\cdots,q$.

Proof: For arbitrary k, $n \geq 2$, let N be an (n,k)-near algebra with basis $\mathcal{A}_n \cup \{R_i | i = 1,2,\cdots,k\}$. By Lemmas 4.5 and 4.8, we may assume that $R_i(\bar{e}_j) = \bar{0}$ for $i = 1,2,\cdots,k$ and $j = 1,2,\cdots,n$ and each R_i is a coordinate function.

Let q_i be the number of non-linear basis functions which are non-zero in the i^{th} coordinate. If there exists a coordinate position, say j, such that $q_j = 0$, then, from the non-linear portion of the basis, choose an i^{th} coordinate function R such that $i \neq j$. By Lemma 4.6, $A_{ji}R = t_1R_1 + t_2R_2 + \cdots + t_kR_k$. Since there are no j^{th} coordinate functions in the set $\{R_i | i = 1,2,\cdots,k\}$, and, for $\bar{x} \in \mathbb{R}^n$, $A_{ji}R(\bar{x}) = R(\bar{x})\bar{e}_j$ is a j^{th} coordinate function, we have that $R(\bar{x}) = \bar{0}$ for all $\bar{x} \in \mathbb{R}^n$. This contradiction implies $q_j \neq 0$ for $j = 1,2,\cdots,n$. Since $q_1 + q_2 + \cdots + q_n = k$, we now need to show that $q_1 = q_2 = \cdots = q_n$; then letting $q_1 = q$, we have $qn = k$.

Since each non-linear basis element is a coordinate function, let us denote these functions in the following fashion:

$\mathcal{R}_1 = \{R_{11}, R_{12}, \cdots, R_{1q_1}\}$ are the 1^{st} coordinate functions,

$\mathcal{R}_2 = \{R_{21}, R_{22}, \cdots, R_{2q_2}\}$ are the 2^{nd} coordinate functions, $\cdots$

$\mathcal{R}_n = \{R_{n1}, R_{n2}, \cdots, R_{nq_n}\}$ are the n^{th} coordinate functions.

If we assume that it is not true that $q_1 = q_2 = \cdots = q_n$, then there exists at least two elements which are not equal.

Without loss of generality, assume $q_1 > q_2$. Let S $= \mathcal{A}_n \cup \mathcal{R}_1 \cup \{A_{21}R_{1j} | R_{1j} \in \mathcal{R}_1\} \cup \mathcal{R}_3 \cup \cdots \cup \mathcal{R}_n$, the original basis with the second coordinate functions $\mathcal{R}_2$ replaced by $A_{21}R_{1j}$, for $j = 1,2,\cdots,q_1$. If $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}A_{ij} + \sum_{j=1}^{q_1} b_{1j}R_{1j}$

$$+ \sum_{j=1}^{q_1} c_{1j}A_{21}R_{1j} + \sum_{i=3}^{n} \sum_{j=1}^{q_i} b_{ij}R_{ij} = 0$$ for any linear combinations of elements from S, then, by using $\bar{e}_j$ for $j = 1,2,\cdots,n$, we have that $a_{ij} = 0$ for all i and j. Also, since each coordinate must be zero, $c_{11}R_{11}(\bar{x}) + \cdots + c_{1q_1}R_{1q_1}(\bar{x}) = 0$ and, hence, $c_{1j} = 0$ for all $j = 1,2,\cdots,q_1$. Similarly, $b_{ij} = 0$ for $i = 1,3,4,\cdots,n$ and $j = 1,2,\cdots,q_i$. Therefore, S is an independent set of functions and the number of such functions is $n^2 + q_1 + q_1 + q_3 + \cdots + q_n > n^2 + q_1 + q_2 + \cdots + q_n = n^2 + k$. This contradiction implies that $q_1 = q_2 = \cdots = q_n = q$.

Choose as a basis the set of linearly independent functions $\mathcal{A}_n \cup \mathcal{R}_1 \cup A_{21}\mathcal{R}_1 \cup \cdots \cup A_{n1}\mathcal{R}_1$ where $A_{i1}\mathcal{R}_1 = \{A_{i1}R_{1j} | j = 1,2,\cdots,q\}$ for $i = 1,2,\cdots,n$. ▯

<u>4.2 Definition.</u> Let S be a non-empty subset of $T_C(\mathbb{R}^n)$. If S is a set of coordinate functions such that S = $S_1 \cup A_{21}S_1 \cup \cdots \cup A_{n1}S_1$ where S_1 is a set of first coordinate functions, then S is said to be a <u>completely symmetric set of coordinate functions</u> generated by S_1.

The preceding lemma states that every (n,k)-near algebra has a basis such that the non-linear portion is a completely symmetric set of coordinate functions generated by a set of first coordinate functions. For convenience in notation, if $\{P_i / i = 1,2,\cdots,q;\ k = nq\}$ is the set of first coordinate functions in the non-linear portion of the basis of an (n,k)-near algebra, then we let $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$ denote the entire basis where $\mathcal{P}_k(P_i)$ is the completely symmetric set of coordinate functions generated by $\{P_i | i = 1,2,\cdots,q;\ k = q_n\}$. We will also assume $P_i(\bar{e}_j) = \bar{0}$.

This characterization of the basis of an (n,k)-near algebra indicates that the properties of the first coordinate functions are shared by all the non-linear basis elements. Thus, we need only prove our results for the first coordinate functions.

<u>4.10 Lemma.</u> Let $k,\ n \in \mathbb{N}$, $k,\ n \geq 2$. If N is an (n,k)-near algebra with basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$, then $P_i(x\bar{e}_j) = \bar{0}$ for $x \geq 0$, $i = 1,2,\cdots,q$ and $j = 1,2,\cdots,n$.

Proof: Let n and k be arbitrary natural numbers, $n,\ k \geq 2$, and let $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$ be a basis with $\{P_i | i = 1,2,\cdots,q\}$ the first coordinate function such that $P_i = (P_{i1},0,0,\cdots,0)$. For each $i = 1,2,\cdots,q$, define the functions $Q_i(x) = P_{i1}(x\bar{e}_1)$ for $x \in \mathbb{R}$. Thus, $Q_i \in T_C(\mathbb{R})$.

Consider the linear space $\langle Q_1,Q_2,\cdots,Q_q \rangle$. For any $x \in \mathbb{R}$ and i, $1 \leq i \leq q$, $Q_i\left(\sum_{j=1}^{q} t_jQ_j\right)(x)$

$$= Q_i\left(\sum_{j=1}^{q} t_jP_{j1}(x\bar{e}_1)\right) = P_{i1}\left(\left(\sum_{j=1}^{q} t_jP_{j1}(x\bar{e}_1)\right)\bar{e}_1\right).$$ However, for

the same i and scalars t_j, $P_i\left(\sum_{j=1}^{q} t_j P_j\right) = \sum_{m=1}^{q} s_{im} P_m$. Thus,

for $x\bar{e}_1 \in \mathbb{R}^n$, $P_i\left(\sum_{j=1}^{q} t_j P_j\right)(x\bar{e}_1) = P_{i1}\left(\left(\sum_{j=1}^{q} t_j P_{j1}(x\bar{e}_1)\right)\bar{e}_1\right)\bar{e}_1$

$$= \left(Q_i\left(\sum_{j=1}^{q} t_j Q_j\right)(x)\right)\bar{e}_1 = \left(\sum_{m=1}^{q} s_{im} P_m\right)(x\bar{e}_1)$$

$$= \left(\sum_{m=1}^{q} s_{im} P_{m1}(x\bar{e}_1)\right)\bar{e}_1 = \left(\sum_{m=1}^{q} s_{im} Q_m(x)\right)\bar{e}_1.$$ Therefore,

$Q_i\left(\sum_{j=1}^{q} t_j Q_j\right)(x) = \left(\sum_{m=1}^{q} s_{im} Q_m\right)(x)$ for all $x \in \mathbb{R}$ and $\langle Q_1, Q_2, \cdots, Q_q \rangle$ is a finite dimensional sub near algebra of $T_C(\mathbb{R})$. We have shown that such a near algebra is positive homogeneous; hence $Q_i(1) = P_{i1}(\bar{e}_1) = 0$ and $P_{i1}(x\bar{e}_1) = Q_i(x) = xQ_i(1) = 0$ for $x \geq 0$. Therefore, $P_i(x\bar{e}_1) = 0$ for $x \geq 0$ and $i = 1,2,\cdots q$.

Finally, by Lemma 4.7, $P_i A_{j1} = t_1 P_1 + \cdots + t_q P_q$ for $i = 1,2,\cdots,q$ and $j = 1,2,\cdots,n$; thus, $P_i A_{j1}(x\bar{e}_1) = P_i(x\bar{e}_j) = t_1 P_1(x\bar{e}_1) + \cdots + t_q P_q(x\bar{e}_1) = \bar{0}$ for all i and j. ▯

<u>4.11 Lemma.</u> Let $k, n \in \mathbb{N}$; $k, n \geq 2$. If N is an (n,k)-near algebra with basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$, then P_i is positive homogeneous for each $i = 1,2,\cdots,q$.

Proof: For arbitrary k and n; $k, n \geq 2$, let N be such an (n,k)-near algebra with basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$. Lemma 4.10 states that $P_i(x\bar{e}_j) = \bar{0}$ for $x \geq 0$, $i = 1,2,\cdots,q$ and $j = 1,2,\cdots,n$. By Lemma 4.7, for $\bar{x} = (x_1, x_2 \ldots, x_n) \in \mathbb{R}^n$ and arbitrary i, $1 \leq i \leq q$, there exist scalars $t_{1j}(\bar{x})$, $t_m(\bar{x}) \in \mathbb{R}$ such that $P_i(x_1A_{11} + x_2A_{21}+\cdots+x_nA_{n1}) = \sum_{j=1}^{n} t_{1j}(\bar{x})A_{1j} + \sum_{m=1}^{q} t_m(\bar{x})P_m$. For arbitrary $\bar{x} \in \mathbb{R}^n$ and $\bar{e}_p$, $p = 2,3,\cdots,n$,

$P_i(x_1A_{11} + x_2A_{21}+\cdots+x_nA_{n1})\bar{e}_p = \bar{0} = t_{1p}(\bar{x})\bar{e}_1 + \bar{0}$.

Thus, $t_{1p}(\bar{x}) = 0$ for all $p = 2,3,\cdots,n$ and $\bar{x} \in \mathbb{R}^n$. Also, for $\bar{x} \in \mathbb{R}^n$, $P_i(x_1A_{11} + x_2A_{21}+\cdots+x_nA_{n1})\bar{e}_1 = P_i(x) = t_{11}(\bar{x})\bar{e}_1$. Since $P_i(\alpha\bar{e}_1) = \alpha P_i(\bar{e}_1) = \bar{0}$ for $\alpha \geq 0$, we have that $p_i(\alpha\bar{x})$

$= P_i(\alpha x_1A_{11} + \alpha x_2A_{21}+\cdots+\alpha x_nA_{n1})\bar{e}_1$

$= P_i(x_1A_{11} + x_2A_{21}+\cdots+x_nA_{n1})(\alpha\bar{e}_1) = t_{11}(\bar{x})(\alpha\bar{e}_1) = \alpha P_i(\bar{x})$

for $\alpha \geq 0$ and $i = 1,2,\cdots,q$. $\square$

The following lemma indicates that a sub near algebra of $T_C(\mathbb{R}^n)$ which contains certain special functions must contain an arbitrarily large number of linearly independent functions. This will show that such a sub near algebra can not be finite dimensional. The special set of functions is given as follows: For $n \geq 2$, let

$\mathcal{K}_n = \{J_{ij} \mid J_{ij}(\bar{x}) = J(x_j)\bar{e}_i$ for $\bar{x} \in \mathbb{R}^n$; $i, j = 1,2\}$

$\cup \{K_{ij} \mid K_{ij}(\bar{x}) = K(x_j)\bar{e}_i$ for $\bar{x} \in \mathbb{R}^n$; $i, j = 1,2\}$ where

J, K are the constructed basis elements of $Lip_p(\mathbb{R})$. $\mathcal{K}_n$ is a set of linearly independent functions for $n \geq 2$. The proof of the lemma consists of actually constructing an arbitrarily large number of linearly independent functions.

<u>4.12 Lemma.</u> For $n \in \mathbb{N}$, $n \geq 2$, there is no finite dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which contains $\mathcal{K}_n$.

Proof: Let n be arbitrary, $n \geq 2$, and assume N is a q-dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which contains $\mathcal{K}_n$. For each $m \in \mathbb{N}$, let $F_m = J_{11}(J_{11} + K_{11} - m(J_{12} + K_{12}))$; thus, $F_m \in N$ for each m. Let $\mathcal{C}_m = \mathcal{K}_n \cup \{F_i | i = 1,2,\cdots,m\}$. We will show, by induction, that $\mathcal{C}_m$ is a linearly independent set of functions for all $m \in \mathbb{N}$ which will contradict the fact that N had dimension q.

For m = 1, if $\mathcal{C}_1$ is not a linearly independent set, then $F_1 = t_{11}J_{11} + t_{12}J_{12} + t_{21}J_{21} + t_{22}J_{22} + s_{11}K_{11} + s_{12}K_{12} + s_{21}K_{21} + s_{22}K_{22}$. $F_1(\overline{e}_1 + \overline{e}_2) = \overline{0} = (t_{11} + t_{12})\overline{e}_1 + (t_{21} + t_{22})\overline{e}_2$ and $F_1(2\overline{e}_1 + \overline{e}_2) = \overline{e}_1 = (2t_{11} + t_{12})\overline{e}_1 + (2t_{21} + t_{22})\overline{e}_2$. These equalities result in the two equations $t_{11} + t_{12} = 0$ and $2t_{11} + t_{12} = 1$ which have solution $t_{11} = 1$ and $t_{12} = -1$. However, $F_1(\overline{e}_1 + 2\overline{e}_2) = \overline{0} = (t_{11} + 2t_{12})\overline{e}_1 + (t_{21} + 2t_{22})\overline{e}_2$ which implies that $t_{11} + 2t_{12} = 0$. Therefore, $\mathcal{C}_1$ is an independent set.

Assume $\mathcal{C}_m$ is an independent set. If $\mathcal{C}_{m+1}$ is a dependent set, then $F_{m+1} = t_{11}J_{11} + \cdots + s_{22}K_{22} + \alpha_1 F_1 + \cdots + \alpha_m F_m$. If $x, y \geq 0$, then $F_{m+1}(x\overline{e}_1 + y\overline{e}_2)$

$$= \begin{cases} (x - (m+1)y)\overline{e}_1, & \text{if } x \geq (m+1)y \\ \overline{0}, & \text{if } x < (m+1)y \end{cases}$$

$$= (t_{11}x + t_{12}y + \alpha_1(x-y) + \cdots + \alpha_m(x-my))\overline{e}_1 + (t_{21}x + t_{22}y)\overline{e}_2$$

$$= (a_m x + b_m y)\overline{e}_1 + (t_{21}x + t_{22}y)\overline{e}_2 \quad \text{where}$$

$a_m = t_{11} + \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and $b_m = t_{12} - \alpha_1 - \alpha_2 - \cdots - \alpha_m$. Let $x = m+1$ and $y = 1$, then $F_{m+1}(x\bar{e}_1 + y\bar{e}_2) = \bar{0}$ $= (a_m(m+1) + b_m)\bar{e}_1 + (t_{21}(m+1) + t_{22})\bar{e}_2$. If $\bar{x} = m+2$ and $y = 1$, then $\bar{e}_1 = (a_m(m+2) + b_m)\bar{e}_1 + (t_{21}(m+2) + t_{22})\bar{e}_2$. These equations imply $a_m = 1$ and $b_m = -(m+1)$. However, if $x = m(m+2)$ and $y = m+1$, then we get $0 = a_m(m)(m+2) + b_m(m+1)$ $= m(m+2) - (m+1)(m+1) = -1$. Therefore, $\mathcal{C}_{m+1}$ is a linearly independent set. This completes the induction and the proof. $\square$

<u>4.13 Lemma.</u> Let $k, n \in \mathbb{N}$; $k, n \geq 2$. If N is an (n,k)-near algebra with basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$, then P_i is homogeneous for each $i = 1,2,\cdots q$.

Proof: For arbitrary k and n, $k, n \geq 2$, let N be an (n,k)-near algebra with basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$ such that each P_i is positive homogeneous and $P_i(\bar{e}_j) = 0$ for $i = 1,2,\cdots,q$ and $j = 1,2,\cdots,n$.

If $P_i(-\bar{e}_1) = \bar{0}$ for each $i = 1,2,\cdots,q$, then, for $x \in \mathbb{R}$, $P_i(x\bar{e}_1) = |x|P_i(\pm\bar{e}_1) = \bar{0}$. By Lemma 4.7, P_iA_{m1} $= t_1P_1 + \cdots + t_qP_q$; hence, $P_iA_{m1}(x\bar{e}_1) = P_i(x\bar{e}_m) = \bar{0}$ for all $i = 1,2,\cdots,q$ and $m = 1,2,\cdots,n$ and all $x \in \mathbb{R}$. This is the condition found in the proof of Lemma 4.11 except in this case it holds for all $x \in \mathbb{R}$; thus, by an argument similar to that of Lemma 4.11, each P_i is homogeneous.

However, if $P_{i_o}(-\bar{e}_1) \neq \bar{0}$ for some i_o, then, by reordering, we may assume $P_1(-\bar{e}_1) = -\bar{e}_1$. $P_1A_{11}(\bar{x}) =$

$= P_1(x_1\bar{e}_1) = \begin{cases} x_1P_1(\bar{e}_1) \\ -x_1P_1(-\bar{e}_1) \end{cases} = \begin{cases} \bar{0} & \text{, if } x_1 \geq 0 \\ x_1\bar{e}_1, & \text{if } x_1 < 0 \end{cases}$. Therefore, $P_1A_{11} = K_{11}$ and $J_{11} = A_{11} - P_1A_{11}$. $P_1A_{12}(\bar{x}) = P_1(x_2\bar{e}_1)$ $= \begin{cases} x_2P_1(\bar{e}_1) \\ -x_2P_1(-\bar{e}_1) \end{cases} = \begin{cases} \bar{0} & \text{, if } x_2 \geq 0 \\ x_2\bar{e}_1, & \text{if } x_2 < 0 \end{cases}$. Therefore, $P_1A_{12} = K_{12}$ and $J_{12} = A_{12} - P_1A_{12}$. Also $A_{21}K_{11} = K_{21}$, $A_{21}J_{11} = J_{21}$, $A_{21}K_{12} = K_{22}$ and $A_{21}J_{12} = J_{22}$.

Therefore, $\mathcal{K}_n \subseteq N$ and, by Lemma 4.12, N can not be finite dimensional. Thus, only the first case can hold and the functions P_i are homogeneous. $\square$

We are now in a position to complete the development of the major result of this paper. We proceed by letting $n = 2$ and, by induction on k, we show that there is no finite dimensional sub near algebra of $T_C(\mathbb{R}^2)$ which contains $\mathcal{B}(\mathbb{R}^2)$. We then complete the induction on n.

<u>4.14 Lemma.</u> For $k \in \mathbb{N}$, if $\{R_1, R_2, \cdots, R_k\}$ is a set of linearly independent, homogeneous, first coordinate functions contained in $T_C(\mathbb{R}^2)$ such that $R_i(1,0) = R_i(0,1) = (0,0)$, then there exist a collection of linearly independent functions $\{P_1, P_2, \cdots, P_k\}$ and a collection of non-zero real numbers $\{z_1, z_2, \cdots, z_k\}$ such that the functions are contained in $\langle R_1, R_2, \cdots, R_k \rangle$, the linear space generated by the R_i's and $P_i(1, z_j) = (\delta_{ij} z_j, 0)$.

Proof: For $k = 1$, let R_1 be such a function. Choose $x \neq 0$, $y \neq 0$ such that $R_1(x,y) = (u,0)$, $u \neq 0$. Let $z_1 = \frac{y}{x}$ and let $P_1 = \frac{y}{u}R_1$. Clearly, $P_1 \in \langle R_1 \rangle$ and $P_1(1, z_1) = (z_1, 0)$.

Assume for $m < k$ that we can construct the collection of functions and the real numbers z_i whenever the functions R_i exist, $i = 1,2,\cdots,m$. Let $\{R_1,R_2,\cdots,R_k\}$ be a collection of linearly independent, homogeneous, first coordinate functions such that $R_i(1,0) = R_i(0,1) = (0,0)$. By the induction hypothesis, there exists k-1 functions $\{Q_1,Q_2,\cdots Q_{k-1}\}$ $\subseteq \langle R_1,\cdots,R_{k-1}\rangle$ and elements $z_1,z_2,\cdots,z_{k-1}$, $z_i \neq 0$, such that $Q_i(1,z_j) = (\delta_{ij}z_j,0)$, $i, j = 1,2,\cdots,k-1$. Let Q_k $= R_k - \sum_{i=1}^{k-1} \frac{R_{k1}(1,z_i)}{z_i} Q_i$, where $R_k = (R_{k1},0)$. Clearly, $Q_k \in \langle R_1,R_2,\cdots,R_k\rangle$. If $\{Q_1,Q_2,\cdots,Q_k\}$ is a dependent set, then $Q_k \in \langle Q_1,Q_2,\cdots,Q_{k-1}\rangle \subseteq \langle R_1,R_2,\cdots,R_{k-1}\rangle$ and, consequently, $R_k \in \langle R_1,R_2,\cdots,R_{k-1}\rangle$. Therefore, $\{Q_1,Q_2,\cdots,Q_k\}$ is a linearly independent set. Also, for arbitrary z_j, $j = 1,2,\cdots,k-1$, $Q_k(1,z_j) = (0,0)$.

Choose $x \neq 0$, $y \neq 0$ such that $Q_k(x,y) = (u,0)$, $u \neq 0$. Let $z_k = \frac{y}{x}$ and let $P_k = \frac{y}{u}Q_k$. Also, let P_i $= Q_i - \frac{x}{u}Q_{i1}(1,z_k)Q_k$, $i = 1,2,\cdots,k-1$, where $Q_i = (Q_{i1},0)$. The set $\{P_1,P_2,\cdots,P_k\}$ is a linearly independent set of functions contained in $\langle Q_1,\cdots,Q_k\rangle \subseteq \langle R_1,\cdots,R_k\rangle$ and a direct calculation shows $P_i(1,z_j) = (\delta_{ij}z_j,0)$ for $i, j = 1,2,\cdots,k$. This completes the induction. $\square$

<u>4.15 Lemma.</u> Let $k \in \mathbb{N}$, $k \geq 2$. If N is a (2,k)-near algebra, then there exists a basis $\mathcal{A}_2 \cup \mathcal{P}_k(P_i)$ such that P_i is homogeneous and $P_i(1,0) = P_i(0,1) = (0,0)$ for each

$i = 1,2,\cdots,q$, $k = 2q$, and there exist non-zero z_i, $i = 1,2,\cdots,q$, such that $P_i(1,z_j) = (\delta_{ij}z_j,0)$.

Proof: For any $k \geq 2$, we may choose, by Lemmas 4.9 and 4.13, a basis $\mathcal{A}_2 \cup \mathcal{R}_k(R_i)$ such that R_i is homogeneous and $R_i(1,0) = R_i(0,1) = (0,0)$. By Lemma 4.14, there exists a collection $\{P_1,P_2,\cdots,P_q\} \subseteq \langle R_1,\cdots,R_q\rangle$ and there exists a set of non-zero real numbers $\{z_1,z_2,\cdots,z_q\}$ such that the functions are linearly independent and $P_i(1,z_j) = (\delta_{ij}z_j,0)$. The desired basis is $\mathcal{A}_2 \cup \mathcal{P}_k(P_i)$. ▯

<u>4.16 Lemma.</u> Let $k \in \mathbb{N}$, $k \geq 2$. If N is a $(2,k)$-near algebra, then there exists a basis $\mathcal{A}_2 \cup \mathcal{P}_k(P_i)$ such that P_i is homogeneous and $P_i(1,0) = P_i(0,1) = P_i(1,1) = (0,0)$ for $i = 1,2,\cdots,q$, $k = 2q$.

Proof: By Lemma 4.15, we may choose a basis $\mathcal{A}_2 \cup \mathcal{P}_k(P_i)$ such that P_i is homogeneous and $P_i(1,0) = P_i(0,1) = (0,0)$. Also, there exist a set $\{z_1,z_2,\cdots,z_q\}$, $z_j \neq 0$, such that $P_i(1,z_j) = (\delta_{ij}z_j,0)$. Let j be arbitrary, $1 \leq j \leq q$, and let $F = A_{11} + A_{22} - A_{21}P_1 - \cdots - A_{21}P_q$; then, $P_jF = t_{11}A_{11} + t_{12}A_{12} + t_1P_1 + \cdots + t_qP_q$ by Lemma 4.7. Using, successively, the elements $(1,0)$, $(0,1)$ and $(1,z_i)$ $i = 1,2,\cdots,q$, we find that $P_jF = 0$. For any $x \in \mathbb{R}$, $P_jF(1,x) = P_j(1,x-P_{11}(1,x)-\cdots-P_{q1}(1,x)) = (0,0)$ where $A_{21}P_i = (0,P_{i1})$. Let $f(x) = x - P_{11}(1,x)-\cdots-P_{q1}(1,x)$; then, f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. If $f(x) = 0$ for all $x \in \mathbb{R}$, then $P_{11}(x,y)+\cdots+P_{q1}(x,y) = x(P_{11}(1,\frac{y}{x})+\cdots+P_{q1}(1,\frac{y}{x})) = x(\frac{y}{x}) = y$, for all $x \neq 0$. However, $P_{11}(0,y)+\cdots+P_{q1}(0,y) = 0$. This contradicts the fact that the sum $P_{11}+\cdots+P_{q1}$ is a continuous function from $\mathbb{R}^2 \rightarrow \mathbb{R}$. Therefore, f is not identically equal to zero.

If there exists $u \in \mathbb{R}$ such that $f(u) = b > 0$, then, for $y \in \mathbb{R}$, $0 \leq y \leq b$, there exists $x \in \mathbb{R}$ such that $f(x) = y$ and $P_j(1,y) = P_j(1,f(x)) = P_jF(1,x) = (0,0)$. For any ε, $0 \leq \varepsilon \leq b$, let $F_1 = A_{11} + \varepsilon A_{21} + A_{22} - A_{21}P_1 - \cdots - A_{21}P_q$. By Lemma 4.7, $P_jF_1 = t_{11}(\varepsilon)A_{11} + t_{12}(\varepsilon)A_{12} + t_1(\varepsilon)P_1 + \cdots + t_q(\varepsilon)P_q$.

Using $(1,0)$, $(0,1)$, and $(1,z_i)$, $i = 1,2,\cdots,q$, and the fact that $P_j(1,y) = (0,0)$, $0 \leq y \leq b$, we find that $P_jF_1 = 0$. Thus, $P_j(1, \varepsilon + f(x)) = P_jF_1(1,x) = (0,0)$ for all $x \in \mathbb{R}$ and for all ε, $0 \leq \varepsilon \leq b$. If $\varepsilon = 0$, then $f(x) + \varepsilon$ assumes all values y, $0 \leq y \leq b$, and $P_j(1,y) = P_j(1,f(x)) = (0,0)$. If $\varepsilon = b$, then $f(x) + \varepsilon = f(x) + b$ assumes all values y, $b \leq y \leq 2b$, and $P_j(1,y) = P_j(1,f(x) + b) = (0,0)$. Therefore, $P_j(1,y) = (0,0)$ for all y, $0 \leq y \leq 2b$.

Assume $P_j(1,x) = (0,0)$ for all x, $0 \leq x \leq mb$ and proceed as above. For ε, $0 \leq \varepsilon \leq mb$, let $F_m = A_{11} + \varepsilon A_{21} + A_{22} - A_{21}P_1 - \cdots - A_{21}P_q$. Then, $P_jF_m = r_{11}(\varepsilon)A_{11} + r_{12}(\varepsilon)A_{12} + r_1(\varepsilon)P_1 + \cdots + r_q(\varepsilon)P_q$ and, as above, using $(1,0)$, $(0,1)$, and $(1,z_i)$, $i = 1,2,\cdots,q$, we have that $P_jF_m = 0$. Therefore, $P_jF_m(1,x) = P_j(1, \varepsilon + f(x)) = (0,0)$ for all $x \in \mathbb{R}$ and all ε, $0 \leq \varepsilon \leq mb$. For $\varepsilon = mb$, $f(x) + \varepsilon = f(x) + mb$ is a continuous function which assumes all the values y, $mb \leq y \leq (m+1)b$; thus, combining this with the induction hypothesis, we have that $P_j(1,y) = (0,0)$ for all $0 \leq y \leq (m+1)b$. Therefore, by induction, $P_j(1,x) = (0,0)$ for all x, $0 \leq x \leq mb$, for all $m \in \mathbb{N}$.

If, on the other hand, $f(u) < 0$ for all $u \in \mathbb{R}$, then let $G = A_{11} - A_{22} + A_{21}P_1 + \cdots + A_{21}P_q$. By Lemma 4.7, $P_jG = s_{11}A_{11} + s_{12}A_{12} + s_1P_1 + \cdots + s_qP_q$. Using (1,0), (0,1) and $(1,z_i)$, $i = 1,2,\cdots,q$, we find that $P_jG = 0$. Thus, $P_jG(1,x) = P_j(1-x+P_{11}(1,x)+P_{21}(1,x)+\cdots+P_{q1}(1,x)) = P_j(1,-f(x))$ for all $x \in \mathbb{R}$. Since $-f(x) > 0$ for all $x \in \mathbb{R}$, we can proceed as above and show that $P_j(1,x) = (0,0)$ for all x, $0 \leq x \leq mv$, for all $m \in \mathbb{N}$ and some $v > 0$.

Therefore, in either case, we can show $P_j(1,1) = (0,0)$ by choosing $m \in \mathbb{N}$ sufficiently large. This implies that $P_i(1,1) = (0,0)$ for $i = 1,2,\cdots,q$.

4.17 Lemma. There is no finite dimensional sub near algebra of $T_C(\mathbb{R}^2)$ which properly contains the linear functions $\mathcal{B}(\mathbb{R}^2)$.

Proof: Proposition 4.3 shows that there is no 5-dimensional sub near algebra of $T_C(\mathbb{R}^2)$ which contains $\mathcal{B}(\mathbb{R}^2)$. We will assume there exists a (4+k)-dimensional sub near algebra, $k \geq 2$, and show that this leads to the contradiction that $P_1 \equiv 0$.

If there is a (4+k)-dimensional sub near algebra of $T_C(\mathbb{R}^2)$ which contains $\mathcal{B}(\mathbb{R}^2)$, $k \geq 2$, then, by Lemma 4.16, there exists a basis $\mathcal{A}_2 \cup \mathcal{P}_k(P_i)$ such that P_i is homogeneous and $P_i(1,0) = P_i(0,1) = P_i(1,1) = (0,0)$ for $i = 1,2,\cdots,q$, $k = 2q$. $\mathcal{P}_k(P_i)$ is a completely symmetric set of coordinate functions generated by the first coordinate functions $\{P_i | i = 1,2,\cdots,q\}$.

By Lemma 4.7, for any $z \in \mathbb{R}$, there exists scalars

$t_{11}(z)$, $t_{12}(z)$ and $t_i(z)$, $i = 1,2,\cdots,q$, such that $P_1(A_{11} + A_{21} + zA_{22}) = t_{11}(z)A_{11} + t_{12}(z)A_{12} + t_1(z)P_1 + \cdots$ $\cdots + t_q(z)P_q$. Using (1,0), (0,1) and the fact that $P_1(1,1)$ $= P(0,u) = (0,0)$ for $u \in \mathbb{R}$, we have that $t_{11}(z) = t_{12}(z) = 0$ for all $z \in \mathbb{R}$. For any $z \in \mathbb{R}$ and $(x,1) \in \mathbb{R}^2$, $P_1(x,x+z)$ $= P_1(A_{11} + A_{21} + zA_{22})(x,1) = t_1(z)P_1(x,1) + \cdots + t_q(z)P_q(x,1)$. Thus, for all $y \in \mathbb{R}$, $P_1(1,y) = P_1(1,1+(y-1)) = t_1(y-1)P_1(1,1) +$ $\cdots + t_q(y-1)P_q(1,1) = (0,0)$. Therefore, for $(u,v) \in \mathbb{R}^2$, if $u \neq 0$, then $P_1(u,v) = uP_1(1,\frac{v}{u}) = (0,0)$ and, if $u = 0$, then $P_1(0,v) = (0,0)$. Thus, $P_1(u,v) = (0,0)$ for all $(u,v) \in \mathbb{R}^2$. ▯

We now state and prove the major result of this paper.

<u>4.7 Theorem.</u> There is no finite dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which properly contains $\mathcal{B}(\mathbb{R}^n)$ for $n \geq 2$.

Proof: We proceed by induction on n. For $n = 2$, the result follows from Lemma 4.17. Assume there is no finite dimensional sub near algebra of $T_C(\mathbb{R}^s)$ which properly contains $\mathcal{B}(\mathbb{R}^s)$ for $2 \leq s < n$ and show this implies there is no finite dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which properly contains $\mathcal{B}(\mathbb{R}^n)$.

If there exists an (n^2+k)-dimensional sub near algebra of $T_C(\mathbb{R}^n)$ which properly contains $\mathcal{B}(\mathbb{R}^n)$, then $k \geq 2$ by Proposition 4.3. Therefore, by Lemmas 4.9 and 4.13, there exists a basis $\mathcal{A}_n \cup \mathcal{P}_k(P_i)$ such that $\mathcal{P}_k(P_i)$ is a completely symmetric set of coordinate functions generated by the first coordinate functions $\{P_i \mid i = 1,2,\cdots,q\}$, $k = nq$, and each P_i is homogeneous with $P_i(\bar{e}_j) = \bar{0}$ for $i = 1,2,\cdots,q$ and $j = 1,2,\cdots,n$. We will proceed to reach the contradiction that $P_1 = 0$.

We first show that $P_i|_{V_r} = 0$ for $i = 1,2,\cdots,q$ and $r = 1,2,\cdots,n-1$ where $V_r = \langle \bar{e}_1,\bar{e}_2,\cdots,\bar{e}_r \rangle$ is the linear subspace of $\mathbb{R}^n$ spanned by the first r basis elements of $\mathbb{R}^n$. If this is not the case, then there exists i_o and p, $2 \leq p \leq n-1$, such that $P_{i_o}|_{V_p} \neq 0$ and we let M $=\{i \mid 1 \leq i \leq q$ and $P_i|_{V_p} \neq 0\}$. Note, the case for $r = 1$ trivially holds since $P_i(x\bar{e}_1) = xP_i(\bar{e}_1) = 0$. By reordering, we may assume, without loss of generality, that $M = \{1,2,\cdots,m\}$ for some m, $1 \leq m \leq q$.

Let $\{\bar{d}_1,\bar{d}_2,\cdots,\bar{d}_p\}$ be the standard basis set for $\mathbb{R}^p$ and let $\mathcal{B}_p = \{B_{ij} \mid i, j = 1,2,\cdots,p\}$ be the standard basis for $\mathcal{B}(\mathbb{R}^p)$. For $i = 1,2,\ldots,m$, define $R_i: \mathbb{R}^p \rightarrow \mathbb{R}^p$ by the formula $R_i(x_1\bar{d}_1+\cdots+x_p\bar{d}_p) = P_{i1}(x_1\bar{e}_1+\cdots x_p\bar{e}_p)\bar{d}_1$. Each R_i is a continuous homogeneous first coordinate function such that $R_i(\bar{d}_j) = \bar{0}$ for $i = 1,2,\cdots,m$ and $j = 1,2,\ldots,p$. Also, R_i is non-zero and non-linear on $\mathbb{R}^p$ since this would imply $P_{i1}(x_1\bar{e}_1+\cdots+x_p\bar{e}_p) = x_1P_{i1}(\bar{e}_1)+\cdots+x_pP_{i1}(\bar{e}_p) = \bar{0}$, for all $x_1\bar{e}_1+\cdots+x_p\bar{e}_p \in V_p$. Let $\mathcal{R} = \mathcal{R}(\{R_i \mid i = 1,2,\cdots,m\})$ be the completely symmetric set of coordinate functions generated by the first coordinate functions $\{R_i \mid i = 1,2,\cdots,m\}$. Let N be the linear subspace of $T_C(\mathbb{R}^p)$ spanned by $\mathcal{B}_p \cup \mathcal{R}$. We wish to show N is a sub near algebra of $T_C(\mathbb{R}^p)$ which properly contains $\mathcal{B}(\mathbb{R}^p)$. This will contradict the induction hypothesis.

Since the spanning set $\mathcal{B}_p \cup \mathcal{R}$ consists of coordinate functions and $\mathcal{R}$ is a completely symmetric set of coordinate functions, we only need to consider closure with respect to multiplication on the left by R_i and B_{ij}. Any linear coordinate function B_{ij} merely shifts coordinates; hence $B_{ij}R_k = 0$ or $B_{ij}R_k \in \mathcal{R}$ for all i, j and k. Thus, for $F \in N$, $B_{ij}F \in N$.

$$\text{Let } F = \sum_{i=1}^{p} \sum_{j=1}^{p} s_{ij}B_{ij} + \sum_{i=1}^{p} \sum_{j=1}^{m} t_{ij}B_{i1}R_j \in N.$$

For an arbitrary element $\overline{x}_d = x_1\overline{d}_1+\cdots+x_p\overline{d}_p \in \mathbb{R}^p$, let $\overline{x}_e = x_1\overline{e}_1+\cdots+x_p\overline{e}_p$ be the corresponding element of $\mathbb{R}^n$. Thus, for arbitrary R_k, $1 \leq k \leq m$,

$$R_kF(\overline{x}_d) = R_k\left(\sum_{i=1}^{p}\left(\sum_{j=1}^{p} s_{ij}x_j + \sum_{j=1}^{m} t_{ij}P_{j1}(\overline{x}_e)\right)\overline{d}_i\right)$$

$$= \left[P_{k1}\left(\sum_{i=1}^{p}\left(\sum_{j=1}^{p} s_{ij}x_j + \sum_{j=1}^{m} t_{ij}P_{j1}(\overline{x}_e)\right)\overline{e}_i\right)\right]\overline{d}_1$$

However, by Lemma 4.7, there exist scalars α_{1i} and α_i such

$$\text{that } P_k\left(\sum_{i=1}^{p} \sum_{j=1}^{p} s_{ij}A_{ij} + \sum_{i=1}^{p} \sum_{j=1}^{m} t_{ij}A_{i1}P_j\right)$$

$$= \sum_{i=1}^{n}\alpha_{1i}A_{1i} + \sum_{i=1}^{q}\alpha_i P_i.$$ Using e_j, $j = p+1, p+2,\cdots,n$,

we get that $\alpha_{ij} = 0$ for $j = p+1, p+2,\cdots,n$. Also, the $\overline{x}_e$ which corresponds to $\overline{x}_d$ belongs to V_p; hence,

$$\sum_{i=1}^{q}\alpha_i P_i(x_e) = \sum_{i=1}^{m}\alpha_i P_i(x_e)$$ since $P_i\big|_{V_p} = 0$ for $i = m+1, m+2,\cdots,q$. Therefore,

$$P_k\left(\sum_{i=1}^{p}\ \sum_{j=1}^{p} s_{ij}A_{ij}(\overline{x}_e) + \sum_{i=1}^{p}\ \sum_{j=1}^{m} t_{ij}A_{i1}P_j(\overline{x}_e)\right)$$

$$= \left[P_{k1}\left(\sum_{i=1}^{p}\left(\sum_{j=1}^{p} s_{ij}x_j + \sum_{j=1}^{m} t_{ij}P_{j1}(\overline{x}_e)\right)\overline{e}_i\right)\right]\overline{e}_1$$

$$= \sum_{i=1}^{p}\alpha_{1i}A_{1i}(\overline{x}_e) + \sum_{i=1}^{m}\alpha_iP_i(\overline{x}_e) = \left(\sum_{i=1}^{p}\alpha_{1i}x_i + \sum_{i=1}^{m}\alpha_iP_{i1}(\overline{x}_e)\right)\overline{e}_1.$$

Setting the coefficients of $\overline{e}_1$ equal,

$$R_kF(\overline{x}_d) = \left[P_{k1}\left(\sum_{i=1}^{p}\left(\sum_{j=1}^{p} s_{ij}x_j + \sum_{j=1}^{m} t_{ij}P_{j1}(\overline{x}_e)\right)\overline{e}_i\right)\right]\overline{d}_1$$

$$= \left(\sum_{i=1}^{p}\alpha_{1i}x_i + \sum_{i=1}^{m}\alpha_iP_{i1}(\overline{x}_e)\right)\overline{d}_1$$

$$= \left(\sum_{i=1}^{p}\alpha_{1i}B_{1i} + \sum_{i=1}^{m}\alpha_iR_i\right)\overline{x}_d.$$

Therefore, $R_kF \in N$ for all $F \in N$ and $k = 1,2,\cdots,m$. This, along with right distributivety in $T_C(\mathbb{R}^p)$ and the completely symmetric property of $\mathcal{R}$, completes the proof that N is a near algebra.

Since none of the R_i can be linear, there exists an R_i such that $\mathcal{B}_p \cup \{R_i\}$ is a linearly independent set. Therefore, N is a finite dimensional sub near algebra of $T_C(\mathbb{R}^p)$, $2 \leq p \leq n-1$, which properly contains $\mathcal{B}(\mathbb{R}^p)$. Thus, $P_i|_{V_r} = 0$ for all $i = 1,2,\cdots,q$ and $r = 1,2,\cdots,n-1$.

Let $r \in \mathbb{N}$ be arbitrary, $1 \leq r \leq n-1$, and let $\{\overline{e}_{s_1},\overline{e}_{s_2},\cdots,\overline{e}_{s_r}\}$ be an arbitrary collection of r distinct standard basis elements of $\mathbb{R}^n$. Also, let $S_r = \langle e_{s_1}, e_{s_2}, \cdots, e_{s_r} \rangle$ be the r-dimensional linear subspace of $\mathbb{R}^n$ spanned by this collection. By Lemma 4.7, for arbitrary i, $1 \leq i \leq q$, there exists scalars t_{1j} and t_j such that

$$P_i(A_{s_1 1} + A_{s_2 2} + \cdots + A_{s_r r}) = \sum_{j=1}^{n} t_{1j}A_{1j} + \sum_{j=1}^{q} t_j P_j.$$ Using

$\bar{e}_m$, $m = 1,2,\cdots,n$, we have that $t_{1j} = 0$ for all $j = 1,2,\cdots,n$. Let $\bar{x} = x_1\bar{e}_{s_1} + x_2\bar{e}_{s_2} + \cdots + x_r\bar{e}_{s_r}$ be an arbitrary element of S_r and let $\bar{x}_e = x_1\bar{e}_1 + x_2\bar{e}_2 + \cdots + x_r\bar{e}_r$ be the corresponding element in $V_r = \langle \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r \rangle$ described above. Then, $P_i(A_{s_1 1} + A_{s_2 2} + \cdots + A_{s_r r})\bar{x}_e = P_i(\bar{x}) = t_1 P_1(\bar{x}_e) + t_2 P_2(\bar{x}_e) + \cdots$ $\cdots + t_q P_q(\bar{x}_e) = \bar{0}$ by the previous discussion. Therefore, $P_i|_S = 0$ for $i = 1,2,\cdots,q$ where S is any r-dimensional linear subspace of $\mathbb{R}^n$ spanned by r distinct standard basis elements of $\mathbb{R}^n$, $1 \leq r \leq n-1$.

We can now complete the proof of the theorem by showing that $P_1 = 0$. Let $\bar{x} = (x_1, x_2, \cdots, x_n)$ be an arbitrary element of $\mathbb{R}^n$ and let $F_x = x_1A_{11} + x_2A_{22} + \cdots + x_{n-1}A_{n-1n-1} + x_nA_{n1}$. By Lemma 4.7, there exist scalars $t_{1j}(\bar{x})$ and $t_j(\bar{x})$ such that $$P_1F_x = \sum_{j=1}^{n} t_{1j}(\bar{x})A_{1j} + \sum_{j=1}^{q} t_j(\bar{x})P_j.$$ Using $\bar{e}_m$, $m = 1,2,\cdots,n$, we have that $P_1F_x(\bar{e}_m) = P_1(x_m\bar{e}_m)$, $m \neq 1$, and $P_1F_x(\bar{e}_1) = P_1(x_1\bar{e}_1 + x_n\bar{e}_n)$. In either case, by the above discussion, $P_1F_x(\bar{e}_m) = \bar{0} = t_{1m}(\bar{x})\bar{e}_m$. Thus, $t_{1m}(\bar{x}) = 0$ for all $\overline{x} \in \mathbb{R}^n$ and $m = 1,2,\cdots,n$. Let $\overline{e} = \overline{e}_1+\overline{e}_2+\cdots+\overline{e}_{n-1}$ $\in V_{n-1}$ and let $\overline{x}$ be an arbitrary element of $\mathbb{R}^n$. Then, $P_1F_x(\overline{e}) = P_1(\overline{x}) = t_1(\overline{x})P_1(\overline{e})+\cdots+t_q(\overline{x})P_q(\overline{e}) = \overline{0}$ since $P_i\big|_{V_{n-1}} = 0$. Therefore, $P_1 = 0$. This completes the induction and the proof. $\square$

The Representation of a Finite Dimensional Near Algebra as a Function Space

We showed, in Proposition 4.2 of section one, that every one-dimensional near algebra can be represented as a function space by identifying the near algebra with the space generated by the coefficient function in the expression $e(te) = (t)e$. This is similar to the construction of a representation for a finite dimensional algebra as a matrix algebra given in [1]. Consider a finite dimensional algebra A with basis $B = \{u_1, u_2, \cdots, u_n\}$. If $x = \sum_{j=1}^{n} t_j u_j$, then

$$xu_i = \sum_{j=1}^{n} t_{ij} u_j \text{ for } i = 1,2,\cdots,n \text{ where } t_{ij} = \sum_{k=1}^{n} t_k r_i^{kj} \text{ and}$$

the r_i^{kj} are multiplication constants determined by

$u_k u_j = \sum_{i=1}^{n} r_i^{kj} u_i$. Let U be the one-columned matrix whose elements are $u_1, u_2, \cdots, u_n$ and let $X = (t_{ij})$ be the matrix of coefficients, then the above equations for xu_i, $i = 1,2,\cdots,n$, become $xU = XU$. This defines a correspondence $x \rightarrow X$ from the algebra A to the set $\mathfrak{M}$ of all matrices X. This correspondence is an algebra homomorphism and the algebra $\mathfrak{M}$ of matrices is called the regular representation of A with respect to the basis $u_1, u_2, \cdots, u_n$. The correspondence is an isomorphism if and only if there is no quantity $x \neq 0$ in A such that $xa = 0$ for every a of A.

Although, at the present time, there is no generalized near algebra of matrices, we can construct a linear space of coefficient functions which we will designate as the linear coefficient function space determined by the given basis. However, unlike the case for an algebra, we have not been able to show that, in general, the linear coefficient space is a near algebra. To accomplish this, we introduce the concept of a basis with an annihilator set; for example, an orthogonal idempotent basis, and prove that the linear coefficient space becomes a near algebra. The resulting representation theorem generalizes the result of Theorem 3.9 and is equivalent to the matrix algebra approach whenever the near algebra is an algebra.

We proceed by first defining the construction of the linear coefficient space and then the special annihilator condition. Repeated use of this condition and the properties of a finite dimensional near algebra results in the principle theorem of this section which states that every n-dimensional near algebra with a basis which has an annihilator set can be represented by a sub near algebra of $T(\mathbb{R}^n)$.

<u>4.3 Definition.</u> Let N be an n-dimensional near algebra and let $B = \{x_1, x_2, \cdots, x_n\}$ be a basis for N. For each $i = 1, 2, \cdots, n$, define the <u>coefficient functions</u> $\lambda_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the following manner: For $\overline{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$ define

$\lambda_i(\bar{w}) = (\lambda_{i1}(\bar{w}), \lambda_{i2}(\bar{w}), \cdots, \lambda_{in}(\bar{w}))$ where the $\lambda_{ij}(\bar{w})$ are the coefficients in the expression $x_i(w_1x_1 + w_2x_2 + \cdots + w_nx_n)$ $= \lambda_{i1}(\bar{w})x_1 + \lambda_{i2}(\bar{w})x_2 + \cdots + \lambda_{in}(\bar{w})x_n$. Let $\mathcal{C}_B =$ $\langle \lambda_1, \lambda_2, \cdots, \lambda_n \rangle$ be the linear space generated by the functions λ_i, $i = 1, 2, \cdots, n$.

4.4 Definition. Let N be an n-dimensional near algebra and let $B = \{x_1, x_2, \cdots, x_n\}$ be a basis for N. If there exists a set of n elements $\{y_1, y_2, \cdots, y_n\} \subseteq N$ such that $x_iy_j = 0$ for $i \neq j$ and $x_iy_i \neq 0$, then $\{y_1, y_2, \cdots, y_n\}$ is called a set of annihilators for the basis B and B is said to be a basis with a set of annihilators.

An orthogonal idempotent basis B, as defined in Chapter II, is an example in which B is its own set of annihilators.

Although, in general, we are unable to show whether or not $\mathcal{C}_B$ is a near algebra, the above condition is sufficient for this result.

4.18 Lemma. If N is an n-dimensional near algebra with basis $B = \{x_1, x_2, \cdots, x_n\}$ which has a set of annihilators $\{y_1, y_2, \cdots, y_n\}$ such that $x_iy_j = \delta_{ij}z_j$, then, for $\bar{w} \in \mathbb{R}^n$ and $\lambda_i \in \mathcal{C}_B$, $\lambda_{ij}(\bar{w})z_j = x_i(w_jz_j)$ and $\lambda_{ij}(\bar{w}) = \lambda_{ij}(\bar{u})$ whenever $w_j = u_j$.

Proof: Let N be such a near algebra with basis $B =$ $\{x_1, x_2, \cdots, x_n\}$ and let $\{y_1, y_2, \cdots, y_n\}$ be the set of annihilators such that $x_iy_j = \delta_{ij}z_j$. Let $\bar{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$ and let λ_i be an arbitrary element of $\mathcal{C}_B$. For any

j, $1 \leq j \leq n$, $\lambda_{ij}(\bar{w})z_j = \sum_{k=1}^{n} \lambda_{ik}(\bar{w})x_k y_j = \left(x_i \sum_{k=1}^{n} w_k x_k\right) y_j$ $= x_i(w_j z_j)$. Let $\bar{u} \in \mathbb{R}^n$ such that $w_j = u_j$; then, $\lambda_{ij}(\bar{w})z_j$ $= x_i(w_j z_j) = x_i(u_j z_j) = \lambda_{ij}(\bar{u})z_j$. Since $z_j \neq 0$, $\lambda_{ij}(\bar{w})$ $= \lambda_{ij}(\bar{u})$. //

<u>4.8 Theorem.</u> If N is an n-dimensional near algebra with basis $B = \{x_1, x_2, \cdots, x_n\}$ which has a set of annihilators, then the linear coefficient space $\mathcal{C}_B$ is an n-dimensional near algebra.

Proof: Let N be such a near algebra with basis $B = \{x_1, x_2, \cdots, x_n\}$ and let $\{y_1, y_2, \cdots, y_n\}$ be the set of annihilators such that $x_i y_j = \delta_{ij} z_j$.

In order to show that the linear space $\mathcal{C}_B$ is a near algebra, it is sufficient, along with right distributivity, to prove that there exists scalars b_{ij} such that

$\lambda_i \left(\sum_{k=1}^{n} a_k \lambda_k\right) = \sum_{k=1}^{n} b_{ik} \lambda_k$ for each $i = 1, 2, \cdots, n$. To this end, let $\sum_{k=1}^{n} a_k \lambda_k$ be an arbitrary element of $\mathcal{C}_B$ and let p and m be arbitrary but fixed natural numbers such that $1 \leq p$, $m \leq n$. Let $\sum_{k=1}^{n} a_k x_k$ be the element in N with the same scalar coefficients and choose scalars b_{pj} such that $x_p \left(\sum_{k=1}^{n} a_k x_k\right) =$

$= \sum_{k=1}^{n} b_{pk}x_k$. Also, let $\bar{w} \in \mathbb{R}^n$ with m^{th} coordinate w_m. Then,

$$\left[x_p\left(\sum_{k=1}^{n} a_k x_k\right)\right](w_m z_m) = x_p\left(\sum_{k=1}^{n}(a_k x_k)w_m z_m\right) =$$

$$= x_p\left(\sum_{k=1}^{n}(a_k \lambda_{km}(\bar{w}))\ z_m\right) = \lambda_{pm}(\bar{u})\ z_m \text{ where } u_j = \sum_{k=1}^{n} a_k \lambda_{kj}(\bar{w}).$$

Also, $\left[x_p\left(\sum_{k=1}^{n} a_k x_k\right)\right](w_m z_m) = \left(\sum_{k=1}^{n} b_{pk}x_k\right)(w_m z_m) =$

$\left(\sum_{k=1}^{n} b_{pk}\lambda_{km}(\bar{w})\right) z_m$. Therefore, equating the two expressions,

$\lambda_{pm}(\bar{u}) = \sum_{k=1}^{n} b_{pk}\lambda_{km}(\bar{w})$. Using the coordinatewise sum of

n-tuples, we can rewrite $\bar{u}$ as the sum $\bar{u} = \left(\sum_{k=1}^{n} a_k \lambda_k\right)(\bar{w})$.

Therefore, $\lambda_{pm}(\bar{u}) = \left(\lambda_{pm}\left(\sum_{k=1}^{n} a_k \lambda_k\right)\right)(\bar{w}) = \left(\sum_{k=1}^{n} b_{pk}\lambda_{km}\right)(\bar{w})$

for all $\bar{w} \in \mathbb{R}^n$ and for arbitrary p and m. Thus, for $\bar{w} \in \mathbb{R}^n$

and i, $1 \leq i \leq n$, $\left(\lambda_i\left(\sum_{k=1}^{n} a_k\ {}_k\right)\right)(\bar{w}) =$

$$= \left(\left[\lambda_{i1}\left(\sum_{k=1}^{n} a_k \lambda_k\right)\right](\bar{w}), \left[\lambda_{i2}\left(\sum_{k=1}^{n} a_k \lambda_k\right)\right](\bar{w}), \cdots, \left[\lambda_{in}\left(\sum_{k=1}^{n} a_k \lambda_k\right)\right](\bar{w})\right) =$$

$= \left(\sum_{k=1}^{n} b_{ik}\lambda_k\right)(\overline{w})$. Hence, $\lambda_i\left(\sum_{k=1}^{n} a_k\lambda_k\right) = \sum_{k=1}^{n} b_{ik}\lambda_k$ for all i and $\mathcal{C}_B$ is a near algebra.

Assume $\sum_{k=1}^{n} a_k\lambda_k = 0$. Then, for all $\overline{w} \in \mathbb{R}^n$ and for all $i = 1,2,\cdots,n$, $\left(\sum_{k=1}^{n} a_k\lambda_{ki}\right)(\overline{w}) = 0$. For each $j = 1,2,\cdots,n$, let $y_j = t_{j1}x_1 + t_{j2}x_2 + \cdots + t_{jn}x_n$ be the expansion of y_j and let $\overline{t}_j = (t_{j1}, t_{j2}, \cdots, t_{jn}) \in \mathbb{R}^n$. We now proceed to show that $a_i = 0$ for all $i = 1,2,\cdots,n$.

Let k be an arbitrary but fixed integer, $1 \leq k \leq n$. Then, for $i = 1,2,\cdots,n$, $x_iy_k = x_i(t_{k1}x_1 + t_{k2}x_2 + \cdots + t_{kn}x_n)$ $= \lambda_{i1}(\overline{t}_k)x_1 + \lambda_{i2}(\overline{t}_k)x_2 + \cdots + \lambda_{in}(\overline{t}_k)x_n = \delta_{ik}z_k$. Since x_ky_k $= z_k \neq 0$, there must exist m, $1 \leq m \leq n$, such that $\lambda_{km}(\overline{t}_k) \neq 0$. For $i \neq k$, $x_iy_k = 0$ and $\lambda_{i1}(\overline{t}_k) = \lambda_{i2}(\overline{t}_k) = \cdots = \lambda_{in}(\overline{t}_k) = 0$. In particular, for all i, $1 \leq i \leq n$, $i \neq k$, $\lambda_{im}(\overline{t}_k) = 0$ for the fixed m. Therefore, $a_1\lambda_{1m}(\overline{t}_k) + a_2\lambda_{2m}(\overline{t}_k) + \cdots + a_n\lambda_{nm}(\overline{t}_k)$ $= a_k\lambda_{km}(\overline{t}_k) = 0$ and $a_k = 0$. Since k was arbitrary, $a_1 = a_2 = \cdots = a_n = 0$. Thus, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is a linearly independent set and $\mathcal{C}_B$ is n-dimensional. ▯

<u>4.9 Theorem.</u> Every finite dimensional near algebra with a basis B which has an annihilator set is near algebra isomorphic to the coefficient function space $\mathcal{C}_B$.

Proof: Let N be such an n-dimensional near algebra with basis $B = \{x_1, x_2, \cdots, x_n\}$ and annihilator set $\{y_1, y_2, \cdots, y_n\}$ such that $x_i y_j = \delta_{ij} z_j$. By Theorem 4.8, $\mathcal{C}_B$ is an n-dimensional near algebra with basis $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. For $x_i \in B$, define $\varphi(x_i) = \lambda_i$ and extend linearly. Clearly, φ is a linear space isomorphism; we wish to show φ preserves multiplication.

Let $x = \sum_{k=1}^{n} t_k x_k$ and $y = \sum_{k=1}^{n} s_k x_k$ be arbitrary elements of N and let $\overline{t} = (t_1, t_2, \cdots, t_n)$; $\overline{s} = (s_1, s_2, \cdots, s_n)$.

Then, $\varphi(xy) = \varphi\left(\sum_{k=1}^{n} t_k x_k y\right) = \sum_{i=1}^{n} \varphi\left(\sum_{k=1}^{n} t_k \lambda_{ki}(\overline{s})\right) x_i$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} t_k \lambda_{ki}(\overline{s})\right) \lambda_i .$$

For the same x, $y \in N$, $\varphi(x)\,\varphi(y) =$

$= \left(\sum_{k=1}^{n} t_k \lambda_k\right)\left(\sum_{i=1}^{n} s_i \lambda_i\right)$. For any $\overline{u} \in \mathbb{R}^n$, let $\overline{w} = \sum_{j=1}^{n} s_j \lambda_j(\overline{u})$.

Then, $\varphi(x)\,\varphi(y)(\overline{u}) = \varphi(x)(\overline{w}) =$

$\left(\sum_{k=1}^{n} t_k \lambda_{k1}(\overline{w}), \sum_{k=1}^{n} t_k \lambda_{k2}(\overline{w}), \cdots, \sum_{k=1}^{n} t_k \lambda_{kn}(\overline{w})\right)$. The desired result will follow, if we compute $\lambda_{ij}(\overline{w})$ for all i, $j = 1, 2, \cdots, n$.

Let i, j be arbitrary natural numbers such that $1 \leq i, j \leq n$ and recall that $w_p = \sum_{k=1}^{n} s_k \lambda_{kp}(\overline{u})$ is the p^{th} coordinate of $\overline{w}$.

$$x_i\left(\sum_{k=1}^{n} w_k x_k\right) y_j = \lambda_{ij}(\bar{w})z_j = x_i(w_j z_j) \text{ where } w_j z_j =$$

$$= \sum_{k=1}^{n} s_k \lambda_{kj}(\bar{u})z_j = \sum_{k=1}^{n}(s_k x_k(u_j z_j)) = \left(\sum_{k=1}^{n} s_k x_k\right)(u_j z_j).$$

Therefore, $\lambda_{ij}(\bar{w})z_j = x_i(w_j z_j) = \left(x_i\left(\sum_{k=1}^{n} s_k x_k\right)\right)(u_j z_j) =$

$$=\left(\sum_{k=1}^{n}\lambda_{ik}(\bar{s})x_k\right)(u_j z_j) = \sum_{k=1}^{n}(\lambda_{ik}(\bar{s})(x_k(u_j z_j))) =$$

$=\left(\sum_{k=1}^{n}\lambda_{ik}(\bar{s})\lambda_{kj}(\bar{u})\right) z_j$. Thus, $\lambda_{ij}(\bar{w}) = \sum_{k=1}^{n}\lambda_{ik}(\bar{s})\lambda_{kj}(\bar{u})$ for all i, j = 1,2,⋯,n. Using this in the expression for $\varphi(x)\varphi(y)(\bar{u})$ we get, after collecting the coefficients for $\lambda_{ij}(\bar{u})$, that $\varphi(x)\varphi(y)(\bar{u}) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} t_k \lambda_{ki}(\bar{s})\right)\lambda_i(\bar{u})$.

Therefore, $\varphi(x)\varphi(y) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} t_k\lambda_{ki}(\bar{s})\right)\lambda_i = \varphi(xy)$ and φ is a near algebra isomorphism from N to $\mathcal{C}_B$. ▯

<u>4.9.1 Corollary.</u> If N is an n-dimensional near algebra with basis $B = \{x_1, x_2, \cdots, x_n\}$ such that $x_i(tx_j) = \delta_{ij} t x_j$ for all $t \in \mathbb{R}$, then N is an algebra.

Proof: Let N be an n-dimensional near algebra with such a basis $B = \{x_1, x_2, \cdots, x_n\}$. If $\bar{u}, \bar{v} \in \mathbb{R}^n$, $\lambda_{ij}(\bar{u}+\bar{v})x_j = x_i(u_j+v_j)x_j = \delta_{ij}(u_j+v_j)x_j = \lambda_{ij}(\bar{u})x_j + \lambda_{ij}(\bar{v})x_j$. Therefore, $\lambda_{ij}(\bar{u}+\bar{v}) = \lambda_{ij}(\bar{u}) + \lambda_{ij}(\bar{v})$. Similarly,

$\lambda_{ij}(t\bar{u}) = x_i(tu_jx_j) = \delta_{ij}(tu_j)x_j = t\,\lambda_{ij}(\bar{u})$ for all $t \in \mathbb{R}$ and $\bar{u} \in \mathbb{R}^n$. Therefore, λ_i is linear for all $i = 1,2,\cdots,n$ and $\mathcal{C}_B$ is an algebra. //

For emphasis we repeat the following result.

4.9.2 Corollary. Every one-dimensional near algebra over $\mathbb{R}$ with non-trivial multiplication is near algebra isomorphic to $\mathcal{C}_B$.

We close this section with two examples and some facts concerning a finite dimensional near algebra with a basis which has a set of annihilators.

First, let V be an n-dimensional linear space with basis $B = \{x_1, x_2, \cdots, x_n\}$. For x, y $\in$ V where $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ and $y = b_1x_1 + b_2x_2 + \cdots + b_nx_n$, define $xy = a_1|b_1|x_1 + a_2|b_2|x_2 + \cdots + a_n|b_n|x_n$. With this product as multiplication, V becomes an n-dimensional near algebra which is not an algebra and $x_ix_j = \delta_{ij}x_j$.

The second example demonstrates that a basis with a set of annihilators is not a necessary condition for the linear coefficient space to be a near algebra. Let N be an n-dimensional linear space with basis $B = \{x_1, x_2, \cdots, x_n\}$. For x, y $\in$ N, define $xy = x$ if $x \neq 0$ and $y \neq 0$ and let $0x = x0 = 0$. N becomes a near algebra with this multiplication and the coefficient functions have the following form. For $\bar{u} \in \mathbb{R}^n$, $\bar{u} \neq \bar{0}$, $\lambda_i(\bar{u}) = \bar{e}_i$ and $\lambda_i(\bar{0}) = \bar{0}$. It is easily verified that $\mathcal{C}_B$ is a near algebra and $N \cong \mathcal{C}_B$. However, for the basis B or any given basis, if $x_1y_2 = 0$ then $y_2 = 0$ and the condition $x_2y_2 \neq 0$ can not be satisfied. Therefore, there is no basis of N which has an annihilator set.

We conclude with the following propositions.

<u>4.4 Proposition.</u> If N is an n-dimensional near algebra with a basis having an annihilator set, then each basis element generates a minimal left module and N is a direct sum of these minimal left modules.

Proof: Let N be such a near algebra with basis B = $\{x_1, x_2, \cdots, x_n\}$ and annihilator set $\{y_1, y_2, \cdots, y_n\}$. Let $M_i = \langle x_i \rangle$ be the linear space generated by x_i for i = 1,2,$\cdots$,n. For $x \in N$, $x(ax_i) = t_1x_1 + t_2x_2 + \cdots + t_nx_n$ and, using the annihilator element y_j, $j \neq i$, we get $x(ax_i)y_j = 0 = t_jx_j$. Therefore, $t_j = 0$ for $j \neq i$ and $x(ax_i) = t_ix_i \in M_i$. Thus, M_i is a left module for each i.

Clearly, $N = M_1 + M_2 + \cdots + M_n$. If $x \in M_i \cap M_j$, $i \neq j$, then $x = tx_i = sx_j$ which imples $x = 0$. Therefore, $N = \bigoplus_{i=1}^{n} M_i$. ∎

<u>4.5 Proposition.</u> If N is an n-dimensional near algebra with a basis having an annihilator set, then there exists a basis $B = \{x_1, x_2, \cdots, x_n\}$ having an annihilator set $\{y_1, y_2, \cdots, y_n\}$ such that $x_iy_j = \delta_{ij}y_j$.

Proof: Let N be such a near algebra with basis $\{u_1, u_2, \cdots, u_n\}$ and annihilator set $\{v_1, v_2, \cdots, v_n\}$. Let i be arbitrary but fixed, $1 \leq i \leq n$. Then, $u_iv_i = b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{in}u_n \neq 0$; thus, there exists k, $1 \leq k \leq n$, such that $b_{ik} \neq 0$. Let $v_i' = a_{ik}(u_kv_k)$ where $v_i = a_{i1}u_1 + \cdots + a_{ik}u_k + \cdots + a_{in}u_n$ and let $w_i = b_{ik}(u_kv_k)$. Thus, $v_i' = \frac{a_{ik}}{b_{ik}}(b_{ik}u_kv_k) = \frac{a_{ik}}{b_{ik}}w_i$. Also

$u_i v_i = u_i(a_{i1}u_1 + \cdots + a_{in}u_n) = b_{i1}u_1 + b_{i2}u_2 + \cdots + b_{in}u_n$ and, multiplying by v_k, we get $u_i(a_{ik}u_k v_k) = u_i v_i' = b_{ik}u_k v_k = w_i \neq 0$. Furthermore, $u_j v_i' = (u_j v_i)v_k = 0v_k = 0$ for $j \neq i$. Therefore, for each i, $1 \leq i \leq n$, we can construct elements v_i' and w_i such that $v_i' = \frac{a_{ik}}{b_{ik}}w_i$ and $\frac{a_{ik}}{b_{ik}} \neq 0$. Let $\alpha_i = \frac{a_{ik}}{b_{ik}}$ and choose a new basis $B = \{x_1, x_2, \cdots, x_n\}$ where $x_i = \alpha_i u_i$. Let $y_i = \alpha_i w_i$; then, $x_i y_j = (\alpha_i u_i)(v_j') = 0$ for $i \neq j$ and $x_i y_i = (\alpha_i u_i)v_i = \alpha_i w_i = v_i' = y_i$. $\square$

CHAPTER V

CONCLUDING QUESTIONS

The purpose of this paper has been to extend the knowledge of near algebras in the area of analysis. In particular, we have studied normed near algebras and function near algebras. In the course of our study several specific, as well as general, open questions have arisen.

In Chapter I we showed that the characterization of a near algebra generated by a non-empty set S is not necessarily all the finite linear combinations of finite products in S. In general, is there a characterization of such a generated near algebra?

Positive homogeneity, finite dimensionality and orthogonal idempotency have played a central role in showing continuity of multiplication in various near algebras. In particular, $\mathrm{Lip}_p(\mathbb{R})$, the positive homogeneous Lipschitz functions on $\mathbb{R}$, forms a finite dimensional positive homogeneous strongly D-normed near algebra with orthogonal idempotent basis. What are the properties of $\mathrm{Lip}_p(E)$ for an arbitrary normed linear space E? Can $\mathrm{Lip}_p(E)$ be given a D-norm? Is there a characterization of $\mathrm{Lip}_{LC}(E)$, the sub near algebra of Lip(E) with continuous left multiplication in the Lipschitz norm topology? Can the positive homogeneous and orthogonal idempotent conditions of section three in

Chapter II be removed to prove that all finite dimensional normed near algebras are D-normed near algebras?

Adjoining an identity to a normed near algebra required the positive homogeneous condition; can this condition be removed? Can an identity be adjoined to a D-normed near algebra? Also, the completion of a normed near algebra required a strong D-normed condition. Can a normed or D-normed near algebra be completed? Finally, does the continuous inverse property of a strongly D-normed near algebra hold for a D-normed near algebra?

In Chapter IV we considered finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ which were not necessarily normed near algebras. Although finite dimensional sub near algebras of arbitrarily large dimension have been constructed, there are no finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ for $n \geq 2$ which properly contain all the linear operators on $\mathbb{R}^n$. Is it necessary that all linear operators be included? Can finite dimensional sub near algebras of $T_C(\mathbb{R}^n)$ of arbitrarily large dimension be constructed which contain Lipschitz functions or only some of the linear functions?

In addition to these questions, there are more general questions which arise. The D-norm condition is not the only condition which insures continuous multiplication in both variables. For example, if $\|xy\| \leq K_y \|x\|$ and $\|bx - by\| \leq K_b \|x - y\|$ for each x, y, and b and for some K_y, $K_b \geq 0$, then multiplication is continuous in both variables. What other norm or weak distributive conditions will insure the continuity of left and right multiplication?

We have restricted our study to real normed near algebras. What does the theory of complex near algebras reveal? What can be said about near algebras with an involution or symmetric near algebras? Can the representation theory be improved and expanded?

We trust that this paper serves as a starting point for the answering of these questions.

BIBLIOGRAPHY

1. Albert, A. Adrian. Modern Higher Algebra. Chicago, University of Chicago Press, 1935.

2. Beidleman, James C. On near-rings and near-ring modules. Doctoral Dissertation, Pennsylvania State University, 1964.

3. ________. Quasi-regularity in near-rings. Math. Zeitschr. 89 (1965), 224-229.

4. ________. A radical for near-ring modules. Michigan Math. J. 12 (1965), 377-383.

5. ________, and Cox, R. H. Topological near-rings. Arch. Math. 18 (1967), 485-492.

6. Berman, Gerald, and Silverman, Robert J. Near-rings. Amer. Math. Monthly 66 (1959), 23-34.

7. ________. Embedding of algebraic systems. Pacific J. Math. 60 (1960), 777-786.

8. Betch, Gerhard. Ein Radikal Für Fastringe. Math. Zeitschr. 78 (1962), 86-90.

9. Blackett, Donald W. Simple and semi-simple near-rings. Proc. Amer. Math. Soc. 4 (1953). 772-785.

10. Brown, Harold D. Near algebras. Illinois J. Math.12 (1968), 215-217.

11. ________. Distributor theory in near algebra. Comm.Pure App. Math. 21 (1958), 535-544.

12. Calkin, J. W. Two-sided ideals and congruences in the ring of bounded linear operators in Hilbert space. Ann. of Math. 42 (1941), 839-873.

13. Day, Mahlon M. Normed Linear Spaces. Berlin, Springer-Verlag, 1959.

14. Deskins, W. E. A radical for near-rings. Proc. Amer. Math. Soc. 5 (1954), 825-827.

15. Dickson, Leonard E. On finite algebras. Nachr. Akad. Wiss Gottingen. (1905), 358-393.

16. Frölich, A. *Distributively generated near-rings I. Ideal theory*. Proc. London Math. Soc. 8 (1958) 76-94.

17. ________. *Distributively generated near-rings II. Representation theory*. London Math. Soc. 8 (1958), 95-108.

18. Hughes, D. R. *A class of non-Desarguesian projective planes*. Canad. J. Math. 9 (1957), 378-388.

19. Kalscheuer, Franz. *Die Bestimmung aller Stetigen Fastkörper, uber dem Körper der rellen Zalhen als Grundkörper*. Abh. Math. Sem. Univ. Hamburg. 13 (1940), 413-435.

20. Kaplansky, I. *Topological rings*. Amer. J. Math. 69 (1947), 153-183.

21. Kelley, John L. *General Topology*. Princeton, Van Nostrand, 1955.

22. Loomis, Lynn H. *An Introduction to Abstract Harmonic Analysis*. Princeton, Van Nostrand, 1953.

23. ________, and Sternberg, S. *Advanced Calculus*. Reading, Addison-Wesley, 1968.

24. MacLane, S., and Birkhoff, G. *Algebra*. New York, MacMillan, 1967.

25. Mendelsohn, N. S. *Non-Desarguesian projective plane geometries*. Cand. J. Math. 8 (1956), 532-562.

26. Naimark, M. A. *Normed Rings*. Groningen, the Netherlands, P. Noordhoff, 1964.

27. Rickart, Charles E. *General theory of Banach Algebras*. Princeton, Van Nostrand, 1960.

28. Wilansky, Albert. *Functional Analysis*. New York, Blais-Dell Publishing Company, 1964.

29. Yamamuro, Sadayuki. *On near-algebras of mappings of Banach spaces*. Proc. Japan Acad. 41 (1965), 889-892.

30. ________. *Ideals and homomorphisms in some near-algebras*. Proc. Japan Acad. 42 (1966), 427-432.

31. ________. *A note on D-ideals in some near-algebras*. J. Aust. Math. Soc. 7 (1967), 129-134.

32. ________. *On the spaces of mappings on Banach spaces*. J. Aust. Math. Soc. 7 (1967), 160-164.

33. Zassenhaus, Hans. *Über Endliche Fastkörper*. Abh. Math. Sem. Univ. Hamburg. 11 (1935-36), 187-220