Fall 1974

H-STRUCTURES ON SP(2), SU(4) AND RELATED SPACES

CURTIS PAUL MURLEY

H-STRUCTURES ON Sp(2), SU(4) AND RELATED SPACES

by

CURTIS P. MURLEY

M.S., University of Oregon, 1968

A THESIS

Submitted to the University of New Hampshire

In Partial Fulfillment of

The Requirements for the Degree of

Doctor of Philosophy

Graduate School

Department of Mathematics

November, 1974

This thesis has been examined and approved.

Thesis director, Arthur H. Copeland, Jr. Prof. of Mathematics

Edward H. Batho, Professor of Mathematics

M. Evans Munroe, Professor of Mathematics

Albert O. Shar, Assistant Professor of Mathematics

Donovan H. Van Osdol, Associate Prof. of Mathematics

November 25, 1974

Date

ACKNOWLEDGEMENTS

Many people made this thesis possible. I particularly wish to express my gratitude to Professor Arthur Copeland, my thesis advisor, whose help, encouragement, patience and hard work enabled me to complete it. I am also indebted to Professor Albert Shar, who provided some key ideas at the right time.

More generally, I want to thank those members of the Department of Mathematics of the University of New Hampshire who provided my graduate instruction and saw fit to recommend the financial support which made my studies possible. In this connection, I want to acknowledge the support provided by a National Defense Education Act Fellowship.

Finally, I am grateful to my wife, without whose encouragement and support none of this would have been possible.

Curtis P. Murley

Raymond, Maine

November, 1974

TABLE OF CONTENTS

ABSTRACT

H-STRUCTURES ON Sp(2), SU(4) AND RELATED SPACES

by

CURTIS P. MURLEY

The classical groups Sp(2) and SU(4) can be realized as the total spaces of fiber bundles over spheres. Associated in a natural way with these fiberings are a number of induced fiberings, some of whose total spaces also support H-structures. In this paper we consider the problem of determining the number of distinct H-structures supported by the classical group SU(4) and by the H-spaces associated with SU(4) and Sp(2).

The technique used to solve this problem involves the notion of localization of topological spaces. In Chapter I, we discuss localization and prove a general theorem which allows us to reduce the H-structure problem for a given space to a similar problem for its associated localized spaces.

The number of H-structures supported by Sp(2) is known. Using this number, the localization technique easily produces a complete solution for the total spaces of the associated fiberings. These computations are carried out in Chapter II.

For SU(4) the problem has not previously been solved. In Chapter III, using the general result from Chapter I, together

with information about the structure of Sp(2) and the generators of certain homotopy groups of spheres, we give a partial solution for SU(4) and its related spaces.

CHAPTER I

LOCALIZATION IN TOPOLOGY

The main results of this paper are obtained using the technique of localizing topological spaces. In this chapter we will discuss localization and some of its properties. We will also prove an important theorem that is most useful in obtaining our main results.

We begin with a brief discussion of algebraic localization in the category $\mathcal{N}$ of nilpotent groups. This is necessary since topological localization is defined in terms of algebraic localization and, further, we will later have occasion to make some computations involving localizations of nilpotent groups.

In what follows $\mathbb{P}$ denotes the set of all primes and $P \subset \mathbb{P}$ is any subset. The symbol $\langle P \rangle$ is used to denote the multiplicative set generated by P and P' will denote $\mathbb{P}-P$.

DEFINITION (1.1) $G \in \mathcal{N}$ is said to be P-local if and only if the n^{th} power map $e^n : G \longrightarrow G$ is an isomorphism for all $n \in \langle P' \rangle$.

Let $\mathbb{Z}_P = \{ {}^m/_n \in \mathbb{Q} \mid n \in \langle P' \rangle \}$ where $\mathbb{Q}$ denotes the set of rational numbers. It is easily seen that $\mathbb{Z}_P$ is P-local and in the special cases $P = \mathbb{P}$ and $P = \emptyset$ we have $\mathbb{Z}_{\mathbb{P}} = \mathbb{Z}$ and $\mathbb{Z}_{\emptyset} = \mathbb{Q}$.

PROPOSITION (1.2) If A is an Abelian group the following are equivalent: (i) A is P-local,

(ii) $A \approx A \otimes \mathbb{Z}_P$ and

(iii) A is a $\mathbb{Z}_P$-module.

We denote by $\mathcal{N}_P$ the full subcategory of $\mathcal{N}$ consisting of P-local nilpotent groups and by $i_P:\mathcal{N}_P \longrightarrow \mathcal{N}$ the inclusion functor.

DEFINITION (1.3) If $B \in \mathcal{N}_P$ then $e \in \mathrm{Hom}_{\mathcal{N}}(G, i_P B)$ is said to P-localize G if and only if given any $H \in \mathcal{N}_P$ and $g \in \mathrm{Hom}_{\mathcal{N}}(G, i_P H)$ there exists a unique $g' \in \mathrm{Hom}_{\mathcal{N}_P}(B, H)$ such that $g = i_P(g') \circ e$.

DEFINITION (1.4) $\varphi \in \mathrm{Hom}_{\mathcal{N}}(G, K)$ is said to be a P-isomorphism if and only if

(i) $g \in \ker\varphi \Rightarrow \exists m \in \langle P' \rangle$ such that $g^m = 1$ and

(ii) for all $k \in K$ there exists $n \in \langle P' \rangle$ and $g \in G$ such that $\varphi(g) = k^n$.

THEOREM (1.5) [Lazard (1954)]. There exists a functor $L_P:\mathcal{N} \to \mathcal{N}_P$. Further, for each $G \in \mathcal{N}$ there exists a morphism $e_P \in \mathrm{Hom}_{\mathcal{N}}(G, i_P L_P(G))$ having the property that e_P P-localizes G.

The functor L_P of the above theorem is called the P-localization functor and the P-local nilpotent group $L_P(G)$, usually denoted by G_P, is called the P-localization of G. It is not difficult to show that for any $G \in \mathcal{N}$, the pair $L_P G$ and e_P are uniquely determined up to isomorphism.

THEOREM (1.6) [Hilton (1973)]. If $H \in \mathcal{N}_P$ and $G \in \mathcal{N}$ then $\varphi \in \mathrm{Hom}_{\mathcal{N}}(G, i_P H)$ P-localizes G if and only if φ is a P-isomorphism.

COROLLARY (1.7) $L_P:\mathcal{N} \longrightarrow \mathcal{N}_P$ is an exact functor.

We now turn to a description of topological localization in the homotopy category of simply connected pointed CW complexes. We will denote this category by $\mathcal{C}$.

DEFINITION (1.8) $X \in \mathcal{C}$ is said to be P-local if and only if $\pi_*(X)$ is P-local.

We denote by $\mathcal{C}_P$ the full subcategory of $\mathcal{C}$ consisting of P-local spaces and by $i_P:\mathcal{C}_P \longrightarrow \mathcal{C}$ the inclusion functor.

DEFINITION (1.9) If $Y\in\mathcal{C}_P$ then $f\in\mathrm{Hom}_{\mathcal{C}}(X, i_PY)$ is said to P-localize X if and only if f is universal with respect to maps from X to P-local spaces, i.e.,

$$f^*:\mathrm{Hom}_{\mathcal{C}}(X, i_PZ) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(i_PY, i_PZ)$$

is an isomorphism for all $Z\in\mathcal{C}_P$. If f P-localizes X then Y is said to be a P-localization of X.

P-local spaces and P-localizations are characterized by the following.

THEOREM (1.10) [Sullivan (1971)]. For $f\in\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the following are equivalent:

(i) f P-localizes X,

(ii) f_* P-localizes integral homology, i.e., $f_*:\tilde{H}_*(X) \longrightarrow \tilde{H}_*(Y)$ P-localizes $\tilde{H}_*(X)$, and

(iii) f_* P-localizes homotopy, i.e., $f_*:\pi_*(X) \longrightarrow \pi_*(Y)$ P-localizes $\pi_*(X)$.

COROLLARY (1.11) For $X\in\mathcal{C}$ the following are equivalent:

(i) $X\in\mathcal{C}_P$,

(ii) $\tilde{H}_*(X)\in\mathcal{N}_P$, and

(iii) $\pi_*(X)\in\mathcal{N}_P$

COROLLARY (1.12) If $f\in\mathrm{Hom}_{\mathcal{C}_P}(X, Y)$ then the following are equivalent:

(i) f is a homotopy equivalence,

(ii) $f_*:\tilde{H}_*(X) \longrightarrow \tilde{H}_*(Y)$ is an isomorphism, and

(iii) $f_*:\pi_*(X) \longrightarrow \pi_*(Y)$ is an isomorphism.

Note that Corollary (1.12) also holds in the more general category $\mathcal{C}$.

To see that P-localizations of spaces $X \in \mathcal{C}$ exist we follow Sullivan (1971) and outline a cellular construction. We begin by describing a P-local n-sphere.

Choose a cofinal sequence $\mathcal{L}$ from $\langle P' \rangle$ and denote the elements of this sequence by $\ell_1, \ell_2, \ldots$. Choose maps $\ell_i : S^n \longrightarrow S^n$ of degree ℓ_i and define the P-local n-sphere, denoted S^n_P, as

$$S^n_P = \underset{\mathcal{L}}{\underrightarrow{\lim}} \; (S^n \xrightarrow{\ell_i} S^n).$$

A P-local CW complex is built inductively from a point or from a P-local 1-sphere by attaching cones over the P-local sphere using maps of the P-local sphere S^n_P into the lower "local skeletons".

A point to notice about the above construction is that since there is no P-local 0-sphere, there is no P-local 1-cell.

<u>THEOREM (1.13)</u> [Sullivan (1971)]. If X is a CW complex with one 0-cell and no 1-cells, then there is a P-local CW complex, denoted $L_P(X)$, and a cellular map $e_P : X \longrightarrow L_P(X)$ such that

(i) e_P induces a bijection between the cells of X and the P-local cells of $L_P(X)$ and

(ii) $e_{P*} : \pi_*(X) \longrightarrow \pi_*(L_P(X))$ P-localizes $\pi_*(X)$.

<u>COROLLARY (1.14)</u> There exists a functor $L_P : \mathcal{C} \longrightarrow \mathcal{C}_P$. Further, for each $X \in \mathcal{C}$ there is a canonical map $e_P \in \mathrm{Hom}_{\mathcal{C}}(X, i_P L_P(X))$ having the property that e_P P-localizes X.

As before, we will write X_P for $L_P(X)$ and call X_P the P-localization of X. Again, note that the universality condition means that X_P is uniquely determined up to homotopy equivalence.

The following proposition, which, among other places, appears in Mimura-Nishida-Toda (1971), shows that localization behaves nicely with respect to some important concepts and constructions of algebraic topology.

PROPOSITION (1.15) In $\mathcal{C}$ P-localization preserves fibrations and cofibrations.

COROLLARY (1.16) If X, $Y \in \mathcal{C}$ then

(i) $(X \times Y)_P \simeq X_P \times Y_P$,

(ii) $(X \vee Y)_P \simeq X_P \vee Y_P$, and

(iii) $(X \wedge Y)_P \simeq X_P \wedge Y_P$

where $X \vee Y$ denotes the "wedge product", i.e., the one point union and $X \wedge Y$ denotes the "smash product", i.e., the quotient space $(X \times Y)/(X \vee Y)$.

THEOREM (1.17) [Mimura-Nishida-Toda (1971)]. For $X \in \mathcal{C}$ let P_i, $i \in I$, be a family of subsets of $\mathbb{P}$ and set $\overline{P} = \bigcap_I P_i$ and $P = \bigcup_I P_i$. If we let $\prod_{X_{\overline{P}}} X_{P_i}$ denote the pull-back of the canonical maps $e_{P_i} : X_{P_i} \longrightarrow X_{\overline{P}}$ then $\prod_{X_{\overline{P}}} X_{P_i} \simeq X_P$.

When $P \subset \mathbb{P}$ is a singleton, $P = \{p\}$, we will denote X_P by $X_{(p)}$.

COROLLARY (1.18) [Mimura-Nishida-Toda (1971)]. $X \in \mathcal{C}$ is homotopy equivalent to $\prod_{X_\emptyset} X_{(p)}$, the pull-back of $e_{(p)} : X_{(p)} \longrightarrow X_\emptyset$ over $X_\emptyset$ for all primes p.

Let $\mathcal{FC}$ denote the subcategory of $\mathcal{C}$ of finite CW complexes, and, as is usual, for topological spaces X and Y let [X, Y] denote the set of homotopy classes of maps from X to Y.

THEOREM (1.19) [Hilton-Mislin-Roitberg (1973)]. Let X, $Y \in \mathcal{FC}$ and let P_i, $\overline{P}$ and P be as in Theorem (1.17). Then

$$[X, Y] \simeq \prod_{[X, Y_{\overline{P}}]} [X, Y_{P_i}] ,$$

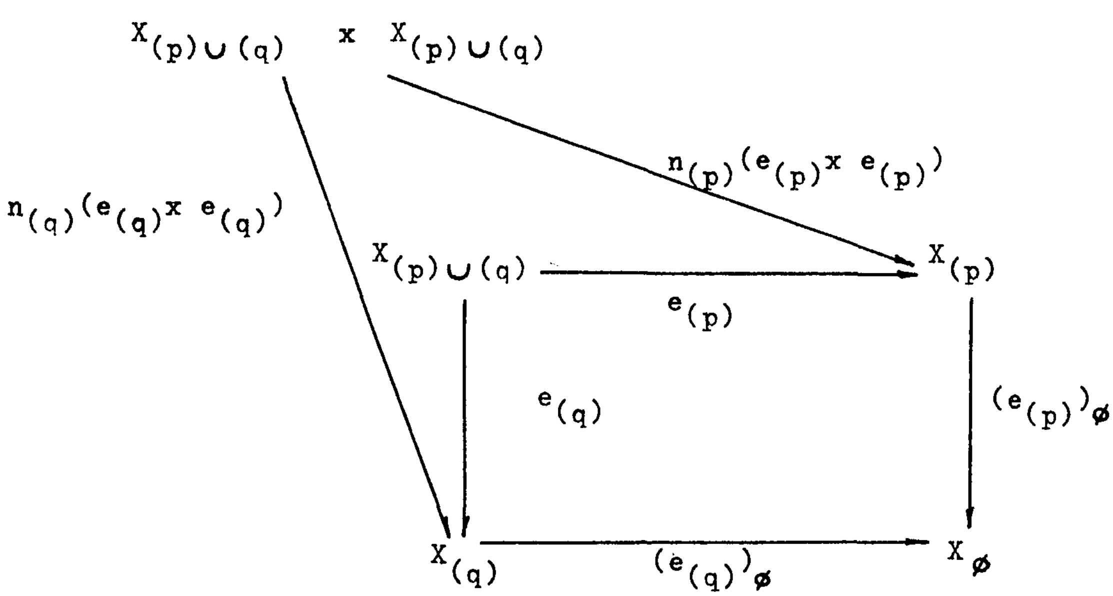
in the category of sets, where $\prod_{[X, Y_{\overline{P}}]} [X, Y_{P_i}]$ is the pull-back of of the maps $e_{P_i *} : [X, Y_{P_i}] \longrightarrow [X, Y_{\overline{P}}]$ over $[X, Y_{\overline{P}}]$.

The following theorem provides a criteria for determining if a topological space is an H-space. This result appears in several slightly differing forms in the literature, see for example Sullivan (1971), Mimura-Nishida-Toda (1971), or Hilton-Mislin-Roitberg (1973). Some of these versions appear to have incorrect proof, Sullivan (1971) for example, although all results are reported to be true. We state and prove a variation of these results in a form which will be useful to us later.

THEOREM (1.20) Let $X \in \mathcal{C}$. If (X,m) is an H-space then m induces a multiplication $m_{(p)}$ on $X_{(p)}$ for all $p \in \mathbb{P}$. Conversely, if $(X_{(p)}, n_{(p)})$ is an H-space for each $p \in \mathbb{P}$ and, further if the multiplications induced on $X_{\emptyset}$ by $n_{(p)}$ and $n_{(q)}$ are equal for all $p, q \in \mathbb{P}$, then X is an H-space.

PROOF Suppose (X, m) is an H-space. Localizing the multiplication m we get a map $m_{(p)}:(X \times X)_{(p)} \longrightarrow X_{(p)}$ for each prime $p \in \mathbb{P}$. However, $(X \times X)_{(p)}$ is homotopic to $X_{(p)} \times X_{(p)}$ and we have a map $m_{(p)}:X_{(p)} \times X_{(p)} \longrightarrow X_{(p)}$ which is easily seen to make $(X_{(p)}, m_{(p)})$ an H-space.

Conversely, consider the following diagram where p and q are primes and the e-maps are the canonical localization maps.



By Theorem (1.17) the rectangle is a pull-back diagram. Let $e_\emptyset : X_{(p)\cup(q)} \longrightarrow X_\emptyset$. Since $\emptyset \subset \{p,q\}$

$$(e_{(p)})_\emptyset \circ e_{(p)} = e_\emptyset \text{ and } (e_{(q)})_\emptyset \circ e_{(q)} = e_\emptyset .$$

Thus $(e_{(p)})_\emptyset \circ [n_{(p)} \circ (e_{(p)} \times e_{(p)})]$

$$= (n_{(p)})_\emptyset \circ [((e_{(p)})_\emptyset \times (e_{(p)})_\emptyset) \circ (e_{(p)} \times e_{(p)})]$$

$$= (n_{(p)})_\emptyset \circ (e_\emptyset \times e_\emptyset)$$

and similarly $(e_{(q)})_\emptyset \circ [n_{(q)} \circ (e_{(q)} \times e_{(q)})]$

$$= (n_{(q)})_\emptyset \circ [((e_{(q)})_\emptyset \times (e_{(q)})_\emptyset) \circ (e_{(q)} \times e_{(q)})]$$

$$= (n_{(q)})_\emptyset \circ (e_\emptyset \times e_\emptyset).$$

But by hypothesis $(n_{(q)})_\emptyset = (n_{(p)})_\emptyset$ hence we have

$$(e_{(p)})_\emptyset \circ [n_{(p)} \circ (e_{(p)} \times e_{(p)})] = (e_{(q)})_\emptyset \circ [n_{(q)} \circ (e_{(q)} \times e_{(q)})].$$

Since $X_{(p)\cup(q)}$ is a pull-back, the above equality means there exists a unique map

$$n_{(p)\cup(q)} : X_{(p)\cup(q)} \times X_{(p)\cup(q)} \longrightarrow X_{(p)\cup(q)}$$

which is easily seen to be a multiplication on $X_{(p)\cup(q)}$.

Inductively one can now define a multiplication, m, on the infinite pull-back over all primes, which by Corollary (1.18) is homotopic to X.

Q.E.D.

If $G \in \mathcal{G}$ is a topological group then by a theorem of G. Whitehead, (1954), [, G] defines a functor [, G]: $\mathcal{HC} \longrightarrow \mathcal{N}$. The following is a useful relationship between algebraic and topological localization.

<u>PROPOSITION (1.21)</u> [Harrison-Scheerer (1972)]. For $X \in \mathcal{HC}$ and $G \in \mathcal{G}$ where G is a topological group there is a natural isomorphism $[X, G]_P \approx [X, G_P]$.

<u>COROLLARY (1.22)</u> If $X \in \mathcal{C}$ then $\pi_*(X_P) \approx \pi_*(X) \otimes \mathbb{Z}_P$.

PROOF For $n \geq 2$, $\pi_n(X) \approx [S^{n-1}, \Omega X]$. But $\Omega X \simeq G(X)$ where $G(X)$ is a topological group, hence corollary is immediate from main proposition if one uses Proposition (1.2) together with the fact that $\pi_n(X)$ is Abelian for $n \geq 2$.

Q.E.D.

The next theorem is a restated localized version of a result due to Copeland (1972). Its proof is the same as that given by Copeland, since the restriction to the category of finitely generated CW complexes that is made by the author in the paper where the result appears is unnecessary.

THEOREM (1.23) [Copeland (1972)]. Let $X \in \mathcal{C}$ be a finite product of spaces $X = X_1 x X_2 x \dots x X_n$ where $X_i \in \mathcal{C}$ and $(X_i)_P$ is an H-space for each $i = 1, 2, \dots, n$ and some set of primes P. For all integers u and s with $1 \leq u, s \leq n$ let $\alpha = (i_1, \dots, i_u, j_1, \dots, j_s)$ be an $(u+s)$-tuple of integers with $1 \leq i_1 < i_2 < \dots < i_u \leq n$ and $1 \leq j_1 < j_2 < \dots < j_s \leq n$. Set $\#\alpha = u + s$, $A_n = \{\alpha \mid 2 \leq \#\alpha \leq 2n\}$ and

$$(M_\alpha)_P = (X_{i_1})_P \wedge \dots \wedge (X_{i_u})_P \wedge (X_{j_1})_P \wedge \dots \wedge (X_{j_s})_P$$

then

$$\#[X_P \wedge X_P, X_P] = \prod_{t=1}^{n} \left(\prod_{\alpha \in A_n} \#[(M_\alpha)_P, (X_t)_P] \right).$$

Finally, we have the following theorem which will be most useful in obtaining our main results. Its proof requires a lemma.

LEMMA (1.24) If $X \in \mathcal{FC}$ and $Y \in \mathcal{C}$ then $[X_P, Y_P] \approx [X, Y_P]$ as sets.

PROOF That $e_P : X \longrightarrow X_P$ induces a surjective function

$$e_P^* : [X_P, Y_P] \longrightarrow [X, Y_P]$$

follows immediately from the fact that e_P P-localizes X. To show that e_P^* is injective one uses the fact that e_P P-localizes X together with the result that $(X \times I)_P \simeq X_P \times I$ which follows from Corollary (1.16).

Q.E.D.

<u>THEOREM (1.25)</u> (Arkowitz, Murley, Shar). Let $X \in \mathcal{C}$ be an H-space having the property that $[X \wedge X, X_\emptyset]$ is trivial. Let $X_1, X_2 \in \mathcal{C}$ be such that for some set $P \subseteq \mathbb{P}$, $X_P \simeq (X_1)_P$ and $X_{P'} \simeq (X_2)_{P'}$. Then $(X_1)_P$ and $(X_2)_{P'}$ are H-spaces and

$$\#[X \wedge X, X] = \#[(X_1)_P \wedge (X_1)_P, (X_1)_P] \# [(X_2)_{P'} \wedge (X_2)_{P'}, (X_2)_{P'}].$$

<u>PROOF</u> By Theorem (1.20) it is easy to see that X_P and $X_{P'}$ are H-spaces so it is immediate that $(X_1)_P$ and $(X_2)_{P'}$ are H-spaces.

By Theorem (1.19) and the hypothesis that $[X \wedge X, X_\emptyset]$ is trivial we see that

$$\#[X \wedge X, X] = \#[X \wedge X, X_P] \# [X \wedge X, X_{P'}].$$

But by Lemma (1.24) and Corollary (1.16 (iii)) we have

$$\#[X \wedge X, X_P] = \#[X_P \wedge X_P, X_P] = \#[(X_1)_P \wedge (X_1)_P, (X_1)_P]$$

and similarly

$$\#[X \wedge X, X_{P'}] = \#[X_{P'} \wedge X_{P'}, X_{P'}] = \#[(X_2)_{P'} \wedge (X_2)_{P'}, (X_2)_{P'}].$$

Q.E.D.

We now proceed to the main results.

CHAPTER II

H-STRUCTURES ON SIMPLY CONNECTED H-SPACES OF TYPE (3,7)

The problem we consider in this chapter is that of determining, up to homotopy, the number of H-structures supported by H-spaces which are total spaces of principal S^3-bundles over S^7. Among these spaces are $S^3 \times S^7$, Sp(2), and the famous Hilton-Roitberg H-space, Hilton Roitberg (1969), which was the first example not involving S^7 of a compact simply connected H-space not of the homotopy type of a Lie group. Later Hilton and Roitberg, Hilton-Roitberg (1970), showed that any simply connected H-space of type (3,7) has the homotopy type of the total space of a principal S^3-bundle over S^7. Thus, the problem we consider is that of determining the number of H-structures that a simply connected H-space of type (3,7) may support.

We are considering principal fibrations of the form

$$\begin{array}{ccc} S^3 & \longrightarrow & X \\ & & \downarrow \\ & & S^7. \end{array}$$

The classifying space for such fibrations is B_S3 and so the number of homotopy classes of such fibrations is in one-to-one correspondence with $[S^7, B_S3] \approx \pi_7(B_S3)$. Using the fact that $\Omega B_S3 \simeq S^3$ we see that $\pi_7(B_S3) \approx \pi_6(\Omega B_S3) \approx \pi_6(S^3) \approx \mathbb{Z}/12$. Thus, there are twelve distinct homotopy classes of such fibrations.

The elements of $\pi_7(B_S3)$ corresponding to the distinct homotopy classes of fibrations are called the characteristic classes of the fibrations. For a fibration $S^3 \longrightarrow X \longrightarrow S^7$ we will denote its characteristic class by $\chi(X)$ and will consider $\chi(X)$ to be an element of $\pi_6(S^3)$ under identification via the canonical isomorphism $\pi_7(B_S3) \approx \pi_6(S^3)$.

Let PB_S3 denote the space of based paths on B_S3 and $p:PB_S3 \longrightarrow B_S3$ be the projection on the terminal point. With this notation $\Omega B_S3 \longrightarrow PB_S3 \xrightarrow{p} B_S3$ is a fibration and any fibration of the form $S^3 \longrightarrow X \longrightarrow S^7$ is induced by $\chi(X)$ as follows:

$$\begin{array}{ccc} S^3 & \longrightarrow & \Omega B_S3 \\ \downarrow & & \downarrow \\ X & \longrightarrow & PB_S3 \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{\chi(X)} & B_S3 \end{array}$$

The two unlabeled horizontal maps in the above diagram are induced by $\chi(X)$ in the usual fashion.

It is known that $Sp(2)$ is the total space of a principal S^3-bundle over S^7 and, further, that $\chi(Sp(2)) = \nu' + \alpha_1(3) \in \pi_6(S^3)$ where ν' and $\alpha_1(3)$ are the Toda, Toda (1962), generators of the direct summands $\mathbb{Z}/4$ and $\mathbb{Z}/3$ respectively of $\pi_6(S^3) \approx \mathbb{Z}/12$. Let $n:S^7 \longrightarrow S^7$ denote a map of degree n and X_n the total space of the principal fibration induced from $S^3 \longrightarrow Sp(2) \longrightarrow S^7$ by n as shown below.

$$\begin{array}{ccccc} S^3 & \longrightarrow & S^3 & & \\ \downarrow & & \downarrow & & \\ X_n & \longrightarrow & Sp(2) & & \\ \downarrow & & \downarrow & & \\ S^7 & \xrightarrow{n} & S^7 & \xrightarrow{\nu' + \alpha_1(3)} & B_S3 \end{array}$$

It is now clear that with this notation $\chi(X_n) = n(\nu' + \alpha_1(3)) = n\nu' + n\alpha_1(3)$ and that X_n, $n = 0,1,2,\ldots,11$, is a complete list of the total spaces of principal fibrations of the form $S^3 \longrightarrow X \longrightarrow S^7$.

Not all of the spaces X_n are of different homotopy type; indeed, the following proposition shows that $X_n \simeq X_m$ if and only if $n = \pm m (\mathrm{mod}\ 12)$.

PROPOSITION (2.1) [Douglas, Hilton, Sigirst (1969)]. Let X_α denote the total space of an S^3-bundle over S^n with $\chi(X_\alpha) = \alpha \in \pi_{n-1}(S^3)$. Then $X_\alpha \simeq X_\beta$ if and only if $\alpha = \pm\beta$.

Thus, we find that there are only seven distinct homotopy types of total spaces X_n, namely those having the following representatives: X_0, X_1, X_2, X_3, X_4, X_5 and X_6. Of these we are interested only in the ones that are H-spaces. The question of deciding which of these spaces are H-spaces has been answered, although there is a minor problem involved.

From our notation it is clear that $X_0 \simeq S^3 \times S^7$ and $X_1 \simeq Sp(2)$ and, hence, are H-spaces. X_5 is the Hilton-Roitberg H-space and Zabrodsky (1971) has shown that X_2 and X_6 are not H-spaces. The problem lies with Stasheff's (1969) proof that both X_3 and X_4 are H-spaces. It seems that his proof of this fact used a result that has subsequently been shown to have had an incorrect proof. However, the result Stasheff used is thought to be true, although a correct proof has not yet appeared. To avoid this difficulty we will argue that both X_3 and X_4 are H-spaces, using Theorem (1.20).

In order to use Theorem (1.20) to show that X_3 and X_4 are H-spaces, we must verify that they satisfy the hypotheses of the theorem, i.e., that $(X_3)_{(p)}$ and $(X_4)_{(p)}$ are H-spaces for each prime p and, further, that the condition on the induced multiplications in $(X_3)_\phi$ and $(X_4)_\phi$ is satisfied. That these facts are indeed true is a consequence of the following lemmas numbered (2.3) and (2.5). Both of these lemmas contain additional information about X_5 which will be needed later.

In order to prove Lemma (2.3) we need the next result, which is a localized version of Lemma (2.3), Mimura-Toda (1964).

<u>LEMMA (2.2)</u> Let $\alpha \in \pi_m(X)$, $m \geq 2$, be of finite order and $X_\alpha \in \mathcal{C}$ be the total space of the fibration $\Omega X \longrightarrow X_\alpha \longrightarrow S^m$ induced by α in the usual fashion. If $n: S^m \longrightarrow S^m$ is a map of degree n then $(X_{n\alpha})_{(p)} \simeq (X_\alpha)_{(p)}$ for all primes p such that $(p,n) = 1$.

<u>PROOF</u> The following diagram commutes where $\bar{n}$ denotes the map induced on total spaces by n.

$$\begin{array}{ccccc}
\Omega X & \xrightarrow{1_{\Omega X}} & \Omega X & & \\
\downarrow & & \downarrow & & \\
E_{n\alpha} & \xrightarrow{\bar{n}} & E_\alpha & & \\
\downarrow & & \downarrow & & \\
S^m & \xrightarrow{n} & S^m & \xrightarrow{\alpha} & X
\end{array}$$

Since localization preserves fibration we may localize the above diagram and consider the resulting homotopy exact sequences.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_{i+1}(S^m_{(p)}) & \longrightarrow & \pi_i((\Omega X)_{(p)}) & \longrightarrow & \pi_i((X_{n\alpha})_{(p)}) & \longrightarrow \\
& \downarrow n_{(p)*} & & \downarrow 1 & & \downarrow \bar{n}_{(p)*} & \\
\cdots \longrightarrow & \pi_{i+1}(S^m_{(p)}) & \longrightarrow & \pi_i((\Omega X)_{(p)}) & \longrightarrow & \pi_i((X_\alpha)_{(p)}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\pi_i(S^m_{(p)}) & \longrightarrow & \pi_{i-1}((\Omega X)_{(p)}) & \longrightarrow & \cdots \\
\downarrow n_{(p)*} & & \downarrow 1 & & \\
\pi_i(S^m_{(p)}) & \longrightarrow & \pi_{i-1}((\Omega X)_{(p)}) & \longrightarrow & \cdots
\end{array}$$

Since $(p,n) = 1$ and $\pi_*(S^m_{(p)}) \approx \pi_*(S^m) \otimes \mathbb{Z}_{(p)}$ we see that $n_{(p)*}$ is an isomorphism. We now conclude from the 5-lemma that $\bar{n}_{(p)*}$ is an isomorphism and thus, since all spaces are CW-complexes, that $(X_{n\alpha})_{(p)} \simeq (X_\alpha)_{(p)}$.

Q.E.D.

LEMMA (2.3) Let p be a prime and $(p)' = \mathbb{P}-\{p\}$, then

(a) $(X_3)_{(3)} \simeq (S^3 x\ S^7)_{(3)}$ and $(X_3)_{(3)'} \simeq Sp(2)_{(3)'}$,

(b) $(X_4)_{(2)} \simeq (S^3 x\ S^7)_{(2)}$ and $(X_4)_{(2)'} \simeq Sp(2)_{(2)'}$, and

(c) $(X_5)_{(p)} \simeq (Sp(2))_{(p)}$ for all $p \in \mathbb{P}$.

PROOF (a) Consider the following diagrams of fibrations and induced fibrations.

$$\begin{array}{ccccc}
S^3 & \xrightarrow{1_{S^3}} & S^3 & \xrightarrow{1_{S^3}} & S^3 \\
\downarrow & & \downarrow & & \downarrow \\
S^3 \times S^7 & \xrightarrow{\bar{4}} & X_3 & \xrightarrow{\bar{3}} & Sp(2) \\
\downarrow & & \downarrow & & \downarrow \\
S^7 & \xrightarrow{4} & S^7 & \xrightarrow{3} & S^7
\end{array}$$

Lemma (2.2) can now be applied to give (a).

(b) and (c) are proved similarly where in the proof of (c) we make use of the fact that $X_5 \simeq X_7$.

Q.E.D.

In what follows we will use the notation $\mu(X)$ to denote the number of distinct homotopy classes of multiplications that a given H-space X will support, and, as before # will be used to denote set cardinality.

Before stating and proving Lemma (2.5) we record the following fundamental result which is needed not only in Lemma (2.5), but forms the basis for later computations.

THEOREM (2.4) [Copeland (1959)]. If $X \in \mathcal{C}$ is an H-space then

$\mu(X) = \#([X \wedge X, X])$.

<u>LEMMA (2.5)</u> $\mu((X_n)_\emptyset) = 1$ for $n = 0,1,3,4$ and 5.

<u>PROOF</u> Since all spaces X_n, $n = 0,1,3,4,5$, are of type $(3,7)$ we know that $(X_n)_\emptyset \simeq K(\mathbb{Q},3) \times K(\mathbb{Q},7)$. Furthermore, we have set bijections,

$$\begin{aligned}[(X_n)_\emptyset \wedge (X_n)_\emptyset, (X_n)_\emptyset] &= [X_n \wedge X_n, K(\mathbb{Q},3) \times K(\mathbb{Q},7)] \\ &= H^3(X_n \wedge X_n;\mathbb{Q}) \oplus H^7(X_n \wedge X_n;\mathbb{Q}).\end{aligned}$$

James-Whitehead (1954, p.205), show that X_n has CW structure $X_n \simeq S^3 \cup e^7 \cup e^{10}$ and, hence for dimensional reasons $H^3(X_n \wedge X_n;\mathbb{Q}) \oplus H^7(X_n \wedge X_n;\mathbb{Q})$ is trivial. Thus, by Theorem (2.4) there is only one homotopy class of multiplications on $(X_n)_\emptyset$.

Q.E.D.

From Lemma (2.3) we see that $(X_3)_{(p)} \simeq (Sp(2))_{(p)}$ for all primes $p \neq 3$ while $(X_3)_{(3)} \simeq (S^3 \times S^7)_{(3)}$. Since both $Sp(2)$ and $S^3 \times S^7$ are H-spaces, $(X_3)_{(p)}$ is an H-space for all primes p by Theorem (1.20). Similarly, one sees that $(X_4)_{(p)}$ is an H-space for all $p \in \mathbb{P}$. Finally, Lemma (2.5) insures that X_3 and X_4 satisfy the condition on induced multiplications in $X_\emptyset$. Thus, the hypotheses of Theorem (1.20) are satisfied and we may conclude that both X_3 and X_4 are H-spaces.

We thus know that there are five distinct homotopy types of simply connected H-spaces of type $(3,7)$, namely $S^3 \times S^7$, $Sp(2)$, X_3, X_4 and X_5.

The basic result used in solving problems concerning the number of H-structures that a given H-space will support is Theorem (2.4) It turns out, however, that computing the order of the algebraic loop $[X \wedge X, X]$ is usually very difficult and comparatively little has been done in the way of specific computations, except for relatively simple spaces. However, using algebraic techniques, Arkowitz and Curjel proved the following general results about Lie groups.

THEOREM (2.6) [Arkowitz-Curjel (1963)]. $\mu(X)$ is infinite for $X = SO(10)$, $SO(14)$, $SO(n)$ for $n \geq 17$, $SU(n)$ for $n \geq 6$, $Sp(n)$ for $n \geq 8$, and the representatives of the exceptional groups E_6 and E_8. $\mu(X)$ is finite for X any other classical group or representative of the other exceptional structures.

This means that in particular Sp(2) has a finite number of non-homotopic multiplications. Mimura subsequently computed $\mu(Sp(2))$ by finding $\#[Sp(2)\wedge Sp(2), Sp(2)]$ via a direct assault on the cell structure of $Sp(2)\wedge Sp(2)$.

THEOREM (2.7) [Mimura (1969)]. $\mu(Sp(2)) = 2^{20}\cdot 3\cdot 5^5\cdot 7$.

The only other homotopy type of the five listed above for which the problem has been solved is $S^3 \times S^7$, which solution follows from Theorem (1.23). Using this result we carry out the computations in Proposition (2.8) below for two reasons, the first being that this result does not seem to appear in the literature and the second being that we will need some details of this computation later.

PROPOSITION (2.8) $\mu(S^3 \times S^7) = 2^{38}\cdot 3^{15}\cdot 5^5\cdot 7$.

PROOF Using Theorem (1.23) with $P = \mathbb{P}$, the set of all primes, we have

$$\#[(S^3 \times S^7)\wedge(S^3 \times S^7),\ S^3 \times S^7] = \prod_{j=3,7} (a_j\ b_j{}^2 c_j\ d_j{}^2 e_j{}^2 f_j)$$

where

$$a_j = \#[S^3\wedge S^7\wedge S^3\wedge S^7,\ S^j] = \#[S^{20},\ S^j] = (\pi_{20}(S^j)),$$

$$b_j = \#[S^3\wedge S^7\wedge S^7,\ S^j] = \#[S^{17},\ S^j] = \#(\pi_{17}(S^j)),$$

$$c_j = \#[S^7\wedge S^7, S^j] = \#[S^{14},\ S^j] = \#(\pi_{14}(S^j)),$$

$d_j = \#[S^3 \wedge S^3 \wedge S^7, S^j] = \#[S^{13}, S^j] = \#(\pi_{13}(S^j))$,
$e_j = \#[S^3 \wedge S^7, S^j] = \#[S^{10}, S^j] = \#(\pi_{10}(S^j))$, and
$f_j = \#[S^3 \wedge S^3, S^j] = \#[S^6, S^j] = \#(\pi_6(S^j))$.

Using Toda's results, Toda (1962), on the homotopy groups of S^3 and S^7 we obtain

$a_3 = 2^4 \cdot 3$, $a_7 = 2 \cdot 3$,
$b_3 = 2 \cdot 3 \cdot 5$, $b_7 = 2^4 \cdot 3$,
$c_3 = 2^4 \cdot 3 \cdot 7$, $c_7 = 2^3 \cdot 3 \cdot 5$,
$d_3 = 2^3 \cdot 3$, $d_7 = 2$,
$e_3 = 3 \cdot 5$, $e_7 = 2^3 \cdot 3$, and
$f_3 = 2^2 \cdot 3$, $f_7 = 1$

which gives the result.

Q.E.D.

We now come to the main result of this section, which together with the known results concerning $S^3 \times S^7$ and Sp(2), Theorem (2.7) and Proposition (2.8), provides a complete solution to the problem of determining the number of H-structures supported by simply connected H-spaces of type (3,7).

First, we prove an algebraic lemma.

LEMMA (2.9) If G is a finite nilpotent group of order n then the order of G_P for any set of primes P is the product of the prime power factors of n for those primes in P.

PROOF We first note that a finite nilpotent group of order n can be expressed as a direct sum of its Sylow p-subgroups, and that algebraic localization preserves sums, Corollary (1.7). The result now follows from the observation that for a p-group H,

$$H_{(q)} \approx \begin{cases} H & \text{if } q = p \\ 1 & \text{otherwise.} \end{cases}$$

This last observation is an immediate consequence of Theorem (1.6) and the definition of p-group.

Q.E.D.

THEOREM (2.10) For the total spaces X_n, n = 3,4 and 5,

(a) $\mu(X_3) = 2^{20}\cdot 3^{15}\cdot 5^5\cdot 7$,

(b) $\mu(X_4) = 2^{38}\cdot 3\cdot 5^5\cdot 7$, and

(c) $\mu(X_5) = \mu(Sp(2)) = 2^{20}\cdot 3\cdot 5^5\cdot 7$.

PROOF Lemmas (2.3) and (2.5) show that we can use Theorem (1.25) to obtain the following:

$$\mu(X_3) = \mu(Sp(2)_{(3)'})\cdot\mu((S^3 x\ S^7)_{(3)}),$$
$$\mu(X_4) = \mu(Sp(2)_{(2)'})\cdot\mu((S^3 x\ S^7)_{(2)}),$$

and

$$\mu(X_5) = \mu(Sp(2)_P)\cdot\mu(Sp(2)_{P'}) \text{ where } P = \{2,3,5,7\}.$$

The proof is now reduced to some simple calculations.

With regard to the calculations dealing with the various localizations of Sp(2), we note that Whitehead (1954) has shown that for a topological group X, the functor [,X] takes values in the category of finitely generated nilpotent groups. Since Sp(2) is indeed a topological group, we have

$$\begin{aligned}\mu(Sp(2)_P) &= \#[Sp(2)_P\wedge Sp(2)_P,\ Sp(2)_P] \\ &= \#[Sp(2)\wedge Sp(2),\ Sp(2)_P] \\ &= \#([Sp(2)\wedge Sp(2),\ Sp(2)]_P)\end{aligned}$$

where the above equalities follow from Theorem (2.4), Corollary (1.16), Lemma (1.24) and Proposition (1.21). Since [Sp(2)∧Sp(2), Sp(2)] is a finite nilpotent group, we may apply Lemma (2.9) to obtain

$$\mu(Sp(2)_{(3)'}) = 2^{20}\cdot 5^5\cdot 7,$$
$$\mu(Sp(2)_{(2)'}) = 3\cdot 5^5\cdot 7,$$

$\mu(Sp(2)_p) = 2^{20}\cdot 3\cdot 5^5\cdot 7$, and

$\mu(Sp(2)_{p'}) = 1$.

The result (c) now follows. To obtain (a) and (b) we must compute

$\mu((S^3 x\ S^7)_{(3)})$ and $\mu((S^3 x\ S^7)_{(2)})$.

To make these computations we use Theorem (1.23) along with the observation that if X is a finite smash of spheres the sum of whose dimensions is $n \geq 2$ and $Y \in \mathcal{C}$ then for any prime p,

$$\#[X_{(p)}, Y_{(p)}] = \#[X, Y_{(p)}] = \#(\pi_n(Y_{(p)})) = \#(\pi_n(Y) \otimes \mathbb{Z}_{(p)}).$$

With this it is an easy matter to see that

$$\mu((S^3 x\ S^7)_{(3)}) = 3^{15}$$

and

$$((S^3 x\ S^7)_{(2)}) = 2^{38}$$

from the computations done in the proof of Proposition (2.8). The results (a) and (b) now are obtained.

Q.E.D.

CHAPTER III

H-STRUCTURES ON SU(3)-BUNDLES OVER S^7

Curtis-Mislin (1970) have shown that all SU(3)-bundles over S^7 are H-spaces. These bundles, $SU(3) \longrightarrow Y \longrightarrow S^7$, are classified by $\pi_7(B_{SU(3)}) \approx \pi_6(SU(3)) \approx \mathbb{Z}/2 \oplus \mathbb{Z}/3$. Let $\alpha+\beta \in \pi_7(B_{SU(3)})$ be a suitable generator where α has order two and β has order three. It is known that SU(4) is an SU(3)-bundle over S^7 and that for suitable choice of a generator, $\chi(SU(4)) = \alpha+\beta$. As in Chapter II, we define a total space Y_n, n = 0,1,2,3,4, or 5, to be the total space of the fibration induced from $SU(3) \longrightarrow SU(4) \longrightarrow S^7$ by a map $n: S^7 \longrightarrow S^7$ of degree n. Again, analogous to the results of Chapter II, we have six total spaces but only four distinct homotopy types, $Y_0 = SU(3) \times S^7$, $SU(4) = Y_1 \simeq Y_5$, $Y_2 \simeq Y_4$ and Y_3.

In this chapter we consider the problem of determining the number of H-structures that each of the homotopy types Y_n, n = 0, 1, 2, 3 will support. Unlike Sp(2) and its associated spaces, nothing is known about this problem and, unfortunately, we will be able to give only a partial solution.

The first proposition of this chapter gives several equivalences that enable us to separate the solvable parts of the problem from the unsolvable.

PROPOSITION (3.1) Let $\mathbb{P}$ denote the set of all primes, then

(a) $(SU(3) \times S^7)_{\mathbb{P}-\{2\}} \simeq (S^3 \times S^5 \times S^7)_{\mathbb{P}-\{2\}}$,

(b) $SU(4)_{(3)} \simeq (Sp(2) \times S^5)_{(3)}$ and

$SU(4)_{\mathbb{P}-\{3,2\}} \simeq (S^3 \times S^5 \times S^7)_{\mathbb{P}-\{3,2\}}$,

(c) $(Y_2)_{(3)} \simeq (Sp(2) \times S^5)_{(3)}$, $(Y_2)_{(2)} \simeq (SU(3) \times S^7)_{(2)}$

and $(Y_2)_{\mathbb{P}-\{3,2\}} \simeq (S^3 \times S^5 \times S^7)_{\mathbb{P}-\{3,2\}}$ and

(d) $(Y_3)_{\mathbb{P}-\{2\}} \simeq (S^3 \times S^5 \times S^7)_{\mathbb{P}-\{2\}}$ and $(Y_3)_{(2)} \simeq SU(4)_{(2)}$.

PROOF $SU(3)$ is the total space of an S^3-bundle over S^5 with $\chi(SU(3))$ having order two. A map of degree two $S^5 \longrightarrow S^5$ then induces the product bundle, $S^3 \longrightarrow S^3 \times S^5 \longrightarrow S^5$, from $S^3 \longrightarrow SU(3) \longrightarrow S^5$ and (a) now follows from Lemma (2.2).

A map $S^7 \longrightarrow S^7$ of degree six induces the product bundle $SU(3) \longrightarrow SU(3) \times S^7 \longrightarrow S^7$ from $SU(3) \longrightarrow SU(4) \longrightarrow S^7$ since $\chi(SU(4))$ has order six, thus the second part of (b) follows. To show that the first part is true, one uses the fact that $SU(4)$ is an $Sp(2)$-bundle over S^5 with $\chi(SU(4)) \in \pi_5(B_{Sp(2)}) \approx \pi_4(Sp(2)) \approx \mathbb{Z}/2$. As in the proof above, one now sees that $SU(4)_{(3)} \simeq (Sp(2) \times S^5)_{(3)}$.

(c) and (d) follow in similar fashion using the fact that if one views Y_2 and Y_3 as $SU(3)$-bundles over S^7 then $\chi(Y_2)$ has order three and $\chi(Y_3)$ has order two.

Q.E.D.

Each of the spaces, Y_n, $n = 0,1,2,3$, is of the type $(3,5,7)$ and, as in the case of the spaces X_n of Chapter II, $[Y_n \wedge Y_n, (Y_n)_\emptyset]$ is trivial for dimensional reasons. Thus, we may use Theorem (1.25) which with the above proposition would give a complete solution to our problem, provided we could compute each of the following numbers:

(i) $\mu((SU(3) \times S^7)_{(2)})$,

(ii) $\mu(SU(4)_{(2)})$,

(iii) $\mu((Sp(2) \times S^5)_{(3)})$,

(iv) $\mu((S^3 \times S^5 \times S^7)_{\mathbb{P}-\{2\}})$, and

(v) $\mu((S^3 \times S^5 \times S^7)_{\mathbb{P}-\{3,2\}})$.

Of the above list (iv) and (v) are easily computed using Theorem (1.23). (iii) is computable but requires a knowledge of the cellular structure of $Sp(2)_{(3)}$ together with information about the generators of the 3-primary homotopy groups of certain spheres. (i) and (ii) are essentially non-computable by the techniques that yield (iii), (iv) and (v). The reason for this is that these methods would require a knowledge of the 2-primary unstable homotopy of certain spheres which is well beyond the range for which it has been computed. We will, however, be able to give a rough upper bound for (i) and (ii), modulo the cardinalities of some undetermined 2-primary homotopy groups of spheres.

We begin our computations with the easiest, namely (iv) and (v).

PROPOSITION (3.2)

(a) $\mu((S^3 \times S^5 \times S^7)_{\mathbb{P}-\{2\}}) = 3^{105} \cdot 5^{30} \cdot 7^9 \cdot 11^5 \cdot 13$ and

(b) $\mu((S^3 \times S^5 \times S^7)_{\mathbb{P}-\{3,2\}}) = 5^{30} \cdot 7^9 \cdot 11^5 \cdot 13$.

PROOF Using Theorem (1.23) with $P = \mathbb{P}-\{2\}$ we have

$$\# \left[(S^3 \times S^5 \times S^7)_P \wedge (S^3 \times S^5 \times S^7)_P, (S^3 \times S^5 \times S^7)_P\right]$$

$$= \prod_{j=3,5,7} \left[\left(\prod_{k=1}^{5} a_{k_j} \right) \left(\prod_{h=1}^{7} b_{h_j}^2 \right) c_j^3 \left(\prod_{m=1}^{4} d_{m_j}^4 \right) e_j^5 \; f_j^6 \right]$$

where the values of the quantities a_{k_j}, b_{h_j}, c_j, d_{m_j}, and e_j are given by the table below.

	j = 3	j = 5	j = 7
$a_{1_j} = \#(\pi_6(S^j)_p)$ =	3	1	1
$a_{2_j} = \#(\pi_{14}(S^j)_p)$ =	$3\cdot 7$	1	$3\cdot 5$
$a_{3_j} = \#(\pi_{16}(S^j)_p)$ =	3	$3^2\cdot 7$	1
$a_{4_j} = \#(\pi_{24}(S^j)_p)$ =	1	$3\cdot 11$	1
$a_{5_j} = \#(\pi_{30}(S^j)_p)$ =	3^2	1	$3^3\cdot 5\cdot 7\cdot 13$
$b_{1_j} = \#(\pi_8(S^j)_p)$ =	1	3	1
$b_{2_j} = \#(\pi_{11}(S^j)_p)$ =	1	1	1
$b_{3_j} = \#(\pi_{12}(S^j)_p)$ =	1	$3\cdot 5$	1
$b_{4_j} = \#(\pi_{19}(S^j)_p)$ =	3	3	1
$b_{5_j} = \#(\pi_{23}(S^j)_p)$ =	1	3	1
$b_{6_j} = \#(\pi_{25}(S^j)_p)$ =	$3\cdot 5\cdot 7$	3	3
$b_{7_j} = \#(\pi_{27}(S^j)_p)$ =	$3\cdot 5$	$3^2\cdot 5$	3
$c_j = \#(\pi_{10}(S^j)_p)$ =	$3\cdot 5$	1	3
$d_{1_j} = \#(\pi_{13}(S^j)_p)$ =	3	1	1
$d_{2_j} = \#(\pi_{17}(S^j)_p)$ =	$3\cdot 5$	1	3
$d_{3_j} = \#(\pi_{18}(S^j)_p)$ =	$3\cdot 5$	3	$3^2\cdot 7$
$d_{4_j} = \#(\pi_{22}(S^j)_p)$ =	$3\cdot 11$	1	$3\cdot 5$
$e_j = \#(\pi_{20}(S^j)_p)$ =	3	$3\cdot 5$	3
$f_j = \#(\pi_{15}(S^j)_p)$ =	1	3^2	1

The cardinalities of the homotopy groups above are obtained from Toda (1962) and Toda (1965). (a) is verified by adding appropriate exponents while (b) follows from (a) by setting $P = \mathbb{P}-\{3,2\}$.

Q.E.D.

In order to compute $\mu((Sp(2) \times S^5)_{(3)})$ we must first give some details about the cellular structure of $Sp(2)_{(3)}$, compute some homotopy groups, and develop some information regarding generators of the homotopy groups of certain spheres. We begin with a result about the cell structure of $Sp(2)_{(3)}$ and some of its related spaces.

Let X_2 be as in Chapter II, i.e., the total space of an S^3-bundle over S^7 having $\chi(X_2) \simeq \alpha_1(3)$, a generator of the 3-primary component of $\pi_6(S^3) \approx \mathbb{Z}/12$. Using Lemma (2.2) it is easy to see that $(X_2)_{(3)} \simeq Sp(2)_{(3)}$.

LEMMA (3.3)

(a) X_2 has cell structure $S^3 \cup_{\alpha_1(3)} e^7 \cup e^{10}$,

(b) for $k \geq 2$, $E^k(X_2) \simeq (S^{k+3} \cup_{\alpha_1(k+3)} e^{k+7}) \vee S^{k+10}$,

(c) $E^k(X_2 \wedge X_2) \simeq (S^{k+6} \cup_{\alpha_1(k+6)} e^{k+10} \vee S^{k+10}) \cup_{E^k\beta} e^{k+14} \vee$
$(S^{k+13} \cup_{\alpha_1(k+13)} e^{k+17}) \vee (S^{k+13} \cup_{\alpha_1(k+13)} e^{k+17}) \vee S^{k+20}$

with $E^k\beta = \widetilde{\alpha}_1(k+9) \vee \alpha_1(k+10)$ where $\widetilde{\alpha}_1(k+9) : S^{k+13} \longrightarrow S^{k+6} \cup_{\alpha_1(k+6)} e^{k+10}$ is a coextension of $\alpha_1(k+9) : S^{k+12} \longrightarrow S^{k+9}$, and

(d) $(X_2 \wedge X_2)/S^6 \simeq (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup_{\alpha_1(13)} e^{17}) \vee (S^{13} \cup_{\alpha_1(13)} e^{17})$.

PROOF (a) is a result due to James-Whitehead (1954, p.205). (b) follows immediately from Mimura (1969, Lemma 2.1(ii)) and (c) and (d) are obtained from Mimura (1969, Proposition 4.1). In the latter three cases the proofs are essentially the same as those given by Mimura, one merely ignores 2-primary generators. This is justified by the fact that Sp(2) has cell structure $S^3 \cup_{\nu' + \alpha_1(3)} e^7 \cup e^{10}$, which differs from that of X_2 only in that the attaching maps have 2-torsion as well as 3-torsion since $\nu' \in \pi_6(S^3)$ has order four.

Q.E.D.

In later computations we will need to know the cardinalities of the homotopy groups $\pi_i((X_2)_{(3)})$ for $8 \leq i \leq 30$. In order to compute these we first need some information about the homotopy of $S^3_{(3)}$ and $S^7_{(3)}$. In what follows the notation for the generators of the homotopy groups of spheres is that of Toda as found in Toda (1962), (1965) and (1966).

LEMMA (3.4)

i =	24	25	26	27	28
$\pi_i(S^3_{(3)}) \approx$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3$	0
generators		$\gamma_2(3)$	$\alpha_6(3)$ $\alpha_1(3)\beta_1(6)\beta_1(16)$	$\bar{u}_3(0,\beta_1\beta_1)$	

i =	29	30
$\pi_i(S^3_{(3)}) \approx$	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$
generators	$p_*\bar{Q}^2(\alpha_5)$ $p_*Q^2(\beta_1\beta_1)$	$\alpha_7(3)$ $p_*\bar{Q}^2(\beta_1\beta_1)$

The following compositions also generate:

(1) $\alpha_1(3)\alpha_5(6) = \pm\gamma_2(3)$,

(2) $\alpha_1(3)\alpha_6'(6) = \pm p_*\bar{Q}^2(\alpha_5)$,

(3) $\alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19) = \pm p_*Q^2(\beta_1\beta_1)$, and

(4) $\bar{u}_3(0,\beta_1\beta_1)\alpha_1(27) = \pm p_*\bar{Q}^2(\beta_1\beta_1)$.

PROOF The entries in the table come from Toda (1966) since $\pi_i(S^3_{(3)}) \approx \pi_i(S^3:3)$ by Corollary (1.22). The only facts requiring proof are that the compositions (1), (2), (3), and (4) also generate.

Using the exact sequence

$$\cdots \longrightarrow \pi_{26}(S^7:3) \xrightarrow{\Delta} \pi_{24}(S^5:3) \xrightarrow{G} \pi_{25}(S^3:3) \longrightarrow \pi_{25}(S^7:3) \longrightarrow \cdots$$

and information concerning Δ and G given in Toda (1962, Proposition 13.3) together with the facts that $\pi_{24}(S^5:3) \approx \mathbb{Z}/3$ has generator $\alpha_5(5)$, $\pi_{26}(S^7:3) \approx \mathbb{Z}/3$ has generator $\alpha_5(7) = E^2\alpha_5(5)$, and $\pi_{25}(S^3:3) \approx \mathbb{Z}/3$, we see that

$$\Delta(\alpha_5(7)) = \Delta(E^2\alpha_5(5)) = 3\alpha_5(5) = 0$$

and

$$G(\alpha_5(5)) = \alpha_1(3)E\alpha_5(5) = \alpha_1(3)\alpha_5(6).$$

Thus, G is an isomorphism and $\alpha_1(3)\alpha_5(6)$ generates $\pi_{25}(S^3:3)$. This proves relation (1). To prove (2) and (3) one first notes the following concerning groups and generators: $\pi_{30}(S^7:3) \approx \mathbb{Z}/9 \oplus \mathbb{Z}/3$ has generators $\alpha_6'(7) = E^2\alpha_6'(5)$ of order 9 and $\alpha_1(7)\beta_1(10)\beta_1(20) = E^2(\alpha_1(5)\beta_1(8)\beta_1(18))$ of order 3, $\pi_{30}(S^5:3) \approx \mathbb{Z}/9 \oplus \mathbb{Z}/3$ has generators $\alpha_6'(5)$ of order 9 and $\alpha_1(5)\beta_1(8)\beta_1(18)$ of order 3.

By Toda (1962, Proposition 13.3),

$$\Delta(\alpha_6'(7)) = \Delta E^2\alpha_6'(5) = 3\alpha_6'(5) \neq 0, \text{ and}$$

$$\Delta(\alpha_1(7)\beta_1(10)\beta_1(20)) = \Delta E^2(\alpha_1(5)\beta_1(8)\beta_1(18)) = 3\alpha_1(5)\beta_1(8)\beta_1(18) = 0.$$

Hence, $G(\alpha_1(5)\beta_1(8)\beta_1(18)) = \alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19)$ generates and since $\alpha_6'(5) \notin \operatorname{im}\Delta$, $G(\alpha_6'(5)) = \alpha_1(3)\alpha_6'(6)$ generates also.

That they map to the indicated generators is seen as follows: By Toda (1965, (2.12)), $H = I \circ H^{(2)}$ and by Toda (1965 (6.3) and Lemma 6.1)

$$H(p_*\bar{Q}^2(\alpha_5)) = I \circ H^{(2)}(p_*\bar{Q}^2(\alpha_5)) = IQ^1(\alpha_6') = I_*I'(\alpha_6'(5)) = 0, \text{ and}$$

$H(p_*\bar{Q}^2(\beta_1\beta_1))=I\circ H^{(2)}(p_*Q^2(\beta_1\beta_1))=IQ^1(\alpha_1\beta_1\beta_1)=I\circ I'(\alpha_1(5)\beta_1(8)\beta_1(18))=0$ since $I\circ I'$ is two steps in exact sequence, Toda (1965, (2.5)). Hence each of the generators of $\pi_{29}(S^3:3)$ is the image of an element of $\pi_{28}(S^5:3)$ under G.

By Toda (1965, (1.3)),

$$H^{(2)}:\pi_{29}(S^3:3)\longrightarrow\pi_{26}(Q^1{}_2:3)$$

is an isomorphism and by Toda (1966, (10.1)),

$$H^{(2)}(\alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19)) = H^{(2)}(\alpha_1(3))\alpha_1(3)\beta_1(6)\beta_1(16).$$

$H^{(2)}(\alpha_1(3))$ generates $\pi_3(Q^1{}_2:3)$ which in Toda's notation, (1965, (6.4)), is $I'\iota_5$ so using Toda (1965, (2.6),(6.3) and Lemma 6.1), we have

$$\begin{aligned}H^{(2)}(\alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19)) &= (I'\iota_5)\alpha_1(3)\beta_1(6)\beta_1(16)\\ &= I'(\iota_5\alpha_1(5)\beta_1(8)\beta_1(18))\\ &= I'(\alpha_1(5)\beta_1(8)\beta_1(18))\\ &= Q^1(\alpha_1\beta_1\beta_1)\\ &= H^{(2)}(p_*Q^2(\beta_1\beta_1)).\end{aligned}$$

Thus $\alpha_1(3)\alpha_1(6)\beta_1(9)\beta_1(19) = \pm p_*Q^2(\beta_1\beta_1)$, and relations (2) and (3) are established.

To see that relation (4) holds, consider the following. By Toda (1966, (11.1)) we have an exact sequence

$$\cdots\longrightarrow\pi_{28}(S^1:3)\xrightarrow{E^2}\pi_{30}(S^3:3)\xrightarrow{H^2}\pi_{27}(Q^1{}_2:3)\longrightarrow\pi_{27}(S^1:3)\longrightarrow\cdots$$

which means that $H^{(2)}:\pi_{30}(S^3:3)\longrightarrow\pi_{27}(Q^1{}_2:3)$ is an isomorphism. By Toda (1965, Lemma 6.1 (i))

$$H^{(2)}(p_*\bar{Q}^2(\beta_1\beta_1))=\pm\bar{Q}^1(\alpha_1\beta_1\beta_1)$$

and from Toda (1966, (10.1) and (11.7)) we have

$$\begin{aligned}H^{(2)}(\bar{u}_3(0,\beta_1\beta_1)\alpha_1(27)) &= H^{(2)}(\bar{u}_3(0,\beta_1\beta_1))\alpha_1(24)\\ &=\pm\bar{Q}^1(\beta_1\beta_1)\alpha_1(24).\end{aligned}$$

In the exact sequence Toda (1965, (2.5))

$$\cdots\longrightarrow\pi_{31}(S^7:3)\xrightarrow{\Delta}\pi_{29}(S^5:3)\xrightarrow{I'}\pi_{27}(Q^1{}_2:3)\xrightarrow{I}\pi_{30}(S^7:3)\longrightarrow\cdots$$

Δ is an isomorphism [Toda (1966, (vi) p.241)] , and so $\ker I \approx 0$.

By definition [Toda, (1965, 6.3 (ii))] ,

$$E^{\infty} I(\pm \bar{Q}^{1}(\alpha_1\beta_1\beta_1)) = \pm \alpha_1\beta_1\beta_1$$

and by Toda (1965, (2.6)) and Toda (1962, (3.4))

$$\begin{aligned} E^{\infty} I(\pm \bar{Q}^{1}(\beta_1\beta_1)\alpha_1(24)) &= \pm \beta_1\beta_1\ E^{\infty}\alpha_1(27) \\ &= \pm \beta_1\beta_1\alpha_1 \\ &= \pm \alpha_1\beta_1\beta_1 . \end{aligned}$$

But E^{∞} is an injection in this case which means

$$\bar{Q}^{1}(\alpha_1\beta_1\beta_1) - [\pm \bar{Q}^{1}(\beta_1\beta_1)\alpha_1(24)] \in \ker I .$$

Thus $\bar{Q}^{1}(\alpha_1\beta_1\beta_1) = \pm \bar{Q}^{1}(\beta_1\beta_1)\alpha_1(24)$ and since $H^{(2)}$ is an injection, relation (4) now follows.

Q.E.D.

<u>LEMMA (3.5)</u>

i =	24	25	26	27	28
$\pi_i(S^7_{(3)}) \approx$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$
generators			$\alpha_5(7)$	$\beta_1(7)\beta_1(17)$	$u_3(1,\beta_1)$

i =	29	30	31
$\pi_i(S^7_{(3)}) \approx$	$\mathbb{Z}/9$	$\mathbb{Z}/9 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3$
generators		$\alpha'_6(7)$ $\alpha_1(7)\beta_1(10)\beta_1(20)$	$p_* Q^4(\beta_1)$

where $u_3(1,\beta_1)\,\alpha_1(28) = \pm p_* Q^4(\beta_1)$.

<u>PROOF</u> As in previous lemma, table entries come from Toda (1965, 1966) while the relation $u_3(1,\beta_1)\alpha_1(28) = \pm p_* Q^4(\beta_1)$ is proven by Toda (1966, p.242).

Q.E.D.

As stated earlier, for the most part all we will need to know about the homotopy groups of $(X_2)_{(3)}$ is their cardinalities; however, where possible in the next lemma we have specified the groups themselves.

LEMMA (3.6)

i=	8	9	10	11	12	13	14	15	16
$\pi_i((X_2)_{(3)}) \approx$	0	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0

i=	17	18	19	20	21	22	23	24
$\pi_i((X_2)_{(3)}) \approx$	0	$\mathbb{Z}/9$	0	0	0	$\mathbb{Z}/3$	0	0

i=	25	26	27	28	29	30
$\pi_i((X_2)_{(3)}) \approx$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9$	$\mathbb{Z}/9$ or $\mathbb{Z}/3 \oplus \mathbb{Z}/3$

PROOF Entries for $8 \leq i \leq 23$ appear in Mimura-Toda (1964) since $(X_2)_{(3)} \simeq Sp(2)_{(3)}$ and $\pi_i(Sp(2)_{(3)}) \approx \pi_i(Sp(2):3)$. The entries for $24 \leq i \leq 30$ are computed from the exact sequences

(3.6.1) $$0 \longrightarrow \operatorname{coker}(\partial:\pi_{i+1}(S^{4n+3}:3) \longrightarrow \pi_i(Sp(n):3)) \longrightarrow \pi_i(Sp(n+1):3) \longrightarrow \ker(\partial:\pi_i(S^{4n+3}:3) \longrightarrow \pi_{i-1}(Sp(n):3)) \longrightarrow 0$$

which are obtained from the fiberings $Sp(n) \longrightarrow Sp(n+1) \longrightarrow S^{4n+3}$. ∂ is the boundary map and for the case n=1 the following table gives the action of $\partial:\pi_{i+1}(S^7:3) \longrightarrow \pi_i(S^3:3)$ on the various generators.

$i =$	25	26	27
$\alpha =$	$\alpha_5(7)$	$\beta_1(7)\beta_1(17)$	$u_3(1,\beta_1)$
$\partial(\alpha) =$	$\pm\gamma_2(3)$	$\alpha_1(3)\beta_1(6)\beta_1(16)$	$\pm\bar{u}_3(0,\beta_1\beta_1)$

$i =$	28	29		30
$\alpha =$		$\alpha_6'(7)$	$\alpha_1(7)\beta_1(10)\beta_1(20)$	$p_*Q^4(\beta_1)$
$\partial(\alpha) =$		$\pm p_*\bar{Q}^2(\alpha_5)$	$\pm p_*Q^2(\beta_1\beta_1)$	$\pm p_*\bar{Q}^2(\beta_1\beta_1)$

Mimura-Toda (1964a, Thm. 5.1) show in general that for $\alpha E\beta \epsilon \pi_i(S^{4n+3})$

(3.6.2) $$\partial(\alpha E\beta) = \partial\alpha\beta \quad .$$

Now $\partial(\iota_7) = \chi(X_2) = \alpha_1(3)$ which is a generator of $\pi_6(S^3:3)$. Thus for $\alpha \epsilon \pi_i(S^7:3)$ which is a suspension element, that is $\alpha = E\beta$ for some $\beta \epsilon \pi_{i-1}(S^6:3)$, we have

$$\begin{aligned} \partial(E\beta) &= \partial(\iota_7 E\beta) \\ &= \partial(\iota_7)\beta \\ &= \alpha_1(3)\beta \ . \end{aligned}$$

For i = 25, 26 and both 29's, α in the above table is a suspension element and the images under ∂ for these entries follow immediately from the above observation and Lemma (3.4). It remains to check the entries for i = 27 and 30.

Unfortunately neither $u_3(1,\beta_1)$ nor $p_*Q^4(\beta_1)$ are suspension elements so the technique used above does not apply. The entry for i = 27 will be needed to establish the entry for i = 30. We will establish the entry for i = 27 by showing that $\pi_{27}((X_2)_{(3)}) \approx \pi_{28}((X_2)_{(3)}) \approx 0$. That this gives $\partial(u_3(1,\beta_1) = \pm\bar{u}_3(0,\beta_1\beta_1)$

follows from the exact sequence

$$\cdots \longrightarrow \pi_{28}(S^3:3) \longrightarrow \pi_{28}((X_2)_{(3)}) \longrightarrow \pi_{28}(S^7:3) \xrightarrow{\partial} \pi_{27}(S^3:3)$$
$$\longrightarrow \pi_{27}((X_2)_{(3)}) \longrightarrow \ker(\partial:\pi_{27}(S^7:3) \longrightarrow \pi_{26}(S^3:3)) \longrightarrow 0$$

by noting that $\pi_{28}(S^3:3) \approx \ker(\partial:\pi_{27}(S^7:3) \longrightarrow \pi_{26}(S^3:3)) \approx 0$ and $\pi_{28}(S^7:3) \approx \pi_{27}(S^3:3) \approx \mathbb{Z}/3$. Also note that this sequence implies that $\pi_{28}((X_2)_{(3)}) \approx \pi_{27}((X_2)_{(3)})$.

Mimura-Toda (1964a) show that $\pi_{28}(Sp(6):3) \approx \pi_{27}(Sp(6):3) \approx 0$. Using exact sequence (3.6.1) with n=2, 4 and 5, i = 27 and 28 together with the facts that $\pi_{29}(S^{11}:3) \approx \pi_{28}(S^{11}:3) \approx \pi_{27}(S^{11}:3) \approx \pi_{28}(S^{19}:3) \approx \pi_{27}(S^{19}:3) \approx \pi_{29}(S^{23}:3) \approx \pi_{28}(S^{23}:3) \approx \pi_{27}(S^{23}:3) \approx 0$ one can see that $\pi_{28}(Sp(5):3) \approx \pi_{27}(Sp(4):3) \approx 0$ and $\pi_{27}(Sp(3):3) \approx \pi_{27}((X_2)_{(3)}) \approx \pi_{28}(Sp(3):3) \approx \pi_{28}((X_2)_{(3)})$.

Consider now the following pieces of exact sequences obtained from the fiberings $Sp(n) \longrightarrow Sp(n+1) \longrightarrow S^{4n+3}$ for n = 3 and 4.

(3.6.3) $\cdots \longrightarrow \pi_{29}(S^{19}:3) \longrightarrow \pi_{28}(Sp(4):3) \longrightarrow \pi_{28}(Sp(5):3) \longrightarrow \cdots$

(3.6.4) $\cdots \longrightarrow \pi_{29}(S^{15}:3) \longrightarrow \pi_{28}(Sp(3):3) \longrightarrow \pi_{28}(Sp(4):3) \longrightarrow$
$\pi_{28}(S^{15}:3) \longrightarrow \pi_{27}(Sp(3):3) \longrightarrow \pi_{27}(Sp(4):3) \longrightarrow \cdots$.

Since $\pi_{28}(Sp(5):3) \approx 0$ and $\pi_{29}(S^{19}:3) \approx \mathbb{Z}/3$ by (3.6.3) it must be that $\pi_{28}(Sp(4):3) \approx \mathbb{Z}/3$ or 0. If $\pi_{28}(Sp(4):3) \approx 0$ then since $\pi_{28}(S^{15}:3) \approx \mathbb{Z}/3$ and $\pi_{29}(S^{15}:3) \approx 0$ we see from (3.6.4) that $\pi_{28}(Sp(3):3) \approx 0$ and $\pi_{27}(Sp(3):3) \approx \mathbb{Z}/3$ which is a contradiction since this means that $\pi_{28}((X_2)_{(3)}) \not\approx \pi_{27}((X_2)_{(3)})$. Thus $\pi_{28}(Sp(4):3) \approx \mathbb{Z}/3$. Consider now the homomorphism $\partial:\pi_{28}(S^{15}:3) \longrightarrow \pi_{27}(Sp(3):3)$ in sequence (3.6.4). By Toda (1962) $\pi_{28}(S^{15}:3) \approx \mathbb{Z}/3$ is generated by $\alpha_1(15)\beta_1(18)$ and thus $\partial(\alpha_1(15)\beta_1(18)) = \partial(\alpha_1(15))\beta_1(17)$ by (3.6.2). But $\partial(\alpha_1(15)) \in \pi_{17}(Sp(3):3)$ which is shown to be trivial by Mimura-Toda (1964a). Thus ∂ is trivial in (3.6.4) and we see that $\pi_{27}(Sp(3):3) \approx 0$. Hence, we have shown that $\pi_{28}((X_2)_{(3)}) \approx \pi_{27}((X_2)_{(3)}) \approx 0$ and have also established the entry in boundary homomorphism table for i = 27.

To establish entry for i = 30 we first note that Mimura-Toda (1964, Thm. 2.5) show in general that $\partial(\alpha\beta) = E^*(E\partial\alpha)\beta)$ where $E^*: \pi_{i+1}(S^4) \longrightarrow \pi_i(S^3)$ has the property that $E^* \circ E = 1$. Using the relations given in Lemmas (3.5) and (3.4) we have

$$\begin{aligned}\partial(p_*Q^4(\beta_1) &= \partial(u_3(1,\beta_1)\alpha_1(28)) \\ &= E^*((E\partial u_3(1,\beta_1))\alpha_1(28)) = E^* \circ E(\bar{u}_3(1,\beta_1\beta_1)\alpha_1(27)) \\ &= \pm\bar{u}_3(0,\beta_1\beta_1)\alpha_1(27) = \pm p_*\bar{Q}^2(\beta_1\beta_1).\end{aligned}$$

Using the table together with Lemmas (3.4) and (3.5) one can easily compute coker∂ and ker∂. This information, when used in exact sequence (3.6.1) with n = 1 establishes results $\pi_i((X_2)_{(3)})$ for i = 24, 25, 26, 29 and 30.

Q.E.D.

The next lemma completes the preliminary information needed to compute $\mu((Sp(2) x\ S^5{}_{(3)})$.

<u>LEMMA (3.7)</u>

i =	8	9	10	11	12	13	14	15	16
$\pi_i(S^5{}_{(3)}) \approx$	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9$	$\mathbb{Z}/9$
generators	$\alpha_1(5)$				$\alpha_2(5)$			$\beta_1(5)$	$\alpha_3'(5)$

i =	17	18	19	20	21	22
$\pi_i(S^5{}_{(3)}) \approx$	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0
generators		$\alpha_1(5)\beta_1(8)$	$p_*\bar{Q}^3(\alpha_1)$	$\alpha_4(5)$		

$i =$	23	24	25	26	27	30
$\pi_i(S^5_{(3)}) \approx$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0
generators	$p_*\bar{Q}^3(\alpha_2)$	$\alpha_5(5)$		$p_*\bar{Q}^3(\beta_1)$	$\gamma_2(5)$	

where the following relations hold:

(1) $\alpha_2(5)\alpha_1(12) = \pm 3\beta_1(5)$,

(2) $\beta_1(5)\alpha_1(15) = \alpha_1(5)\beta_1(8)$,

(3) $\alpha_3'(5)\alpha_1(16) = \pm p_*\bar{Q}^3(\alpha_1)$,

(4) $p_*\bar{Q}^3(\alpha_2)\alpha_1(23) = 0$,

(5) $\alpha_5(5)\alpha_1(24) = \pm 3\gamma_2(5)$, and

(6) $\alpha_4(5)\alpha_1(20) = 0$.

PROOF The tabular entries can be obtained directly from Toda (1962) and Toda (1966). We will prove that the six relations hold below.

Mimura (1967) shows that $\alpha_2(3)\alpha_1(10) = \pm\alpha_1(3)\alpha_2(6)$. Toda (1962, Lemma 13.8) shows that $\alpha_1(5)\alpha_1(8) = \pm 3\beta_1(5)$, thus

$$\alpha_2(5)\alpha_1(12) = E^2(\alpha_2(3)\alpha_1(10)) = \pm E^2(\alpha_1(3)\alpha_2(6)) = \pm\alpha_1(5)\alpha_2(8)$$

and (1) follows.

$E^\infty : \pi_{18}(S^5) \longrightarrow$ (3-primary component of stable 13-stem) is an injection. The stable generators anti-commute, Toda (1962, (3.4)), hence

$$E^\infty(\beta_1(5)\alpha_1(15)) = \beta_1\alpha_1 = \alpha_1\beta_1 = E^\infty(\alpha_1(5)\beta_1(8))$$ and

(2) holds.

To show that relation (3) holds it suffices to show that $\alpha_3'(5)\alpha_1(16) \neq 0$ since the group involved is $\mathbb{Z}/3$.

Consider the homomorphisms

$$\beta_1(19)^* : \pi_{19}(S^5:3) \longrightarrow \pi_{29}(S^5:3)$$

and

$$\alpha_1(26)^*:\pi_{26}(S^5:3) \longrightarrow \pi_{29}(S^5:3).$$

$$\begin{aligned}\beta_1(19)^*(\alpha_3'(5)\alpha_1(16)) &= \alpha_3'(5)\alpha_1(16)\beta_1(19)\\ &= \alpha_3'(5)\beta_1(16)\alpha_1(26)\\ &= \alpha_1(26)^*(\alpha_3'(5)\beta_1(16)).\end{aligned}$$

By Toda (1966, p.242) we see that $p_*\overline{Q}^3(\beta_1)\alpha_1(26)$ generates $\pi_{29}(S^5:3)$ and since $p_*\overline{Q}^3(\beta_1)$ generates $\pi_{26}(S^5:3)$ we conclude that $\alpha_1(26)^*$ is an isomorphism. Thus, to show that $\alpha_3'(5)\alpha_1(16) \neq 0$ it suffices to show that $\alpha_3'(5)\beta_1(16) \neq 0$.

By Toda (1966, (10.1) and Lemma 11.5)

$$\begin{aligned}H^{(2)}(\alpha_3'(5)\beta_1(16)) &= H^{(2)}(\alpha_3'(5))\beta_1(13)\\ &= \overline{Q}^2(\alpha_1)\beta_1(13\end{aligned}$$

where $H^{(2)}(\alpha_3'(5)\beta_1(16)) \in \pi_{23}(Q_2^3:3)$. By Toda (1965, (6.4)) $\pi_{23}(Q_2^3:3) \approx \mathbb{Z}/3$ and has generator $\overline{Q}^2(\alpha_1\beta_1)$.

By definition,

$$E^\infty I(\overline{Q}^2(\alpha_1\beta_1)) = \alpha_1\beta_1$$

and from Toda (1965, (2.6))

$$\begin{aligned}E^\infty I(\overline{Q}^2(\alpha_1)\beta_1(13)) &= E^\infty\left[(I(\overline{Q}^2(\alpha_1))\beta_1(16)\right]\\ &= E^\infty(I\overline{Q}^2(\alpha_1))E^\infty\beta_1(16)\\ &= \alpha_1\beta_1.\end{aligned}$$

From the exact sequence Toda (1966, (11.2)) we see that

$$I:\pi_{23}(Q_2^3:3) \longrightarrow \pi_{26}(S^{13}:3)$$

is an isomorphism and since E^∞ is an injection we see that

$$\overline{Q}^2(\alpha_1)\beta_1(13) = \overline{Q}^2(\alpha_1\beta_1).$$

This implies that $\alpha_3'(5)\beta_1(16) \neq 0$ and we have demonstrated (3).

By Toda (1966, (10.1)) and Toda (1965, Lemma 6.1 (iii)) we have

$$H^{(2)}(p_*\bar{Q}^3(\alpha_2)\alpha_1(23)) = H^{(2)}(p_*\bar{Q}^3(\alpha_2))\alpha_1(20)$$
$$= \pm Q^2(\alpha_3')\alpha_1(20).$$

From the exact sequence Toda (1966, (11.1)) and the fact that $\pi_{24}(S^3:3)$ is trivial we see that

$$H^{(2)}:\pi_{26}(S^5:3) \longrightarrow \pi_{23}(Q_2^3:3)$$

is an injection.

Similarly, from the exact sequence Toda (1966, (11.2)) and the fact that $\pi_{25}(S^{11}:3)$ is trivial we see that

$$I:\pi_{23}(Q_2^3:3) \longrightarrow \pi_{26}(S^{13}:3)$$

is an injection. But

$$I(Q^2(\alpha_3')\alpha_1(20)) = I(Q^2(\alpha_3'))\alpha_1(23)$$
$$= I(I'(\alpha_3(11))\alpha_1(23)$$
$$= 0$$

by Toda (1965, (2.6)), definition of the generator $Q^2(\alpha_3')$, and the fact that $I \circ I'$ is two steps in an exact sequence. Thus

$$\pm Q^2(\alpha_3')\alpha_1(20) = 0$$

and (4) follows.

$\alpha_5(5)\alpha_1(24) \in \pi_{27}(S^5:3)$ and $E:\pi_{27}(S^5:3) \longrightarrow \pi_{28}(S^6:3)$ is an injection. Using Toda (1962, Proposition 3.1) we have

$$E(\alpha_5(5)\alpha_1(24)) = \alpha_5(6)\alpha_1(25)$$
$$= E^3\alpha_5(3)E^{22}\alpha_1(3)$$
$$= -E^3\alpha_1(3)E^6\alpha_5(3)$$
$$= E(-\alpha_1(5)\alpha_5(8)).$$

Thus $\alpha_5(5)\alpha_1(24) = -\alpha_1(5)\alpha_5(8) = -E^2(\alpha_1(3)\alpha_5(6))$. By Lemma (3.4) relation (1),

$$\alpha_1(3)\alpha_5(6) = \pm\gamma_2(3)$$

and by Toda (1966, (11.8))

$$E^2(\gamma_2(3) = 3\gamma_2(5)$$

thus $\alpha_5(5)\alpha_1(24) = \pm 3\gamma_2(5)$ and (5) is established.

Finally, as above one can show that

$$\alpha_4(5)\alpha_1(20) = -\alpha_1(5)\alpha_4(8).$$

But $\alpha_1(5)\alpha_4(8) = E^2(\alpha_1(3)\alpha_4(6))$ with $\alpha_1(3)\alpha_4(6) \in \pi_{21}(S^3:3)$. Toda (1966) shows that $\pi_{21}(S^3:3) \approx \mathbb{Z}/3$ and that it is generated by an element whose double suspension is trivial, hence $\alpha_4(5)\alpha_1(20) = 0$ and we have (6).

Q.E.D.

We are now ready to compute $\mu((Sp(2) \times S^5)_{(3)})$.

<u>PROPOSITION (3.8)</u>

$$\mu((Sp(2) \times S^5)_{(3)}) = 3^{46}$$

<u>PROOF</u> Since $(Sp(2)\times S^5)_{(3)} \simeq Sp(2)_{(3)} \times S^5_{(3)} \simeq (X_2)_{(3)} \times S^5_{(3)}$ it suffices to compute $\mu((X_2 \times S^5)_{(3)})$. Using Theorem (1.23) we have

$$\mu((X_2 \times S^5)_{(3)}) = \#\left[(X_2 \times S^5)_{(3)} \wedge (X_2 \times S^5)_{(3)},\ (X_2 \times S^5)_{(3)}\right]$$

$$= \left(\prod_{i=1}^{6} a_i\right)\left(\prod_{j=1}^{6} b_j^{\,2}\right)$$

where

$$a_1 = \#[X_2 \wedge S^5 \wedge X_2 \wedge S^5,\ (X_2)_{(3)}] = \#[E^{10}(X_2 \wedge X_2),\ (X_2)_{(3)}] = 3^2$$

$$a_2 = \#[X_2 \wedge S^5 \wedge X_2 \wedge S^5,\ S^5_{(3)}] = \#[E^{10}(X_2 \wedge X_2),\ S^5_{(3)}] = 3^8$$

$$a_3 = \#[X_2 \wedge X_2,\ (X_2)_{(3)}] = 3^1$$

$$a_4 = \#[X_2 \wedge X_2,\ S^5_{(3)}] = 3^1$$

$$a_5 = \#[S^5 \wedge S^5,\ (X_2)_{(3)}] = \#(\pi_{10}((X_2)_{(3)})) = 3^1$$

$$a_6 = \#[S^5 \wedge S^5,\ S^5_{(3)}] = \#(\pi_{10}(S^5_{(3)})) = \#(\pi_{10}(S^5:3)) = 3^0$$

$$b_1 = \#[X_2 \wedge S^5 \wedge X_2,\ (X_2)_{(3)}] = \#[E^5(X_2 \wedge X_2),\ (X_2)_{(3)}] = 3^7$$

$$b_2 = \#[X_2 \wedge S^5 \wedge X_2,\ S^5_{(3)}] = \#[E^5(X_2 \wedge X_2),\ S^5_{(3)}] = 3^5$$

$$b_3 = \#[X_2 \wedge S^5 \wedge S^5,\ (X_2)_{(3)}] = \#[E^{10}(X_2),\ (X_2)_{(3)}] = 3^0$$

$$b_4 = \#[X_2 \wedge S^5 \wedge S^5,\ (S^5)_{(3)}] = \#[E^{10}(X_2),\ S^5_{(3)}] = 3^1$$

$$b_5 = \#[X_2 \wedge S^5,\ (X_2)_{(3)}] = \#[E^5(X_2),\ (X_2)_{(3)}] = 3^0$$
$$b_6 = \#[X_2 \wedge S^5,\ S^5_{(3)}] = \#[E^5(X_2),\ S^5_{(3)}] = 3^4.$$

The final result follows by adding exponents so it remains to verify the twelve values a_i and b_j for $1 \leq i \leq 6$.

Since $(X_2)_{(3)} \simeq Sp(2)_{(3)}$, $a_3 = 3^1$ follows from Mimura's (1969) computation of $\mu(Sp(2))$. $a_5 = 3^1$ and $a_6 = 3^0$ comes from Lemmas (3.6) and (3.7).

By Lemma (3.3 (ii)) we have

$$E^5(X_2) \simeq S^8 \cup_{\alpha_1(8)} e^{12} \vee S^{15}$$

and

$$E^{10}(X_2) \simeq S^{13} \cup_{\alpha_1(13)} e^{17} \vee S^{20}.$$

Thus

$$[E^5(X_2),\ S^5_{(3)}] \approx [S^8 \cup_{\alpha_1(8)} e^{12},\ S^5_{(3)}] \oplus \pi_{15}(S^5_{(3)}),$$

$$[E^5(X_2),\ (X_2)_{(3)}] \approx [S^8 \cup_{\alpha_1(8)} e^{12},\ (X_2)_{(3)}] \oplus \pi_{15}((X_2)_{(3)}),$$

$$[E^{10}(X_2),\ S^5_{(3)}] \approx [S^{13} \cup_{\alpha_1(13)} e^{17},\ S^5_{(3)}] \oplus \pi_{20}(S^5_{(3)}),$$

and

$$[E^{10}(X_2),\ (X_2)_{(3)}] \approx [S^{13} \cup_{\alpha_1(13)} e^{17},\ (X_2)_{(3)}] \oplus \pi_{20}((X_2)_{(3)}).$$

Letting Z represent $S^5_{(3)}$ or $(X_2)_{(3)}$, from the cofibrations

$$S^{11} \xrightarrow{\alpha_1(8)} S^8 \longrightarrow S^8 \cup_{\alpha_1(8)} e^{12} \longrightarrow S^{12} \xrightarrow{\alpha_1(9)} S^9 \longrightarrow \cdots$$

and

$$S^{16} \xrightarrow{\alpha_1(13)} S^{13} \longrightarrow S^{13} \cup_{\alpha_1(13)} e^{17} \longrightarrow S^{17} \xrightarrow{\alpha_1(14)} S^{14} \longrightarrow \cdots$$

we obtain exact sequences

$$\pi_{11}(Z) \xleftarrow{\alpha_1(8)^*} \pi_8(Z) \longleftarrow [S^8 \cup_{\alpha_1(8)} e^{12},\ Z] \longleftarrow \pi_{12}(Z) \xleftarrow{\alpha_1(9)^*} \pi_9(Z) \longleftarrow \cdots$$

and

$$\pi_{16}(Z) \xleftarrow{\alpha_1(13)^*} \pi_{13}(Z) \longleftarrow [S^{13} \cup_{\alpha_1(13)} e^{17},\ Z] \longleftarrow \pi_{17}(Z) \xleftarrow{\alpha_1(14)^*} \pi_{14}(Z) \longleftarrow.$$

Now $\pi_{11}(S^5_{(3)}) \approx \pi_9(S^5_{(3)}) \approx 0$ and $\pi_8(S^5_{(3)}) \approx \pi_{12}(S^5_{(3)}) \approx \mathbb{Z}/3$ means $\#[S^8 \cup_{\alpha_1(8)} e^{12}, S^5_{(3)}] = 3^2$ which together with $\pi_{15}(S^5_{(3)}) \approx \mathbb{Z}/9$ implies $b_6 = 3^4$.

Similarly, $\pi_8((X_2)_{(3)}) \approx \pi_{12}((X_2)_{(3)}) \approx \pi_{15}((X_2)_{(3)}) \approx 0$ gives $b_5 = 3^0$, and likewise $\pi_{13}(Z) \approx \pi_{17}(Z) \approx \pi_{20}((X_2)_{(3)}) \approx 0$ together with $\pi_{20}(S^5_{(3)}) \approx \mathbb{Z}/3$ gives $b_4 = 3^1$ and $b_3 = 3^0$.

To compute a_4 we use the cofibering

$$S^6 \longrightarrow X_2 \wedge X_2 \longrightarrow (X_2 \wedge X_2)/S^6 \longrightarrow S^7 \longrightarrow E(X_2 \wedge X_2) \longrightarrow \cdots$$

which gives an exact sequence

$$\pi_6(S^5_{(3)}) \longleftarrow [X_2 \wedge X_2, S^5_{(3)}] \longleftarrow [(X_2 \wedge X_2)/S^6, S^5_{(3)}] \longleftarrow \pi_7(S^5_{(3)}) \longleftarrow \cdots$$

which, since $\pi_6(S^5_{(3)}) \approx \pi_7(S^5_{(3)}) \approx 0$, means that

$$a_4 = \#[(X_2 \wedge X_2)/S^6, S^5_{(3)}].$$

By Lemma (3.3 (iv)) we see that

$$[(X_2 \wedge X_2)/S^6, S^5_{(3)}] \approx [(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}), S^5_{(3)}] \oplus$$
$$[S^{13} \cup_{\alpha_1(13)} e^{17}, S^5_{(3)}] \oplus [S^{13} \cup_{\alpha_1(13)} e^{17}, S^5_{(3)}].$$

But $[S^{13} \cup_{\alpha_1(13)} e^{17}, S^5_{(3)}] \approx 0$ so we have

$$[(X_2 \wedge X_2)/S^6, S^5_{(3)}] \approx [(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}), S^5_{(3)}].$$

As before, we use the appropriate cofibering to get an exact sequence

$$\pi_{19}(S^5_{(3)}) \oplus \pi_{13}(S^5_{(3)}) \longleftarrow \pi_{10}(S^5_{(3)}) \oplus \pi_{10}(S^5_{(3)}) \longleftarrow$$
$$[(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}), S^5_{(3)}] \longleftarrow \pi_{20}(S^5_{(3)}) \oplus$$
$$\pi_{14}(S^5_{(3)}) \longleftarrow \pi_{11}(S^5_{(3)}) \oplus \pi_{11}(S^5_{(3)}) \longleftarrow \cdots \quad .$$

Using the fact that $\pi_{10}(S^5_{(3)}) \approx \pi_{11}(S^5_{(3)}) \approx \pi_{14}(S^5_{(3)}) \approx 0$ and $\pi_{20}(S^5_{(3)}) \approx \mathbb{Z}/3$ we see that

$$[(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}),\ S^5_{(3)}] \approx \mathbb{Z}/3$$

and so $a_4 = 3^1$.

By Lemma (3.3 (ii)),

$$E^5(X_2 \wedge X_2) \simeq [(S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19}]$$
$$\vee (S^{18} \cup_{\alpha_1(18)} e^{22}) \vee (S^{18} \cup_{\alpha_1(18)} e^{22}) \vee S^{25}$$

hence

$$b_1 = \#[(S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19},\ (X_2)_{(3)}] \cdot$$
$$(\#[S^{18} \cup_{\alpha_1(18)} e^{22},\ (X_2)_{(3)}])^2 \cdot \#(\pi_{25}((X_2)_{(3)}))$$

and

$$b_2 = \#[(S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19},\ S^5_{(3)}] \cdot$$
$$(\#[S^{18} \cup_{\alpha_1(18)} e^{22},\ S^5_{(3)}])^2 \cdot \#(\pi_{25}(S^5_{(3)})).$$

From Lemmas (3.6) and (3.7) we have

$$\#(\pi_{25}((X_2)_{(3)})) = \#(\pi_{25}(S^5_{(3)})) = 3^1.$$

The cofibering

$$S^{21} \xrightarrow{\alpha_1(18)} S^{18} \longrightarrow S^{18} \cup_{\alpha_1(18)} e^{22} \longrightarrow S^{22} \xrightarrow{\alpha_1(19)} S^{19} \longrightarrow \cdots\cdots$$

gives an exact sequence

$$\pi_{21}(Z) \xleftarrow{\alpha_1(18)^*} \pi_{18}(Z) \longleftarrow [S^{18} \cup_{\alpha_1(18)} e^{22},\ Z] \longleftarrow$$
$$\pi_{22}(Z) \xleftarrow{\alpha_1(19)^*} \pi_{19}(Z) \longleftarrow \cdots\cdots$$

which since $\pi_{21}(Z) \approx \pi_{22}(S^5_{(3)}) \approx \pi_{19}((X_2)_{(3)}) \approx 0$, $\pi_{18}((X_2)_{(3)}) \approx \mathbb{Z}/9$, $\pi_{22}((X_2)_{(3)}) \approx \pi_{18}(S^5_{(3)}) \approx \mathbb{Z}/3$ produces the cardinalities

$$\#[S^{18} \cup_{\alpha_1(18)} e^{22},\ (X_2)_{(3)}] = 3^3$$

and

$$\#[S^{18} \cup_{\alpha_1(18)} e^{22},\ S^5_{(3)}] = 3^1.$$

From the cofibering

$$S^{18} \longrightarrow (S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \longrightarrow (S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19} \longrightarrow S^{19} \longrightarrow (S^{12} \cup_{\alpha_1(12)} e^{16} \vee S^{16}) \longrightarrow \cdots\cdots$$

we get an exact sequence

(3.8.1) $$\pi_{18}(Z) \xleftarrow{(E^5\beta)^*} [S^{11} \cup_{\alpha_1(11)} e^{15}, Z] \oplus \pi_{15}(Z) \longleftarrow [(S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19}, Z] \longleftarrow \pi_{19}(Z) \xleftarrow{(E^6\beta)^*} [(S^{12} \cup_{\alpha_1(12)} e^{16}, Z] \oplus \pi_{16}(Z) \longleftarrow \cdots\cdots .$$

The exact cofibration sequence

(3.8.2) $$\pi_{14}(Z) \xleftarrow{\alpha_1(11)^*} \pi_{11}(Z) \longleftarrow [S^{11} \cup_{\alpha_1(11)} e^{15}, Z] \xleftarrow{r^*} \pi_{15}(Z) \xleftarrow{\alpha_1(12)^*} \pi_{12}(Z) \longleftarrow \cdots\cdots$$

is needed to compute the group

$$[S^{11} \cup_{\alpha_1(11)} e^{15}, Z] \oplus \pi_{15}(Z).$$

For $Z = (X_2)_{(3)}$ we have $\pi_{11}((X_2)_{(3)}) \approx \pi_{15}((X_2)_{(3)}) \approx \pi_{19}((X_2)_{(3)}) \approx 0$ and we get $[S^{11} \cup_{\alpha_1(11)} e^{15}, (X_2)_{(3)}] \oplus \pi_{15}((X_2)_{(3)}) \approx 0$ from (3.8.2) Thus exact sequence (3.8.1) gives

$$[(S^{11} \cup_{\alpha_1(11)} e^{15} \vee S^{15}) \cup_{E^5\beta} e^{19}, (X_2)_{(3)}] \approx 0$$

and we have

$$b_1 = 3^7.$$

The computation for $Z = S^5{}_{(3)}$ is not as simple because of the existence of non-trivial homotopy groups. We begin by determining $[S^{11} \cup_{\alpha_1(11)} e^{15}, S^5{}_{(3)}]$.

From Lemma (3.7) we have $\pi_{11}(S^5{}_{(3)}) \approx 0$, $\pi_{15}(S^5{}_{(3)}) \approx \mathbb{Z}/9$ with generator $\beta_1(5)$ and $\pi_{12}(S^5{}_{(3)}) \approx \mathbb{Z}/3$ with generator $\alpha_2(5)$. The group in question is thus seen to be isomorphic to coker$\alpha_1(12)^*$ which is isomorphic to $\mathbb{Z}/3$ since

$$\alpha_1(12)^*(\alpha_2(5)) = \alpha_2(5)\alpha_1(12) = \pm 3\beta_1(5)$$

by Lemma (3.7 relation 1). Thus we see that

$$[S^{11}\cup_{\alpha_1(11)}e^{15},\ S^5{}_{(3)}] \approx \mathbb{Z}/3$$

and has generator

$$r^*(\beta_1(5))$$

where r^* is induced by the collapsing map $r: S^{11}\cup_{\alpha_1(11)}e^{15} \longrightarrow S^{15}$.

Since $\pi_{19}(S^5{}_{(3)}) \approx \mathbb{Z}/3$ with generator $p_*\bar{Q}^3(\alpha_1)$ and $\pi_{18}(S^5{}_{(3)}) \approx \mathbb{Z}/3$ with generator $\alpha_1(5)\beta_1(8)$, to compute the group

$$[(S^{11}\cup_{\alpha_1(11)}e^{15}\vee S^{15})\cup_{E^5\beta}e^{19},\ S^5{}_{(3)}]$$

from the exact sequence (3.8.1) we must determine $\ker(E^5\beta)^*$ and $\operatorname{coker}(E^6\beta)^*$.

By Lemma (3.3 (iii)), $E^6\beta \simeq \widetilde{\alpha_1}(15)\vee\alpha_1(16)$ and so $(E^6\beta)^* \approx (\widetilde{\alpha_1}(15))^* \oplus \alpha_1(16)^*$. But $\alpha_1(16)^*$ applied to the generator $\alpha_3'(5)$ of $\pi_{16}(S^5{}_{(3)})$ is $\alpha_3'(5)\alpha_1(16)$ which generates $\pi_{19}(S^5{}_{(3)})$ by relation (3) of Lemma (3.7). Thus $\operatorname{coker}(E^6\beta)^* \approx 0$.

Similarly, $E^5\beta \simeq \widetilde{\alpha_1}(14)\vee\alpha_1(15)$ and $(E^5\beta)^* \approx (\widetilde{\alpha_1}(14))^* \oplus \alpha_1(15)^*$. Thus $\ker(E^5\beta)^*$ is determined by the images $(\widetilde{\alpha_1}(14))^*(r^*(\beta_1(5)))$ and $\alpha_1(15)^*(\beta_1(5))$. Now

$$\alpha_1(15)^*(\beta_1(5)) = \beta_1(5)\alpha_1(15) = \pm\alpha_1(5)\beta_1(8)$$

by relation (2) of Lemma (3.7). Thus we see that $\ker\alpha_1(15)^* \approx \mathbb{Z}/3$. For $\widetilde{\alpha_1}(14)^*$ we get

$$\widetilde{\alpha_1}(14)^*(r^*(\beta_1(5))) = r^*(\beta_1(5))\widetilde{\alpha_1}(14) = \beta_1(5)r\widetilde{\alpha_1}(14).$$

By Toda (1962, (1.18)) we have

$$r\widetilde{\alpha_1}(14) = E\alpha_1(14) = \alpha_1(15)$$

which means

$$\widetilde{\alpha}_1(14)^*(r^*(\beta_1(5))) = \beta_1(5)\alpha_1(15) = \pm\alpha_1(5)\beta_1(8)$$

by relation (2) Lemma (3.7). Thus $\ker(\widetilde{\alpha}_1(14)^*) \approx 0$ and we conclude

$$\ker(E^5\beta)^* \approx \mathbb{Z}/9.$$

Thus, from exact sequence (3.8.1) we see that

$$[(S^{11}\cup_{\alpha_1(11)}e^{15}\vee S^{15})\cup_{E^5\beta}e^{19},\ S^5_{(3)}] \approx \mathbb{Z}/3$$

and we have shown that

$$b_2 = 3^5.$$

As in the above computations for b_1 and b_2 it is similarly seen that

$$a_1 = \#[(S^{16}\cup_{\alpha_1(16)}e^{20}\vee S^{20})\cup_{E^{10}\beta}e^{24},\ (X_2)_{(3)}]\cdot$$
$$(\#[S^{23}\cup_{\alpha_1(23)}e^{27},\ (X_2)_{(3)}])^2\cdot\#(\pi_{30}((X_2)_{(3)}))$$

and

$$a_2 = \#[(S^{16}\cup_{\alpha_1(16)}e^{20}\vee S^{20})\cup_{E^{10}\beta}e^{24},\ S^5_{(3)}]\cdot$$
$$(\#[S^{23}\cup_{\alpha_1(23)}e^{27},\ S^5_{(3)}])^2\cdot\#(\pi_{30}((S^5)_{(3)})).$$

Beginning with the right-most cardinalities in the above products we have

$$\#(\pi_{30}((X_2)_{(3)})) = 3^2$$

and

$$\#(\pi_{30}(S^5_{(3)})) = 3^0$$

from Lemmas (3.6) and (3.7).

The exact cofibration sequence

$$\pi_{26}(Z) \xleftarrow{\alpha_1(23)^*} \pi_{23}(Z) \longleftarrow [S^{23}\cup_{\alpha_1(23)}e^{27},\ Z] \longleftarrow \pi_{27}(Z) \xleftarrow{\alpha_1(24)^*} \pi_{24}(Z) \longleftarrow$$

together with the fact that $\pi_{23}((X_2)_{(3)}) \approx \pi_{27}((X_2)_{(3)}) \approx 0$ gives

$$\#[S^{23}\cup_{\alpha_1(23)}e^{27},\ (X_2)_{(3)}] = 3^0.$$

For $Z = S^5_{(3)}$ all relevant homotopy groups in the above exact sequence are non-trivial so we must determine $\ker\alpha_1(23)^*$ and $\operatorname{coker}\alpha_1(24)^*$.

From Lemma (3.7) we have the following information about groups and generators:

$$\pi_{24}(S^5_{(3)}) \approx \mathbb{Z}/3, \text{ generator } \alpha_5(5)$$
$$\pi_{27}(S^5_{(3)}) \approx \mathbb{Z}/9, \text{ generator } \gamma_2(5) \text{ and}$$
$$\pi_{23}(S^5_{(3)}) \approx \mathbb{Z}/3, \text{ generator } p_*\bar{Q}^3(\alpha_2).$$

By relations (4) and (5) of Lemma (3.7) we have

$$\alpha_1(23)^*(p_*\bar{Q}^3(\alpha_2)) = p_*\bar{Q}^3(\alpha_2)\alpha_1(23) = 0$$

and

$$\alpha_1(24)^*(\alpha_5(5)) = \alpha_5(5)\alpha_1(24) = \pm 3\gamma_2(5)$$

and we thus conclude that

$$\ker\alpha_1(23)^* \approx \operatorname{coker}\alpha_1(24)^* \approx \mathbb{Z}/3$$

which means

$$\#[S^{23}\cup_{\alpha_1(23)}e^{27}, S^5_{(3)}] = 3^2.$$

The cofibration sequence

(3.8.3)
$$\pi_{23}(Z) \xleftarrow{(E^{10}\beta)^*} [S^{16}\cup_{\alpha_1(16)}e^{20}, Z] \oplus \pi_{20}(Z) \longleftarrow$$
$$[(S^{16}\cup_{\alpha_1(16)}e^{20}\vee S^{20})\cup_{E^{10}\beta}e^{24}, Z] \longleftarrow \pi_{24}(Z) \longleftarrow$$
$$[S^{17}\cup_{\alpha_1(17)}e^{21}, Z] \oplus \pi_{21}(Z) \longleftarrow \cdots$$

is used to determine the remaining factors of a_1 and a_2. To use this sequence we must first compute

$$[S^{16}\cup_{\alpha_1(16)}e^{20}, Z]$$

and

$$[S^{17}\cup_{\alpha_1(17)}e^{21}, Z].$$

These groups are obtained from the exact sequence

(3.8.4)
$$\pi_{19}(Z) \xleftarrow{\alpha_1(16)^*} \pi_{16}(Z) \xleftarrow{i^*} [S^{16} \cup_{\alpha_1(16)} e^{20}, Z]$$
$$\xleftarrow{r^*} \pi_{20}(Z) \xleftarrow{\alpha_1(17)^*} \pi_{17}(Z) \longleftarrow [S^{17} \cup_{\alpha_1(17)} e^{21}, Z]$$
$$\longleftarrow \pi_{21}(Z) \longleftarrow \cdots$$

where i* is induced from inclusion and r* by collapsing map.

For $Z = (X_2)_{(3)}$ computations are easy since $\pi_{16}((X_2)_{(3)}) \approx \pi_{20}((X_2)_{(3)}) \approx \pi_{21}((X_2)_{(3)}) \approx 0 \approx \pi_{24}((X_2)_{(3)})$ which in (3.8.4) and (3.8.3) easily give

$$[(S^{16} \cup_{\alpha_1(16)} e^{20} \vee S^{20}) \cup_{E^{10}\beta} e^{24}, (X_2)_{(3)}] \approx 0,$$

that is

$$\#[(S^{16} \cup_{\alpha_1(16)} e^{20} \vee S^{20}) \cup_{E^{10}\beta} e^{24}, (X_2)_{(3)}] = 3^0.$$

At this point we may conclude that $a_1 = 3^2$.

For $Z = S^5_{(3)}$ we have $\pi_{21}(S^5_{(3)}) \approx \pi_{17}(S^5_{(3)}) \approx 0$ and from (3.8.4) we see that

$$[S^{17} \cup_{\alpha_1(17)} e^{21}, S^5_{(3)}] \approx 0.$$

$\pi_{20}(S^5_{(3)}) \approx \mathbb{Z}/3$ with generator $\alpha_4(5)$, $\pi_{17}(S^5_{(3)}) \approx 0$ and $\ker\alpha_1(16)^*$ $\approx \mathbb{Z}/3$ since $\alpha_1(16)^*$ applied to the generator $\alpha_3'(5)$ of $\pi_{16}(S^5_{(3)}) \approx \mathbb{Z}/9$ generates $\pi_{19}(S^5_{(3)}) \approx \mathbb{Z}/3$. We also note at this point that since $3\alpha_3'(5) = \alpha_3(5)$ we may use $\alpha_3(5)$ as a generator of $\ker\alpha_1(16)^*$. In any event, we conclude from (3.8.4) that

$$[S^{16} \cup_{\alpha_1(16)} e^{20}, S^5_{(3)}] \approx \begin{cases} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \\ \text{or} \\ \mathbb{Z}/9 \end{cases}$$

and that both $r^*(\alpha_4(5))$ and $\overline{\alpha_3(5)}$ are non-zero where one or both may be generators. The symbol $\overline{\alpha_3(5)}$ represents an element having the property that $i^*(\overline{\alpha_3(5)}) = \alpha_3(5)$, i.e. $\overline{\alpha_3(5)}$ is an extension of $\alpha_3(5)$.

Using exact sequence (3.8.3) we can determine the cardinality of

$$[(S^{16} \cup_{\alpha_1(16)} e^{20} \vee S^{20}) \cup_{E^{10}\beta} e^{24},\ S^5{}_{(3)}]$$

by noting that it is equal to the product

$$\#(\ker(E^{10}\beta)^*) \cdot \#(\pi_{24}(S^5{}_{(3)})).$$

By Lemma (3.3 (iii)),

$$E^{10}\beta = \widetilde{\alpha_1}(19) \vee \alpha_1(20)$$

so

$$\ker(E^{10}\beta)^* \approx \ker\widetilde{\alpha_1}(19)^* \oplus \ker\alpha_1(20)^*.$$

By relation (6) of Lemma (3.7),

$$\alpha_1(20)^*(\alpha_4(5)) = \alpha_4(5)\alpha_1(20) = 0$$

so

$$\ker\alpha_1(20)^* \approx \pi_{20}(S^5{}_{(3)}) \approx \mathbb{Z}/3$$

i.e. $\#(\ker\alpha_1(20)^*) = 3^1$.

We now show that $\#(\ker\widetilde{\alpha_1}(19)^*) = 3^2$ by demonstrating that both $r^*(\alpha_4(5))\ \widetilde{\alpha_1}(19)$ and $\overline{\alpha_3(5)}\ \widetilde{\alpha_1}(19)$ are trivial.

We first consider the composition $\overline{\alpha_3(5)}\widetilde{\alpha_1}(19)$. By Toda (1962, Proposition (1.7)) the set of all compositions $\{\overline{\alpha_3(5)}\widetilde{\alpha_1}(19)\}$ is equal the secondary composition

$$\{\alpha_3(5),\ \alpha_1(16),\ \alpha_1(19)\}$$

which is a coset of $\pi_{20}(S^5:3)\ \alpha_1(20) \oplus \alpha_3(5)\ \pi_{23}(S^{16}:3)$. However, $\pi_{20}(S^5:3)\ \alpha_1(20) \approx 0$ by relation (6) of Lemma (3.7) and $\alpha_3(5)\pi_{23}(S^{16}:3) \approx 0$ since $\alpha_3(5)\alpha_2(16) = E^2(\alpha_3(3)\alpha_2(14))$ and Toda (1966) observes that the generator of $\pi_{23}(S^5:3)$ is not in the image of E^2. Thus the coset $\{\alpha_3(5),\ \alpha_1(16),\ \alpha_1(19)\}$ contains the single element $\overline{\alpha_3(5)\widetilde{\alpha_1}(19)}$. We may now use Toda (1962, Proposition (1.9)) to conclude that

$$r^*(\overline{\alpha_3(5)\widetilde{\alpha_1}(19)}) = \alpha_3(5)\overline{\alpha_1(16)}$$

where as before r^* is induced by collapsing map

$$r: S^{19}\cup_{\alpha_1(19)} e^{23} \longrightarrow S^{23}$$

and $\overline{\alpha_1(16)}$ is an extension of $\alpha_1(16)$ to $S^{19}\cup_{\alpha_1(19)} e^{23}$.

Using the cofibration sequence

$$\pi_{22}(S^5_{(3)}) \xleftarrow{\alpha_1(19)^*} \pi_{19}(S^5_{(3)}) \xleftarrow{i^*} [S^{19}\cup_{\alpha_1(19)} e^{23},\ \ S^5_{(3)}]$$

$$\xleftarrow{r^*} \pi_{23}(S^5_{(3)}) \xleftarrow{\alpha_1(20)^*} \pi_{20}(S^5_{(3)}) \longleftarrow \cdots$$

and observing that $\alpha_1(20)^*$ is trivial we see that r^* is an injection. Thus, to show that $\overline{\alpha_3(5)\widetilde{\alpha_1}(19)} = 0$ it suffices to show that

$$\alpha_3(5)\overline{\alpha_1(16)} = 0.$$

That this is indeed the case comes from the following observations:

(1) $\alpha_3(5)\alpha_1(16) = (3\alpha_3'(5))\alpha_1(16) = 3(\alpha_3'(5)\alpha_1(16)) = 0$

by relation (3) of Lemma (3.7),

(2) $\alpha_3(5)\overline{\alpha_1(16)}$ may be chosen to be $\overline{\alpha_3(5)\alpha_1(16)}$ since their restrictions to S^{19} are equal, and finally

(3) $\overline{\alpha_3(5)\alpha_1(16)}$ is an extension of a trivial map and hence may be chosen to be trivial.

Finally, as to the triviality of $r^*(\alpha_4(5))\widetilde{\alpha_1}(19)$, we note that

$$r^*(\alpha_4(5))\widetilde{\alpha_1}(19) = \alpha_4(5)\ r\ \widetilde{\alpha_1}(19) = \alpha_4(5)\ E\alpha_1(19)$$
$$= \alpha_4(5)\alpha_1(20) = 0$$

by Toda (1962, (1.18)) and relation (6) Lemma (3.7).

Thus

$$\#[(S^{16}\cup_{\alpha_1(16)} e^{20} \vee S^{20}) \cup_{E^{10}\beta} e^{24},\ S^5_{(3)}] = 3^4$$

and we conclude that

$$a_2 = 3^8. \qquad \text{Q.E.D.}$$

The next, and final lemma will be used to compute an upper bound for the numbers $\mu(SU(4)_{(2)})$ and $\mu((SU(3)\times S^7)_{(2)})$.

LEMMA (3.9)

Let n_i, $i = 1,2,\ldots,k$ be a sequence of positive integers greater than 1 and X be any space, then

$$\#[S^{n_1} \cup_{\alpha_{n_1}} e^{n_2} \cup_{\alpha_{n_2}} e^{n_3} \cup_{\alpha_{n_3}} \cdots \cup_{\alpha_{n_{k-1}}} e^{n_k}, X] \leq \prod_{i=1}^{k} \#(\pi_{n_i}(X)).$$

PROOF

For k = 2 the inequality $\#[S^{n_1} \cup_{\alpha_{n_1}} e^{n_2}, X] \leq \#(\pi_{n_1}(X)) \cdot \#(\pi_{n_2}(X))$ follows from the cofibration sequence

$$\pi_{n_2-1}(X) \xleftarrow{\alpha_{n_1}^*} \pi_{n_1}(X) \longleftarrow [S^{n_1} \cup_{\alpha_{n_1}} e^{n_2}, X] \longleftarrow \pi_{n_2}(X) \xleftarrow{(E\alpha_{n_1})^*} \pi_{n_1+1}(X) \longleftarrow \cdots$$

since $\#(\ker\alpha_{n_1}^*) \leq \#(\pi_{n_1}(X))$ and $\#(\operatorname{coker} E\alpha_{n_1}^*) \leq \#(\pi_{n_2}(X))$.

The result for any integer $k \geq 2$ now follows easily by induction.

Q.E.D.

For a given X and set of primes P we will use the symbol $\mu_P(X)$ to denote the number of distinct homotopy classes of multiplications that the space X_P will support.

The main and final result of this chapter is the following theorem.

THEOREM (3.10) Let $\mathbb{P}$ denote the set of all primes, then

(a) $\mu_{\mathbb{P}-\{2\}}(SU(4)) = 3^{46}\cdot 5^{30}\cdot 7^{9}\cdot 11^{5}\cdot 13,$

(b) $\mu_{\mathbb{P}-\{2\}}(SU(3)\times S^{7}) = 3^{105}\cdot 5^{30}\cdot 7^{9}\cdot 11^{5}\cdot 13,$

(c) $\mu_{\mathbb{P}-\{2\}}(Y_2) = 3^{46}\cdot 5^{30}\cdot 7^{9}\cdot 11^{5}\cdot 13,$

(d) $\mu_{\mathbb{P}-\{2\}}(Y_3) = 3^{105}\cdot 5^{30}\cdot 7^{9}\cdot 11^{5}\cdot 13,$

(e) $\mu_{\{2\}}(SU(4)) = \mu_{\{2\}}(Y_3) \leq 2^{210}\cdot C_{27,3}^{2}\cdot C_{30,3}\cdot C_{30,5}\cdot C_{30,7},$ and

(f) $\mu_{\{2\}}(SU(3)\times S^{7}) = \mu_{\{2\}}(Y_2) \leq 2^{235}\cdot C_{27,3}^{2}\cdot C_{30,3}\cdot C_{30,5}\cdot C_{30,7}$

where $C_{i,j} = \#(\pi_i(S^j:2))$.

PROOF (a), (b), (c) and (d) follow immediately from Theorem (1.25) and Propositions (3.1), (3.2) and (3.8).

Since Y_n has cellular structure

$$S^3 \cup e^5 \cup e^7 \cup e^8 \cup e^{10} \cup e^{12} \cup e^{15}$$

it is easy to see that $Y_n \wedge Y_n$ has cells in the numbers and dimensions given by the lattice below.

18	20	22	23	25	27	30
15	17	19	20	22	24	27
13	15	17	18	20	22	25
11	13	15	16	18	20	23
10	12	14	15	17	19	22
8	10	12	13	15	17	20
6	8	10	11	13	15	18

We may use Lemma (3.9) to obtain the following inequality.

(3.10.1)

$$\mu_{\{2\}}(SU(4)) \leq \#(\pi_6(SU(4):2)) \cdot [\#(\pi_8(SU(4):2))]^2 \cdot [\#(\pi_{10}(SU(4):2))]^3 \cdot [\#(\pi_{11}(SU(4):2))]^2 \cdot [\#(\pi_{12}(SU(4):2))]^2 \cdot [\#(\pi_{13}(SU(4):2))]^4 \cdot \#(\pi_{14}(SU(4):2)) \cdot [\#(\pi_{15}(SU(4):2))]^6 \cdot \#(\pi_{16}(SU(4):2)) \cdot [\#(\pi_{17}(SU(4):2))]^4 \cdot [\#(\pi_{18}(SU(4):2))]^4 \cdot [\#(\pi_{19}(SU(4):2))]^2 \cdot [\#(\pi_{20}(SU(4):2))]^5 \cdot [\#(\pi_{22}(SU(4):2))]^4 \cdot [\#(\pi_{23}(SU(4):2))]^2 \cdot \#(\pi_{24}(SU(4):2)) \cdot [\#(\pi_{25}(SU(4):2))]^2 \cdot [\#(\pi_{27}(SU(4):2))]^2 \cdot \#(\pi_{30}(SU(4):2)).$$

The cardinalities of all the groups involved, with the exception of the last four can be found in Mimura-Toda (1964). We may estimate the last four using the exact fibration sequence

$$\cdots \longrightarrow \pi_i(SU(3):2) \longrightarrow \pi_i(SU(4):2) \longrightarrow \pi_i(S^7:2) \longrightarrow \cdots$$

wherein we estimate $\pi_i(SU(3):2)$ for $i = 24$, 25, 27 and 30 using the fibration sequence

$$\cdots \longrightarrow \pi_i(S^3:2) \longrightarrow \pi_i(SU(3):2) \longrightarrow \pi_i(S^5:2) \longrightarrow \cdots \quad .$$

For SU(3) we get

$$\#(\pi_{24}(SU(3):2)) \leq \#(\pi_{24}(S^3:2)) \cdot \#(\pi_{24}(S^5:2)) = 2 \cdot 2^4 = 2^5,$$

$$\#(\pi_{25}(SU(3):2)) \leq \#(\pi_{25}(S^3:2)) \cdot \#(\pi_{25}(S^5:2)) = 2 \cdot 2^3 = 2^4,$$

$$\#(\pi_{27}(SU(3):2)) \leq \#(\pi_{27}(S^3:2)) \cdot \#(\pi_{27}(S^5:2)) = C_{27,3} \cdot 2^3,$$

and

$$\#(\pi_{30}(SU(3):2)) \leq \#(\pi_{30}(S^3:2)) \cdot \#(\pi_{30}(S^5:2)) = C_{30,3} \cdot C_{30,5}$$

where the numerical values of the cardinalities involved are obtained from Toda (1962), Mimura-Toda (1963), and Mimura (1965). [See Appendix I for tabulation of cardinalities.]

Combining the above results for SU(3) with the fact that

$$\#(\pi_{24}(S^7:2)) = 2^4,$$

$$\#(\pi_{25}(S^7:2)) = 2^4,$$

and

$$\#(\pi_{27}(S^7:2)) = 2^3$$

we have

$$\#(\pi_{24}(SU(4):2)) \leq 2^9,$$

$$\#(\pi_{25}(SU(4):2)) \leq 2^8$$

and

$$\#(\pi_{27}(SU(4):2)) \leq C_{27,3} \cdot 2^6.$$

Inequality (3.10.1) now becomes

$$\mu_{\{2\}}(SU(4)) \leq 2^0 \cdot (2^3)^2 \cdot (2^4)^3 \cdot (2^2)^2 \cdot (2^2)^2 \cdot (2^2)^4 \cdot 2^5 \cdot (2^4)^6 \cdot 2^7 \cdot (2^6)^4 \cdot (2^3)^4 \cdot (2^3)^2 \cdot (2^3)^5 \cdot (2^8)^4 \cdot (2^7)^2 \cdot 2^9 \cdot (2^8)^2 \cdot C_{27,3}^2 \cdot (2^6)^2 \cdot C_{30,3} \cdot C_{30,5} \cdot C_{30,7}$$

$$= 2^{210} \cdot C_{27,3}^2 \cdot C_{30,3} \cdot C_{30,5} \cdot C_{31,7}$$

and we have demonstrated (e). [See Appendix I for a tabulation of cardinalities.]

In a similar fashion one can check that (f) holds.

Q.E.D.

LIST OF REFERENCES

Arkowitz, M. and Curjel, C. 1963. On the number of multiplications of an H-space. Topology. 2: 205-8.

Copeland, A.H., Jr. 1959. Binary operations on sets of mapping classes. Mich. Math. J. 6: 7-23.

__________________, 1972. H-structures on products. Proc. Am. Math. Soc. 32: 611-14.

Curtis, M. and Mislin, G. 1970. Two new H-spaces. Bull. Am. Math. Soc. 76: 851-2.

Douglas, R., Hilton, P., and Sigirst, F. 1969. H-spaces. Category Theory, Homology Theory and Their Applications II. Lec. Notes in Math. v.92. New York and Berlin: Springer-Verlag.

Harrison, J. und Scheerer, H. 1972. Zur homotopietheorie von abbildungen in homotopie-assoziative H-räume. Archiv. der Math. 23: 319-23.

Hilton, P. 1973. Localization and cohomology of nilpotent groups. Math. Zeit. 132: 263-86.

Hilton, P., Mislin, G., and Roitberg, J. 1973. Homotopical localization. Proc. Lon. Math. Soc. 3rd. series 26: 693-706.

Hilton, P. and Roitberg, J. 1969. On principal S^3-bundles over spheres. Ann. of Math. 90: 91-107.

__________________________, 1970. On the classification problem for H-spaces of rank two. Comment. Math. Helv. 45:506-16.

James, I. and Whitehead, J.H.C. 1954. Homotopy theory of sphere bundles over spheres. Proc. Lon. Math. Soc. 4:196-218.

Lazard, M. 1954. Sur les groupes nilpotent et les anneaux de Lie. Ann. Sci. École Norm. Sup. 71: 101-190.

Mimura, M. 1965. On the generalized Hopf homomorphism and the higher composition Part II $\pi_{n+i}(S^n)$ for i = 21 and 22. J. Math. Kyoto Univ. 4: 301-26.

_________, 1967. The homotopy groups of Lie groups of low rank. J. Math. Kyoto Univ. 6: 131-76.

_________, 1969. On the number of multiplications of SU(3) and Sp(2). Trans. Am. Math. Soc. 146: 473-92.

Mimura, M., Nishida, G., and Toda, H. 1971. Localization of CW-complexes and its applications. J. Math. Soc. Japan. 23: 593-624.

Mimura, M. and Toda, H. 1963. The (n+20th) homotopy groups of n-spheres. J. Math. Kyoto Univ. 3: 37-58.

_____________________, 1964. Homotopy groups of SU(3), SU(4) and Sp(2). J. Math Kyoto Univ. 3: 217-50.

_____________________, 1964a. Homotopy groups of symplectic groups. J. Math. Kyoto Univ. 3: 251-73.

Stasheff, J. 1969. Manifolds of the homotopy type of (non-Lie) groups. Bull. Am. Math. Soc. 146: 998-1000.

Sullivan, D. 1971. Geometric Topology. rev. ed. Part I. Cambridge, Mass.: M.I.T. Press.

Toda, H. 1962. Composition Methods in Homotopy Groups of Spheres. Annals of Math Studies. v. 49. Princeton, N.J.: Princeton University Press.

_______, 1965. On iterated suspensions I. J. Math. Kyoto Univ. 5: 87-142.

_______, 1966. On iterated suspensions II. J. Math. Kyoto Univ. 5: 209-50.

Whitehead, G.W. 1954. On mappings into group-like spaces. _Comment. Math. Helv._ 28: 320-28.

Zabrodsky, A. 1971. On sphere extensions of classical Lie groups. _Proceeding of Symposia in Pure Math._ v. 22. Providence, R.I.: American Mathematics Society.

APPENDIX I

i	$\#(\pi_i(S^3))$	$\#(\pi_i(S^5))$	$\#(\pi_i(S^7))$	$\#(\pi_i(SU(3)))$	$\#(\pi_i(SU(4)))$
6	$2^2 \cdot 3^1$	2^1	1	$2^1 \cdot 3^1$	1
7	2^1	2^1	∞	1	∞
8	2^1	$2^3 \cdot 3^1$	2^1	$2^2 \cdot 3^1$	$2^3 \cdot 3^1$
9	3^1	2^1	2^1	3^1	2^1
10	$3^1 \cdot 5^1$	2^1	$2^3 \cdot 3^1$	$2^1 \cdot 3^1 \cdot 5^1$	$2^4 \cdot 3^1 \cdot 5^1$
11	2^1	2^1	1	2^2	2^2
12	2^2	$2^1 \cdot 3^1 \cdot 5^1$	1	$2^2 \cdot 3^1 \cdot 5^1$	$2^2 \cdot 3^1 \cdot 5^1$
13	$2^3 \cdot 3^1$	2^1	2^1	$2^1 \cdot 3^1$	2^2
14	$2^4 \cdot 3^1 \cdot 7^1$	2^3	$2^3 \cdot 3^1 \cdot 5^1$	$2^3 \cdot 3^1 \cdot 7^1$	$2^5 \cdot 3^1 \cdot 5^1 \cdot 7^1$
15	2^2	$2^4 \cdot 3^2$	2^3	$2^2 \cdot 3^2$	$2^4 \cdot 3^2$
16	$2^1 \cdot 3^1$	$2^5 \cdot 3^2 \cdot 7^1$	2^4	$2^3 \cdot 3^3 \cdot 7^1$	$2^7 \cdot 3^2 \cdot 7^1$
17	$2^1 \cdot 3^1 \cdot 5^1$	2^3	$2^4 \cdot 3^1$	$2^2 \cdot 3^1 \cdot 5^1$	$2^6 \cdot 5^1$
18	$2^1 \cdot 3^1 \cdot 5^1$	$2^2 \cdot 3^1$	$2^4 \cdot 3^2 \cdot 7^1$	$2^2 \cdot 3^2 \cdot 5^1$	$2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$
19	$2^2 \cdot 3^1$	$2^2 \cdot 3^1$	1	$2^3 \cdot 3^2$	$2^3 \cdot 3^1$
20	$2^4 \cdot 3^1$	$2^2 \cdot 3^1 \cdot 5^1$	$2^1 \cdot 3^1$	$2^3 \cdot 3^2 \cdot 5^1$	$2^3 \cdot 3^1 \cdot 5^1$

APPENDIX I--Continued

i	$\#(\pi_i(S^3))$	$\#(\pi_i(S^5))$	$\#(\pi_i(S^7))$	$\#(\pi_i(SU(3)))$	$\#(\pi_i(SU(4)))$
21	$2^4 \cdot 3^1$	2^2	$2^5 \cdot 3^1$	$2^1 \cdot 3^1$	2^5
22	$2^3 \cdot 3^1 \cdot 11^1$	2^4	$2^6 \cdot 3^1 \cdot 5^1$	$2^2 \cdot 3^1 \cdot 11^1$	$2^8 \cdot 3^1 \cdot 5^1 \cdot 11^1$
23	2^2	$2^5 \cdot 3^1$	2^4	$2^3 \cdot 3^1$	$2^7 \cdot 3^1$
24	2^1	$2^4 \cdot 3^1 \cdot 11^1$	2^4		
25	$2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$	$2^3 \cdot 3^1$	$2^4 \cdot 3^1$		
26	$C_{26,3} \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1$	$2^2 \cdot 3^1$	$2^4 \cdot 3^1 \cdot 11^1$		
27	$C_{27,3} \cdot 3^1 \cdot 5^1$	$2^3 \cdot 3^2 \cdot 5^1$	$2^3 \cdot 3^1$		
28	$C_{28,3}$	$C_{28,5} \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 13^1$	$2^2 \cdot 3^1$		
29	$C_{29,3} \cdot 3^2$	$C_{29,5} \cdot 3^1 \cdot 5^1$	$2^6 \cdot 3^2$		
30	$C_{30,3} \cdot 3^2$	$C_{30,5}$	$C_{30,7} \cdot 3^3 \cdot 5^1 \cdot 7^1 \cdot 13^1$		