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ON REDEI SUBGROUPS CONTAINING THEIR CENTRALIZERS

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by

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ABSTRACT

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by

ROBERT E. MCDONALD

Three classes of finite p-groups for $p > 2$ are considered in this thesis, each class being defined by conditions which are placed on the minimal nonabelian subgroups of a group. The nonabelian subgroups for which each proper subgroup is abelian are called Redei groups.

Class $\mathcal{Z}$ is the collection of nonabelian p-groups for which the centralizer of a Redei subgroup is its center. Class $\mathcal{R}$ is the collection of nonabelian p-groups for which each Redei subgroup is the centralizer of its center. A third class $\mathcal{Z}^*$ is the collection of all nonabelian p-groups for which all subgroups for some given index are Redei. The three classes are shown to satisfy the following inclusion relationships. $\mathcal{R} \subset \mathcal{Z}^* \subset \mathcal{Z}$. All three classes are subgroup inherited.

The two properties on Redei subgroups which define class $\mathcal{Z}$ and class $\mathcal{R}$, respectively, are shown to extend to all nonabelian subgroups. For $G \in \mathcal{Z}$, $Z(N) = C_G(N)$ for each nonabelian subgroup N of G iff $Z(R) = C_G(R)$ for each Redei subgroup R of G .

For $G \in \mathcal{R}$, $N = C_G(Z(N))$ for each nonabelian subgroup N of G iff $R = C_G(Z(R))$ for each Redei subgroup R of G . A result of this second equivalence is that for $G \in \mathcal{R}$, $(Z(N): Z(M)) = (M: N)$ for each pair of nonabelian subgroups N , M such that $N \subsetneq M \subsetneq G$.

It is shown that a group in class $\mathcal{R}$ is either a metacyclic group or a nonmetacyclic Redei group. This leads to a characterization of non-Redei metacyclic p-groups as groups for which each Redei subgroup is the centralizer of its center.

It is found that class $\mathcal{J}^*$ is "almost equal" to class $\mathcal{R}$ in that, for $p > 3$, there is only one type of group in $\mathcal{J}^*$ that is not in $\mathcal{R}$. These groups have order p^5 . For $p = 3$, there is also only one type of group in $\mathcal{J}^*$ that is not in $\mathcal{R}$. Each of these groups has its Redei subgroups of order p^4 . As a consequence, a second characterization is obtained for non-Redei metacyclic p-groups as groups for which there is an abelian subgroup of order p^4 and for which each subgroup of index p^i is Redei if there is one Redei subgroup of that index.

The minimal non-Redei subgroups of class $\mathcal{J}$ are identified. These are precisely the nonabelian groups for which there is at most one abelian maximal subgroup and for which each subgroup of index p^2 is abelian.

INTRODUCTION

Several means by which a group's structure is controlled by its minimal nonabelian subgroups are investigated for finite nonabelian p-groups for $p > 2$. The nonabelian p-groups for which each proper subgroup is abelian have been examined by Miller-Moreno [19] and have been completely described in terms of defining relations by Redei [20]. Such nonabelian p-groups are minimal nonabelian subgroups of a group and are called Redei groups throughout this study.

One common approach to the problem of classifying nonabelian groups is to stress the type of abelian subgroups possessed by the groups. While surveying the literature with regard to this approach it has become apparent that attention continually reverts to the existence and the properties of the Redei subgroups. This raises the question as to whether or not Redei subgroups can assume more than passive control over the structure of a group. In this regard, a new approach has been employed whereby the dominant role in examining group structures is assigned to the Redei subgroups. This can be accomplished by placing conditions on the minimal nonabelian subgroups whereas the previous approach concentrated on choosing specific types of abelian subgroups.

The next few examples about well known results illustrate the effectiveness of Redei groups in this previous approach.

Example 1: Burnside [6] classified the nonabelian groups of order p^4 by examining the group structure through the maximal cyclic subgroups of the group. Huppert [15] derived the same classification, but through the maximal abelian subgroups of the group. In the cases where the group was not itself a Redei group, both authors were required to examine the Redei subgroups of order p^3.

Example 2: The paper of Gaschütz [9] revived interest in the properties of the Frattini subgroup $\Phi(G)$, the intersection of the maximal subgroups of a group G. Hobby [14] proved that $\Phi(G)$ for a nonabelian p-group G cannot have a cyclic center unless $\Phi(G)$ is itself cyclic. This extended Burnside's result which used the commutator subgroup G' rather than $\Phi(G)$ in this statement. In order to prove his result it was necessary for Hobby to show that no Redei group of order p^3 can be the Frattini subgroup of a group.

Example 3: Several significant works dealing with simple groups (e.g., Feit and Thompson [7], Thompson [21], and Gorenstein [10]) continually refer to a theorem of P. Hall [11]. For clarification two definitions are first introduced. For the product $G = ER$ to be a central product, the properties $E \cap R \subseteq Z(G)$ and $E \subseteq C_G(R)$ must be satisfied where $C_G(R)$ is the centralizer in G of R. In addition, a group E is defined as extraspecial if $E' = \Phi(G) = Z(G)$ and the order of E' is p. In his theorem Hall has classified a nonabelian p-group G with no noncyclic characteristic abelian subgroups as a central product ER where $E = 1$ or E is extraspecial and R is cyclic or $p = 2$ and R is isomorphic to the generalized dihedral, the generalized quaternion or the symmetric group of order 2^m for $m \geq 4$. To obtain a complete classification of these groups, Gorenstein [10] has elaborated on the class of extraspecial p-groups. He has proved that an extraspecial p-group G

has order p^{2r+1} and is the central product of $r \geq 1$ Redei subgroups of order p^3 . Indeed, here are two classes of groups, the extraspecial groups and the groups in Hall's theorem, where the structure of the groups is described in terms of Redei subgroups.

Moreover, the proof of Gorenstein's result is also dependent upon Redei subgroups and their hold over the structure of the group. For this reason an outline of the proof is included. Two results are needed.

(1) $P = CC_p(C)$ if C is an extraspecial subgroup of the p-group P such that $[P,C] \subseteq Z(G)$. $[P,C]$ is the subgroup of P generated by the commutators $[p,c] = p^{-1}c^{-1}pc$ for any $p \in P$ and any $c \in C$.

(2) Each extraspecial group G has a Redei subgroup R of order p^3 for which $[G,R] \subseteq [G,G] = G' = Z(G)$.

At this time it should be noted that each Redei group of order p^3 is an extraspecial group. By the application of (1) and (2) to an extraspecial group G , $G = RC_G(R)$ where R is a Redei subgroup of order p^3 . If $C_G(R)$ is abelian, then $C_G(R) \subseteq Z(G) = Z(R)$ so that $G = R$. If, on the other hand, $C_G(R)$ is nonabelian, then $R \subset G$ properly. It can be shown that $C_G(R)$ is also an extraspecial group. So, an inductive argument can now be applied to arrive at the desired result. Although the Redei subgroups of order p^3 control the direction of the proof, it is significant to note that the role of the Redei subgroups is nevertheless passive since their existence follows from the group structure under examination.

Example 4: In line with the approach which stresses the role of abelian subgroups is the problem of whether or not a group possesses a normal abelian subgroup of a given order when this same group has an abelian subgroup of that order. As a partial answer, Alperin [1] showed that if a group has an abelian subgroup of index p^3 then the group has a normal abelian subgroup of index p^3. Konvisser [17] generalized the above to show that if a normal subgroup N of a group G has an abelian subgroup of index p^2, then N has an abelian subgroup of index p^2 that is normal in G.

One aspect of this problem revolves around the structure of a group with an abelian maximal subgroup. This structure is derived from the power imparted to the Redei subgroups by the abelian maximal subgroups. If the group is not itself a Redei group, then there are Redei subgroups properly contained in G. If G has two abelian maximal subgroups, it can be shown that each Redei subgroup is a normal subgroup of G and that G can be written as a central product involving a Redei subgroup. If G has only one abelian maximal subgroup, then G is a product of this abelian maximal subgroup and of any Redei subgroup. This product is not a central product. However, the G-normalizer of the Redei subgroup R has nilpotent class at most three; that is, if N is the normalizer, then the commutators $a^{-1}b^{-1}ab \in Z(N)$ for $a \in N$ and $b \in N'$.

These four examples are some indication of how Redei subgroups influence a group's structure when the group possesses particular abelian subgroups. In contrast, this thesis introduces a new aspect of the Redei subgroups' hold over the group's structure—namely, through the classification of groups according to properties possessed by their Redei subgroups.

Somewhat natural conditions to be examined for the minimal nonabelian subgroups in this approach are suggested by the maximal abelian subgroups which, in concept, are dual to the Redei subgroups. It is well known [15, p. 302] that $M = C_G(M)$ for each maximal abelian subgroup M of a group G. Since each abelian subgroup is its own center, the above equality can be written as either $Z(M) = C_G(M)$ or $M = C_G(Z(M))$. These two forms provide the motivation behind two of the properties which are placed on the Redei subgroups in this work:

(1) $Z(R) = C_G(R)$ for each Redei subgroup R, and

(2) $R = C_G(Z(R))$ for each Redei subgroup R.

Altogether, there are three properties placed on the Redei subgroups, each of which defines a class of finite nonabelian p-groups for $p > 2$. Class $\mathcal{Z}$ and class $\mathcal{R}$ are defined by properties (1) and (2), respectively:

$$\mathcal{Z} = \{G \mid Z(R) = C_G(R) \text{ for each Redei subgroup } R \text{ of } G\} \text{ and}$$
$$\mathcal{R} = \{G \mid R = C_G(Z(R)) \text{ for each Redei subgroup } R \text{ of } G\} .$$

The third class $\mathcal{Z}^*$ is defined by the property:

(3) If R is a Redei subgroup of G of index p^i, then each subgroup of index p^i is a Redei group.

There are five chapters in this thesis. Chapter I contains the prerequisite definitions and fundamental theorems needed throughout this work.

The groups in $\mathcal{R}$ are classified in Chapter II. For this purpose a fourth class $\mathcal{R}^*$ is introduced by means of the property:

(4) $N = C_G(Z(N))$ for each nonabelian subgroup N of G.

This property is a stronger version of property (2).

A group is defined as metacyclic if there is a cyclic normal subgroup whose factor group is also cyclic. One of the main results of Chapter II shows that a nonabelian group $G \in \mathcal{R}$ is either a metacyclic group or a nonmetacyclic Redei group. Another outcome identifies the class of nonabelian metacyclic groups as a subclass of $\mathcal{R}^*$. Consequences of these two results are equality of class $\mathcal{R}$ and $\mathcal{R}^*$ and a new characterization for the class of nonabelian metacyclic groups.

Corresponding to the method of Alperin [1] where he considered large abelian subgroups of a group, the investigation of class $\mathcal{Z}$ in Chapter III considers groups from class $\mathcal{Z}$ with a Redei maximal subgroup. It is first determined that a group from this subclass of $\mathcal{Z}$ is one of two types:

(i) a group for which each subgroup of index p^2 is abelian,or

(ii) a group with no abelian maximal subgroup but with a Redei subgroup which is not a maximal subgroup.

It is shown that the groups in (ii) are affected by the groups in (i). Also, the groups in (i) with an abelian maximal subgroup are completely characterized.

The investigation started in Chapter III is continued in Chapter IV. The groups of type (i) for which each maximal subgroup is Redei are described. These are precisely the groups in $\mathcal{Z}^*$ for which each subgroup of index p^2 is abelian. It is found that class $\mathcal{Z}^*$ is "almost equal" to class $\mathcal{R}$. Furthermore, a second characterization of non-Redei metacyclic p-groups is obtained.

Chapter V contains a summary of the results presented here and identifies unanswered related questions that provide a basis for future investigations with respect to Redei subgroups in p-groups.

NOTATION

Only finite p-groups for primes $p > 2$ are considered.

$\|G\|$	- order of the group G
$a \in G$	- a is an element of G
$< a_1, a_2, \cdots, a_n >$	- the subgroup generated by the elements $a_1, a_2, \cdots, a_n$
$A \subseteq B$	- A is a subset of B
$A \sim B$	- the set of elements from A that are not elements of B
$A \leq G$	- A is a subgroup of G
$A < G$	- A is a proper subgroup of G
AB	- product of A and B; $< ab \mid a \in A$ and $b \in B >$
$(G:A)$	- the index of A in G
$A \triangleleft G$	- A is a normal subgroup of G
$[a,b]$	- the commutator, $a^{-1}b^{-1}ab$, of the elements a and b
$[A,B]$	- $< [a,b] \mid a \in A, b \in B >$
G'	- the commutator subgroup of G, $[G,G]$
G_i	- the i^{th} term of the lower central series where $G_1 = G$ and $G_i = [G, G_{i-1}]$
class of $G = c$	- $G_c \neq 1$ but $G_{c+1} = 1$
$Z(G)$	- the center of G
$\Phi(G)$	- the Frattini subgroup of G; the intersection of all the maximal subgroups of G
$\Omega_1(G)$	- the subgroup of G generated by the elements of order p; $< g \mid g \in G$ and $g^p = 1 >$
$\upsilon_1(G)$	- the subgroup of G generated by the p^{th} powers of elements of G; $< g^p \mid g \in G >$

$C_G(N)$ - the G-centralizer of $N; < g | g \in G \text{ and } [g,n] = 1 >$

$\mathcal{N}_G(N)$ - the G-normalizer of $N; < g | g \in G \text{ and } g^{-1}Ng = N >$

type (m,n) - refers to an abelian group which is the direct product of a cyclic group of order p^m and a cyclic group of order p^n

CHAPTER I

FUNDAMENTAL CONCEPTS

In this chapter the relationships between the classes of groups under investigation are established, and essential background material is provided. It should be emphasized that minimal nonabelian groups are called Redei groups and that only finite p-groups for primes $p > 2$ are considered.

Definition 1.1: Class $\mathcal{Z}$ is the collection of all nonabelian groups G which satisfy the property:

(1.1) $Z(R) = C_G(R)$ for each Redei subgroup R of G .

Definition 1.2: Class $\mathcal{R}$ is the collection of all nonabelian groups G which satisfy the property:

(1.2) $R = C_G(Z(R))$ for each Redei subgroup R of G .

Proposition 1.1: $\mathcal{R} \subseteq \mathcal{Z}$. Moreover, both $\mathcal{R}$ and $\mathcal{Z}$ are subgroup inherited.

Proof: $C_G(M) \leq C_G(N)$ when $N \leq M \leq G$. In particular $C_G(R) \leq C_G(Z(R))$ for each Redei subgroup R of G . If $G \in \mathcal{R}$, then $C_G(Z(R)) = R$. Thus $C_G(R) \leq R$. Hence $G \in \mathcal{Z}$.

Let N be a nonabelian subgroup of $G \in \mathcal{R}$. Let R be a Redei subgroup of N . R is also a Redei subgroup of G , so $C_G(Z(R)) = R$. Thus $C_N(Z(R)) = N \cap C_G(Z(R)) = N \cap R = R$. Hence $N \in \mathcal{R}$. This proves that $\mathcal{R}$ is subgroup inherited.

The proof that $\mathcal{Z}$ is subgroup inherited is similar.

Some basic properties of Redei subgroups are established in the next two results.

<u>Proposition 1.2</u>: If R is a Redei group, then

(a) $Z(R)$ has index p^2 in R, and

(b) $Z(R) = \Phi(R)$.

Proof: Each maximal subgroup of R is abelian. If there is a maximal subgroup A of R such that $Z(R) \not\leq A$, then $G = AZ(R)$. Consequently, G is abelian; a contradiction is reached. Thus $Z(R) \leq \Phi(G)$.

Let M and M^* be any two maximal subgroups of R . $R = MM^*$. This implies that $M \cap M^*$ has index p^2 in R and that $M \cap M^* \leq Z(R)$. Hence $Z(R) = \Phi(R) = M \cap M^*$.

THEOREM 1.1: If G is a nonabelian group which is not Redei, then G has at least two Redei subgroups.

Proof: Let R be a Redei subgroup of G . Let $M \leq G$ such that $(M:R) = p$. If M has at least two Redei subgroups, then G has at least two Redei subgroups. Therefore, without loss of generality, it can be assumed that R is a maximal subgroup of the group G .

Suppose, further, that R is the only Redei subgroup of G . Since G is not a cyclic group, it has at least two maximal subgroups. Also, the number of subgroups of a given order is congruent to 1 , modulo p [15, p. 314]. From the fact that $p > 2$, it follows that the number of maximal subgroups of G is at least three. In particular, G has at least two abelian maximal subgroups A and A^* . $G = AA^*$ so that $A \cap A^* \leq Z(G)$. However, $(G : Z(G)) \geq p^2$ which implies that $Z(G) = A \cap A^*$. Therefore, $G/Z(G)$ has order p^2 . Now $G/Z(G)$ has at least two abelian subgroups $A/Z(G)$ and $A^*/Z(G)$ of order p . Hence, $G/Z(G)$ is elementary abelian, that is, abelian of type (1,1). This implies that $G/Z(G)$ has $p+1$ subgroups of order p . Thus, G has at least $p+1$ abelian maximal subgroups, each corresponding to a subgroup of $G/Z(G)$. But each abelian maximal subgroup contains $Z(G)$ (otherwise G itself would be abelian), so G has exactly $p+1$ abelian maximal subgroups. Thus, G has $p+2$ maximal subgroups in all, which contradicts the fact that the number must be congruent to 1 , mod p . Hence, G must have at least two Redei subgroups.

For each group in class $\mathcal{R}$, there is a special relationship between its Redei subgroups and its maximal abelian subgroups.

THEOREM 1.2: If $G \in \mathcal{R}$, then

(a) each maximal abelian subgroup of G is contained in a Redei subgroup of G , and

(b) each maximal subgroup of a Redei group is a maximal abelian subgroup of G .

Proof: (a) Let A be a maximal abelian subgroup of G. Let $M \leq G$ such that $(M:A) = p$. Since A is a maximal abelian subgroup of G, M is nonabelian. M contains a Redei subgroup R. $M = AR$ so that $p = (M:A) = (R: A \cap R)$. By Proposition 1.1, $Z(R) \leq A \cap R \leq A$. This implies that $A \leq C_G(Z(R)) = R$. Thus $M = R$. Therefore, each maximal abelian subgroup of G is contained in a Redei subgroup.

(b) Let R be a Redei subgroup of G and let M be a maximal subgroup of R. Let A be a maximal abelian subgroup of G which contains M. By Proposition 1.1, $Z(R) < M \leq A$. Thus $A \leq C_G(Z(R)) = R$. So, $M = A$. Hence, each maximal subgroup of a Redei subgroup of G is a maximal abelian subgroup of G.

Theorem 1.2 can be improved to the following result.

THEOREM 1.3: If $G \in \mathcal{R}$, then there exists an integer $i \geq 0$ such that each subgroup of G of index p^i is a Redei group.

Proof: The theorem is trivially satisfied for Redei groups, in which case $i = 0$. Therefore, let $G \in \mathcal{R}$ be a non-Redei group. By Theorem 1.2(a), G has no abelian maximal subgroup. Inductively, assume that each proper nonabelian subgroup of G satisfies the theorem. This is possible since $\mathcal{R}$ is subgroup inherited.

Let R be a Redei subgroup of G. Let N be a subgroup of G such that $(G:N) = (G:R)$. If there is a maximal subgroup M of G which contains both N and R, then by the inductive hypothesis N is a Redei subgroup. Assume, therefore, that there is no maximal subgroup containing both R and N . Let M and M^* be maximal subgroups of G such that $R \leq M$, $N \leq M^*$, $R \not\leq M^* \cap M$ and $N \not\leq M^* \cap M$. Let $N^* \leq M^* \cap M$ such that $(G:N^*) = (G:R)$. The inductive hypothesis applied to M yields that N^* is a Redei subgroup of G . Then, from the application of the inductive hypothesis to M^* , it follows that N is a Redei group. This completes the proof.

In the above theorem, once the index of one Redei subgroup is known for a group in class $\mathcal{R}$, it follows that each subgroup with that same index is also Redei. Thus, property (1.2), by which class $\mathcal{R}$ is defined, leads to another condition on Redei subgroups.

Definition 1.3: Class $\mathcal{Z}^*$ is the collection of all nonabelian groups G which satisfy the property:

(1.3) If R is a Redei subgroup of G and $(G:R) = p^i$, then each subgroup of index p^i is a Redei subgroup of G .

Proposition 1.3: $\mathcal{R} \subseteq \mathcal{Z}^* \subseteq \mathcal{Z}$. In addition, class $\mathcal{Z}^*$ is subgroup inherited.

Proof: Theorem 1.3 shows that $\mathcal{R} \subseteq \mathcal{Z}^*$.

Let G be a group in $\mathcal{Z}^*$. Let R be a Redei subgroup of G and let M be a maximal subgroup of R . By property (1.3) M is a maximal abelian subgroup of G , so that $M = C_G(M)$. However, $M < R$ implies that $C_G(R) \leq C_G(M) = M < R$. Hence $G \in \mathcal{Z}$.

$\mathcal{Z}^*$ is clearly subgroup inherited.

<u>Proposition 1.4</u>: Let G be a nonabelian group.

(a) [4] If N is a normal subgroup of G of order p^2, then $C_G(N)$ is a normal subgroup of index at most p.

(b) If N is a normal abelian subgroup of type (2,1), then $C_G(N)$ has index at most p in G.

(c) If $|G| = p^4$, then G has an abelian maximal subgroup.

Proof: (a) Each element x of G generates an automorphism τ_x of N since N is normal in G. The mapping $x \to \tau_x$ is a homomorphism of G into Aut (N), the group of automorphisms of N, with kernel $C_G(N)$. Thus, $C_G(N)$ is normal in G and $G/C_G(N)$ is isomorphic to a subgroup A of Aut (N). Therefore, A is a p-group; the order of A divides the order of Aut (N). Now $|\text{Aut }(N)| = p(p-1)$ if N is cyclic and $|\text{Aut }(N)| = p(p^2-1)(p-1)$ if N is elementary abelian. Hence $|A|$ divides p and $(G\colon C_G(N)) \leq p$.

(b) By a result of Miller [18], Aut (2,1) is isomorphic to Aut (1,1). As in (a), $G/C_G(N)$ is isomorphic to a subgroup A of Aut (N). Thus $|A|$ divides p and $(G\colon C_G(N)) \leq p$.

(c) If G is a Redei group, then G has an abelian maximal subgroup. Suppose, therefore, that G is not a Redei group. If $Z(G)$ has order p^2, then there is a maximal subgroup M containing $Z(G)$ and $(M\colon Z(G)) = p$. It follows that M is abelian. On the other hand, if $|Z(G)| = p$, then there is a normal subgroup N of G of order p^2 with $Z(G) < N$. From part (a), $(G\colon C_G(N)) = p$. But then, $(C_G(N)\colon N) = p$. However, $N \leq Z(C_G(N))$ which implies that $C_G(N)$ is abelian. Hence G always has an abelian maximal subgroup.

Both inclusions in part (a) of Proposition 1.3 are, in fact, proper inclusions. Part (c) of Proposition 1.4 indicates that a group in $\mathcal{J}^* \sim \mathcal{R}$ has order greater than p^4 . An example of such a group is given in Chapter IV. The following example shows that $\mathcal{J}^* \subset \mathcal{J}$.

Example 1.1: If $G = C_3 \wr C_3$, the standard restricted wreath product of the cyclic group of order 3 by the cyclic group of order 3, then [2, p. 30] $|G| = 3^4$ and $|Z(G)| = |C_3| = 3$. By Proposition 1.1, G is not Redei. By Proposition 1.4(c), G has an abelian maximal subgroup. Thus $G \notin \mathcal{J}^*$.

Suppose that R is a Redei subgroup of G for which $C_G(R) \nless R$. $|R| = 3^3$ and $G = RC_G(R)$. Since R is nonabelian, $C_G(R) < G$. Now $R \cap C_G(R) \leq Z(G)$. Also, $3 = (G:R) = (C_G(R): R \cap C_G(R))$. It then follows that $|C_G(R)| = 3^2$, so that $C_G(R)$ is abelian. Hence, $C_G(R) \leq Z(G)$; a contradiction is reached. Therefore, for each Redei subgroup R , $C_G(R) < R$. Consequently, $G \in \mathcal{J}$.

The rest of the chapter contains necessary background material. If the result is known, its proof has been omitted.

THEOREM 1.4: [20] If G is a group for which each proper subgroup is abelian, then $G = \langle a,b \rangle$ has defining relations:

$$(1.4) \quad a^{p^m} = b^{p^n} = 1 , \; b^{-1}ab = a^{1+p^{m-1}} \qquad (m \geq 2, n \geq 1); \quad \text{or}$$

$$(1.5) \qquad a^{p^m} = b^{p^n} = c^p = 1 , \; b^{-1}ab = ac .$$

If G satisfies (1.4), then $|G| = p^{m+n}$. If G satisfies (1.5), then $|G| = p^{m+n+1}$.

Proposition 1.5: If $[a,b]$ commutes with a , then $[a^n,b] = [a,b]^n$ for every integer n .

Definition 1.4: A group is called regular if

$$(xy)^p = x^p y^p \prod_i d_i^p \quad \text{for each pair } x, y \in G \text{ where } d_i \in \langle x,y \rangle' .$$

THEOREM 1.5: If G is a Redei group, then
(a) G' has order p ,
(b) G is a regular group.

Proof: (a) Let $x, y \in G$. $\Phi(G) = G'\upsilon_1(G)$ [15, p. 272] . By Proposition 1.1, $G' \leq Z(G)$. By Proposition 1.5, $[x,y]^p = [x^p,y]$. However, $\upsilon_1(G) \leq Z(G)$ so that $[x^p,y] = 1$. Hence $|G'| = p$.

(b) Because G' is cyclic, it can be concluded that G is regular [15, p. 322].

THEOREM 1.6: [13] If G is a regular group, then
(a) $|G/\Omega_1(G)| = |\upsilon_1(G)|$,
(b) each element of $\upsilon_1(G)$ is the p^{th} power of an element of G and each element of $\Omega_1(G)$ has order p .

THEOREM 1.7: [13] (a) If the class of G is less than p , then G is a regular group.

(b) If G is a two-generator 3-group in which G' is not cyclic, then G is not a regular group.

THEOREM 1.8: Let G be a Redei group; then
(a) $\upsilon_1(G) = \Phi(G)$ when G has defining relations (1.4),
(b) $(\Phi(G): \upsilon_1(G)) = p$ and $G' \cap \upsilon_1(G) = 1$ when G has defining relations (1.5).

<u>Definition 1.5</u>: A group G is called metacyclic if and only if there exists a normal subgroup N such that both N and G/N are cyclic.

THEOREM 1.9: [16] G is a metacyclic group if and only if $|G/\upsilon_1(G)| \leq p^2$. Moreover, each metacyclic group is a regular group.

By reason of Proposition 1.2 and Theorem 1.8, a Redei group with defining relations (1.4) satisfies Theorem 1.9, so it is a metacyclic group.

THEOREM 1.10: [3] Let G be a group for which all proper subgroups are metacyclic but G itself is not. Then, G is one of the following.

(a) G is elementary abelian of order p^3 .
(b) G is the nonabelian group of order p^3 and exponent p .
(c) G is a 3-group of class 3 and order 3^4 .

Proposition 1.6: [5] If G is a nonabelian group with two generators, then $\Phi(G')G_3$ is the only maximal subgroup of G' which is normal in G .

Proposition 1.7: [5] If G is a nonmetacyclic group with two generators, then $G/\Phi(G')G_3$ has defining relations $[a,b] = c$, $a^{p^m} = b^{p^n} = c^p = 1$, $[a,c] = [b,c] = 1$ in terms of two generators a,b where G/G' is abelian of type (m,n) . This factor group is the Redei group with defining relations (1.5).

CHAPTER II

REDEI SUBGROUPS AS CENTRALIZERS OF THEIR CENTERS

The relation $C_G(N) \leq C_G(R)$ for $R \leq N \leq G$ implies that property (1.1) which defines class $\mathcal{Z}$ can be extended to all nonabelian subgroups of G; that is,

$$Z(N) = C_G(N) \text{ for each nonabelian subgroup } N \text{ of } G \text{ if and only if}$$

$$Z(R) = C_G(R) \text{ for each Redei subgroup } R \text{ of } G .$$

The question is raised as to whether or not property (1.2) which defines class $\mathcal{R}$ can also be extended to all nonabelian groups. An affirmative answer is given in this chapter. Moreover, the structure of the groups in class $\mathcal{R}$ is completely determined.

<u>Definition 2.1</u>: Class $\mathcal{R}^*$ is the collection of all nonabelian groups G which satisfy the property:

$$(2.1) \quad N = C_G(Z(N)) \quad \text{for each nonabelian subgroup } N \text{ of } G .$$

Property (2.1) is a stronger version of property (1.2). Thus $\mathcal{R}^* \subseteq \mathcal{R}$. Also, each subgroup of a group in $\mathcal{R}^*$ is a group in $\mathcal{R}^*$.

<u>Proposition 2.1</u>: If $G \varepsilon \mathcal{R}^*$, then

(a) $Z(N) < Z(M)$ if $M < N \leq G$ and M is nonabelian,

(b) $Z(M) \neq Z(N)$ if $(G:M) = (G:N)$ and $M \neq N$.

Proof: (1) Since $Z(M) < M < N$, then $Z(N) \leq C_G(Z(M)) = M$. Thus $Z(N) \leq Z(M)$. If $Z(N) = Z(M)$, then $M = C_G(Z(M)) = C_G(Z(N)) = M$; a contradiction is reached. Hence $Z(N) < Z(M)$.

(2) $\mathcal{R}^* \subseteq \mathcal{R}$. By Theorem 1.3, M and N are both abelian or both nonabelian. If both are abelian, then $Z(M) = M \neq N = Z(N)$. On the other hand, suppose that both are not abelian. If $Z(M) = Z(N)$, then $M = C_G(Z(M)) = C_G(Z(N)) = N$, which contradicts $M \neq N$. Thus, $Z(M) \neq Z(N)$ when neither M nor N is abelian.

Closer examination of part (a) of Proposition 2.1 yields the following stronger result.

THEOREM 2.1: $G \varepsilon \mathcal{R}^*$ if and only if $(Z(M): Z(N)) = (N:M)$ whenever M is nonabelian and $M \leq N \leq G$.

Proof: Assume that $(Z(M): Z(N)) = (N:M)$ whenever M is a nonabelian subgroup of G and $M \leq N \leq G$. Let $B < G$. $B \leq C_G(Z(B))$. By hypothesis, $Z(C_G(Z(B))) \leq Z(B)$. However, $Z(B) \leq B \leq C_G(Z(B))$ so that $Z(B) \leq Z(C_G(Z(B)))$. Thus, $Z(B) = Z(C_G(Z(B)))$. According to the hypothesis, it follows that $B = C_G(Z(B))$. Hence, $G \varepsilon \mathcal{R}^*$.

The proof for the other direction will proceed by induction on the index of the Redei subgroups in G. This is possible by Theorem 1.3.

Let $G \in \mathcal{R}^*$. If G is a Redei group, then the condition is trivially satisfied. Therefore, let G be a non-Redei group and let $(G:R) = p^i$, $i > 0$, for each Redei subgroup R of G . By induction, assume that any group in $\mathcal{R}^*$, for which the index of the Redei subgroups is less than p^i , satisfies the theorem in the direction under consideration.

Let M be a nonabelian subgroup of G and let $M \leq N \leq G$. If $N < G$, then $(N:R) < p^i$ where R is a Redei subgroup of N . $N \in \mathcal{R}^*$; so, by the induction hypothesis, $(Z(M): Z(N)) = (N:M)$. It remains to show that $(Z(M): Z(G)) = (G:M)$. Let B be a maximal subgroup of G such that $M \leq B$. $B \in \mathcal{R}^*$. $(B:R) < p^i$ for each Redei subgroup R of B . Then, $(Z(M): Z(B)) = (B:M)$ by the inductive hypothesis. By Proposition 2.1(a), $Z(G) < Z(B)$. It must next be shown that $(Z(B): Z(G)) = p$; then,

$$(Z(M): Z(G)) = (Z(M): Z(B))(Z(B): Z(G)) = (B:M)(G:B) = (G:M) ,$$

which will complete the proof.

Let B be a maximal subgroup of G . Since G is not cyclic, G has another maximal subgroup B^* . By Theorem 1.3, both B and B^* are nonabelian. By Proposition 2.1, $Z(B) \neq Z(B^*)$, $Z(G) < Z(B)$ and $Z(G) < Z(B^*)$. Thus,

$$Z(G) \leq Z(B) \cap Z(B^*) \ . \tag{2.2}$$

Two cases arise. In the first case, B and B^* are themselves Redei groups. Then, by Proposition 1.1, $Z(B) = \Phi(B)$ and $Z(B^*) = \Phi(B^*)$, each having index p^2 in B and B^* , respectively. Thus, $(G: Z(B)) =$

$(G: Z(B^*)) = p^3$. Now $\Phi(B) \leq \Phi(G) \leq B \cap B^*$, and $\Phi(B^*) \leq \Phi(G) \leq B \cap B^*$ [15, p. 273]. Since $Z(B) \neq Z(B^*)$, it follows that $\Phi(B) < \Phi(G)$ and that $\Phi(B^*) < \Phi(G)$. These, in turn, imply that $\Phi(G) = B \cap B^*$. Then, $\Phi(G)$ has index p^2 in G; so, $\Phi(G)$ is a maximal abelian subgroup of G . Also,

$$Z(B^*) < Z(B)Z(B^*) = \Phi(B)\Phi(B^*) \leq \Phi(G) .$$

Consequently, $Z(B)Z(B^*) = \Phi(G)$. Thus,

$$p = (\Phi(G): Z(B^*)) = (Z(B): Z(B) \cap Z(B^*)) .$$

However, $G = BB^*$. Thus, $Z(B) \cap Z(B^*) \leq Z(G)$. By (2.2), $Z(G) = Z(B) \cap Z(B^*)$. Hence, $(Z(B): Z(G)) = p$.

In the second case, B and B^* are not Redei groups. Then, $B \cap B^*$, which has index p^2 in G , is not abelian. By the inductive hypothesis, $(Z(B \cap B^*): Z(B^*)) = (B: B \cap B^*) = p$. Since $Z(B) \neq Z(B^*)$, then $Z(B \cap B^*) = Z(B)Z(B^*)$. Therefore, $(Z(B): Z(B) \cap Z(B^*)) = p$. But $G = BB^*$ which implies that $Z(B) \cap Z(B^*) \leq Z(G)$. By (2.2) $Z(G) = Z(B) \cap Z(B^*)$. Hence $(Z(B): Z(G)) = p$.

The proof is now complete.

Each Redei group is a member of both $\mathcal{R}^*$ and $\mathcal{R}$. If $G \in \mathcal{R}$ and if G has a maximal subgroup that is a Redei subgroup, then by Theorem 1.3, each maximal subgroup is a Redei subgroup. Consequently, $G \in \mathcal{R}^*$. Thus, the restrictions of class $\mathcal{R}$ and class $\mathcal{R}^*$ to their minimal non-Redei groups coincide. The next theorem describes these groups and provides the initial step toward the classification of groups in $\mathcal{R}$.

THEOREM 2.2: If $G \in \mathcal{R}$ and if G is minimal with the property that it is not a Redei group, then G is metacyclic with order greater than p^5 .

Proof: Let R be a Redei subgroup of G . If $(G:R) > p$, then there exists a maximal subgroup M of G containing R . However, $M \in \mathcal{R}$ and M is not Redei, which contradicts the fact that G is a minimal non-Redei group. Thus, each Redei subgroup of G is a maximal subgroup. By the remarks preceding the theorem each maximal subgroup of G is a Redei subgroup, and $G \in \mathcal{R}^*$. By Proposition 1.2, $(G: Z(R)) = p^3$ and $Z(R) = \Phi(R)$ for each Redei subgroup R of G . By Theorem 2.1,

$$(G: Z(G)) = p^4 \tag{2.3}$$

Also, from Proposition 2.1, distinct Redei subgroups have distinct Frattini subgroups. It follows that

$$(G: \Phi(G)) = p^2 , \tag{2.4}$$

so that $\Phi(G)$ is a maximal abelian subgroup of G .

Proposition 1.4(c) implies that $|G| > p^4$. If $|G| = p^5$, then, from (2.3) and (2.4), $|Z(G)| = p$, $Z(R) = p^2$ for each Redei subgroup R of G , and $|\Phi(G)| = p^3$. Since $\Phi(G)$ is maximal abelian in G , $\Phi(G) = C_G(\Phi(G))$. $\Phi(G)$ contains the distinct centers of the maximal subgroups so that $\Phi(G)$ is not cyclic. If $\Phi(G)$ is abelian of type $(2,1)$, then by Proposition 1.4(b), $(G: C_G(\Phi(G))) = p$. Thus, $\Phi(G) < C_G(\Phi(G))$ which is a contradiction to the fact that $\Phi(G)$ is a maximal abelian subgroup of G . Hence, $\Phi(G)$ is elementary abelian; so, $\Phi(G) \leq \Omega_1(G)$.

Now $\Phi(G)$ is a maximal subgroup of each Redei subgroup R of G. Thus $|\Omega_1(R)| \geq p^3$. From the fact that R is regular (Theorem 1.5) and from Theorem 1.6, it follows that $|\upsilon_1(R)| \leq p$. However, as a result of Theorem 1.8 $|\upsilon_1(R)| \geq p$. Hence, $|\upsilon_1(R)| = p$. Moreover, by Theorem 1.5, $|R'| = p$. Both $\upsilon_1(R)$ and R', as characteristic subgroups of R, are normal in G. This implies that both $\upsilon_1(R)$ and R' intersect $Z(G)$ nontrivially. Therefore, $Z(G) = \upsilon_1(R) = R'$. As a consequence of $\Phi(G) = G'\upsilon_1(G)$, $|\Phi(G)| = p$; this is a contradiction to $|\Phi(G)| = p^3$. Hence, $|G| > p^5$.

The proof that G is metacyclic will be by contradiction. Therefore, suppose that G is not metacyclic. By Theorem 1.10 there exists a subgroup of G that is not metacyclic. Because the metacyclic property is subgroup inherited, there is then a maximal subgroup R of G that is not metacyclic. By Theorem 1.9, $(R: \upsilon_1(R)) > p^2$. Since R is Redei, then from Proposition 1.2, $(R: \Phi(R)) = p^2$. From Theorem 1.9, $|R'| = p$. These last two relations imply that $R' \cap \upsilon_1(R) = 1$, so $(R: \upsilon_1(R)) = p^3$. Moreover, R' is characteristic in R which implies that $R' \leq Z(G)$. If $\upsilon_1(R) \leq Z(G)$, then $Z(R) = \Phi(R) \leq Z(G)$; this is a contradiction to Proposition 2.1. Hence,

$$\upsilon_1(R) \not\leq Z(G) . \tag{2.5}$$

Suppose that $\upsilon_1(G) = \upsilon_1(R)$. From Theorem 1.8, it follows that $(M: \upsilon_1(M)) \leq p^3$ for each Redei subgroup M of G. Thus $\upsilon_1(G) = \upsilon_1(M)$ for each maximal subgroup M of G. This implies that $\upsilon_1(G) \leq Z(M)$ for each maximal subgroup M of G. By Theorem 2.1, $(Z(M): Z(G)) = p$ for each maximal subgroup of G. Then, $Z(G)$ is the intersection of the centers of the maximal subgroups; so $\upsilon_1(G) \leq Z(G)$. However, this contradicts (2.5). So,

(2.6) $$\upsilon_1(R) < \upsilon_1(G) .$$

Now from Theorem 1.9 $(G: \upsilon_1(G)) > p^2$. However, $(G: \upsilon_1(R)) = p^4$. Consequently,

(2.7) $$(G: \upsilon_1(G)) = p^3 .$$

Suppose that $Z(G) < \upsilon_1(G)$. M' is a characteristic subgroup of M for each maximal subgroup M of G which implies that $M' \leq Z(G)$ for each maximal subgroup M of G . Then, $Z(M) = \Phi(M) = M'\upsilon_1(M) \leq \upsilon_1(G)$ for each maximal subgroup M . But $(M: Z(M)) = p^3$ so that $Z(M) = \upsilon_1(G)$. This is a contradiction to Proposition 2.1(b). Hence, $Z(G) \not< \upsilon_1(G)$. Consequently, $\upsilon_1(G)$ is not the center of any maximal subgroup. Furthermore,

(2.8) $$M' \not< \upsilon_1(G) \text{ for each maximal subgroup } M \text{ of } G .$$

Assume that there are two maximal subgroups R_1 and R_2 such that $R_1' \neq R_2'$. Then $|R_1'R_2'| = p^2$. Denote $R_1'R_2'$ by A . If G/A is abelian, then $G' < A$. In particular, $M' < A$ for each maximal subgroup M of G . Suppose that G/A is not abelian. G/A has two abelian maximal subgroups R_1/A and R_2/A . Thus, $Z(G/A) = R_1/A \cap R_2/A$ and $Z(G/A)$ has index p^2 in G/A . Also, $\Phi(G/A) \leq Z(G/A)$. However, $(G/A: \Phi(G/A)) = p^2$ since

$$(G/A: \Phi(G/A)) = (G/A: \Phi(G)/A) = (G: \Phi(G)) .$$

Hence, $Z(G/A) = \Phi(G/A)$. It then follows that each maximal subgroup of G/A is abelian. Therefore $M' \leq A$ for each maximal subgroup M of G .

Suppose that the commutators are distinct for distinct maximal subgroups. Since $(G: \Phi(G)) = p^2$, then G has $p + 1$ maximal subgroups. However, A is elementary abelian of order p^2 , so that A has $p + 1$ subgroups of order p . Thus each subgroup of A of order p is the commutator subgroup of a maximal subgroup. From (2.8) it follows that $A \cap \upsilon_1(G) = 1$. However, (2.4) and (2.7) imply that $A \cap \upsilon_1(G) \neq 1$; a contradiction is reached. Thus G has at least two maximal subgroups M_1 and M_2 for which $M_1' = M_2'$. Denote this subgroup M_1' by A^* . By assumption, $R_1' \neq R_2'$. Thus G/A^* is not abelian. However, G/A^* has two abelian maximal subgroups M_1/A^* and M_2/A^* . Therefore $Z(G/A^*)$ has index p^2 in G/A^* and $Z(G/A^*) = \Phi(G/A^*)$. Consequently each maximal subgroup of G/A^* is abelian. In particular, $R_1' = R_2' = A$; a contradiction is reached. Hence, the commutator subgroups of the maximal subgroups are equal. Denote this subgroup of G of order p by R^* .

If $G' = R^*$, then $G' \leq Z(G)$. By Proposition 1.5, $[x^p,y] = [x,y]^p$. But $[x,y]^p = 1$. It follows that $\upsilon_1(G) \leq Z(G)$. Then $\Phi(G) \leq Z(G)$ which contradicts (2.3) and (2.4). Thus $|G'| \geq p^2$. Now each maximal subgroup of G/R^* is abelian, but G/R^* is not abelian; so, G/R^* is a Redei group. Hence, $|G'| = p^2$. But then, from (2.2) and (2.7), $|G' \cap \upsilon_1(G)| = p$.

Suppose $G' \not\leq Z(G)$. $R^* \leq Z(G)$, which implies that $G' \cap Z(G) = R^*$. In addition, $G' \cap \upsilon_1(G) \lhd G$. Thus $G' \cap \upsilon_1(G) \leq Z(G)$. Therefore, $G' \cap \upsilon_1(G) = G' \cap Z(G) = R^*$. Thus $M' < \upsilon_1(G)$ for each maximal subgroup M , which contradicts (2.8). Thus $G' < Z(G)$.

By Proposition 1.5, $[x,y]^p = [x^p,y]$. If G' is elementary abelian, then $[x^p,y] = 1$. Therefore, $\upsilon_1(G) \leq Z(G)$, which contradicts (2.6). G' is thus a cyclic subgroup. Then, $R^* = G' \cap \upsilon_1(G)$ which contradicts (2.8).

Since all possible avenues have been exhausted, it follows that the group G must be a metacyclic group. This completes the proof.

Example 2.1: Given $G = \langle a,b | a^{p^4} = b^{p^2} = 1, [a,b] = a^{p^2} \rangle$. This group is used to illustrate Theorem 2.2 and to show that there are groups in class $\mathcal{R}$ which are not Redei groups.

For this group, $|G| = p^6$; $\upsilon_1(G) = \langle a^p \rangle\langle b^p \rangle$, and $\upsilon_1(G)$ has index p^2 in G . By Theorem 1.9, G is metacyclic. Also, $G' = \langle a^{p^2} \rangle$ and $G' \leq Z(G)$.

Let $M = \langle a \rangle\langle b^p \rangle$. Let $M^* = \langle a^p \rangle\langle b \rangle$. The relations $b^{-p}ab^p = a^{1+p^3}$ and $b^{-1}a^pb = a^{p+p^3}$ imply that $M' = M^{*\prime} = \langle a^{p^3} \rangle$. Let $A = \langle a^{p^3} \rangle$. G/A is metacyclic but is not abelian. M/A and M^*/A are abelian maximal subgroups, so that $Z(G/A) = M/A \cap M^*/A = \Phi(G/A)$. Each maximal subgroup of G/A is abelian; A is the commutator subgroup of each nonabelian maximal subgroup. Because $Z(G)$ is a characteristic subgroup, $Z(G) \leq \Phi(G)$; in particular, $Z(G) \leq Z(M)$ and $Z(G) \leq Z(M^*)$. But $Z(M) = \langle a^p \rangle$ and $Z(M^*) = \langle a^{p^2} \rangle\langle b^p \rangle$ so that $Z(G) \leq \langle a^{p^2} \rangle$. Hence $Z(G) = \langle a^{p^2} \rangle = G'$.

Suppose that G has an abelian maximal subgroup A^* . Since $Z(M) \leq \Phi(G)$, then $Z(M) < A^*$. Thus, $Z(M) \leq Z(G)$, which contradicts the fact that $Z(G) = \langle a^{p^2} \rangle < Z(M)$. Therefore, each maximal subgroup of G is nonabelian.

Since $B' = A \leq Z(G)$ for each maximal subgroup B , then in view of Proposition 1.5 $[x^p,y] = [x,y]^p = 1$. It follows that $\upsilon_1(B) \le Z(B)$, and consequently, $\Phi(B) \le Z(B)$. Since G is metacyclic, then B is metacyclic and $(B: \Phi(B)) = p^2$. Therefore, $Z(B) = \Phi(M)$ which implies that each maximal subgroup of B is abelian. B is thus a Redei subgroup. Moreover, $Z(G) < Z(B)$ which leads to $B = C_G(Z(B))$. Hence, $G \in \mathcal{R}$.

THEOREM 2.3: If $G \in \mathcal{R}$ and G is not a Redei group, then G is metacyclic.

Proof: By Theorem 1.3, an inductive proof on the index of the Redei subgroups can be applied. Let R be a Redei subgroup of G and let $(G:R) = p^k$. Since G is not a Redei group, $k \geq 1$. If $k = 1$, then G is metacyclic by Theorem 2.2. Also, $|G| \ge p^6$. Therefore, let $k > 1$. Let M be a maximal subgroup of G . $M \in \mathcal{R}$. For each Redei subgroup R of M , $(M:R) = p^{k-1}$. By the inductive hypothesis, M is a metacyclic group. Hence, each proper subgroup of G is metacyclic. Because of Theorem 2.2, $|G| \ge p^7$. Therefore, it follows from Theorem 1.10 that G is itself metacyclic.

The above theorem places all non-Redei groups of class $\mathcal{R}$ in the class of metacyclic groups. The next theorem leads to the reverse inclusion.

THEOREM 2.4: If G is a nonabelian metacyclic p-group, then $G \in \mathcal{R}^*$.

Proof: Let G be a counterexample of minimal order. G is not a Redei group and $|G| \geq p^4$. There exists a subgroup M_0 such that $M_0 < C_G(Z(M_0))$.

Suppose that G has a nonabelian subgroup N_0 such that $(G:N_0) > p$. For any nonabelian subgroup N of G such that $(G:N) > p$, there is a maximal subgroup N^* such that $N < N^*$. N^* is metacyclic. By hypothesis, $N^* \in \mathcal{R}^*$. Thus, $N = C_{N^*}(Z(N)) = N^* \cap C_G(Z(N))$. If $C_G(Z(N)) = G$, then $N = N^*$; a contradiction has been reached. Therefore, $C_G(Z(N)) < G$. Denote $C_G(Z(N))$ by B. B is metacyclic and nonabelian. By hypothesis, $B \in \mathcal{R}^*$; so, $N = C_B(Z(N)) = C_G(Z(N)) \cap B = B$. Thus,

(2.9) $N = C_G(Z(N))$ whenever N is nonabelian and $(G:N) > p$.

In particular, $N_0 = C_G(Z(N_0))$. Furthermore $(G:M_0) = p$. Thus, $G = C_G(Z(M_0))$ which implies that

$$Z(M_0) \leq Z(G) . \tag{2.10}$$

By Theorem 1.9, $(G: \Phi(G)) = p^2$. If N_0^* is a maximal subgroup of G containing N_0, then $N_0^* \in \mathcal{R}^*$ and $(N_0^* : \Phi(G)) = p < (N_0^*:N_0)$. By Theorem 1.3, $\Phi(G)$ is nonabelian. By (2.9), $\Phi(G) = C_G(Z(\Phi(G)))$. Thus $Z(G) \leq \Phi(G) < M_0$. From (2.10) it follows that $Z(M_0) = Z(G)$.

Let M be any maximal subgroup of G other than M_0. $Z(M_0) = Z(G) < \Phi(G) < M$ which implies that $Z(M_0) \leq Z(M)$. Now $M \in \mathcal{R}^*$ since $\Phi(G)$ is nonabelian. But then, by Theorem 2.1, $(Z(\Phi(G)): Z(M)) = p$. In the same way, $(Z(\Phi(G)): Z(M_0)) = p$. Thus

$Z(M) = Z(M_0)$. Therefore,

(2.11) $Z(G) = Z(M)$ for each maximal subgroup M of G .

From Theorems 1.9 and 1.6 it follows that $|\Omega_1(G)| = p^2$. If $\Omega_1(G) \not\leq Z(G)$, then $C_G(\Omega_1(G))$ is a maximal subgroup of G by Proposition 1.4(a). By (2.11) $Z(C_G(\Omega_1(G)))= Z(G)$. However, $\Omega_1(G) < Z(C_G(\Omega_1(G)))$. Thus, $\Omega_1(G) < Z(G)$ which is a contradiction. Therefore, $\Omega_1(G) \leq Z(G)$, and accordingly, $Z(G)$ is not cyclic.

Now G' is cyclic since G is metacyclic. There is then a subgroup A of $Z(G)$ such that $|A| = p$ and $A \not\leq G'$. G/A is nonabelian and metacyclic. Since $|G/A| < |G|$, $G/A \in \mathcal{R}^*$. Let $Z^* \leq G$ such that $Z^*/A = Z(G/A)$. $[Z^*,G] \leq A \cap G' = 1$, so that $Z^* \leq Z(G)$. Hence $Z(G) = Z^*$, that is, $Z(G/A) = Z(G)/A$.

Let M/A be a maximal subgroup of G/A . M/A is nonabelian. By Theorem 2.1, $(Z(M/A): Z(G/A)) = p$. **Let** $M^* < M$ such that $M^*/A = Z(M/A)$. Thus $(M^*: Z(G)) = p$. However, $[M^*,M] \leq A \cap G' = 1$; so, $M^* \leq Z(M)$. By (2.11), $M^* \leq Z(G)$; a contradiction is reached. Hence, G has no nonabelian subgroup of index greater than p . Consequently, M_0 is a Redei group, and $Z(M_0) \leq Z(G)$.

Since M_0 is nonabelian, then

(2.12) $$Z(G) < \mho_1(G) = \Phi(G) .$$

Thus, $C_G(\mho_1(G)) < G$. $\mho_1(G)$ is abelian, however; and, this implies both that $(C_G(\mho_1(G)): \mho_1(G)) \leq p$ and that $C_G(\mho_1(G))$ is abelian. Let $y \in C_G(\mho_1(G))$. For each $x \in G$, $[x^p,y] = 1$. Now G is regular so that $[x^p,y] = [x,y^p]$, [12, p. 185]. Thus,

$$\mho_1(C_G(\mho_1(G))) \leq Z(G) \ . \tag{2.13}$$

Suppose that $\mho_1(C_G(\mho_1(G))) \leq Z(M_0)$. Let $g \in G$. $g \in M^*$ for some maximal subgroup M^* of G . If M^* is abelian, then $M^* = C_G(\mho_1(G))$. But then $g^p \in \mho_1(M^*) = \mho_1(C_G(\mho_1(G))) \leq Z(M_0)$. If, on the other hand, M^* is nonabelian, then $g^p \in \mho_1(M^*) = \Phi(M^*) = Z(M^*)$. Now $Z(M_0) = \mho_1(M_0) < \Phi(G) < M^* < C_G(Z(M_0))$ so that $Z(M_0) \leq Z(M^*)$. Since both M_0 and M^* are Redei groups, $|Z(M_0)| = |Z(M^*)|$. Thus $Z(M_0) = Z(M^*)$ so that $g^p \in Z(M_0)$. Therefore, $\mho_1(G) \leq Z(G)$, which contradicts (2.12). Therefore, $\mho_1(C_G(\mho_1(G))) \not\leq Z(M_0)$. Since $(\mho_1(G): \mho_1(M_0)) = p$, it follows that $\mho_1(G) = Z(M_0)\mho_1(C_G(\mho_1(G)))$. Thus, from (2.13) $\mho_1(G) \leq Z(G)$ which contradicts (2.12).

Since no other cases are possible, no such minimal counterexample exists.

A new characterization of nonabelian metacyclic p-groups is obtained as a corollary to the above theorem.

Corollary 2.4.1: If G is nonabelian and is not Redei, then G is metacyclic if and only if $G \in \mathcal{R}$ and G has an abelian subgroup of order p^4 .

Proof: If G is a nonabelian metacyclic group which is not a Redei group, then $G \in \mathcal{R}^* \subseteq \mathcal{R}$ by Theorem 2.4. Since G is not Redei, there exists a Redei subgroup $R < G$. Furthermore, there exists a subgroup R^* of G such that $R < R^* \leq G$ and $(R^*: R) = p$. R^* is a minimal non-Redei group which belongs to $\mathcal{R}$. In view of Theorem 2.3,

$|R^*| \geq p^6$. Moreover, if M is a maximal subgroup of R , then M is abelian. But $|M| \geq p^4$. Hence G has an abelian subgroup of order p^4 .

The converse is Theorem 2.3.

Corollary 2.4.2: $\mathcal{R}^* = \mathcal{R}$.

Proof: Let $G \in \mathcal{R}$. If G is either abelian or a Redei group, then $G \in \mathcal{R}^*$. Therefore, suppose that G is a non-Redei group in $\mathcal{R}$. By Theorem 2.3, G is a metacyclic group. By Theorem 2.4, $G \in \mathcal{R}^*$. Thus $\mathcal{R} \subseteq \mathcal{R}^*$.

If a group $G \in \mathcal{R}$, then Corollary 2.4.2 states that each non-abelian subgroup of G is a centralizer in G , namely, the centralizer of its center. This result supplements the investigation of Gaschütz [8] where he has dealt with groups, for which each subgroup is a centralizer.

Furthermore, Theorem 2.3 and Corollary 2.4.1 give a complete description of the groups in class $\mathcal{R}$ as either metacyclic or nonmetacyclic Redei. The metacyclic groups are described by defining relations in the following theorem.

THEOREM 2.5: [15] If G is a metacyclic group then G has the defining relations:

$$a^{p^m} = 1 \ , \ b^{p^n} = a^{p^\ell} \ , \ b^{-1}ab = a^k \tag{2.14}$$

where $\ell \geq 0$, $k^{p^n} \equiv 1 \pmod{p^m}$ and $p^\ell(k-1) \equiv 0 \pmod{p^m}$. Furthermore,

$|G| = p^{a+b}$.

A group in class $\mathcal{R}$ has either the defining relations (1.5) or the defining relations (2.14).

CHAPTER III

REDEI SUBGROUPS WHICH CONTAIN THEIR CENTRALIZERS

Unlike the situation for a group from class $\mathcal{R}$ where all Redei subgroups have the same index, no relationship is known for the Redei subgroups of a group from class $\mathcal{J}$. For this reason, attention is first focused upon groups from class $\mathcal{J}$ which have a Redei maximal subgroup. The particular groups that have an abelian maximal subgroup as well as a Redei maximal subgroup are characterized in this chapter.

Proposition 3.1: If $G \in \mathcal{J}$ and G is not a Redei group, then G has at most one abelian maximal subgroup.

Proof: Suppose that G has two abelian maximal subgroups, A and A^* . From $G = AA^*$, it follows that $Z(G) = A \cap A^*$ and that $(G: Z(G)) = p^2$. Since $G \in \mathcal{J}$, then $Z(G) \leq Z(R)$ for each Redei subgroup R . However, $(R: Z(R)) = p^2$, for each Redei subgroup by Proposition 1.1. Thus, $G = R$, which is a contradiction. Hence, G has at most one abelian maximal subgroup.

Proposition 1.4(c) states that each nonabelian, non-Redei group of order p^4 has at least one abelian maximal subgroup. The group in Example 1.1 is one such group, and it has been shown that this group is in fact an element in $\mathcal{J}$. This group is, therefore, included in the following proposition.

Proposition 3.2: If $|G| = p^4$ and if G has exactly one abelian maximal subgroup, then $G \in \mathfrak{Z}$.

Proof: Let A be the abelian maximal subgroup of G . Let M be any nonabelian maximal subgroup. $|M| = p^3$ and M is a Redei group. Thus, $M' = \Phi(M) = Z(M)$ with order p . Since A is abelian, $Z(G) < A$.

Suppose there exists an M_0 , nonabelian, such that $C_G(M_0) \neq Z(M_0)$. Thus $C_G(M_0) \not< M_0$. If $C_G(M_0)$ is a maximal subgroup of G , then $G = M_0 C_G(M_0)$ and $M_0 \cap C_G(M_0)$ has order p^2 . But $M_0 \cap C_G(M_0)$ is then abelian, which forces $M_0 \cap C_G(M_0) \leq Z(M_0)$; a contradiction is reached. Therefore, $C_G(M_0)$ has index p^2 in G . However, $C_G(M_0)$ is then abelian. Since $G = M_0 C_G(M_0)$, it follows that $C_G(M_0) = Z(G)$. Now, if N is any other subgroup of index p^2 different from $Z(G)$, then N is abelian. This forces $Z(G)N$ to be abelian. Thus, $A = Z(G)N$ and A contains every subgroup of index p^2 . Furthermore, each nonabelian maximal subgroup contains exactly one maximal subgroup, its intersection with A . The nonabelian subgroup is then cyclic; and a contradiction is reached.

Therefore, $C_G(M_0) = Z(M_0)$ for each nonabelian maximal subgroup, and $G \in \mathfrak{Z}$.

Proposition 3.3: If $G \in \mathfrak{Z}$ and if G has both a Redei maximal subgroup and an abelian maximal subgroup, then for each Redei subgroup R of G , R is maximal in G and $Z(G) = Z(R)$.

Proof: Let R_0 be the Redei maximal subgroup and let M_0 be the abelian maximal subgroup. Then $G = M_0R_0$. $M_0 \cap R_0$ is, therefore, a maximal subgroup of R_0 . Since $Z(R_0) < M_0 \cap R_0 < M_0$, it follows that $Z(R_0) \leq Z(G)$. However, $Z(G) \leq Z(R_0)$ since $G \in \mathcal{J}$. Thus $Z(G) = Z(R_0)$. But $(R_0 : Z(R_0)) = p^2$ since R_0 is Redei. This implies that $Z(G)$ has index p^3 in G . Consequently, $Z(G) = Z(R)$ for each Redei subgroup R , and R is maximal in G .

The groups from $\mathcal{J}$ with a Redei maximal subgroup can be separated into two types:

(i) groups with each Redei subgroup as a maximal subgroup, or

(ii) groups with a Redei subgroup that is not maximal.

Because of Proposition 3.3, no maximal subgroup of a group of type (ii) is abelian.

<u>Proposition 3.4</u>: Let $G \in \mathcal{J}$. Let G have a Redei maximal subgroup R_0 and a maximal subgroup that is nonabelian and non-Redei. Then, each nonabelian, non-Redei subgroup N of G has exactly one abelian maximal subgroup, and all Redei subgroups of N have the same index in N . Furthermore, $(Z(R): Z(G)) \leq p$ for each Redei subgroup R of G different from R_0 .

Proof: Let N be a nonabelian, non-Redei proper subgroup of G . Let R be a Redei subgroup of N . Since $G = R_0N$, then $R_0 \cap N$ is a maximal abelian subgroup of N . Because class $\mathcal{J}$ is subgroup inherited, $N \in \mathcal{J}$. However, by Proposition 3.1 $R_0 \cap N$ is the only abelian maximal subgroup of N .

Let R be a Redei subgroup of N. Then $N = (R_0 \cap N)R$, which implies that $R_0 \cap R$ is a maximal subgroup of R. Therefore, $Z(R) < \Phi(R) < R_0 \cap R \leq R_0 \cap N$. Hence, $Z(R) \leq Z(N)$. But from (1.1) $Z(N) \leq Z(R)$. Hence, $Z(N) = Z(R)$. Now $(R: Z(R)) = p^2$ for each Redei subgroup R of N from which it follows that all Redei subgroups of N have the same index in N.

To prove the index relation for Redei subgroups, let R^* be any Redei subgroup of G and let N^* be a maximal subgroup of G containing R^*. By the first part of the proof, $Z(N^*) = Z(R^*)$. Thus $N^* \leq C_G(Z(R^*))$. If $C_G(Z(R^*)) = G$, then $Z(R^*) = Z(G)$. On the other hand, let $N^* = C_G(Z(R^*))$. Since $N^* \cap R_0$ is a maximal abelian subgroup of both N^* and R_0, then $Z(R_0)Z(R^*) \leq N^* \cap R_0$. But $Z(R^*) \not\leq Z(R_0)$, which forces the equality $Z(R_0)Z(R^*) = N^* \cap R_0$. Hence, $(Z(R^*): Z(R_0) \cap Z(R^*)) = p$. However, $Z(R_0) \cap Z(R^*) = Z(G)$. This leads to the conclusion that $(Z(R): Z(G)) \leq p$ for each Redei subgroup R of G different from R_0.

The above theorem shows that if a group from class $\mathcal{J}$ with a Redei maximal subgroup is of type (ii), then no proper nonabelian, non-Redei subgroup of G can belong to either class $\mathcal{J}^*$ or class $\mathcal{R}$. Therefore, any group in $\mathcal{J} \sim \mathcal{J}^*$ has only the groups of type (i), with an abelian maximal subgroup, as minimal non-Redei subgroups.

THEOREM 3.1: If $G \in \mathcal{J}$ and if G has both an abelian maximal subgroup and a Redei maximal subgroup, then G is one of the following nonmetacyclic groups.

(1) $G = R_0 < c \mid c^{3^2} = 1 >$ where $R_0 = < a,b \mid a^{3^2} = b^3 = 1,\ [a,b] = a^3 >$,

$[c,b] = 1,\ [a,c] = b$ and $c^3 = a^{3u}$ where $u \not\equiv 0 \pmod 3$. $|G| = 3^4$.

(2) $G = M_0 < b \mid b^p = 1 >$ where $M_0 = < a_1 \mid a_1^p = 1 > \otimes < a_2 \mid a_2^p = 1 > \otimes < a_3 \mid a_3^p = 1 >$,

$[a_1,b] = a_2$, $[a_2,b] = a_3$. $|G| = p^4$.

(3) $G = M_0 < c \mid c^3 = 1 >$ where $M_0 = < a,b \mid a^{3^2} = b^3 = 1,\ [a,b] = 1 >$,

$[c,a] = b$ and $[c,b] = a^6$. $|G| = 3^4$.

(4) $G = R_0 < c \mid c^3 = 1 >$ where $R_0 = < a,b \mid a^{3^2} = b^3 = 1,\ [a,b] = a^3 >$,

$[c,a] = b$ and $[b,c] = 1$. $|G| = 3^4$.

(5) $G = R_0 < b \mid b^p = 1 >$ where $R_0 = < a,c \mid a^{p^m} = c^p = d^p = 1,\ d = [a,c], [a,d]=[c,d]=1 >$

$[a,b] = c$, $[b,c] = [b,d] = 1$. $|G| = p^{m+3} \geq p^5$.

(6) $G = R_0 < c \mid c^p = 1 >$ where $R_0 = < a,b \mid a^{p^m} = b^p = 1,\ [a,b] = a^{p^{m-1}} >$,

$[a,c] = b$ and $[b,c] = 1$. $|G| = p^{m+2} \geq p^4$.

(7) $G = M_0 < c \mid c^p = 1 >$ where $M_0 = < a,b \mid a^{p^2} = b^p = 1,\ [a,b] = 1 >$,

$[a,c] = b$ and $[b,c] = a^{rp}$ where r is either 1 or a quadratic non-residue for $p > 3$ and $r = 1$ for $p = 3$. $|G| = p^4$.

(8) $G = R_0 < d \mid d^{p^2} = 1 >$ where $R_0 = < a,b \mid a^{p^m} = b^p = c^p = 1,\ c=[a,b], [a,c]=[b,c]= 1 >$,

$[a,d] = b,\ [b,d] = 1$ and $c = d^{rp}$ for $r \not\equiv 0 \pmod p$. $|G| = p^{m+3} \geq p^5$.

(9) $G = R_0 < c \mid c^p = 1 >$ where $R_0 = < a,b \mid a^{p^m} = b^p = 1,\ [a,b] = a^{p^{m-1}} >$,

$[b,c] = 1$ and $[a,c] = a^p$. $|G| = p^{m+2} \geq p^4$.

(10) $G = R_0 < c \mid c^p = 1 >$ where $R_0 = < a,b \mid a^{p^m} = b^{p^2} = 1, [a,b] = a^{p^{m-1}} >$,
$[a,c] = b^p$ and $[c,b] = 1$. $|G| = p^{m+3} \geq p^5$.

(11) $G = R_0 < d \mid d^{p^2}=1>$ where $R_0 = < a,b \mid a^{p^m}=b^{p^2}=c^p=1,\ c=[a,b],[a,c]=[b,c]=1 >$,
$d^p = c$, $[d,b] = 1$ and $[d,a] = c^r z^{sp}$ for $r = 0$ or 1 and
$s \not\equiv 0 (\mathrm{mod}\ p)$. $|G| = p^{m+4} \geq p^6$.

Each of the next seven lemmas constitutes a part of the proof of Theorem 3.1. The basic hypotheses are the same for all of these lemmas. For convenience these hypotheses are summarized as follows.

(3.1) $G \in \mathcal{J}$. G has a Redei maximal subgroup R_0 , an abelian maximal subgroup M_0 , and a nonmetacyclic maximal subgroup.

Lemma 3.1: Given hypotheses (3.1). Then the index of $\Phi(G)$ in G is either p^2 or p^3 . In the case where $(G: \Phi(G)) = p^2$, then both $|G'| = p^2$ and all the nonabelian subgroups have the same commutator subgroup of order p . When $(G: \Phi(G)) = p^3$, then G is a regular group.

Proof: Since R_0 is maximal in G , $(G: \Phi(R_0)) = p^3$. From $\Phi(R_0) \leq \Phi(G)$ it follows that $(G: \Phi(G)) \leq p^3$. Since G is not cyclic, $(G: \Phi(G)) \geq p^2$. Thus, $p^2 \leq (G: \Phi(G)) \leq p^3$.

Suppose that $(G: \Phi(G)) = p^2$. To show the commutators are the same, consider G/R_0' . G/R_0' has two abelian maximal subgroups R_0/R_0' and M_0/R_0' , which implies that $Z(G/R_0') = \Phi(G/R_0')$. Then each maximal subgroup of G/R_0' is abelian. Since $|R_0'| = p$, R_0' is the commutator subgroup of each nonabelian maximal subgroup of G .

If G/R_0' is abelian, then $G' = R_0' \leq Z(R_0)$. Thus, by Proposition 3.3, $G' \leq Z(G)$. By Proposition 1.5, $[x^p,y] = [x,y]^p = 1$ for each pair of elements x,y from G . Therefore, $\upsilon_1(G) \leq Z(G)$. From $\Phi(G) = G'\upsilon_1(G)$, it now follows that $\Phi(G) \leq Z(G)$. This is a contradiction to Proposition 3.3 by which $Z(G) = \Phi(R_0) < \Phi(G)$. Hence, G/R_0' is a Redei group, which leads to $|G'| = p^2$.

If $(G: \Phi(G)) = p^3$, then $\Phi(G) = Z(G)$. Since $G' \leq \Phi(G) = Z(G)$, the class of G is 2 . By Theorem 1.7(a), G is regular.

Lemma 3.2: Given hypotheses (3.1), then $p^3 \leq |\Omega_1(G)| \leq p^4$.

Proof: By hypothesis, there is a nonmetacyclic maximal subgroup N_0 . By Theorem 1.9, $|N_0/\upsilon_1(N_0)| \geq p^3$. Since N_0 is either Redei or abelian, then N_0 is regular. Consequently, by Theorem 1.6, $|\Omega_1(N_0)| \geq p^3$. But $\Omega_1(N_0) \leq \Omega_1(G)$ so that $|\Omega_1(G)| \geq p^3$.

For the other inequality in the conclusion, two cases are derived from Lemma 3.1, which states $p^2 \leq (G: \Phi(G)) \leq p^3$. First, if $(G: \Phi(G)) = p^3$, then G is regular. By Theorem 1.8, $(N: \upsilon_1(N)) \leq p^3$ for each nonabelian maximal subgroup. Thus, $(G: \upsilon_1(N)) \leq p^4$ for each nonabelian maximal subgroup. Since $\upsilon_1(N) \leq \upsilon_1(G)$, then $(G: \upsilon_1(G)) \leq p^4$. From regularity, it follows that $|\Omega_1(G)| \leq p^4$. On the other hand, if $(G: \Phi(G)) = p^2$, then by Lemma 2.1, $|G'| = p^2$. By Proposition 1.6, $\Phi(G')G_3$ is the only maximal subgroup of G' which is normal in G . Thus $\Phi(G')G_3 = R_0'$. But, Proposition 1.7 implies that G/R_0' is a nonmetacyclic Redei group. This in turn implies that $(G/R_0': \upsilon_1(G/R_0')) = p^3$ by Theorem 1.8. From the regularity of Redei groups, $|\Omega_1(G/R_0')| = p^3$.

Now $R_0' \leq \Omega_1(G)$. Thus $\Omega_1(G)/R_0' \leq \Omega_1(G/R_0')$, from which it can be concluded that $|\Omega_1(G)| \leq p^4$.

Hence, $p^3 \leq |\Omega_1(G)| \leq p^4$.

Lemma 3.3: Assume hypotheses (3.1). If $(G\colon \Phi(G)) = p^2$ and if $|\Omega_1(G)| = p^4$, then either $G = \Omega_1(G)$ or $\Phi(G)\Omega_1(G)$ is a maximal subgroup. If $G = \Omega_1(G)$, then G is the group of type (2), type (3), or type (4). If $G \neq \Omega_1(G)$, then G is the group of type (5).

Proof: If $\Omega_1(G) \leq \Phi(G)$, then $\Omega_1(G) \leq R_0$. But then, $\Omega_1(G) = \Omega_1(R_0)$. However, R_0 is regular which, from Theorems 1.8 and 1.6, implies that $|\Omega_1(R_0)| \leq p^3$; a contradiction is reached. Hence, either $G = \Omega_1(G)$ or $\Phi(G)\Omega_1(G)$ is a maximal subgroup of G .

Case 1: $G = \Omega_1(G)$. By Lemma 3.1 and Proposition 3.3, $\Phi(G) = G'$ and $|Z(G)| = p$. Thus, $G_3 = [G,G'] = Z(G)$ and the class of G is 3 .

If $p > 3$, then G is regular by Theorem 1.7(a). This implies that $\upsilon_1(G) = 1$ by Theorem 1.6. Thus, the abelian maximal subgroup M_0 has the form

$$M_0 = \langle a_1 | a_1^p = 1 \rangle \otimes \langle a_2 | a_2^p = 1 \rangle \otimes \langle a_3 | a_3^p = 1 \rangle .$$

It may be assumed that $G' = \langle a_2, a_3 \rangle$ and that $Z(G) = \langle a_3 \rangle$. Let b be any element in $G \sim M_0$. Then,

$$G = M_0 \langle b | b^p = 1 \rangle , \; [a_1,b] = a_2^r a_3^s , \text{ and } [a_2,b] = a_3^t .$$

From Lemma 3.1, $G/Z(G)$ is nonabelian, which implies that $r \not\equiv 0 (\mathrm{mod}\ p)$. Since $a_2 \notin Z(G)$, $t \not\equiv 0 (\mathrm{mod}\ p)$. Let

$$\bar{a}_2 = a_2^{rs} \quad \text{and} \quad \bar{a}_3 = a_3^{rt} .$$

Then, $G' = \langle \bar{a}_2, \bar{a}_3 \rangle$ and $[a_1, b] = \bar{a}_2$. Also, by Proposition 1.5, $[\bar{a}_2, b] = \bar{a}_3$. Thus,

$$G = M_0 \langle b | b^p = 1 \rangle \text{ where } M_0 = \langle a_1 | a_1^p = 1 \rangle \otimes \langle \bar{a}_2 | \bar{a}_2^p = 1 \rangle \otimes \langle \bar{a}_3 | \bar{a}_3^p = 1 \rangle ,$$

$$[a_1, b] = \bar{a}_2 \quad \text{and} \quad [\bar{a}_2, b] = \bar{a}_3 .$$

This is the group of type (2) in Theorem 3.1.

Now consider $p = 3$. The nonmetacyclic maximal subgroup N_0, which is either Redei or abelian, has order 3^3. Then, $\upsilon_1(N_0) = 1$ by Theorem 1.9. This implies that $\Omega_1(N_0) = N_0$. But G', which has order p^2, is contained in N_0 so that G' is elementary abelian. If $\upsilon_1(G) = 1$, then G automatically satisfies the definition of regularity. However, by Theorem 1.7(a), G is not a regular group. Hence, $\upsilon_1(G) \neq 1$. Since G is not metacyclic, it then follows that $\upsilon_1(G) = Z(G)$. This implies that G must have a metacyclic maximal subgroup.

By Proposition 1.4(a), the abelian maximal subgroup M_0 is the G-centralizer of G'. M_0 is either metacyclic or nonmetacyclic.

Suppose first that M_0 is metacyclic. Thus,

$$M_0 = \langle a, b | a^{3^2} = b^3 = 1 , [a,b] = 1 \rangle .$$

Since G' is elementary abelian, then

$$G' = \Omega_1(M_0) = \langle a^3 \rangle \otimes \langle b \rangle \quad \text{and} \quad Z(G) = \langle a^3 \rangle .$$

There exists an element c in $G \sim M_0$ such that $c^3 = 1$ (otherwise, $\Omega_1(G) \leq M_0$) . Hence, $G = M_0\langle c|c^3 = 1 \rangle$. Now $G'\langle c \rangle$ is a non-metacyclic group; denote $G'\langle c \rangle$ by N_0 . If $[c,b] = 1$, then $Z(G) = \langle a^3 \rangle \otimes \langle b \rangle$, which is a contradiction to $|Z(G)| = p$. Hence, $[c,b] \neq 1$. But $[c,b] \in [G,G'] = Z(G)$, which implies that

$$[c,b] = a^{\alpha 3} \quad \text{where} \quad \alpha \not\equiv 0 (\text{mod } 3) .$$

If $[c,a] \in Z(G)$, then it follows that $G' = Z(G)$, which is a contradiction to $|G'| = p^2$. Consequently, $[c,a] \in G' \sim \langle a^3 \rangle$, so that

$$[c,a] = a^{\gamma 3}b^{\beta} \quad \text{where} \quad \beta \not\equiv 0(\text{mod } 3) .$$

There exists a δ such that $\beta\delta \equiv 1(\text{mod } 3)$. Let $a_1 = a^{\delta}$ and $b_1 = a^{\gamma\delta 3}b$. By Proposition 1.5, $[c,a_1] = b_1$, $[a_1,b] = 1$, and $[c,b_1] = a^{\alpha\beta 3} = a_1^{\alpha\beta^2 3}$. It is possible, then, to drop the subscripts, so the group is defined by

$$G = M_0\langle c|c^3 = 1 \rangle \text{ where } M_0 = \langle a,b|a^{3^2} = b^3 = [a,b] = 1 \rangle ,$$

(3.2)

$$[c,a] = b \quad \text{and} \quad [c,b] = a^{r3} \quad \text{for} \quad r \not\equiv 0(\text{mod } 3) .$$

The relations in (3.2) lead to

$$c^y b^x = a^{xyr3} b^x c^y \quad \text{and} \quad c^y a^x = a^{x + y(y+1)xr3/2}$$

which in turn give

$$(3.3) \quad d^3 = a^{[1 + t(t+1)r/2 + 2t(2t+1)r/2]3q} \quad \text{where} \quad d = a^q b^s c^t \quad .$$

Suppose that $d^3 = 1$. Then,

$$[1 + (5t^2+3t)r/2]q \equiv 0 (\text{mod } 3) \; .$$

If $r = 1$ and $q \not\equiv 0(\text{mod } 3)$, then $1 + (5t^2+3t)/2 \equiv 0(\text{mod } 3)$. It follows that $t^2 \equiv 2(\text{mod } 3)$, which has no solution. If $r = 1$ and $q \equiv 0(\text{mod } 3)$, then $d \in N_0 = G' < c >$, which implies that $\Omega_1(G) = N_0$. This is a contradiction to $G = \Omega_1(G)$. For $r = 2$, consider the element ac which is not an element of N_0 . By (3.3), $(ac)^3 = 1$. Consequently, $\Omega_1(G) = G$. Hence, in defining relations (3.2), $r = 2$. This group is the group of type (3) in Theorem 3.1.

For the other possibility where M_0 is not a metacyclic group, it follows that G has a metacyclic maximal subgroup R_1 . Let

$$R_1 = < a,b | a^{3^2} = b^3 = 1 \; , \; [a,b] = a^3 > \; .$$

Since G' is elementary abelian, then

$$G' = < a^3 > \otimes < b > \quad \text{and} \quad Z(G) = < a^3 > \quad .$$

There exists an element $c \in M_0 \sim N_0$ such that $c^3 = 1$. Then $[b,c] = 1$. If $[c,a] \in Z(G)$, then $G' = Z(G)$ which is a contradiction to $|G'| = p^2$. Thus,

$$[b,c] = 1 \text{ and } [c,a] = b^s a^{r3} \text{ where } s \not\equiv 0 (\text{mod } 3) .$$

There exists a t such that $st \equiv 1 (\text{mod } 3)$. Since $[[c,a], c] = 1$, Proposition 1.5 implies that $[c^t,a] = ba^{rt3}$. Let $c_1 = c^t$ and $b_1 = ba^{rt3}$. Then, it is possible to drop the subscripts to get the defining relations:

$$G = R_1 < c | c^3 = 1 > \text{ where } R_1 = < a,b | a^{3^2} = b^3 = 1 , [a,b] = a^3 >$$
$$[c,a] = b \text{ and } [b,c] = 1 .$$

This is the group of type (4) in Theorem 3.1.

<u>Case 2</u>: $\Phi(G)\Omega_1(G)$ is a maximal subgroup of G. Then, $\Omega_1(G) = \Omega_1(\Phi(G)\Omega_1(G))$. From Theorems 1.6 and 1.8, $|\Omega_1(R)| \leq p^3$ for each Redei subgroup R. Thus, $\Phi(G)\Omega_1(G) = M_0$, the abelian maximal subgroup of G.

Since $(M_0: \Phi(G)) = p$, then $\Phi(G) \cap \Omega_1(G)$ is a maximal subgroup of $\Omega_1(G)$ and has order p^3. Moreover, $\Omega_1(G)$ is abelian which implies that $\Phi(G) \cap \Omega_1(G) = \Omega_1(\Phi(G))$. In view of the fact that $\Omega_1(\Phi(G)) \leq M$ for each maximal subgroup M of G, it follows that $\Omega_1(\Phi(G)) \leq \Omega_1(M)$. Thus, each nonabelian maximal subgroup is nonmetacyclic.

By Lemma 3.1, G/R_0' is a Redei group. From Propositions 1.6

and 1.7,

$$G/R_0' = < \bar{a},\bar{b} \mid \bar{a}^{p^m} = \bar{b}^{p^n} = \bar{c}^p = 1,\ \bar{c} = [\bar{a},\bar{b}],\ [\bar{a},\bar{c}] = [\bar{b},\bar{c}] = 1 > \quad .$$

If $m, n \geq 2$, then $\Omega_1(G/R_0') = < \bar{a}^{p^{m-1}} > \otimes < \bar{b}^{p^{n-1}} > \otimes < \bar{c} > \leq \Phi(G)/R_0'$. However, $\Omega_1(G)/R_0' \leq \Omega_1(G/A)$ so that $\Omega_1(G) \leq \Phi(G)$. This is a contradiction. Thus, one of m,n must be 1; say $n = 1$. There exists an element $b \in \Omega_1(G) \sim \Phi(G)$ such that $bR_0' = \bar{b}$; $b^p = 1$. Now $\bar{a} \in M/R_0'$ for some maximal subgroup M of G . There exists an element $a \in M$ such that $aR_0' = \bar{a}$. Then $a^{p^m} \in R_0'$. If $M = M_0$, then $G/R_0' = < \bar{a},\bar{b} > = M_0/R_0'$, which is a contradiction. Thus, $M \neq M_0$. Since M is nonmetacyclic and Redei, $M' \cap \upsilon_1(M) = 1$. However, $M' = R_0'$ so that $a^{p^m} = 1$. Let $c = [a,b]$. Then $cR_0' = \bar{c}$. Both b and c are elements in M_0 from which it follows that $[b,c] = 1$. Furthermore, $[a,c] \neq 1$ (otherwise $G' = R_0'$) . Let $d = [a,c]$. Hence,

$$G = R_1 < b \mid b^p = 1 > \quad \text{where}$$

$$R_1 = < a,c \mid a^{p^m} = c^p = d^p = 1,\ d = [a,c],\ [a,d] = [c,d] = 1 > ,$$

$$[a,b] = c \quad \text{and} \quad [b,c] = [b,d] = 1 \ .$$

This is the group of type (5) in Theorem 3.1.

Lemma 3.4: Given hypotheses (3.1). If $(G: \Phi(G)) = p^2$ and if $|\Omega_1(G)| = p^3$, then either $\Phi(G)\Omega_1(G)$ is a maximal subgroup or

$\Omega_1(G) \leq \Phi(G)$. If $\Omega_1(G) \leq \Phi(G)$, then G is the group of type (8). If $\Omega_1(G) \not\leq \Phi(G)$, then G is the group of type (6) or type (7).

Proof: For the nonmetacyclic maximal subgroup N_0 , $|\Omega_1(N_0)| = p^3 = |\Omega_1(G)|$. This implies that $\Omega_1(G) < G$. Thus, either $\Omega_1(G) < \Phi(G)$ or $\Phi(G)\Omega_1(G)$ is maximal.

Case 1: Let $\Phi(G)\Omega_1(G)$ be a maximal subgroup of G . Then, $\Phi(G)\Omega_1(G) = N_0$. N_0 is thus the only nonmetacyclic maximal subgroup of G . Let M be a metacyclic maximal subgroup. If $\Omega_1(M) \not\leq \Phi(G)$, then $\Phi(G)\Omega_1(M)$ is maximal in G . But then, $M = \Phi(G)\Omega_1(M) \leq \Phi(G)\Omega_1(G) = N_0$, which is a contradiction to N_0 nonmetacyclic. Thus $\Omega_1(M) \leq \Phi(G)$ for each metacyclic maximal subgroup M , that is, $\Omega_1(M) \leq \Omega_1(\Phi(G))$.

Let R be a nonabelian maximal subgroup different from N_0 . Then R is metacyclic and $\upsilon_1(R) = \Phi(R)$ where $(R: \upsilon_1(R)) = p^2$. G , however, is not metacyclic, which implies that $(G: \upsilon_1(G)) \geq p^3$. From $\upsilon_1(R) \leq \upsilon_1(G)$, it follows that $\upsilon_1(G) = \upsilon_1(R) = Z(R)$. In particular,

$$(3.4) \qquad \upsilon_1(G) = Z(G) .$$

Suppose first of all that $|G| > p^4$. If N_0 is nonabelian, then $N_0' \cap \upsilon_1(N_0) = 1$ by Theorem 1.8, and $|N_0'\Omega_1(\upsilon_1(N_0))| \geq p^2$. However, $N_0'\Omega_1(\upsilon_1(N_0)) \leq \Phi(N_0) \leq \Phi(G)$, so that $N_0'\Omega_1(\upsilon_1(N_0)) \leq \Omega_1(\Phi(G))$. But $|\Omega_1(\Phi(G))| = p^2$ since $\Omega_1(G) \not\leq \Phi(G)$. Thus, $N_0'\Omega_1(\upsilon_1(N_0))$ is an elementary abelian subgroup of order p^2 which is contained in $\Phi(G)$. Therefore, $N_0'\Omega_1(\upsilon_1(N_0)) = \Omega_1(\Phi(G))$. Since $\Omega_1(M) \leq \Omega_1(\Phi(G))$ for a metacyclic maximal subgroup M , it follows that $\Omega_1(M) = N_0'\Omega_1(\upsilon_1(N_0))$.

Since N_0 is nonabelian, $\Phi(N_0) = Z(N_0) = Z(G)$. Consequently, $\Phi(N_0) = \mho_1(G)$. Now, $(\Phi(G) : \mho_1(G)) = p$, which implies that $G' \not\leq \Phi(N_0)$. Furthermore, since $G' \leq R$ and since $\Omega_1(R) = N_0'\Omega_1(\mho_1(N_0)) \leq \Phi(N_0) = \mho_1(G)$, then $G' \not\leq \Omega_1(R)$. Therefore, G' is cyclic. Let $G' = < c | c^{p^2} = 1 >$. Then $c^p \in \Omega_1(G')$. By Lemma 3.1, $N_0' = \Omega_1(G')$. Since $G' \leq N_0$, $c^p \in \mho_1(N_0)$. Thus, $N_0' \cap \mho_1(N_0) \neq 1$, which contradicts the fact that N_0 is nonmetacyclic. Hence, N_0 must be abelian, that is, $N_0 = M_0$.

By (3.4), $\mho_1(G)$ has index p^3 in G . This implies that $\mho_1(G) = \mho_1(M)$ for each metacyclic maximal subgroup M , whereby $\mho_1(M_0) \leq \mho_1(M)$. Moreover, $G' \not\leq \mho_1(G)$ from which it can be concluded that $G' \not\leq \mho_1(M_0)$. If $G' \cap \mho_1(M_0) = 1$, then G' , as a subgroup of M_0 , must be elementary abelian of order p^2 . Then, $G'\Omega_1(\mho_1(M_0)) \leq \Phi(G)$. But $|G'\Omega_1(\mho_1(M_0))| \geq p^3$ from which it follows that $\Omega_1(M_0) = G'\Omega_1(\mho_1(M_0) \leq \Phi(G)$. This is a contradiction to $N_0 = M_0 = \Phi(G)\Omega_1(G)$. Thus, $G' \cap \mho_1(M_0) \neq 1$. By Lemma 3.1, $G' \cap \mho_1(M_0) = R_0'$.

Consider now G/R_0' . Denote this by $\overline{G}$. Since $\Omega_1(G) \not\leq \Phi(G)$, then $\Omega_1(\overline{G}) \leq \Phi(\overline{G})$. From Lemma 3.1 and Propositions 1.6 and 1.7, $\overline{G}$ is of the form $< \overline{a},\overline{b} | \overline{a}^{p^m} = \overline{b}^p = \overline{c}^p = 1, \overline{c} = [\overline{a},\overline{b}], [\overline{a},\overline{c}] = [\overline{b},\overline{c}] = 1 >$. Let a and b be elements of G such that $aR_0' = \overline{a}$ and $bR_0' = \overline{b}$. Let $c = [a,b]$. Then $cR_0' = \overline{c}$ and $c^p \in R_0'$. In addition, $\overline{M}_0 = \overline{\Phi(G)\Omega_1(G)} \leq \Phi(\overline{G})\Omega_1(\overline{G})$. If $\overline{M} < \Phi(\overline{G})\Omega_1(\overline{G})$, then $\overline{G} = \Omega_1(\overline{G})$. Since $\overline{G}$ is Redei and nonmetacyclic, it follows that $\mho_1(\overline{G}) = 1$. Thus, $m = 1$ and $|\overline{G}| = p^3$. This is a contradiction to $|G| > p^4$. Hence, $\overline{M}_0 = \Phi(\overline{G})\Omega_1(\overline{G})$. Since $\overline{b} \in \overline{M}_0$, then $b \in M_0$. By Proposition 1.5, this leads to $c^p = [a,b]^p = [a,b^p]$. But, $b^p \in \mho_1(G) = Z(G)$ which implies that $[a,b^p] = 1$. In particular, $c^p = 1$. Consequently, G' is elementary abelian. From $|\Omega_1(\Phi(G))| = p^2$ and from $G' \not\leq \mho_1(G)$, it follows that $|\Omega_1(\upsilon_1(G))| = p$. This, in turn, implies that $\upsilon_1(G)$ is cyclic.

It is now possible to describe the metacyclic maximal subgroups of G . Let M be such a subgroup. Then,

$$M = \langle a,b \mid a^{p^m} = b^p = 1,\ [a,b] = a^{p^{m-1}} \rangle .$$

There exists an element $c \in M_0 \sim M$ such that $c^p = 1$. Then,

$$G = M\langle c \mid c^p = 1 \rangle .$$

Both $[b,c]$ and $[a,c]$ belong to G' . Also, $b \in \Omega_1(M) \leq \Phi(G)$, so that $b \in \Omega_1(\Phi(G)) = G'$. But then, $b \in M_0$. Hence,

$$[b,c] = 1 \text{ and } [a,c] = a^{rp^{m-1}} b^s ,\ s \not\equiv 0(\text{mod } p) .$$

Let $c_1 = c^t$ where t is such that $ts \equiv 1(\text{mod } p)$. Since $[a,c] \in M_0$, then $[[a,c], c] = 1$. By Proposition 1.5, $[a,c]^t = [a,c^t]$. Thus, $[a,c_1] = a^{trp^{m-1}}b$.

Let $b_1 = a^{trp^{m-1}}b$, then $b_1^p = 1$ and $b_1 \notin Z(G)$ (otherwise, $b \in Z(G)$) . Now $[a,b_1] = [a,a^{trp^{m-1}}b] = a^{p^{m-1}}$ since $a^{trp^{m-1}} \in Z(G) = \upsilon_1(G)$. Also $[b_1,c_1] = 1$ and $b_1 \notin \Phi(M)$. Thus,

$$M = \langle a,b_1 \mid a^{p^m} = b_1^p = 1,\ [a,b_1] = a^{p^{m-1}} \rangle \text{ and } G = M\langle c_1 \mid c_1^p = 1 \rangle$$

$$\text{where } [a,c_1] = b_1 \text{ and } [b_1,c_1] = 1 .$$

This is the group of type (6) in Theorem 3.1.

Now suppose that $|G| = p^4$. Then, $N_0 = \Omega_1(N_0) = \Omega_1(G)$, $Z(G) = R_0' = \upsilon_1(G)$ with order p , and G' has order p^2 . Since N_0

is regular (either abelian or Redei), then $\upsilon_1(N_0) = 1$. It follows from $G' \leq N_0$ that G' is elementary abelian of order p^2 .

Suppose that N_0 is abelian. There exists then a nonabelian metacyclic maximal subgroup M with defining relations

$$M = < a,b|a^{p^2} = b^p = 1 \text{ , } [a,b] = a^p > .$$

There exists an element $c \in N_0 \sim M$. By the argument that has been used for $|G| > p^4$, it follows that

$$G = M< c|c^p = 1 > \text{ where } [a,c] = b \text{ and } [b,c] = 1 \text{ ,}$$

which is the group of type (6) in Theorem 3.1.

If, on the other hand, N_0 is nonabelian, then the abelian maximal subgroup M_0 is metacyclic and has defining relations

$$M_0 = < a|a^{p^2} = 1 > \otimes < b|b^p = 1 > .$$

Then $G' = < a^p > \otimes < b >$ and $Z(G) = R_0' = \upsilon_1(G) = < a^p >$. There exists an element $c \in N_0 \sim M_0$ and

$$G = [M_0]< c|c^p = 1 > .$$

Since both b and c belong to N_0 , $[b,c] \in N_0' = R_0' = \upsilon_1(G) = < a^p >$. Also, $[a,c] \in G' \sim Z(G)$. Thus,

$$[b,c] = a^{rp} \text{ , } r \not\equiv 0 (\text{mod } p) \text{ and } [a,c] = a^{sp}b^t \text{ , } t \not\equiv 0 (\text{mod } p) \quad .$$

Let $b_1 = a^{sp}b^t$. Then, $b_1^p = 1$ and $b_1 \in M_0 \sim \upsilon_1(M_0)$. Then $M_0 = < a|a^{p^2} = 1 > \otimes < b_1|b_1^p = 1 >$. $[b_1,c] = [a^{sp}b^t,c] = [b^t,c]$. Now

$[b,c] \in N_0' = Z(N_0)$ so by Lemma 1.3, $[b^t,c] = [b,c]^t$. Denote tr by u. Then $u \not\equiv 0 (\mathrm{mod}\ p)$ and $[b_1,c] = a^{up}$.

It is now possible to drop the subscripts to get $G = M_0 < c | c^p = 1 >$ where $M_0 = < a,b | a^{p^2} = b^p = 1 , [a,b] = 1 >$, $[b,c] = a^{up}$ where $u \not\equiv 0 (\mathrm{mod}\ p)$ and $[a,c] = b$. These relations give rise to $[b,c^\alpha] = a^{\alpha up}$ and $[a,c^\alpha] = a^{\alpha(\alpha-1)p/2} b^\alpha$. For $\alpha \not\equiv 0 (\mathrm{mod}\ 3)$, $G = M_0 < c^\alpha >$. Thus, for $c_1 = c^\alpha$, and $b_1 = a^{\alpha(\alpha-1)p/2} b^\alpha$, then

$$(3.5) \quad G = M_0 < c_1 > \text{ where } M_0 = < a,b_1 | a^{p^2} = b_1^p = 1, [a,b_1] = 1 > ,$$

$$[a,c_1] = b_1 \quad \text{and} \quad [b_1,c_1] = a^{\alpha^2 up} .$$

Now $u \not\equiv 0 (\mathrm{mod}\ p)$ so there exists v such that $vu \equiv 1 (\mathrm{mod}\ p)$. If u is a quadratic residue, then v is a quadratic residue, that is, there exists a k such that $k^2 u \equiv 1$. If u is not a quadratic residue and t is not a quadratic residue, then from $tvu = t$ it follows that tv is a quadratic residue. Then there is a k such that $k^2 u = t$. Thus for $s \in \{1,t\}$ where t is a quadratic nonresidue, there exists a k such that $k^2 u = s$. Hence, it can be assumed, without loss of generality, that $\alpha^2 u$ is either 1 or a quadratic nonresidue.

Since $Z(G) < G'$, G has class 3. If $p > 3$, then from Proposition 1.7(a), G is regular; it follows that a group will exist for each value of r. However, if $p = 3$, then the proof of Lemma 3.3 has shown that $\Omega_1(G) = G$ when the quadratic nonresidue $\alpha^2 u$ is equal to 2. Hence, for $p = 3$, $\alpha^2 u$ can only be 1.

The groups satisfying (3.5) are the groups of type (7) in Theorem 3.1.

Case 2: $\Omega_1(G) \le \Phi(G)$. Then, $\Omega_1(M) = \Omega_1(G)$ for each maximal subgroup M , whereby each nonabelian maximal subgroup is nonmetacyclic. By Proposition 1.6, R_0' is the only maximal subgroup of G' which is normal in G . Since $M' \cap \upsilon_1(M) = 1$ for each nonabelian maximal subgroup M , it follows that $G' \cap \upsilon_1(M) = 1$. Now by Lemma 3.2, $|G'| = p^2$. If G' is cyclic, then $M' \le \upsilon_1(M)$ which is a contradiction to $G' \cap \upsilon_1(M) = 1$. Thus, G' is elementary abelian. This implies that $\upsilon_1(M)$ is cyclic. Consequently, for the nonabelian Redei subgroup R_0 ,

$$R_0 = \langle a,b | a^{p^m} = b^p = c^p = 1 , c = [a,b] , [b,c] = [a,c] = 1 \rangle \quad .$$

To determine the relation between G' and $\upsilon_1(G)$ suppose that $G' \cap \upsilon_1(G) = 1$. Since $(\Phi(G): \Phi(M)) = p$ for each nonabelian maximal subgroup M , then $\upsilon_1(G) = \upsilon_1(M)$; so $\upsilon_1(G)$ is cyclic. Let $G/G' = \langle \bar{x} \rangle \otimes \langle \bar{y} \rangle$ where $\bar{x}^{p^\alpha} = 1$ and $\bar{y}^{p^\beta} = 1$. Let x and y be elements from G such that $xG' = \bar{x}$ and $yG' = \bar{y}$. Then $x^{p^\alpha} \in G'$ and $y^{p^\beta} \in G'$ where $\alpha, \beta \ge 1$. Since $G' \cap \upsilon_1(G) = 1$, $x^{p^\alpha} = y^{p^\beta} = 1$. However, since $\Omega_1(G) \le \Phi(G)$, then $\alpha, \beta \ge 2$. Thus, $\Phi(G/G') = \langle \bar{x}^p \rangle \otimes \langle \bar{y}^p \rangle$. Now $\Phi(G/G')$ is isomorphic to $\upsilon_1(G)$, which is cyclic; accordingly, either $x^p \in G'$ or $y^p \in G'$. Thus, one of α or $\beta = 1$; a contradiction is reached. Hence, $G' \cap \upsilon_1(G) \ne 1$.

In view of Proposition 1.6, $R_0' \le \upsilon_1(G)$. Then $(G: \upsilon_1(G)) = p^3$. It now follows that G/R_0' is nonmetacyclic and Redei. $\Phi(R_0/R_0')$ is isomorphic to $\upsilon_1(R_0)$ and has index p^2 in R_0/R_0' . Since R_0/R_0' is abelian, $\Omega_1(R_0/R_0')$ has order p^2 . But, since G/R_0' is nonmetacyclic, $\Omega_1(G/R_0')$ has order p^3 . Thus, there exists an element $\bar{d} \in G/R_0'$

such that $\bar{d}^p = 1$ and $G/R_0' = (R_0/R_0') < \bar{d}^p >$. Then, there exists a $d \in G \sim R_0$ such that $dR_0' = \bar{d}$. Hence, $d^{p^2} = 1$ and $d^p \in R_0' = < c >$. It may be assumed that $d^p = c$ (If not, then $d^p = c^\alpha$ where $\alpha \not\equiv 0 (\mathrm{mod}\ p)$. There exists a γ such that $\gamma\alpha \equiv 1(\mathrm{mod}\ p)$ and $(d^\gamma)^p = c$. Then d can be replaced by d^γ). Consequently,

$$G = R_0 < d | d^p = 1 > \text{ where } d^p = c \quad .$$

Moreover, there is a maximal subgroup M such that $d \in M$. If M is nonabelian, then $d^p = R_0' \cap \upsilon_1(M) = M' \cap \upsilon_1(M) = 1$, which is a contradiction. Thus, M is abelian, that is, $M = M_0$. Since $b \in \Omega_1(G) \leq M_0$, then

$$[b,d] = 1 \ .$$

If $[a,d] \in R_0'$, then it follows from $G/R_0' = (R_0 < d >)/R_0'$ that G/R_0' is abelian, which is a contradiction. Thus,

$$[a,d] = b^r d^{sp} \ , \quad r \not\equiv 0(\mathrm{mod}\ p) \quad .$$

Let $d_1 = d^t$ where t is such that $tr \equiv 1(\mathrm{mod}\ p)$. $[a,d] \in M_0$, so that $[[a,d], d] = 1$. Then by Proposition 1.5, $[a,d_1] = bd_1^{sp}$. Let $b_1 = bd_1^{sp}$. Then $b_1 \notin Z(G) = \Phi(R_0)$ and $b_1 \in \Omega_1(G) \leq \Phi(G)$. Also, $[a,b_1] = d_1^{rp}$. Hence,

$$G = R_0 < d_1 | d_1^p = 1 > \text{ where } < a,b_1 | a^{p^m} = b_1^p = c^p = 1, c = [a,b_1], [a,c] = [b_1,c] = 1 >$$

$$[a,d_1] = b_1,\ [b_1,d_1] = 1 \text{ and } c = d_1^{rp} \text{ for } r \not\equiv 0(\mathrm{mod}\ p) \ .$$

This is the group of type (8) in Theorem 3.1.

Lemma 3.5: Given hypotheses (3.1). If $(G: \Phi(G)) = p^3$, then $|\Omega_1(G)| = p^3$.

Proof: By Lemma 3.2, $p^3 \leq |\Omega_1(G)| \leq p^4$. Suppose that $|\Omega_1(G)| = p^4$. By Lemma 3.1, G is regular. Then, $(G: \mho_1(G)) = p^4$ and $(\Phi(G): \mho_1(G)) = p$. Since $(M: \mho_1(G)) = p^3$ for each nonabelian maximal subgroup M , then $\mho_1(G) = \mho_1(M)$ and M is nonmetacyclic. Now, $M' \cap \mho_1(G) = M' \cap \mho_1(M) = 1$ so that $\overline{G} = G/\mho_1(G)$, which has order p^4 , has precisely one abelian maximal subgroup $M_0/\mho_1(G) = \overline{M}_0$. $\overline{G}'$ has order p and $\overline{G}' = \overline{M}' = Z(\overline{M})$ for each nonabelian maximal subgroup $\overline{M}$ of $\overline{G}$.

Let $x \in G \sim M_0$; then $\Phi(\overline{G})<\overline{x}>$ has order p^2 , is elementary abelian, and is normal in $\overline{G}$. Denote $\Phi(\overline{G})<\overline{x}>$ by $\overline{A}$. By Proposition 1.4(a), $(\overline{G}: C_{\overline{G}}(\overline{A})) \leq p$; also, $\overline{A} \leq Z(C_{\overline{G}}(\overline{A}))$. If $\overline{G} = C_{\overline{G}}(\overline{A})$, then $\overline{x} \in Z(\overline{G})$. It follows from $\overline{G} = \overline{M}_0<\overline{x}>$ that $\overline{G}$ is abelian, which is a contradiction. Thus $C_{\overline{G}}(\overline{A})$ is maximal in $\overline{G}$. Since $(C_{\overline{G}}(\overline{A}): \overline{A}) = p$, $C_{\overline{G}}(\overline{A})$ is abelian. Consequently, $C_{\overline{G}}(\overline{A}) = \overline{M}_0$, which implies that $x \in M_0$; a contradiction is reached. Hence, $|\Omega_1(G)| = p^3$.

Lemma 3.6: Given hypotheses (3.1). If $(G: \Phi(G)) = p^3$ and if $G' = R_0'$, then $\Phi(G)\Omega_1(G)$ is a maximal subgroup of G . Furthermore, G is the group of type (9).

Proof: $\Omega_1(G)$ is clearly a proper subgroup of G . Thus, three cases must be considered.

Case 1: Suppose that $\Omega_1(G) \leq \Phi(G)$. Then each maximal subgroup of G is nonmetacyclic. Let $x \in G \sim M_0$. There exists a maximal subgroup R to which x belongs. R is nonabelian, which implies that $R' \cap \mho_1(R) = 1$. Let $\overline{G} = G/\mho_1(R_0)$, which has order p^4. $\Phi(\overline{G}) < \overline{x} >$ has order p^2, is elementary abelian, and is normal in $\overline{G}$. Let $\overline{A} = \Phi(\overline{G}) < \overline{x} >$. If $\overline{A} \leq Z(\overline{G})$, then $\overline{A} \leq Z(\overline{R})$ since $\overline{A} < \overline{R}$; consequently, $\overline{R}$ is abelian, which is a contradiction to $R' \cap \mho_1(R) = 1$. Hence, $\overline{A} \not\leq Z(\overline{G})$. By Proposition 1.4(a), $C_{\overline{G}}(\overline{A})$ has index p in $\overline{G}$. But $\overline{A} \leq Z(C_{\overline{G}}(\overline{A}))$, whence $C_{\overline{G}}(\overline{A})$ is abelian. Thus, $\overline{x} \in \overline{M}_0$ from which it follows that $x \in M_0$; a contradiction is reached. Thus $\Omega_1(G) \not\leq \Phi(G)$.

Case 2: Suppose that $\Phi(G)\Omega_1(G)$ has index p^2 in G. Then $\Phi(G)\Omega_1(G)$ is abelian and nonmetacyclic. Denote $\Phi(G)\Omega_1(G)$ by A. There exists a maximal subgroup A^* which contains A. A^* is nonmetacyclic since A is nonmetacyclic.

The regularity of G implies that $(G: \mho_1(G)) = p^3$. Thus, $G' \leq \mho_1(G)$. If each element x, for which $x^p \in G'$, is contained in A, then $G' \leq \mho_1(A) \leq \mho_1(A^*)$. If A^* were nonabelian, then $A^{*\prime} \cap \mho_1(A^*) = G' \cap \mho_1(A^*) \neq 1$, which is a contradiction. Therefore, A^* is abelian. If, on the other hand, there exists an $x \in G$ such that $x \not\in A$ but such that $x^p \in G'$, then $A< x >$ is maximal in G, is nonmetacyclic, and $G' \leq \mho_1(A< x >)$. By the same argument used to show that A^* was abelian, it can be concluded that $A< x >$ is abelian. In either situation, A is contained in the abelian maximal subgroup M_0.

Suppose that there exists a nonmetacyclic nonabelian maximal subgroup N_0. $N_0' \cap \mho_1(N_0) = 1$. Let $\overline{G} = G/\mho_1(N_0)$, which has order p^4. $\overline{G}'$ has order p and $\overline{G}$ has precisely one abelian maximal subgroup $\overline{M}_0$. Now, $\mho_1(\overline{G}) = \overline{\mho_1(G)}$, which has order p. From the regularity of

$\overline{G}$, $|\Omega_1(\overline{G})| = p^3$. Then, $\overline{N}_0 = \Omega_1(\overline{G})$. Let $x \in N_0 \sim \Phi(G)$. $x^p \in \mho_1(N_0)$, but $\overline{x} \notin Z(\overline{N}_0) = Z(\overline{G}) = \Phi(\overline{G})$. By Proposition 1.4(a), $C_{\overline{G}}(\Phi(\overline{G})< \overline{x} >)$ has index p . Thus, it follows that $C_{\overline{G}}(\Phi(\overline{G})< \overline{x} >)$ is abelian, that is, $C_{\overline{G}}(\Phi(\overline{G})< \overline{x} >) = \overline{M}_0$. In view of the fact that x is arbitrary in $N_0 \sim \Phi(G)$, $\overline{M}_0 = C_{\overline{G}}(\overline{x})$ for each $x \in N_0 \sim \Phi(G)$. However, for $x \in \Phi(G) < M_0$, $\overline{M}_0 = C_{\overline{G}}(\overline{x})$. Thus, $\overline{M}_0 = C_{\overline{G}}(\overline{N}_0)$, which implies that $\overline{M}_0 \cap \overline{N}_0 = Z(\overline{N}_0)$. Hence, $\overline{N}_0$ is abelian, and a contradiction is reached. Then, M_0 is the only nonmetacyclic maximal subgroup, and M_0 is the only maximal subgroup containing A .

Let R be any metacyclic maximal subgroup. $\Omega_1(R) = \Omega_1(\Phi(G))$, so that R has the defining relations $R =$ $< a,b|a^{p^m} = b^{p^n} = 1, [a,b] = a^{p^{m-1}} , m , n \geq 2 >$. $G' = < a^{p^{m-1}} >$ and $\Phi(G) = \Phi(R) = < a^p > \otimes < b^p >$. There exists an $x \in M_0 \sim R$ such that $x^p = 1$. Then, $A = \Phi(G)< x >$. Now $A< b >$ is a maximal subgroup of G . Thus, $A< b > = M_0$. However, $A< a >$ is a maximal subgroup of G; so, $A< a > = M_0$. Hence, $R = < a,b > = M_0$, which is a contradiction. Consequently, $(G: \Phi(G)\Omega_1(G)) \neq p^2$.

<u>Case 3</u>: $\Phi(G)\Omega_1(G)$ is a maximal subgroup of G . Denote $\Phi(G)\Omega_1(G)$ by M . Then, $\Omega_1(G) = \Omega_1(M)$. Since $|\Omega_1(G)| = p^3$, M is not metacyclic. Now, $|\Phi(G) \cap \Omega_1(G)| = p^2$. Since G is regular, then by Theorem 1.6, each element of $\Omega_1(G)$ has order p . Therefore, $\Omega_1(\Phi(G)) = \Phi(G) \cap \Omega_1(G)$, which implies that $\Phi(G)$ is cyclic. Let $y \in \Phi(G)$ such that $y^p = 1$. $G'< y >$ is both elementary abelian and normal in G , so that by Proposition 1.4(a), $(G: C_G(G'< y >)) \leq p$. Since $y \notin Z(G)$, $C_G(G'< y >)$ is a maximal subgroup of G . But, $\Phi(G)< y > \leq Z(C_G(G'< y >))$ and $(C_G(G'< y >): \Phi(G)< y >) = p$, from which it follows that $C_G(G'< y >)$ is abelian, that is, $C_G(G'< y >) = M_0$.

Since y is arbitrary, $\Omega_1(G) \leq M_0$. Thus, $\Phi(G)\Omega_1(G) = M_0$.

Let R be a maximal subgroup of G , $R \neq M_0$. $R \cap \Omega_1(G)$ has order p^2. Therefore, R is metacyclic with $\Phi(R) = \Phi(G)$ cyclic. Then, $R = < a,b | a^{p^m} = b^p = 1 , [a,b] = a^{p^{m-1}} >$. There exists an element $c \in M_0 \sim R$ such that $c^p = 1$ and $G = R< c >$. Since $b \in \Omega_1(G) \leq M_0$, $[b,c] = 1$. Moreover, $G' \leq < a >$, whereby $< a > \triangleleft G$. Now, $< a >$ has index p^2 in G; then, $< a >< c >$ is a maximal subgroup of G which is not abelian. Thus, $[a,c] = a^{rp^{m-1}}$. Let t be such that $tr \equiv 1 (\mathrm{mod}\ p)$ and let $c_1 = c^t$. $G' < Z(G)$ so that by Proposition 1.5 $[a,c_1] = [a,c^t] = [a,c]^t = a^{trp^{m-1}} = a^{p^{m-1}}$. Hence,

$$G = R< c_1 | c_1^p = 1 > \text{ where } R = < a,b | a^{p^m} = b^p = 1,\ [a,b] = a^{p^{m-1}} > ,$$

$$[a,c_1] = a^{p^{m-1}} \text{ and } [b,c_1] = 1 .$$

This is the group of type (9) in Theorem 3.1.

Lemma 3.7: Given hypotheses (3.1). If $(G: \Phi(G)) = p^3$ and if $R_0' < G'$, then either $\Omega_1(G) \leq \Phi(G)$ or $\Phi(G)\Omega_1(G)$ has index p^2 . If $\Omega_1(G) \leq \Phi(G)$, then G is the group of type (11). If $\Omega_1(G) \not\leq \Phi(G)$, then G is the group of type (10).

Proof: Let A be a normal subgroup of G' , minimal with the property that $M' < A$ for each nonabelian maximal subgroup M . Each maximal subgroup of G/A is abelian, so that G/A is either Redei or abelian. If G/A is Redei, then $(G/A: \Phi(G/A)) = p^2$, which contradicts the hypothesis that $(G: \Phi(G)) = p^3$. Thus G/A is abelian. Hence,

$G' = A$. This implies that

(3.6) G has at least two maximal subgroups with distinct commutator subgroups.

Furthermore, G' is elementary abelian.

Since G is regular, then by Theorem 1.6, each element of $\Omega_1(G)$ has order p . Therefore, since $G' \leq \Omega_1(G)$, $|\Phi(G) \cap \Omega_1(G)| \geq p^2$ and $(\Omega_1(G): \Phi(G) \cap \Omega_1(G)) \leq p$. This implies that $(\Phi(G)\Omega_1(G): \Phi(G)) \leq p$ and that $\Phi(G)\Omega_1(G)$ is abelian.

Case 1: $\Phi(G)\Omega_1(G)$ has index p^2 in G . Denote $\Phi(G)\Omega_1(G)$ by A_0 . $\Phi(G) \cap \Omega_1(G)$ has order p^2 and by Theorem 1.6 equals $\Omega_1(\Phi(G))$. Since $|G'| \geq p^2$, it follows that $G' = \Omega_1(\Phi(G))$.

There exists a maximal subgroup M such that $\Omega_1(G) \not< M$. Then, $|\Omega_1(M)| = p^2$, which implies that M is metacyclic. Thus, $\upsilon_1(M) = \Phi(G)$. For each maximal subgroup M^* of M , $G' \leq \Phi(G) = \upsilon_1(M) < M^*$ so that $M^* \triangleleft G$. Hence, $M^*\Omega_1(G)$ is a maximal nonmetacylic subgroup of G . M has $p+1$ maximal subgroups in all, each of which determines a nonmetacyclic maximal subgroup of G . G then has at least $p+1$ nonmetacyclic maximal subgroups, of which at most one could be abelian. Thus, let R be a nonabelian, nonmetacyclic maximal subgroup of G . Since $\Omega_1(R) \not\leq \Phi(G) = \Phi(R)$, then R is of the form

$$R = < a,b | a^{p^m} = b^p = c^p = 1,\ c = [a,b],\ [a,c] = [b,c] = 1 > .$$

Suppose that the abelian maximal subgroup M_0 is metacyclic. Then, $\upsilon_1(M_0) = \Phi(R) = < a^p > \otimes < c >$. By Theorem 1.6, there exist elements x and a_1 in M_0 such that $x^p = c$ and $a_1^p = a^p$. However, $x \not\in R$ so that $M_0 = < x > \otimes < a_1 >$. Let $R^* = \Phi(G)< x >$. $R^* \triangleleft G$ and $(G:R^*) = p^2$. Also, $b \not\in R^*$, since R^* is a subgroup of the metacyclic group M_0. Thus, $R^*< b >$ is a maximal subgroup of G, is nonmetacyclic, and is nonabelian. In view of the fact that $b \not\in \Phi(G) = \Phi(R^*< b >)$, then $|\Omega_1(\Phi(R^*< b >))| = p^2$. Since $(R^*< b >)' \cap \upsilon_1(R^*< b >) = 1$, it follows that $\upsilon_1(R^*< b >)$ is cyclic. But $x^p \in \upsilon_1(R^*< b >)$ and $a^{p^2} \in \upsilon_1(R^*< b >)$ so that $x^p \in \Omega_1(\upsilon_1(R^*< b >)) = < a^{p^{m-1}} >$. Thus, $c \in < a^{p^{m-1}} >$, which is a contradiction unless $m = 1$. Consequently, $|R| = p^3$, $|G| = p^4$ and $|G'| \leq |\Phi(G)| = p$; a contradiction is reached. Therefore, the abelian maximal subgroup M_0 of G is nonmetacyclic.

Consider $\overline{G} = G/\upsilon_1(R)$, which is nonabelian of order p^4. $\overline{R}$ is a nonabelian maximal subgroup of $\overline{G}$. Since $|\upsilon_1(\overline{G})| = p$, then by the regularity of $\overline{G}$, it follows that $\Omega_1(\overline{G}) = \Omega_1(\overline{R}) = \overline{R}$. Let R^* be a nonmetacyclic, nonabelian maximal subgroup different from R. If $R^{*\prime} \leq \upsilon_1(R)$, then $\overline{R}^* \cap \overline{M}_0 = Z(\overline{G})$. Then, $\overline{A}_0 = Z(\overline{G})$. But, $\overline{A}_0 \leq \overline{R}$, so that $\overline{R}$ is abelian, which is a contradiction. Hence, $R^{*\prime} \not\leq \upsilon_1(R)$ and $\overline{R}^*$ is nonabelian. Moreover, $\Omega_1(\overline{R}^*) = \overline{R}^* \cap \Omega_1(\overline{G})$. Thus, $|\Omega_1(\overline{R}^*)| = p^2$, from which it follows that $\overline{R}^*$ is metacyclic. Therefore, there exists an element x in R^* such that $< \overline{x}^p > = \overline{R}^{*\prime} = Z(\overline{R}^*)$. Since $\overline{x} \not\in Z(\overline{R}^*)$, $\overline{x} \not\in Z(\overline{G})$. By Proposition 1.4(a), $C_{\overline{G}}(< \overline{x} >)$ is maximal in $\overline{G}$. But $< \overline{x} > \leq Z(C_{\overline{G}}(< \overline{x} >))$; thus, $C_{\overline{G}}(< \overline{x} >)$ is abelian. Since $\overline{R}^* = < \overline{x} >< \overline{b} >$, $\overline{b} \not\in C_{\overline{G}}(< \overline{x} >)$. However, $\overline{b} \in \overline{\Omega_1(G)}$, which implies that $C_{\overline{G}}(< \overline{x} >)$ is not the image of any nonmetacyclic maximal subgroup of G. There exists, then, a metacyclic maximal subgroup M such that $\overline{M} = C_{\overline{G}}(< \overline{x} >)$. Thus, $M' = < a^{p^{m-1}} >$.

From $G = M_0M$, it follows that $Z(G/M')$ has index p^2 and that $Z(G/M') \not< R_0/M'$. There exists a $y \in M$ such that $y \notin R$ and $y^p = c \in R'$. $G'< y >$ is abelian of type (2,1) but $y \notin Z(G)$. By Proposition 1.4(b), $C_G(G'< y >)$ is a maximal subgroup of G . $\Phi(G)< y > \leq Z(C_G(G'< y >))$, which implies that $C_G(G'< y >)$ is abelian, that is, $C_G(G'< y >) = M_0$. Thus, $y \in M_0 \cap M$. If α is the natural homomorphism α: $G \to G/M'$, then $y^\alpha \in Z(G/M')$. Furthermore, $[a^\alpha,y^\alpha] = 1$, $[b^\alpha,y^\alpha] = 1$, and $[c^\alpha,y^\alpha] = 1$. Since both b and y belong in M_0 , $[b,y] = 1$. $[a,y] \in M' = < a^{p^{m-1}} >$ so that $[a,y] = a^{rp^{m-1}}$, $r \not\equiv 0(\text{mod } p)$ (otherwise $G' = R_0'$) . Let s be such that $rs \equiv 1(\text{mod } p)$ and let $y_1 = y^s$. Then $y_1^p = y^{sp} = c^s$. Let $c_1 = c^s$ and let $b_1 = b^s$. Since $[a,b] = c \in G' \leq Z(G)$ and $[a,y] \in G' \leq Z(G)$, then by Proposition 1.5 $[a,b_1] = c_1$ and $[a,y_1] = a^{p^{m-1}}$. Thus,

$$G = S < b_1 | b_1^p = 1 > \text{ where } S = < a,y_1 | a^{p^m} = y_1^{p^2} = 1, [a,y_1] = a^{p^{m-1}} > ,$$

$$[a,b_1] = y_1^p \text{ and } [b_1,y_1] = 1 .$$

This is the group of type (10) in Theorem 3.1.

Case 2: $\Omega_1(G) \leq \Phi(G)$. Then, each maximal subgroup of G is nonmetacyclic. From the regularity of G , $(G: \mho_1(G)) = p^3$ and $\Phi(G) = \mho_1(G)$. Let R_0 and R_1 be two nonabelian maximal subgroups, with $R_0' \neq R_1'$. Two such subgroups exist by (3.6). Consider $\overline{G} = G/R_0'$. This factor group has two abelian maximal subgroups $\overline{R}_0$ and $\overline{M}_0$, so that $Z(\overline{G})$ has index p^2 in $\overline{G}$. It follows that $\overline{G}$ has at least p+1 abelian maximal subgroups. Let A_1 be a maximal subgroup of R_1 .

$R_0' \leq \Phi(G) = \Phi(R_1) \leq A_1$, and $Z(\overline{G})\,\overline{A}_1$ is abelian maximal in $\overline{G}$. Thus, each maximal subgroup of R_1 determines an abelian maximal subgroup of $\overline{G}$. Conversely, if $\overline{M}$ is an abelian maximal subgroup of $\overline{G}$, then $Z(\overline{G}) \leq \overline{M}$ (otherwise, $\overline{G}$ would be abelian). $\overline{M} \cap \overline{R}_1$ is maximal in $\overline{R}_1$, which is nonabelian and Redei. Thus, $(\overline{M} \cap \overline{R})Z(\overline{G}) = Z(\overline{G})\overline{R}_1 \cap \overline{M} = \overline{G} \cap \overline{M} = \overline{M}$. Hence, there is a one-to-one correspondence between the maximal abelian subgroups of $\overline{G}$ and the maximal subgroups of R_1 , that is, $\overline{G}$ has exactly $p+1$ abelian maximal subgroups. Therefore, G has exactly p nonabelian maximal subgroups with the commutator subgroup R_0' . In view of the fact that R_0 is arbitrary, the nonabelian subgroups of G are separated into classes where the subgroups in each class all have the same commutator subgroup. Moreover, there are p subgroups in each class. Since there are p^2+p+1 maximal subgroups [15, p. 311] of which p^2+p are nonabelian, there are $p+1$ distinct classes, that is, $p+1$ distinct commutator subgroups.

Let N_0 be a nonabelian maximal subgroup. $N_0 \cap M_0$ has index p^2 in G and is normal in G . There exists a $y \in N_0 \sim M_0$ such that $y^p \neq 1$. $N_0 = (M_0 \cap N_0)\langle y \rangle$ and $G = M_0\langle y \rangle$. Also, there exists an $x \in M_0 \sim N_0$ such that $x^p \neq 1$, $M_0 = (M_0 \cap N_0)\langle x \rangle$ and $G = N_0\langle x \rangle$. If $[x,y] = 1$, then $x \in Z(G) = \Phi(G) \leq N_0$, which is a contradiction. Thus, $[x,y] \neq 1$. Now, $N_0'\langle [x,y] \rangle \leq G'$ and $|N_0'\langle [x,y] \rangle| \leq p^2$. Since $G' \leq Z(G)$, then $N_0'\langle [x,y] \rangle \lhd G$. Let α be the natural mapping $\alpha\colon G \to G/N_0'\langle [x,y] \rangle$. $G^\alpha = G/N_0'\langle [x,y] \rangle$. $G^\alpha = M_0^\alpha\langle y^\alpha \rangle = ((M_0 \cap N_0)^\alpha\langle x^\alpha \rangle)\langle y^\alpha \rangle$. But, $(M_0 \cap N_0)^\alpha\langle y^\alpha \rangle = N_0^\alpha$, which is abelian. This implies that $\langle y^\alpha \rangle \leq C_{G^\alpha}(M_0 \cap N_0)^\alpha$. Also, since $[x,y] \in N_0'\langle [x,y] \rangle$, then $\langle y^\alpha \rangle \leq C_{G^\alpha}(\langle x^\alpha \rangle)$. Thus, $\langle y^\alpha \rangle \leq C_{G^\alpha}((M_0 \cap N_0)^\alpha\langle x^\alpha \rangle) = C_{G^\alpha}(M_0^\alpha)$. Since M_0^α is abelian, it follows that $G^{\alpha} = M_0^{\alpha} < y^{\alpha} >$ is abelian, that is, $G' \leq N_0' < [x,y] >$. In addition, since $|G'| \geq p^2$, then $[x,y] \not\leq N_0'$ and $G' = N_0'[x,y]$.

Since G is regular, then by Theorem 1.6, each element of $\upsilon_1(G)$ is the p^{th} power of an element of G. Since $\Phi(N_0) = \upsilon_1(G)$, there exists a $z \in G$ such that $< z^p > = N_0'$. However, $N_0' \cap \upsilon_1(N_0) = 1$ whereby $z \not\in N_0$. Now, $G' < z >$ is normal in G, is abelian of type (2,1) but is not contained in $Z(G) = \Phi(G)$. Thus, $C_G(G' < z >)$ is a maximal subgroup of G by Proposition 1.4(b). Furthermore, $Z(G) < z >$ has index p in $C_G(G' < z >)$ and $Z(G) < z > \leq Z(C_G(G' < z >))$. These imply that $C_G(G' < z >)$ is abelian. Thus, $M_0 = C_G(G' < z >)$; and accordingly, $N_0' \leq \upsilon_1(M_0)$. From the fact that N_0 is an arbitrary nonabelian subgroup of G, it follows that $G' \leq \upsilon_1(M_0)$.

For each subgroup Z_i (for $i = 1,2,\cdots,p+1$) of G' of order p, there exists, by the argument used in the paragraph above, an element $z_i \in M_0 \sim \Phi(G)$ such that $z_i^{p^2} = 1$, and such that $Z_i = < z_i^p >$. Let $M_0^* = < z_i | i = 1,2,\cdots,p+1 >$. Then, M_0^* is abelian but is not contained in $\Phi(G)$. The exponent of M_0^* is p^2, and $G' = \Omega_1(M_0^*) = \upsilon_1(M_0^*)$. Therefore, from the regularity of G, $(M_0^* : \upsilon_1(M_0^*)) = p^2$, $|M_0^*| = p^4$ and M_0^* has exactly $p+1$ maximal subgroups. Each maximal subgroup of M_0^* corresponds to a distinct commutator subgroup of a nonabelian maximal subgroup of G; that is, $G' < z_i >$, for each $i = 1,2,\cdots,p+1$, is a maximal subgroup of M_0^*. If $\Phi(G)M_0^*$ is properly contained in M_0, then $\Phi(G) \cap M_0^*$ is a maximal subgroup of M_0^*. But then, $\Phi(G) \cap M_0^* = G' < z_i >$, where $< z_i^p > = Z_i \leq G'$ for some $i = 1,2,\cdots,p+1$. Consequently, $z_i \in \Phi(G)$, which is a contradiction. Thus, $\Phi(G)M_0^* = M_0$, and M_0^* is contained in no nonabelian maximal subgroup of G. Then, for a nonabelian maximal subgroup N_0 , $G = N_0M_0^*$. $(M_0^*: N_0 \cap M_0^*) = p$, so that $N_0 \cap M_0^*$ is a maximal subgroup of M_0^* . This implies that $N_0 \cap M_0^* = G'<z>$ for some $z \in M_0 \sim \Phi(G)$ of order p^2 . Hence, $z \in N_0$, and $<z^p> = N_1'$ for some nonabelian maximal subgroup N_1 different from N_0 .

Suppose that $N_0 = <a,b \mid a^{p^m} = b^{p^n} = c^p = 1,\ c = [a,b],\ [a,c] = [b,c] = 1>$. Since $\Omega_1(G) = \Omega_1(N_0) \leq \Phi(N_0)$, then both $m, n \geq 2$. Let $z = a^r b^s c^t$. $z \notin \Phi(G)$, so that at least one of $r, s \not\equiv 0 (\mathrm{mod}\ p)$; say $s \not\equiv 0(\mathrm{mod}\ p)$. $1 = z^{p^2} = (a^r b^s)^{p^2}$. By Definition 1.4, $(a^r b^s)^{p^2} = a^{rp^2} b^{sp^2} \prod_i d_i^{p^2}$ where $d_i \in <a,b>'$. Thus, $(a^r b^s)^{p^2} = a^{rp^2} b^{sp^2} = 1$, from which it follows that $a^{rp^2} = b^{-sp^2}$. Then, $r \equiv 0(\mathrm{mod}\ p^{m-2})$ and $s \equiv 0(\mathrm{mod}\ p^{n-2})$. But then, $s \equiv 0(\mathrm{mod}\ p)$, which is a contradiction unless $n = 2$. Therefore, $z = a^{r^* p^{m-2}} b^s c^t$ and $b = (a^{-r^* p^{m-2}} z c^{-t})^{-s}$ for some r^* . Let $c_1 = [a,z] = c^s$. It follows that

$$N_0 = <a,z \mid a^{p^m} = z^{p^2} = c_1^p = 1,\ c_1 = [a,z],\ [a,c_1] = [z,c_1] = 1> \quad .$$

By the argument used earlier in the proof, there exists an element $\omega \in M_0 \sim N_0$ for which $\omega^{p^2} = 1$ and $\omega^p = c_1$. Then,

$$G = N_0 <\omega \mid \omega^{p^2} = 1> \quad .$$

Since both z and ω are in M_0 ,

$$[z,\omega] = 1 \ .$$

Now, $[\omega,a] \in G' = <z^p, c_1>$ but $[\omega,a] \notin <c_1>$ (otherwise $<\omega> \triangleleft G$ and $<\omega><a>$ is a nonabelian subgroup of index p^2 in G).

Let

$$[\omega,a] = c_1^u z^{vp} \quad \text{where} \quad v \neq 0 .$$

If $u \not\equiv 0 (\mathrm{mod}\ p)$, then there exists an x such that $xu \equiv 1 (\mathrm{mod}\ p)$. Let

$$a_1 = a^x \quad \text{and} \quad z_1 = z^u .$$

Then $[a_1,z_1] \in N_0' \leq Z(G)$, so that by Proposition 1.5, $[a_1,z_1] = c_1$. Then

$$N_0 = \langle\, a,z_1 | a_1^{p^m} = z_1^{p^2} = c_1^p = 1,\ c_1 = [a_1,z_1],\ [a_1,c_1] = [z_1,c_1] = 1 \,\rangle .$$

Also, $[\omega,a_1] \in G' \leq Z(G)$, which implies that $[\omega,a_1] = c_1 z_1^{x^2 vp}$. Hence,

(3.8)
$$G = N_0 \langle\, \omega | \omega^{p^2} = 1 \,\rangle \quad \text{where} \quad \omega^p = c_1 \quad \text{and}$$
$$[\omega,a_1] = c_1 z^{kp} \quad \text{for} \quad k \not\equiv 0 (\mathrm{mod}\ p) .$$

If $u = 0$, then

(3.9)
$$G = N_0 \langle\, \omega | \omega^{p^2} = 1 \,\rangle \text{ where } N_0 = \langle\, a,z | a^{p^m} = z^{p^2} = c^p = 1,\ c = [a,z] \,\rangle ,$$
$$\omega^p = c \quad \text{and} \quad [\omega,a] = z^{rp} \quad \text{for} \quad r \not\equiv 0 (\mathrm{mod}\ p) .$$

The groups in (3.8) and (3.9) are the groups of type (10) in Theorem 3.1.

Proof of THEOREM 3.1: Let M_0 be the abelian maximal subgroup and let R_0 be the Redei maximal subgroup. By Proposition 1.1,

$Z(R_0) \leq \Phi(G) \leq M_0$. It follows from $G = R_0M_0$ that $Z(R_0) \leq Z(G)$. But $Z(G) \leq Z(R_0)$, since $G \in \mathcal{J}$. Thus, $Z(G) = Z(R_0)$ and $(G: Z(G)) = p^3$. Furthermore, by Proposition 3.3, each Redei subgroup R is a maximal subgroup and $Z(R) = Z(G)$.

If G is metacyclic, then by Theorem 2.4, $G \in \mathcal{R}^*$; so, each maximal subgroup is a Redei group. This is a contradiction to the fact that M_0 is abelian. Hence, G is not a metacyclic group.

Two possible situations arise. Either each maximal subgroup of G is metacyclic or there is at least one nonmetacyclic maximal subgroup. The case where there is a nonmetacyclic maximal subgroup has been examined in Lemmas 3.1 through 3.7 , and the groups have been characterized.

For the case where each maximal subgroup is metacyclic, by Theorem 1.10, it follows that G is a 3-group of class 3 and order 3^4 . By Theorem 1.9 $(G: \upsilon_1(G)) = p^3$. Thus, $\upsilon_1(G) = Z(G) = Z(R) = \upsilon_1(R)$ for each Redei subgroup. Also, since the class of G is 3 , $|G'| = 3^2$.

If there exists an element $x \notin G \sim G'$ such that $x^p = 1$, then there exists a maximal subgroup M such that $x \notin M$. $|\Omega_1(M)| = 3^2$. Thus, $|\Omega_1(M)< x >| = 3^3$. But $\Omega_1(\Omega_1(M)< x >) = \Omega_1(M)< x >$, so that $\Omega_1(M)< x >$ is not metacyclic, which is a contradiction. Hence, $G' = \Omega_1(G)$. Then, $\upsilon_1(G) = < a^3 >$ and $G' = < a^3,b >$. There is an element $c \in M_0 \sim R_0$ such that $c^{3^2} = 1$. Then,

$$G = R_0< c|c^{3^2} = 1 > , [b,c] = 1 \text{ and } c^3 = a^{3r} \text{ for } r \not\equiv 0(\text{mod } 3) .$$

Now, $[a,c] = b^sa^{3t}$ where $s \not\equiv 0(\text{mod } 3)$. There exists s^* such that

$ss^* \equiv 1 (\mathrm{mod}\ 3)$. Let

$$c_1 = c^{s^*} \quad \text{and} \quad b_1 = bx^{3st} \ .$$

Then, by Proposition 1.5, $[a,c_1] = b_1$. Also, $[a,b_1] = a^3$, and $[b_1,c_1] = 1$. Hence,

$$G = R_0 < c_1 | c_1^{3^2} = 1 > \text{ where } R_0 = < a,b_1 | a^{3^2} = b^3 = 1, \ [a,b_1] = a^3 > \ ,$$

$$[c_1,b_1] = 1, \ [a,c_1] = b_1 \ \text{ and } \ c_1^3 = a^{3u} \ \text{ where } \ u \not\equiv 0 (\mathrm{mod}\ 3) \ .$$

This is the group of type (1) in Theorem 3.1.

Each of the eleven types has been constructed from a specific case which relates the characteristic subgroup $\Omega_1(G)$ to the characteristic subgroup $\Phi(G)$. It is clear, then, that these groups are pairwise nonisomorphic.

It should be noted that the group of type (2) in Theorem 3.1 is the group $C_3 \wr C_3$ of Example 1.1.

CHAPTER IV

ALL SUBGROUPS OF A FIXED INDEX ARE REDEI

In this chapter the groups in $\mathcal{Z}^* \sim \mathcal{R}$ with a Redei maximal subgroup are characterized. It follows as a consequence of this characterization that class $\mathcal{R}$ is "almost equal" to class $\mathcal{Z}$. Furthermore, for $p > 2$ a second characterization is found for the nonabelian metacyclic groups.

THEOREM 4.1: If $G \in \mathcal{Z}^* \sim \mathcal{R}$ and if G has a Redei maximal subgroup, then

(a) G has a nonmetacyclic maximal subgroup,

(b) $Z(G) = \upsilon_1(G)$,

(c) G' is elementary abelian of order p^3 ,

(d) $(G: \Phi(G)) = p^2$,

(e) no two Redei subgroups have the same commutator subgroup.

Proof: Since G has a Redei maximal subgroup, then by Definition 1.3, each maximal subgroup of G is Redei. Since $G \notin \mathcal{R}$, there exists a Redei subgroup R_0 such that $R_0 < C_G(Z(R_0))$. Then, $G = C_G(Z(R_0))$ and $Z(R_0) \leq Z(G)$. But Proposition 1.3 implies that $Z(G) \leq Z(R_0)$. Thus, $Z(G)$ has index p^3 in G . In addition, it follows that $Z(G) = Z(R)$ for each Redei subgroup R of G .

If all the maximal subgroups of G are metacyclic, then by Theorem 1.10 either G is itself metacyclic or $|G| = 3^4$. However, if G is metacyclic, then by Theorem 2.4, $G \in \mathcal{R}^*$, which is a contradiction to $G \not\in \mathcal{R}$. If, on the other hand, $|G| = 3^4$, then G has an abelian maximal subgroup by Proposition 1.4(c), which is a contradiction to $G \in \mathcal{J}^*$. Hence, G has at least one nonmetacyclic maximal subgroup, say N_0 . Since N_0 is Redei, then $(N_0: \upsilon_1(N_0)) = p^3$ and $N_0' \cap \upsilon_1(N_0) = 1$.

To show that $Z(G) = \upsilon_1(G)$, first suppose that $\upsilon_1(G) = \upsilon_1(N_0)$. Since each maximal subgroup M is a Redei group and since $(M: \upsilon_1(M)) \leq p^3$, it then follows that $\upsilon_1(G) = \upsilon_1(M)$. Thus, $(M: \upsilon_1(M)) = p^3$ for each maximal subgroup M . Also, $N_0' \cap \upsilon_1(G) = 1$, which implies that $G/\upsilon_1(G)$ is nonabelian. By Proposition 1.4(c), $G/\upsilon_1(G)$ has an abelian maximal subgroup $M_0/\upsilon_1(G)$. Then, $M_0' \leq \upsilon_1(G) = \upsilon_1(M_0)$; consequently, $\Phi(M_0) = \upsilon_1(M_0)$, and $(M_0: \Phi(M_0)) = p^3$. This is a contradiction to the fact that M_0 is a Redei subgroup. Therefore, $\upsilon_1(N_0) < \upsilon_1(G)$. Moreover, for each $x \in G$, there exists a maximal subgroup M such that $x \in M$. Then $x^p \in \upsilon_1(M) \leq \Phi(M) = Z(M) = Z(G)$. Thus, $\upsilon_1(G) \leq Z(G)$. Since $(Z(G): \upsilon_1(N_0)) = p$, then $\upsilon_1(G) = Z(G)$.

Now $\upsilon_1(N_0)$, as a characteristic subgroup of N_0 , is normal in G . $G/\upsilon_1(N_0)$ is nonabelian of order p^4 , but by Proposition 1.4(c), $G/\upsilon_1(N_0)$has an abelian maximal subgroup $M_0/(\upsilon_1(N_0))$. Since $N_0' \cap \upsilon_1(N_0) = 1$, it follows that $N_0'M_0'$ has order p^2 . Denote $N_0'M_0'$ by A . A is normal in G since both N_0' and M_0' are characteristic subgroups of N_0 and M_0, respectively. G/A has at least two abelian maximal subgroups N_0/A and M_0/A . Also, $Z(G/A)$ has index p^2 in G/A .

If G/A is abelian, then $G' = A$. But $A \leq Z(G)$ which implies that the class of G is 2 . By Theorem 1.7(a), G is regular. Then, by Theorem 1.6(b), each element in $\upsilon_1(G)$ is the p^{th} power of an element in G . In particular, since $N_0' \leq \upsilon_1(G) \sim \upsilon_1(N_0)$, there exists an $x \in G \sim N_0$ such that $N_0' = \langle x^p \rangle$. In view of the fact that $G' \leq Z(G)$,

$G'\langle x\rangle \triangleleft G$; then $G'\langle x\rangle$ is abelian of type (2,1). By Proposition 1.4(b), $C_G(G'\langle x\rangle)$ has index at most p in G. Now, since $Z(G) = Z(N_0)$, then $x \notin Z(G)$; therefore, $C_G(G'\langle x\rangle)$ is a maximal subgroup of G. However, $Z(G)\langle x\rangle \leq C_G(G'\langle x\rangle)$ with index p. It follows that $C_G(G'\langle x\rangle)$ is an abelian maximal subgroup, which is a contradiction. Thus,

(4.1) $\quad G/A$ is nonabelian and $|G'| \geq p^3$.

Suppose that $(G\colon \Phi(G)) = p^3$, that is, $\Phi(G) = \mho_1(G) = Z(G)$. Then, $G' \leq Z(G)$ and the class of G is 2. By Theorem 1.7(a), G is a regular group.

Let $\bar{A}$ be a subgroup of G' minimal with the property that it contains the commutator subgroup of each of the maximal subgroups. $A \leq \bar{A} \leq G'$. $\bar{A} \leq Z(G)$ so that $\bar{A} \triangleleft G$. Each maximal subgroup of $G/\bar{A}$ is abelian, which implies that $G/\bar{A}$ is either an abelian group or a Redei group. If $G/\bar{A}$ is a Redei group, then $\Phi(G/\bar{A})$ has index p^2 in $G/\bar{A}$. But, $(G\colon \Phi(G)) = (G/\bar{A}\colon \Phi(G)/\bar{A}) = (G/\bar{A}\colon \Phi(G/\bar{A}))$. This is a contradiction to $(G\colon \Phi(G)) = p^3$. Thus, $G/\bar{A}$ is abelian and $\bar{A} = G'$.

Since G/A is nonabelian, $A < \bar{A}$. Therefore, there exists a maximal subgroup R of G such that $R' \not\leq A$. Since $G' < Z(G)$, then AR' is elementary abelian of order p^3. Also, $AR' \leq N_0$, which implies that $AR' \leq \Omega_1(N_0)$. Let S be a maximal subgroup of G different from R, N_0, and M_0. $AR'S' \leq \bar{A} < N_0$ and $AR'S'$ is elementary abelian so that $AR'S' \leq \Omega_1(N_0)$. However, $|\Omega_1(N_0)| = p^3$ since N_0 is nonmetacyclic. Therefore, $AR'S' = AR'$. Since S is any maximal subgroup, AR' is a minimal subgroup of G' containing the commutator subgroup of each maximal subgroup. Hence, $AR' = \bar{A} = G'$. Thus, $|G'| = p^3$ and G'

is elementary abelian.

$G' < M$ for each maximal subgroup so that $G' \leq \Omega_1(M) \leq \Omega_1(G)$. However, since G is a regular group and since $(G\colon \mho_1(G)) = p^3$, then $|\Omega_1(G)| = p^3$. Thus, $G' = \Omega_1(M) = \Omega_1(G)$. It follows, therefore, that $(M\colon \mho_1(M)) = p^3$ and that M is nonmetacyclic. Each maximal subgroup, then, is a nonmetacyclic subgroup.

The fact that $G/\Phi(G)$ is elementary abelian of order p^3 implies that $G/\Phi(G)$ has p^2+p+1 maximal subgroups [15, p. 311]. Thus, G has p^2+p+1 maximal subgroups. If $M' = N'$ for two distinct maximal subgroups M and N , then by Theorem 1.6(b), there exists an $x \in G$ such that $M' = \langle x^p \rangle$. Both M and N are nonmetacyclic, which implies that $x \notin M$ and $x \notin N$; in particular, $x \notin Z(G) \leq M \cap N$. However, $\langle x \rangle \lhd M$ and $\langle x \rangle \lhd N$. It follows that $\langle x \rangle \lhd G$. By Proposition 1.4(b), $C_G(\langle x \rangle)$ is a maximal subgroup of G . However, $Z(G)\langle x \rangle \leq Z(C_G(\langle x \rangle))$ and $(C_G(\langle x \rangle)\colon Z(G)\langle x \rangle) = p$, whereby $C_G(\langle x \rangle)$ is an abelian maximal subgroup of G . This is a contradiction to $G \in \mathcal{J}^*$. Hence, no two maximal subgroups of G have the same commutator subgroup.

Now, G/N_0' has precisely one abelian maximal subgroup, and p^2+p nonabelian maximal subgroups. Let M_0 be a maximal subgroup different from N_0 . $M_0' \leq \mho_1(G)$, so there exists, by Theorem 1.6, an element $x \in G$ such that $\langle x^p \rangle = M_0'$. In addition, M_0 is non-metacyclic which implies that $x \notin M_0$. Denote G/N_0' by $\overline{G}$. $\overline{x}$ has order p^2 since $M_0' \neq N_0'$. Moreover, $\overline{G}'$ has order p^2 and $\overline{G}'\langle \overline{x} \rangle$ has order p^3 . Since $G' \leq Z(G)$, it follows that $\overline{G}'\langle \overline{x} \rangle$ is abelian of type (2,1). By Proposition 1.4(b), $C_{\overline{G}}(\overline{G}'\langle \overline{x} \rangle)$ has index at most p . If $\overline{G}'\langle \overline{x} \rangle \leq Z(\overline{G})$, then $Z(\overline{G}) = \overline{Z(G)\langle x \rangle}$, which has index p^2 in $\overline{G}$.

This implies that $Z(\overline{G})$ is contained in no maximal subgroup other than $\overline{N}_0$. Let $\overline{y}$ be any element of $\overline{G} \sim Z(\overline{G})$. If $\overline{y}^p = 1$, then $Z(\overline{G})<\overline{y}>$ is maximal in $\overline{G}$; so $Z(\overline{G})<\overline{y}> = \overline{N}_0$. If $\overline{y}^p \neq 1$, then since $\Phi(\overline{G}) \leq Z(\overline{G})$, $Z(\overline{G})<\overline{y}>$ is maximal in $\overline{G}$, and $Z(\overline{G})<\overline{y}> = \overline{N}_0$. Thus $\overline{G} = \overline{N}_0$, which is a contradiction. Hence, $\overline{G}'<\overline{x}> \not\leq Z(\overline{G})$ from which it follows that $C_{\overline{G}}(\overline{G}'<\overline{x}>)$ is maximal in $\overline{G}$. However, $Z(\overline{G})<\overline{x}> \leq Z(C_{\overline{G}}(\overline{G}'<\overline{x}>))$ and $(C_{\overline{G}}(\overline{G}'<\overline{x}>):Z(\overline{G})<\overline{x}>) = p$. Consequently, $C_{\overline{G}}(\overline{G}'<\overline{x}>)$ is abelian. This implies that $x \in N_0$, that is, $M_0' \leq \mho_1(N_0)$. Since M_0 is arbitrary, $M' \leq \mho_1(N_0)$ for each of the p^2+p maximal subgroups different from N_0.

Since $N_0' \cap \mho_1(N_0) = 1$, $\Omega_1(\mho_1(N_0))$ is elementary abelian of order p^2. Then, $\Omega_1(\mho_1(N_0))$ has $p+1$ subgroups of order p. Therefore, G has at most $p+2$ distinct commutator subgroups associated with the maximal subgroups. This contradicts the fact that G has p^2+p+1 maximal subgroups with distinct commutator subgroups when $(G: \Phi(G)) = p^3$. Hence, $(G: \Phi(G)) = p^2$, which proves part (d).

From (4.1), it now follows that G/A is a Redei group, that is, $|G'| = p^3$. Since $\mho_1(G) = Z(G)$ and since $(\Phi(G): \mho_1(G)) = p$, then $G' \not< Z(G)$. However, G' is abelian. If G' is abelian of type (2,1), then by Proposition 1.4(b), $C_G(G')$ is a maximal subgroup of G. But then, $Z(G)G' \leq Z(C_G(G'))$ and $(C_G(G'): Z(G)G') = p$. These force $C_G(G')$ to be an abelian maximal subgroup, which is a contradiction. Hence G' is elementary abelian of order p^3.

For the last part of the theorem let M and N be any two maximal subgroups such that $M' = N'$. Then G/N' has two abelian maximal subgroups M/N' and N/N'. It follows that $Z(G/N')$ has index p^2 in G/N' and that $Z(G/N') = \Phi(G/N')$. Therefore, each maximal subgroup of G/N' is abelian, whence G/N' is itself abelian or Redei. In either case, $|G'| \leq p^2$, which is a contradiction. Hence, no two maximal subgroups have the same commutator subgroup.

THEOREM 4.2: If $G \in \mathcal{J} \sim \mathcal{R}$, if $|G| \geq p^6$ for $p > 3$ and if each subgroup of index p^2 is abelian, then G has an abelian maximal subgroup.

Proof: Since $G \notin \mathcal{R}$, G is not itself a Redei group. There exists a Redei subgroup $R_0 < G$ such that $R_0 < C_G(Z(R_0))$. By hypothesis, R_0 is a maximal subgroup of G and $G = C_G(Z(R_0))$. It follows, therefore, that $Z(R_0) = Z(G)$. By Proposition 3.1, G has at most one abelian maximal subgroup.

Suppose that G has no abelian maximal subgroups, that is, $G \in \mathcal{J}^*$. By Theorem 4.1, G has a nonmetacyclic maximal subgroup N_0, $|G'| = p^3 = (G: Z(G))$, $Z(G) = \upsilon_1(G)$, $(G: \Phi(G)) = p^2$, and G' is elementary abelian. Now, $G' \nleq Z(G)$, but $(G': G' \cap Z(G)) = (G'Z(G): Z(G)) = (\Phi(G): Z(G)) = p$. Therefore, $G_3 = [G,G'] \leq Z(G)$ and the class of G is 3. Thus, G is a regular group by Theorem 1.7(a). By Theorem 1.6, $|\Omega_1(G)| = (G: \upsilon_1(G)) = p^3$. Hence, $G' = \Omega_1(G)$. Also by Theorem 1.6, each element of $\upsilon_1(G)$ is the p^{th} power of an element of G.

Let M be any maximal subgroup of G different from N_0. Since $M' \leq \Phi(M) = Z(M) = Z(G) = \upsilon_1(G)$, there exists an $x \in G$ such that $M' = < x^p >$. $|G'< x >| = p^4$, which by hypothesis, implies that $G'< x >$ is an abelian subgroup of G. Denote G/N_0' by $\overline{G}$. By Theorem 4.1(e), $\overline{N}_0$ is the only abelian maximal subgroup of $\overline{G}$.

Then $\overline{G'\langle x \rangle}$ is an abelian subgroup of $\overline{G}$ of order p^3 and of type (2,1). By Proposition 1.4(b), $(\overline{G}: C_{\overline{G}}(\overline{G'\langle x \rangle})) \leq p$. If $\overline{G'\langle x \rangle} \leq Z(\overline{G})$, then $\overline{G'\langle x \rangle} \leq \overline{N}_0$. If, on the other hand, $\overline{G'\langle x \rangle} \not\leq Z(\overline{G})$, then $C_{\overline{G}}(\overline{G'\langle x \rangle})$ is a maximal subgroup of G . It follows from $(C_{\overline{G}}(\overline{G'\langle x \rangle}) : \overline{G'Z(G)}) = p$ that $C_{\overline{G}}(\overline{G'\langle x \rangle})$ is abelian. Thus, $\overline{G'\langle x \rangle}$ is always contained in $\overline{N}_0$. Consequently, $x \in N_0$ and $M_0' \leq \upsilon_1(N_0)$. Since M is an arbitrary maximal subgroup different from N_0 , then the commutator subgroup of each nonabelian subgroup different from N_0 is contained in $\upsilon_1(N_0)$.

Now, $\upsilon_1(N_0) \lhd G$ and $|G/\upsilon_1(N_0)| = p^4$. $G/\upsilon_1(N_0)$ has the nonabelian maximal subgroup $N_0/\upsilon_1(N_0)$. However, every other maximal subgroup of $G/\upsilon_1(N_0)$ is abelian. Hence $G/\upsilon_1(N_0)$ has precisely one Redei subgroup, which contradicts Theorem 1.1. Therefore, it is impossible for each maximal subgroup to be nonabelian. So, $G \in \mathcal{J} \sim \mathcal{J}^*$, that is, G has an abelian maximal subgroup.

The above theorem shows that, for $p > 3$, the minimal non-Redei groups of class $\mathcal{J} \sim \mathcal{R}$ with order greater than p^5 are actually groups of class $\mathcal{J} \sim \mathcal{J}^*$. These groups are included in Theorem 3.1. To obtain a complete classification of the groups in $\mathcal{J} \sim \mathcal{R}$ with each subgroup of index p^2 abelian, there remains for $p > 3$ only the classification of the groups in $\mathcal{J}^* \sim \mathcal{R}$ of order less than or equal to p^5 . From Proposition 1.4(c), it follows that only groups of order p^5 must be considered.

THEOREM 4.3: If $|G| = p^5$ for $p > 3$ and if $G \in \mathcal{J}^* \sim \mathcal{R}$, then $$G = M_0 < c \mid c^{p^2} = 1 >$$

(4.2) where $M_0 = < a,b \mid a^{p^2} = b^p = 1,\ d = [a,b] >$,

$[a,c] = b,\ c^p = d^s$ for $s \not\equiv 0 \pmod p$ and $[b,c] = a^{rp}c^{tp}$

where r is 1 or a quadratic nonresidue mod p .

Proof: By hypothesis, G is not Redei. By Proposition 1.4(c), G has at least one abelian subgroup of order p^3 . Thus by Definition 1.4, each subgroup of order p^3 is abelian and each subgroup of order p^4 is Redei. By Theorem 4.1, G has a nonmetacyclic maximal subgroup N_0 , G' is elementary abelian of order p^3 , $Z(G) = \upsilon_1(G)$, and G/N_0' has precisely one abelian maximal subgroup N_0/N_0' . $G/N_0' \notin \mathcal{Z}^*$, but by Proposition 3.2, $G/N_0' \in \mathcal{Z}$.

Now, $G_3 = [G,G'] \leq Z(G)$ which implies that G has class 3 . Thus, by Theorem 1.7(a), G is regular. But then, from Theorem 1.6, it follows that $G' = \Omega_1(G)$. Also, since $N_0' \leq Z(N_0) = Z(G) = \upsilon_1(G)$, there exists an $x \in G \sim N_0$ such that $< x^p > = N_0'$ (Theorem 1.6). Let $\overline{G} = G/N_0'$, which has order p^4 . Now $\overline{x} \in \Omega_1(\overline{G})$, and $\overline{G'< x >} \leq \Omega_1(\overline{G})$. Moreover, since $\overline{G}$ is regular, then $|\Omega_1(\overline{G})| = (\overline{G}\colon \upsilon_1(\overline{G})) = p^3$. Thus, $\Omega_1(\overline{G})$ is a maximal subgroup of G . Since $x \notin N_0$, $\Omega_1(\overline{G})$ is not abelian. By Lemma 3.4, $\overline{G}$ is then the group of type (7) in Theorem 3.1. Hence,

$\overline{G} = \overline{N}_0 < \overline{c} \mid \overline{c}^p = 1 >$ where $\overline{N}_0 = < \overline{a},\overline{b} \mid \overline{a}^{p^2} = \overline{b}^p = [\overline{a},\overline{b}] = 1 >$
$[\overline{a},\overline{c}] = \overline{b}$ and $[\overline{b},\overline{c}] = \overline{a}^{rp}$ for $r = 1$ or a quadratic nonresidue mod p .

There exist elements a,b from N_0 such that $aN_0' = \overline{a}$ and $bN_0' = \overline{b}$, and there exists an element $c \in G \sim N_0$ such that $cN_0' = \overline{c}$. Let $d = [a,b]$. Then $c^p \in \langle d \mid d^p = 1 \rangle = N_0'$. Since $c \notin G' = \Omega_1(G)$, $c^p \neq 1$. Thus $c^p = d^s$ where $s \not\equiv 0 (\text{mod } p)$.

Now, $[a,c] = bd^u$ for some u . Let $b_1 = bd^u$. Then $\overline{b}_1 = \overline{b}$ and $[a,b_1] = d$. Since $[\overline{b},\overline{c}] = \overline{a}^{rp}$, then $[b_1,c] = a^{rp}d^t = a^{rp}c^{tp}$ for some t .

The groups described in Theorem 4.3 provide examples which show that class $\mathcal{J}^*$ properly contains class $\mathcal{R}$. The next few theorems lead to the conclusion that for $p > 3$, these groups are the only groups in $\mathcal{J}^*$ which are not in $\mathcal{R}$.

THEOREM 4.4: If $p > 3$ and if $G \in \mathcal{J}^* \sim \mathcal{R}$, then $|G| \geq p^5$ and each Redei subgroup has order p^4 .

Proof: Since $G \in \mathcal{J}^* \sim \mathcal{R}$, there exists an integer $i \geq 1$ such that each subgroup of index p^i is a Redei group. By Proposition 1.4(c), $|G| \geq p^5$.

Suppose that G has an abelian subgroup of order p^4 . Then, each subgroup of order p^4 is abelian. Since G is not a Redei group, it follows that $|G| \geq p^6$. Let R_0 be a Redei subgroup of G . $|R_0| \geq p^5$. There exists a subgroup R^* of G such that $R_0 < R^*$ and $(R^*: R) = p$. $\mathcal{J}^*$ is a subgroup inherited class, so that by Theorem 4.2, R^* has an abelian maximal subgroup, which is a contradiction to R^* non-Redei. Thus, G has no abelian subgroup of order p^4 .

THEOREM 4.5: If $G \in \mathcal{J} \sim \mathcal{R}$ and if $|G| \geq p^6$ for $p > 3$, then $G \notin \mathcal{J}^*$.

Proof: Suppose that $G \in \mathcal{Z}^* \sim \mathcal{R}$. By Theorem 4.4, each Redei subgroup of G has order p^4 . Each subgroup of G of order p^6 also belongs to $\mathcal{Z}^*$. In particular, by Theorem 2.2 $G \not\in \mathcal{R}$. Therefore, if a contradiction is reached for a subgroup of order p^6 , then a contradiction is also reached for G . Without loss of generality, it may thus be assumed that $|G| = p^6$.

Since each subgroup of order p^4 is Redei, then there is at least one normal Redei subgroup R_0 . $Z(R_0)$, as a characteristic subgroup of R_0 , is normal in G; $G/Z(R_0)$ has order p^4 . By Proposition 1.4(c), $G/Z(R_0)$ has an abelian maximal subgroup $M_0/Z(R_0)$. Thus, $M_0' \leq Z(R_0)$. By Theorem 2.2 $M_0 \not\in \mathcal{R}$. But then, as a consequence of Theorem 4.1, $M_0' \not\leq Z(G)$, and a contradiction has been reached. Hence, $G \not\in \mathcal{Z}^*$.

Corollary 4.5.1: If $p > 3$ and if $G \in \mathcal{Z}^* \sim \mathcal{R}$, then G is a group described by (4.2).

The next theorem characterizes for $p = 3$ the groups in $\mathcal{Z}^* \sim \mathcal{R}$ with each Redei subgroup as a maximal subgroup. This result, along with Theorems 2.2, 3.1, and 4.3, gives a complete classification of the groups in $\mathcal{Z}$, for which each subgroup of index p^2 is an abelian subgroup.

THEOREM 4.6: If $p = 3$, if $G \in \mathcal{Z}^* \sim \mathcal{R}$, and if G has a Redei maximal subgroup, then $|G| = p^5$ and G is one of the following.

(1) $G = N_0 \langle c \mid c^{3^2}=1 \rangle$ where $N_0 = \langle a,b \mid a^{3^2}=b^3=d^3=1,\ d=[a,b] \rangle$ is a nonmetacyclic group, $[a,c] = b$, $c^3 = d^s$ for some $s \not\equiv 0 \pmod 3$ and $[b,c] = a^3c^{t3}$ for some t .

(2) $G = N_0 < c \mid c^{3^2}=1 >$ where $N_0 = < a,b \mid a^{3^2}=b^3=d^3=1,\ d=[a,b] >$ is a nonmetacyclic group, $[c,a] = b$, $c^3 = d^s$ for some $s \not\equiv 0 \pmod 3$ and $[c,b] = a^6c^{t3}$ for some t .

Proof: Since G has a Redei maximal subgroup, then each maximal subgroup of G is Redei. It follows from Proposition 1.4(c), that $|G| \geq p^5$. By Theorem 4.1 G' is elementary abelian of order p^3 , $Z(G) = \mho_1(G)$, and $(G: \Phi(G)) = p^2$. By Theorem 1.7(b), G is not a regular group. Since $G' \leq \Omega_1(R)$ for each Redei subgroup R , then it follows both that each maximal subgroup of G is nonmetacyclic and that $G' = \Omega_1(R)$. Moreover, since each element is in some maximal subgroup, then $\Omega_1(G) = G'$.

Let N_0 be a maximal subgroup of G . Denote G/N_0' by $\overline{G}$. $\overline{G}$ is a nonabelian group, which by Theorem 4.1 has precisely one abelian maximal subgroup $\overline{N}_0$. Since $N_0' \cap \mho_1(N_0) = 1$, $\overline{N}_0$ is metacyclic. Let $\overline{M}$ be any subgroup of $\overline{G}$ of index p^2 such that $\overline{M} \not\leq \overline{N}_0$. $\overline{M} \leq C_{\overline{G}}(\overline{M})$. If $\overline{M} < C_{\overline{G}}(\overline{M})$, then either $C_{\overline{G}}(\overline{M}) = \overline{G}$ or $C_{\overline{G}}(\overline{M})$ is a maximal subgroup of $\overline{G}$. If $C_{\overline{G}}(\overline{M}) = \overline{G}$, then $\overline{M} \leq Z(\overline{G})$ which is a contradiction. If $C_{\overline{G}}(\overline{M})$ is a maximal subgroup, then $C_{\overline{G}}(\overline{M})$ is abelian, which is a contradiction. Thus, $\overline{M}$ is a maximal abelian subgroup of $\overline{G}$. This implies that $C_{\overline{G}}(\overline{R}) \leq C_{\overline{G}}(\overline{M}) = \overline{M}$ when $\overline{M} \leq \overline{R}$. Hence, $\overline{G} \in \mathcal{Z}$.

By Lemma 3.2, $3^3 \leq |\Omega_1(\overline{G})| \leq 3^4$. Now, $\overline{G}' \leq \Omega_1(\overline{G})$ and $|G'| = p^2$. If there exists $\overline{x} \in \Phi(G) \sim \overline{G}'$ such that $\overline{x}^3 = 1$, then there exists an $x \in \Phi(G)$ such that $x^{3^2} = 1$ and $x^3 \in N_0'$. But $x \in N_0$, which implies that $N_0' \leq \mho_1(N_0)$, which is a contradiction. Thus, $\overline{G}' = \Omega_1(\Phi(\overline{G}))$ and $\Phi(\overline{G}) < \Phi(\overline{G})\Omega_1(\overline{G})$.

Case 1: $|\Omega_1(\overline{G})| = 3^3$. Then $\Phi(\overline{G})\Omega_1(\overline{G})$ is maximal in $\overline{G}$. Since $\overline{N}_0$ is metacyclic, Lemma 3.4 implies that $\overline{G}$ is the group of type (7) in Theorem 3.1, that is,

$$\overline{G} = \overline{N}_0 < \overline{c} \mid \overline{c}^3 = 1 > \text{ where } \overline{N}_0 = < \overline{a},\overline{b} \mid \overline{a}^{3^2} = \overline{b}^3 = [\overline{a},\overline{b}] = 1 > \ ,$$
$$[\overline{a},\overline{c}] = \overline{b} \text{ and } [\overline{b},\overline{c}] = \overline{a}^3 \ .$$

Moreover, $|G| = 3^5$.

Let a and b be elements of N_0 such that $aN_0' = \overline{a}$ and $bN_0' = \overline{b}$. Let $c \in G$ such that $cN_0' = \overline{c}$. Since $G' = \Omega_1(G) \leq N_0$, then $c^3 \neq 1$. Thus, $N_0' = < c^3 >$. Since $N_0' \cap \upsilon_1(N_0) = 1$, then $a^{3^2} = b^3 = 1$. Let $d = [a,b]$. Then $c^3 = d^s$ for some $s \not\equiv 0 (\text{mod } 3)$. Now $[a,c] = bd^u$ for some u . Let $b_1 = bd^u$. Then $\overline{b}_1 = \overline{b}$ and $[a,b_1] = d$. Since $[\overline{b},\overline{c}] = \overline{a}^3$, $[b_1,c] = a^3c^{t3}$ for some t .

Case 2: $\Omega_1(\overline{G}) = 3^4$. Then, since $|\Omega_1(\Phi(G))| = 3^2$, it follows that $\Omega_1(\overline{G}) = \overline{G}$. Since $\overline{N}_0$ is metacyclic, it follows from Lemma 3.3 that $\overline{G}$ is the group of type (3) in Theorem 3.1. Thus,

$$\overline{G} = \overline{N}_0 < \overline{c} \mid \overline{c}^3 = 1 > \ , \ \overline{N}_0 = < \overline{a},\overline{b} \mid \overline{a}^{3^2} = \overline{b}^3 = [\overline{a},\overline{b}] = 1 >$$
$$[\overline{c},\overline{a}] = \overline{b} \text{ and } [\overline{c},\overline{b}] = \overline{a}^6 \ .$$

Also, $|G| = 3^5$.

Let a and b be elements of N_0 such that $aN_0' = \overline{a}$ and $bN_0' = \overline{b}$. Let $c \in G \sim N_0$ such that $cN_0' = \overline{c}$. Then $c^3 \neq 1$ and $a^{3^2} = b^3 = 1$. Let $d = [a,b]$. Consequently, $c^3 = d^s$ for some $s \not\equiv 0(\text{mod } 3)$. Since $[c,a] = bd^u$ for some u , let $b_1 = bd^u$. Therefore, $\overline{b}_1 = b$ and $[a,b_1] = d$. Since $[\overline{c},\overline{b}_1] = \overline{a}^6$, then $[c,b_1] = a^6c^{t3}$

for some t .

Corollary 4.6.1: If $p = 3$ and if $G \in \mathcal{Z}^* \sim \mathcal{R}$, then each Redei subgroup has order 3^4 .

Proof: Let R be a Redei subgroup of G . There exists a subgroup R^* of G such that $(R^*: R) = 3$. By Theorem 4.6, $|R^*| = 3^5$. Hence, R has order 3^4 . By the definition of $\mathcal{Z}^*$, each Redei subgroup of G has order 3^4 .

A new characterization is obtained for non-Redei metacyclic groups.

THEOREM 4.7: A p-group G is a non-Redei metacyclic group if and only if G has an abelian subgroup of order p^4 and there exists an integer $i \geq 1$ such that each subgroup of G of index p^i is Redei.

Proof: Suppose that G is a non-Redei metacyclic group. In view of Corollary 2.4.1, G has an abelian subgroup of order p^4 and $G \in \mathcal{R}$. Thus $G \in \mathcal{Z}^*$.

Conversely, suppose that G is non-Redei, that G has an abelian subgroup of order p^4 and that $G \in \mathcal{Z}^*$. Then $|G| \geq p^6$. If $p > 3$, then by Theorem 4.4, $G \in \mathcal{R}$. If $p = 3$, then by Corollary 4.6.1 , $G \in \mathcal{R}$. Thus, by Corollary 2.4.1, G is metacyclic.

CHAPTER V

SUMMARY

Three classes of finite p-groups for $p > 2$ have been considered in this thesis, each class having been defined by conditions which are placed on the minimal nonabelian subgroups of a group. The role of the Redei subgroups has therefore been emphasized, and the resulting influence of these conditions on the structure of the group has been examined. Several results of this work are summarized here together with directions for future investigation with respect to Redei subgroups of a group.

The study of the two classes $\mathcal{R}$ and $\mathcal{Z}^*$ has led to two new characterizations for the class of metacyclic groups in terms of Redei subgroups. As mentioned previously, both extraspecial groups and groups with no noncyclic metacyclic characteristic subgroups can be described by means of their Redei subgroups. The question is raised as to whether or not other known classes of nonabelian groups can be defined in terms of their Redei subgroups. Such information could provide alternate areas for the application of these classes.

All nonabelian groups of class $\mathcal{Z}$, for which each subgroup of index p^2 is abelian and for which there is at most one abelian maximal subgroup, have been characterized in Theorems 2.2, 3.1, 4.3, and 4.6. The number of different types shows the extensive nature of class $\mathcal{Z}$. One should recall, however, that the formulation of this class was motivated by a property inherent to the maximal abelian subgroups of a group. It is not surprising, then, that the carry-over of the property to Redei subgroups would result in a collection of groups whose properties need to be more sharply defined. Classes $\mathcal{R}$ and $\mathcal{Z}^*$ are two steps in this general direction.

Theorem 1.1 points out that each non-Redei group has at least two Redei subgroups. It is natural then to ask about the total number of such subgroups and the corresponding conjugate classes of such groups. In this connection, since the metacyclic property is invariant, the question can be raised about the structure of groups for which each Redei subgroup is metacyclic. Information of this sort would extend the result of Blackburn (Theorem 1.10) which classifies groups for which each subgroup is metacyclic. A partial answer to this question is supplied in the present work. Theorems 3.1 and 4.1 indicate that there is only one type of minimal non-Redei group from class $\mathcal{Z} \sim \mathcal{R}$ which has each Redei subgroup as a metacyclic group. This same question could be posed with respect to the nonmetacyclic Redei subgroups.

In both class $\mathcal{R}$ and class $\mathcal{Z}$, it is seen that there must always exist a normal Redei subgroup. One then wonders if there must always be a normal Redei subgroup. If not, one could look for conditions which would ensure the existence of such normal subgroups. In this regard, there is also raised the question about the structure of a group which possesses characteristic Redei subgroups; two examples would be to consider $\Phi(G)$ and G' as Redei subgroups.

It should be noticed that Redei subgroups are nilpotent of class 2. Each Redei subgroup, then, is contained in a subgroup maximal with the property of having class 2. An investigation in this area would clarify the position of such subgroups in the theory of finite groups.

In particular, one could consider groups for which the Redei subgroups are maximal with the property of having class 2. The theorems here, dealing with the minimal non-Redei groups, provide some information in this respect.

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